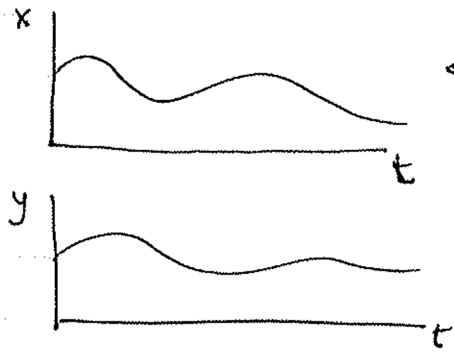
Nov 15, 2010 Part I

$\boxed{\text{Phase Plane diagrams.}}$

Given a lin. sys of ODEs, $\frac{d\vec{x}}{dt} = M\vec{x}$, ie $\frac{dx}{dt} = a_{11}x + a_{12}y$

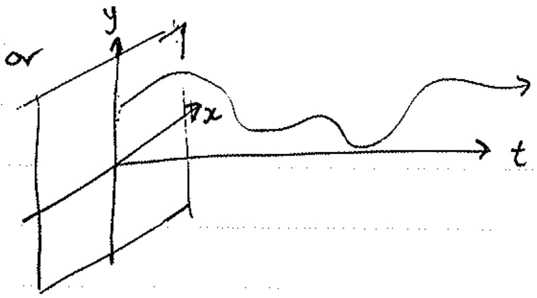
$\frac{dy}{dt} = a_{21}x + a_{22}y$

Here are Ways to represent the solns of 2x2 sys:

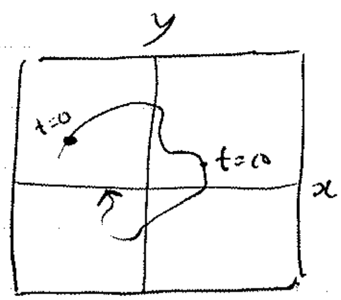


← one graph for each component, $x(t), y(t)$

or



or, as a combined curve in xyt space.
for each t there is a point $(t, x(t), y(t))$



Suppress t and plot $(x(t), y(t))$ in xy plane. This is a parametric curve with t as a parameter.
As t evolves, we trace out some curve. The coordinates of any point on this solution curve (also called trajectory) represent the values of x and y at time t.

This is called a phase plane diagram.

We can often draw this diagram without "solving" the full system, using an idea like direction field.

Remember: $\vec{x}(t) = \begin{pmatrix} x \\ y \end{pmatrix}$ is a point or position vector

$\frac{d\vec{x}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ is a "velocity" vector: it has to be tangent to any solution curve locally

ODE relates points to veloc. vectors, allowing us to sketch the phase plane diagram.

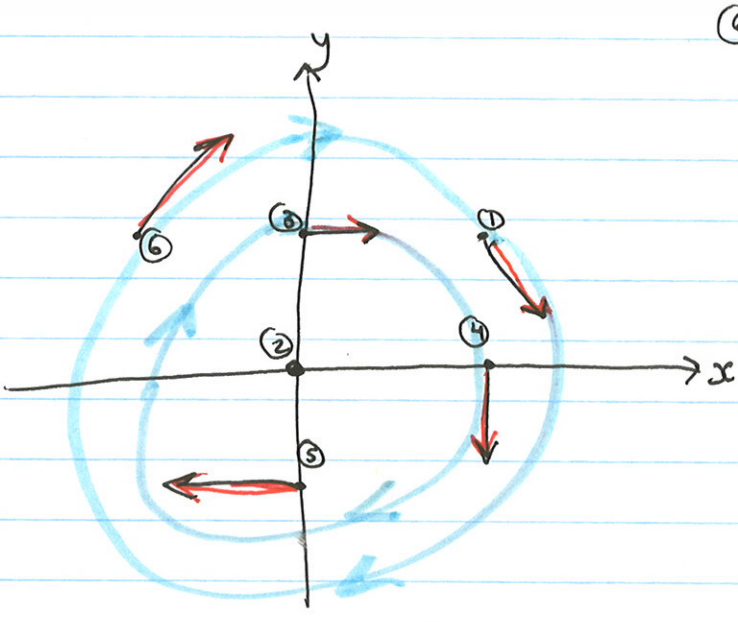
Idea is similar to that of "direction field" we saw in 1st order ODE.

Example:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -3x \end{cases}$$

can tabulate points and corresponding 'velocity vectors'

	point (x,y)	velocity vector (shown in red) $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (y, -3x)$
1	$(1, 1)$	$(1, -3)$
2	$(0, 0)$	$(0, 0)$
3	$(0, 1)$	$(1, 0)$
4	$(1, 0)$	$(0, -3)$
5	$(0, -1)$	$(-1, 0)$
6	$(-1, 1)$	$(1, 3)$



= sketch of soln curves *

We can continue filling in such a table and plot arrows. We will get a picture that looks something like this.

It looks like there is a "circulation" around the point $(0,0)$.
At $(0,0)$ there is no flow.
$(0,0)$ is a <u>fixed point</u> (also called <u>steady state</u> of this system.

(Sometimes we can make a shortcut, and obtain many arrows all at once) ← to be discussed later.

* but note that we need other information to see that these curves are <u>closed</u> loops (rather than spirals) : e.g. that eigenvalues are <u>pure imag</u>.

Nov 15, 2010 Part II

$\boxed{\text{Lin Sys ODEs with Complex Roots}}$

Systems of 1st order ODE's Cont'd : Complex roots
(≡ complex eigenvalues)

Suppose $\frac{d\vec{x}(t)}{dt} = M \cdot \vec{x}(t)$ is such that the matrix M

has complex eigenvalues, i.e. char. eqn,

$r^2 - \beta r + \gamma = 0$ has $\beta^2 - 4\gamma < 0$

so $r_{1,2} = \sigma \pm \mu i$ ← complex conjugate roots.

Note that $\sigma = Re(r_1, r_2)$, $\mu = Im(r_1, ')$

then, in terms of β and γ $\sigma = \beta/2$

$$\mu = \sqrt{|\beta^2 - 4\gamma|}\ /2$$
$$= \sqrt{4\gamma - \beta^2}/2$$

The eigenvectors of M are similarly found to be complex conjug.

$$\vec{v}_i = \vec{a} \pm \vec{b} i$$

$$\vec{v}_1 = \begin{pmatrix} \frac{r_1 - a_{22}}{a_{21}} \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} \frac{r_2 - a_{22}}{a_{21}} \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sigma - a_{22} + \mu i}{a_{21}} \\ 1 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} \frac{\sigma - a_{22} - \mu i}{a_{21}} \\ 1 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} \frac{\sigma - a_{22}}{a_{21}} \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{\mu}{a_{21}} \\ 0 \end{pmatrix} i \qquad \vec{v}_2 = \begin{pmatrix} \frac{\sigma - a_{22}}{a_{21}} \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{\mu}{a_{21}} \\ 0 \end{pmatrix} i$$

$$\vec{v}_1 = \vec{a} + \vec{b} i \qquad \vec{v}_2 = \vec{a} - \vec{b} i$$

$$\vec{a} = \begin{pmatrix} \frac{\sigma - a_{22}}{a_{21}} \\ 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} \frac{\mu}{a_{21}} \\ 0 \end{pmatrix}$$

Accordingly, when roots are complex, we get complex valued solutions of form

$$\tilde{\vec{x}}_1(t) = (\vec{a} + i\vec{b})\, e^{(\sigma + i\mu)t}$$

and $$\tilde{\vec{x}}_2(t) = (\vec{a} - i\vec{b})\, e^{(\sigma - i\mu)t}$$

We will rewrite each of these in the form of real and imaginary parts, and then form appropriate superposition(s) so as to get real valued solutions.

Recall: $$e^{(\sigma + i\mu)t} = e^{\sigma t}\left(\cos(\mu t) + i\sin(\mu t)\right), \quad i^2 = -1$$

Then

$$\begin{aligned}
\tilde{\vec{x}}_1 &= (\vec{a} + i\vec{b})\, e^{(\sigma + i\mu)t} \\
&= (\vec{a} + i\vec{b})\, e^{\sigma t}(\cos(\mu t) + i\sin(\mu t)) \\
&= \underbrace{\left[(\vec{a}\cos(\mu t) - \vec{b}\sin(\mu t))e^{\sigma t}\right]}_{\vec{u}(t)} + i\underbrace{\left[(\vec{b}\cos(\mu t) + \vec{a}\sin(\mu t))e^{\sigma t}\right]}_{\vec{v}(t)} \\
&= \vec{u}(t) + i\,\vec{v}(t)
\end{aligned}$$

$$\begin{aligned}
\tilde{\vec{x}}_2 &= (\vec{a} - i\vec{b})\, e^{(\sigma - i\mu)t} \\
&= (\vec{a} - i\vec{b})\, e^{\sigma t}\left(\cos(\mu t) - i\sin(\mu t)\right) \\
&= \underbrace{\left[(\vec{a}\cos(\mu t) - \vec{b}\sin(\mu t))e^{\sigma t}\right.}_{\vec{u}(t)} - i\left[(\vec{b}\cos(\mu t) + \vec{a}\sin(\mu t))e^{\sigma t}\right] \\
&= \vec{u}(t) - i\,\vec{v}(t)
\end{aligned}$$

I.e. we found that

$$\begin{aligned}
\tilde{\vec{x}}_1(t) &= \vec{u}(t) + i\,\vec{v}(t) \\
\tilde{\vec{x}}_2(t) &= \vec{u}(t) - i\,\vec{v}(t)
\end{aligned}$$

are two (lin. indep) complex conj solns.

(where $\vec{u}(t)$ and $\vec{v}(t)$ are shown above)

To constuct real valued solns, form the superpositions

$$\vec{X}_1(t) = \frac{\tilde{\vec{X}}_1(t) + \tilde{\vec{X}}_2(t)}{2} = \vec{u}(t)$$

$$\vec{X}_2(t) = \frac{\tilde{\vec{X}}_1(t) - \tilde{\vec{X}}_2(t)}{2i} = \vec{v}(t)$$

Then we can express the general solution as

$$\vec{x}(t) = c_1 \vec{u}(t) + c_2 \vec{v}(t)$$

where $\vec{u}(t)$, $\vec{v}(t)$ are as found on previous page.

Remarks ① We still need to discuss the fact that $\vec{u}(t)$, $\vec{v}(t)$ form a fundamental set of solns. (We'll use a generalization of the Wronskian) ← to be done in future.

② The constants c_1, c_2 will be obtained from initial cond's. See example;

Example: p409 # 5

$$\vec{x}'(t) = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x} \qquad a_{11} = 1,\ a_{12} = -1,\ a_{21} = 5,\ a_{22} = -3$$

$$\beta = a_{11} + a_{22} = -2$$

$$\gamma = a_{11}a_{22} - a_{12}a_{21} = -3 + 5 = 2$$

$$r_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = \underbrace{-1}_{\sigma=-1} \pm i \quad \leftarrow \text{eigenvalues} \qquad \mu = 1$$

eigenvectors:

for $r_1 = -1 + i$ $\quad \vec{v}_1 = \begin{pmatrix} \frac{-1+i+3}{5} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} + i \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} = \vec{a} + \vec{b} i$

for $r_2 = -1 - i$ $\quad \vec{v}_2 = \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} - i \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix} = \vec{a} - \vec{b} i$

$$\vec{a} = \begin{pmatrix} \frac{2}{5} \\ 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} \frac{1}{5} \\ 0 \end{pmatrix}$$

$$\vec{u}(t) = e^{\sigma t}\left(\vec{a}\cos(t) - \vec{b}\sin(t)\right) = e^{\overset{\sigma=-1}{\sigma t}} \begin{pmatrix} \frac{2}{5}\cos(t) - \frac{1}{5}\sin(t) \\ \cos(t) \end{pmatrix}$$

$$\vec{v}(t) = e^{\sigma t}\left(\vec{b}\cos(t) + \vec{a}\sin(t)\right) = e^{\sigma t} \begin{pmatrix} \frac{1}{5}\cos(t) + \frac{2}{5}\sin(t) \\ \sin(t) \end{pmatrix}$$

general soln:

$$\vec{x}(t) = c_1 \vec{u}(t) + c_2 \vec{v}(t)$$

$$= e^{-t}\left[c_1 \begin{pmatrix} 2\cos(t) - \sin(t) \\ 5\cos(t) \end{pmatrix} + c_2 \begin{pmatrix} \cos(t) + 2\sin(t) \\ 5\sin(t) \end{pmatrix} \right]$$

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{-t}\left[c_1(2\cos t - \sin(t)) + c_2(\cos(t) + 2\sin(t))\right] \\ e^{-t}\left[c_1 5\cos t + c_2 5\sin(t)\right] \end{pmatrix}$$

Suppose given I.C.'s $\vec{x}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Note : at $t=0$
$e^{\sigma t} = 1$
$\cos\mu t = 1$
$\sin\mu t = 0$

Can use this to find the constants c_1, c_2!

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^0 [c_1 \cdot (2-0) + c_2(1+0)] \\ e^0 [c_1 \cdot 5 + c_2 \cdot 0] \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ 5c_1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 1 &= 2c_1 + c_2 \\ -1 &= 5c_1 \end{aligned} \quad \Rightarrow \quad c_1 = -1/5 \qquad c_2 = \frac{7}{5}$$

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} -\frac{1}{5}(2\cos t - \sin t) + \frac{7}{5}(\cos t + 2\sin t) \\ -\frac{1}{5} \cdot 5\cos t + \frac{7}{5} \cdot 5 \sin t \end{pmatrix}$$

$$= e^{-t} \begin{pmatrix} \cos t + 3\sin t \\ -\cos t + 7\sin t \end{pmatrix}$$

Remark: Because there are several ways of forming eigenvectors, the solution could come out in various forms,

For example we could have chosen

for $r_1 = -1+i$ $\qquad \vec{v}_1 = \begin{pmatrix} 1 \\ \frac{r_1 - a_{11}}{a_{12}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-1+i-1}{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

for $r_2 = -1-i$ $\qquad \vec{V}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\vec{u}(t) = e^{-t}\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin t\right) = e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}$$

$$\vec{V}(t) = e^{-t}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin t\right) = e^{-t}\begin{pmatrix} \sin t \\ -\cos t + 2\sin t \end{pmatrix}$$

$$\vec{x}(t) = c_1 \vec{u}(t) + c_2 \vec{V}(t)$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t}\begin{pmatrix} c_1 \cos t + c_2 \sin t \\ 2c_1 \cos t + c_1 \sin t - c_2 \cos t + 2c_2 \sin t \end{pmatrix}$$

$\leftarrow$ this gen'l soln looks a lot different, but it is equivalent to our previous results

IC's: $\vec{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 \\ 2c_1 - c_2 \end{pmatrix}$

$c_1 = 1$

$-c_2 = -1 - 2c_1 = -3$

$c_2 = 3$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t}\begin{pmatrix} \cos t + 3\sin t \\ 2\cos t + \sin t - 3\cos t + 6\sin t \end{pmatrix} = e^{-t}\begin{pmatrix} \cos t + 3\sin t \\ -\cos t + 7\sin t \end{pmatrix}$$

$\uparrow$

Once we use the IC's, the soln obtained is the same, no matter what method for eigenvectors was used.