Examples for Sept 12

Integrating factor for 1st order ODEs (linear)

Example 1 $\quad \frac{dy}{dt} = a - by \qquad y(0) = 0 \qquad a, b > 0$ constants

rewrite as

$$\frac{dy}{dt} + by = a$$

Then integr. factor is $\mu(t) = e^{\int b\,dt} = e^{bt}$

$$\frac{d\,ye^{bt}}{dt} = ae^{bt}$$

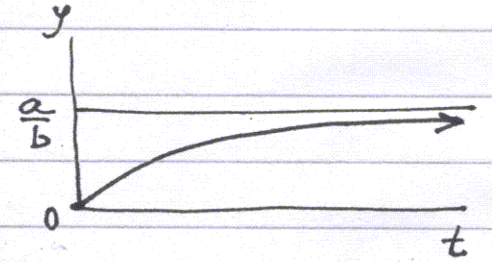
$$ye^{bt} = \int ae^{bt} + C' = \frac{a}{b}e^{bt} + C$$

$$y = e^{-bt}\left(\frac{a}{b}e^{bt} + C'\right) = \frac{a}{b} + C'e^{-bt}$$

initial cond^{n} $\Rightarrow y(0) = 0 = \frac{a}{b} + C' \quad \Rightarrow \quad C' = -\frac{a}{b}$

so $\quad y \equiv y(t) = \frac{a}{b} - \frac{a}{b}e^{-bt} = \frac{a}{b}(1 - e^{-bt})$

behaviour:



Remark: From any initial condition, the solns to this ODE approach a steady state (constant) value, i.e.

$$y \to \frac{a}{b} \quad \text{as} \quad t \to \infty$$

Example 2: Use an integrating factor to solve

$$ty' + 2y = t^2 - t + 1 \qquad y(1) = \frac{1}{2} \qquad t > 0$$

Soln: Rewrite: $y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$ *

$$\mu(t) = \exp\left[\int \frac{2}{t}\,dt\right] = \exp\left[2\ln t\right] = e^{\ln t^2}$$

So $\mu(t) = t^2$ ← multiply both sides of * by this

$$\underbrace{t^2\left[y' + \frac{2}{t}y\right]}_{\shortparallel} = t^2\left[t - 1 + \frac{1}{t}\right]$$

$$\frac{d}{dt}\left[y t^2\right] = t^3 - t^2 + t$$

integrate $\quad y t^2 = \int (t^3 - t^2 + t)\,dt + C$

$$= \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C$$

The general solution is $\Rightarrow \quad y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}$

Use the initial condition $\quad y(1) = \frac{1}{2} \Rightarrow$

$$\frac{1}{2} = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + \frac{C}{1}$$

$$\Rightarrow C = \frac{1}{12}$$

So soln is

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t}$$