Sols to HW5

Problem 3

(a) Compute transform of $f(t) = Kt$

$u = t \quad dv = e^{-st}dt$

$du = dt \quad v = \frac{e^{-st}}{-s}$

Soln: $\mathcal{L}\{f(t)\} = \int_0^\infty K t e^{-st} dt = K \int_0^\infty \underbrace{t}_{u} \underbrace{e^{-st} dt}_{dv}$

$\boxed{\int u\,dv = uv - \int v\,du}$

$$= K\left[t \frac{e^{-st}}{-s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt \right]$$

$$= \frac{K}{s}\left[-te^{-st}\Big|_0^\infty - \frac{1}{s} e^{-st}\Big|_0^\infty \right]$$

$$= \frac{K}{s}\left[(0) - \frac{1}{s}[0 - 1] \right] = \frac{K}{s^2} \qquad s > 0$$

where we have used: $\lim_{t\to\infty} -te^{-st} = 0$ $\quad$ $\lim_{t\to\infty} e^{-st} = 0$

(b) $f(t) = e^{-t/\tau}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-t/\tau} e^{-st} dt = \int_0^\infty e^{-(s+\frac{1}{\tau})t} dt$$

$$= \frac{e^{-(s+1/\tau)t}}{-(s+1/\tau)}\Big|_0^\infty = \underbrace{\lim_{t\to\infty} \frac{e^{-(s+1/\tau)t}}{-(s+1/\tau)}}_{0} - \frac{1}{-(s+1/\tau)}$$

$$= \frac{1}{s+1/\tau}$$

Problem 3(c)

Compute $\mathcal{L}\{\sin(at)\}$ and $\mathcal{L}\{\cos(at)\}$
(Note: you get both from same calculation).

$$L_1 \equiv \mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)\,dt = -\frac{e^{-st}\sin(at)}{s}\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\,dt$$

$\boxed{\text{(integrate by parts with } u = \sin(at) \quad dv = e^{-st}dt)}$

$$= \underset{\text{provided } s>0}{0} + \frac{a}{s}\underbrace{\int_0^\infty e^{-st}\cos(at)\,dt}_{\mathcal{L}\{\cos(at)\}}$$

$$L_2 \equiv \mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st}\cos(at)\,dt = -\frac{e^{-st}\cos(at)}{s}\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)\,dt$$

$$= \underset{\text{provided } s>0}{0} + \frac{1}{s} - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)\,dt$$

Get two equations:

$$\left.\begin{aligned} L_1 &= \frac{a}{s}L_2 \\ L_2 &= \frac{1}{s} - \frac{a}{s}L_1 \end{aligned}\right\} \quad L_1 = \frac{a}{s}\left(\frac{1}{s} - \frac{a}{s}L_1\right) \Rightarrow$$

$$L_1\left(1 + \frac{a^2}{s^2}\right) = \frac{a}{s^2} \qquad L_1 = \frac{a/s^2}{1 + a^2/s^2} = \frac{a}{s^2+a^2}$$

similarly $L_2 = \frac{s}{a}L_1 \Rightarrow L_2 = \frac{s}{s^2+a^2}$

$$\Rightarrow \quad \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \qquad s>0$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \qquad s>0$$

Let $f(t)$ be some (acceptable) function, and let

$M_n = \mathcal{L}\{f^{(n)}(t)\}$ be the Laplace Transform of its n-th derivative (assuming this is also an acceptable function)

(b) show that $M_n = s M_{n-1} - f^{(n-1)}(0)$

[Note about notation: $f^{(n)}(t)$ is not a power. It represents n'th deriv].

Soln:

$$M_n = \mathcal{L}\{f^{(n)}(t)\} = \int_0^\infty \underbrace{f^{(n)}(t)}_{dv} \underbrace{e^{-st}}_{u} \underbrace{dt}_{dv}$$

integrate by parts
$u = e^{-st}$ $du = -s e^{-st}$
$dv = f^{(n)}(t)dt$
$v = f^{(n-1)}(t)$

$$= e^{-st} f^{(n-1)}(t) \Big|_0^\infty - \int_0^\infty f^{(n-1)}(t)(-s e^{-st})dt$$

$$= \underbrace{\lim_{t\to\infty} e^{-st} f^{(n-1)}(t)}_{} - e^0 f^{(n-1)}(0) + s \int_0^\infty f^{(n-1)} e^{-st} dt$$

assumes $f^{(n-1)}$ is of exponential order

$$= 0 - f^{(n-1)}(0) + s \underbrace{\mathcal{L}\{f^{(n-1)}(t)\}}_{= M_{n-1}}$$

$$\therefore \boxed{M_n = s M_{n-1} - f^{(n-1)}(0)}$$

(c) Follows by repeated use of this link e.g.

$$M_n = s \overbrace{M_{n-1}} - f^{(n-1)}(0)$$

$$M_n = s\left(s M_{n-2} - f^{(n-2)}(0)\right) - f^{(n-1)}(0)$$

$$= s\left(s\left(M_{n-3} - f^{(n-3)}(0)\right) - f^{(n-2)}(0)\right) - f^{(n-1)}(0))$$

$$= \vdots$$

$$= s^n M_0 - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots f^{(n-1)}(0)$$

Problem 5

Find $\mathcal{L}\{f(t)\}$ for the function $f(t) = \begin{cases} 0 & 0 \le t < 3 \\ 2 & t \geq 3 \end{cases}$

Soln: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^3 e^{-st} \cdot 0\,dt + \int_3^\infty 2e^{-st}\,dt$

$$= 0 + 2\frac{e^{-st}}{-s}\Big|_3^\infty = -\frac{2}{s}\Big[\underbrace{\lim_{t\to\infty} e^{-st}}_{\substack{= 0 \\ \text{provided} \\ s>0}} - e^{-3t}\Big]$$

$$= -\frac{2}{s}\left(-e^{-3t}\right)$$

$$= \frac{2e^{-3t}}{s}$$

"Shifting (translation) Theorem:

Show that $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ where $F(s) = \mathcal{L}\{f(t)\}$

Soln: $\mathcal{L}\{e^{at}f(t)\} = \int_0^\infty e^{at} f(t) e^{-st}\,dt = \int_0^\infty f(t)\, e^{(a-s)t}\,dt$

$$= \int_0^\infty f(t) e^{-(s-a)t}\,dt = F(s-a)$$

Examples

Use this to compute

$$\mathcal{L}\{e^{5t} t^3\} \quad \text{and} \quad \mathcal{L}\{e^{-2t}\cos 4t\}$$

Soln

$\mathcal{L}\{t^3\}_{s\to s-5}$

$= \frac{3!}{s^4}\Big|_{s\to s-5} = \frac{6}{(s-5)^4}$

$\mathcal{L}\{\cos t\}_{s\to s+2}$

$= \frac{s}{s^2+16}\Big|_{s\to s+2}$

$= \frac{(s+2)}{(s+2)^2+16}$

(c) Find the inverse Laplace Transform for $F(s) = \frac{1}{6}\cdot\frac{1}{(s-1)^3}$

Soln: this looks like a shifted power fn.

recall $\mathcal{L}\{t^2\} = \frac{2!}{s^3}$ so $\mathcal{L}\{\frac{t^2}{2!}\} = \frac{1}{s^3}$

$\mathcal{L}\{\frac{t^2}{6\cdot 2!}\} = \frac{1}{6}\cdot\frac{1}{s^3}$ Now to get $(s-1)^3$ in denom,

we need $\mathcal{L}\{\frac{e^t t^2}{6\cdot 2!}\} = \frac{1}{6}\cdot\frac{1}{(s-1)^3} \leftarrow \frac{1}{12}\mathcal{L}\{e^t t^2\}$

Problem 6

(a) Solve using Laplace transform:

$$y''(t) - 3y'(t) + 2y = 12e^{4t} \qquad y(0)=1, \; y'(0)=0$$

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{12e^{4t}\}$$

$$\left(s^2F(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0}\right) - 3\left(sF(s) - \underbrace{y(0)}_{1}\right) + 2F(s) = 12\frac{1}{s-4}$$

$$s^2F(s) - s - 3sF(s) + 3 + 2F(s) = \frac{12}{s-4}$$

$$F(s)\left(s^2-3s+2\right) = \frac{12}{s-4} + s - 3$$

factors: $(s-2)(s-1)$

$$F(s) = \frac{1}{(s-2)(s-1)}(s-3) + \frac{12}{(s-4)(s-2)(s-1)}$$

$$= \frac{(s-3)(s-4)+12}{(s-4)(s-2)(s-1)}$$

$$= \frac{s^2-7s+24}{(s-4)(s-2)(s-1)}$$

partial fractions

$$= \frac{A}{(s-4)} + \frac{B}{(s-2)} + \frac{C}{(s-1)}$$

$$= \frac{A(s-2)(s-1) + B(s-4)(s-1) + C(s-4)(s-2)}{(s-4)(s-2)(s-1)}$$

Find A, B, C so that

$$s^2-7s+24 = A(s-2)(s-1) + B(s-4)(s-1) + C(s-4)(s-2)$$

let $s \to 1$ to see that $\quad 1-7+24 = C\cdot(1-4)(1-2)$

let $s \to 2$ $\quad 4-14+24 = B(2-4)(2-1)$

let $s \to 4$ $\quad 16-28+24 = A(4-2)(4-1)$

$\rightarrow$ $C = 6$, $B = -7$, $A = 2$

$$F(s) = \frac{2}{s-4} - \frac{7}{s-2} + \frac{6}{s-1}$$

$$y(t) = 2e^{4t} - 7e^{2t} + 6e^{t}$$

6(b)

Solve $y'' + y' - 2y = 4e^t + 1$ $\qquad y(0) = 1, \; y'(0) = 0$

$$\mathcal{L}\{y'' + y' - 2y\} = \mathcal{L}\{4e^t + 1\}$$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = 4\mathcal{L}\{e^t\} + \mathcal{L}\{1\}$$

$$\left[s^2F(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0}\right] + \left[sF(s) - \underbrace{y(0)}_{=1}\right] - 2F(s) = \frac{4}{s-1} + \frac{1}{s}$$

$$s^2F(s) - s + sF(s) - 1 - 2F(s) = \frac{4}{s-1} + \frac{1}{s}$$

$$F(s)\left(s^2 + s - 2\right) - (s+1) = \frac{4}{s-1} + \frac{1}{s}$$

$$F(s) = \left(\frac{1}{s^2 + s - 2}\right)\left[\frac{4}{s-1} + \frac{1}{s} + s + 1\right]$$

Now we want to find the inverse transform, i.e. get $y(t)$
But to do so, need to write $F(s)$ in a form where we can easily use look-up table of functions and their Laplace transform

Steps: (1) Factor denominator fully :

$$F(s) = \frac{1}{(s+2)(s-1)}\left[\frac{4}{s-1} + \frac{1}{s} + s + 1\right]$$

(2) Rewrite this in the partial fraction form

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)}$$

(3) Find A, B, C, D (constants) $\leftarrow$ (HW5)

(4) Look up the functions in table.

Method 1 (the hard way)

Solution : The common denominator for $F(s)$ is

$$F(s) = \frac{4s + (s-1) + (s+1)(s-1)s}{(s+2)(s-1)^2 s}$$

$$= \frac{s^3 + 4s - 1}{(s+2)(s-1)^2 s}$$

$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)}$$

these two expressions have to match for every s

$$= \frac{A(s+2)(s-1)^2 + B(s-1)^2 s + C s(s+2) + D s(s-1)(s+2)}{(s+2)(s-1)^2 s}$$

$$= \frac{A[s^3-3s+2] + B[s^3-2s^2+s] + C(s^2+2s) + D[s^3+s^2-2s]}{(s+2)(s-1)^2 s}$$

$$= \frac{s^3[A+B+D] + s^2[-2B+C+D] + s[-3A+B+2C-2D] + 2A}{(s+2)(s-1)^2 s}$$

match "like terms"

$$\left.\begin{array}{lll} s^3 & : & A+B+D=1 \\ s^2 & : & -2B+C+D=0 \\ s & : & -3A+B+2C-2D=4 \\ 1 & : & 2A=-1 \end{array}\right\} \rightarrow \text{solve for } A, B, C, D$$

Find

$A=-\frac{1}{2}$, $B=\frac{17}{18}$, $C=\frac{4}{3}$ $D=\frac{5}{9}$

This is really painful and results in a huge mess.

See better approach next page

Method 2: The numerators have to match for all s values. 'Plug in' useful s values that "knock out" most of the terms.

$$s^3 + 4s - 1 = A(s+2)(s-1)^2 + B(s-1)^2 s + Cs(s+2) + Ds(s-1)(s+2)$$

let $s \to 1$

then

$$1 + 4 - 1 = C \cdot 1 \cdot 3 \quad \Rightarrow \quad C = 4/3$$

let $s \to -2$

then

$$-8 - 8 - 1 = B(-2-1)^2(-2)$$

$$-17 = B \cdot 9 \cdot (-2) \quad \Rightarrow \quad B = \frac{17}{18}$$

let $s \to 0$

then

$$-1 = A(2)(-1)^2 = A = -\frac{1}{2}$$

In this case, we still did not find D, but we could do so using any one of the equations e.g. $A + B + D = 1 \Rightarrow$

$$D = 1 - A - B = 1 + \frac{1}{2} - \frac{17}{18} = \frac{5}{9}$$

$$\Rightarrow \quad A = -\frac{1}{2}, \quad B = \frac{17}{18}, \quad C = \frac{4}{3} \quad D = \frac{5}{9}$$

$$F(s) = -\frac{1}{2}\left(\frac{1}{s}\right) + \frac{17}{18}\left(\frac{1}{s+2}\right) + \frac{4}{3}\left(\frac{1}{(s-1)^2}\right) + \frac{5}{9}\left(\frac{1}{s-1}\right)$$

Now use table of Laplace transforms to invert each part

$$\mathcal{L}^{-1}(F(s)) = y(t) = -\frac{1}{2} \cdot 1 + \frac{17}{18} e^{-2t} + \frac{4}{3} t e^{t} + \frac{5}{9} e^{t}$$

6(c)

Solve $y''+4y'-5y=te^t \qquad y(0)=1 \qquad y'(0)=0$

$$\mathcal{L}\{y''\}+4\mathcal{L}\{y'\}-5\mathcal{L}\{y\}=\mathcal{L}\{te^t\}$$

table or integr.

$$\left(s^2F(s)-s\underbrace{y(0)}_{1}-\underbrace{y'(0)}_{0}\right)+4\left(sF(s)-\underbrace{y(0)}_{1}\right)-5F(s)=\frac{1}{(s-1)^2}$$

$$s^2F(s)-s+4sF(s)-4-5F(s)=\frac{1}{(s-1)^2}$$

$$(s^2+4s-5)F(s)=\frac{1}{(s-1)^2}+s+4$$

$(s+5)(s-1)$

$$F(s)=\frac{1}{(s+5)(s-1)}\left[\frac{1}{(s-1)^2}+s+4\right]$$

$$=\frac{1}{(s+5)(s-1)^3}+\frac{s+4}{(s+5)(s-1)}$$

$$=\frac{s^3+2s^2-7s+5}{(s+5)(s-1)^3}$$

Partial fractions

$$=\frac{A}{(s-1)^3}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)}+\frac{D}{(s+5)}$$

$$s^3+2s^2-7s+5=A(s+5)+B(s-1)(s+5)+C(s-1)^2(s+5)+D(s-1)^3 \quad (*)$$

let $s\to 1$:

$1+2-7+5=A(6) \qquad A=1/6$

$s\to -5$

$-125+50+35+5=D(-6)^3 \qquad D=35/216$

we still need to find C, B so expand *

$$s^3+2s^2-7s+5=(C+D)s^3+(B+3C-3D)s^2+(4B-9C+A+3D)s+(5A+5C-5B-D)$$

s^3 term: $C+D=1$ so

$C=1-D=181/216$ etc, etc

$$F(s)=\frac{1}{6}\cdot\frac{1}{(s-1)^3}-\frac{1}{36}\cdot\frac{1}{(s-1)^2}+\frac{181}{216}\cdot\frac{1}{(s-1)}+\frac{35}{216}\cdot\frac{1}{s+5}$$

Now use the shift Thm on the first two terms to get

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{6}\cdot e^{t}\frac{t^2}{2} - \frac{1}{36}e^{t}t + \frac{181}{216}e^{t} + \frac{35}{216}e^{-5t}$$