Some examples of 2nd order linear ODEs with complex and repeated roots

1 Oscillatory solutions

1.1 Pure oscillations, no damping

Let us consider the equation

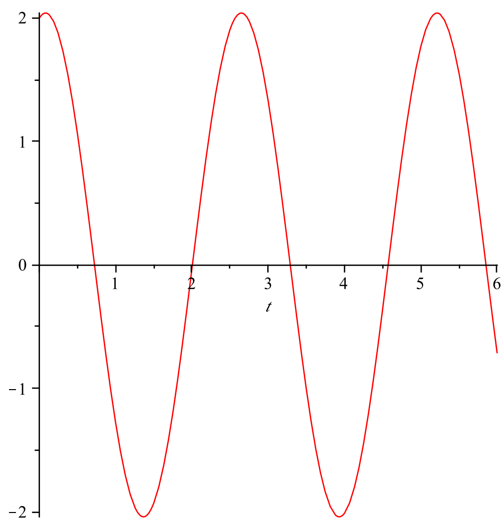
$$y'' + 6y = 0, \quad y(0) = 2, y'(0) = 1$$

The characteristic equation is $r^2 + 6 = 0$ which has complex conjugate roots $r = i\sqrt{6}$. The roots have zero real part and $\sqrt{6}$ imaginary part and thus solutions are of the form

$$y(t) = c_1 \cos(\sqrt{6}t) + c_2 \sin(\sqrt{6}t)$$

We can use the initial conditions to solve for the constants $2 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$ so $c_1 = 2$. Also, $y'(t) = -c_1\sqrt{6}\sin(\sqrt{6}t) + c_2\sqrt{6}\cos(\sqrt{6}t)$ so $1 = y'(0) = -c_1\sqrt{6}\sin(0) + c_2\sqrt{6}\cos(0) = c_2\sqrt{6}$ so $c_2 = \sqrt{6}/6$. The solution is thus

$$y(t) = 2\cos(\sqrt{6}t) + \frac{\sqrt{6}}{6}\sin(\sqrt{6}t)$$



plot of the periodic solution

1.2 With damping

Now let us see what happens when we add damping. We modify the problem slightly to include damping with coefficient 2, i.e. consider the initial value problem

$$y'' + 2y' + 6y = 0, \quad y(0) = 2, y'(0) = 1$$

Find the solution.

Begin by finding the characteristic equation: $r^2 + 2r + 6 = 0$ which has complex conjugate roots

$$r = -1 \pm i\sqrt{5}.$$

Solutions have the form

$$y(t) = c_1 \exp(\sigma t) \cos(\omega t) + c_2 \exp(\sigma t) \sin(\omega t)$$

where σ, ω are the real and imaginary parts of r. We have that $\sigma = -1$ and $\omega = \sqrt{5}$ so that

$$y(t) = c_1 \exp(-t) \cos(\sqrt{5}t) + c_2 \exp(-t) \sin(\sqrt{5}t)$$

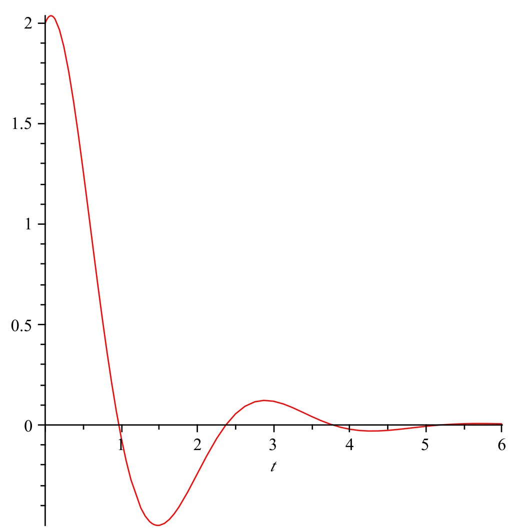
Use the initial conditions to find the constants c_1, c_2. We have a system of two equations for these as follows:

$$c_1 = 2, -c_1 + c_2\sqrt{5} = 1$$

We find that $c_1 = 2, c_2 = 3/5\sqrt{5}$ Thus

$$y(t) = 2\,\mathrm{e}^{-t} \cos\left(\sqrt{5}t\right) + 3/5\,\sqrt{5}\mathrm{e}^{-t} \sin\left(\sqrt{5}t\right)$$

Let's look at the plot



plot of the solution in the case with damping

2 Repeated roots

2.1 Example A

Now consider the equation

$$y'' + 2y' + 1y = 0, \quad y(0) = 1, y'(0) = 3$$

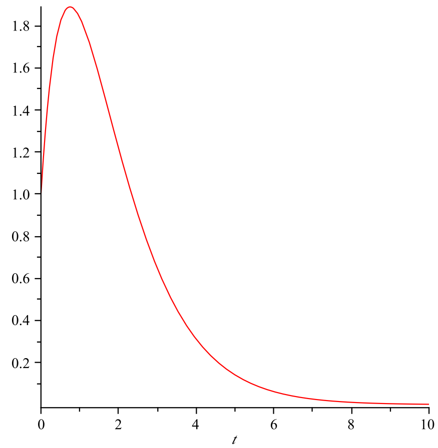
The characteristic equation is $r^2 + 2r + 1 = (r+1)(r+1) = 0$ which has roots $r = -1, -1$, i.e. repeated real roots. The solution is then in the form

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

The initial conditions lead to the equations for c_1, c_2: $c_1 = 1, -c_1 + c_2 = 3$ so $c_2 = 4$ and we have

$$y(t) = e^{-t} + 4te^{-t}.$$

The graph of this solution looks as follows:



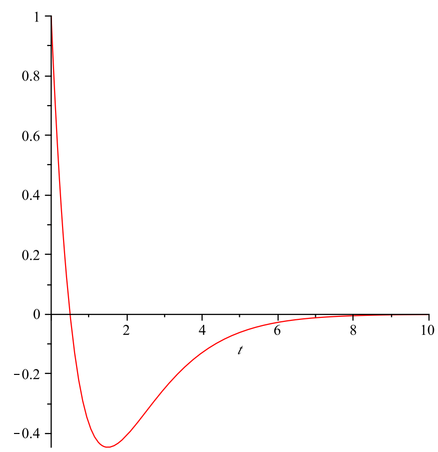
plot of a solution with repeated roots of the characteristic eqn. and $y(0) = 1, y'(0) = 3$

2.2 Example B

Now consider the same equation, but with a different pair of initial conditions, i.e.

$$y'' + 2y' + 1y = 0, \quad y(0) = 1, y'(0) = -3$$

We get the same general solution, but the two constants have to satisfy $c_1 = 1, -c_1 + c_2 = -3$ which leads to $c_1 = 1, c_2 = -2$ and the solution is then $y(t) = e^{-t} - 2te^{-t}$, which is shown in the next plot.



plot of a solution with repeated roots of the characteristic eqn. and $y(0) = 1, y'(0) = -3$