Dec 1, 2010 Last Class of M265 Review and exam strategy

The University of British Columbia

Final Examination - December 2007

Mathematics 265

Section 101

we'll use a past exam as focus of discussion

Closed book examination Time: 2.5 hours

Last Name: ____________ **First:** ____________ **Signature** ____________

Student Number ____________

Special Instructions:

- Be sure that this examination has 11 pages. Write your name on top of each page.

-

no formula sheet, sorry.

- In case of an exam disruption such as a fire alarm, leave the exam papers in the room and exit quickly and quietly to a pre-designated location.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

1		10
2		10
3		20
4		15
5		15
6		15
7		15
Total		100

[10] **1.** Find all solutions of $y' - 2xy^2 = 0$. ← What kind of problem is this? First or Second order? Linear or Nonlinear ?

indep variable is x (we used $\underline{t}$ a lot)

one derivative only $\Leftrightarrow$ first order

y^2 term $\Leftrightarrow$ Nonlinear !!

Only hope ~~i$~~ if we can separate variables

$$\frac{dy}{dx} = 2xy^2$$

$$\frac{dy}{y^2} = 2x\,dx$$

yes! it'll work

$$\int y^{-2}\,dy = \int 2x\,dx + C_1$$

← Don't forget this arbitrary integration constant at this step (Not later on!!)

integrate carefully! (power rule $\int y^n dy = \frac{y^{n+1}}{n}$

$$-y^{-1} = \cancel{2}\frac{x^2}{\cancel{2}} + C_1$$

$$-\frac{1}{y} = x^2 + C_1$$

$$y = -\frac{1}{x^2 + C_1}$$

initial cond's

[10] **2.** Solve the initial value problem $xy' = x^3 - 2y$, $\boxed{y(1) = 0.}$

What about this one?
First / Second order?
Lin / Non lin?

- Note $y = y(x)$ dependent variable appears only in y' or $2y$ terms,
- problem is a linear ODE (despite x^3, since x is indep. var.)

- Non constant coeffs but Linear $\rightarrow$ try integrating factor
  - first put in standard form! $\Leftarrow$ $y' + b(x)y = f(x)$

$$y' + \frac{2}{x}y = x^2 \qquad \text{so: } \boxed{b(x) = \frac{2}{x}}$$

integrating factor:

$$\mu(x) = \exp \int b(x)\,dx = \exp \int \frac{2}{x}\,dx = \underbrace{\exp(2 \ln x)}$$

don't forget this exp!

Simplify!! otherwise have a mess!

$$\mu(x) = \exp \ln x^2 = e^{\ln x^2} = x^2$$

e and $\ln$ are inverse functions. So expression simplifies!

$$\mu(x)\left[y' + \frac{2}{x}y\right] = \mu(x)\,x^2$$

$$\frac{d}{dx}\left[\mu(x)\,y\right] = \mu(x)\,x^2$$

$$\frac{d}{dx}\left[x^2 y\right] = x^2 \cdot x^2 = x^4$$

$$x^2 y = \int x^4\,dx + C_1$$

don't forget this! constant!

the rest is "easy"

Use $y(1) = 0$ to find C_1 at the end.

[20] **3.** Consider the initial value problem

2nd order lin ODE, constant coefs.

$$y'' + ay' + by = 0, \qquad y(0) = 3,\ y'(0) = 5.$$

The differential equation has as a fundamental set of solutions $\{y_1(t), y_2(t)\}$, where $y_1(t) = e^{-t}$. The Wronskian of y_1 and y_2 is $W(t) = 4e^{2t}$.

(a) Solve for $y_2(t)$.

(b) Determine the values of the constants a and b.

(c) Solve the initial value problem.

Oh wow! This seems totally from outer space. No clue what to do here... But lets take a stab at it.

(a) They tell me one soln is $y_1(t) = e^{-t}$.

I don't know $y_2(t)$, but I'm going to expect it to be something like

$$y_2(t) = e^{r_2 t}$$ (because of 2nd order lin. ODE above)

Wronskian... hmm, let's see now... isn't that

$$W = \det\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t)\end{bmatrix}$$ .. Let's put in the forms of y_1 and y_2

then

$$W = \det\begin{bmatrix} e^{-t} & e^{r_2 t} \\ -e^{-t} & r_2 e^{r_2 t}\end{bmatrix}.$$ I can calculate it! (in terms of some unknown root r_2)

$$= e^{-t} r_2 e^{r_2 t} - (-e^{-t} e^{r_2 t})$$

$$= e^{-t}e^{r_2 t}(r_2+1) = e^{-t+r_2 t}(r_2+1) = e^{t(r_2-1)}(r_2+1)$$

Oh, OK! $\quad 4e^{2t} = (r_2+1)e^{(r_2-1)t}$ so $r_2 + 1 = 4$ and $r_2 - 1 = 2$ $\quad r_2 = 3$ ✓

Thus $\quad y_2(t) = e^{3t}$

(b) From (a) I know that $r=-1, r=3$ gotta be roots of char. eqn, i.e. $(r+1)(r-3) = 0$ is char. eqn.!!

$r^2 - 2r - 3 = 0$ so $a = -2$, $b = -3$

(c) $y_1(t) = e^{-t}$, $y_2(t) = e^{3t}$ $\Rightarrow$ gen'l soln $y(t) = c_1 e^{-t} + c_2 e^{3t}$

I can use I.C.'s to find c_1, c_2.

[15] 4. The homogeneous differential equation

$$t^2y'' - 2ty' + 2y = 0,$$

defined over the open interval $0.5 < t < 2$, has a non-trivial solution $y_1 = t^2$.

(a) Use reduction of order to find a second solution y_2.

(b) Show that y_1 and y_2 form a fundamental set of solutions.

(c) Find the particular solution that satisfies the initial conditions $y(1) = 3$ and $y'(1) = 4$.

We did not include this topic in our discussions this term.

(Luckily, since the instructor is reasonable, we wouldn't be expected to solve this problem.)

Extra space (if needed)

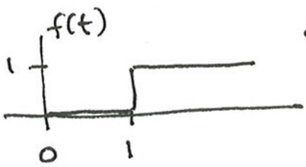
nonhomogeneous

[15] **5.** Solve the initial value problem

$$y'' + 2y' + 5y = f(t), \qquad y(0) = 1,\ y'(0) = -1,$$

the forcing function ("input") is discontin. I'd be nuts to try anything other than Laplace Tr !!

where



sketch this ←

$$f(t) = \begin{cases} 0 & \text{if } t < 1; \\ 1 & \text{if } 1 \le t. \end{cases}$$

This is a unit step function $f = u_1(t)$ or $H(t-1)$ (heaviside step fn)

LAPLACE TR: $\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{u_1(t)\}$ ← Table (supplied)

Careful here! the prof. made some errors! don't make same mistakes!)

$$\left[s^2F(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{-1}\right] + 2\left[sF(s) - \underbrace{y(0)}_{1}\right] + 5F(s) = \frac{e^{-s}}{s}$$

$$\left[s^2 + 2s + 5\right]F(s) - s + 1 - 2 = \frac{e^{-s}}{s}$$

$$F(s) = \frac{s+1}{s^2+2s+5} + \frac{e^{-s}}{s(s^2+2s+5)}$$

Inverting (the hardest part)
Will denom. factor? → if so, do it ✗ (won't work in this case)
→ if not, use "complete the square" ✓

$$\rightarrow \quad s^2 + 2s + 5 = (s+1)^2 + 4$$

$$\text{so} \quad F(s) = \frac{(s+1)}{(s+1)^2+4} + \frac{e^{-s}}{s[(s+1)^2+4]}$$

Need partial fractions to break this up

looks like a shifted entry for cosine

$$= \left.\frac{s}{s^2+4}\right|_{s \to s+1} + \left(\frac{A}{s} + \frac{Bs+C}{(s+1)^2+4}\right)e^{-s}$$

Find A, B, C

$$= \cos(2t)\cdot e^{t} + A\,u_1(t) + \cdots\cdots$$

(a bit more work to finish this off)

Extra space (if needed)

[15] **6.** Solve the initial value problem

$$x_1' = x_1 - x_2$$
$$x_2' = 5x_1 - 3x_2$$

sys of 2 1st order linear ODES. We can write it in matrix form

with $\underbrace{x_1(0) = 1,\ x_2(0) = 3}_{\text{IC's}}$. Describe the behaviour of the solution as $t \to \infty$.

$$\frac{d\vec{X}}{dt} = M\vec{x} \qquad M = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$$

<u>eigenvalues</u>
$\det(M - rI) = 0$

$$\begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix} = (1-r)(-3-r) + 5 = r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

case of COMPLEX ROOTS! $\quad r = \sigma \pm \mu i \quad \sigma = -1 \quad \mu = 1$

neg. real part $\Rightarrow$ decaying oscillations

<u>eigenvectors</u>
$(M - rI)\vec{v} = 0$

$$\begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1-r)v_1 - v_2 = 0$
take, e.g. $v_1 = 1 \Rightarrow v_2 = 1 - r$

$$\vec{v} = \begin{pmatrix} 1 \\ 1-r \end{pmatrix}$$

so $\quad r_{1,2} = -1 \pm i \qquad \vec{V}_{1,2} = \begin{pmatrix} 1 \\ 1 + 1 \mp i \end{pmatrix}$

$$r_1 = -1 + i \qquad \vec{V}_1 = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{a}} + i \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\vec{b}}$$

must find the real and imag. parts !! to get vectors $\vec{a}, \vec{b}$

Soln:
$$\vec{X} = \vec{V}e^{rt} = (\vec{a} + \vec{b}i)e^{\sigma t}(\cos\mu t + i\sin\mu t)$$
$$= \underbrace{e^{\sigma t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t)}_{\text{"}\vec{u}(t)\text{"}} + \underbrace{e^{\sigma t}(\vec{a}\sin\mu t + \vec{b}\cos\mu t)\, i}_{\text{"}v(t)\text{"}\cdot i}$$

real valued
Genl soln
$$\vec{X}(t) = c_1\vec{u}(t) + c_2\vec{v}(t)$$

put in $\sigma = -1$, $\mu = 1$, $\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

use ICs to solve for c_1, c_2

a bit more work needed to complete these rote steps.

Extra space (if needed)

[15] 7. Find a fundamental matrix for the system of equations

eigenvalues: (roots of char eqn:) det $(M - rI) = 0$

this just means a matrix whose columns are the two solns.

$$x' = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} x.$$

det $\begin{pmatrix} 1-r & -2 \\ 2 & 5-r \end{pmatrix} = 0$

$0 = (1-r)(5-r) + 4 = r^2 - 6r + 9 = (r-3)^2$

$\boxed{\text{Repeated roots:}}$ $r = 3$

eigenvectors : $(M - rI) \cdot \vec{v} = 0$ $\quad \begin{pmatrix} 1-r & -2 \\ 2 & 5-r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$r = 3 \Rightarrow \quad \begin{pmatrix} -2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad -2v_1 - 2v_2 = 0$

pick, e.g., $v_1 = 1$ then $v_2 = -1$ so $\vec{V} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

one soln is $\vec{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$ ← only one eigenvector

Make up 2nd soln $\vec{X}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \underline{\underline{t}} e^{3t} + \begin{pmatrix} a \\ b \end{pmatrix} e^{3t}$

... Some work req'd here to find $\begin{pmatrix} a \\ b \end{pmatrix}$ so that $\vec{X}_2$ is a soln.

Fundam matrix is then

$$\left[\vec{X}_1 \ \vec{X}_2 \right] = \begin{bmatrix} e^{3t} & (t+a)e^{3t} \\ -e^{3t} & (-t+b)e^{3t} \end{bmatrix}$$

(where a, b found by subst. $\vec{X}_2$ into ODE system and finding set of 2 eqns for a and b.)