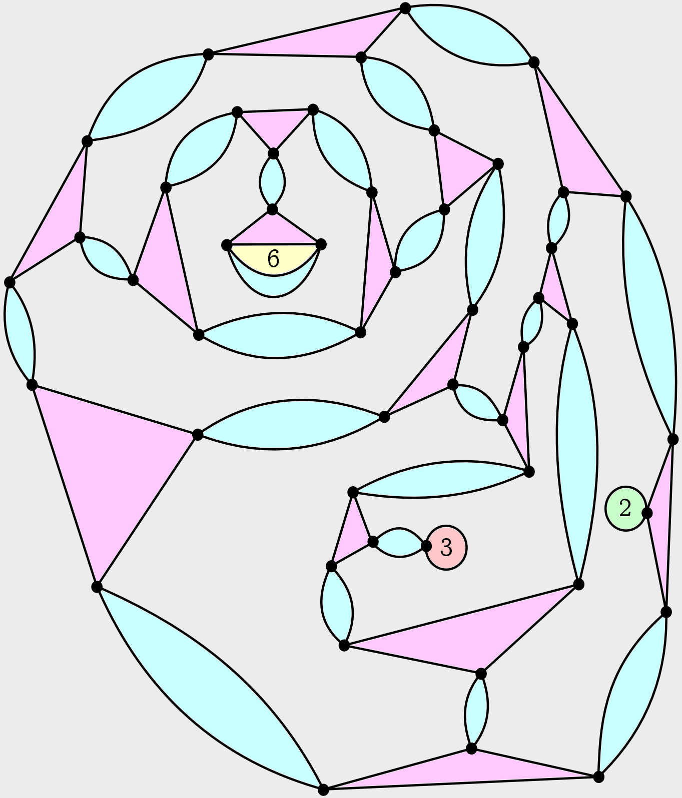
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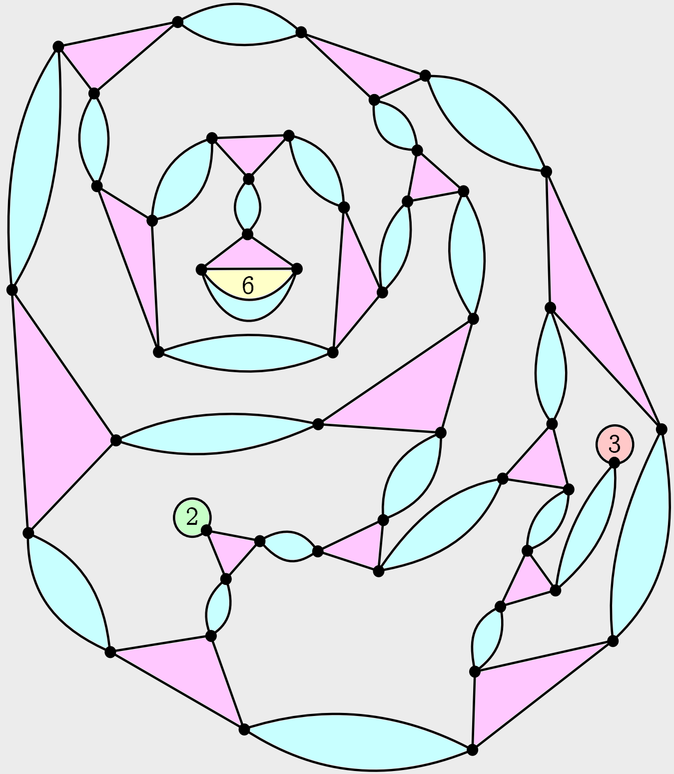
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Orbifold covers

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July 26, 2021. For a fixed positive integer n, how many integers x, y satisfy $n = x^2 + xy + y^2$? This question has a fun geometric interpretation; consider an infinite honeycomb, subdivided into equilateral triangles tiling the plane. Algebraically, the vertices of this triangular tiling are elements of the polynomial ring $\mathbb{Z}[\alpha] \subset \mathbb{C}$ where $\alpha = \frac{1+\sqrt{3}i}{2}$, and the norm of $x + y\alpha \in \mathbb{Z}[\alpha]$ is $x^2 + xy + y^2$, so that solutions to this equation correspond to vertices of the triangular tiling of distance $\sqrt{n}$ from the origin. It is possible to classify such points algebraically, but here is a topological approach.

Take a vertex w of the triangular tiling with a distance of $\sqrt{n}$ from the origin. The tiling may be expanded and rotated so that a vertex adjacent to the origin is carried to w; algebraically this is a self-embedding $f_w\colon \mathbb{Z}[\alpha] \to \mathbb{Z}[\alpha]$ given by multiplication by w (see [1] for more on sub-lattices of the hexagonal lattice).

Now quotient the plane by all symmetries of the triangular tiling, collapsing it into a sphere $S^2(2, 3, 6)$ with a cone point of order 6 coming from the vertices, a cone point of order 3 coming from the center of the triangles, and a cone point of order 2 coming from the midpoints of the edges. The map f_w then descends to a self-covering $S^2(2, 3, 6) \to S^2(2, 3, 6)$ of degree n.

The postcard shows two such coverings of degree 49 corresponding to $x = 5, y = 3$ (left) and $x = 7, y = 0$ (right). In order to draw these, start with a pair of tangent circles on $S^2(2, 3, 6)$, one containing the order 2 cone point and one containing the order 3 cone point. Lifting these to the 49-fold covers gives the planar graphs shown. The bigons correspond to edges, the dodecagons correspond to vertices, and the triangles correspond to faces of the triangular tiling.

As an application, these orbifold coverings produce coverings of Seifert fibered 3-manifolds with that base orbifold; see [2]. For example, [3] shows that 24/5 surgery on the trefoil $T(2, 3)$ is Seifert fibered over $S^2(2, 3, 6)$ so that $S^3_{24/5}(T(2, 3))$ is 49-fold covered by $S^3_{1176/197}(T(2, 3))$ in two different ways. Generalizing this gives:

Theorem. *There are pairs of Dehn surgeries on the trefoil with arbitrarily many distinct covering maps between them.*

[1] Bernstein, Sloane & Wright. On sublattices of the hexagonal lattice. *Discrete Math*. 1997.
[2] Boyle. On the virtual cosmetic crossing conjecture. *New York J. Math*. 2018.
[3] Moser. Elementary surgery along a torus knot. *Pacific J. Math*. 1971.

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