Math 120 Practice problems

1. Use the $\epsilon - \delta$ definition of the limit to prove that

$$\lim_{x \to 1} x^3 + 2x + 1 = 4.$$

2. Use the $\epsilon - \delta$ definition of the limit to prove that

$$\lim_{x \to 0} e^{(x^2)} = 1.$$

3. Define

$$f(x) = \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Prove that for every $x \in \mathbb{R}$, f is *not* differentiable at x.

4. a. Compute the zeroth, first, and second term (plus error term) of the Taylor expansion of $\arcsin(x)$ around the point $c = 0$; i.e. when you apply Taylor's theorem, you should have $n = 3$.

b. Use part a to prove that $\arcsin(1/2) \leq \frac{1}{2} + \frac{1}{9\sqrt{3}}$.

5. Evaluate

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^4}.$$

Justify any steps that you make.

6. Let f be a function that is increasing on $[0, 1]$. Prove that for every $c \in (0, 1]$, $\lim_{x \to c^-} f(x)$ exists (and is a real number).

7. Let $f(x) = \sin(\log(-x))$ and let $S = \{x \in D(f) \colon f(x) = 0\}$. Does S have a least upper bound? If so, find it (and prove that your answer is correct). If not, prove that S does not have a least upper bound.

8. Let $f(x)$ be a function that is differentiable at every point $x \in \mathbb{R}$ and whose range is a subset of $(0, \infty)$. Prove that the curves $y = f(x)$ and $y = \log(f(x))$ have horizontal tangent lines at exactly the same values of x.

9. Let $f(x)$ and $g(x)$ be functions that are two-times differentiable for all $x \in \mathbb{R}$, and suppose $f(x)g(x) = 1$ for all $x \in \mathbb{R}$. Show that the following relation holds for all $x \in \mathbb{R}$ where the denominators are non-zero

$$\frac{f''(x)}{f'(x)} - 2\frac{f'(x)}{f(x)} - \frac{g''(x)}{g'(x)} = 0.$$

10. Prove that if F_1 and F_2 are anti-derivatives for the function f on the interval (a, b), then $F_1(x) - F_2(x)$ is a constant.

Differential Equations

Since we haven't had any homework problems on differential equations, here are a bunch to help you study.

11. Carbon dating is a method used to determine how long ago an organism died. There are three isotopes of carbon that are common on earth: carbon-12 and carbon-13 (which have half-lives of ∞ years), and carbon-14 (which has a half-life of 5730 years). Most of the carbon on earth is carbon-12, but some is carbon-14 (we'll forget about carbon-13 for this problem; it might as well be carbon-12). When an organism is alive, it absorbs carbon-12, carbon-13, and carbon-14 equally easily, so the ratio of carbon-12 to carbon-14 is the same inside all living organisms. For simplicity, we will assume that this ratio has also been the same at all points in the past. Once an organism dies, however, it stops absorbing new carbon, so the carbon-14 inside the organism decays into carbon-12. If a fossil has %3 as much carbon-14 as a modern living organism does, how old is the fossil?

12. Solve the initial value problem

$$\begin{aligned} &y'(x) + xy(x) = 0, \\ &y(0) = 1. \end{aligned}$$

13. Solve the initial value problem

$$\begin{aligned} &y'(x) = \sin(x), \\ &y(0) = 17. \end{aligned}$$

14. Solve the initial value problem

$$\begin{cases} y'(x) - 3y(x) = e^{2x}, \\ y(0) = 0. \end{cases}$$

What is the largest interval containing 0 in which the solution is defined?

15. Solve the initial value problem

$$\begin{cases} y'(x) - xy(x) = x^3, \\ y(0) = 0. \end{cases}$$

What is the largest interval containing 0 in which the solution is defined?

16. Consider the initial value problem

$$\begin{cases} y'(x) = y(x)^{1/3}, \\ y(0) = 0. \end{cases}$$

Prove that $y_1(x) = 0$, $y_2(x) = \left(\frac{2}{3}x\right)^{3/2}$ and $y_3(x) = -\left(\frac{2}{3}x\right)^{3/2}$ are all solutions to the initial value problem. This should surprise you.

Remark This example shows that the initial value problem $y'(x) = f(x, y(x)),\ y(x_0) = C_0$ might not have a unique solution if the function f is "badly behaved". In this example, the function $f(x, y) = y^{1/3}$ is not differentiable with respect to y.