18.100B Problem Set 8

Due Thursday April 14 by 2:30pm. When solving homework problems, you may cite the theorems proved in class. However, you may not cite theorems from Apostol that were not discussed/proved in class unless noted in the problem description.

Part A

1 (3 points). In class, we defined what it means for a set $S \subset \mathbb{R}$ to have measure 0 (to be pedantic, we actually defined what it means for a set to have *Lebesgue* measure 0; there are other measures as well). Give an example of a set $S \subset (0,1)$ that has measure 0, and a function $f\colon S \to \mathbb{R}$, so that $f(S)$ does not have measure 0.

2 (5 points). We say that a function $f\colon (a,b) \to \mathbb{R}$ is *absolutely continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ with the following property: let $(a_1,b_1),(a_2,b_2)\ldots,(a_k,b_k)$ be a finite set of disjoint subintervals of (a,b), such that $\sum_{i=1}^{k}(b_i - a_i) < \delta$. Then $\sum_{i=1}^{k}|f(b_i) - f(a_i)| < \epsilon$. Absolute continuity is stronger that uniform continuity, i.e. every absolutely continuous function is uniformly continuous, but the converse need not hold.

Prove that if $f : (a,b) \to \mathbb{R}$ is absolutely continuous, and if $S \subset (a,b)$ has measure 0, then $f(S)$ has measure 0.

(I'm sure if you look hard enough online, you can find a proof to copy. But don't—figure it out for yourself!)

3 (5 points). Prove that if $f : (a,b) \to \mathbb{R}$ is absolutely continuous, then f has bounded variation on (a,b) (note: in class we defined what it means for a function $f : [a,b] \to \mathbb{R}$ to have bounded variation. If f is instead defined on an open interval (a,b), then we say f has bounded variation if there exists a number A so that for all sets of numbers $\{x_0,\ldots,x_k\}$ with $a < x_0 < x_1 < \ldots < x_k < b$, we have $\sum_{i=1}^{k}|f(x_i) - f(x_{i-1})| < A$. Equivalently, for every closed interval $[c,d] \subset (a,b)$, the restriction of f to the closed interval $[c,d]$ has variation at most A.)

4 (3 points). Is the converse to problem 3 true? Prove it or provide a counter-example.

Part B

Let (M,d) be a metric space and let $S \subset M$. A point $x \in S$ is called *isolated* if there is a $r > 0$ so that $B(x,r) \cap S = \{x\}$. A set S is called *perfect* if it is closed and has no isolated points.

A set S is said to be *nowhere dense* (in M) if $\operatorname{int}\bar{S} = \emptyset$ (recall the definition of int and $\bar{S}$ from HW 2 and 3, respectively).

5 (5 points). Recall the definition of the Cantor set, discussed in lecture: define $C_0 = [0,1]$; this is a union of $2^0 = 1$ closed intervals, each of length $3^0 = 1$. For each $i = 1,2,\ldots$, define C_i to be the union of 2^i closed intervals, each of length 3^{-i} obtained by removing the middle third of each of the intervals from C_{i-1}. Define $\mathcal{C} = \bigcap_{i=0}^{\infty} C_i$.

Prove that $\mathcal{C}$ is closed. Then prove that it is perfect and nowhere dense.

6. In this problem, we will construct what is known as a "fat" Cantor set. Define $C_0^f = [0,1]$; this is a union of $2^0 = 1$ closed intervals. For each $i = 1, \ldots,$ define C_i^f to be the union of 2^i closed intervals obtained by removing an open interval of length 2^{-2i} from the middle of each of the 2^{i-1} intervals from C_{i-1}^f. Define $\mathcal{C}^f = \bigcap_{i=0}^{\infty} C_i^f$.

(A 5 points) Prove that $\mathcal{C}^f$ is a perfect, nowhere dense set.

(B 5 points) Prove that $\mathcal{C}^f$ does *not* have measure 0; this is why it is called a fat Cantor set.