LIOUVILLE THEOREMS FOR STABLE SOLUTIONS OF BIHARMONIC PROBLEM

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ABSTRACT. We prove some Liouville type results for stable solutions to the biharmonic problem $\Delta^2 u = u^q$, $u > 0$ in $\mathbb{R}^n$ where $1 < q < \infty$. For example, for $n \geq 5$, we show that there are no stable classical solution in $\mathbb{R}^n$ when $\frac{n+4}{n-4} < q \leq \left(\frac{n-8}{n}\right)_+^{-1}$.

1. INTRODUCTION

Consider classical solutions to the following biharmonic equation

$$\Delta^2 u = u^q, \quad u > 0 \text{ in } \mathbb{R}^n \tag{1.1}$$

where $n \geq 5$ and $q > 1$. Define

$$\Lambda_u(\phi) := \int_{\mathbb{R}^n} |\Delta\phi|^2 dx - q\int_{\mathbb{R}^n} u^{q-1}\phi^2 dx, \quad \forall\, \phi \in H^2(\mathbb{R}^n). \tag{1.2}$$

The Morse index of a classical solution to (1.1), $\mathrm{ind}(u)$ is defined as the maximal dimension of all subspaces of $E_{\mathbb{R}^n} := H^2(\mathbb{R}^n)$ such that $\Lambda_u(\phi) < 0$ in $E_{\mathbb{R}^n} \setminus \{0\}$. Similarly, we consider also classical solutions $\Delta^2 u = u^q$ on a proper domain $\Omega \neq \mathbb{R}^n$, and define its Morse index with

$$\Lambda_{u,\Omega}(\phi) := \int_{\Omega} |\Delta\phi|^2 dx - q\int_{\Omega} u^{q-1}\phi^2 dx, \quad \forall\, \phi \in E_\Omega := H_0^2(\Omega). \tag{1.3}$$

A solution u is said stable if $\Lambda_u(\phi) \geq 0$ for any test function $\phi \in E_\Omega$. Clearly, u is stable if and only if its Morse index is equal to zero.

In this paper, we prove the following classification results.

Theorem 1.1. *Let $n \geq 5$.*

(i) *For $n \leq 8$ and any $1 < q < \infty$, the equation (1.1) has no stable solution.*
(ii) *For $n \geq 9$, there exists $\epsilon_n > 0$ such that for any $1 < q < \frac{n}{n-8} + \epsilon_n$, the equation (1.1) has no stable solution.*

In the second order case, the finite Morse index solutions to the corresponding nonlinear problem

$$\Delta u + |u|^{q-1}u = 0 \text{ in } \mathbb{R}^n, \quad q > 1 \tag{1.4}$$

have been completely classified by Farina [4]. One main result of [4] is that nontrivial finite Morse index solutions to (1.4) exist if and only if $q \geq p_{JL}$ and $n \geq 11$, or $q = \frac{n+2}{n-2}$ and $n \geq 3$. Here p_{JL} is the so-called Joseph-Lundgren exponent, see [8].

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In the fourth order case, the nonexistence of positive solutions to (1.1) are showed if $q < \frac{n+4}{n-4}$, and all entire solutions are classified if $q = \frac{n+4}{n-4}$, see [12, 19]. More precisely, when $q = \frac{n+4}{n-4}$ and $n \geq 5$, any classical solution to (1.1) is in the form

$$\widetilde{u}(x) = \frac{c_n \lambda^{\frac{n-4}{2}}}{(1 + \lambda^2 |x - x_0|^2)^{\frac{n-4}{2}}}, \quad \text{with } \ x_0 \in \mathbb{R}^n, \lambda > 0.$$

It was proved by Rozenblum (see [11, 15]) that when $n \geq 5$, the number of negative eigenvalues with multiplicity for the operator $(\Delta^2 - V)$ is bounded by

$$C_n \int_{\mathbb{R}^n} |V(x)|^{\frac{n}{4}} dx.$$

Using this, it is easy to check that $\widetilde{u}$ is a finite Morse index solution of (1.1) with the critical exponent.

So our results concern essentially the supercritical case, $n \geq 5$ and $q > \frac{n+4}{n-4}$. As far as we know, there are no results on the classification of entire solutions to (1.1) with finite Morse index and supercritical exponent q. Therefore Theorem 1.1 is a first step towards the understanding of stable solutions of fourth order problems. We note that only recently the radially symmetric solutions to (1.1) are studied in [5, 6, 9]. The radial entire solutions are shown to have the layer structure if and only if $q \geq p_{JL}^4$ and $n \geq 13$ where p_{JL}^4 stands for the corresponding Joseph-Lundgren exponent to Δ^2 (see [5, 6]). Theorem 1.1 classifies stable solutions to (1.1) in dimensions $n \leq 8$ and shows the nonexistence of stable solution for some special cases with $n \geq 9$. There is still a big gap to fill in towards a complete classification.

Our proof borrows crucially an idea from Cowan-Esposito-Ghoussoub [2], who proved the regularity of extremal solutions for fourth order problems in bounded domains. They made a key observation by using a nice result of Souplet [18]. Here we also rely crucially on some results of Souplet [18]. The key argument is to use two different test functions: the first one is u itself, and the other one is $v = -\Delta u$. We believe that further exploration of this idea may help to give the complete classification of stable solutions to (1.1).

At the end, we show some classification results on the half space or compactness results for stable solutions to $\Delta^2 u = \lambda(u+1)^p$ on bounded domain (see section 3).

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2. Proof of Theorem 1.1

We organize our proof into three steps.

- Step 1. Non existence of stable solution with $1 < q < \left(\frac{n-8}{n}\right)_+^{-1}$.
- Step 2. Non existence of stable solution with $q = \frac{n}{n-8}$ for $n \geq 9$.
- Step 3. Non existence of stable solution with q slightly larger than $\frac{n}{n-8}$ with $n \geq 9$.

2.1. **Step 1.** According to Theorem 3.1 of [19], $v := -\Delta u > 0$ in $\mathbb{R}^n$ since $q > 1$. Rewrite then (1.1) as a system

$$\Delta u + v = 0, \quad \Delta v + u^q = 0, \quad u > 0, \ v > 0 \ \text{ in } \ \mathbb{R}^n. \tag{2.1}$$

We recall several crucial estimates. First, following the idea in [13, 18], we have

Lemma 2.1. *If there exists a stable positive solution to* (1.1) *or* (2.1)*, there exists a bounded stable positive solution u to* (1.1) *such that $v = -\Delta u$ is also bounded in $\mathbb{R}^n$.*

We can prove this lemma by contradiction and proceed exactly as for Theorem 4.3 in [13] (see also Remark 1.1 in [18]). Indeed, if no bounded stable positive solution exists for (2.1), we have the estimate $u(x) \le C_{n,q} d(x, \partial\Omega)^{-\alpha}$ for any stable solution $\Delta^2 u = u^q$ in $\Omega \neq \mathbb{R}^n$, here $\alpha = \frac{4}{q-1}$ and $C_{n,q}$ depends only on n and $q > 1$. Therefore no stable entire solution to (1.1) could exist in $\mathbb{R}^n$, which contradicts the hypothesis.

The main reason for the estimate $u(x) \le C_{n,q} d(x, \partial\Omega)^{-\alpha}$ comes from the following fact: The scaling argument used in [13] does not affect the stability of solutions. Let $u_\lambda(x) := \lambda^\alpha u(\lambda x + x_0)$ with $\lambda > 0$, $x_0 \in \Omega$, there hold $\Delta^2 u_\lambda = u_\lambda^q$ in Ω_λ and

$$\Lambda_{u_\lambda, \Omega_\lambda}(\phi) = \lambda^{4-n} \Lambda_{u,\Omega}(\psi) \quad \text{where} \ \ \Omega_\lambda = \frac{\Omega - x_0}{\lambda}, \ \psi(y) = \phi\left(\frac{y - x_0}{\lambda}\right).$$

Let $\alpha = \frac{4}{q-1}$. By Lemma 2.4 of [18], for any solution of (2.1), there exists $C > 0$ such that

$$\int_{B_R} u dx \le C R^{n-\alpha}, \quad \int_{B_R} u^q dx \le C R^{n - q\alpha}, \quad \forall\, R > 0. \tag{2.2}$$

Here and in the following, B_R stands for the ball of radius R centered at the origin. Another important estimate is the following comparison between u and v (see Lemma 2.7 in [18]):

$$\text{As } u \text{ is bounded,} \quad v^2 \ge \frac{2}{q+1} u^{q+1} \ \text{ in } \ \mathbb{R}^n. \tag{2.3}$$

We need also the following identities:

Lemma 2.2. *For any $\xi, \eta \in C^4(\mathbb{R}^n)$, we have*

$$\Delta\xi \Delta(\xi\eta^2) - [\Delta(\xi\eta)]^2 = -4(\nabla\xi\cdot\nabla\eta)^2 - \xi^2(\Delta\eta)^2 + 2\xi\Delta\xi|\nabla\eta|^2 - 4\xi\Delta\eta\nabla\xi\cdot\nabla\eta.$$

and

Lemma 2.3. *For any $\xi \in C^4(\mathbb{R}^n)$ and $\eta \in C_0^\infty(\mathbb{R}^n)$, there hold*

$$\begin{aligned} \int_{\mathbb{R}^n} (\Delta^2\xi)\xi\eta^2 dx = & \int_{\mathbb{R}^n} [\Delta(\xi\eta)]^2 dx + \int_{\mathbb{R}^n} \left[-4(\nabla\xi\cdot\nabla\eta)^2 + 2\xi\Delta\xi|\nabla\eta|^2\right] dx \\ & + \int_{\mathbb{R}^n} \xi^2 \Big[2\nabla(\Delta\eta)\cdot\nabla\eta + (\Delta\eta)^2\Big] dx, \end{aligned} \tag{2.4}$$

$$\int_{\mathbb{R}^n} |\nabla\xi|^2 |\nabla\eta|^2 dx = \int_{\mathbb{R}^n} \left[\xi(-\Delta\xi)|\nabla\eta|^2 + \frac{1}{2}\xi^2\Delta(|\nabla\eta|^2)\right] dx. \tag{2.5}$$

Proof. The proof of Lemma 2.2 is done by direct verification. The equality (2.5) follows from

$$\frac{1}{2}\Delta(\xi^2) = \xi\Delta\xi + |\nabla\xi|^2.$$

On the other hand, a simple integration by parts yields

$$
\begin{aligned}
2\int_{\mathbb{R}^n} \xi\nabla\xi\cdot\nabla\eta\Delta\eta dx &= -\int_{\mathbb{R}^n}\xi^2 \mathrm{div}\left(\Delta\eta\nabla\eta\right)dx \\
&= -\int_{\mathbb{R}^n}\xi^2\left[(\Delta\eta)^2+\nabla\eta\cdot\nabla(\Delta\eta)\right]dx.
\end{aligned} \tag{2.6}
$$

By Lemma 2.2,

$$
\begin{aligned}
\int_{\mathbb{R}^n}(\Delta^2\xi)\xi\eta^2 dx &= \int_{\mathbb{R}^n}\Delta\xi\Delta\left(\xi\eta^2\right)dx \\
&= \int_{\mathbb{R}^n}\left[\Delta(\xi\eta)\right]^2 dx - 4\int_{\mathbb{R}^n}(\nabla\xi\cdot\nabla\eta)^2 dx - \int_{\mathbb{R}^n}\left[\xi^2(\Delta\eta)^2+2\xi\Delta\xi|\nabla\eta|^2\right]dx \\
&\quad - 4\int_{\mathbb{R}^n}\xi\nabla\xi\cdot\nabla\eta\Delta\eta dx
\end{aligned}
$$

The equality (2.4) is straightforward using (2.6). □

From (2.4) and (1.1), for any $\eta \in C_0^\infty(\mathbb{R}^n)$, there holds

$$
\begin{aligned}
&\int_{\mathbb{R}^n}[\Delta(u\eta)]^2 dx - \int_{\mathbb{R}^n}u^{q-1}(u\eta)^2 dx \\
&= 4\int_{\mathbb{R}^n}(\nabla u\nabla\eta)^2 dx - 2\int_{\mathbb{R}^n}u\Delta u|\nabla\eta|^2 dx - \int_{\mathbb{R}^n}u^2\left[2\nabla(\Delta\eta)\cdot\nabla\eta+(\Delta\eta)^2\right]dx.
\end{aligned}
$$

In the following, we denote C, C' as various generic positive constants which are independent on u, they could be changed from one line to another. Using stability condition $\Lambda_u(\phi)\geq 0$ with $\phi = u\eta$, we obtain the following estimate.

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\left[(\Delta(u\eta))^2+u^{q+1}\eta^2\right]dx \\
&\leq C\int_{\mathbb{R}^n}\left[|\nabla u|^2|\nabla\eta|^2+u|\Delta u||\nabla\eta|^2+u^2|\nabla\left(\Delta\eta\right)\cdot\nabla\eta|+u^2(\Delta\eta)^2\right]dx.
\end{aligned} \tag{2.7}
$$

Moreover, as

$$
\Delta(u\eta) = -v\eta + 2\nabla u\cdot\nabla\eta + u\Delta\eta,
$$

by (2.7) and Young's inequality (recalling that $v = -\Delta u > 0$ in $\mathbb{R}^n$),

$$
\int_{\mathbb{R}^n}\left[v^2\eta^2+u^{q+1}\eta^2\right]dx \leq C\int_{\mathbb{R}^n}\left[uv|\nabla\eta|^2+|\nabla u|^2|\nabla\eta|^2+u^2|\nabla\left(\Delta\eta\right)\cdot\nabla\eta|+u^2(\Delta\eta)^2\right]dx.
$$

Applying (2.5) with $\xi = u$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\left[(v^2\eta^2+u^{q+1}\eta^2\right]dx \\
&\leq C\int_{\mathbb{R}^n}uv|\nabla\eta|^2 dx + C\int_{\mathbb{R}^n}u^2\left[|\nabla(\Delta\eta)\cdot\nabla\eta|+\left|\Delta\left(|\nabla\eta|^2\right)\right|+(\Delta\eta)^2\right]dx.
\end{aligned} \tag{2.8}
$$

Take $\eta = \varphi^m$ with $m > 2$, it follows that

$$
\begin{aligned}
\int_{\mathbb{R}^n}uv|\nabla\eta|^2 dx &= m^2\int_{\mathbb{R}^n}uv\varphi^{2(m-1)}|\nabla\varphi|^2 dx \\
&\leq \frac{1}{2C}\int_{\mathbb{R}^n}(v\varphi^m)^2 dx + C\int_{\mathbb{R}^n}u^2\varphi^{2(m-2)}|\nabla\varphi|^4 dx.
\end{aligned}
$$

Now let us choose φ_1 a cut-off function verifying $0\leq\varphi_1\leq 1$, $\varphi_1 = 1$ for $|x|<1$ and $\varphi_1 = 0$ for $|x|>2$. Substituting the above inequality into (2.8) with $\varphi = \varphi_1(x/R)$ for $R>0$ and $\eta = \varphi^m$, we arrive at

$$\int_{\mathbb{R}^n} (v\varphi^m)^2 dx + \int_{\mathbb{R}^n} u^{q+1}\varphi^{2m} dx \le CR^{-4}\int_{\mathbb{R}^n} u^2\varphi^{2(m-2)}dx. \tag{2.9}$$

We claim:

$$\int_{B_R} u^2 dx \le CR^{n-2\alpha}, \quad \forall\, R>0. \tag{2.10}$$

When $q>2$, the above estimate follows from Hölder's inequality using (2.2) while for $q=2$, it is just the second estimate in (2.2). If $q\in(1,2)$, fix $m>\frac{2}{q-1}$, by Hölder's inequality and (2.9), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} u^2\varphi^{2(m-2)}dx &\le \left(\int_{\mathbb{R}^n} u^q\varphi^{2m-\frac{4}{q-1}}dx\right)^{q-1}\left(\int_{\mathbb{R}^n} u^{q+1}\varphi^{2m}dx\right)^{2-q}\\ &\le C\left(\int_{B_{2R}} u^q dx\right)^{q-1}\left(R^{-4}\int_{\mathbb{R}^n} u^2\varphi^{2(m-2)}dx\right)^{2-q},\end{aligned}$$

hence

$$\int_{\mathbb{R}^n} u^2\varphi^{2(m-2)}dx \le CR^{-\frac{4(2-q)}{q-1}}\int_{B_{2R}} u^q dx.$$

Using (2.2), there holds

$$\int_{B_R} u^2 dx \le \int_{\mathbb{R}^n} u^2\varphi^{2(m-2)}dx \le CR^{-\frac{4(2-q)}{q-1}}\int_{B_{2R}} u^q dx \le C'R^{n-q\alpha}R^{-\frac{4(2-q)}{q-1}} = C'R^{n-2\alpha},$$

so the claim (2.10) is proved. Combining (2.9) and (2.10),

$$\int_{\mathbb{R}^n} \left(v^2+u^{q+1}\right)\varphi^{2m}dx \le CR^{n-4-2\alpha}. \tag{2.11}$$

Next we make use of the stability condition again, but this time with the test function $\phi=v\eta$. By equations (2.1), we have

$$\Delta^2 v = -\Delta(u^q) = qu^{q-1}v - q(q-1)u^{q-2}|\nabla u|^2. \tag{2.12}$$

Multiplying (2.12) by $v\eta^2$, similarly as for (2.7), by (2.4) and (2.5),

$$\begin{aligned}0 &\le \int_{\mathbb{R}^n}\left[(\Delta(v\eta))^2 - qu^{q-1}(v\eta)^2\right]dx\\ &\le -q(q-1)\int_{\mathbb{R}^n} u^{q-2}|\nabla u|^2 v\eta^2 dx + C\int_{\mathbb{R}^n} v|\Delta v||\nabla\eta|^2 dx\\ &\quad + C\int_{\mathbb{R}^n} v^2\left[|\nabla(\Delta\eta)\cdot\nabla\eta| + \left|\Delta\left(|\nabla\eta|^2\right)\right| + |\Delta\eta|^2\right]dx\\ &\le -q(q-1)\int_{\mathbb{R}^n} u^{q-2}|\nabla u|^2 v\eta^2 dx\\ &\quad + C\int_{\mathbb{R}^n} vu^q|\nabla\eta|^2 dx + C\int_{\mathbb{R}^n} v^2\left[|\nabla(\Delta\eta)\cdot\nabla\eta| + \left|\Delta\left(|\nabla\eta|^2\right)\right| + |\Delta\eta|^2\right]dx.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{\mathbb{R}^n} u^{q-2}|\nabla u|^2 v\eta^2 dx\\ &\le C\int_{\mathbb{R}^n} vu^q|\nabla\eta|^2 dx + C\int_{\mathbb{R}^n} v^2\left[|\nabla(\Delta\eta)\cdot\nabla\eta| + \left|\Delta\left(|\nabla\eta|^2\right)\right| + |\Delta\eta|^2\right]dx.\end{aligned} \tag{2.13}$$

Furthermore, for any C^1 function H, integration by parts yields

$$\int_{\mathbb{R}^n} H(u)(-\Delta u)\eta^2 dx = \int_{\mathbb{R}^n} H'(u)|\nabla u|^2\eta^2 dx + \int_{\mathbb{R}^n} H(u)\nabla u\cdot\nabla(\eta^2)dx. \tag{2.14}$$

Following an idea of Cowan-Esposito-Ghoussoub [2], set $H(u) = u^{\frac{3q-1}{2}}$, then

$$\int_{\mathbb{R}^n} u^{\frac{3q-1}{2}} v\eta^2 dx \le C\int_{\mathbb{R}^n} u^{\frac{3q-3}{2}}|\nabla u|^2\eta^2 dx + C\int_{\mathbb{R}^n} u^{\frac{3q+1}{2}}\left|\Delta(\eta^2)\right| dx.$$

Recall that $v \ge Cu^{\frac{q+1}{2}}$, we conclude, using (2.13) and (2.14),

$$\begin{aligned}
\int_{\mathbb{R}^n} u^{2q}\eta^2 dx &\le C\int_{\mathbb{R}^n} u^{\frac{3q-1}{2}} v\eta^2 dx\\
&\le C\int_{\mathbb{R}^n} u^{\frac{3q-3}{2}}|\nabla u|^2\eta^2 dx + C\int_{\mathbb{R}^n} u^{\frac{3q+1}{2}}\left|\Delta(\eta^2)\right| dx\\
&\le C\int_{\mathbb{R}^n} u^{q-2}|\nabla u|^2 v\eta^2 dx + C\int_{\mathbb{R}^n} vu^q\left|\Delta(\eta^2)\right| dx\\
&\le C\int_{\mathbb{R}^n} vu^q\Big(|\nabla\eta|^2 + \left|\Delta(\eta^2)\right|\Big)dx\\
&\quad + C\int_{\mathbb{R}^n} v^2\Big[|\nabla(\Delta\eta)\cdot\nabla\eta| + \left|\Delta\left(|\nabla\eta|^2\right)\right| + |\Delta\eta|^2\Big]dx.
\end{aligned} \tag{2.15}$$

As before, let $\eta = \varphi^m$ with large m and $\varphi = \varphi_1(x/R)$ for $R > 0$. Similarly to the derivation of inequality (2.9), we get from (2.15) and (2.11),

$$\int_{\mathbb{R}^n} (u^q\varphi^m)^2 dx \le CR^{-4}\int_{B_{2R}\setminus B_R} v^2 dx \le CR^{n-8-2\alpha}. \tag{2.16}$$

If $1 < q < \left(\frac{n-8}{n}\right)_+^{-1}$, $n-8-2\alpha < 0$. So when u is stable, letting $R\to\infty$, we deduce $u\equiv 0$ in $\mathbb{R}^n$ by (2.16). This proves the nonexistence of stable solution to (1.1) for $1 < q < \left(\frac{n-8}{n}\right)_+^{-1}$.

2.2. **Step 2.** Here we show the nonexistence of stable solutions with $q = \frac{n}{n-8}$ for $n \ge 9$. Our proof is based on the nonexistence of fast decay solutions with supercritical exponent.

Proposition 2.4. *Let $n \ge 5$, $q > \frac{n+4}{n-4}$ and $\alpha = \frac{4}{q-1}$. Then the system* (2.1) *has no classical solution verifying*

$$u(x) = o\left(|x|^{-\alpha}\right),\ \ v(x) = o\left(|x|^{-2-\alpha}\right) \quad \textit{as } |x|\to\infty. \tag{2.17}$$

Proof. Suppose that such a solution u exists. Let w be the Emden-Fowler transformation of u, i.e. $w(t,\sigma) = r^\alpha u(r\sigma)$ for any $t = \ln r\in\mathbb{R}$, and $\sigma\in\mathbb{S}^{n-1}$ the standard unit sphere of $\mathbb{R}^n$. Direct calculation yields

$$r^{2+\alpha}\Delta u = w_{tt} + (n-2-2\alpha)w_t - \alpha(n-2-\alpha)w + \Delta_{\mathbb{S}^{n-1}}w$$

where $\Delta_{\mathbb{S}^{n-1}}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. Applying again this formula,

$$\begin{aligned}
w^q = r^{4+\alpha}u^q = r^{4+\alpha}\Delta^2 u &= w_{tttt} + K_3 w_{ttt} + K_2 w_{tt} + K_1 w_t + K_0 w\\
&\quad + \Delta^2_{\mathbb{S}^{n-1}}w + 2\Delta_{\mathbb{S}^{n-1}}w_{tt} + K_5\Delta_{\mathbb{S}^{n-1}}w_t + K_6\Delta_{\mathbb{S}^{n-1}}w
\end{aligned} \tag{2.18}$$

where K_i are constants depending on α and n, for example

$$K_5 = K_3 = (2n-8-4\alpha), \quad K_6 = -\Big[(\alpha+2)(n-4-\alpha) + \alpha(n-2-\alpha)\Big].$$

In particular, we have (see [6] for K_i, $0\le i\le 4$)

$$K_1 < 0,\ \ K_3 = K_5 > 0, \quad \forall\, n\ge 5,\ q > \frac{n+4}{n-4}. \tag{2.19}$$

Set

$$E(w) = \int_{\mathbb{S}^{n-1}} \left(\frac{w^{q+1}}{q+1} - \frac{K_0}{2}w^2 - \frac{K_2}{2}w_t^2 - K_3 w_{tt}w_t + \frac{w_{tt}^2}{2} - w_{ttt}w_t\right) d\sigma$$
$$+ \int_{\mathbb{S}^{n-1}} \left(\frac{K_6}{2}\left|\nabla_{\mathbb{S}^{n-1}} w\right|^2 + \left|\nabla_{\mathbb{S}^{n-1}} w_t\right|^2 - \frac{1}{2}\left|\Delta_{\mathbb{S}^{n-1}} w\right|^2\right) d\sigma.$$

Multiplying the equation (2.18) with w_t, we get from (2.19)

$$\frac{\mathrm{d}}{\mathrm{d}t}E(w)(t) = \int_{\mathbb{S}^{n-1}} \left(K_1 w_t^2 - K_5\left|\nabla_{\mathbb{S}^{n-1}} w_t\right|^2 - K_3 w_{tt}^2\right) d\sigma \le 0.$$

By the decay conditions (2.17),

$$-\Delta u = v, \quad -\Delta v = u^q = o\left(|x|^{-4-\alpha}\right) \text{ as } |x| \to \infty.$$

The standard elliptic estimates imply then

$$\lim_{|x|\to+\infty} |x|^{k+\alpha}|\nabla^k u(x)| = 0, \text{ for } 1 \le k \le 4 \quad \text{so that } \lim_{t\to\infty} \|w(t,\cdot)\|_{C^3(\mathbb{S}^{n-1})} = 0. \tag{2.20}$$

Therefore $\lim_{t\to\infty} E(w) = 0$. We have also $\lim_{t\to-\infty} E(w) = 0$ because u is regular at the origin. Finally we conclude

$$\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}} \left(K_1 w_t^2 - K_5\left|\nabla_{\mathbb{S}^{n-1}} w_t\right|^2 - K_3 w_{tt}^2\right) d\sigma dt = 0.$$

So $w_t \equiv 0$, hence $w \equiv 0$ as $\lim_{t\to-\infty} w = 0$, but this contradicts the positivity of u. □

Back to Theorem 1.1. Suppose that u is a stable solution of (1.1), we may assume again u is bounded, recall (2.11) and (2.16).

$$\int_{B_R} v^2 dx \le CR^{n-4-2\alpha}, \quad \int_{B_R} u^{2q} dx \le CR^{n-8-2\alpha}, \quad \forall\, R > 0. \tag{2.21}$$

Applying now the Sobolev embedding of H^2,

$$\|v\|_{L^{p_*}(B_R)}^2 \le C(n)\left(\|\Delta v\|_{L^2(B_R)}^2 + R^{-4}\|v\|_{L^2(B_R)}^2\right), \quad \text{where } p_* = \frac{2n}{n-4}.$$

Combining with (2.21), there exists $C > 0$ such that for any $R > 0$,

$$\|v\|_{L^{p_*}(B_R)}^2 \le CR^{n-8-2\alpha}. \tag{2.22}$$

As $q = \frac{n}{n-8}$, we have $n - 8 - 2\alpha = 0$. The above estimate means just

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-4}} dx < \infty. \tag{2.23}$$

Now we are ready to prove the fast decay of u and v. Instead to use the Harnack argument in [16] (see [4]), let us recall a special case of Theorem 4.4 in [10]: For any $p \in [2, \infty)$, there exists $\epsilon(p) > 0$ such that if $\Delta w + \rho w = 0$ in B_2 with $\|\rho\|_{L^{\frac{n}{2}}(B_2)} \le \epsilon(p)$, we have

$$\|w\|_{L^p(B_1)} \le C\|w\|_{L^2(B_2)} \tag{2.24}$$

where the constant C depending only on p and n.

Let $x_0 \in \mathbb{R}^n$ with $|x_0| > 4R_0$ and $R = \frac{|x_0|}{4}$, consider the function $w(y) = v(x_0 + Ry)$. Then w satisfies $\Delta w + \rho w = 0$ where

$$\rho(y) = R^2\frac{u^q}{v}(x_0 + Ry).$$

Using (2.3), $0 < \rho(y) \leq CR^2 u^{\frac{q-1}{2}}(x_0+Ry) \leq C'R^2 v^{\frac{q-1}{q+1}}(x_0+Ry)$. As $q = \frac{n}{n-8}$, $\frac{n}{2+\alpha} = \frac{q-1}{q+1} \times \frac{n}{2} = p_*$. Therefore, by (2.23)

$$\int_{B_2} |\rho|^{\frac{n}{2}} dx \leq C \int_{B_2} R^n v^{p_*}(x_0 + Ry)dy = C \int_{B_{2R}(x_0)} v^{p_*} dx \to 0, \quad \text{when} \ \ |x_0| \to \infty.$$

From (2.24) and Hölder's inequality, we derive that for any $p \geq 2$, as $|x_0| \to \infty$,

$$\begin{aligned} \|v\|_{L^p(B_R(x_0))} = R^{\frac{n}{p}} \|w\|_{L^p(B_1)} \leq CR^{\frac{n}{p}} \|w\|_{L^2(B_2)} &\leq C'R^{\frac{n}{p}} \|w\|_{L^{p_*}(B_2)} \\ &\leq C'R^{\frac{n}{p} - \frac{n}{p_*}} \|v\|_{L^{p_*}(B_{2R}(x_0))} \\ &= o\left(R^{-2-\alpha+\frac{n}{p}}\right). \end{aligned} \tag{2.25}$$

Using classical elliptic estimates (see Theorem 8.17 of [7]), there exists $C > 0$ such that

$$\sup_{B_{\frac{R}{2}}(x_0)} v \leq C\left[R^{-\frac{n}{2}} \|v\|_{L^2(B_R(x_0))} + R\|\Delta v\|_{L^n(B_R(x_0))}\right]. \tag{2.26}$$

It is clear that $\widetilde{q} = \frac{2nq}{q+1} \geq 2$ and $|\Delta v|^n = u^{nq} \leq Cv^{\widetilde{q}}$. Thanks to (2.25), when $|x_0| \to \infty$,

$$R\|\Delta v\|_{L^n(B_R(x_0))} \leq CR\|v\|^{\frac{\widetilde{q}}{n}}_{L^{\widetilde{q}}(B_R(x_0))} = o\left(R^{1+\left(-2-\alpha+\frac{n}{\widetilde{q}}\right)\frac{\widetilde{q}}{n}}\right) = o\left(R^{-2-\alpha}\right).$$

Substituting the above estimate into (2.26), applying (2.25) with $p = 2$, we conclude then

$$v(x) = o\left(|x|^{-2-\alpha}\right) \quad \text{as} \ \ |x| \to \infty.$$

We get also $u(x) = o\left(|x|^{-\alpha}\right)$ at infinity by (2.3), hence the decay estimate (2.17) holds, we reach then a contradiction seeing Proposition 2.4.

2.3. **Step 3.** Here we will prove that no stable solution exists for exponent q slightly higher than $\frac{n}{n-8}$ if $n \geq 9$. The main idea is a blow up argument.

Suppose that the claim (ii) of Theorem 1.1 does not hold, there exist then a sequence $\delta_j > 0$, $\delta_j \to 0$ and a sequence of stable solutions u_j to (1.1) with $q_j = \frac{n}{n-8} + \delta_j$. Lemma 2.1 permits to assume that u_j and $v_j = -\Delta u_j$ are bounded in $\mathbb{R}^n$. Choose $\lambda_j > 0$ such that

$$\frac{1}{\|v_j\|_\infty} = \lambda_j^{\frac{4}{q_j-1}+2}.$$

Let $\widetilde{u}_j(x) = \lambda_j^{\frac{4}{q_j-1}} u_j(\lambda_j x)$, so $\Delta^2 \widetilde{u}_j = \widetilde{u}_j^{q_j}$, $\widetilde{v}_j := -\Delta\widetilde{u}_j$ satisfies $\|\widetilde{v}_j\|_\infty = 1$. Up to a translation, we assume also $\widetilde{v}_j(0) \in (\frac{1}{2}, 1]$. Using (2.3) to $\widetilde{u}_j$, we have also $\|\widetilde{u}_j\|_\infty \leq C$.

By standard elliptic theory, there is a subsequence still denoted by $\widetilde{u}_j$ which tends to a bounded nonnegative function u_* in $C^k_{loc}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$, so $\Delta^2 u_* = u_*^{\frac{n}{n-8}}$ in $\mathbb{R}^n$. As u_j are stable, it is easy to see that u_* is stable (taking the limit in (1.2) with $\widetilde{u}_j$ and q_j). Finally, since $-\Delta u_* \geq 0$ in $\mathbb{R}^n$ and $-\Delta u_*(0) = \lim \widetilde{v}_j(0) > 0$, u_* is nontrivial, hence positive in $\mathbb{R}^n$. This is impossible by the previous step, the claim (ii) is then proved. □

3. Some applications

As we have mentioned yet, the nonexistence result of entire stable solution yields immediately (with blow-up and scaling argument as in [13, 18])

Corollary 3.1. *Assume that Ω is a proper subdomain of $\mathbb{R}^n$ and u is a classical, positive and stable solution of $\Delta^2 u = u^q$ in Ω where $1 < q < \infty$ if $n \le 8$; or $1 < q < \frac{n}{n-8} + \epsilon_n$ if $n \ge 9$ with ϵ_n in Theorem 1.1. Then*

$$u(x) \le C_{n,q} d(x, \partial\Omega)^{-\alpha}, \ |\Delta u(x)| \le C_{n,q} d(x, \partial\Omega)^{-\alpha-2} \quad \textit{where} \ \ \alpha = \frac{4}{q-1},$$

the constant C depends only on q and n.

Consider now

$$(3.1) \qquad \begin{cases} \Delta^2 u = u^q & \text{in } \mathbb{R}^n_+ = \mathbb{R}_+ \times \mathbb{R}^{n-1}, \ \ n \ge 2 \\ u > 0, -\Delta u > 0 & \text{in } \mathbb{R}^n_+ \\ u = -\Delta u = 0 & \text{on } \{0\} \times \mathbb{R}^{n-1}. \end{cases}$$

The following result is due to Dancer (Theorem 2 in [3], see also Theorem 10 in [17]).

Lemma 3.2. *Suppose that u is a classical solution of (3.1) such that u and $-\Delta u$ are bounded in $\mathbb{R}^n_+$, then $\partial_{x_1} u > 0$ and $-\partial_{x_1}\Delta u > 0$ in $\mathbb{R}^n_+$.*

Therefore, under the condition of this lemma, $w(y) = \lim_{x_1 \to \infty} u(x_1, y)$ exists for all $y \in \mathbb{R}^{n-1}$ and $\Delta^2 w = w^p$ in $\mathbb{R}^{n-1}$. It is not difficult to see that if w is unstable, then $\mathrm{ind}(u)$ is infinite. Indeed, let $\psi \in C_0^\infty(\mathbb{R}^{n-1})$ verify $\Lambda_{w,\mathbb{R}^{n-1}}(\psi) < 0$, we choose $\zeta \in C^\infty(\mathbb{R})$, $\mathrm{supp}(\zeta) \subset [1, 2]$ and denote $\phi_R(x) = \zeta(x_1/R)\psi(\widetilde{x})$ where $\widetilde{x} = (x_2, \ldots, x_n)$ and $R > 0$. Obviously $\phi_R \in C_0^\infty(\mathbb{R}^n)$. By the locally uniform convergence w.r.t. $\widetilde{x}$ of u to w when $x_1 \to \infty$, we check readily that $\Lambda_u(\phi_R) < 0$ for R large enough, and similarly $\Lambda_u(\phi_R(x + \mu e_1)) < 0$ for any $a > 0$. Therefore, taking a suitable sequence $\mu_k \to \infty$ with R fixed, we observe easily that $\mathrm{ind}(u) = \infty$. In other words, if $\mathrm{ind}(u) < \infty$, then w must be stable. This enable us the following classification result.

Theorem 3.3. *Let u be a classical solution of (3.1) with $n \ge 2$. Assume moreover u and $-\Delta u$ are bounded. Then $\mathrm{ind}(u) = \infty$, when $q > 1$ and $n \le 9$; or $1 < q < \frac{n-1}{n-9} + \epsilon_{n-1}$ and $n \ge 10$. Here $\epsilon_k > 0$ are given by Theorem 1.1.*

Finally, Consider the bounded domain situation with polynomial growth:

$$(P_\lambda) \qquad \begin{cases} \Delta^2 u = \lambda(u+1)^q & \text{in a bounded smooth domain } \Omega \subset \mathbb{R}^n, \ \ n \ge 1 \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that there exists a critical value $\lambda^* > 0$ depending on $q > 1$ and Ω such that

- If $\lambda \in (0, \lambda^*)$, (P_λ) has a minimal and classical solution which is stable;
- If $\lambda = \lambda^*$, a unique weak solution, called the extremal solution u^* exists for (P_{λ^*});
- No weak solution of (P_λ) exists whenever $\lambda > \lambda^*$.

In the same spirit of Corollary 3.1, we can prove

Theorem 3.4. *There exists $\widetilde{\epsilon}_n > 0$ such that the extremal solution u^*, the unique solution of (P_{λ^*}) is bounded provided that*

$$n \le 8, \ \ q > 1 \quad \textit{or} \quad n \ge 9, \ \ 1 < q < \frac{n}{n-8} + \widetilde{\epsilon}_n.$$

Here we need just to consider stable minimal solutions u_λ to (P_λ) since $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$, so the conclusion comes from contradiction with (ii) of Theorem 1.1 or Theorem 3.3, whenever the blow up occurs, we omit the detail. The case $1 < q < \left(\frac{n-8}{n}\right)_+^{-1}$ was proved in [2] by different approach.

References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure. Appl. Math.* 12 (1959), 623-727.

[2] C. Cowan, P. Esposito and N. Ghoussoub, Regularity of extremal solutions in fourth order nonlinear eigenvalue problems on general domains, *DCDS-A* 28 (2010), 1033-1050.

[3] E.N. Dancer, Moving plane methods for systems on half spaces, *Math. Ann.* 342 (2008), 245-254.

[4] A. Farina, On the classification of solutions of the Lane-Emden equaation on unbouned domains of $\mathbb{R}^N$, *J. Math. Pures Appl.* 87 (2007), 537-561.

[5] A. Ferrero, H.-Ch. Grunau and P. Karageorgis, Supercritical biharmonic equations with power-like nonlinearity, *Ann. Mat. Pura Appl.* 188 (2009), 171-185.

[6] F. Gazzola and H.-Ch. Grunau, Radial entire solutions for supercritical biharmonic equations, *Math. Ann.* 334 (2006), 905-936.

[7] Gilbarg and Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd Edition, Springer-Verlag.

[8] C. Gui, W.M. Ni and X.F. Wang, On the stability and instability of positive steady states of a semilinear heat equation in $\mathbf{R}^n$, *Comm. Pure Appl. Math.* Vol. XLV (1992), 1153-1181.

[9] Z. Guo and J. Wei, Qualitative properties of entire radial solutions for a biharmonic equation with supcritical nonlinearity, *Proc. American Math. Soc.* (2010) to appear.

[10] Q. Han and F.H. Lin, Elliptic Partial Differential Equations, Courant Lecture Notes, New York University, (1997).

[11] D. Levin, On an analogue of the Rozenblum-Lieb-Cwikel inequality for the biharmonic operator on a Riemannian manifold, *Math. Res. Letters* 4 (1997), 855-869.

[12] C.S. Lin, A classification of solutions to a conformally invariant equation in $\mathbb{R}^4$, *Comm. Math. Helv.* 73 (1998), 206-231.

[13] P. Polácik, P. Quittner and P. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part I: Elliptic systems, *Duke Math. J.* 139 (2007), 555-579.

[14] W. Reichel and T. Weth, A priori bounds and a Liouville theorem on a half space for higher-order elliptic Dirichlet problems, *Math. Z.* 261 (2009) 805-827.

[15] G.V. Rozenblum, The distribution of the discrete spectrum for singular differential operators, *Dokl. Akad. SSSR* 202 (1972), 1012-1015.

[16] J. Serrin, Local behavior of solutions of quasi-linear equations, *Acta Math.* 111 (1964), 247-302.

[17] B. Sirakov, Existence results and a priori bounds for higher order elliptic equations and systems, *J. Math. Pures Appl.* 89 (2008), 114-133.

[18] P. Souplet, The proof of the Lane-Emden conjecture in four space dimensions, *Adv. Math.* 221 (2009), 1409-1427.

[19] J. Wei and X. Xu, Classification of solutions of high order conformally invariant equations, *Math. Ann.* 313(2) (1999), 207-228.

[20] C. Wang and D. Ye, Some Liouville theorems for Hénon type elliptic equations, *J. Func. Anal.* 262 (2012), 1705-1727.

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