AN ELLIPTIC PROBLEM WITH CRITICAL GROWTH AND LAZER-MCKENNA CONJECTURE

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ABSTRACT. We consider an elliptic problem of Ambrosetti-Prodi type involving critical Sobolev exponent on a bounded smooth domain. We show that if the domain has some symmetry, the problems have infinitely many solutions, thereby obtaining a stronger result than the Lazer-McKenna conjecture.

1. INTRODUCTION

Elliptic problems of Ambrosetti-Prodi type have the following form:

$$\begin{cases} -\Delta u = g(u) - \bar{s}\varphi_1(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $g(t)$ satisfies $\lim_{t\to-\infty} \frac{g(t)}{t} = \nu < \lambda_1$, $\lim_{t\to+\infty} \frac{g(t)}{t} = \mu > \lambda_1$, λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition and $\varphi_1 > 0$ is the first eigenfunction. Here $\mu = +\infty$ and $\nu = -\infty$ are allowed. It is well-known that the location of μ, ν with respect to the spectrum of $(-\Delta, H_0^1(\Omega))$ plays an important role in the multiplicity of solutions for problem (1.1). See for example [1], [6, 7], [16]–[18], [21]–[24], [29]–[32]. In the early 1980s, Lazer and McKenna conjectured that if $\mu = +\infty$ and $g(t)$ does not grow too fast at infinity, (1.1) has an unbounded number of solutions as $\bar{s} \to +\infty$. See [22].

In this paper, we will consider the following special case:

$$\begin{cases} -\Delta u = u_+^{2^*-1} + \lambda u - \bar{s}\varphi_1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in R^N with C^2 boundary, $N \geq 3$, $\lambda < \lambda_1$, $\bar{s} > 0$, $u_+ = \max(u, 0)$ and $2^* = 2N/(N-2)$.

It is easy to see that (1.2) has a negative solution

$$\underline{u}_{\bar{s}} = -\frac{\bar{s}}{\lambda_1 - \lambda}\varphi_1,$$

if $\lambda < \lambda_1$. Moreover, if $\underline{u}_{\bar{s}} + u$ is a solution of (1.2), then u satisfies

$$\begin{cases} -\Delta u = \big(u - s\varphi_1\big)_+^{2^*-1} + \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $s = \frac{\bar{s}}{\lambda_1 - \lambda} > 0$.

Let us recall some recent results on the Lazer-McKenna conjecture related to (1.3). Firstly, Dancer and the second author proved in [10, 11] that the Lazer-McKenna conjecture is true if $\lambda \in (-\infty, \lambda_1)$, if the critical exponent in (1.3) is replaced by sub-critical one. In the critical case, it was proved in [25, 26, 34] that if $N \geq 6$ and $\lambda \in (0, \lambda_1)$, then (1.3) has unbounded number of solutions as $s \to +\infty$. The solutions constructed for (1.3) concentrate either at the maximum points of the first eigenfunction [25], or at some boundary points of the domain [34] as $s \to +\infty$. On the other hand, Druet proves in [19] that the conditions $N \geq 6$ and $\lambda \in (0, \lambda_1)$ are necessary for the existence of the peak-solutions constructed in [25, 34]. More precisely, the result in [19] states that if $N = 3, 4, 5$, or $N \geq 6$ and $\lambda \leq 0$, then (1.3) has no solution u_s, such that the energy of u_s is bounded as $s \to +\infty$. This result suggests that it is more difficult to find solutions for (1.3) in the lower dimensional cases $N = 3, 4, 5$, or in the case $\lambda \leq 0$ and $N \geq 6$.

Note that all the results just mentioned state that (1.3) has more and more solutions as the parameter $s \to +\infty$. But for fixed $s > 0$, it is hard to estimate how many solutions (1.3) has.

In this paper, we will deal with (1.3) in the lower dimensional cases $N = 4, 5, 6$, or $N \geq 7$ and $\lambda \leq 0$, assuming that the domain Ω satisfies the following symmetry condition:

- $(S1)$: If $x = (x_1, \cdots, x_N) \in \Omega$, then, for any $\theta \in [0, 2\pi]$, $(r\cos\theta, r\sin\theta, x_3, \cdots, x_N) \in \Omega$, where $r = \sqrt{x_1^2 + x_2^2}$;
- $(S2)$: If $x = (x_1, \cdots, x_N) \in \Omega$, then, for any $3 \leq i \leq N$, $(x_1, x_2, \cdots, -x_i, \cdots, x_N) \in \Omega$.

The main result of this paper is the following:

Theorem 1.1. *Suppose that Ω satisfies $(S1)$ and $(S2)$. Assume that one of the following conditions holds:*

(i) *$N = 4, 5$, $\lambda < \lambda_1$ and $s > 0$;*
(ii) *$N = 6$, $\lambda < \lambda_1$ and $s > |\lambda| s_0$ for some $s_0 > 0$, which depends on Ω only;*
(iii) *$N \geq 7$, $\lambda = 0$ and $s > 0$.*

Then, (1.3) *has infinitely many solutions.*

The result in Theorem 1.1 is stronger than the Lazer-McKenna conjecture. Note that in Theorem 1.1, the constant s is fixed. This gives a striking contrast to the results in [25, 34], where s is regarded as a parameter which needs to tend to infinity in order to obtain the results there. We are not able to obtain similar result for the cases $N = 3$, and $N \geq 7$ and $\lambda < 0$. But we have the following weaker result for $N \geq 7$ and $\lambda < 0$, which gives a positive answer to the Lazer–McKenna conjecture in this case:

Theorem 1.2. *Suppose that Ω satisfies $(S1)$ and $(S2)$, and $N \geq 7$, $\lambda < \lambda_1$. Then, the number of the solutions for* (1.3) *is unbounded as $s \to +\infty$.*

Problem (1.3) is a bit tricky in the case $N = 3$. When $s = 0$, Brezis and Nirenberg [5] proved that (1.3) has a least energy solution if $\lambda \in (0, \lambda_1)$, while for $N = 3$, this result holds only if $\lambda \in (\lambda^*, \lambda_1)$ for some $\lambda^* > 0$ (if Ω is a ball, $\lambda^* = \frac{\lambda_1}{4}$). The main reason for this difference is that the function defined in (1.4) does not decay fast enough if $N = 3$.

Similarly, the main reason that we are not able to prove Theorem 1.1 for $N = 3$ is that the function defined in (1.7) does not decay fast enough.

In the Lazer and McKenna conjecture, the parameter s is large. Let us now consider the other extreme case: $s \to 0+$. Using the same argument as in [5], we can show that for $\lambda \in (\lambda^*, \lambda_1)$, $\lambda^* = 0$ if $N = 4$, $\lambda^* > 0$ if $N = 3$, (1.3) has a least energy solution if $s > 0$ is small. We can obtain more in the case $N = 3$.

Theorem 1.3. *Suppose that Ω satisfies $(S1)$ and $(S2)$, and $N = 3$, $\lambda < \lambda_1$. Then, the number of the solutions for (1.3) is unbounded as $s \to 0+$.*

Note that the result in Theorem 1.3 is not trivial, because if $\lambda < \lambda^*$, we can not find even one solution by using the method in [5]. Moreover, we show that (1.3) has more and more solutions as $s \to 0+$ for all $\lambda < \lambda_1$ if $N = 3$.

The readers can refer to [4, 8, 9, 15] for results on the Lazer-McKenna conjecture for other type of nonlinearities.

Before we close this section, let us outline the proof of Theorems 1.1 and 1.2 and discuss the conditions imposed in these two theorems.

For any $\bar{x} \in R^N$, $\mu > 0$, denote

$$U_{\bar{x},\mu}(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} \frac{\mu^{(N-2)/2}}{(1 + \mu^2 |y - \bar{x}|^2)^{(N-2)/2}}. \tag{1.4}$$

Then, $U_{\bar{x},\mu}$ satisfies $-\Delta U_{\bar{x},\mu} = U_{\bar{x},\mu}^{2^*-1}$. In this paper, we will use the following notation: $U = U_{0,1}$.

Let

$$\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}, \quad \mu = \frac{\Lambda}{\varepsilon}, \quad \Lambda \in [\delta, \delta^{-1}]$$

and $k \geq k_0$, where $\delta > 0$ is a small constant, and $k_0 > 0$ is a large constant, which is to be determined later.

Using the transformation $u(y) \mapsto \varepsilon^{-\frac{N-2}{2}} u\big(\frac{y}{\varepsilon}\big)$, we find that (1.3) becomes

$$\begin{cases} -\Delta u = \big(u - s\varepsilon^{\frac{N-2}{2}} \varphi_1(\varepsilon y)\big)_+^{2^*-1} + \lambda \varepsilon^2 u, & \text{in} \quad \Omega_\varepsilon, \\ u = 0, & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{1.5}$$

where $\Omega_\varepsilon = \{y : \ \varepsilon y \in \Omega\}$. Let

$$\Phi_\varepsilon(y) = \varepsilon^{\frac{N-2}{2}} \varphi_1(\varepsilon y).$$

For $\xi \in \Omega_\varepsilon$, we define $W_{\Lambda,\xi}$ as the unique solution of

$$\begin{cases} -\Delta W - \lambda \varepsilon^2 W = U_{\Lambda,\xi}^{2^*-1} & \text{in} \quad \Omega_\varepsilon, \\ W = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{1.6}$$

Define

$$H_s = \big\{u : u \in H^1(\Omega_\varepsilon), u \text{ is even in } y_h, h = 2, \cdots, N,$$
$$u(r\cos\theta, r\sin\theta, y'') = u(r\cos(\theta + \frac{2\pi j}{k}), r\sin(\theta + \frac{2\pi j}{k}), y''), j = 1, \dots, k-1\big\},$$

and

$$x_j = \big(\frac{r}{\varepsilon}\cos\frac{2(j-1)\pi}{k}, \frac{r}{\varepsilon}\sin\frac{2(j-1)\pi}{k}, 0\big), \quad j = 1, \cdots, k,$$

where 0 is the zero vector in R^{N-2}.

Let

$$W_{r,\Lambda}(y) = \sum_{j=1}^{k} W_{\Lambda,x_j}. \tag{1.7}$$

We are going to construct a solution for (1.3), which is close to $W_{r,\Lambda}$ for some suitable Λ and r and large k.

Theorem 1.1 is a direct consequence of the following result:

Theorem 1.4. *Under the same conditions as in Theorem 1.1, there is an integer $k_0 > 0$, such that for any integer $k \ge k_0$, (1.5) has a solution u_k of the form*

$$u_k = W_{r_k,\Lambda_k}(y) + \omega_k,$$

where $\omega_k \in H_s$, and as $k \to +\infty$, $r_k \to r_0 > 0$, $\Lambda_k \to \Lambda_0 > 0$, $\|\omega_k\|_{L^\infty} \to 0$.

On the other hand, if $N \ge 7$ and $\lambda < 0$, we have the following weaker result:

Theorem 1.5. *Suppose that $N \ge 7$ and $\lambda < \lambda_1$. Then there is a large constant $s_0 > 0$, such that for any $s > s_0$, and integer k satisfying $s^{\frac{(2-2\theta)(N-4)}{(N-6)(N-2)}} \le k \le s^{\frac{(2-\theta)(N-4)}{(N-6)(N-2)}}$, where $\theta > 0$ is a fixed small constant, (1.5) has a solution $u_{k,s}$ of the form*

$$u_{k,s} = W_{r_k,\Lambda_k}(y) + \omega_{k,s},$$

where $\omega_{k,s} \in H_s$, and as $s \to +\infty$, $r_k \to r_0 > 0$, $\Lambda_k \to \Lambda_0 > 0$, $\|\omega_{k,s}\|_{L^\infty} \to 0$.

Since $s^{\frac{(2-\theta)(N-4)}{(N-6)(N-2)}} - s^{\frac{(2-2\theta)(N-4)}{(N-6)(N-2)}} \to +\infty$ as $s \to +\infty$, Theorem 1.2 is a direct consequence of Theorem 1.5. Let us point out that in the case $N \ge 7$ and $\lambda \in (0, \lambda_1)$, the solutions in Theorem 1.5 are different from those constructed in [25, 34], where the energy of the solutions remains bounded as $s \to +\infty$.

It is easy to see that Theorem 1.3 is a direct consequence of the following result:

Theorem 1.6. *Suppose that $N = 3$ and $\lambda < \lambda_1$. Then there is a small constant $s_1 > 0$ and a large constant $k_0 > 0$ (independent of s), such that for any $s \in (0, s_1)$, and integer k satisfying*

$$k_0 \le k \le C s^{-\frac{2\tau}{1-2\tau}}, \tag{1.8}$$

for some $\tau \in (0, \frac{4}{11})$, *then* (1.5) *has a solution* $u_{k,s}$ *of the form*

$$u_{k,s} = W_{r_k,\Lambda_k}(y) + \omega_{k,s},$$

where $\omega_{k,s} \in H_s$, *and as* $s \to 0$, $r_k \to r_0 > 0$, $\Lambda_k \to \Lambda_0 > 0$, $\|\omega_{k,s}\|_{L^\infty} \to 0$.

Let make a few remarks on the conditions imposed on Theorems 1.1 and 1.2.

It is easy to see that the first eigenfunction $\varphi_1 \in H_s$ In this paper, we denote

$$\bar\varphi(r) = \varphi_1(r, 0).$$

The functional corresponding to (1.5) is

$$I(u) = \frac{1}{2}\int_{\Omega_\varepsilon} \big(|Du|^2 - \lambda\varepsilon^2 u^2\big) - \frac{1}{2^*}\int_{\Omega_\varepsilon} \big(u - s\Phi_\varepsilon\big)_+^{2^*}, \quad u \in H_s.$$

Let Γ be a connected component of the set $\Omega \cap \{x_3 = \cdots = x_N = 0\}$. Then, by $(S1)$, there are $r_2 > r_1 \geq 0$, such that

$$\bar\Gamma = \Big\{r_1 \leq \sqrt{x_1^2 + x_2^2} \leq r_2\Big\}.$$

If $N = 4, 5$, then $\frac{N-2}{2} < 2$. We obtain from Proposition A.3,

$$I(W_{r,\Lambda}) = k\Big(A_0 + \frac{A_2 s\bar\varphi(r)\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3\varepsilon^{N-2}k^{N-2}}{\bar r^{N-2}\Lambda^{N-2}} + O\big(\varepsilon^{\frac{(N-2)(1+\sigma)}{2}}\big)\Big). \tag{1.9}$$

It is easy to see that the function

$$r^{\frac{N-2}{2}}\bar\varphi(r), \quad r \in [r_1, r_2], \tag{1.10}$$

has a maximum point r_0, satisfying $r_0 \in (r_1, r_2)$, since $r_i^{\frac{N-2}{2}}\bar\varphi(r_i) = 0$, $i = 1, 2$. As a result,

$$\frac{A_2 s\bar\varphi(r)}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3}{\bar r^{N-2}\Lambda^{N-2}}, \quad (r, \lambda) \in (r_1, r_2) \times (\delta, \delta^{-1}),$$

has a maximum point (r_0, Λ_0), where

$$\Lambda_0 = \Big(\frac{2A_3}{A_2 s r_0^{N-2}\bar\varphi(r_0)}\Big)^{\frac{2}{N-2}},$$

for any fixed $s > 0$. Thus, $I(W_{r,\Lambda})$ has a maximum point in $(r_1, r_2) \times (\delta, \delta^{-1})$, if $k > 0$ is large.

If $N = 6$, then $\frac{N-2}{2} = 2$. Thus, we find from Proposition A.3,

$$I(W_{r,\Lambda}) = k\Big(A_0 + \big(-\lambda A_1 + A_2 s\bar\varphi(r)\big)\frac{\varepsilon^2}{\Lambda^2} - \frac{A_3\varepsilon^4 k^4}{\bar r^4\Lambda^4} + O\big(\varepsilon^{2+\sigma}\big)\Big), \tag{1.11}$$

It is easy to see that we can always choose a constant $s_0 > 0$, such that if $s > |\lambda| s_0$, then the function

$$g(r) = r^2(A_2 s\bar\varphi(r) - A_1\lambda), \quad r \in [r_1, r_2], \tag{1.12}$$

has a maximum point r_0, satisfying $g(r_0) > 0$, $r_0 \in (r_1, r_2)$. As a result,

$$\frac{-\lambda A_1 + A_2 s\bar{\varphi}(r)}{\Lambda^2} - \frac{A_3}{\bar{r}^4\Lambda^4}, \quad (r, \lambda) \in (r_1, r_2) \times (\delta, \delta^{-1}),$$

has maximum point (r_0, Λ_0), where

$$\Lambda_0 = \Big(\frac{2A_3}{(-\lambda A_1 + A_2 s\bar{\varphi}(r_0))r_0^4}\Big)^{\frac{1}{2}},$$

for any fixed $s > 0$. Thus, $I(W_{r,\lambda})$ has a maximum point in $(r_1, r_2) \times (\delta, \delta^{-1})$, if $k > 0$ is large.

If $N \geq 7$ and $\lambda = 0$, then Proposition A.3 gives

$$I(W_{r,\Lambda}) = k\Big(A_0 + \frac{A_2 s\bar{\varphi}(r)\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3\varepsilon^{N-2}k^{N-2}}{\bar{r}^{N-2}\Lambda^{N-2}} + O\big(\varepsilon^{\frac{(N-2)(1+\sigma)}{2}}\big)\Big), \tag{1.13}$$

So, we are in the same situation as the case $N = 4, 5$.

On the other hand, if $N \geq 7$, then $\frac{N-2}{2} > 2$. Thus $\varepsilon^{\frac{N-2}{2}}$ is a higher order term of ε^2. Thus if $\lambda < 0$, then for each fixed $s > 0$, we have

$$I(W_{r,\Lambda}) = k\Big(A_0 - \frac{\lambda A_1\varepsilon^2}{\Lambda^2} - \frac{A_3\varepsilon^{N-2}k^{N-2}}{\bar{r}^{N-2}\Lambda^{N-2}} + O\big(\varepsilon^{2+\sigma}\big)\Big), \tag{1.14}$$

But

$$-\frac{\lambda A_1}{\Lambda^2} - \frac{A_3}{\bar{r}^{N-2}\Lambda^{N-2}}, \quad (r, \lambda) \in (r_1, r_2) \times (\delta, \delta^{-1}),$$

does not have a critical point even if $\lambda < 0$. So, we don't know whether $I(W_{r,\Lambda})$ has a critical point. Thus, to obtain a solution for (1.3), we need to let s change so that

$$\varepsilon^2 << s\varepsilon^{\frac{N-2}{2}}, \quad \varepsilon << 1. \tag{1.15}$$

If (1.15) holds, then

$$I(W_{r,\Lambda}) = k\Big(A_0 + \frac{A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3\varepsilon^{N-2}k^{N-2}}{\bar{r}^{N-2}\Lambda^{N-2}} + O\big((s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\big)\Big). \tag{1.16}$$

So, we are in a similar situation as $\lambda = 0$. Note the (1.15) implies

$$k << s^{\frac{2(N-4)}{(N-2)(N-6)}}, \quad k >> s^{\frac{1}{N-2}},$$

which gives an upper bound for k. So, in this case, we are not able to obtain the existence of infinitely many solutions even if $s > 0$ is large.

In the case $N = 3$, for fixed $s > 0$, some estimates which are valid for $N \geq 4$ may not be true due to the slow decay of the function $W_{r,\Lambda}$. Under the condition $s \leq Ck^{-\frac{1}{2\tau}+1}$ for some $\tau \in (0, \frac{4}{11})$, we can recover all these estimates. But the condition $s \leq Ck^{-\frac{1}{2\tau}+1}$ imposes an upper bound (1.8) for the number of bubbles k.

The energy of the solutions obtained in Theorems 1.4 and 1.5 is very large because k must be large. This result is in consistence of the result in [19].

Finally, let us point out that the eigenvalue φ_1 is not essential in this paper. We can replace φ_1 by any function φ, satisfying $\varphi > 0$ in Ω, $\varphi = 0$ on $\partial\Omega$ and $\varphi \in H_s$.

We will use the reduction argument as in [2, 3], [12]– [14], [27, 28] and [36] to prove the main results of this paper. Unlike those papers, where a parameter always appears in some form, in Theorem 1.4, s is a fixed positive constant. To prove Theorem 1.4, the number of the bubbles k is used as a parameter to carry out the reduction. Similar idea has been used in [33, 35].

2. The reduction

In this section, we will reduce the problem of finding a k-peak solution for (1.3) to a finite dimension problem.

Let

$$\|u\|_* = \sup_y \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}} \Big)^{-1} |u(y)|, \tag{2.1}$$

and

$$\|f\|_{**} = \sup_y \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \Big)^{-1} |f(y)|, \tag{2.2}$$

where $\tau \in (0,1)$ is a constant, such that

$$\sum_{j=2}^k \frac{1}{|x_j - x_1|^\tau} \le C. \tag{2.3}$$

Recall that $\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}$, and

$$\sum_{j=2}^k \frac{1}{|x_j - x_1|^\tau} \le C\varepsilon^\tau k^\tau \sum_{j=2}^k \frac{1}{j^\tau} \le C\varepsilon^\tau k.$$

In order to achieve (2.3), we need to choose τ according to whether $s > 0$ is fixed or not. We choose τ as follows:

$$\tau = \begin{cases} \frac{1}{2}, & \text{in Theorems 1.4 and 1.5;} \\ \text{the number in (1.8)}, & \text{in Theorem 1.6.} \end{cases} \tag{2.4}$$

Let

$$Y_{i,1} = \frac{\partial W_{\Lambda,x_i}}{\partial \Lambda}, \qquad Z_{i,1} = -\Delta Y_{i,1} - \lambda\varepsilon^2 Y_{i,1} = (2^*-1)U_{\Lambda,x_i}^{2^*-2} \frac{\partial U_{\Lambda,x_i}}{\partial \Lambda},$$

and

$$Y_{i,2}=\frac{\partial W_{\Lambda,x_i}}{\partial r},\qquad Z_{i,2}=-\Delta Y_{i,2}-\lambda\varepsilon^2 Y_{i,2}=(2^*-1)U_{\Lambda,x_i}^{2^*-2}\frac{\partial U_{\Lambda,x_i}}{\partial r}.$$

We consider

$$\begin{cases}-\Delta\phi_k-\lambda\varepsilon^2\phi_k-(2^*-1)\big(W_{r,\Lambda}-s\Phi_\varepsilon\big)_+^{2^*-2}\phi_k=h+\sum\limits_{j=1}^{2}\sum\limits_{i=1}^{k}c_jZ_{i,j}, \quad \text{in } \Omega_\varepsilon,\\ \phi_k\in H_s,\\ <\sum\limits_{i=1}^{k}Z_{i,j},\phi_k>=0, \quad j=1,2,\end{cases} \tag{2.5}$$

for some number c_j, where $<u,v>=\int_{\Omega_\varepsilon}uv$.

We need the following result, whose proof is standard.

Lemma 2.1. *Let f satisfy $\|f\|_{**}<\infty$ and let u be the solution of*

$$-\Delta u-\lambda\varepsilon^2u=f \quad in \quad \Omega_\varepsilon, \qquad u=0 \quad on \quad \partial\Omega_\varepsilon,$$

where $\lambda<\lambda_1$. Then we have

$$|u(x)|\le C\int_{\Omega_\varepsilon}\frac{|f(y)|}{|x-y|^{N-2}}dy.$$

Next, we need the following lemma to carry out the reduction.

Lemma 2.2. *Assume that ϕ_k solves (2.5) for $h=h_k$. If $\|h_k\|_{**}$ goes to zero as k goes to infinity, so does $\|\phi_k\|_*$.*

Proof. We argue by contradiction. Suppose that there are $k\to+\infty$, $h=h_k$, $\Lambda_k\in[\delta,\delta^{-1}]$, and ϕ_k solving (2.5) for $h=h_k$, $\Lambda=\Lambda_k$, with $\|h_k\|_{**}\to 0$, and $\|\phi_k\|_*\ge c'>0$. We may assume that $\|\phi_k\|_*=1$. For simplicity, we drop the subscript k.

By Lemma 2.1,

$$\begin{aligned}|\phi(y)|\le&C\int_{\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}W_{r,\Lambda}^{2^*-2}|\phi(z)|\,dz\\&+C\int_{\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}\Big(|h(z)|+|\sum_{j=1}^{2}\sum_{i=1}^{k}c_jZ_{i,j}(z)|\Big)\,dz\end{aligned} \tag{2.6}$$

Using Lemma B.4 and B.5, there is a strictly positive number θ such that

$$\begin{aligned}&\Big|\int_{\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}W_{r,\Lambda}^{2^*-2}\phi(z)\,dz\Big|\\ \le&C\|\phi\|_*\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.\end{aligned} \tag{2.7}$$

It follows from Lemma B.3 that

$$\begin{aligned}
&\Big|\int_{\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}h(z)\,dz\Big|\\
\le& C\|h\|_{**}\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\,dz\\
\le& C\|h\|_{**}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}},
\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
&\Big|\int_{\Omega_\varepsilon}\frac{1}{|z-y|^{N-2}}\sum_{i=1}^{k}Z_{i,j}(z)\,dz\Big|\\
\le& C\sum_{i=1}^{k}\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\frac{1}{(1+|z-x_i|)^{N+2}}\,dz\\
\le& C\sum_{i=1}^{k}\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.
\end{aligned}\tag{2.9}$$

Next, we estimate c_j. Multiplying (2.5) by $Y_{1,l}$ and integrating, we see that c_j satisfies

$$\Big\langle\sum_{j=1}^{2}\sum_{i=1}^{k}Z_{i,j},Y_{1,l}\Big\rangle c_j=\Big\langle-\Delta\phi-\lambda\varepsilon^2\phi-(2^*-1)W_{r,\Lambda}^{2^*-2}\phi,Y_{1,l}\Big\rangle-\Big\langle h,Y_{1,l}\Big\rangle.\tag{2.10}$$

It follows from Lemma B.2 that

$$\begin{aligned}
\big|\big\langle h,Y_{1,l}\big\rangle\big|\le& C\|h\|_{**}\int_{\mathbb{R}^N}\frac{1}{(1+|z-x_1|)^{N-2-\beta}}\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N+2}{2}+\tau}}\,dz\\
\le& C\|h\|_{**},
\end{aligned}$$

since $\beta>0$ can be chosen as small as desired.

On the other hand,

$$\begin{aligned}
&\Big\langle-\Delta\phi-\lambda\varepsilon^2\phi-(2^*-1)W_{r,\Lambda}^{2^*-2}\phi,Y_{1,l}\Big\rangle\\
=&\Big\langle-\Delta Y_{1,l}-\lambda\varepsilon^2Y_{1,l}-(2^*-1)W_{r,\Lambda}^{2^*-2}Y_{1,l},\phi\Big\rangle\\
=&(2^*-1)\Big\langle U_{\Lambda,x_1}^{2^*-2}\partial_l U_{\Lambda,x_1}-W_{r,\Lambda}^{2^*-2}Y_{1,l},\phi\Big\rangle,
\end{aligned}\tag{2.11}$$

where $\partial_l=\partial_\Lambda$ if $l=1$, $\partial_l=\partial_r$ if $l=2$.

By Lemmas B.1,

$$|\phi(y)|\le C\|\phi\|_*.$$

We consider the cases $N \geq 6$ first. Note that $\frac{4}{N-2} \leq 1$ for $N \geq 6$. Using Lemmas A.1 and B.2, noting that

$$|W_{r,\Lambda}^{2^*-2} - W_{\Lambda,x_1}^{2^*-2}| \leq \sum_{j=2}^{k} W_{\Lambda,x_j}^{2^*-2},$$

and

$$\varepsilon \leq \frac{C}{1+|z-x_1|},$$

we obtain

$$\begin{aligned}
&\Big|\big\langle U_{\Lambda,x_1}^{2^*-2}\partial_l U_{\Lambda,x_j} - W_{r,\Lambda}^{2^*-2} Y_{1,l}, \phi\big\rangle\Big| \\
\leq & C\|\phi\|_* \int_{\Omega_\varepsilon} \frac{1}{(1+|z-x_1|)^{N-2-\beta}} \sum_{i=2}^{k} \frac{1}{(1+|z-x_i|)^{4-\beta}}\, dz \\
& + C\|\phi\|_* \int_{\Omega_\varepsilon} U_{\Lambda,x_1}^{2^*-2}\big(\varepsilon^{N-2} + \frac{|\lambda|\varepsilon^2}{(1+|y-x_j|)^{N-4-\beta}}\big) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}} \\
& + C\|\phi\|_* \int_{\Omega_\varepsilon} U_{\Lambda,x_1}\big(\varepsilon^{N-2} + \frac{|\lambda|\varepsilon^2}{(1+|y-x_j|)^{N-4-\beta}}\big)^{2^*-2} \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}} \\
\leq & C\|\phi\|_* \sum_{j=2}^{k} \frac{1}{|x_1-x_j|^{1+\sigma}} + o(1)\|\phi\|_* = o(1)\|\phi\|_*.
\end{aligned} \tag{2.12}$$

For $N = 3,4,5$, we have $\frac{4}{N-2} > 1$. By Lemmas B.1, B.2,

$$
\begin{aligned}
&\Big|\big\langle U_{\Lambda,x_1}^{2^*-2}\partial_l U_{\Lambda,x_j}-W_{r,\Lambda}^{2^*-2}Y_{1,l},\phi\big\rangle\Big| \\
\le & C\int_{\Omega_\varepsilon} W_{\Lambda,x_1}^{2^*-3}\sum_{j=2}^k W_{\Lambda,x_j}|Y_{1,l}\phi| + C\int_{\Omega_\varepsilon}\big(\sum_{j=2}^k W_{\Lambda,x_j}\big)^{\frac{4}{N-2}}|Y_1\phi| \\
&+C\|\phi\|_*\int_{\Omega_\varepsilon} U_{\Lambda,x_1}^{2^*-2}\big(\varepsilon^{N-2}+\frac{|\lambda|\varepsilon^2}{(1+|y-x_j|)^{N-4-\beta}}\big)\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}} \\
&+C\|\phi\|_*\int_{\Omega_\varepsilon} U_{\Lambda,x_1}\big(\varepsilon^{N-2}+\frac{|\lambda|\varepsilon^2}{(1+|y-x_j|)^{N-4-\beta}}\big)^{2^*-2}\sum_{j=1}^k\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}} \qquad (2.13)\\
\le & C\|\phi\|_*\int_{\Omega_\varepsilon}\frac{1}{(1+|z-x_1|)^{4-\beta}}\sum_{j=2}^k\frac{1}{(1+|z-x_j|)^{N-2-\beta}} \\
&+C\int_{\Omega_\varepsilon}\big(\sum_{j=2}^k U_{\Lambda,x_j}^{1-\beta}\big)^{\frac{4}{N-2}}|Y_{1,l}\phi|+o(1)\|\phi\|_* \\
\le & C\|\phi\|_*\int_{\Omega_\varepsilon}\frac{1}{(1+|z-x_1|)^{N-2-\beta}}\big(\sum_{j=2}^k U_{\Lambda,x_j}^{1-\beta}\big)^{\frac{4}{N-2}}\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
&+o(1)\|\phi\|_*.
\end{aligned}
$$

Let

$$
\Omega_j=\big\{y=(y',y'')\in\Omega_\varepsilon:\ \big\langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\big\rangle\ge\cos\frac{\pi}{k}\big\}.
$$

If $y\in\Omega_1$, then

$$
\begin{aligned}
\sum_{j=2}^k U_{\Lambda,x_j}^{1-\beta}\le&\frac{1}{(1+|y-x_1|)^{N-2-\tau-(N-2)\beta-\theta}}\sum_{j=2}^k\frac{1}{|x_j-x_1|^{\tau+\theta}}\\
=&o(1)\frac{1}{(1+|y-x_1|)^{N-2-\tau-(N-2)\beta-\theta}},
\end{aligned}
$$

and

$$
\sum_{i=1}^k\frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\le\frac{C}{(1+|y-x_1|)^{\frac{N-2}{2}}}.
$$

So, we obtain

$$
\begin{aligned}
&\int_{\Omega_1} \frac{1}{(1+|z-x_1|)^{N-2-\beta}}\Bigl(\sum_{j=2}^k U_{\Lambda,x_j}^{1-\beta}\Bigr)^{\frac{4}{N-2}} \sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\
=&o(1)\int_{\Omega_1} \frac{1}{(1+|z-x_1|)^{N+\frac{N+2}{2}-\frac{4(\tau+\theta)}{N-2}-4\beta}} = o(1),
\end{aligned}
$$

since $\frac{N+2}{2}-\frac{4(\tau+\theta)}{N-2}-4\beta>0$, if $\beta>0$ and $\theta>0$ are small.

If $y\in\Omega_l$, $l\ge 2$, then

$$
\sum_{j=2}^k U_{\Lambda,x_j}^{1-\beta} \le \frac{C}{(1+|y-x_l|)^{N-2-\tau-(N-2)\beta}},
$$

and

$$
\sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \le \frac{C}{(1+|y-x_l|)^{\frac{N-2}{2}}}.
$$

As a result,

$$
\begin{aligned}
&\int_{\Omega_l} \frac{1}{(1+|z-x_1|)^{N-2}}\Bigl(\sum_{j=2}^k U_{\Lambda,x_j}^{1-\beta}\Bigr)^{\frac{4}{N-2}} \sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\
=\le& C\int_{\Omega_l} \frac{1}{(1+|z-x_1|)^{N-2}}\frac{1}{(1+|y-x_l|)^{4-4\beta-\frac{4\tau}{N-2}+\frac{N-2}{2}}}\\
\le& \frac{C}{|x_l-x_1|^{\frac{N+2}{2}-\frac{4\tau}{N-2}-\theta-4\beta}},
\end{aligned}
$$

where $\theta>0$ is a fixed small constant.

Since $\tau=\frac12$ for $N\ge 4$, and $\tau<\frac12$ for $N=3$, we find that for $\theta>0$ and $\beta>0$ small, $\frac{N+2}{2}-\frac{4\tau}{N-2}-\theta-4\beta>\tau$. Thus

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \frac{1}{(1+|z-x_1|)^{N-2}}\Bigl(\sum_{j=2}^k U_{\Lambda,x_j}^{1-\beta}\Bigr)^{\frac{4}{N-2}} \sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}\\
\le& o(1)+C\sum_{l=2}^k \frac{1}{|x_l-x_1|^{\frac{N+2}{2}-\frac{4\tau}{N-2}-\theta}} = o(1).
\end{aligned}
$$

So, we have proved

$$
\Bigl|\bigl\langle U_{\Lambda,x_1}^{2^*-2}\partial_l U_{\Lambda,x_j} - W_{r,\Lambda}^{2^*-2}Y_1,\phi\bigr\rangle\Bigr| = o(1)\|\phi\|_*.
$$

But there is a constant $\bar c>0$,

$$\Big\langle\sum_{j=1}^{2}\sum_{i=1}^{k} Z_{i,j}, Y_{1,l}\Big\rangle = \bar{c}\delta_{lj} + o(1).$$

Thus we obtain that

$$c_l = o(\|\phi\|_*) + O(\|h\|_{**}).$$

So,

$$\|\phi\|_* \le \Big(o(1) + \|h_k\|_{**} + \frac{\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}}\Big). \tag{2.14}$$

Since $\|\phi\|_* = 1$, we obtain from (2.14) that there is $R > 0$, such that

$$\|\phi(y)\|_{B_R(x_i)} \ge c_0 > 0, \tag{2.15}$$

for some i. But $\bar{\phi}(y) = \phi(y - x_i)$ converges uniformly in any compact set of $\mathbb{R}^N_+$ to a solution u of

$$\Delta u + (2^* - 1)U_{\Lambda,0}^{2^*-2} u = 0 \tag{2.16}$$

for some $\Lambda \in [\delta, \delta^{-1}]$, and u is perpendicular to the kernel of (2.16). So, $u = 0$. This is a contradiction to (2.15).

□

From Lemma 2.2, using the same argument as in the proof of Proposition 4.1 in [12], we can prove the following result :

Proposition 2.3. *There exists $k_0 > 0$ and a constant $C > 0$, independent of k, such that for all $k \ge k_0$ and all $h \in L^\infty(\Omega_\varepsilon)$, problem (2.5) has a unique solution $\phi \equiv L_k(h)$. Besides,*

$$\|L_k(h)\|_* \le C\|h\|_{**}, \qquad |c_j| \le C\|h\|_{**}. \tag{2.17}$$

Moreover, the map $L_k(h)$ is C^1 with respect to Λ.

Now, we consider

$$\begin{cases} -\Delta\big(W_{r,\Lambda} + \phi\big) - \lambda\varepsilon^2(W_{r,\Lambda} + \phi) = \big(W_{r,\Lambda} + \phi - s\Phi_\varepsilon\big)_+^{2^*-1} + \sum_{j=1}^{2}\sum_{i=1}^{k} c_j Z_{i,j}, \quad \text{in } \Omega_\varepsilon, \\ \phi \in H_s, \\ < \sum_{i=1}^{k} Z_{i,j}, \phi >= 0, \quad j = 1, 2. \end{cases} \tag{2.18}$$

We have

Proposition 2.4. *There is an integer $k_0>0$, such that for each $k\ge k_0$, $r_1\le r\le r_2$, $\delta\le\Lambda\le\delta^{-1}$, where δ is a fixed small constant,* (2.18) *has a unique solution ϕ, satisfying*

$$\|\phi\|_*\le C(s\varepsilon^{\frac{N-2}{2}})^{\frac{1}{2}+\sigma}+C|\lambda|\varepsilon^{1+\sigma},$$

where $\sigma>0$ is a fixed small constant. Moreover, $\Lambda\to\phi(\Lambda)$ is C^1.

Rewrite (2.18) as

$$\begin{cases}-\Delta\phi-\lambda\varepsilon^2\phi-(2^*-1)(W_{r,\Lambda}-s\Phi_\varepsilon)_+^{2^*-2}\phi=N(\phi)+l_k+\sum\limits_{j=1}^{2}\sum\limits_{i=1}^{k}c_jZ_{i,j}, \quad \text{in }\Omega_\varepsilon,\\ \phi\in H_s,\\ <\sum\limits_{i=1}^{k}Z_{i,j},\phi>=0,\ j=1,2,\end{cases}\tag{2.19}$$

where

$$N(\phi)=\big(W_{r,\Lambda}-s\Phi_\varepsilon+\phi\big)_+^{2^*-1}-(W_{r,\Lambda}-s\Phi_\varepsilon)_+^{2^*-1}-(2^*-1)(W_{r,\Lambda}-s\Phi_\varepsilon)_+^{2^*-2}\phi,$$

and

$$l_k=\Big(W_{r,\Lambda}^{2^*-1}-\sum_{j=1}^{k}U_{\Lambda,x_j}^{2^*-1}\Big)+(W_{r,\Lambda}-s\Phi_\varepsilon)_+^{2^*-1}-W_{r,\Lambda}^{2^*-1}.$$

In order to use the contraction mapping theorem to prove that (2.19) is uniquely solvable in the set that $\|\phi\|_*$ is small, we need to estimate $N(\phi)$ and l_k.

Lemma 2.5. *We have*

$$\|N(\phi)\|_{**}\le C\|\phi\|_*^{\min(2^*-1,2)}.$$

Proof. We have

$$|N(\phi)|\le\begin{cases}C|\phi|^{2^*-1}, & N\ge6;\\ C\big(W_{r,\Lambda}^{\frac{6-N}{N-2}}\phi^2+|\phi|^{2^*-1}\big), & N=3,4,5.\end{cases}$$

Firstly, we consider $N\ge6$. We have

$$
\begin{aligned}
|N(\phi)| \leq & C\|\phi\|_*^{2^*-1}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} \\
\leq & C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\tau}}\Big)^{\frac{4}{N-2}} \\
\leq & C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}},
\end{aligned}
\tag{2.20}
$$

where we use the inequality

$$
\sum_{j=1}^k a_j b_j \leq \Big(\sum_{j=1}^k a_j^p\Big)^{\frac{1}{p}}\Big(\sum_{j=1}^k b_j^q\Big)^{\frac{1}{q}}, \quad \frac{1}{p}+\frac{1}{q}=1, a_j, b_j \geq 0, j=1,\dots,k,
$$

and

$$
\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\tau}} \leq C + \sum_{j=2}^k \frac{C}{|x_1-x_j|^{\tau}} \leq C.
$$

which follows from Lemma B.1.

For $N=3,4,5$, similarly to the case $N\geq 6$, we have

$$
\begin{aligned}
& |N(\phi)| \\
\leq & C\|\phi\|_*^2\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N-2-\beta}}\Big)^{\frac{6-N}{N-2}}\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^2 \\
& + C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \\
\leq & C\|\phi\|_*^2\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1} + C\|\phi\|_*^{2^*-1}\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \\
\leq & C\|\phi\|_*^2\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
\tag{2.21}
$$

□

Next, we estimate l_k.

Lemma 2.6. *We have*

$$
\|l_k\|_{**} \leq C\big(s\varepsilon^{\frac{N-2}{2}}\big)^{\frac{1}{2}+\sigma} + C|\lambda|\varepsilon^{1+\sigma},
$$

where $\sigma>0$ is a fixed small constant.

Proof. Recall

$$\Omega_j = \big\{ y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \rangle \ge \cos \frac{\pi}{k} \big\}.$$

From the symmetry, we can assume that $y \in \Omega_1$. Then,

$$|y - x_j| \ge |y - x_1|, \quad \forall\, y \in \Omega_1.$$

Thus, for $y \in \Omega_1$, by Lemma A.1,

$$\begin{aligned} |l_k| \le & \frac{C}{(1+|y-x_1|)^{4-\beta}} \sum_{j=2}^k \frac{1}{(1+|y-x_j|)^{N-2-\beta}} + C\Big(\sum_{j=2}^k \frac{1}{(1+|y-x_j|)^{N-2-\beta}} \Big)^{2^*-1} \\ & + C \sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{4-\beta}} \big(\varepsilon^{N-2} + \frac{|\lambda| \varepsilon^2}{(1+|y-x_j|)^{N-4-\beta}} \big) \\ & + C W_{r,\Lambda}^{2^*-1-\frac{1}{2}-\frac{2\sigma}{N-2}} s^{\frac{1}{2}+\frac{2\sigma}{N-2}} \varepsilon^{\frac{N-2}{4}+\sigma}. \end{aligned} \tag{2.22}$$

Here, we have used the inequality: for any bounded $a > 0$ and $b > 0$, $\alpha \in (0, 1]$:

$$\big| (a-b)_+^{2^*-1} - a^{2^*-1} \big| \le C a^{2^*-1-\alpha} b^{\alpha}.$$

Let us estimate the first term of (2.22). Using Lemma B.2, we obtain

$$\begin{aligned} & \frac{1}{(1+|y-x_1|)^{4-\beta}} \frac{1}{(1+|y-x_j|)^{N-2-\beta}} \\ \le & C \Big(\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} + \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \Big) \frac{1}{|x_j - x_1|^{\frac{N+2}{2}-\tau-2\beta}} \\ \le & C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \frac{1}{|x_j - x_1|^{\frac{N+2}{2}-\tau-2\beta}}, \quad j > 1. \end{aligned} \tag{2.23}$$

Since $\frac{N+2}{2} - \tau - 2\beta > 1$, we find

$$\begin{aligned} & \frac{1}{(1+|y-x_1|)^{4-\beta}} \sum_{j=2}^k \frac{1}{(1+|y-x_j|)^{N-2-\beta}} \\ \le & C \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}} \big(k\varepsilon \big)^{\frac{N+2}{2}-\tau-2\beta} \le C \big(s \varepsilon^{\frac{N-2}{2}} \big)^{\frac{1}{2}+\sigma} \frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}. \end{aligned} \tag{2.24}$$

Here we have used

$$\big(k\varepsilon \big)^{\frac{N+2}{2}-\tau-2\beta} = O\Big(\big(s \varepsilon^{\frac{N-2}{2}} \big)^{\frac{1}{2}+\sigma} \Big), \tag{2.25}$$

for some small $\sigma > 0$.

In fact, if $s>0$ is fixed (as in Theorem 1.4), then $k=\frac{1}{\sqrt{\varepsilon}}$ and $\tau=\frac{1}{2}$. As a result,

$$\big(k\varepsilon\big)^{\frac{N+2}{2}-\tau-2\beta}=O\big(\varepsilon^{\frac{N+2}{4}-\frac{\tau}{2}-\beta}\big)=O\big(\varepsilon^{\frac{N-2}{4}+\sigma}\big).$$

So, we obtain (2.25).

If $N\ge 7$, then $\tau=\frac{1}{2}$, and

$$s^{\frac{(2-2\theta)(N-4)}{(N-6)(N-2)}}\le k\le s^{\frac{(2-\theta)(N-4)}{(N-6)(N-2)}}. \tag{2.26}$$

But

$$\big(k\varepsilon\big)^{\frac{N+2}{2}-\tau-2\beta}=\Big(\frac{s^{\frac{2}{N-2}}}{k}\Big)^{\frac{N+2}{2}-\tau-2\beta}=\frac{s^{\frac{N+1-4\beta}{N-2}}}{k^{\frac{N+1-4\beta}{2}}}$$

and

$$\big(s\varepsilon^{\frac{N-2}{2}}\big)^{\frac{1}{2}+\sigma}=\big(\frac{s^2}{k^{N-2}}\big)^{\frac{1}{2}+\sigma}$$

Thus, we see that (2.25) is equivalent to

$$s^{\frac{3-4\beta}{N-2}-2\sigma}\le Ck^{\frac{3}{2}-2\beta-(N-2)\sigma}. \tag{2.27}$$

Using (2.26), we find (2.27) holds.

For $N=3$, $k=\frac{s}{\sqrt{\varepsilon}}$. Thus,

$$(k\varepsilon)^{\frac{5}{2}-\tau-2\beta}=(s\varepsilon^{\frac{1}{2}})^{\frac{5}{2}-\tau-2\beta}\le C(s\varepsilon^{\frac{1}{2}})^{\frac{1}{2}+\sigma}.$$

So, we obtain (2.25).

Now, we estimate the second term of (2.22).

Using Lemma B.2 again, we find for $y\in\Omega_1$,

$$\begin{aligned}&\frac{1}{(1+|y-x_j|)^{N-2-\beta}}\le\frac{1}{(1+|y-x_1|)^{\frac{N-2-\beta}{2}}}\frac{1}{(1+|y-x_j|)^{\frac{N-2-\beta}{2}}}\\ \le&\frac{C}{|x_j-x_1|^{\frac{N-2}{2}-\beta-\frac{N-2}{N+2}\tau}}\Big(\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}+\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}\Big)\\ \le&\frac{C}{|x_j-x_1|^{\frac{N-2}{2}-\beta-\frac{N-2}{N+2}\tau}}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}.\end{aligned} \tag{2.28}$$

Suppose that $N\ge 5$. Then $\frac{N-2}{2}-\beta-\frac{N-2}{N+2}\tau>1$ since $\tau<1$. Then (2.28) gives for $y\in\Omega_1$

$$
\begin{aligned}
&\Bigl(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2-\beta}}\Bigr)^{2^*-1}\\
\le& C\bigl(k\varepsilon\bigr)^{\frac{N+2}{2}-\tau-(2^*-1)\beta}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}=C\bigl(s\varepsilon^{\frac{N-2}{2}}\bigr)^{\frac{1}{2}+\sigma}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
\tag{2.29}
$$

If $N=3,4$, then (2.28) gives

$$
\begin{aligned}
&\Bigl(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{N-2-\beta}}\Bigr)^{2^*-1}\\
\le& C\bigl(k\varepsilon^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau-\beta}\bigr)^{2^*-1}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}\\
=& Ck^{\frac{N+2}{N-2}}\varepsilon^{\frac{N+2}{2}-\tau-(2^*-1)\beta}\frac{1}{(1+|y-x_1|)^{\frac{N+2}{2}+\tau}}.
\end{aligned}
\tag{2.30}
$$

If $N=4$, then

$$
k^{\frac{N+2}{N-2}}\varepsilon^{\frac{N+2}{2}-\tau-(2^*-1)\beta}=k^3\varepsilon^{3-\frac{1}{2}-(2^*-1)\beta}\le C\varepsilon^{1-(2^*-1)\beta}\le C\varepsilon^{\frac{1}{2}+\sigma}.
$$

Hence for $N=4$,

$$
\left(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^{2}}\right)^{2^*-1}\le\sum_{i=1}^{k}\frac{C\varepsilon^{\frac{N-2}{4}+\sigma}}{(1+|y-x_i|)^{\frac{N+2}{2}+\tau}}.
$$

For $N=3$, we have

$$
k^5\varepsilon^{\frac{5}{2}-\tau-(2^*-1)\beta}=k^{2\tau+2(2^*-1)\beta}s^{5-2\tau-2(2^*-1)\beta}.
$$

But

$$
\bigl(s\varepsilon^{\frac{1}{2}}\bigr)^{\frac{1}{2}+\sigma}=\frac{s^{1+2\sigma}}{k^{\frac{1}{2}+\sigma}}.
$$

So, $k^5\varepsilon^{\frac{5}{2}-\tau-(2^*-1)\beta}\le C(s\varepsilon^{\frac{1}{2}})^{\frac{1}{2}+\sigma}$ is equivalent to

$$
k\le Cs^{-\frac{8-4\tau-4\sigma-4(2^*-1)\beta)}{1+4\tau+2\sigma+4(2^*-1)\beta)}}
\tag{2.31}
$$

Since $k\le Cs^{-\frac{2\tau}{1-2\tau}}$, we see that (2.31) is valid if

$$
\frac{8-4\tau}{1+4\tau}>\frac{2\tau}{1-2\tau}.
$$

Thus, if $\tau\in(0,\frac{4}{11})$, (2.31) holds. Hence for $N=3$, we also have

$$\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-x_j|)^2}\Big)^{2^*-1}\le\sum_{i=1}^{k}\frac{C(s\varepsilon^{\frac{N-2}{2}})^{\frac12+\sigma}}{(1+|y-x_i|)^{\frac{N+2}{2}+\tau}}.$$

Note that for $y\in\Omega_1$,

$$W_{r,\Lambda}(y)\le\frac{C}{(1+|y-x_1|)^{N-2-\tau-\beta}}.$$

We claim that

$$(\frac{N+2}{N-2}-\frac12-\frac{2\sigma}{N-2})(N-2-\tau)\ge\frac{N+2}{2}+\tau, \tag{2.32}$$

if $N\ge 3$.

In fact, (2.32) is equivalent to

$$\tau<\frac{4(N-2)}{3N+2},$$

which is true, since $\tau=\frac12$ if $N\ge4$, $\tau<\frac{4}{11}$ if $N=3$.

Thus, we obtain

$$s^{\frac12+\frac{2\sigma}{N-2}}\varepsilon^{\frac{N-2}{4}+\sigma}W_{r,\Lambda}^{\frac{N+2}{N-2}-\frac12-\frac{2\sigma}{N-2}}\le Cs^{\frac12+\frac{2\sigma}{N-2}}\varepsilon^{\frac{N-2}{4}+\sigma}\frac{C}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.$$

Finally,

$$\begin{aligned}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^4}\frac{|\lambda|\varepsilon^2}{(1+|y-x_j|)^{N-4-\beta}}=&\sum_{j=1}^{k}\frac{|\lambda|\varepsilon^2}{(1+|y-x_j|)^{N-\beta}}\\ \le C|\lambda|\varepsilon^2\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}},&\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^4}\varepsilon^{N-2}\le C\varepsilon^{N-2-\frac{N-6}{2}-\tau}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\\ =&C\varepsilon^{\frac{N+2}{2}-\tau}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\le C(k\varepsilon)^{\frac{N+2}{2}-\tau}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\\ \le&C(s\varepsilon^{\frac{N-2}{2}})^{\frac12+\sigma}\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}.\end{aligned}$$

Combining all the above estimates, we obtain the result.

□

Now, we are ready to prove Proposition 2.4.

Proof of Proposition 2.4. Let us recall that

$$\varepsilon = \frac{s^{\frac{2}{N-2}}}{k^2}.$$

Let

$$E_N = \Big\{u : u \in C(\Omega_\varepsilon), \|u\|_* \le \sqrt{s}\varepsilon^{\frac{N-2}{4}}, \int_{\Omega_\varepsilon} \sum_{i=1}^k Z_{i,j}\phi = 0, j = 1, 2\Big\}$$

Then, (2.19) is equivalent to

$$\phi = A(\phi) =: L(N(\phi)) + L(l_k).$$

Now we prove that A is a contraction map from E_N to E_N. Using Lemma 2.5, we have

$$\begin{aligned}
&\|A\phi\|_* \le C\|N(\phi)\|_{**} + C\|l_k\|_{**} \le C\|\phi\|_*^{\min(2^*-1,2)} + C\|l_k\|_{**} \\
\le& C(\sqrt{s}\varepsilon^{\frac{N-2}{4}})^{\min(2^*-1,2)} + C\|l_k\|_{**} \\
\le& C(\sqrt{s}\varepsilon^{\frac{N-2}{4}})^{1+\sigma} + C\|l_k\|_{**}.
\end{aligned} \tag{2.33}$$

Thus, by Lemma 2.6, we find that A maps E_N to E_N.

Next, we show that A is a contraction map.

$$\|A(\phi_1) - A(\phi_2)\|_* = \|L(N(\phi_1)) - L(N(\phi_2))\|_* \le C\|N(\phi_1) - N(\phi_2)\|_{**}.$$

Using

$$|N'(t)| \le \begin{cases} C|t|^{2^*-2}, & N \ge 6; \\ C\big(W^{\frac{6-N}{N-2}}|\phi| + |\phi|^{2^*-2}\big), & N = 3, 4, 5, \end{cases}$$

we can prove that

$$\begin{aligned}
&\|A(\phi_1) - A(\phi_2)\|_* \le C\|N(\phi_1) - N(\phi_2)\|_{**} \\
\le& C\big(\|\phi_1\|_*^{\min(1,2^*-2)} + \|\phi_2\|_*^{\min(1,2^*-2)}\big)\|\phi_1 - \phi_2\|_* \le \frac{1}{2}\|\phi_1 - \phi_2\|_*.
\end{aligned}$$

Thus, A is a contraction map.

It follows from the contraction mapping theorem that there is a unique $\phi \in E_N$, such that

$$\phi = A(\phi).$$

Moreover, it follows from (2.33) that

$$\|\phi\|_* \le C(\sqrt{s}\varepsilon^{\frac{N-2}{4}})^{1+\sigma} + C\|l_k\|_{**}.$$

So, the estimate for $\|\phi\|_*$ follows from Lemma 2.6.

□

3. Proof of the Main Results

Let

$$F(r,\Lambda)=I\big(W_{r,\Lambda}+\phi\big),$$

where ϕ is the function obtained in Proposition 2.4, and let

$$I(u)=\frac{1}{2}\int_{\Omega_\varepsilon}(|Du|^2-\lambda\varepsilon^2u^2)-\frac{1}{2^*}\int_{\Omega_\varepsilon}(u-s\Phi_\varepsilon)_+^{2^*}.$$

Using the symmetry, we can check that if Λ is a critical point of $F(\Lambda)$, then $W_{r,\Lambda}+\phi$ is a solution of (1.3).

Proposition 3.1. *We have*
We have

$$F(r,\Lambda)=k\Big(A_0+\frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}}-\frac{A_3k^{N-2}\varepsilon^{N-2}}{r^{N-2}\Lambda^{N-2}}+O\big((s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}+(k\varepsilon)^{(N-2)(1+\sigma)}\big)\Big),\quad N=3,4;$$

and

$$\begin{aligned}F(r,\Lambda)=k\Big(A_0-\frac{A_1\lambda\varepsilon^2}{\Lambda^2}+\frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}}-\frac{A_3k^{N-2}\varepsilon^{N-2}}{r^{N-2}\Lambda^{N-2}}\\+O\Big(|\lambda|\varepsilon^{2+\sigma}+\big(s\varepsilon^{\frac{N-2}{2}}\big)^{1+\sigma}+(k\varepsilon)^{(N-2)(1+\sigma)}\Big)\Big),\quad N\ge 5.\end{aligned}$$

where the constant $A_i>0, i=0,1,2$ are positive constants, which are given in Proposition A.3.

Proof. There is $t\in(0,1)$, such that

$$\begin{aligned}F(r,\Lambda)&=I(W_{r,\Lambda})+\big\langle I'\big(W_{r,\Lambda}\big),\phi\big\rangle+\frac{1}{2}D^2I\big(W_{r,\Lambda}+t\phi\big)(\phi,\phi)\\
=&I(W_{r,\Lambda})-\int_{\Omega_\varepsilon}l_k\phi+\int_{\Omega_\varepsilon}\big(|D\phi|^2+\varepsilon^2\mu\phi^2-(2^*-1)\big(W_{r,\Lambda}-s\Phi_\varepsilon+t\phi\big)_+^{2^*-2}\phi^2\big)\\
=&I(W_{r,\Lambda})-(2^*-1)\int_{\Omega_\varepsilon}\Big(\big(W_{r,\Lambda}-s\Phi_\varepsilon+t\phi\big)_+^{2^*-2}-(W_{r,\Lambda}-s\Phi_\varepsilon)_+^{2^*-2}\Big)\phi^2+\int_{\Omega_\varepsilon}N(\phi)\phi\\
=&I(W_{r,\Lambda})-(2^*-1)\int_{\Omega_\varepsilon}\Big(\big(W_{r,\Lambda}-s\Phi_\varepsilon+t\phi\big)_+^{2^*-2}-(W_{r,\Lambda}-s\Phi_\varepsilon)_+^{2^*-2}\Big)\phi^2\\
&+O\Big(\int_{\Omega_\varepsilon}|N(\phi)||\phi|\Big).\end{aligned}\tag{3.1}$$

But

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} |N(\phi)||\phi| \\
\le & C\|N(\phi)\|_{**}\|\phi\|_* \int_{\Omega_\varepsilon} \sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}}.
\end{aligned} \tag{3.2}
$$

Using Lemma B.2, we find

$$
\begin{aligned}
&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \sum_{i=1}^k \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
=&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+2\tau}} + \sum_{j=1}^k \sum_{i\ne j} \frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}} \frac{1}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau}} \\
\le&\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+2\tau}} + C\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+\frac{1}{2}\tau}} \sum_{i=2}^k \frac{1}{|x_i-x_1|^{\frac{3}{2}\tau}} \\
\le& C\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{N+\frac{1}{2}\tau}},
\end{aligned}
$$

Thus, we obtain

$$
\int_{\Omega_\varepsilon} |N(\phi)||\phi| \le Ck\|N(\phi)\|_{**}\|\phi\|_* \le Ck\|\phi\|_*^2 \le Ck\Big(|\lambda|\varepsilon^{2+\sigma} + \big(s\varepsilon^{\frac{N-2}{2}}\big)^{1+\sigma}\Big).
$$

Now

$$
\big(W_{r,\Lambda} - s\Phi_\varepsilon + t\phi\big)_+^{2^*-2} - \big(W_{r,\Lambda} - s\Phi_\varepsilon\big)_+^{2^*-2} = \begin{cases} O\big(|\phi|^{2^*-2}\big), & N\ge 6; \\ O\big(W_{r,\Lambda}^{\frac{6-N}{N-2}}|\phi| + |\phi|^{2^*-2}\big), & N=3,4,5. \end{cases}
$$

Thus, we have

$$
\begin{aligned}
&\Big|\int_{\Omega_\varepsilon} \Big(\big(W_{r,\Lambda} - s\Phi_\varepsilon + t\phi\big)^{2^*-2} - \Big(\big(W_{r,\Lambda} - s\Phi_\varepsilon\big)^{2^*-2}\Big)\phi^2\Big| \\
\le & C\|\phi\|_*^{2^*} \int_{\Omega_\varepsilon} \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*},
\end{aligned}
$$

if $N\ge 6$. If $N=3,4,5$, noting that $N-2 > \frac{N-2}{2}+\tau$, we obtain

$$
\begin{aligned}
&\Big|\int_{\Omega_\varepsilon}\Big(\big(W_{r,\Lambda}-s\Phi_\varepsilon+t\phi\big)^{2^*-2}-\Big(\big(W_{r,\Lambda}-s\Phi_\varepsilon\big)^{2^*-2}\Big)\phi^2\Big|\\
\le& C\int_{\Omega_\varepsilon}W_{r,\Lambda}^{\frac{6-N}{N-2}}|\phi|^3+C\int_{\Omega_\varepsilon}|\phi|^{2^*}\le\|\phi\|_*^3\int_{\Omega_\varepsilon}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}.
\end{aligned}
$$

Let $\bar\eta>0$ small. Using Lemma B.2, if $y\in\Omega_1$, then

$$
\begin{aligned}
&\sum_{j=2}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\\
\le&\sum_{j=2}^k\frac{1}{(1+|y-x_1|)^{\frac{N-2}{4}+\frac12\tau}}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{4}+\frac12\tau}}\\
\le& C\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac12\bar\eta}}\sum_{j=2}^k\frac{1}{|x_j-x_1|^{\tau-\frac12\bar\eta}}\le C\varepsilon^{-\bar\eta}\frac{1}{(1+|y-x_1|)^{\frac{N-2}{2}+\frac12\bar\eta}}.
\end{aligned}
$$

As a result,

$$
\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}\le C\varepsilon^{-2^*\bar\eta}\frac{1}{(1+|y-x_1|)^{N+2^*\frac12\bar\eta}},\quad y\in\Omega_1.
$$

Thus

$$
\int_{\Omega_\varepsilon}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{2^*}\le Ck\varepsilon^{-2^*\bar\eta}.
$$

So, we have proved

$$
\begin{aligned}
&\Big|\int_{\Omega_\varepsilon}\Big(\big(W_{r,\Lambda}-s\Phi_\varepsilon+t\phi\big)^{2^*-2}-\Big(\big(W_{r,\Lambda}-s\Phi_\varepsilon\big)^{2^*-2}\Big)\phi^2\Big|\\
\le& Ck\varepsilon^{-2^*\bar\eta}\|\phi\|_*^{\min(3,2^*)}\le Ck\varepsilon^{-2^*\bar\eta}\Big(|\lambda|\varepsilon^{1+\sigma}+\big(s\varepsilon^{\frac{N-2}{2}}\big)^{\frac12+\sigma}\Big)^{\min(3,2^*)}\qquad(3.3)\\
\le& Ck\Big(|\lambda|\varepsilon^{2+\sigma}+\big(s\varepsilon^{\frac{N-2}{2}}\big)^{1+\sigma}\Big)
\end{aligned}
$$

Combining (3.1), (3.2) and (3.3), we find

$$
F(r,\Lambda)=I(W_{r,\Lambda})+kO\Big(|\lambda|\varepsilon^{2+\sigma}+\big(s\varepsilon^{\frac{N-2}{2}}\big)^{1+\sigma}\Big).\qquad(3.4)
$$

□

Proof of Theorems 1.4, 1.5 and 1.6. We just need to prove that $F(r,\Lambda)$ has a critical point.

Firstly, we consider the cases $N\neq6$. It follows from (3.4) and Proposition A.3 that

$$F(r,\Lambda)=k\Big(A_0+\frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}}-\frac{A_3k^{N-2}\varepsilon^{N-2}}{r^{N-2}\Lambda^{N-2}}\\ +O\Big((k\varepsilon)^{(N-2)(1+\sigma)}+\big(s\varepsilon^{\frac{N-2}{2}}\big)^{1+\sigma}\Big)\Big).$$

Let

$$\bar F(r,\Lambda)=\frac{A_2\bar\varphi(r)}{\Lambda^{(N-2)/2}}-\frac{A_3}{r^{N-2}\Lambda^{N-2}},\quad (r,\Lambda)\in[r_1,r_2]\times[\delta,\delta^{-1}].$$

Then, $\bar F(r,\Lambda)$ has a maximum point at (r_0,Λ_0), where

$$\Lambda_0=\Big(\frac{2A_3}{A_2r_0^{N-2}\bar\varphi(r_0)}\Big)^{\frac{2}{N-2}},$$

and r_0 is a maximum point of $r^{\frac{N-2}{2}}\bar\varphi(r)=r^{\frac{N-2}{2}}\varphi_1(r,0)$. So, if $\delta>0$ is small, (r_0,Λ_0) is an interior point of $[r_1,r_2]\times[\delta,\delta^{-1}]$. Thus, if $k>0$ is large, $F(r,\Lambda)$ attains its maximum in the interior of $[r_1,r_2]\times[\delta,\delta^{-1}]$. As a result, $F(r,\Lambda)$ has a critical point in $[r_1,r_2]\times[\delta,\delta^{-1}]$.

If $N=6$, then

$$F(r,\Lambda)=k\Big(A_0+\frac{-\lambda A_1\varepsilon^2+A_2\bar\varphi(r)s\varepsilon^2}{\Lambda^2}-\frac{A_3k^4\varepsilon^4}{r^4\Lambda^4}\\ +O\big((k\varepsilon)^{4(1+\sigma)}+(s\varepsilon^2)^{1+\sigma}\big)\Big).$$

Let

$$\bar F(r,\Lambda)=\frac{-\lambda A_1s^{-1}+A_2\bar\varphi(r)}{\Lambda^2}-\frac{A_3}{r^4\Lambda^4},\quad (r,\Lambda)\in[r_1,r_2]\times[\delta,\delta^{-1}].$$

It is easy to see that there is an $s_0>0$, such that if $s>|\lambda|s_0$, then

$$\tilde\varphi(r)=:r^{\frac{N-2}{2}}\big(-\lambda A_1s^{-1}+A_2\bar\varphi(r)\big),\quad r\in[r_1,r_2]$$

has a maximum point $r_0\in(r_1.r_2)$ and $\tilde\varphi(r_0)>0$. Then, $\bar F(r,\Lambda)$ has a maximum point at (r_0,Λ_0), where

$$\Lambda_0=\Big(\frac{2A_3}{r_0^4\tilde\varphi(r_0)}\Big)^{\frac12}.$$

So, we can prove that $F(r,\Lambda)$ has a critical point in $[r_1,r_2]\times[\delta,\delta^{-1}]$.

□

Appendix A

In this section, we will expand $I(W_{r,\Lambda})$. We always assume that $d(\bar{x}_j, \partial\Omega) \geq c_0 > 0$, where $\bar{x}_j = \varepsilon x_j$. Denote

$$\bar{\varphi}(r) = \varphi_1(r, 0).$$

First, let us recall that $W_{\Lambda,\xi}$ is the solution of

$$\begin{cases} -\Delta W - \lambda\varepsilon^2 W = U_{\Lambda,\xi}^{2^*-1} & \text{in} \quad \Omega_\varepsilon, \\ W = 0 & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{A.1}$$

Let

$$\psi_{\Lambda,\xi} = U_{\Lambda,\xi} - W_{\Lambda,\xi}.$$

Then,

$$\begin{cases} -\Delta \psi_{\Lambda,\xi} - \lambda\varepsilon^2 \psi_{\Lambda,\xi} = -\lambda\varepsilon^2 U_{\Lambda,\xi} & \text{in} \quad \Omega_\varepsilon, \\ \psi_{\Lambda,\xi} = U_{\Lambda,\xi}, & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{A.2}$$

To calculate $I(W_{r,\Lambda})$, we need to estimate $\psi_{\Lambda,\xi}$.

Decompose $\psi_{\Lambda,\xi}$ as follows

$$\psi_{\Lambda,\xi} = \psi_{\Lambda,\xi,1} + \psi_{\Lambda,\xi,2},$$

where $\psi_{\Lambda,\xi,1}$ is the solution of

$$\begin{cases} -\Delta \psi_{\Lambda,\xi,1} - \lambda\varepsilon^2 \psi_{\Lambda,\xi,1} = -\lambda\varepsilon^2 U_{\Lambda,\xi} & \text{in} \quad \Omega_\varepsilon, \\ \psi_{\Lambda,\xi} = 0, & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{A.3}$$

and $\psi_{\Lambda,\xi,2}$ is the solution of

$$\begin{cases} -\Delta \psi_{\Lambda,\xi,2} - \lambda\varepsilon^2 \psi_{\Lambda,\xi,2} = 0, & \text{in} \quad \Omega_\varepsilon, \\ \psi_{\Lambda,\xi} = U_{\Lambda,\xi}, & \text{on} \quad \partial\Omega_\varepsilon. \end{cases} \tag{A.4}$$

Since

$$U_{\Lambda,\xi} \leq C\varepsilon^{N-2}, \quad \text{on} \ \ \partial\Omega_\varepsilon,$$

it is easy to see that

$$|\psi_{\Lambda,\xi,2}| \leq C\varepsilon^{N-2}. \tag{A.5}$$

Let $\bar{\psi}_{\Lambda,\xi,\varepsilon}$ be the solution of

$$\begin{cases} -\Delta \psi - \lambda\varepsilon^2 \psi = U_{\Lambda,\xi} & \text{in} \quad \Omega_\varepsilon, \\ \psi = 0, & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{A.6}$$

Then, we can check that

$$|\bar\psi_{\Lambda,\xi,\varepsilon}(y)| \le \frac{C\ln^m(2+|y-\xi|)}{(1+|y-\xi|)^{N-4}}, \tag{A.7}$$

where $m=1$ if $N=4$, otherwise, $m=0$. Thus, we have

Lemma A.1. *We have*

$$\psi_{\Lambda,\xi} = -\lambda\varepsilon^2\bar\psi_{\lambda,\xi,\varepsilon} + O(\varepsilon^{N-2}).$$

where $\bar\psi_{\lambda,\xi,\varepsilon}$ is the solution of (A.6). *Moreover,*

$$|W_{\Lambda,\xi}| \le C|\ln\varepsilon|^m U_{\Lambda,\xi},$$

where $m=1$ if $N=4$, otherwise, $m=0$.

Proof. We only need to show

$$|W_{\Lambda,\xi}| \le C|\ln\varepsilon|^m U_{\Lambda,\xi},$$

which follows from (A.7) and $\varepsilon \le \frac{C}{1+|y-\xi|}$. □

Proposition A.2. *We have*

$$I\big(W_{\Lambda,x_j}\big) = A_0 + \frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} + O\Big((s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\Big), \quad N=3,4,$$

and

$$I\big(W_{\Lambda,x_j}\big) = A_0 - \frac{A_1\lambda\varepsilon^2}{\Lambda^2} + \frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} + O\Big(|\lambda|\varepsilon^{2+\sigma} + \big(s\varepsilon^{\frac{N-2}{2}}\big)^{1+\sigma}\Big), \quad N\ge 5;$$

where

$$A_0 = \frac12\int_{R^N}|DU|^2 - \frac{1}{2^*}\int_{R^N}U^{2^*}, \quad A_2 = \int_{R^N}U^{2^*-1},$$
$$A_1 = \frac12\int_{R^N}U^2, \quad N\ge 5,$$

and σ is some positive constant.

Proof. Write

$$I(u) = \tilde I(u) - \frac{1}{2^*}\int_{\Omega_\varepsilon}\big((u-s\Phi_\varepsilon)_+^{2^*} - |u|^{2^*}\big),$$

where

$$\tilde I(u) = \frac12\int_{\Omega_\varepsilon}|Du|^2 - \frac12\lambda\varepsilon^2\int_{\Omega_\varepsilon}u^2 - \frac{1}{2^*}\int_{\Omega_\varepsilon}|u|^{2^*}.$$

By Lemma A.1, we have

$$
\begin{aligned}
\tilde I(W_{\Lambda,x_j}) &= \frac{1}{2}\int_{\Omega_\varepsilon} U_{\Lambda,x_j}^{2^*-1} W_{\Lambda,x_j} - \frac{1}{2^*}\int_{\Omega_\varepsilon} W_{\Lambda,x_j}^{2^*} \\
&= A_0 + \frac{1}{2}\int_{\Omega_\varepsilon} U_{\Lambda,x_j}^{2^*-1}\psi_{\Lambda,x_j} + O\Big(\int_{\Omega_\varepsilon} U_{\Lambda,x_j}^{2^*-1-\sigma}\psi_{\Lambda,x_j}^{1+\sigma}\Big) \\
&= A_0 + \frac{1}{2}\int_{\Omega_\varepsilon} U_{\Lambda,x_j}^{2^*-1}\psi_{\Lambda,x_j} + O\big(|\lambda|\varepsilon^{2(1+\sigma)} + \varepsilon^{(N-2)(1+\sigma)}\big).
\end{aligned}
\tag{A.8}
$$

On the other hand,

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \big(W_{\Lambda,x_j} - s\Phi_\varepsilon\big)_+^{2^*} - \int_{\Omega_\varepsilon} (W_{\Lambda,x_j})^{2^*} \\
=& -2^*\int_{R^N} U^{2^*-1} s\varepsilon^{\frac{N-2}{2}}\Lambda_j^{-\frac{N-2}{2}}\bar\varphi(r) + O\Big(\big(s\varepsilon^{\frac{N-2}{2}}\big)^{1+\sigma}\Big).
\end{aligned}
\tag{A.9}
$$

For $N = 3, 4$, by Lemma A.1 and (A.7),

$$
\int_{\Omega_\varepsilon} U_{\Lambda,x_j}^{2^*-1}\psi_{\Lambda,x_j} = O(\varepsilon^{N-2} + \varepsilon^2) = O\Big(\big(s\varepsilon^{\frac{N-2}{2}}\big)^{1+\sigma}\Big).
\tag{A.10}
$$

Here, we have used $\varepsilon = \frac{s^2}{k^2} = \frac{1}{k}s\sqrt{\varepsilon} = (s\sqrt{\varepsilon})^{1+\sigma}$ if $N = 3$. So, the result for $N = 3, 4$ follows from (A.8)–(A.10).

Suppose that $N \ge 5$. Let $\bar\psi_{\Lambda,\xi}$ be the solution of

$$
\begin{cases}
-\Delta\psi = U_{\Lambda,\xi} & \text{in } \ R^N, \\
\psi(|y|) \to 0, & \text{as } |y| \to +\infty.
\end{cases}
\tag{A.11}
$$

Then,

$$
|\bar\psi_{\Lambda,\xi}| \le \frac{C}{(1+|y-\xi|)^{N-4}},
$$

and

$$
|\bar\psi_{\Lambda,\xi} - \bar\psi_{\Lambda,\xi,\varepsilon}| \le \frac{C\varepsilon^2 \ln^m(2+|y-\xi|)}{1+|y-\xi|)^{N-6}},
$$

where $m = 1$ if $N = 6$, otherwise, $m = 0$. Thus,

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} U_{\Lambda,x_j}^{2^*-1}\psi_{\Lambda,x_j} = -\lambda\varepsilon^2\int_{R^N} U_{\Lambda,x_j}^{2^*-1}\bar\psi_{\Lambda,x_j} + O\big(\varepsilon^{N-2} + |\lambda|\varepsilon^4|\ln\varepsilon|\big) \\
=& -\lambda\varepsilon^2\int_{R^N} U^2 + O\big(\varepsilon^{N-2} + |\lambda|\varepsilon^4|\ln\varepsilon|\big).
\end{aligned}
\tag{A.12}
$$

So we obtain the result for $N \ge 5$.

□

Proposition A.3. *We have*

$$I\big(W_{r,\Lambda}\big) = k\Big(A_0 + \frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}} - \frac{A_3k^{N-2}\varepsilon^{N-2}}{r\Lambda^{N-2}} + O\big((k\varepsilon)^{(N-2)(1+\sigma)} + (s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}\big)\Big), \quad N = 3, 4;$$

and

$$I\big(W_{r,\lambda}\big) = k\Big(A_0 - \frac{A_1\lambda\varepsilon^2}{\Lambda^2} + \frac{A_2\bar\varphi(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}} - \frac{A_3k^{N-2}\varepsilon^{N-2}}{r^{N-2}\Lambda^{N-2}} + O\Big((k\varepsilon)^{(N-2)(1+\sigma)} + |\lambda|\varepsilon^{2+\sigma} + \big(s\varepsilon^{\frac{N-2}{2}}\big)^{1+\sigma}\Big)\Big), \quad N \ge 5.$$

Proof. By using the symmetry, we have

$$\begin{aligned}
&\int_{\Omega_\varepsilon} |DW_{r,\Lambda}|^2 - \lambda\varepsilon^2 \int_{\Omega_\varepsilon} W_{r,\Lambda}^2 = \sum_{j=1}^k \sum_{i=1}^k \int_{\Omega_\varepsilon} U_{\Lambda,x_i}^{2^*-1} W_{\Lambda,x_j} \\
=&k\Big(\int_{\Omega_\varepsilon} U_{\Lambda,x_1}^{2^*} + \int_{\Omega_\varepsilon} U_{\Lambda,x_1}^{2^*-1}\psi_{\Lambda,x_1} + \sum_{i=2}^k \int_{\Omega_\varepsilon} U_{\Lambda,x_1}^{2^*-1} U_{\Lambda,x_i} + O\big(\sum_{i=2}^k \frac{1}{|x_i - x_1|^{N-2+\sigma}}\big)\Big) \\
=&k\Big(\int_{R^N} U^{2^*} + \int_{\Omega_\varepsilon} U_{\Lambda,x_1}^{2^*-1}\psi_{\Lambda,x_1} + \sum_{i=2}^k \frac{B_0}{\Lambda^{N-2}|x_i - x_1|^{N-2}} \\
&\qquad + O\big(\sum_{i=2}^k \frac{1}{|x_i - x_1|^{N-2+\sigma}}\big)\Big),
\end{aligned} \tag{A.13}$$

where $B_0 > 0$ is a constant.

Let

$$\Omega_j = \big\{y = (y', y'') \in \Omega_\varepsilon : \big\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|}\big\rangle \ge \cos\frac{\pi}{k}\big\}.$$

Then,

$$|y - x_i| \ge |y - x_j|, \qquad \forall y \in \Omega_j.$$

We have

$$
\begin{aligned}
&\frac{1}{2^*}\int_{\Omega_\varepsilon}\big(W_{r,\Lambda}-s\Phi_\varepsilon\big)_+^{2^*}=\frac{k}{2^*}\int_{\Omega_1}\big(W_{r,\Lambda}-s\Phi_\varepsilon\big)_+^{2^*}\\
=&\frac{k}{2^*}\Big(\int_{\Omega_1}\big(W_{\Lambda,x_1}-s\Phi_\varepsilon\big)_+^{2^*}+2^*\int_{\Omega_1}\sum_{i=2}^k\big(W_{\Lambda,x_1}-s\Phi_\varepsilon\big)_+^{2^*-1}W_{\Lambda,x_i}\\
&\quad+O\Big(\int_{\Omega_1}W_{\Lambda,x_1}^{2^*-2}(\sum_{i=2}^k W_{\Lambda,x_i})^2\Big)\Big)\\
=&\frac{k}{2^*}\Big(\int_{R^N}U^{2^*}-2^*\int_{\Omega_\varepsilon}U_{\Lambda,x_1}^{2^*-1}\psi_{\Lambda,x_1}-\frac{2^*A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{(N-2)/2}}+2^*\int_{\Omega_1}\sum_{i=2}^k U_{\Lambda,x_1}^{2^*-1}U_{\Lambda,x_i}\\
&\quad+O\Big(\int_{\Omega_1}U_{\Lambda,x_1}^{2^*-2}s\Phi_\varepsilon\sum_{i=2}^k U_{\Lambda,x_i}+\int_{\Omega_1}U_{\Lambda,x_1}^{2^*-2}(\sum_{i=2}^k U_{\Lambda,x_i})^2+(s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}+|\lambda|\varepsilon^{2+\sigma}\Big)\Big)\\
=&\frac{k}{2^*}\Big(\int_{R^N}U^{2^*}-2^*\int_{\Omega_\varepsilon}U_{\Lambda,x_1}^{2^*-1}\psi_{\Lambda,x_1}-\frac{2^*A_2\bar{\varphi}(r)s\varepsilon^{\frac{N-2}{2}}}{\Lambda^{\frac{N-2}{2}}}+\sum_{i=2}^k\frac{2^*B_0}{\Lambda^{N-2}|x_i-x_1|^{N-2}}\\
&\quad+O\Big((k\varepsilon)^{(N-2)(1+\sigma)}+(s\varepsilon^{\frac{N-2}{2}})^{1+\sigma}+|\lambda|\varepsilon^{2+\sigma}\Big)\Big).
\end{aligned}
\tag{A.14}
$$

Since

$$
|x_j-x_1|=2|x_1|\sin\frac{2(j-1)\pi}{k},\quad j=2,\dots,k,
$$

we can prove

$$
\sum_{j=2}^k\frac{1}{|x_j-x_1|^{N-2}}=B_4(\varepsilon k)^{N-2}+O\Big((k\varepsilon)^{(1+\sigma)(N-2)}\Big). \tag{A.15}
$$

Thus, the result follows from (A.13), (A.14) and (A.15).

□

Appendix B

Firstly, we gives a few lemmas, whose proof can be found in [35, 33].

Lemma B.1. *For any $\alpha>0$,*

$$
\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^\alpha}\le C\Big(1+\sum_{j=2}^k\frac{1}{|x_1-x_j|^\alpha}\Big),
$$

where $C>0$ is a constant, independent of k.

For each fixed i and j, $i \neq j$, consider the following function

$$g_{ij}(y) = \frac{1}{(1+|y-x_j|)^\alpha} \frac{1}{(1+|y-x_i|)^\beta}, \tag{B.1}$$

where $\alpha \geq 1$ and $\beta \geq 1$ are two constants. Then, we have

Lemma B.2. *For any constant $0 \leq \sigma \leq \min(\alpha, \beta)$, there is a constant $C > 0$, such that*

$$g_{ij}(y) \leq \frac{C}{|x_i - x_j|^\sigma} \Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}} \Big).$$

Lemma B.3. *For any constant $0 < \sigma < N-2$, there is a constant $C > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}}\, dz \leq \frac{C}{(1+|y|)^\sigma}.$$

Let us recall that

$$\varepsilon = \frac{S^{\frac{2}{N-2}}}{k^2}.$$

For the constant $\tau \in (0,1)$ defined is (2.4),

$$\sum_{j=2}^k \frac{1}{|x_j - x_1|^\tau} \leq C\varepsilon^\tau k^\tau \sum_{j=2}^k \frac{1}{j^\tau} \leq C\varepsilon^\tau k \leq C,$$

and for any $\theta > 0$,

$$\sum_{j=2}^k \frac{1}{|x_j - x_1|^{\tau+\theta}} = o(1).$$

Lemma B.4. *Suppose that $N \geq 4$. There is a small $\theta > 0$, such that*

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{r,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^k \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\, dz \\
\leq& C \sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}},
\end{aligned}$$

where $W_{r,\lambda}$ is defined in (1.7).

Proof. Recall that

$$\Omega_j = \big\{ y = (y', y'') \in \Omega_\varepsilon : \big\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \big\rangle \geq \cos\frac{\pi}{k} \big\}.$$

For $z \in \Omega_1$, we have $|z - x_j| \geq |z - x_1|$. Using Lemma B.2, we obtain

$$\sum_{j=2}^{k}\frac{1}{(1+|z-x_j|)^{N-2-\beta}}\le\frac{1}{(1+|z-x_1|)^{\frac{1}{2}(N-2-\beta)}}\sum_{j=2}^{k}\frac{1}{(1+|z-x_j|)^{\frac{1}{2}(N-2-\beta)}}$$
$$\le\frac{C}{(1+|z-x_1|)^{N-2-\beta-\tau}}\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\tau}}\le\frac{C}{(1+|z-x_1|)^{N-2-\beta-\tau}},$$

Thus,

$$W_{r,\Lambda}^{\frac{4}{N-2}}(z)\le\frac{C}{(1+|z-x_1|)^{4-\frac{4(\tau+\beta)}{N-2}}}.$$

As a result, for $z\in\Omega_1$, using Lemma B.2 again, we find that for $\theta>0$ small,

$$W_{r,\Lambda}^{\frac{4}{N-2}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}$$
$$\le\frac{C}{(1+|z-x_1|)^{2+\frac{N-2}{2}+\tau+2-\tau-\frac{4(\tau+\beta)}{N-2}}}.$$

Since $\theta=:2-\tau-\frac{4(\tau+\beta)}{N-2}>0$ if $N\ge4$ and $\beta>0$ is small, we obtain

$$\int_{\Omega_1}\frac{1}{|y-z|^{N-2}}W_{r,\Lambda}^{\frac{4}{N-2}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz$$
$$\le\int_{\Omega_1}\frac{1}{|y-z|^{N-2}}\frac{C}{(1+|z-x_1|)^{2+\frac{N-2}{2}+\tau+\theta}}\,dz\le\frac{C}{(1+|y-x_1|)^{\frac{N-2}{2}+\tau+\theta}},$$

which gives

$$\int_{\Omega_\varepsilon}\frac{1}{|y-z|^{N-2}}W_{r,\Lambda}^{\frac{4}{N-2}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz$$
$$=\sum_{i=1}^{k}\int_{\Omega_i}\frac{1}{|y-z|^{N-2}}W_{r,\Lambda}^{\frac{4}{N-2}}(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\,dz$$
$$\le\sum_{i=1}^{k}\frac{C}{(1+|y-x_i|)^{\frac{N-2}{2}+\tau+\theta}}.$$

□

The above proof does not work for $N=3$ because

$$2-\tau-\frac{4\tau}{N-2}<0 \tag{B.2}$$

if $N=3$ and $\tau=\frac{1}{2}$. The choice of $\tau\in(0,1)$ should ensure

$$\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^{\tau}}\le C\varepsilon^{\tau}k\le C.$$

The above relation shows that τ can be chosen smaller if ε becomes smaller, which in turn will make $2-\tau-\frac{4\tau}{N-2}>0$. Noting that $\varepsilon=\frac{s^2}{k^2}$, we find that if $s\to 0+$, then $\varepsilon=o(\frac{1}{k^2})$. We have

Lemma B.5. *Suppose that $N=3$, the parameter $s>0$ and the integer k satisfy*

$$s\le Ck^{-\frac{1}{2\tau}+1},$$

for some $\tau\in(0,\frac{2}{5})$. Then, there is a small $\theta>0$, such that

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{1}{|y-z|}W_{r,\Lambda}^4(z)\sum_{j=1}^{k}\frac{1}{(1+|z-x_j|)^{\frac{1}{2}+\tau}}\,dz\\ \le& C\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{1}{2}+\tau+\theta}}.\end{aligned}$$

Proof. The proof of this lemma is similar to that of Lemma B.4. We only need to use that for $\tau<\frac{2}{5}$,

$$2-5\tau>0,$$

and

$$\varepsilon^{\tau}k=s^{2\tau}k^{1-2\tau}\le C.$$

Thus, we omit the details.

□

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