Solutions with transition layer and spike in an inhomogeneous phase transition model

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Abstract

We consider the following singularly perturbed elliptic problem

$$\varepsilon^2\triangle\tilde{u} + \big(\tilde{u} - a(\tilde{y})\big)(1-\tilde{u}^2) = 0 \quad \text{in } \Omega, \quad \frac{\partial\tilde{u}}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $-1 < a(\tilde{y}) < 1$, ε is a small parameter, n denotes the outward normal of $\partial\Omega$. Assume that $\Gamma = \{\, \tilde{y} \in \Omega : a(\tilde{y}) = 0\,\}$ is a simple closed and smooth curve contained in Ω in such a way that Γ separates Ω into two disjoint components $\Omega_+ = \{\tilde{y} \in \Omega : a(\tilde{y}) > 0\}$ and $\Omega_- = \{\tilde{y} \in \Omega : a(\tilde{y}) < 0\}$ and $\frac{\partial a}{\partial \nu} > 0$ on Γ, where ν is the outer normal to Ω_-. We will show the existence of a solution u_ε with a transition layer near Γ and a downward spike near the maximum points of $a(\tilde{y})$ whose profile looks like

$$u_\varepsilon \to C < 1 \text{ at a point } P_\varepsilon, \quad u_\varepsilon \to 1 \text{ in } \Omega_+ \setminus P_\varepsilon, \quad u_\varepsilon \to -1 \text{ in } \Omega_-, \qquad \text{as } \varepsilon \to 0.$$

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1 Introduction

Let Ω be a bounded and smooth domain in $\mathbb{R}^2$. In gradient theory of phase transition it is common to seek for critical points in $H^1(\Omega)$ of an energy of the form

$$J_\varepsilon(\tilde{u}) = \frac{\varepsilon}{2}\int_\Omega |\nabla\tilde{u}|^2 + \frac{1}{\varepsilon}\int_\Omega W(\tilde{y}, \tilde{u}) \tag{1.1}$$

where $W(\tilde{y},\cdot)$ is a double-well potential with exactly two strict local minimizers at $\tilde{u}=-1$ and $\tilde{u}=+1$, which as well correspond to trivial local minimizers of J_ε in $H^1(\Omega)$. For simplicity of exposition we shall restrict ourselves to a potential of the form

$$W(\tilde{y},u)=\int_{-1}^{u}(s^2-1)(s-a(\tilde{y}))\,\mathrm{d}s, \tag{1.2}$$

for a smooth function $a(\tilde{y})$ with

$$-1<a(\tilde{y})<1 \text{ for all } \tilde{y}\in\bar{\Omega}.$$

Critical points of J_ε correspond to solutions of the problem

$$\varepsilon^2\Delta\tilde{u}+\big(\tilde{u}-a(\tilde{y})\big)(1-\tilde{u}^2)=0 \quad \text{in } \Omega, \quad \frac{\partial\tilde{u}}{\partial n}=0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where ε is a small parameter, n denotes the outward normal of $\partial\Omega$. Function $\tilde{u}$ represents a continuous realization of the phase present in a material confined to the region Ω at the point x which, except for a narrow region, is expected to take values close to $+1$ or -1. Of interest are of course non-trivial steady state configurations in which the antiphases coexist. For further reference, we denote $f(\tilde{y},\tilde{u})=\big(\tilde{u}-a(\tilde{y})\big)(1-\tilde{u}^2)$.

The case $a\equiv 0$ corresponds to the standard Allen-Cahn equation [6]

$$\varepsilon^2\Delta\tilde{u}+\tilde{u}(1-\tilde{u}^2)=0 \quad \text{in } \Omega, \quad \frac{\partial\tilde{u}}{\partial n}=0 \quad \text{on } \partial\Omega, \tag{1.4}$$

for which extensive literature on transition layer solution is available, see for instance [4, 8, 26, 33, 34, 42, 43, 44, 46, 47, 50, 51, 52, 53, 56], and the references therein for these and related issues.

In this paper, we consider the inhomogeneous Allen-Cahn equation, i.e, problem (1.3). Let us assume that $\Gamma=\{\,\tilde{y}\in\Omega : a(\tilde{y})=0\,\}$ is a simple, closed and smooth curve in Ω which separates the domain into two disjoint components

$$\Omega=\Omega_-\cup\Gamma\cup\Omega_+ \tag{1.5}$$

such that

$$a(\tilde{y})>0 \text{ in } \Omega_+,\ a(\tilde{y})<0 \text{ in } \Omega_-,\ \frac{\partial a}{\partial\nu}>0 \text{ on } \Gamma \tag{1.6}$$

where ν is the outer normal to Ω_-. Observe in particular that for the potential (1.2), we have

$$W(\tilde{y},-1)<W(\tilde{y},1) \text{ in } \Omega_+, \quad W(\tilde{y},+1)<W(\tilde{y},-1) \text{ in } \Omega_-.$$

Thus, if one consider a global minimizer u_ε for J_ε, which exists by standard arguments, it should be such that its value want to minimize $W(\tilde{y},u)$, namely, u_ε should intuitively achieve as $\varepsilon\to 0$,

$$u_\varepsilon\to -1 \text{ in } \Omega_+ \quad u_\varepsilon\to +1 \text{ in } \Omega_-. \tag{1.7}$$

A solution u_ε to problem (1.3) with these properties was constructed, and precisely described, by Fife and Greenlee[25] via matched asymptotics. Super-subsolutions were later used by Angenent, Mallet-Paret and Peletier in the one dimensional case (see [7]) for construction and classification of stable solutions. Radial solutions were found variationally by Alikakos and Simpson in [5]. These results were extended by del Pino in [14] for general (even non smooth) interfaces in any dimension, and further constructions have been done recently by Dancer and Yan [13] and Do Nascimento [18]. In particular, it was proved in [13] that solutions with the asymptotic behavior like (1.7) are typically minimizer of J_ε. Related results can be found in [1, 2].

On the other hand, a solution exhibiting a transition layer in the *opposite direction*, namely

$$u_\varepsilon \to +1 \text{ in } \Omega_+ \ , \qquad u_\varepsilon \to -1 \text{ on } \Omega_- \qquad \text{as } \varepsilon \to 0, \tag{1.8}$$

has been believed to exist for many years. Hale and Sakamoto [31] established the existence of this type of solution in the one-dimensional case, while this was done for the radial case in [15], see also [12]. The layer with the asymptotics in (1.8) in this scalar problem is meaningful in describing pattern-formation for reaction-diffusion systems such as Gierer-Meinhardt with saturation, see [15, 24, 49, 54, 55] and the references therein.

Recently this problem has been completely solved by del Pino-Kowalczyk-Wei [17] (in the two dimensional domain case) and Mahmoudi-Malchiodi-Wei [36] (in the higher dimensional case). More precisely, in [17], M. del Pino, M. Kowalczyk and J. Wei proved the existence of a transition layer solution H_ε in the opposite direction, namely,

$$H_\varepsilon \to +1 \text{ in } \Omega_+, \quad H_\varepsilon \to -1 \text{ in } \Omega_-. \tag{1.9}$$

In fact, defining

$$\lambda_* = \frac{1}{3\pi^2 \displaystyle\int_{\mathbb{R}} H_x^2 \,\mathrm{d}x} \left[\int_\Gamma \sqrt{\frac{\partial a}{\partial \nu}} \right]^2, \tag{1.10}$$

where $H(x)$ is the unique heteroclinic solution of

$$H'' + H - H^3 = 0 \quad \text{in } \mathbb{R}, \ H(0) = 0, \ H(\pm\infty) = \pm 1,$$

they proved

Theorem 1.1. *Given $c > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ satisfying the gap condition*

$$|k^2 \varepsilon - \lambda_*| \geq c\sqrt{\varepsilon} \text{ for all } k \in \mathbb{N}, \tag{1.11}$$

problem (1.3) has a solution H_ε satisfying

$$H_\varepsilon \to +1 \text{ in } \Omega_+, \ H_\varepsilon \to -1 \text{ in } \Omega_-, \tag{1.12}$$

as $\varepsilon \to 0$. Moreover, the transition layer locates at Γ_ε which will collapse to the curve Γ as $\varepsilon \to 0$.

□

Since $-1 < a(\tilde{y}) < 1$ in Ω, there exists a maximum $0 < b < 1$ of the function $a(\tilde{y})$ attained at a point $P \in \Omega_+$. It is well-known that the following problem has a unique solution $U \equiv U_b$, which is nondegenerate,

$$\triangle U + g(U) = 0, U > 0 \text{ in } \mathbb{R}^2, \quad \max_{y \in \mathbb{R}^2} U(y) = U(0), \quad U(y) \to 0 \text{ as } |y| \to \infty, \tag{1.13}$$

where $g(U) = U\,[U-(1-b)]\,(2-U)$. Here, by nondegeneracy, we mean that the kernel in $H^1(\mathbb{R}^2)$ of the linearized operator $-\triangle - g^{'}(\cdot)$ is spanned by $\{\frac{\partial U}{\partial y_1}, \frac{\partial U}{\partial y_2}\}$. Moreover, U is radially symmetric and there is a constant $c_0 > 0$ such that

$$U(x) \le e^{-c_0|x|} \quad \text{as } |x| \to \infty. \tag{1.14}$$

We will use the function U to add a downward interior spike layer, near P, to the transition layer in (1.9) and show the existence of solutions with *both transition layer and spike.*

The following is the main result of this paper:

Theorem 1.2. *There exists ε_0 such that for all $\varepsilon < \varepsilon_0$ satisfying the gap condition (1.11), problem (1.3) has an interface solution u_ε satisfying*

$$u_\varepsilon \to 1 - U(0) \ \text{at } P_\varepsilon, \quad u_\varepsilon \to +1 \ \text{in } \Omega_+ \setminus P_\varepsilon, \quad u_\varepsilon \to -1 \ \text{in } \Omega_-, \qquad \text{as } \varepsilon \to 0, \tag{1.15}$$

where P_ε locates near the maximum point P of the function $a(\tilde{y})$. Near Γ, u_ε takes the form

$$u_\varepsilon(\tilde{y}) = H\left(\frac{dist(\tilde{y}, \Gamma_\varepsilon)}{\varepsilon}\right)(1 + o(1)).$$

Near P_ε, u_ε takes the form

$$u_\varepsilon(\tilde{y}) = \left[1 - U\left(\frac{dist(\tilde{y}, P_\varepsilon)}{\varepsilon}\right)\right](1 + o(1)).$$

□

Solutions with sharp layers as well as spikes were obtained in some papers, but they are all different from the case in this paper. There are results on the Allen-Cahn model (1.4) with finite or infinite Dirichlet boundary values, which asserts the existence of a slution of the form $w_\varepsilon + v_\varepsilon$ with w_ε a stable boundary layered solution and v_ε concentrating at a "most centered" point of the domain, see [30], [10], [48], [11] and [23]. In [12], Dancer and Yan constructed solutions of (1.3) with Dirichlet boundary conditions of the form $w_\varepsilon + v_\varepsilon$, where w_ε is a stable boundary layered solution and v_ε concentrates at an interior point (or at several interior points). These are very different from the situation here because the spectrum for w_ε is positive and uniformly away from

0. The related results in [20] and [21] are the same in the nature. For the coexistence of unstable transition layers and spikes, in [19], Y. Du considered the following problem

$$\varepsilon^2 \Delta \tilde{u} + (\tilde{u} - a(|x|))(1 - \tilde{u}^2) = 0 \text{ in } B_1(0), \quad \frac{\partial \tilde{u}}{\partial n} = 0 \text{ on } \partial B_1(0) \tag{1.16}$$

where $B_1(0)$ is the unit ball in $\mathbb{R}^N$, $N \geq 2$, and proved Theorem 1.2 for all $N \geq 2$. The advantage of problem (1.16) is that the layered solution is *radially symmetric* and its spectrum can be computed explicitly, see [22]. Here, the domain Ω and $a(x)$ are more general. Hence it is more difficult to get the spectral gap estimates.

The main difficulties in the proof of Theorem 1.2 come from the *highly unstable resonance phenomena* of the layer H_ε as well as the *interaction* between the layer and the spike. The spectral gap between all eigenvalues(close to zero) are very small, which leads to "near non-invertibility" of the corresponding linearized operator at H_ε. To overcome the difficulties, we will prove Theorem 1.2 by the reduction method which consists of two steps in the sequel. (A further extension of this method to higher dimensional case is underway.)

Step 1: We analyze the following linearized eigenvalue problem

$$\varepsilon^2 \Delta \phi + f_u(\tilde{y}, H_\varepsilon)\phi + \lambda \phi = 0 \text{ in } \Omega, \quad \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial \Omega \tag{1.17}$$

where H_ε is the transition layer solution constructed in Theorem 1.1 of [17]. In section 2, we show that there exists a spectral gap of the size $O(\varepsilon^{3/2})$ for all small ε satisfying (1.11). (By the same calculation in [22], we can show that the smallest eigenvalue for H_ε is $-\mu_0 \varepsilon + o(\varepsilon)$ for some positive constant μ_0, although we do not include the proof in this paper.)

Step 2: We use *localized energy method* to construct solutions of transition layers with downward spikes with profile looks like $H_\varepsilon + U(\frac{x - P_\varepsilon}{\varepsilon})$. In fact, in order to decompose the interaction between the transition layer and the single spike, we also apply the gluing technique ([16]) and then solve a system of projected problems in sections 3, 4, 5. After that, we locate the spike by the localized energy method in section 6.

The localized energy method was introduced in [27] in dealing with spikes. The advantage of such method is that it can be applied to subcritical, critical and supercritical problems as long as the limiting solutions is well analyzed. See also [28],[29].

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2 Estimates on the eigenvalues of the linearized eigenvalue problem at H_ε

Let H_ε be the solution constructed in Theorem 1.1. In this section, we study the associated linearized eigenvalue problem

$$\varepsilon^2\Delta\phi + f_u(\tilde{y}, H_\varepsilon)\phi + \lambda_\varepsilon\phi = 0 \ \text{ in } \Omega,\ \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \partial\Omega, \tag{2.1}$$

with $f_u(\tilde{y}, H_\varepsilon) = 1 - 3H_\varepsilon^2 + 2a(\tilde{y})H_\varepsilon$. The main result in this section is to show that, for all small ε satisfying the gap condition (1.11), we have the spectral gap estimate $|\lambda_\varepsilon| \geq C\varepsilon^{\frac{3}{2}}$, which will be stated in Proposition 2.9 in subsection 2.3.

2.1 Local coordinate and some notations

Let $\ell = |\Gamma|$ denote the total length of Γ. We consider natural parametrization $\gamma(\theta)$ of Γ with positive orientation, where θ denotes arclength parameter measured from a fixed point of Γ. Let $\nu(\theta)$ denote the outer unit normal to Γ, pointing to the interior of Ω_+. Points $\tilde{y}$, which are δ_0-close to Γ for sufficiently small δ_0, can be represented in the form

$$\tilde{y} = \gamma(\theta) + t\,\nu(\theta), \quad |t| < \delta_0,\ \theta \in [0, \ell), \tag{2.2}$$

where the map $\tilde{y} \mapsto (t, \theta)$ is a local diffeomorphism. By slight abuse of notation we denote $a(t,\theta)$ to actually mean $a(\tilde{y})$ for $\tilde{y}$ in (2.2). Any curve Γ_η sufficiently close to Γ can be parameterized as

$$\tilde{y} = \gamma(\theta) + \eta(\theta)\nu(\theta),$$

where η is a smooth, $\ell-$periodic function with small $L^\infty-$norm. For a small positive constant c, define the c-neighborhood Γ_c of Γ by

$$\Gamma_c = \{\,(t,\theta)|\, - c < t < c,\ \theta \in [0, \ell)\,\}.$$

The function $dist(\cdot\,,\,\cdot)$ is the signed distance along the outer normal to Γ.

In this coordinate, near Γ the metric can be parameterized as

$$g_{t,\theta} = \mathrm{d}t^2 \ + \ (1+kt)^2\mathrm{d}\theta^2,$$

and the Laplacian operator is

$$\triangle_{t,\theta} = \frac{\partial^2}{\partial t^2} + \frac{1}{(1+kt)^2}\frac{\partial^2}{\partial\theta^2} + \frac{k}{1+kt}\frac{\partial}{\partial t} - \frac{k' t}{(1+kt)^3}\frac{\partial}{\partial\theta},$$

where $k(\theta)$ is the curvature of Γ.

Stretching variable $\tilde{y} = \varepsilon y$, we denote by $\frac{P}{\varepsilon}$ the maximum point, P, of $a(\tilde{y})$ in $\bar{\Omega}$, by Γ/ε the curve Γ, by Ω_ε the domain Ω and $\mathcal{U}(y) = H_\varepsilon(\varepsilon y)$ after rescaling.

Setting up new coordinate $(\xi,\, z)$ near Γ/ε in Ω_ε,

$$y = \frac{\gamma(\varepsilon z)}{\varepsilon} + \xi\,\nu(\varepsilon z), \quad |\xi| < \delta_0/(20\varepsilon),\ z \in [0, \ell/\varepsilon), \tag{2.3}$$

the metric and Laplacian operator can be written as

$$\begin{aligned} g_{\xi,z} &= \mathrm{d}\xi^2 + \big(1+\varepsilon k\xi\big)^2 \mathrm{d}z^2, \\ \triangle_{\xi,z} &= \frac{\partial^2}{\partial \xi^2} + \frac{1}{[1+\varepsilon k\xi]^2}\frac{\partial^2}{\partial z^2} + \frac{\varepsilon\, k}{1+\varepsilon k\xi}\frac{\partial}{\partial \xi} - \frac{\varepsilon^2\, k'\,\xi}{[1+\varepsilon k\xi]^3}\frac{\partial}{\partial z}. \end{aligned}$$

Let $x = \xi - f(\varepsilon z)$, where $f(\theta)$ is the function to be defined in (2.5). Then we have in the (x, z) coordinate

$$\begin{aligned} g_{x,z} &= \mathrm{d}x^2 + \big(1+\varepsilon k(x+f)\big)^2 \mathrm{d}z^2, \\ \triangle_{x,z} &= \frac{\partial^2}{\partial x^2} + \frac{1}{[1+\varepsilon k(x+f)]^2}\frac{\partial^2}{\partial z^2} - \frac{2\varepsilon\, f'}{[1+\varepsilon k(x+f)]^2}\frac{\partial^2}{\partial x \partial z} \\ &\quad + \frac{\varepsilon\, k}{1+\varepsilon k(x+f)}\frac{\partial}{\partial x} + \frac{\varepsilon^2\,(f')^2}{[1+\varepsilon k(x+f)]^2}\frac{\partial^2}{\partial x^2} - \frac{\varepsilon^2\, f''}{[1+\varepsilon k(x+f)]^2}\frac{\partial}{\partial x} \\ &\quad - \frac{\varepsilon^2\, k'(x+f)}{[1+\varepsilon k(x+f)]^3}\frac{\partial}{\partial z} - \frac{\varepsilon^3\, k'(x+f)f'}{[1+\varepsilon k(x+f)]^3}\frac{\partial}{\partial x}. \end{aligned}$$

For further reference, we introduce the following lemma, whose complete proofs can be found in [45] and [34].

Lemma 2.1. *Let $\phi \in H^1(-\delta/\varepsilon,\, \delta/\varepsilon)$ be functions such that*

$$\int_{-\delta/\varepsilon}^{\delta/\varepsilon} H_x \phi \,\mathrm{d}x = o(1), \quad \text{as } \varepsilon \to 0.$$

Then there exists a constant $\delta^* > 0$ such that

$$\int_{-\delta/\varepsilon}^{\delta/\varepsilon} \Big[\, |\phi_x|^2 - (1-3H^2)\phi^2 \,\Big]\,\mathrm{d}x \ \geq\ \delta^* \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \phi^2 \,\mathrm{d}x.$$

□

2.2 The unstable interface solution H_ε

In [17], M. Del Pino, M. Kowalczyk and J. Wei proved Theorem 1.1 and constructed the transition layer of the form

$$H_\varepsilon = \Phi_\varepsilon + w_\varepsilon, \tag{2.4}$$

where

$$\Phi_\varepsilon = \begin{cases} +1, & y \in \Omega_+ \setminus \Gamma_{6\delta}, \\ H\Big(\frac{dist(y,\Gamma_\varepsilon)}{\varepsilon}\Big) + \phi_1\Big(\frac{dist(y,\Gamma_\varepsilon)}{\varepsilon}, \theta\Big) + \Psi\Big(\frac{dist(y,\Gamma_\varepsilon)}{\varepsilon}, \frac{\theta}{\varepsilon}\Big), & y \in \Gamma_{3\delta}, \\ -1, & y \in \Omega_- \setminus \Gamma_{6\delta}, \end{cases}$$

where $\delta < \delta_0/100$ is a fixed positive constant, the function $dist$ is the signed distance along the outer normal to Γ.

More precisely, we can get the following information from the construction procedure.

- In (t,θ) coordinate, the location of the interface Γ_ε can be described as

$$\gamma(\theta) + \varepsilon f(\theta)\nu(\theta), \;\; \theta \in [0,\ell), \tag{2.5}$$

where $f(\theta) = f_0(\theta) + f_1(\theta)$ with

$$f_0(\theta) = c_0\,\frac{k(\theta)}{a_t(0,\theta)},$$

for some positive constant

$$c_0 = \frac{3}{4}\int_{\mathbb{R}} H_x^2\,\mathrm{d}x,$$

and f_1 satisfying

$$||f_1||_a \;\;\equiv\;\; ||f_1||_{L^\infty(0,\ell)} + \sqrt{\varepsilon}\,||f_1^{'}||_{L^2(0,\ell)} + \varepsilon\,||f_1^{''}||_{L^2(0,\ell)} \le \varepsilon, \tag{2.6}$$

so that

$$||f_1||_{L^\infty}(0,\ell) \le \varepsilon, \;\; ||f_1^{'}||_{L^2(0,\ell)} \le \sqrt{\varepsilon}, \;\; ||f_1^{''}||_{L^2(0,\ell)} \le 1.$$

By Sobolev emmbedding, it also holds

$$||f_1^{'}||_{L^\infty}(0,\ell) \le \sqrt{\varepsilon}.$$

- The function $\phi_1(x,\theta)$ can be defined as

$$\phi_1(x,\theta) = \varepsilon a_t(0,\theta)H_1(x) + \varepsilon k(\theta)H_2(x) \tag{2.7}$$

where $H_1(x)$ is the unique odd function satisfying

$$-H_{1,xx} - H_1 + 3H^2H_1 = -x(1-H^2),\; H_1(\pm\infty) = 0,$$

and $H_2(x)$ is the unique even function satisfying

$$-H_{2,xx} - H_2 + 3H^2H_2 = H_x - c_0(1-H^2),\; H_2(\pm\infty) = 0.$$

- The function Ψ was extended identically zero in the region $\Omega \setminus \Gamma_{6\delta}$, while, in the local coordinate $(x,\, z)$ near Γ/ε, $\Psi(x,z)$ is a solution to the following problem

$$\begin{aligned}
&\Psi_{xx} \,+\, \Psi_{zz} \,+\, [\,1-3(H+\phi_1)^2\,]\Psi \,+\, 2a\Big(\varepsilon(x+f),\varepsilon z\Big)\,\big[H+\phi_1\big]\,\Psi \\
&\qquad\qquad + B_1(\Psi) \,=\, -E_1 \,-\, N(\Psi) \,+\, O(\varepsilon^3), \\
&\Psi(x,0) = \Psi(x,\ell/\varepsilon), \quad \Psi_z(x,0) = \Psi_z(x,\ell/\varepsilon), \quad \int_{-\infty}^{\infty} \Psi(x,z) H_x(x) dx = 0,
\end{aligned}$$

where $B_1(\cdot)$ is a differential operator which is a high order perturbation of $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ in the differential operator $\triangle_{x,z}$,

$$\begin{aligned}
E_1 \;\;&=\;\; -\,\varepsilon a_t(0,\varepsilon z)\, f_1\, (1-H^2) \,-\, \varepsilon^2\Big[\,k^2 x H_x - |f^{'}|^2 H_{xx} + a_{tt} f x (1-H^2)\,\Big] \\
&\qquad -\varepsilon^2\Big[\,k^2 f H_x + f^{''} H_x + \frac{a_{tt}}{2}(x^2+f^2)(1-H^2)\,\Big] \\
&\qquad -3H(\phi_1)^2 \,+\, O(\varepsilon^3), \\
N(\Psi) \;\;&=\;\; -\,3\phi_1\Psi^2 \,-\, \Psi^3 \,+\, a\big(\varepsilon(x+f),\varepsilon z\big)\Psi^2.
\end{aligned}$$

Moreover, there exist positive constants D, γ_0, independent of ε, such that

$$\begin{aligned}
\Big|\Big|\,\Psi\,\Big|\Big|_{H^2(\mathbb{R}\times(-\frac{1}{\varepsilon},\frac{1}{\varepsilon}))} \;&\le\; D\varepsilon^{\frac{3}{2}}, \\
\Bigg|\Bigg|\,\Big|\Psi\big(\tfrac{dist(y,\Gamma_\varepsilon)}{\varepsilon},\,\tfrac{\theta}{\varepsilon}\big)\Big| \,+\, \Big|\nabla\Psi\big(\tfrac{dist(y,\Gamma_\varepsilon)}{\varepsilon},\,\tfrac{\theta}{\varepsilon}\big)\Big|\,\Bigg|\Bigg|_{L^\infty(t>\delta)} \;&\le\; D\varepsilon^{\frac{3}{2}}\, e^{\frac{-\delta\,\gamma_0}{2\varepsilon}}.
\end{aligned} \tag{2.8}$$

- Finally, the correction term w_ε has estimate

$$\Big|\Big|\, w_\varepsilon \,\Big|\Big|_{L^\infty(\Omega)} \;\le\; De^{\frac{-\delta}{\varepsilon}}. \tag{2.9}$$

2.3 The spectral gap of the linearized eigenvalue problem

Taking rescaling $\tilde{y} = \varepsilon y$ and writing $\mathcal{U}(y) = H_\varepsilon(\varepsilon y)$, the eigenvalue problem (2.1) takes the form

$$\Delta\psi + f_u(\mathcal{U})\psi + \lambda\psi = 0 \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{2.10}$$

with $f_u(\mathcal{U}) = 1 - 3\mathcal{U}^2 + 2a(\varepsilon y)\mathcal{U}$.

It is well known that (2.10) has a sequence of eigenvalues $\lambda_1^\varepsilon < \lambda_2^\varepsilon \le \cdots \le \lambda_i^\varepsilon \le \cdots$ where $\lambda_i^\varepsilon \to \infty$ as $i \to \infty$. For $i \ge 2$, the eigenvalue λ_i^ε corresponds to a finite number of linearly independent sign-changing eigenfunctions which span a finite dimensional space E_i^ε. Note that we have $E_1^\varepsilon = \text{span}\{\psi_1^\varepsilon\}$. Denote $m_i^\varepsilon = \dim(E_i^\varepsilon)$ and suppose that there exists j such that $\lambda_j^\varepsilon < 0$, while $\lambda_{j+1}^\varepsilon \ge 0$, then $m^\varepsilon \equiv \sum_{i=1}^{j} m_i^\varepsilon$ is called the *Morse index* of H_ε. The Morse index gives the dimension of the unstable manifold of H_ε as a steady-state solution of the parabolic problem corresponding to (1.3).

In order to estimate the Morse index and construct solutions of (1.3) which are perturbation of H_ε with downward sharp spikes, we need to obtain good estimates to all λ_i^ε (called *critical eigenvalues*) which are close to zero for all small ε satisfying the gap condition (1.11).

It is easy to see that the eigenvalue problem (2.10) is related to the bilinear form $< L\psi,\, \phi >$ defined by

$$< L\psi,\, \phi >= \int_{\Omega_\varepsilon} \left[\nabla\phi \cdot \nabla\psi \;-\; f_u(\mathcal{U})\phi\psi \right] \mathrm{d}y,\; \forall\, \phi, \psi \in H^1(\Omega_\varepsilon).$$

In fact, $(\lambda,\, \psi)$ is an eigenvalue/eigenfunction to (2.10) if and only if $\psi \in H^1(\Omega_\varepsilon)$ and

$$< L\psi,\, \phi >= \lambda(\psi,\, \phi),\;\; \forall\, \phi \in H^1(\Omega_\varepsilon).$$

Here and in the sequel, $(\cdot,\, \cdot)$ stands for the $L^2(\Omega_\varepsilon)$ inner product. Observe that the principal eigenvalue of (2.10) is the infimum of $< L\psi,\, \psi >$ in $H^1(\Omega_\varepsilon)$ subject to $||\psi||_2 = 1$.

Since away from the curve Γ/ε, $f_u(\mathcal{U})$ is uniformly negative, it is reasonable to believe that the mass of the eigenfunctions corresponding to all critical eigenvalues concentrates near Γ/ε. That is, one needs to study the behavior of L near Γ/ε where the local coordinate $(x,\, z)$ is well-defined. In fact we will only need to estimate the gaps between the critical eigenvalues in further application to constructing a downward sharp spike. All of these will be clarified in following subsections, which follows the method in [34].

We make domain decomposition as $\Omega_\varepsilon = \Omega_{0\varepsilon} \cup \Omega_{1\varepsilon} \cup \Omega_{2\varepsilon} \cup \Omega_{3\varepsilon}$ with

$$\begin{aligned}
\Omega_{0\varepsilon} &= \{\text{all points in } \Omega_\varepsilon \text{ with distance to the point } \frac{P}{\varepsilon} \text{ less than } \delta/\varepsilon\,\},\\
\Omega_{1\varepsilon} &= \{\text{all points in } \Omega_\varepsilon \text{ with distance to the curve } \Gamma/\varepsilon \text{ less than } \delta/\varepsilon\,\},\\
\Omega_{2\varepsilon} &= \{\text{all points in } \Omega_\varepsilon \text{ with distance to the curve } \Gamma/\varepsilon \text{ less than } 6\delta/\varepsilon\,\},\\
\Omega_{3\varepsilon} &= \Omega_\varepsilon \setminus \Omega_{2\varepsilon}.
\end{aligned}$$

In (x, z) coordinate, we locally choose

$$\psi^0(x, z) = H_x \,+\, \varepsilon a_t(0, \varepsilon z) H_{1,x} \,+\, \varepsilon k(\varepsilon z) H_{2,x} \,+\, \Psi_x,$$

and define two function spaces $\mathcal{F}$ and $\mathcal{F}^\perp$ by

$$\mathcal{F} \;=\; \{\, \psi \in H^1(\Omega_\varepsilon) : \psi = 0 \text{ in } \Omega_{3\varepsilon},\; \psi(x, z) = \psi^0(x, z)\Theta^*(\varepsilon z) \text{ in } \Omega_{1\varepsilon}\, \}, \tag{2.11}$$

$$\mathcal{F}^\perp \;=\; \{\, \psi \in H^1(\Omega_\varepsilon) : \psi = 0 \text{ in } \Omega_{3\varepsilon},\; (\psi,\, \phi) = 0,\; \forall\, \phi \in \mathcal{F}\, \}. \tag{2.12}$$

Direct computations give the following lemma

Lemma 2.2. *For any function of the form $\psi^0\Theta^* \in \mathcal{F}$, we have*

$$\begin{aligned}\triangle_{x,z}(\psi^0\Theta^*) \;&+\; f_u(\mathcal{U})\psi^0\Theta^* \\ &= \varepsilon a_t(1-H^2)\Theta^* \;-\; \varepsilon f_0 k[H_{xx}+2c_0 HH_x]\Theta^* \;+\; \varepsilon^2 H_x\Theta^*_{zz} \\ &\quad -2\varepsilon^2 f' H_{xx}\Theta^*_z \;+\; \varepsilon k H_{xx}\Theta^* \;+\; 2\varepsilon HH_x a_t f\Theta^* \;-\; 2\varepsilon a_t f_1 HH_x\Theta^* \\ &\quad +\varepsilon^2 k^2 H_x\Theta^* \;+\; \varepsilon^2 a_{tt} f(1-H^2)\Theta^* \;+\; \varepsilon^2 a_{tt} x(1-H^2)\Theta^* \\ &\quad -2\varepsilon^2 a_t^2 HH_1\Theta^* \;-\; 2\varepsilon^2 a_t f_0 kHH_2\Theta^* \;+\; O(\varepsilon^3).\end{aligned}$$

We omit the proof of this lemma. □

2.3.1 The bilinear form restricted on the space $\mathcal{F}$

In this subsection we study the restriction of $< L\psi,\, \phi >$ on the space $\mathcal{F}$. In fact we study the following eigenvalue problem: finding $(\lambda,\, \psi)$ with $\psi \in \mathcal{F}$ and

$$< L\psi,\, \phi >= \lambda(\psi,\, \phi),\ \forall\, \phi \in \mathcal{F}. \tag{2.13}$$

Define for $\psi_1,\, \psi_2 \in \mathcal{F}$

$$B(\psi_1,\, \psi_2) = \Big(-\triangle_{x,z}\psi_1 - f_u(\mathcal{U})\psi_1,\ \psi_2\Big)_{L^2(\Omega_{2\varepsilon})}.$$

The following is a corollary of Lemma 2.2.

Lemma 2.3. *For every $\psi_1 = \psi^0\Theta^*$, $\psi_2 = \psi^0\Theta^{**} \in \mathcal{F}$, we have*

$$\begin{aligned}B(\psi_1,\, \psi_2) \;&=\; -\frac{4}{3}\int_0^\ell a_t(0,\theta)\Theta^*\Theta^{**}\,\mathrm{d}\theta \;-\; \varepsilon\alpha_0\int_0^\ell \Theta^*_{\theta\theta}\Theta^{**}\,\mathrm{d}\theta \;+\; \varepsilon\int_0^\ell P(\theta)\Theta^*\Theta^{**}\,\mathrm{d}\theta \\ &\quad +O(\varepsilon^2)\int_0^\ell \Big[\, |\Theta^*|^2 \;+\; |\Theta^{**}|^2 \;+\; |\Theta^*_z|^2 \;+\; |\Theta^{**}_z|^2 \,\Big]\,\mathrm{d}\theta, \\ \big(\psi_1,\ \psi_2\big)_{L^2(\Omega_{2\varepsilon})} \;&=\; \frac{\alpha_0}{\varepsilon}\int_0^\ell \Theta^*\Theta^{**}\,\mathrm{d}\theta \;+\; \alpha_0\int_0^\ell k_0(\theta)f_0(\theta)\Theta^*\Theta^{**}\,\mathrm{d}\theta \\ &\quad +\alpha_1\int_0^\ell a_t(0,\theta)\Theta^*\Theta^{**}\,\mathrm{d}\theta,\end{aligned}$$

where $P(\theta)$ is a bounded function of the variable θ in $(0,\ell)$ and

$$\alpha_0 = \int_{\mathbb{R}} H_x^2\,\mathrm{d}x, \quad \alpha_1 = \int_{\mathbb{R}} H_x\, H_{1,x}\,\mathrm{d}x.$$

□

Let $\mathcal{F}_{n-1}$, $n = 1, 2, \cdots$, denote the collection of all $n-1$ dimensional subspaces of $\mathcal{F}$. We define

$$\mu_n \;=\; \max_{S\in\mathcal{F}_{n-1}}\ \min_{\psi\in S^\perp} \frac{B(\psi,\, \psi)}{||\psi\,||^2_{L^2(\Omega_{2\varepsilon})}}.$$

From lemma 2.3, there exist positive constants C_1, C_2 and C_0 such that

$$\begin{aligned}
\mu_n &\leq C_1\,\varepsilon \max_{S\in F_{n-1}} \min_{\psi\in S^\perp} \frac{J(\Theta,\,\Theta)\,+\,C_0\,\varepsilon^2||\Theta||^2_{H^1(0,\ell)}}{||\Theta\,\sqrt{\gamma_4}\,||^2_{L^2}},\\
\mu_n &\geq C_2\,\varepsilon \max_{S\in F_{n-1}} \min_{\psi\in S^\perp} \frac{J(\Theta,\,\Theta)\,-\,C_0\,\varepsilon^2||\Theta||^2_{H^1(0,\ell)}}{||\Theta\,\sqrt{\gamma_4}\,||^2_{L^2}}.
\end{aligned}$$

where

$$\begin{aligned}
J(\Theta,\,\Theta) &= -\frac{4}{3}\int_0^\ell a_t(0,\theta)\Theta^2\,\mathrm{d}\theta\,-\,\varepsilon\alpha_0\int_0^\ell \Theta_{\theta\theta}\Theta\,\mathrm{d}\theta\,+\,\varepsilon\int_0^\ell P(\theta)\Theta^2\,\mathrm{d}\theta,\\
\gamma_4(\theta) &= \frac{4}{3}\,a_t(0,\theta)/\alpha_0 > 0.
\end{aligned}$$

Hence, to characterize the number μ_n close to zero, we should consider the following associated *geometric eigenvalue problem*

$$-\varepsilon\Theta^{''} - \gamma_4(\theta)\Theta \,=\, \Lambda\,\gamma_4(\theta)\,\Theta \quad \text{in } (0,\ell), \tag{2.14}$$

$$\Theta(0) \,=\, \Theta(\ell),\ \Theta^{'}(0) \,=\, \Theta^{'}(\ell). \tag{2.15}$$

It's well known that (2.14)-(2.15) has sequence of different eigenvalues $\Lambda_1 < \Lambda_2 < \cdots$ with corresponding eigenfunctions $\Theta_1, \Theta_2, \cdots$. It is obvious that $\Lambda_1 < 0$ because of the positivity of γ_4 and Θ_1 can be chosen positive. Moreover, all critical eigenvalues of the geometric eigenvalue problem have good estimates.

Lemma 2.4. *For all small ε satisfies the gap condition (1.11), we have the following spectrum gap estimates of the geometric problem (2.14)-(2.15): there exist positive constants C independent of ε and $N^\varepsilon \in \mathbb{N}$, $N^\varepsilon \to \infty$ as $\varepsilon \to 0$, such that*

$$\begin{aligned}
\Lambda_n &\leq -C\sqrt{\varepsilon}, \quad \textit{for all } n = 1, 2, \cdots, N^\varepsilon,\\
\Lambda_n &\geq C\sqrt{\varepsilon}, \quad \textit{for all } n = N^\varepsilon + 1, N^\varepsilon + 2, \cdots.
\end{aligned}$$

Proof. By the following Liouville transformation

$$\ell_0 = \int_0^\ell \sqrt{\gamma_4(\theta)}\,\mathrm{d}\,\theta, \quad t = \frac{\pi}{\ell_0}\int_0^\theta \sqrt{\gamma_4(\theta)}\,\mathrm{d}\,\theta,\ t\in[0,\pi),$$

$$\overline{\Psi}(\theta) = \gamma_4(\theta)^{-\frac{1}{4}}, \quad e(t) = \Theta(\theta)/\,\overline{\Psi}, \quad q(t) = \frac{\ell_0^2\,\overline{\Psi}^{''}}{\pi^2\,\overline{\Psi}^{2}\,\gamma_4(\theta)},$$

Λ satisfies the following eigenvalue problem

$$-e^{''} - q(t)\,e = \frac{\ell_0^2}{\pi^2\,\varepsilon}\,(1+\Lambda)\,e \quad \text{in } (0,\pi),\ e(0) \,=\, e(\pi),\ e^{'}(0) = e^{'}(\pi).$$

Now consider the following auxiliary eigenvalue problem

$$-y^{''} - q(t)y = \xi y,\ 0 < t < \pi,\ y(0) \,=\, y(\pi),\ y^{'}(0) = y^{'}(\pi).$$

The result in [35] shows that, as $n \to \infty$

$$\sqrt{\xi_n} = 2n + O(\frac{1}{n^3}).$$

Hence, if ε is small then we have, as $n \to \infty$,

$$\begin{aligned}\Lambda_n &= \frac{4n^2\pi^2\varepsilon}{\ell_0^2} - 1 + O(\frac{\varepsilon}{n^2}) \\ &= \frac{4\pi^2}{\ell_0^2}(n^2\varepsilon - \lambda_*) + O(\frac{\varepsilon}{n^2}).\end{aligned}$$

The last formula together with the gap condition (1.11) implies the estimates in the lemma.

□

Corollary 2.5. *For all small ε satisfying the gap condition (1.11), there exists a constant* $\mathbf{C}$ *such that for all special functions $\psi = \psi^0\Theta_n \in F$,*

$$\begin{aligned}B(\psi,\psi) &\leq -\mathbf{C}\sqrt{\varepsilon}\int_0^\ell |\Theta_n|^2\,\mathrm{d}\theta + O(\varepsilon^2)\int_0^\ell \Big[|\Theta_n|^2 + |\Theta_{n,z}|^2\Big]\,\mathrm{d}\theta, \quad n \leq N^\varepsilon, \\ B(\psi,\psi) &\geq \mathbf{C}\sqrt{\varepsilon}\int_0^\ell |\Theta_n|^2\,\mathrm{d}\theta + O(\varepsilon^2)\int_0^\ell \Big[|\Theta_n|^2 + |\Theta_{n,z}|^2\Big]\,\mathrm{d}\theta, \quad n \geq N^\varepsilon + 1.\end{aligned}$$

□

The main result of this subsection is the gap estimates of μ_n which are close to zero.

Lemma 2.6. *For all small ε satisfying the gap condition (1.11), there exists a positive constant C independent of ε such that*

$$\begin{aligned}\mu_n &\leq -C\varepsilon^{3/2}, \quad \textit{for all } n = 1, 2, \cdots, N^\varepsilon, \\ \mu_n &\geq C\varepsilon^{3/2}, \quad \textit{for all } n = N^\varepsilon + 1, N^\varepsilon + 2, \cdots.\end{aligned}$$

Proof. It follows from Corollary 2.5 and the definition of μ_n. □

2.3.2 The bilinear form restricted on the space $F \oplus F^\perp$

Lemma 2.7. *There exists a constant $\nu_0 > 0$, independent on ε such that*

$$B(\psi^\perp, \psi^\perp) > \nu_0 \int_{\Omega_{1\varepsilon}} |\psi^\perp|^2, \ \forall\, \psi^\perp \in F^\perp.$$

Proof. Assume that for each positive integer n there exists $\psi_n^\perp \in F^\perp$ with $||\psi_n^\perp||_2 = 1$ such that

$$B(\psi_n^\perp, \psi_n^\perp) \leq \frac{1}{n}. \tag{2.16}$$

We then have for each function $\Theta = \Theta(z)$

$$\int_{\Omega_{2\varepsilon}} \psi_n^\perp \cdot \psi_0\Theta\,\mathrm{d}y = 0,$$

which leads to

$$\int_{-\delta/\varepsilon}^{\delta/\varepsilon} \psi_n^\perp H_x \, \mathrm{d}x \;=\; O(\varepsilon) \quad \text{almost everywhere in } (0,\ell).$$

Similarly, we get

$$\int_{-\delta/\varepsilon}^{\delta/\varepsilon} \left|\psi_n^\perp\right|^2 \;<\; \infty, \qquad \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \left|\psi_{n,x}^\perp\right|^2 \;<\; \infty \quad \text{almost everywhere in } (0,\ell).$$

Hence, using Lemma 2.1, we then have a conclusion that there exists a constant $\delta^* > 0$ independent of ε and n such that

$$\int_{-\delta/\varepsilon}^{\delta/\varepsilon} \Big[\left|\psi_{n,x}^\perp\right|^2 \;-\; (1-3H^2)\left|\psi_n^\perp\right|^2 \Big] \, \mathrm{d}x \;\geq\; \delta^* \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \left|\psi_{n,x}^\perp\right|^2 \mathrm{d}x,$$

which will imply that

$$\begin{aligned} \int_0^\ell \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \Big[\left|\psi_{n,x}^\perp\right|^2 \;-\; (1-3H^2)\left|\psi_n^\perp\right|^2 \Big] \;&\geq\; \delta^* \int_0^\ell \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \left|\psi_{n,x}^\perp\right|^2 \\ &=\; \frac{\delta^*}{\varepsilon} \, ||\psi_n^\perp||^2_{L^2(\Omega_{2\varepsilon})} \;+\; O(1). \end{aligned} \tag{2.17}$$

On the other hand,

$$B(\psi_n^\perp, \, \psi_n^\perp) \;\geq\; \varepsilon \int_0^\ell \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \Big[\left|\psi_{n,x}^\perp\right|^2 \;-\; (1-3H^2)\left|\psi_n^\perp\right|^2 \Big] + O(1). \tag{2.18}$$

Therefore, combining (2.16)-(2.18) we obtain

$$\frac{1}{n} \;\geq\; \delta^*$$

which will lead a contradiction if we let $n \to \infty$. □

Let $\Theta_1, \cdots, \Theta_{N^\varepsilon}$ be the eigenfunctions of (2.14)-(2.15) and define N^ε dimensional subspace

$$\overline{S} \;=\; \mathrm{span}\, \{\, \psi^0 \Theta_i, \; i = 1, 2, \cdots, N^\varepsilon \,\}.$$

Lemma 2.8. *For any functions $\psi \in \mathcal{F} \cap \overline{S}^{\,\perp}$, $\phi^\perp \in \mathcal{F}^\perp$, for any small ε satisfying the gap condition (1.11) and for any small constant $\varrho \in (0, 1/2)$, the following estimates hold*

$$\left|B(\psi, \, \phi^\perp)\right| \;\;\leq\;\; \varepsilon^\varrho B(\psi, \, \psi) \;+\; C\varepsilon^{2+\varrho} \int_{\Omega_{2\varepsilon}} |\psi|^2 \;+\; \int_{\Omega_{2\varepsilon}} \Big[\varepsilon^\varrho \, |\nabla \phi^\perp|^2 \;+\; \frac{\nu_0}{4} \, |\phi^\perp|^2 \Big],$$

$$B(\psi + \phi^\perp, \, \psi + \phi^\perp) \;\;\geq\;\; B(\psi, \, \psi)\Big[1 + O(\varepsilon^{1/4})\Big] \;+\; C\varepsilon^{9/4} \, ||\psi||_{L^2(\Omega_{2\varepsilon})} \;+\; \frac{\nu_0}{2} \, ||\phi^\perp||_{H^1(\Omega_{2\varepsilon})}.$$

Proof. The results can be easily derived from previous Lemma 2.6 and Lemma 2.7.

□

Let $\sum_{n-1}$ denote the collection of $n-1$ dimensional subspace of $L^2(\Omega_{1\varepsilon})$. We define

$$\mu_n^* \;\;=\;\; \max_{S \in \sum_{n-1}} \; \min_{\Psi \in S^\perp} \frac{B(\Psi, \, \Psi)}{||\, \Psi \,||^2_{L^2(\Omega_{1\varepsilon})}},$$

and then our main result in this subsection can be stated as follows.

Proposition 2.9. *For all small ε satisfying the gap condition (1.11), there exists a positive constant C independent of ε such that*

$$\begin{aligned}\mu_n^* &\le -C\varepsilon^{3/2}, \quad \textit{for all } n = 1, 2, \cdots, N^\varepsilon,\\ \mu_n^* &\ge C\varepsilon^{3/2}, \quad \textit{for all } n = N^\varepsilon + 1,\, N^\varepsilon + 2, \cdots.\end{aligned}$$

Proof. For $\Psi \in L^2(\Omega_{1\varepsilon})$, we extend it as zero outside $\Omega_{2\varepsilon}$ and make decomposition $\Psi = \psi + \phi^\perp$, where $\psi \in \mathcal{F}$, $\phi^\perp \in \mathcal{F}^\perp$.

If $\Psi \in \overline{S}^\perp$, then

$$0 = \int_{\Omega_{2\varepsilon}} \Psi \cdot \psi^0 \Theta_i = \int_{\Omega_{2\varepsilon}} (\psi + \phi^\perp)\psi^0\Theta_i = \int_{\Omega_{2\varepsilon}} \psi \cdot \psi^0 \Theta_i, \ \forall\, i = 1, 2, \cdots, N^\varepsilon,$$

that is, the component $\psi \in \overline{S}^\perp$. Hence, from Lemma 2.8, we obtain

$$\begin{aligned}\mu^*_{N^\varepsilon+1} &\ge \min_{\Psi\in\overline{S}^\perp,\, ||\Psi||=1} B(\Psi,\, \Psi)\\ &\ge \min_{\Psi\in\overline{S}^\perp,\ ||\psi||+||\phi^\perp||=1} \Big\{ B(\psi,\, \psi)\cdot\big(1 + O(\varepsilon^{1/4})\big) \,-\, \varepsilon^{9/4}\int_{\Omega_{2\varepsilon}} \psi^2 \,+\, \frac{\nu_0}{2}\int_{\Omega_{2\varepsilon}} |\phi^\perp|^2 \Big\}\\ &\ge C\varepsilon^{3/2}.\end{aligned}$$

On the other hand, to show the validity of the first inequality by contradiction argument, we assume that

$$\mu^*_{N^\varepsilon} \,>\, -\mathbf{C}\,\varepsilon^{3/2},$$

where the positive constant $\mathbf{C}$ is given in Corollary 2.5. So for all small $\delta > 0$, there exists a subspace $\underline{S}$ of $N^\varepsilon - 1$ dimension such that

$$\min_{\Psi\in\underline{S}^\perp,\, ||\Psi||=1} B(\Psi,\, \Psi) \,\ge\, -\mathbf{C}\,(\varepsilon^{3/2} - \delta).$$

Suppose that $\underline{S} = \operatorname{span}\{\, \psi^0\xi_i + \phi_i^\perp,\ i = 1, 2, \cdots, N^\varepsilon - 1\,\}$ (some of the components $\xi_i^{'}$s may be zeros). It follows that there exists a vector $\tilde{\psi} = \psi^0\tilde{\xi} \,\in\, \underline{S}^\perp$ such that

$$||\tilde{\psi}|| = 1 \ \text{ and } \ \int_{\Omega_{2\varepsilon}} \psi^0\xi_i \cdot \psi^0\tilde{\xi} = 0 \ \text{ for } i = 1, 2, \cdots, N^\varepsilon - 1.$$

Thus $\tilde{\psi} \in (S_{n-1})^\perp$ for some $S_{n-1} = \operatorname{span}\{\, \psi^0\xi_i,\ \xi_i \ne 0\,\}$, $n \le N^\varepsilon - 1$ and we can assume that

$$\begin{aligned}\min_{\psi\in(S_{n-1})^\perp,\, ||\psi||=1} B(\psi,\, \psi) &= B(\tilde{\psi}, \tilde{\psi})\\ &\ge \min_{\Psi\in\underline{S}^\perp,\, ||\Psi||=1} B(\Psi,\, \Psi).\end{aligned}$$

Finally, we get, from Corollary 2.5

$$-\mathbf{C}\,(\varepsilon^{3/2} - \delta) \,\le\, \min_{\psi\in(S_{n-1})^\perp,\, ||\psi||=1} B(\psi,\, \psi) \,\le\, -\mathbf{C}\,\varepsilon^{3/2} + O(\varepsilon^3).$$

It leads to a contradiction. Therefore,

$$\mu^*_{N^\varepsilon} \le -\mathbf{C}\,\varepsilon^{3/2}.$$

□

3 The gluing procedure

The next few sections will be devoted to the construction of a solution of the form $H_\varepsilon + v_\varepsilon$ where $v_\varepsilon \in H^1(\Omega)$ concentrates near the maximum point P of the function $a(\tilde y)$ on $\bar\Omega$. By scaling $(\tilde y_1, \tilde y_2) = \varepsilon(y_1, y_2)$, problem (1.3) becomes

$$\triangle_y u + (u - a(\varepsilon y))(1-u^2) = 0 \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{3.1}$$

As stated in subsection 2.1, we denoted by $\frac{P}{\varepsilon}$ the maximum point P and $\mathcal{U}(y) = H_\varepsilon(\varepsilon y)$ after rescaling.

We will use a gluing technique (as in [16]), plus localized energy method([27]), to decompose the interaction of the downward spike and the interface layer and hence reduce the full problem (3.1) in Ω_ε to a simple system of PDEs.

Fix $\gamma_0 \in (0, 1/2)$, denote

$$\mathcal{Z}^\varepsilon \equiv \left\{ \sigma = (\sigma_1, \sigma_2) \in \Omega_\varepsilon : \operatorname{dist}(\sigma, \frac{P}{\varepsilon}) < \varepsilon^{\gamma_0 - 1} \right\},$$

and for all $\sigma \in \mathcal{Z}^\varepsilon$, define a spike at σ by

$$U_\sigma(y) = U(y - \sigma), \tag{3.2}$$

where U is defined in (1.13). Let $\delta < \delta_0/100$ be a fixed number, where δ_0 is a constant defined in (2.2). We consider a smooth cut-off function $\eta_\delta(t)$ where $t \in \mathbb{R}_+$ such that $\eta_\delta(t) = 1$ for $0 \le t \le \delta$ and $\eta(t) = 0$ for $t > 2\delta$. Set $\chi^\varepsilon_\delta(r) = \eta_\delta(\varepsilon r)$ and $\eta^\varepsilon_\delta(s) = \eta_\delta(\varepsilon|s|)$, where r is the distance to the point $\frac{P}{\varepsilon}$ and s is the normal coordinate to Γ/ε. We define our global approximation to be simply

$$W(y) = \mathcal{U}(y) \; - \; \chi^\varepsilon_{3\delta}(r) U_\sigma(y) \quad \text{for } y \in \Omega_\varepsilon. \tag{3.3}$$

In the coordinate (y_1, y_2) introduced in (3.1), W is a function defined on Ω_ε which is extended globally as $\mathcal{U}$ beyond the $6\delta/\varepsilon$-neighborhood of $\frac{P}{\varepsilon}$.

For $u = W + \hat\phi$ where $\hat\phi$ globally defined in Ω_ε, denote

$$S(u) = \triangle_y u \; + \; (u - a(\varepsilon y))\,(1-u^2) \quad \text{in } \Omega_\varepsilon.$$

Then u satisfies (3.1) if and only if

$$\widetilde{\mathcal{L}}(\hat\phi) = -\widetilde{E} + \widetilde{N}(\hat\phi) \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial \hat\phi}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{3.4}$$

where

$$\widetilde{\mathcal{L}}(\hat{\phi}) = \triangle_y \hat{\phi} + \big[1 - 3W^2 + 2a(\varepsilon y) W \big] \hat{\phi},$$

$$\widetilde{N}(\hat{\phi}) = (\hat{\phi})^3 + 3W(\hat{\phi})^2 - a(\varepsilon y)(\hat{\phi})^2, \quad \widetilde{E} = S(W).$$

For further application, it is of importance to carry out the evaluation of the accuracy of the error. Namely,

$$\begin{aligned} \widetilde{E} &= S(W) \\ &= \varepsilon^2(\triangle_y \chi_{3\delta}^\varepsilon) U_\sigma + 2\varepsilon(\nabla_y \chi_{3\delta}^\varepsilon)\cdot(\nabla_y U_\sigma) + \chi_{3\delta}^\varepsilon U_\sigma \big[U_\sigma^2 - (\chi_{3\delta}^\varepsilon)^2 U_\sigma^2 + 2\chi_{3\delta}^\varepsilon \mathcal{U} U_\sigma - 2U_\sigma \big] \\ &\quad + \chi_{3\delta}^\varepsilon U_\sigma (1 - \mathcal{U}^2) + \chi_{3\delta}^\varepsilon U_\sigma \big[2(1-b) - 2(\mathcal{U} - a(\varepsilon y))\mathcal{U} + \chi_{3\delta}^\varepsilon(\mathcal{U} - a(\varepsilon y)) U_\sigma - (1-b) U_\sigma \big]. \end{aligned}$$

In particular, we have the form of errors near Γ/ε and $\frac{P}{\varepsilon}$ as

$$\begin{aligned} \widetilde{E}_2 &\equiv \eta_\delta^\varepsilon \widetilde{E} = 0, \\ \widehat{E}_2 &\equiv \chi_\delta^\varepsilon \widetilde{E} \\ &= 3\chi_\delta^\varepsilon U_\sigma(1-\mathcal{U}^2) + 3\chi_\delta^\varepsilon U_\sigma^2 \big[\chi_{3\delta}^\varepsilon \mathcal{U} - 1 \big] \\ &\quad + 2\chi_\delta^\varepsilon U_\sigma \big[a(\varepsilon y)\mathcal{U} - b \big] + \chi_\delta^\varepsilon U_\sigma^2 \big[b - \chi_{3\delta}^\varepsilon a(\varepsilon y) \big]. \end{aligned}$$

Note that the errors $\widehat{E}_2$ can be defined on the whole plane $\mathbb{R}^2$ and $\overline{E}_2$ on the region $\Omega_{2\varepsilon}$ by trivial extension. Moreover,

$$\begin{aligned} (1 - \chi_\delta^\varepsilon - \eta_\delta^\varepsilon)\widetilde{E} &= \varepsilon^2(\triangle_y \chi_{3\delta}^\varepsilon) U_\sigma + 2\varepsilon(\nabla_y \chi_{3\delta}^\varepsilon)\cdot(\nabla_y U_\sigma) + (1-\chi_\delta^\varepsilon)\chi_{3\delta}^\varepsilon U_\sigma(1-\mathcal{U}^2) \\ &\quad + (1-\chi_\delta^\varepsilon)\chi_{3\delta}^\varepsilon U_\sigma \big[U_\sigma^2 - (\chi_{3\delta}^\varepsilon)^2 U_\sigma^2 + 2\chi_{3\delta}^\varepsilon U_\sigma \mathcal{U} - 2U_\sigma \big] \\ &\quad + (1-\chi_\delta^\varepsilon)\chi_{3\delta}^\varepsilon U_\sigma \big[\chi_{3\delta}^\varepsilon U_\sigma(\mathcal{U} - a(\varepsilon y)) - (1-b)U_\sigma \big] \\ &\quad + (1-\chi_\delta^\varepsilon)\chi_{3\delta}^\varepsilon U_\sigma \big[2(1-b) - 2(\mathcal{U} - a(\varepsilon y))\mathcal{U} \big]. \end{aligned}$$

It is easy to derive the following decay estimates

$$||\widehat{E}_2||_{L^2(\mathbb{R}^2)} \le C\varepsilon^2, \tag{3.5}$$

$$|\widehat{E}_2| \le e^{-\gamma/\varepsilon} \quad \text{for } \mathrm{dist}(y, \frac{P}{\varepsilon}) \ge \delta/\varepsilon, \tag{3.6}$$

$$||(1 - \chi_\delta^\varepsilon - \eta_\delta^\varepsilon)\widetilde{E}||_{L^\infty} \le e^{-\gamma/\varepsilon}. \tag{3.7}$$

Moreover, all above errors are continuous functions of the parameter $\sigma \in \mathcal{Z}^\varepsilon$.

We further separate $\hat{\phi}$ in the following form

$$\hat{\phi} = \chi_{3\delta}^\varepsilon(r)\phi_2 + \eta_{3\delta}^\varepsilon(s)\phi_3 + \psi$$

where we assume that ϕ_3 is defined in $\Omega_{2\varepsilon}$ and ϕ_2 is defined in the whole plane $\mathbb{R}^2$. Obviously, problem (3.4) is equivalent to the following system

$$
\begin{aligned}
&\chi_{3\delta}^{\varepsilon}\Big[\,\triangle_y\phi_2 \,+\, \big(1-3W^2+2a(\varepsilon y)W\,\big)\phi_2\,\Big]\\
&\qquad\qquad = \chi_{\delta}^{\varepsilon}\Big[\,\widetilde{N}\big(\chi_{3\delta}^{\varepsilon}\phi_2+\eta_{3\delta}^{\varepsilon}\phi_3+\psi\big) \,-\, \widetilde{E} \,-\, 3(1-W^2)\psi \,+\, 2aW\psi\,\Big], \qquad (3.8)\\
&\eta_{3\delta}^{\varepsilon}\Big[\,\triangle_y\phi_3 \,+\, \big(1-3W^2+2a(\varepsilon y)W\,\big)\phi_3\,\Big]\\
&\qquad\qquad = \eta_{\delta}^{\varepsilon}\Big[\,\widetilde{N}\big(\chi_{3\delta}^{\varepsilon}\phi_2+\eta_{3\delta}^{\varepsilon}\phi_3+\psi\big) \,-\, \widetilde{E} \,-\, 3(1-W^2)\psi\,\Big], \qquad (3.9)\\
&\triangle_y\psi \,-\, 2\Big[1-\big(1-\chi_{\delta}^{\varepsilon}\big)aW\,\Big]\psi \,+\, 3\big(1-\eta_{\delta}^{\varepsilon}-\chi_{\delta}^{\varepsilon}\,\big)(1-W^2)\psi\\
&\qquad\qquad = -\,\varepsilon^2(\triangle_y\eta_{3\delta}^{\varepsilon})\phi_3 \,-\, 2\varepsilon(\nabla_y\eta_{3\delta}^{\varepsilon})(\nabla_y\phi_3) - \varepsilon^2(\triangle_y\,\chi_{3\delta}^{\varepsilon})\phi_2 \,-\, 2\varepsilon(\nabla_y\,\chi_{3\delta}^{\varepsilon})(\nabla_y\phi_2) \qquad (3.10)\\
&\qquad\qquad\quad + \big(1-\chi_{\delta}^{\varepsilon}-\eta_{\delta}^{\varepsilon}\,\big)\widetilde{N}\big(\chi_{3\delta}^{\varepsilon}\phi_2+\eta_{3\delta}^{\varepsilon}\phi_3+\psi\big) \,-\, \big(1-\chi_{\delta}^{\varepsilon}-\eta_{\delta}^{\varepsilon}\,\big)\widetilde{E},
\end{aligned}
$$

where ψ is defined in Ω_ε and satisfies the homogeneous Neumann boundary condition.

The key observation is that, after solving (3.10), the problem can be transformed to the following nonlinear problem involving the parameter ψ

$$
\begin{aligned}
\widehat{\mathcal{L}}(\phi_2) &= \chi_{\delta}^{\varepsilon}\Big[\,\widetilde{N}\big(\chi_{3\delta}^{\varepsilon}\phi_2+\eta_{3\delta}^{\varepsilon}\phi_3+\psi\big) \,-\, \widetilde{E} \,-\, 3(1-W^2)\psi \,+\, 2aW\psi\,\Big], \qquad (3.11)\\
\overline{\mathcal{L}}(\phi_3) &= \eta_{\delta}^{\varepsilon}\Big[\,\widetilde{N}\big(\chi_{3\delta}^{\varepsilon}\phi_2+\eta_{3\delta}^{\varepsilon}\phi_3+\psi\big) \,-\, \widetilde{E} \,-\, 3(1-W^2)\psi\,\Big]. \qquad (3.12)
\end{aligned}
$$

Notice that the operators $\widehat{\mathcal{L}}$ and $\overline{\mathcal{L}}$ in Ω_ε may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of $\frac{P}{\varepsilon}$ and Γ/ε respectively.

Firstly, we solve, given a small ϕ_2 and ϕ_3, problem (3.10) for ψ. Assume now that ϕ_2 and ϕ_3 satisfy the following decay property

$$
\begin{aligned}
\big|\nabla\phi_2(y)\big| + \big|\phi_2(y)\big| \le e^{-\gamma/\varepsilon} \ \text{ if } \ \mathrm{dist}(y, \frac{P}{\varepsilon}) > \delta/\varepsilon, \qquad (3.13)\\
\big|\nabla\phi_3(y)\big| + \big|\phi_3(y)\big| \le e^{-\gamma/\varepsilon} \ \text{ if } \ \mathrm{dist}(y, \Gamma/\varepsilon) > \delta/\varepsilon, \qquad (3.14)
\end{aligned}
$$

for certain constant $\gamma > 0$. The solvability can be done in the following way. Let us observe that $(1-\chi_{\delta}^{\varepsilon}-\eta_{\delta}^{\varepsilon})(1-W^2)$ is exponentially small and

$$\min_{y\in\bar{\Omega}_\varepsilon} 2\big[\,1-\big(1-\chi_{\delta}^{\varepsilon}\big)a(\varepsilon y)W\,\big] > 0.$$

Then the problem

$$\triangle_y\psi - 2\Big(1-\big(1-\chi_{\delta}^{\varepsilon}\big)aW\Big)\psi + 3(1-\chi_{\delta}^{\varepsilon}-\eta_{\delta}^{\varepsilon})(1-W^2)\psi = h \quad \text{in } \Omega_\varepsilon, \qquad \frac{\partial\psi}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon,$$

has a unique bounded solution ψ whenever $||h||_\infty \le +\infty$. Moreover,

$$||\psi||_\infty \le C||h||_\infty.$$

Since $\widetilde{N}$ is power-like with power greater than one, a direct application of contraction mapping principle yields that (3.10) has a unique (small) solution $\psi = \psi(\phi_2, \phi_3)$ with

$$||\psi(\phi_2,\phi_3)||_{L^\infty} \leq C\varepsilon\Big[\,||\phi_2||_{L^\infty(r>\delta/\varepsilon)} + ||\nabla\phi_2||_{L^\infty(r>\delta/\varepsilon)} + ||\phi_3||_{L^\infty(|s|>\delta/\varepsilon)} + ||\nabla\phi_3||_{L^\infty(|s|>\delta/\varepsilon)} + e^{-\delta/\varepsilon}\Big], \tag{3.15}$$

where $r > \delta/\varepsilon$ denotes the complement in Ω_ε of δ/ε-neighborhood of P_ε and $|s| > \delta/\varepsilon$ denotes the complement in Ω_ε of δ/ε-neighborhood of Γ/ε. Moreover, the nonlinear operator ψ satisfies a Lipschitz condition of the form

$$||\psi(\phi_2,\phi_3) - \psi(\tilde{\phi}_2,\tilde{\phi}_3)||_{L^\infty} \leq C\varepsilon\Big[\,||\phi_2 - \tilde{\phi}_2||_{L^\infty(r>\delta/\varepsilon)} + ||\nabla\phi_2 - \nabla\tilde{\phi}_2||_{L^\infty(r>\delta/\varepsilon)} + ||\phi_3 - \tilde{\phi}_3||_{L^\infty(|s|>\delta/\varepsilon)} + ||\nabla\phi_3 - \nabla\tilde{\phi}_3||_{L^\infty(|s|>\delta/\varepsilon)}\Big]. \tag{3.16}$$

Therefore, from the above discussion, the full problem has been reduced to solving the following (nonlocal) problem (for given $\psi = \psi(\phi_2, \phi_3)$)

$$\widehat{\mathcal{L}}_2(\phi_2) = \chi_\delta^\varepsilon\Big[\,\widetilde{N}\big(\chi_{3\delta}^\varepsilon\phi_2 + \eta_{3\delta}^\varepsilon\phi_3 + \psi\big) - \widetilde{E} - 3(1-W^2)\psi + 2aW\psi\Big], \tag{3.17}$$

$$\overline{\mathcal{L}}_2(\phi_3) = \eta_\delta^\varepsilon\Big[\,\widetilde{N}\big(\chi_{3\delta}^\varepsilon\phi_2 + \eta_{3\delta}^\varepsilon\phi_3 + \psi\big) - \widetilde{E} - 3(1-W^2)\psi\Big], \tag{3.18}$$

for $\phi_2 \in H^2(\mathbb{R}^2)$ satisfying condition (3.13) and $\phi_3 \in H^2(\Omega_{2\varepsilon})$ satisfying condition (3.14). Here the operators $\widehat{\mathcal{L}}_2$ and $\overline{\mathcal{L}}_2$ in may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of $\frac{P}{\varepsilon}$ and Γ/ε respectively.

The definitions of these operators can be showed as follows. $\widehat{\mathcal{L}}_2$ is an operator by

$$\widehat{\mathcal{L}}_2(\phi_2) = \triangle_y\phi_2 + \big[\,(6-2b)U_\sigma - 3U_\sigma^2 + 2(b-1)\,\big]\phi_2 + \chi(r)B_2(\phi_2), \quad y \in \mathbb{R}^2, \tag{3.19}$$

where $\chi(r)$ with $r = \mathrm{dist}(y, \frac{P}{\varepsilon})$ is a smooth cut-off function which equals 1 for $0 \leq r < 10\delta$ and vanishes identically for $r > 20\delta$ and B_2 is the operator defined by

$$B_2(\phi_2) = \big(1 - 3W^2 + 2a(\varepsilon y)W\big)\phi_2 - \big[\,(6-2b)U_\sigma - 3U_\sigma^2 + 2(b-1)\,\big]\phi_2,$$

while the operator $\overline{\mathcal{L}}_2$ can be defined as the following

$$\overline{\mathcal{L}}_2(\phi_3) = \triangle_y\phi_3 + \big(1 - 3\mathcal{U}^2 + 2a(\varepsilon y)\mathcal{U}\big)\phi_3, \quad y \in \Omega_{2\varepsilon}. \tag{3.20}$$

Rather than solving problem (3.17)-(3.18) directly, we deal with the following system: given $\sigma \in \mathcal{Z}^\varepsilon$, finding functions $\phi_2 \in H^2(\mathbb{R}^2)$, $\phi_3 \in H^2(\Omega_{2\varepsilon})$ and constants $c(\sigma)$, $d(\sigma)$ such that

$$\widehat{\mathcal{L}}_2(\phi_2) = \widehat{N}_2(\phi_2,\phi_3) - \widehat{E}_2 + c(\sigma)\,\chi_\delta^\varepsilon U_{\sigma,y_1} + d(\sigma)\,\chi_\delta^\varepsilon U_{\sigma,y_2} \quad \text{in } \mathbb{R}^2, \tag{3.21}$$

$$\phi_2(y) \to 0 \quad \text{as } |y| \to \infty, \qquad \int_{\mathbb{R}^2} \phi_2(y)U_{\sigma,y_i}(y)\,\mathrm{d}y = 0,\ i = 1,2, \tag{3.22}$$

$$\overline{\mathcal{L}}_2(\phi_3) = \overline{N}_2(\phi_2,\phi_3) - \overline{E}_2 \quad \text{in } \Omega_{2\varepsilon}, \quad \phi_3 = 0 \quad \text{on } \partial(\Omega_{2\varepsilon}), \tag{3.23}$$

where

$$\widehat{E}_2 = \chi_\delta^\varepsilon \widetilde{E}, \quad \overline{E}_2 = \eta_\delta^\varepsilon \widetilde{E} \equiv 0,$$

$$\overline{N}_2(\phi_2,\phi_3) = \eta_\delta^\varepsilon \Big[\widetilde{N}\big(\chi_{3\delta}^\varepsilon \phi_2 + \eta_{3\delta}^\varepsilon \phi_3 + \psi(\phi_2,\phi_3)\big) \ - \ 3(1-W^2)\psi(\phi_2,\phi_3) \Big],$$

$$\widehat{N}_2(\phi_2,\phi_3) = \chi_\delta^\varepsilon \Big[\widetilde{N}\big(\chi_{3\delta}^\varepsilon \phi_2 + \eta_{3\delta}^\varepsilon \phi_3 + \psi(\phi_2,\phi_3)\big) \ - \ 3(1-W^2)\psi(\phi_2,\phi_3) \ + \ 2aW\psi(\phi_2,\phi_3) \Big].$$

For simplicity of notations, we write ϕ_2, ϕ_3 instead of stating the dependence on the parameter σ in above formulas and in the sequel.

In Proposition 5.1, we will prove that (3.21)-(3.23) problem has a unique solution (ϕ_2,ϕ_3) whose norm is controlled by the L^2-norm of $\widetilde{E}$. Moreover, ϕ_2 and ϕ_3 will satisfy (3.13)-(3.14). After this has been done, our task is to solve an algebra equation to choose suitable $\sigma \in \mathcal{Z}^\varepsilon$ such that the constants $c(\sigma)$ and $d(\sigma)$ are zero, whence we finish the proof of Theorem 1.2.

4 The invertibility of operators $\overline{\mathcal{L}}_2$ and $\widehat{\mathcal{L}}_2$

For the purpose of the resolution of the projected problem (3.21)-(3.23), we consider the invertibility of the linear operators $\overline{\mathcal{L}}_2$ and $\widehat{\mathcal{L}}_2$ in this section. Denote

$$L(\phi) \equiv \triangle_y \phi \ + \ \big[\,(6-2b)U_\sigma - 3U_\sigma^2 + 2(b-1)\,\big]\phi$$

and then we consider the following problem

$$L(\phi) \ = \ h \ + \ c(\sigma)\,\chi_\delta^\varepsilon U_{\sigma,y_1} \ + \ d(\sigma)\,\chi_\delta^\varepsilon U_{\sigma,y_2} \quad \text{in } \mathbb{R}^2, \tag{4.1}$$

$$\phi(y) \to 0 \quad \text{as } |y| \to \infty, \qquad \int_{\mathbb{R}^2} \phi(y) U_{\sigma,y_i}(y)\,\mathrm{d}y = 0, \ i = 1,2. \tag{4.2}$$

Lemma 4.1. *There exist constants $c(\sigma)$, $d(\sigma)$ with respect to h such that the problem (4.1)-(4.2) has a unique solution $\phi \equiv \widehat{T}_1(h)$. Moreover,*

$$||\phi||_{H^2(\mathbb{R}^2)} \le C\,||h||_{L^2(\mathbb{R}^2)},$$

where the constant C does not depend on h and ε.

Proof. Setting

$$c(\sigma) = -\frac{\int_{\mathbb{R}^2} \chi_\delta^\varepsilon\, h(y)\, U_{\sigma,y_1}(y)\,\mathrm{d}y}{\int_{\mathbb{R}^2} \chi_\delta^\varepsilon\, U_{\sigma,y_1}^2(y)\,\mathrm{d}y}, \quad d(\sigma) = -\frac{\int_{\mathbb{R}^2} \chi_\delta^\varepsilon\, h(y)\, U_{\sigma,y_2}(y)\,\mathrm{d}y}{\int_{\mathbb{R}^2} \chi_\delta^\varepsilon\, U_{\sigma,y_2}^2(y)\,\mathrm{d}y},$$

and applying Fredholm's alternative, we can find the existence and uniqueness of the solution ϕ.

□

Let $\widehat{\mathcal{L}}_2$ be the operator defined in $H^2(\mathbb{R}^2)$ by (3.19). In this section, we study the following linear problem: given $\sigma \in \mathcal{Z}^\varepsilon$, for $h \in L^2(\mathbb{R}^2)$, finding function ϕ and constants $c(\sigma)$, $d(\sigma)$ such that

$$\widehat{\mathcal{L}}_2(\phi) \;=\; h \,+\, c(\sigma)\,\chi^\varepsilon_\delta U_{\sigma,y_1} \,+\, d(\sigma)\,\chi^\varepsilon_\delta U_{\sigma,y_2} \quad \text{in } \mathbb{R}^2, \tag{4.3}$$

$$\phi(y)\to 0 \quad \text{as } |y|\to\infty, \qquad \int_{\mathbb{R}^2}\phi(y)\,U_{\sigma,y_i}(y)\,\mathrm{d}y = 0,\ \ i=1,2. \tag{4.4}$$

Proposition 4.2. *If* δ *in the definition of* $\widehat{\mathcal{L}}_2$ *is chosen small enough and* $h\in L^2(\mathbb{R}^2)$*, then there exists a constant* $C>0$*, independent of* ε*, such that for all small* ε*, the problem (4.3)-(4.4) has a unique solution* $\phi=\widehat{T}_2(h)$ *which satisfies*

$$||\phi||_{H^2(\mathfrak{S})} \le C||h||_{L^2(\mathfrak{S})}. \tag{4.5}$$

Moreover, if h *has compact supports contained in* $|x|\le 20\delta/\varepsilon$*, then*

$$\big|\phi(x,\sigma)\big| + \big|\nabla\phi(x,\sigma)\big| \le e^{-2\delta/\varepsilon}\,||\phi||_{L^\infty} \quad \textit{for } |x|>40\delta/\varepsilon. \tag{4.6}$$

Proof. We write the problem as the form

$$L(\phi) \;=\; -\,\chi(r)B_2(\phi) \,+\, h \,+\, c(\sigma)\,\chi^\varepsilon_\delta U_{\sigma,y_1} \,+\, d(\sigma)\,\chi^\varepsilon_\delta U_{\sigma,y_2} \quad \text{in } \mathbb{R}^2,$$

$$\phi(y)\to 0 \quad \text{as } |y|\to\infty, \qquad \int_{\mathbb{R}^2}\phi(y)U_{\sigma,y_i}(y)\,\mathrm{d}y = 0,\ \ i=1,2.$$

Let

$$\varphi = \widehat{T}_1\Big(h-\chi B_2(\phi)\Big)$$

where $\widehat{T}_1$ is the bounded operator defined by Lemma 4.1. We can use contraction mapping theorem to solve the problem. The key point is that the operator

$$\chi B_2(\phi) \;=\; \chi\,\big(\,1-3W^2+2a(\varepsilon y)W\,\big)\phi \;-\; \chi\,\big[\,(6-2b)U_\sigma - 3U_\sigma^2 + 2(b-1)\,\big]\phi,$$

is small in the sense that

$$||B_2(\phi)||_{L^2(\mathbb{R}^2)} \;\le\; C\delta||\phi||_{H^2(\mathbb{R}^2)}.$$

Hence, the results can be derived by the invertibility conclusion of Lemma 4.1 if we choose δ sufficiently small.

□

Finally, consider the following problem

$$\overline{\mathcal{L}}_2(\phi) = \Delta_y\phi \,+\, \big(1-3\mathcal{U}^2+2a(\varepsilon y)\mathcal{U}\big)\phi \;=\; h \quad \text{in } \Omega_{2\varepsilon}, \qquad \phi=0 \quad \text{on } \partial\Omega_{2\varepsilon}. \tag{4.7}$$

Proposition 4.3. *If* ε *satisfies the gap condition (1.11), then there exists a unique solution* $\phi\equiv\overline{T}_2(h)$ *to (4.7) which satisfies the following estimate*

$$||\phi||_{H^2(\Omega_{2\varepsilon})} \le C\,\varepsilon^{-\frac{3}{2}}||h||_{L^2(\Omega_{2\varepsilon})}.$$

Furthermore, if h *has compact support in the region* $\{\,dist(y,\Gamma/\varepsilon)\le\delta/\varepsilon\,\}$*, then*

$$\big|\nabla\phi(y)\big| + \big|\phi(y)\big| \le e^{-\gamma/\varepsilon} \quad \textit{for} \quad dist(y,\Gamma/\varepsilon)>\delta/\varepsilon.$$

Proof. The existence and priori estimate is an easy corollary of Proposition 2.9. Since h is supported on $\{\,\mathrm{dist}(y,\Gamma/\varepsilon)\le\delta/\varepsilon\,\}$, then ϕ satisfies a problem of the form

$$\triangle_y\phi-(1+o(1))\phi=0\qquad\text{on }\mathrm{dist}(y,\Gamma/\varepsilon)\ge\delta/\varepsilon,$$

with zero boundary condition. Hence, the validity of decay can be showed easily.

□

5 Solving the nonlinear projected problem

In this section, we will solve the nonlinear system (3.21)-(3.23). Let $\widehat{T}_2,\overline{T}_2$ be the bounded operators defined by Proposition 4.2 and Proposition 4.3 respectively. Then the projected problem (3.21)-(3.23) is equivalent to the following fixed point problem

$$\phi_2=\widehat{T}_2\big(\,\widehat{N}_2(\phi_2,\phi_3)-\widehat{E}_2\,\big)\equiv\widehat{\mathcal{A}}(\phi_2,\phi_3),\tag{5.1}$$

$$\phi_3=\overline{T}_2\big(\,\overline{N}_2(\phi_2,\phi_3)-\overline{E}_2\,\big)\equiv\overline{\mathcal{A}}(\phi_2,\phi_3).\tag{5.2}$$

We collect some useful facts to find the domain of the operator $(\widehat{\mathcal{A}},\,\overline{\mathcal{A}})$ such that it becomes a contraction mapping. Since $\widehat{E}_2=\chi^{\varepsilon}_{\delta}\widetilde{E}$, from the estimates (3.5)-(3.7)

$$||\widehat{E}_2||_{L^2(\mathbb{R}^2)}\ \le\ c_*\,\varepsilon^{3/2},\tag{5.3}$$

$$||\widehat{E}_2(\sigma_1)-\widehat{E}_2(\sigma_2)||_{L^2(\mathbb{R}^2)}\ \le\ C\varepsilon^{3/2}|\sigma_1-\sigma_2|.\tag{5.4}$$

Moreover, $\overline{E}_2=\eta^{\varepsilon}_{\delta}\widetilde{E}=0$. The operators $\widehat{T}_2,\overline{T}_2$ have useful properties: assume that $\hat{h}$ has a support contained in $\{\,\mathrm{dist}(y,\frac{P}{\varepsilon})\le\delta/\varepsilon\,\}$ and $\bar{h}$ has a support contained in $\{\,\mathrm{dist}(y,\Gamma/\varepsilon)\le\delta/\varepsilon\,\}$, then $\phi_2=\widehat{T}_2(\hat{h})$ satisfies the estimate

$$\big|\phi_2\big|\ +\ \big|\nabla\phi_2\big|\ \le\ ||\phi_2||_{L^\infty}\ e^{-2\delta/\varepsilon}\quad\text{for}\,\mathrm{dist}(y,\frac{P}{\varepsilon})\ge\delta/\varepsilon,\tag{5.5}$$

and also $\phi_3=\overline{T}_2(\bar{h})$ satisfies the estimate

$$\big|\phi_3\big|\ +\ \big|\nabla\phi_3\big|\ \le\ ||\phi_3||_{L^\infty}\ e^{-2\delta/\varepsilon}\quad\text{for}\,\mathrm{dist}(y,\Gamma/\varepsilon)\ge\delta/\varepsilon.\tag{5.6}$$

Recall that the operator $\psi(\phi_2\,\phi_3)$ satisfies, as seen directly from its definition

$$\begin{aligned}||\psi(\phi_2,\phi_3)||_{L^\infty}\ \le\ C\varepsilon\Big[\,&||\phi_2||_{L^\infty(r>\delta/\varepsilon)}+||\nabla\phi_2||_{L^\infty(r>\delta/\varepsilon)}\\&+||\phi_3||_{L^\infty(|s|>\delta/\varepsilon)}+||\nabla\phi_3||_{L^\infty(|s|>\delta/\varepsilon)}+e^{-\delta/\varepsilon}\Big],\end{aligned}\tag{5.7}$$

and a Lipschitz condition of the form

$$\begin{aligned}||\psi(\phi_2,\phi_3)-\psi(\tilde\phi_2,\tilde\phi_3)||_{L^\infty}\ \le\ C\varepsilon\Big[\,&||\phi_2-\tilde\phi_2||_{L^\infty(r>\delta/\varepsilon)}+||\nabla\phi_2-\nabla\tilde\phi_2||_{L^\infty(r>\delta/\varepsilon)}\\&+||\phi_3-\tilde\phi_3||_{L^\infty(|s|>\delta/\varepsilon)}+||\nabla\phi_3-\nabla\tilde\phi_3||_{L^\infty(|s|>\delta/\varepsilon)}\Big].\end{aligned}\tag{5.8}$$

Now, the facts above will allow us to construct a region where contraction mapping principle applies and then solve the problem (3.21)-(3.23). Consider the following closed, bounded subset

$$\mathfrak{D} = \left\{ \begin{array}{l|l} \phi_2 \in H^2(\mathbb{R}^2) & ||\phi_2||_{H^2(\mathbb{R}^2)} \le \tau\varepsilon^{3/2}, \quad ||\phi_3||_{H^2(\Omega_{2\varepsilon})} \le \tau\varepsilon^{3/2}, \\ \phi_3 \in H_0^2(\Omega_{2\varepsilon}) & \phi_2 \text{ and } \phi_3 \text{ satisfy properties (5.5) and (5.6)respectively.} \end{array} \right\} \tag{5.9}$$

We claim that if the constant τ is sufficiently large, then the map $\left(\widehat{\mathcal{A}}, \overline{\mathcal{A}}\right)$ defined in (5.1)-(5.2) is a contraction mapping from $\mathfrak{D}$ into itself. Let us analyze the Lipschitz character of the nonlinear operators (involved in $\widehat{\mathcal{A}}, \overline{\mathcal{A}}$) defined on the domain $\mathfrak{D}$. Arguing as in [16] and using the Lipschitz dependence of ψ on ϕ_2, ϕ_3, it can be derived

$$\begin{aligned} &||\widehat{N}_2(\phi_2, \phi_3) - \widehat{N}_2(\widetilde{\phi}_1, \widetilde{\phi}_3)||_{L^2(\mathbb{R}^2)} \\ &\qquad \le C\left(\varepsilon^3\tau^2 + \varepsilon^{\frac{3}{2}}\tau\right)\left[\,||\phi_2 - \widetilde{\phi}_1||_{H^2(\mathbb{R}^2)} + ||\phi_3 - \widetilde{\phi}_3||_{H^2(\Omega_{2\varepsilon})}\right]. \end{aligned} \tag{5.10}$$

A similar Lipschitz property for the operator $\overline{N}_2$ holds.

Now, we can find the solution to (5.1)-(5.2) in the sequel. Let $(\phi_2, \phi_3) \in \mathfrak{D}$ and $\nu_1 = \widehat{\mathcal{A}}(\phi_2, \phi_3)$, $\nu_3 = \overline{\mathcal{A}}(\phi_2, \phi_3)$, then from (5.3)

$$\begin{aligned} ||\nu_1||_{H^2(\mathbb{R}^2)} \;&\le\; ||\widehat{T}_2||\left[\; c_*\varepsilon^{3/2} + C\tau^3\varepsilon^{9/2} + C\tau^2\varepsilon^3\;\right], \\ ||\nu_3||_{H^2(\Omega_{2\varepsilon})} \;&\le\; ||\overline{T}_2||\left[\; C\tau^3\varepsilon^{9/2} + C\tau^2\varepsilon^3\;\right]. \end{aligned}$$

Choosing any number $\tau > \max\left(c_*||\widehat{T}_2||,\, \varepsilon^{\frac{3}{2}}||\overline{T}_2||\right)$, we get that for small ε

$$||\nu_1||_{H^2(\mathbb{R}^2)} \;\le\; \tau\varepsilon^{3/2}, \quad ||\nu_3||_{H^2(\Omega_{2\varepsilon})} \;\le\; \tau\varepsilon^{3/2}.$$

From (5.5) and (5.6)

$$\begin{aligned} \Big|\Big||\nu_1| + |\nabla\nu_1|\Big|\Big|_{L^\infty} \;&\le\; ||\nu_1||_\infty\, e^{\frac{-2\delta}{\varepsilon}} \;\le\; ||\nu_1||_{H^2(\mathbb{R}^2)}\, e^{\frac{-\delta}{\varepsilon}} \quad \text{for}\,\mathrm{dist}(y, \frac{P}{\varepsilon}) \ge \delta/\varepsilon, \\ \Big|\Big||\nu_3| + |\nabla\nu_3|\Big|\Big|_{L^\infty} \;&\le\; ||\nu_3||_\infty\, e^{\frac{-2\delta}{\varepsilon}} \;\le\; ||\nu_3||_{H^2(\Omega_{2\varepsilon})}\, e^{\frac{-\delta}{\varepsilon}} \quad \text{for}\,\mathrm{dist}(y, \Gamma/\varepsilon) \ge \delta/\varepsilon. \end{aligned}$$

Therefore, $(\nu_1, \nu_3) \in \mathfrak{D}$. $(\widehat{\mathcal{A}}, \overline{\mathcal{A}})$ is clearly a contraction thanks to (5.10) and we can conclude that (5.1)-(5.2) has a unique solution in $\mathfrak{D}$.

The error $\widehat{E}_2$ and the operator $\widehat{T}_2$ itself carry σ as parameter. For future reference, we should consider their Lipschitz dependence on the parameter. (5.4) is just the formula about the Lipschitz dependence of error $\widehat{E}_2$ on the parameter. The other task can be realized by careful and direct computations of all terms involved in the differential operator which will show this dependence is indeed Lipschitz with respect to the H^2-norm (for all ε).

For the linear operator $\widehat{T}_2$, we have the following Lipschitz dependence

$$||\widehat{T}_2(\sigma_1) - \widehat{T}_2(\sigma_2)|| \le C\varepsilon|\sigma_1 - \sigma_2|.$$

Moreover, the operator $\widehat{N}_2$ also has Lipschitz dependence on σ. It is easily checked that for $(\phi_2, \phi_3) \in \mathfrak{D}$ we have, with obvious notation

$$||\widehat{N}_{2,\sigma_1}(\phi_2,\phi_3) - \widehat{N}_{2,\sigma_2}(\widetilde{\phi}_2,\widetilde{\phi}_3)||_{L^2(\mathbb{R}^2)} \le C\varepsilon^{5/2}\,|\sigma_1 - \sigma_2|.$$

Hence, from the fixed point characterization we get that

$$||\phi_2(\sigma_1) - \phi_2(\sigma_2)||_{H^2(\mathbb{R}^2)} \le C\varepsilon^{3/2}\,|\sigma_1 - \sigma_2|, \tag{5.11}$$

$$||\phi_3(\sigma_1) - \phi_3(\sigma_2)||_{H^2(\Omega_{2\varepsilon})} \le C\varepsilon^{3/2}\,|\sigma_1 - \sigma_2|. \tag{5.12}$$

Proposition 5.1. *There is a number $\tau > 0$ such that for all ε small enough satisfying the gap condition (1.11) and any given parameter $\sigma \in \mathcal{Z}^\varepsilon$, problem (3.21)-(3.23) has a unique solution $(\phi_2, \phi_3) \equiv (\phi_2(\sigma)\,\phi_3(\sigma))$ which satisfies*

$$||\phi_2||_{H^2(\mathbb{R}^2)} \le \tau\varepsilon^{3/2}, \quad ||\phi_3||_{H^2(\Omega_{2\varepsilon})} \le \tau\varepsilon^{3/2},$$
$$\big|\phi_2\big| + \big|\nabla\phi_2\big| \le ||\phi_2||_{L^\infty}\; e^{-2\delta/\varepsilon} \quad \textit{for } dist(y, \frac{P}{\varepsilon}) \ge \delta/\varepsilon,$$
$$\big|\phi_3\big| + \big|\nabla\phi_3\big| \le ||\phi_3||_{L^\infty}\; e^{-2\delta/\varepsilon} \quad \textit{for } dist(y, \Gamma/\varepsilon) \ge \delta/\varepsilon.$$

Moreover, the functions ϕ_2 and ϕ_3 depend Lipschitz-continuously on the parameter σ in the sense of the estimate (5.11)-(5.12).

□

6 Localized energy method and the proof of Theorem 1.2

In this section, we will show the existence of $\sigma_\varepsilon \in \mathcal{Z}^\varepsilon$ such that $c(\sigma_\varepsilon) = d(\sigma_\varepsilon) = 0$ and prove the validity of Theorem 1.2 by localized energy method. Rewrite the equation (3.21) as the following form for $v = \chi^\varepsilon_{3\delta}U_\sigma + \phi_2$

$$\widehat{\mathcal{L}}_2(v) + B_3(v) + \mathcal{E}_0 + \mathcal{P}(\phi_3,\psi,v) + \mathcal{Q}(\phi_3,\psi) = c(\sigma)\,\chi^\varepsilon_\delta U_{\sigma,y_1} + d(\sigma)\,\chi^\varepsilon_\delta U_{\sigma,y_2}, \tag{6.1}$$

where

$$\begin{aligned} B_3(v) &= \chi^\varepsilon_\delta\big[-3U_\sigma^2 + 6WU_\sigma - 2aU_\sigma\big]v + \chi^\varepsilon_\delta\big[3U_\sigma - 3W + a\big]v^2 - v^3, \\ \mathcal{E}_0 &= \chi^\varepsilon_\delta U_\sigma\big(u_\sigma - (1-b)\big)(2 - U_\sigma) - \chi(r)\big(1 - 3W^2 + 2aW\big)U_\sigma \\ &\quad + 2\chi^\varepsilon_\delta(a\mathcal{U} - b)U_\sigma + \chi^\varepsilon_\delta(b-a)U_\sigma^2 - 3\chi^\varepsilon_\delta WU_\sigma^2 + \chi^\varepsilon_\delta aU_\sigma^2 + \chi^\varepsilon_\delta U_\sigma^3. \end{aligned}$$

The operators $\mathcal{P}$ and $\mathcal{Q}$ are high order. Now, we introduce the following energy functional corresponding to (6.1)

$$
\begin{aligned}
K(v) \;=\;& \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2\,\mathrm{d}y \;-\; \frac{1}{2}\int_{\mathbb{R}^2} \big[\,(6-2b)U_\sigma - 3U_\sigma^2 + 2(b-1)\,\big](1-\chi(r))v^2\,\mathrm{d}y \\
&-\frac{1}{2}\int_{\mathbb{R}^2} \chi(r)\big(1-3W^2+2aW\big)v^2\,\mathrm{d}y \;+\; \int_{\mathbb{R}^2}\int_0^v B_3(s)\,\mathrm{d}s\,\mathrm{d}y \\
&+\int_{\mathbb{R}^2} \mathcal{E}_0 v\,\mathrm{d}y \;+\; \int_{\mathbb{R}^2}\int_0^v \mathcal{P}(\phi_3,\psi,s)\,\mathrm{d}s\,\mathrm{d}y \;+\; \int_{\mathbb{R}^2}\int_0^v \mathcal{Q}(\phi_3,\psi)v\,\mathrm{d}y,
\end{aligned}
$$

and define the function

$$
M(\sigma)\;\equiv\; K\big(\,\chi_{3\delta}^{\varepsilon}U_\sigma + \phi_2(\sigma)\,\big),\;\forall\,\sigma\in\mathcal{Z}^\varepsilon. \tag{6.2}
$$

From the results in previous sections, we know that $M(\sigma)$ is a C^1 continuous function in the variable $\sigma\in\mathcal{Z}^\varepsilon$. Moreover, direct computation gives that

$$
M(\sigma) \;=\; c_1 \,+\, c_2\,(b-a(\sigma)) \,+\, O(\varepsilon) \tag{6.3}
$$

where

$$
\begin{aligned}
c_1 \;&=\; 4(b-1)\int_{\mathbb{R}^2} U^2\,\mathrm{d}y \;+\; \big(\frac{5}{2}-\frac{17}{6}b\big)\int_{\mathbb{R}^2} U^3\,\mathrm{d}y \;-\; \frac{17}{4}\int_{\mathbb{R}^2} U^4\,\mathrm{d}y,\\
c_2 \;&=\; \frac{1}{3}\int_{\mathbb{R}^2} U^3\,\mathrm{d}y \;-\; \int_{\mathbb{R}^2} U^2\,\mathrm{d}y.
\end{aligned}
$$

Lemma 6.1. *There exists a critical pint σ_ε in the interior of $\mathcal{Z}^\varepsilon$ of the function $M(\sigma)$.*

Proof. Since $0<U<2$, it is easy to see that $c_2<0$. Now we consider

$$
\max\,\{M(\sigma)\,:\,\sigma\in\mathcal{Z}^\varepsilon\}\,. \tag{6.4}
$$

Let $\sigma_\varepsilon\in\overline{\mathcal{Z}^\varepsilon}$ be the maximum point of the problem (6.4) and $\sigma^*\in\mathcal{Z}^\varepsilon$ be the point such that $a(\sigma^*)=b$. Then we have

$$
M(\sigma_\varepsilon)\;\geq\; M(\sigma^*) = c_1 + o(\varepsilon). \tag{6.5}
$$

On the other hand, if $\varrho>0$ is suitably small, then $|a(\sigma)-b|$ is small for any $|\sigma-\sigma^*|<\varrho$. Thus, we have

$$
M(\sigma_\varepsilon)\;\leq\; c_1 + (c_2+\tau_\varrho)\,(b-a(\sigma_\varepsilon)) + o(\varepsilon), \tag{6.6}
$$

where $\tau_\varrho>0$ and $\tau_\varrho\to 0$ as $\varrho\to 0$. Hence, we obtain from (6.5) and (6.6),

$$
o_\varepsilon(1)\;\leq\;(c_2+\tau_\varrho)\,(b-a(\sigma_\varepsilon))\;\leq\;0,
$$

where $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$, which implies that

$$
b-a(\sigma_\varepsilon)\to 0 \quad \text{as } \varepsilon\to 0.
$$

Therefore, σ_ε is an interior point of $\mathcal{Z}^\varepsilon$.

□

As a result of previous lemma, by standard argument as Proposition 6.2, we can easily deduce that $c(\sigma_\varepsilon) = d(\sigma_\varepsilon) = 0$ and we really solve the problem (3.1). Other results in Theorem 1.2 can be derived from the construction of the solutions.

We include the following proposition only for completeness, although it is well-known.

Proposition 6.2. *For any $\varepsilon > 0$, if $\sigma_\varepsilon \in \mathcal{Z}^\varepsilon$ is a interior critical point of $M(\sigma)$, then*

$$c(\sigma_\varepsilon) = d(\sigma_\varepsilon) = 0.$$

Proof. Denote $I' = K'(\chi^\varepsilon_{3\delta} U_\sigma + \phi_2(\sigma))$ the derivative of K at $\chi^\varepsilon_{3\delta} U_\sigma + \phi_2(\sigma)$, then it suffices to show that $(I', v)_{L^2} = 0$ for $v = U_{\sigma, y_1}$ and U_{σ, y_2}. Since σ_ε is an interior critical point of $M(\sigma)$, we have

$$D_{\sigma_i} M(\sigma_\varepsilon) = \Big(I', \chi^\varepsilon_{3\delta} U_{\sigma,\sigma_i} + D_{\sigma_i}\phi_2(\sigma_\varepsilon) \Big)_{L^2} = 0.$$

Therefore

$$c(\sigma_\varepsilon) + \Big(I', D_{\sigma_1}\phi_2(\sigma_\varepsilon) \Big)_{L^2} = 0, \quad d(\sigma_\varepsilon) + \Big(I', D_{\sigma_2}\phi_2(\sigma_\varepsilon) \Big)_{L^2} = 0.$$

To analyze above formulas, make the following decomposition,

$$D_{\sigma_i}\phi_2(\sigma_\varepsilon) = A_i + \sum_{j=1}^{2} c^\varepsilon_{ij} U_{\sigma,\sigma_j},$$

where the component A_i of $D_{\sigma_i}\phi_2(\sigma_\varepsilon)$ perpendicular to the space spanned by $\{U_{\sigma,\sigma_1}, U_{\sigma,\sigma_2}\}$ and then we get

$$(1 + c^\varepsilon_{11})\, c(\sigma_\varepsilon) \ + \ c^\varepsilon_{12}\, d(\sigma_\varepsilon) = 0, \quad c^\varepsilon_{21}\, c(\sigma_\varepsilon) \ + \ (1 + c^\varepsilon_{22})\, d(\sigma_\varepsilon) = 0. \tag{6.7}$$

Hence it is of crucial to estimate the coefficients c_{ij}'s.

We differentiate $\big(\phi_2(\sigma_\varepsilon), U_{\sigma,\sigma_i} \big)_{L^2} = 0$ for $i = 1, 2$ and obtain

$$c^\varepsilon_{i,j} = \big(D_{\sigma_j}\phi_2(\sigma_\varepsilon), U_{\sigma,\sigma_i} \big)_{L^2} \ = \ -\big(\phi_2(\sigma_\varepsilon), U_{\sigma,\sigma_i\sigma_j} \big)_{L^2} \quad \text{for } i, j = 1, 2.$$

Since $U_{\sigma,\sigma_i z_j}$ has exponential decay, we can easily check that

$$\big| \big(\phi_2(\sigma_\varepsilon), U_{\sigma,\sigma_i\sigma_j} \big)_{L^2} \big| \ \le \ C ||\phi_{\sigma_\varepsilon}||_{L^2} \cdot ||U_{L^2}|| \le C e^{-\gamma/\varepsilon},$$

due to our estimate for ϕ_2. Hence,

$$c^\varepsilon_{ij} = O(e^{-\gamma/\varepsilon}) \quad \text{for } i, j = 1, 2,$$

which lead to that the constants $c(\sigma_\varepsilon) = d(\sigma_\varepsilon) = 0$ is the only solution to (6.7).

□

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