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TODA SYSTEM AND INTERIOR CLUSTERING LINE CONCENTRATION FOR A SINGULARLY PERTURBED NEUMANN PROBLEM IN TWO DIMENSIONAL DOMAIN

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Abstract We consider the equation $\varepsilon^2\Delta\tilde{u}-\tilde{u}+\tilde{u}^p=0$ in a bounded, smooth domain Ω in $\mathbb{R}^2$ under homogeneous Neumann boundary conditions. Let Γ be a segment contained in Ω, connecting orthogonally the boundary, *non-degenerate* and *non-minimal* with respect to the curve length. For any given integer $N\geq 2$ and for small ε away from certain critical numbers, we construct a solution exhibiting N interior layers at mutual distances $O(\varepsilon|\ln\varepsilon|)$ whose center of mass collapse onto Γ at speed $O(\varepsilon^{1+\mu})$ for small positive constant μ as $\varepsilon\to 0$. Asymptotic location of these layers is governed by a Toda system.

1. **Introduction.**

We consider the following problem

$$\varepsilon^2\Delta\tilde{u}-\tilde{u}+\tilde{u}^p = 0 \text{ and } \tilde{u}>0 \quad \text{in } \Omega\subset\mathbb{R}^n, \tag{1.1}$$

$$\frac{\partial\tilde{u}}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, ε is a small parameter, ν denotes the outward normal of Ω and $p>1$.

Problem (1.1)-(1.2) is known as stationary equation of Keller-Segel system in chemotaxis [23]. It can also be viewed as a limiting stationary equation of Gierer-Meinhardt system in biological pattern formation [11]. Problem (1.1)-(1.2) has been studied extensively in recent years. See the review papers [30, 31] for more details.

In the pioneer papers [23], [32] and [33], under the condition that p is subcritical, i.e., $1<p<\frac{n+2}{n-2}$ when $n\geq 3$ and $1<p<+\infty$ when $n=2$, Lin, Ni and Takagi established the existence of a least-energy solution U_ε of (1.1)-(1.2) and showed that, for ε sufficiently small, U_ε has only one local maximum point $P_\varepsilon\in\partial\Omega$. Moreover,

$$H(P_\varepsilon)\to\max_{P\in\partial\Omega}H(P) \quad \text{as } \varepsilon\to 0,$$

where $H(\cdot)$ is the mean curvature of $\partial\Omega$. Such a solution is called boundary spike-layer.

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Since then, many paper investigated further the solutions of (1.1)-(1.2) concentrating at one or multiple points of $\bar{\Omega}$. (These solutions are called spike-layers.) A general principle is that the location of interior spike-layers is determined by the distance function from the boundary. We refer the reader to the articles [2], [7],[4],[13],[14], [12],[22], [37],[39] and references therein. On the other hand, the boundary spike-layers are related to the mean curvature of $\partial\Omega$. This aspect is discussed in the papers [3],[6],[15],[21],[38], [40],[41] and references therein. A good review of the subject is to be found in [30, 31].

However, in all the papers mentioned above, the concentration set is zero-dimensional. The question of constructing higher-dimensional concentration sets has been investigated only in recent years. It has been conjectured in [30] that for any $1 \leq k \leq n-1$, problem (1.1)-(1.2) has a solution U_ε which concentrates on a k-dimensional subset of $\bar{\Omega}$.

In [27] and [28], Malchiodi and Montenegro proved that for $n \geq 2$, there exists a sequence of numbers $\varepsilon_k \to 0$ such that problem (1.1)-(1.2) has a solution U_{ε_k} which concentrates at boundary $\partial\Omega$ (or any component of $\partial\Omega$). (Recently, Wang and Wei [36] showed that the same results hold for any ε sufficiently small.) In [25, 26], Malchiodi showed the concentration phenomena for (1.1)-(1.2) along a closed non-degenerate geodesic of $\partial\Omega$ in three-dimensional smooth bounded domain Ω. As for the conjecture in [30], the general result by F. Mahmoudi and A. Malchiodi, [24], gives a full answer about the concentration phenomenon on the boundary. They prove a full general concentration of solutions along k-dimensional $(1 \leq k \leq n-2)$ non-degenerate minimal submanifolds of the boundary for $n \geq 3$ and $1 < p < \frac{n-k+2}{n-k-2}$.

In [42], we considered the case of concentration on line segments lying in a two-dimensional domain Ω. More precisely, let Γ be a line segment lying in $\Omega \subset \mathbb{R}^2$ and after translation and rotation, Γ is contained in the $\tilde{y}_1 = 0$ axis in the $(\tilde{y}_1, \tilde{y}_2)$ coordinates. Γ intersects $\partial\Omega$ at exactly two points, saying, γ_1, γ_0 and at these points $\Gamma \perp \partial\Omega$. The boundary $\partial\Omega$ can be represented as $\varphi_1(\tilde{y}_1)$ and $\varphi_0(\tilde{y}_1)$ near γ_1, γ_0 respectively. The lines $\tilde{y}_2 = \varphi_i(0), i = 1, 2$ are tangent to Ω. Moreover, after rescaling, we can always assume $|\Gamma| = 1$, i.e. $\varphi_1(0) - \varphi_0(0) = 1$. Let $-k_1$ and k_0 be the curvatures of the boundary $\partial\Omega$ at the points γ_1 and γ_0 respectively, where

$$k_1 = \varphi_1^{''}(0), \quad k_0 = \varphi_0^{''}(0). \tag{1.3}$$

We define a geometric eigenvalue problem

$$-f^{''}(\theta) \;=\; \lambda\, f(\theta), \quad 0 < \theta < 1, \tag{1.4}$$

$$f^{'}(1) + k_1 f(1) \;=\; 0, \quad f^{'}(0) + k_0 f(0) \;=\; 0. \tag{1.5}$$

We say that Γ is *non-degenerate* if (1.4)-(1.5) does not have a zero eigenvalue [9, 19]. This is equivalent to the following condition:

$$k_0 - k_1 + k_0 k_1 |\Gamma| \neq 0. \tag{1.6}$$

Under the condition (1.6), we proved the existence of solution concentrating along an interior curve Γ_ε near the line segment Γ, provided ε is small and away from certain critical numbers. The curve Γ_ε will collapse to Γ as $\varepsilon \to 0$.

For the above Γ, we assume in this paper the following more restrictive condition, called as *non-degenerate* and *non-minimal* condition

$$\begin{aligned} &k_1 < 0, \quad k_0 > 0, \\ &k_0 - k_1 + k_0\, k_1\, |\Gamma| > 0 \\ &-1 < \frac{k_0 + k_1}{k_1 k_0 |\Gamma|} < 1. \end{aligned} \tag{1.7}$$

The first condition means that the domain looks locally strictly convex near γ_1 and γ_0. The third condition sets a bound on the degree of "asymmetry", while the second one not only gives non-degeneracy, but also requires that exactly one negative eigenvalue of problem (1.4)-(1.5) is present. (An example of curve Γ satisfying (1.7) is the short axis of an ellipse.) After the statement of our conditions on Γ, we also make the assumption that

$$n = 2 \text{ and } p \geq 2, \tag{1.8}$$

and show that if Γ is *non-degenerate* and *non-minimal*, then for any integer $N > 1$, there exists a solution exhibiting N concentration layers with mutual distance $O(\varepsilon|\ln\varepsilon|)$, and these N layers collapse to Γ, as $\varepsilon \to 0$.

Before stating the main result, we introduce two functions w and Z. Let w be the unique (even) solution to

$$w^{''} - w + w^p = 0 \text{ and } w > 0 \text{ in } \mathbb{R},\ w^{'}(0) = 0,\ w(\pm\infty) = 0. \tag{1.9}$$

It is well known that the associated linearized eigenvalue problem,

$$h^{''} - h + pw^{p-1}h = \lambda h \text{ in } \mathbb{R},\ \int_{\mathbb{R}} h^2 = 1,\ h \in H^1(\mathbb{R}), \tag{1.10}$$

possesses a unique positive eigenvalue λ_0 with a unique even and positive eigenfunction Z.

The following is the main result of this paper.

Theorem 1.1. *Assume (1.8) holds and Γ satisfies the nondegenerate and non-minimality condition (1.7). Then for each $N > 1$ and all sufficiently small ε satisfying the following gap condition:*

$$|\lambda_0 - k^2\pi^2\varepsilon^2| \geq \tilde{c}\,\varepsilon, \quad \forall\, k \in \mathbb{N}. \tag{1.11}$$

where $\tilde{c} > 0$ is a small constant, there exists a solution u_ε to (1.1)-(1.2) with exactly N concentration layers at mutual distances $O(\varepsilon|\ln\varepsilon|)$. In addition, the center of mass for N concentration layers collapse to Γ at speed $O(\varepsilon^{1+\mu})$ for some small positive constant μ.
More precisely, in the $(\tilde{y}_1, \tilde{y}_2)$ coordinates u_ε has the form

$$u_\varepsilon(\tilde{y}_1, \tilde{y}_2) \sim \sum_{k=1}^{N} w\left(\frac{\tilde{y}_1 - \varepsilon f_k(\tilde{y}_2)}{\varepsilon}\right)$$

where the f_k's satisfy

$$\|f_k\|_\infty \leq C|\ln\varepsilon|^2,\ \min_{1\leq k\leq N-1}(f_{k+1} - f_k) > 2|\ln\varepsilon|,\ \sum_{k=1}^{N} f_k = \varepsilon^\mu, \tag{1.12}$$

and solve the Toda system, for $k = 1, \ldots, N$,

$$\varepsilon^2 f_k'' - a_0\,[e^{-(f_k - f_{k-1})} - e^{-(f_{k+1} - f_k)}] = 0 \quad \textit{in } (0,1), \tag{1.13}$$

$$f_k'(0) + k_0\, f_k(0) = 0, \qquad f_k'(1) + k_1\, f_k(1) = 0, \tag{1.14}$$

for a universal constant $a_0 > 0$, *with the conventions* $f_0 = -\infty$, $f_{N+1} = \infty$.

Let us comment on related results and the difficulties and main steps in proving Theorem 1.1.

The geometric eigenvalue problem (1.4) also appeared in the study of transitional layer for the following Allen-Cahn equation

$$\varepsilon^2 \Delta u + u - u^3 = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.15}$$

In an interesting paper [16], using $\Gamma-$convergence, Kohn and Sternberg constructed local minimizers to (1.15) with transition layer near a straight line segment contained in Ω which locally minimizes length among all curves nearby with endpoints lying on $\partial\Omega$. Later, M. Kowalczyk [19] extended the construction to non-minimizing line segments. More precisely, assuming that Γ satisfies (1.6), for *all* ε sufficiently small he constructed a solution u_ε whose zero set Γ_ε converges to Γ as $\varepsilon \to 0$. In [35], Pacard and Ritore constructed transition layer solutions to (1.15) near minimal submanifold on a closed Riemannian manifold.

In a recent paper [9], M. del Pino, Kowalczyk and Wei constructed clustered line concentrations near a nondegenerate and non-minimal line segment for the Allen-Cahn equation (1.15). The location of the layers is governed by the Toda system (1.13).

To explain in a few words the difficulties we have encountered, let us assume for the moment that $\Omega = (-\infty, +\infty) \times [0, 1]$ is an infinite strip. In term of the stretched coordinates $(s, z) = \varepsilon^{-1}(\tilde{y}_1, \tilde{y}_2)$ the equation would look near the curve approximately like

$$v_{ss} + v_{zz} - v + v^p = 0, \ (s, z) \in \mathfrak{S} \equiv \mathbb{R} \times (0, \frac{1}{\varepsilon}), \quad \frac{\partial v}{\partial z} = 0 \text{ on } \partial\mathfrak{S}.$$

The effect of curvature and of the boundary conditions are here neglected. The linearization of this problem around the profile $w(s)$ becomes

$$\phi_{zz} + \phi_{ss} - \phi + pw^{p-1}\phi = 0, \ (s, z) \in \mathfrak{S}, \quad \frac{\partial v}{\partial z} = 0 \text{ on } \ \partial\mathfrak{S}.$$

Functions of the form

$$\phi^1 = w_s(s) \cos(k\pi\varepsilon z), \quad \phi^2 = Z(s) \cos(k\pi\varepsilon z), \tag{1.16}$$

are eigenfunctions associated to eigenvalues respectively $-k^2\varepsilon^2$ and $\lambda_0 - k^2\varepsilon^2$. Many of these numbers are *small* and thus *"near non-invertibility"* of the linear operator occurs. These two effects, combined in principle orthogonally because of the L^2-orthogonality of Z and w_s, are actually coupled through the smaller order terms neglected.

In [1, 18, 19, 35], related singular perturbation problems, involving the Allen-Cahn equation (1.15), the translation effect ϕ^1 have been successfully treated through successive improvements of the approximation and fine spectral analysis of the actual linearized operator. The principle is simple: the better the approximation, higher the chances of a correct inversion of the linearized operator to obtain a contraction mapping formulation of the problem. In [27, 28] resonance phenomena similar to the "ϕ^2-effect" has been faced in the Neumann problem involving whole boundary concentration. In [24]-[25] this boundary concentration on a $k-$dimensional minimal surface of the boundary, involving both ϕ^1 and ϕ^2 effects, has been treated via arbitrary high order approximations.

In [8], M. del, Pino, M. Kowalczyk and J. Wei constructed the curve concentrations for nonlinear Schrödinger equations

$$\varepsilon^2 \triangle U - V(x)U + U^p = 0,\ U \in H^1(\mathbb{R}^2),\ U > 0.$$

There, they faced the coupled effect ϕ^1 and ϕ^2. They introduced a sort of *infinite-dimensional Liapunov-Schmidt reduction method* which is close in spirit to that of finite dimensional Liapunov-Schmidt reduction method of Floer and Weinstein [10] and provides substantial simplification and flexibility to deal with larger resonance and coupling of the non-invertibility of the linearized operator. Their idea is to solve first a natural projected problem where the linear operator is uniformly invertible, the resolution of the full problem becomes reduction to a nonlinear, nonlocal second order system of differential equations, which turns out to be directly solvable thanks to the assumptions made on the curve.

The main difficulty in our paper will come from the coupling of ϕ^1, ϕ^2, the boundary condition and the interaction between layers. In [8], the error term is of order $O(\varepsilon^2)$, while here the error term is $O(\varepsilon)$ since the stretching of the boundary conditions gives $\frac{\partial \phi}{\partial z} + O(\varepsilon)$. However, the spectrum gap in (1.11) is also $O(\varepsilon)$ which creates additional difficulty. Worse than that, the spectrum gap caused by ϕ^2 and the boundary corrections are *strongly* coupled. We overcome these difficulties by first using successive improvements of the approximation and then perform the infinite-dimensional reduction to reduce the problem to a system of $2N$ coupled nonlinear ODEs. The reduced ODEs involve coefficients of both fast and slow variables (see (7.37)-(7.38)). A careful analysis of Fourier modes is needed to ensure the invertibility.

The organization of the paper is as follows: In Section 2, we set up the problem and find a approximate solution, taking into account of the curvature contributions. We perform a gluing procedure in Section 3 and study the linear theory in Section 4. In Section 5, we solve the nonlinear problem, using infinite-dimensional Liapunov-Schmidt reduction procedure. We derive a reduced system of $2N$ equations including a Toda system and N nonlinear ODEs in Section 6. Finally we use Schauder fixed point theory to solve the reduced nonlinear ODE.

Throughout the paper, $A_\varepsilon \sim B_\varepsilon$ means that there exists fixed constants C_1, C_2 such that $C_1 \leq \frac{A_\varepsilon}{B_\varepsilon} \leq C_2$ for ε small.

2. **The Ansatz.**

In the sequel, w is the even function defined in (1.9). In fact

$$w(x) = C_p \left\{ \exp\Big[\frac{(p-1)\,x}{2}\Big] + \exp\Big[\frac{-(p-1)\,x}{2}\Big] \right\}^{\frac{-2}{p-1}}.$$

Z is the even eigenfunction defined in the eigenvalue problem (1.10), associated to the unique positive eigenvalue λ_0. It is easy to see that for $|x| \gg 1$

$$w(x) = C_p\, e^{-|x|} - \frac{2C_p}{p-1}\, e^{-p|x|} + O(e^{-(2p-1)|x|}), \tag{2.1}$$

$$w'(x) = -C_p\, e^{-|x|} + \frac{2\,p\,C_p}{p-1}\, e^{-p|x|} + O(e^{-(2p-1)|x|}), \tag{2.2}$$

$$Z(x) = C_p'\, e^{-(p+1)|x|} - \frac{2(p+1)C_p'}{p-1}\, e^{-2p|x|} + O(e^{-(3p-1)|x|}), \tag{2.3}$$

where

$$C_p = \Big[\frac{(p+1)}{2}\Big]^{\frac{1}{p-1}}, \quad C_p^{'} = \Big[\frac{(p+1)}{2}\Big]^{\frac{p+1}{p-1}} \Big[\int_{\mathbb{R}} w^{p+1}\,\mathrm{d}x\Big]^{-\frac{1}{2}}.$$

Let $\mathfrak{S}$ be the infinite strip:

$$\mathfrak{S} = \Big\{-\infty < x < \infty, \;\; 0 < z < \frac{1}{\varepsilon}\Big\}.$$

2.1. **Approximate solution.** We first formulate our problem in conveniently chosen coordinate system. We recall that we may assume that the segment Γ is given by

$$\Gamma = \{(\tilde{y}_1, \tilde{y}_2) \mid \tilde{y}_1 = 0,\, 0 < \tilde{y}_2 < 1\}\,.$$

We also assume that near the endpoints of the segment, $\partial\Omega$ is described as the graph of two smooth functions, let us say respectively $\tilde{y}_2 = \varphi_0(\tilde{y}_1)$, $\tilde{y}_2 = \varphi_1(\tilde{y}_1)$, with

$$\varphi_0(0) = 0, \quad \varphi_1(0) = 1, \quad \varphi_0'(0) = 0 = \varphi_1'(0)\,.$$

We also denote $-k_1$ and k_0 the curvatures of the boundary $\partial\Omega$ at the points γ_1 and γ_0 respectively, where

$$k_1 = \varphi_1^{''}(0), \quad k_0 = \varphi_0^{''}(0). \tag{2.4}$$

Let us consider the scaling $u(y) = \tilde{u}(\varepsilon y)$. Problem (1.1)-(1.2) is thus equivalent to

$$\Upsilon(u) \equiv \Delta_y u + F(u) = 0 \quad \text{in } \Omega_\varepsilon, \tag{2.5}$$

$$\Xi(u) \equiv \frac{\partial u}{\partial \nu_y} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{2.6}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Here and in what follows we denote

$$F(u) \equiv -u + u^p.$$

For some small, fixed number δ_0, we can parametrize all points $y \in \Omega_\varepsilon$ with $|y_1| < \delta_0\varepsilon^{-1}$ by means of coordinates (x, z) which straighten the boundary as follows:

$$\begin{aligned} x &= y_1, \\ z &= y_2 - \eta_\sigma(\varepsilon y_2)\,\varphi_0(\varepsilon y_1)/\varepsilon \\ &\quad - \eta_\sigma(1 - \varepsilon y_2)\,\big[\varphi_1(\varepsilon y_1) - 1\big]/\varepsilon. \end{aligned} \tag{2.7}$$

Here $\eta_\sigma(s) = \eta(\sigma^{-1}s)$ where η is a smooth cut-off function such that

$$\eta(s) = 1 \text{ for } |s| < 1 \text{ and } \eta(s) = 0 \text{ for } |s| > 2.$$

For our purpose it is important to compute Laplacian in these coordinates. We call

$$Y(x, z) = \Big(Y_1(x, z),\, Y_2(x, z)\Big)$$

the inverse of the transformation defined by (2.7). We have that

$$\Delta_y = \Delta_{x,z} + B_{x,z},$$

where $B_{x,z}$ is a second order differential operator with small coefficients:

$$B_{x,z} = B_{22}(x, z)\,\frac{\partial^2}{\partial z^2} + B_{21}(x, z)\,\frac{\partial^2}{\partial x \partial z} + \varepsilon B_{20}(x, z)\,\frac{\partial}{\partial z}, \tag{2.8}$$

with

$$\begin{aligned} B_{22}(x,z) &= O(\sigma^{-2}\varepsilon^2|x|^2)\,\tilde\eta_\sigma(\varepsilon Y_2),\\ B_{21}(x,z) &= 2\varepsilon x\Big[-k_0\,\eta_\sigma(\varepsilon Y_2)\;-\;k_1\,\eta_\sigma(1-\varepsilon Y_2)\Big]\\ &\qquad + \Big[O(\sigma^{-1}\varepsilon|x|)+O(\sigma^{-2}\varepsilon^2|x|^2)\Big]\,\tilde\eta_\sigma(\varepsilon Y_2),\\ B_{20}(x,z) &= -k_0\,\eta_\sigma(\varepsilon Y_2)\;-\;k_1\,\eta_\sigma(1-\varepsilon Y_2)\\ &\qquad + \Big[O(\sigma^{-1}\varepsilon|x|)+O(\sigma^{-2}\varepsilon^2|x|^2)\Big]\,\tilde\eta_\sigma(\varepsilon Y_2), \end{aligned}$$

and

$$\tilde\eta_\sigma(s)=\eta_\sigma(s)+\eta_\sigma(1-\varepsilon s).$$

In particular, $B_{x,z}$ is a small perturbation of Δ provided that σ is chosen sufficiently small. In fact we will choose

$$\sigma=\varepsilon^{\frac{1}{8}}.$$

We also have for $z=0$,

$$\frac{\partial}{\partial\nu_y}\;=\;\frac{-1-|\varphi_0'(\varepsilon x)|^2}{(1+|\varphi_0'(\varepsilon x)|^2)^{1/2}}\,\frac{\partial}{\partial z}\;+\;\frac{\varphi_0'(\varepsilon x)}{(1+|\varphi_0'(\varepsilon x)|^2)^{1/2}}\,\frac{\partial}{\partial x},$$

with a similar formula near $z=1/\varepsilon$. In the sequel we will write

$$\frac{\partial}{\partial\nu_y}\;=\;(-1)^{\ell+1}\frac{\partial}{\partial z}\;+\;b_\ell(x,z)\cdot\nabla_{x,z},\quad \ell=0,1,\quad z=\ell/\varepsilon.$$

For a fixed integer $N>1$, we assume that the location of the N concentration layers are characterized by functions $z=f_j(\varepsilon x)$, $1\le j\le N$ in the coordinate (x,z). These functions will be assumed to satisfy

$$f_j:(0,1)\to\mathbb{R}, \tag{2.9}$$

$$||f_j||_{H^2(0,1)}\;<\;C|\ln\varepsilon|^2, \tag{2.10}$$

$$f_{j+1}(\zeta)-f_j(\zeta)\;>\;2|\ln\varepsilon|-4\ln|\ln\varepsilon|. \tag{2.11}$$

For convenience of the notation we will set

$$f_0(\zeta)=-\delta_0/\varepsilon-f_1(\zeta)\quad\text{and}\quad f_{N+1}(\zeta)=\delta_0/\varepsilon-f_N(\zeta).$$

Set

$$w_j(x,z)=w\big(x-f_j(\varepsilon z)\big),\qquad Z_j(x,z)=Z\big(x-f_j(\varepsilon z)\big),$$

and define **the first approximate solution** to (2.5)–(2.6) by

$$u_0(x,z)=\sum_{j=1}^{N}w_j(x,z)\,.$$

2.2. **Accuracy of the approximation.** Our first goal is to compute the errors of approximation in a δ_0/ε neighborhood of $\frac{\Gamma}{\varepsilon}$, namely the quantities

$$\Upsilon(u_0)=\Delta_y u_0+F(u_0),\qquad \Xi(u_0)=\frac{\partial u_0}{\partial\nu_y}. \tag{2.12}$$

We shall do this in (x,z)-coordinates. Thus we estimate the first term

$$E_0\equiv(\Delta_{x,z}+B_{x,z})u_0+F(u_0)\equiv E_{01}+E_{02}$$

where

$$\Delta_{x,z}u_0=\sum_{j=1}^N w''(x-f_j)-\varepsilon^2\sum_{j=1}^N f_j''w'(x-f_j)+\varepsilon^2\sum_{j=1}^N (f_j')^2 w''(x-f_j).$$

Then, taking into account (2.8), we get

$$\begin{aligned}E_{01} &= \Delta_{x,z}u_0+B_{x,z}\,u_0\\ &= \sum_{j=1}^N w''(x-f_j)\Big[1-\varepsilon\, f_j'\,B_{21}(x,z)\Big]-\varepsilon^2\sum_{j=1}^N f_j''\,w'(x-f_j)\Big[1+B_{22}(x,z)\Big]\\ &\quad+\varepsilon^2\sum_{j=1}^N (f_j')^2\,w''(x-f_j)\Big[1+B_{22}(x,z)\Big]-\varepsilon^2\sum_{j=1}^N f_j'\,w'(x-f_j)\,B_{20}(x,z).\end{aligned}$$

We now turn to computing the term E_{02}. For every fixed k, $1\le k\le N$, we consider the following set

$$A_k=\left\{\,(x,z)\in\mathfrak{S}\mid\ \frac{f_{k-1}(\varepsilon z)+f_k(\varepsilon z)}{2}\ \le\ x\ \le\ \frac{f_k(\varepsilon z)+f_{k+1}(\varepsilon z)}{2}\,\right\}.$$

For $(x,z)\in A_k$, we write

$$\begin{aligned}E_{02} &= F(w_k)+F'(w_k)(u_0-w_k)+\frac12 F''(w_k)(u_0-w_k)^2+\max_{j\neq k}O(e^{-3|f_j-x|})\\ &= \sum_{j=1}^N F(w_j)\ +\ p(w_k)^{p-1}(u_0-w_k)-\sum_{j\neq k}(w_j)^p\\ &\qquad\qquad+\Big[\,\frac12 F''(w_k)(u_0-w_k)^2+\max_{j\neq k}O(e^{-3|f_j-x|})\Big]\\ &\equiv \sum_{j=1}^N F(w_j)\ +\ E_{020,k}\ +\ E_{021,k}\ +\ E_{022,k}.\end{aligned}\tag{2.13}$$

It follows then for $(x,z)\in A_k$, $k=1,\dots,N$:

$$\begin{aligned}E_0 &= -\varepsilon\sum_{j=1}^N f_j'\,w''(x-f_j)\,B_{21}(x,z)\ -\ \varepsilon^2\sum_{j=1}^N f_j''\,w'(x-f_j)\Big[1+B_{22}(x,z)\Big]\\ &\quad+\varepsilon^2\sum_{j=1}^N (f_j')^2\,w''(x-f_j)\Big[1+B_{22}(x,z)\Big]\ -\ \varepsilon^2\sum_{j=1}^N f_j'\,w'(x-f_j)\,B_{20}(x,z)\\ &\quad+E_{020,k}\ +\ E_{021,k}\ +\ \max_{j\neq k}O(e^{-2|f_j-x|}).\end{aligned}$$

From the above expressions for E_0 we see that, given the bounds for f_k's in (2.10)-(2.11), denoting by $\chi_{A_k}(x)$ the characteristic function of the set A_k, we have

$$E_0(x,z)=\sum_{k=1}^N \chi_{A_k}\Big[\,O(\varepsilon^2|\ln\varepsilon|^2)e^{-|f_k-x|}\ +E_{020,k}+\ O(1)\max_{j\neq k}e^{-2|f_j-x|}\Big].$$

Next we estimate the accuracy of the Neumann boundary condition for u_0. Again in (x,z) coordinates

$$\begin{aligned} E_{0b}(z=0) \equiv\ & \frac{-1-|\varphi_0'(\varepsilon x)|^2}{(1+|\varphi_0'(\varepsilon x)|^2)^{1/2}}\cdot\frac{\partial u_0}{\partial z} + \frac{\varphi_0'(\varepsilon x)}{(1+|\varphi_0'(\varepsilon x)|^2)^{1/2}}\cdot\frac{\partial u_0}{\partial x} \\ =\ & \varepsilon\sum_{j=1}^{N}\Big[\, f_j'(0) + k_0 f_j(0)\Big]\frac{\partial w_j}{\partial x} + \varepsilon\sum_{j=1}^{N} k_0\Big(x - f_j(0)\Big)\frac{\partial w_j}{\partial x} \\ & +\varepsilon^2\sum_{j=1}^{N}\Big[O(|x-f_j(0)|^2) + O(|f_j(0)|^2)\Big]\frac{\partial w_j}{\partial x} \\ & +\varepsilon^3\sum_{j=1}^{N}\Big[O(|x-f_j(0)|^2) + O(|f_j(0)|^2)\Big] f_j'(0)\frac{\partial w_j}{\partial x}. \end{aligned}$$

A similar formula holds for $E_{0b}(z=1/\varepsilon)$. Thus we see that it is natural to take the following boundary conditions for f_j

$$\begin{aligned} f_j'(0)+k_0 f_j(0) &= 0, \\ f_j'(1)+k_1 f_j(1) &= 0, \end{aligned} \qquad j=1,\ldots,N. \tag{2.14}$$

We assume the validity of these conditions in the sequel.

For further application, the evaluation of the first approximation is of importance. Using the condition (2.10)-(2.11), a tedious computation implies,

$$\Big\| E_0 \Big\|_{L^2(S_{\delta_0/\varepsilon})} = O(\varepsilon^{\frac{3}{2}}|\ln\varepsilon|^q) \tag{2.15}$$

where

$$S_{\delta_0/\varepsilon} = \{-\delta_0/\varepsilon < x < \delta_0/\varepsilon,\ 0<z<1/\varepsilon\}.$$

In fact, we should handle the terms $E_{020,k}$ carefully. If $p\geq 2$, then from (2.1)-(2.3), we obtain the estimate

$$\begin{aligned} \big[\, w_k(x,z)\big]^{p-1}\cdot w_{k+1}(x,z) &\leq Ce^{-(p-1)|x-f_k(z)|}\cdot e^{-|x-f_{k+1}(z)|} \\ &\leq Ce^{-|x-f_k(z)|}\cdot e^{-|x-f_{k+1}(z)|} \\ &\leq Ce^{-|f_{k+1}(z)-f_k(z)|}. \end{aligned}$$

Similar estimates hold for other terms like $\big[\, w_k(x,z)\big]^{p-1}\cdot w_j(x,z)$ for all $j\neq k$. Hence, there hold, for $k=1,\cdots,N$

$$\begin{aligned} ||E_{020,k}||^2_{L^2(A_k)} &\leq p^2\int_{A_k}|w_k|^{2p-2}(u_0-w_k)^2\,\mathrm{d}x\mathrm{d}z \\ &\leq C\int_{A_k}\Big[e^{-2|f_{k+1}(z)-f_k(z)|} + e^{-2|f_k(z)-f_{k-1}(z)|}\Big]\mathrm{d}x\mathrm{d}z. \end{aligned}$$

From the above formula, we use (2.11) to get

$$||E_{020,k}||^2_{L^2(A_k)} = O(\varepsilon^{\frac{3}{2}}|\ln\varepsilon|^q).$$

On the boundary, the error can be written as

$$E_{0b}(z) = O(\varepsilon)\sum_{k=1}^{N}\Big(x - f_k(\varepsilon z)\Big)\, e^{-|x-f_k(\varepsilon z)|} \quad \text{for } z=0,\ \varepsilon^{-1}.$$

As a matter of fact, we actually have that E_{0b} can be extended as an $H^1(S_{\delta_0/\varepsilon})$ function satisfying

$$\|E_{0b}\|_{H^1(S_{\delta_0/\varepsilon})} = O(\varepsilon^{\frac{1}{2}}).$$

The discrepancy between the order of approximation in the interior and on the boundary (E_0 and E_{0b} respectively) makes it necessary to improve the original approximation u_0 and eliminate the $O(\varepsilon)$-part of the error on the boundary. In the language of formal asymptotic expansions one can say that we need to find a boundary layer expansion of our solution. We shall do this in the next section, as well as reformulating the full problem in a convenient way by the gluing procedure.

3. Improvement of Approximation and the Gluing.

3.1. The boundary layer problem. We notice that function $u_0(x, z)$, i.e. the first approximation constructed near $\frac{\Gamma}{\varepsilon}$, can be easily extended to the whole strip $\mathfrak{S}$.

We will construct an improvement in approximation by first solving the following problem

$$\Delta\Phi - \Phi + pw^{p-1}\Phi = \rho(\varepsilon z) Z \quad \text{in } \mathfrak{S}, \quad \frac{\partial \Phi}{\partial \nu} = g \quad \text{on } \partial\mathfrak{S}, \tag{3.1}$$

$$\int_{\mathbb{R}} \Phi(x, z)\, w_x \,\mathrm{d}x = 0, \qquad \int_{\mathbb{R}} \Phi(x, z)\, Z \,\mathrm{d}x \;=\; 0, \tag{3.2}$$

where g is any $H^1(\mathfrak{S})$-extension of the boundary error term $-xw_x \in H^1(\mathbb{R})$. Let us take for instance

$$g(x, z) = -e^{-z}\, xw_x(x)\, \tilde{\eta}(2\varepsilon z),$$

with a suitable smooth cutoff function $\tilde{\eta}$, in such a way that g is an even function in the variable x for each z, and satisfies the estimate

$$\|g\|_{H^1(\mathfrak{S})} \le C,$$

and the boundary constraints

$$g(x, 0) = -xw_x, \quad g(x, 1/\varepsilon) = 0,$$

with C independent of ε.

Lemma 3.1. *There exists a function $\rho(\zeta)$ in $L^2(0, 1)$ with the bound*

$$\|\rho\|_{L^2(0,1)} \le C\varepsilon^{\frac{1}{2}}, \tag{3.3}$$

such that problem (3.1)–(3.2) has a unique solution $\Phi \in H^2(\mathfrak{S})$ which is even in x for each z. Besides, there is a constant $C > 0$ such that for all small ε,

$$\|\Phi\|_{H^2(\mathfrak{S})} \le C. \tag{3.4}$$

In addition there exist constants $0 < \varrho < 1/4$, $\mu > 0$ and $C_\varrho > 0$ such that the following estimate holds:

$$|\Phi(x, z)| + |\nabla\Phi(x, z)| + |D^2\Phi(x, z)| \le C_\varrho\, e^{-[\,(1-\varrho)|x| + \mu z\,]}. \tag{3.5}$$

We will give the proof of this lemma at the end of Section 4. □

Let Φ be the function defined by Lemma 3.1 and set

$$\phi_j^*(x, z) \;=\; k_0\, \Phi\Big(x - f_j(\varepsilon z),\, z\Big) \;+\; k_1\, \Phi\Big(x - f_j(\varepsilon z),\, \frac{1}{\varepsilon} - z\Big).$$

We define the approximate solution of the following problem

$$\begin{aligned}\Delta\phi + F'(w)\phi &= 0 \quad \text{in } \mathfrak{S}\\ \phi_z(x,\frac{1}{\varepsilon}) &= \varepsilon\sum_{j=1}^{N} k_1\Big(x - f_j(1)\Big)\frac{\partial w_j}{\partial x},\\ \phi_z(x,0) &= \varepsilon\sum_{j=1}^{N} k_0\Big(x - f_j(0)\Big)\frac{\partial w_j}{\partial x},\end{aligned}$$

by the formula

$$\phi^*(x,z) = \varepsilon\sum_{j=1}^{N}\phi_j^*(x,z).$$

The next goal is to show that $\phi^*(x,z)$ is the right boundary layer correction. Define **the second approximate solution** to (2.5)–(2.6) by

$$u_1 = u_0 + \phi^*.$$

The new error in the interior of $\mathfrak{S}$ can be written as follows

$$E_1 = E_0 + \Big[(\Delta_{x,z} + B_{x,z})\phi^* + F'(u_0)\phi^*\Big] + \mathcal{N}(\phi^*)$$

where

$$\mathcal{N}(\phi^*) = F(u_0+\phi^*) - F(u_0) - F'(u_0)\phi^*, \tag{3.6}$$

$$\begin{aligned}&\Delta_{x,z}\phi^* + F'(u_0)\phi^* + B_{x,z}\phi^*\\ &= \varepsilon\sum_{j=1}^{N}\big[\,\Delta_{x,z}\phi_j^* + F'(w_j)\phi_j^*\,\big] + \varepsilon p\sum_{j=1}^{N}(\,u_0^{p-1} - w_j^{p-1}\,)\phi_j^* \;+\; B_{x,z}\phi^*\\ &\equiv E_{11} + E_{12} + E_{13}.\end{aligned}$$

We fix an integer k and consider the error in the set A_k, as in the previous section.

$$\begin{aligned}&\Delta\phi_j^* \;+\; F'(w_j)\phi_j^*\\ &= k_0\Big[\rho(\varepsilon z)Z_j \;-\; 2\varepsilon f_j^{'}(\varepsilon z)\Phi_{xz}\big(x - f_j(\varepsilon z), z\big)\\ &\quad -\;\varepsilon^2 f_j^{''}(\varepsilon z)\Phi_x\big(x - f_j(\varepsilon z), z\big) \;+\; \varepsilon^2\big(f_j^{'}(\varepsilon z)\big)^2\Phi_{xx}\big(x - f_j(\varepsilon z), z\big)\Big]\\ &\quad +\; k_1\Big[\rho(1-\varepsilon z)Z_j \;-\; 2\varepsilon f_j^{'}(\varepsilon z)\Phi_{xz}\big(x - f_j(\varepsilon z), \frac{1}{\varepsilon} - z\big)\\ &\quad -\;\varepsilon^2 f_j^{''}(\varepsilon z)\Phi_x\big(x - f_j(\varepsilon z), \frac{1}{\varepsilon} - z\big) \;+\; \varepsilon^2\big(f_j^{'}(\varepsilon z)\big)^2\Phi_{xx}\big(x - f_j(\varepsilon z), \frac{1}{\varepsilon} - z\big)\Big].\end{aligned}$$

Combining above formula and the decay estimate (3.5), we get

$$\begin{aligned}E_{11} &= \varepsilon\sum_{j=1}^{N}\Big[\,k_0\rho(\varepsilon z) + k_1\rho(1-\varepsilon z)\,\Big]\,Z_j \;+\; O(\varepsilon^2)\,|\ln\varepsilon|^2\,e^{-(1-\varrho)|f_k - x|}\\ &\equiv E_{110} \;+\; O(\varepsilon^2)\,|\ln\varepsilon|^2\,e^{-(1-\varrho)|f_k - x|}.\end{aligned} \tag{3.7}$$

Term E_{12} is estimated using (3.5) by

$$|E_{12}| \le C\varepsilon\max_{j\ne k} e^{-(1-\varrho)|f_j - x|}\Big[e^{-\mu z} + e^{-\mu(1/\varepsilon - z)}\Big].$$

Therefore, the following lemma is readily checked.

Lemma 3.2. *With the notation of the previous section we have*

$$\begin{aligned} E_1 &\equiv \Upsilon(u_1) \\ &= O(\varepsilon^2|\ln\varepsilon|^2)\sum_{k=1}^{N}\chi_{A_k}\Big[e^{-|f_k-x|}+\varepsilon e^{-(1-\varrho)|f_k-x|}\Big] \\ &\quad+\sum_{k=1}^{N}\chi_{A_k}\Big[E_{020,k}+E_{021,k}\Big] + E_{110}+O(1)\sum_{j=1}^{N}\chi_{A_k}\max_{j\neq k} e^{-2|f_j-x|} \\ &\quad+O(\varepsilon)\sum_{j=1}^{N}\chi_{A_k}\max_{j\neq k} e^{-(1-\varrho)|f_j-x|}\Big[e^{-\mu z}+e^{-\mu(1/\varepsilon-z)}\Big], \end{aligned}$$

where $E_{020,k}$, $E_{021,k}$ and E_{110} are defined by (2.13), (3.7) respectively. Moreover,

$$\Big\|E_1-E_{110}\Big\|_{L^2(\mathfrak{S})}\le C\,\varepsilon^{3/2}|\ln\varepsilon|^2. \tag{3.8}$$

Similar decay estimate holds for $E_{1b}\equiv\Xi(u_1)$. In addition there is an extension of E_{1b} to the whole strip $\mathfrak{S}$ such that

$$\|E_{1b}\|_{H^1(\mathfrak{S})}\le C\varepsilon^{3/2}|\ln\varepsilon|^2. \tag{3.9}$$

Proof. The remaining terms $B_{x,z}\phi^*$ and $\mathcal{N}(\phi^*)$ are easily seen to be smaller than the ones we have just considered. Estimate (3.8) follows immediately from (3.8). Obviously (3.9) is an easy consequence of the construction. □

To improve the approximate solution to (2.5)-(2.6) while still keeping the terms of order $O(\varepsilon^2|\ln\varepsilon|^2)$, we need to introduce new parameters $\mathbf{e}=(e_1,...,e_N)$ with each $e_j:(0,1)\to\mathbb{R}$ satisfying

$$||e_j||_b=||e_j||_{L^\infty(0,1)}+\varepsilon||e_j^{'}||_{L^2(0,1)}+\varepsilon^2||e_j^{''}||_{L^2(0,1)}\le C\,\varepsilon^{\tau} \tag{3.10}$$

where τ is a positive constant to determined. Define

$$\phi_j^{**}\;=\;e_j(\varepsilon z)Z_j,\quad \phi^{**}\;=\;\sum_{j=1}^{N}\varepsilon\phi_j^{**},$$

and choose our **basic approximate solution** to (2.5)–(2.6) by

$$u_2=u_1+\phi^{**}. \tag{3.11}$$

We have the basic error in the interior of $\mathfrak{S}$ as

$$E_2=E_1+\Big[(\Delta_{x,z}+B_{x,z})\phi^{**}+F'(u_0+\phi^*)\phi^{**}\Big]+\mathcal{N}(\phi^{**}),$$

where

$$\mathcal{N}(\phi^{**})=F(u_0+\phi^*+\phi^{**})-F(u_0+\phi^*)-F'(u_0+\phi^*)\phi^{**}, \tag{3.12}$$

$$\begin{aligned} &\Delta_{x,z}\phi^{**}\;+\;F'(u_0+\phi^*)\phi^{**}+B_{x,z}\phi^{**} \\ &=\varepsilon\sum_{j=1}^{N}\big[\,\Delta_{x,z}\phi_j^{**}+F'(w_j)\phi_j^{**}\,\big]+\varepsilon p\sum_{j=1}^{N}\Big((u_0+\phi^*)^{p-1}-w_j^{p-1}\Big)\phi_j^{**}+B_{x,z}\phi^{**} \\ &\equiv E_{21}+E_{22}+E_{23}. \end{aligned} \tag{3.13}$$

For further reference, we set

$$\begin{aligned} E_{21} &= \varepsilon \sum_{j=1}^{N} (\varepsilon^2 e_j'' + \lambda_0 e_j) Z_j + \varepsilon^3 \sum_{j=1}^{N} (f_j')^2 e_j Z_j'' \\ &\quad - \sum_{j=1}^{N} \Big(2\varepsilon^3 e_j' f_j' Z_j' + \varepsilon^3 e_j f_j'' Z_j' \Big) \\ &\equiv E_{210} + E_{211} + E_{212}. \end{aligned} \tag{3.14}$$

Term E_{22} in the set A_k is estimated by (2.1)-(2.3) and condition (3.10)

$$|E_{22}| \leq C\varepsilon \max_{j\neq k} e^{-|f_j - x|}.$$

As before, we also take the following boundary conditions for e_j

$$\begin{aligned} e_j'(0) + k_0 e_j(0) &= 0, \\ e_j'(1) + k_1 e_j(1) &= 0, \end{aligned} \qquad j = 1, \ldots, N. \tag{3.15}$$

Similarly as Lemma 3.2, we can prove that

Lemma 3.3. *With the notation of the previous section we have*

$$\begin{aligned} E_2 &\equiv \Upsilon(u_2) \\ &= O(\varepsilon^2 |\ln\varepsilon|^2) \sum_{k=1}^{N} \chi_{A_k} \Big[e^{-|f_k - x|} + \varepsilon e^{-(1-\varrho)|f_k - x|} \Big] \\ &\quad + \sum_{k=1}^{N} \chi_{A_k} \Big[E_{020,k} + E_{021,k} \Big] + E_{110} + O(1) \sum_{j=1}^{N} \chi_{A_k} \max_{j\neq k} e^{-2|f_j - x|} \\ &\quad + O(\varepsilon) \sum_{j=1}^{N} \chi_{A_k} \max_{j\neq k} e^{-(1-\varrho)|f_j - x|} \Big[e^{-\mu z} + e^{-\mu(1/\varepsilon - z)} \Big]. \end{aligned}$$

Moreover, in the $L^2(\mathfrak{S})$ norm

$$\Big\| E_2 - E_{110} - E_{210} \Big\|_{L^2(\mathfrak{S})} \leq C\, \varepsilon^{3/2} |\ln\varepsilon|^2, \tag{3.16}$$

where E_{210} is defined by (3.14).
Similar decay estimate holds for $E_{2b} = \Xi(u_2)$. In addition there is an extension of $\Xi(u_2)$ to the whole strip $\mathfrak{S}$ such that

$$\|E_{2b}\|_{H^1(\mathfrak{S})} \leq C\varepsilon^{3/2} |\ln\varepsilon|^2. \tag{3.17}$$

□

3.2. **The gluing procedure.** We will now reduce the original problem which is defined in Ω_ε to a problem defined in the strip $\mathfrak{S}$. This will be done by using a gluing procedure similar to that in [8].

For $\delta = \varepsilon^{1/6}$, we consider a smooth cut-off function $\eta_\delta(t)$ where $t \in \mathbb{R}_+$ such that

$$\eta_\delta = 1 \text{ if } t < \delta \ \text{ and } \eta_\delta = 0 \ \text{ if } t > 2\delta. \tag{3.18}$$

Denote as well $\eta_\delta^\varepsilon(s) = \eta_\delta(\varepsilon|s|)$, where s is the normal coordinate to $\frac{\Gamma}{\varepsilon}$. We define our first global approximation to be simply

$$W = \eta_{3\delta}^\varepsilon(s)\, u_2. \tag{3.19}$$

extended globally as 0 beyond the $6\delta/\varepsilon$-neighborhood of $\frac{\Gamma}{\varepsilon}$.

We look for a solution of (2.5)–(2.6) of the form $u = W + \tilde\phi$. Then

$$\Upsilon(W+\tilde\phi)=0 \quad \text{in } \Omega_\varepsilon, \quad \Xi(W+\tilde\phi)=0 \quad \text{on } \partial\Omega_\varepsilon$$

if and only if

$$\begin{aligned}\tilde L(\tilde\phi) &= -\tilde E - \tilde N(\tilde\phi) \quad \text{in } \Omega_\varepsilon, &(3.20)\\ \Xi(\tilde\phi) &= \tilde E_b \quad \text{on } \partial\Omega_\varepsilon, &(3.21)\end{aligned}$$

where we have denoted

$$\tilde L(\tilde\phi) = \triangle_y\tilde\phi - \tilde\phi + pW^{p-1}\tilde\phi, \quad \tilde N(\tilde\phi) = (W+\tilde\phi)^p - W^p - pW^{p-1}\tilde\phi,$$

and

$$\tilde E = \Upsilon(W), \quad \tilde E_b = -\Xi(W).$$

For simplicity of notation, we replace the error terms E_{020} and E_{110} corresponding to the second approximate solution u_2 in $\mathfrak{S}$ by the error terms $\tilde E_{020}$ and $\tilde E_{110}$ corresponding to approximate solution W in Ω_ε respectively.

We further decompose $\tilde\phi$ in the following form:

$$\tilde\phi = \eta^\varepsilon_{3\delta}\phi + \psi,$$

where, in coordinates (x, z), we assume that ϕ is defined in the whole strip $\mathfrak{S}$. Substituting in (3.20) we find

$$\tilde L(\eta^\varepsilon_{3\delta}\phi) + \tilde L(\psi) = -\tilde E - \tilde N(\eta^\varepsilon_{3\delta}\phi + \psi).$$

We achieve this if the pair (ϕ, ψ) satisfies the following nonlinear coupled system:

$$\tilde L(\phi) = \eta^\varepsilon_\delta\Big[-\widetilde N(\eta^\varepsilon_{3\delta}\phi+\psi) - \widetilde E - pW^{p-1}\psi\Big] \qquad \text{in } \mathfrak{S}, \tag{3.22}$$

$$(-1)^{\ell+1}\frac{\partial\phi}{\partial z} + \eta^\varepsilon_{3\delta} b_\ell(x,z)\cdot\nabla_{x,z}\phi = -\eta^\varepsilon_\delta\tilde E_b \qquad \text{on } \partial\mathfrak{S}, \tag{3.23}$$

and

$$\begin{aligned}\triangle_y\psi - \psi + (1-\eta^\varepsilon_\delta)pW^{p-1}\psi &= \varepsilon^2(\triangle_y\eta^\varepsilon_{3\delta})\phi + 2\varepsilon(\nabla_y\eta^\varepsilon_{3\delta})(\nabla_y\phi)\\ &\quad -(1-\eta^\varepsilon_\delta)\widetilde N(\eta^\varepsilon_\delta\phi+\psi) - (1-\eta^\varepsilon_\delta)\tilde E \quad \text{in } \Omega_\varepsilon, &(3.24)\\ \frac{\partial\psi}{\partial\nu} &= -(1-\eta^\varepsilon_\delta)\tilde E_b - \phi\frac{\partial\eta^\varepsilon_{3\delta}}{\partial\nu} \quad \text{on } \partial\Omega_\varepsilon, &(3.25)\end{aligned}$$

where ϕ is defined globally on $\mathfrak{S}$ and ψ is defined in Ω_ε. Note that the operator in the strip $\mathfrak{S}$ may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of the curve.

What we want to do next is to reduce the problem to one in the strip. To do this, we solve, given a small ϕ, problem (3.24)–(3.25) for ψ.

Let us observe that W is exponentially small for $|s| > \delta/\varepsilon$, where s is the normal coordinate to $\frac{\Gamma}{\varepsilon}$, then the problem

$$\triangle\psi - \big[1-(1-\eta^\varepsilon_\delta)pW^{p-1}\big]\psi = h \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial\psi}{\partial\nu} = g \quad \text{on } \partial\Omega_\varepsilon,$$

has a unique bounded solution ψ whenever

$$\|h\|_{L^\infty(\Omega_\varepsilon)}, \quad \|g\|_{L^\infty(\partial\Omega_\varepsilon)} < +\infty.$$

Moreover,

$$\|\psi\|_\infty \le C\big[\,\|h\|_{L^\infty(\Omega_\varepsilon)} + \|g\|_{L^\infty(\partial\Omega_\varepsilon)}\,\big].$$

Assume now that ϕ satisfies the following decay condition for some $\gamma > 0$

$$\big|\nabla\phi(x,z)\big| + \big|\phi(x,z)\big| \le e^{-\gamma\frac{\delta}{\varepsilon}} \quad \text{for } |x| > \frac{\delta}{\varepsilon}. \tag{3.26}$$

Since $\tilde{N}$ has a power-like behavior with power greater than one, a direct application of contraction mapping principle yields that problem (3.24)–(3.25) has a unique (small) solution $\psi = \psi(\phi)$ with

$$\begin{aligned} \|\psi(\phi)\|_\infty \;\le\; & C\left[\, \|(1-\eta_\delta^\varepsilon)\tilde{E}\|_{L^\infty(\Omega_\varepsilon)} + \|(1-\eta_\delta^\varepsilon)\tilde{E}_b\|_{L^\infty(\partial\Omega_\varepsilon)}\,\right] \\ & + C\,\varepsilon\delta^{-1}\left[\, \|\phi\|_{L^\infty(|x|>\delta\varepsilon^{-1})} + \|\nabla\phi\|_{L^\infty(|x|>\delta\varepsilon^{-1})}\,\right], \end{aligned} \tag{3.27}$$

where with some abuse of notation by $\{\,|x| > \delta/\varepsilon\,\}$ we denote the complement of δ/ε-neighborhood of $\frac{\Gamma}{\varepsilon}$. The nonlinear operator ψ satisfies a Lipschitz condition of the form

$$\begin{aligned} \|\psi(\phi_1)-\psi(\phi_2)\|_\infty \;\le\; & C\,\varepsilon\,\delta^{-1}\Big[\, \|\phi_1-\phi_2\|_{L^\infty(|x|>\delta\varepsilon^{-1})} \\ & + \|\nabla(\phi_1-\phi_2)\|_{L^\infty(|x|>\delta\varepsilon^{-1})}\,\Big]. \end{aligned} \tag{3.28}$$

The full problem has been reduced to solving the (nonlocal) problem in the cylindrical strip $\mathfrak{S}$

$$L_2(\phi) = -\eta_\delta^\varepsilon\tilde{E} - \eta_\delta^\varepsilon\tilde{N}(\phi+\psi(\phi)) - \eta_\delta^\varepsilon pW^{p-1}\psi(\phi) \tag{3.29}$$

for a $\phi \in H^2(\mathfrak{S})$ satisfying condition (3.26). Here L_2 denotes a linear operator that coincides with $\tilde{L}$ on the region $\{\,|x| < \frac{6\delta}{\varepsilon}\,\}$.

We shall define this operator next. The operator $\tilde{L}$ for $|x| < \frac{6\delta}{\varepsilon}$ can be extended in coordinates (x,z) to functions ϕ defined in the entire strip $\mathfrak{S}$ as follows:

$$L_2(\phi) = \Delta_{x,z}\phi - \phi + pW^{p-1}\phi + \eta_{6\delta}^\varepsilon B_{x,z}(\phi). \tag{3.30}$$

Rather than solving problem (3.22)–(3.23) directly, we shall do it in steps. Setting

$$\begin{aligned} \mathbf{f} \;&=\; (f_1,\dots,f_N),\ f_j \text{ satisfies bounds } (2.10)-(2.11), \\ \mathbf{e} \;&=\; (e_1,\dots,e_N),\ e_j \text{ satisfies bound } (3.10) \\ \mathbf{c} \;&=\; (c_1,\dots,c_N),\ c_j \in L^2(0,1), \\ \mathbf{d} \;&=\; (d_1,\dots,d_N),\ d_j \in L^2(0,1), \\ N_2(\phi) \;&=\; \tilde{N}\big(\phi+\psi(\phi)\big) + pW^{p-1}\psi(\phi), \end{aligned} \tag{3.31}$$

we consider the following projected problem in $H^2(\mathfrak{S})$: given parameters $\mathbf{f}$ and $\mathbf{e}$, finding functions $\phi \in H^2(\mathfrak{S})$ and $\mathbf{c}$, $\mathbf{d}$ such that

$$L_2(\phi) \;=\; -\eta_\delta^\varepsilon\tilde{E} \;-\eta_\delta^\varepsilon N_2(\phi) \;+\; \eta_\delta^\varepsilon\sum_{j=1}^N\Big[\, c_j(\varepsilon z)\,w_{j,x} + d_j(\varepsilon z)\,Z_j\Big] \quad \text{in } \mathfrak{S}, \tag{3.32}$$

$$(-1)^{\ell+1}\frac{\partial\phi}{\partial z} + \eta_{3\delta}^\varepsilon b_\ell(x,z)\cdot\nabla_{x,z}\phi = -\eta_\delta^\varepsilon\tilde{E}_b \quad \text{on } \partial\mathfrak{S}, \tag{3.33}$$

$$\int_{\mathbb{R}} \phi(x,z)\,w_{j,x}(x,z)\,\mathrm{d}x = 0, \quad 0<z<\frac{1}{\varepsilon},\ j=1,\dots,N, \tag{3.34}$$

$$\int_{\mathbb{R}} \phi(x,z)\,Z_j(x,z)\,\mathrm{d}x = 0, \qquad 0<z<\frac{1}{\varepsilon},\ j=1,\dots,N. \tag{3.35}$$

We will prove that this problem has a unique solution whose norm is controlled by the L^2 norm of $\eta_\delta^\varepsilon(\tilde{E}-\tilde{E}_{110}-\tilde{E}_{210})$ and H^1 norm of the suitable extension of $\eta_\delta^\varepsilon\tilde{E}_b$. After this has been done, our task is to adjust the parameters $\mathbf{f}$ and $\mathbf{e}$ in such a way that $\mathbf{c}$ and $\mathbf{d}$ are identically zero. Finally, this turns out to be equivalent to solving a nonlocal, nonlinear second order differential equations for $\mathbf{f}$ and $\mathbf{e}$ with Robin boundary conditions. As we will see this system is solvable in a region where the bound (2.10)-(2.11) and (3.10) hold. We will carry out this program in the next sections. To solve (3.32)-(3.35) we need to investigate invertibility of L_2 in L^2-H^2 setting under the boundary and orthogonality conditions.

4. **Linear Theory.** This section will be devoted to the resolution of the basic linear problem, which we define next. Recall (3.19), let us consider the linear operator, defined on functions $\phi \in H^2(\mathfrak{S})$ as

$$L(\phi) = \Delta\phi - \phi + pW^{p-1}\phi.$$

Given functions $h \in L^2(\mathfrak{S})$, $g \in H^1(\mathfrak{S})$ and Λ_j, $\Theta_j \in H^2(0, 1/\varepsilon)$ for $j = 1, \ldots, N$, we consider the problem of finding $\phi \in H^2(\mathfrak{S})$ such that for certain functions c_j, $d_j \in L^2(0, 1)$, $j = 1, \ldots, N$, we have

$$L(\phi) \;=\; h + \sum_{j=1}^{N} c_j(\varepsilon z)\, w_{j,x} + \sum_{j=1}^{N} d_j(\varepsilon z)\, Z_j \quad \text{in } \mathfrak{S}, \quad \frac{\partial \phi}{\partial \nu} = g \quad \text{on } \partial\mathfrak{S}, \tag{4.1}$$

$$\int_{-\infty}^{\infty} \phi(x, z)\, w_{j,x}(x, z)\, \mathrm{d}x = \Lambda_j(z), \quad 0 < z < \frac{1}{\varepsilon}\,, j = 1, \ldots, N, \tag{4.2}$$

$$\int_{-\infty}^{\infty} \phi(x, z)\, Z_j(x, z)\, \mathrm{d}x = \Theta_j(z), \quad 0 < z < \frac{1}{\varepsilon}\,, j = 1, \ldots, N. \tag{4.3}$$

For simplicity of notation, let

$$\Lambda = (\Lambda_1, ..., \Lambda_N), \quad \Theta = (\Theta_1, ..., \Theta_N).$$

Our main result in this section is the following.

Proposition 4.1. *There exists a constant $C > 0$, independent of ε and uniform for the parameters $\mathbf{f}$ and $\mathbf{e}$ in (3.31) such that for all small ε problem (4.1)-(4.3) has a solution $\phi = T(h, g, \Lambda, \Theta)$, which defines a linear operator of its arguments and satisfies the estimate*

$$\|\phi\|_{H^2(\mathfrak{S})} \;\le\; C\Big[\, \|h\|_{L^2(\mathfrak{S})} + \|g\|_{H^1(\mathfrak{S})} + \sum_{j=1}^{N} \|\Lambda_j\|_{H^2(0,\frac{1}{\varepsilon})} + \sum_{j=1}^{N} \|\Theta_j\|_{H^2(0,\frac{1}{\varepsilon})} \Big].$$

For the proof of Proposition 4.1 we need the validity of a priori estimates and existence result for a simpler problem. Given $\tilde{h} \in L^2(\mathfrak{S})$, $\tilde{g} \in H^1(\mathfrak{S})$, let us consider the operator

$$L_0(\tilde{\phi}) = \Delta\tilde{\phi} - \tilde{\phi} + pw^{p-1}\tilde{\phi}$$

and the problem

$$L_0(\tilde{\phi}) \;=\; \tilde{h} \quad \text{in } \mathfrak{S}, \quad \frac{\partial \tilde{\phi}}{\partial \nu} = \tilde{g} \quad \text{on } \partial\mathfrak{S}, \tag{4.4}$$

$$\int_{\mathbb{R}} \tilde{\phi}(x, z)\, w_x(x)\, \mathrm{d}x = \tilde{\Lambda}_0(z), \quad 0 < z < \frac{1}{\varepsilon}\,, \tag{4.5}$$

$$\int_{\mathbb{R}} \tilde{\phi}(x, z)\, Z(x)\, \mathrm{d}x = \tilde{\Theta}_0(z), \quad 0 < z < \frac{1}{\varepsilon}\,, \tag{4.6}$$

where

$$\|\tilde{\Lambda}_0\|_{H^2(0,1/\varepsilon)} \le C, \qquad \|\tilde{\Theta}_0\|_{H^2(0,1/\varepsilon)} \le C. \tag{4.7}$$

Lemma 4.1. *There exists a constant $C > 0$, independent of ε such that solutions of (4.4)-(4.6) with $\tilde{\Lambda}_0$ and $\tilde{\Theta}_0$ satisfying (4.7) have the estimate*

$$\|\tilde{\phi}\|_{H^2(\mathfrak{S})} \le C\,[\,\|\tilde{h}\|_{L^2(\mathfrak{S})} + \|\tilde{g}\|_{H^1(\mathfrak{S})} + \|\tilde{\Lambda}_0\|_{H^2(0,1/\varepsilon)} + \|\tilde{\Theta}_0\|_{H^2(0,1/\varepsilon)}\,].$$

Proof. Let ϕ_0 be the solution of

$$\Delta\phi_0 - \phi_0 = 0, \quad \text{in } \mathfrak{S}, \quad \frac{\partial \phi_0}{\partial \nu} = \tilde{g}, \quad \text{on } \partial\mathfrak{S},$$

and set $\tilde{\phi} = \bar{\phi} - \phi_0$, then $\bar{\phi}$ is a solution to a similar problem, except that it has homogeneous Neumann boundary condition, with all nonhomogeneous terms replaced by $\bar{h}$, $\bar{\Lambda}_0$, $\bar{\Theta}_0$ with bounds like

$$\|\bar{h}\|_{L^2(\mathfrak{S})} \le C\,[\,\|\tilde{h}\|_{L^2(\mathfrak{S})} + \|\tilde{g}\|_{H^1(\mathfrak{S})}\,],$$

$$\|\bar{\Lambda}_0\|_{H^2(0,1/\varepsilon)} \le C\,[\,\|\tilde{\Lambda}_0\|_{H^2(0,1/\varepsilon)} + \|\tilde{g}\|_{H^1(\mathfrak{S})}\,],$$

$$\|\bar{\Theta}_0\|_{H^2(0,1/\varepsilon)} \le C\,[\,\|\tilde{\Theta}_0\|_{H^2(0,1/\varepsilon)} + \|\tilde{g}\|_{H^1(\mathfrak{S})}\,].$$

To prove the general case it suffices to apply the following argument with

$$\phi = \bar{\phi} - \frac{\bar{\Lambda}_0(z)}{\int_{\mathbb{R}} w_x^2}\, w_x(x) - \frac{\bar{\Theta}_0(z)}{\int_{\mathbb{R}} Z^2}\, Z(x).$$

Then ϕ satisfies a problem of the same form with homogeneous Neumann boundary condition and orthogonality condition replaced by $\Lambda_0 = 0, \Theta_0 = 0$ as well as $\bar{h}$ replaced by a function h with $L^2(\mathfrak{S})$ norm bounded by

$$\|h\|_{L^2(\mathfrak{S})} \le C\,[\,\|\tilde{h}\|_{L^2(\mathfrak{S})} + \|\tilde{g}\|_{H^1(\mathfrak{S})} + \|\tilde{\Lambda}_0\|_{H^2(0,1/\varepsilon)} + \|\tilde{\Theta}_0\|_{H^2(0,1/\varepsilon)}\,].$$

Let us consider Fourier series decompositions for h and ϕ of the form

$$\phi(x,z) = \sum_{k=0}^{\infty} \phi_k(x)\cos(\pi k \varepsilon z), \quad h(x,z) = \sum_{k=0}^{\infty} h_k(x)\cos(\pi k \varepsilon z).$$

Then we have the validity of the equations

$$-k^2\pi^2\varepsilon^2\phi_k + \mathcal{L}_0(\phi_k) = h_k, \quad x \in \mathbb{R}, \tag{4.8}$$

and conditions

$$\int_{-\infty}^{\infty} \phi_k\, w_x\, \mathrm{d}x = 0, \quad \int_{-\infty}^{\infty} \phi_k\, Z\, \mathrm{d}x = 0, \tag{4.9}$$

for all k. We have denoted here

$$\mathcal{L}_0(\phi_k) = \phi_{k,xx} - \phi_k + p w^{p-1}\phi_k.$$

Let us consider the bilinear form in $H^1(\mathbb{R})$ associated to the operator $\mathcal{L}_0$, namely

$$B(\psi,\psi) = \int_{\mathbb{R}} [\,|\psi_x|^2 + |\psi|^2 - p w^{p-1}|\psi|^2\,]\,\mathrm{d}x.$$

Since (4.9) holds uniformly in k we conclude that

$$C\,[\,\|\phi_k\|^2_{L^2(\mathbb{R})} + \|\phi_{k,x}\|^2_{L^2(\mathbb{R})}\,] \le B(\phi_k,\phi_k) \tag{4.10}$$

for a constant $C > 0$ independent of k. Using this fact and equation (4.8) we find the estimate

$$(1 + \pi^4 k^4 \varepsilon^4)\|\phi_k\|^2_{L^2(\mathbb{R})} + \|\phi_{k,x}\|^2_{L^2(\mathbb{R})} \le C\|h_k\|^2_{L^2(\mathbb{R})}.$$

Moreover, we see from (4.8) that ϕ_k satisfies an equation of the form

$$\phi_{k,xx} - \phi_k = \tilde{h}_k, \quad x \in \mathbb{R}$$

where $\|\tilde{h}_k\|_{L^2(\mathbb{R})} \le C\|h_k\|_{L^2(\mathbb{R})}$. Hence it follows that additionally we have the estimate

$$\|\phi_{k,xx}\|^2_{L^2(\mathbb{R})} \le C\|h_k\|^2_{L^2(\mathbb{R})}. \tag{4.11}$$

Adding up estimates (4.10), (4.11) in k we conclude that

$$\|D^2\phi\|^2_{L^2(\mathfrak{S})} + \|D\phi\|^2_{L^2(\mathfrak{S})} + \|\phi\|^2_{L^2(\mathfrak{S})} \le C\|h\|^2_{L^2(\mathfrak{S})}.$$

The final estimate of $\tilde{\phi}$ can be easily derived. □

We consider now the following problem: given $h \in L^2(\mathfrak{S})$, $g \in H^1(\mathfrak{S})$, finding functions $\phi \in H^2(\mathfrak{S})$, $c, d \in L^2(0,1)$ such that

$$L_0(\phi) = h + c(\varepsilon z)w_x + d(\varepsilon z)Z \quad \text{in } \mathfrak{S}, \tag{4.12}$$

$$\frac{\partial \phi}{\partial \nu} = g \quad \text{on } \partial\mathfrak{S}, \tag{4.13}$$

$$\int_{\mathbb{R}} \phi(x,z)\, w_x(x)\,\mathrm{d}x = \Lambda_0(z), \quad 0 < z < \frac{1}{\varepsilon}, \tag{4.14}$$

$$\int_{\mathbb{R}} \phi(x,z)\, Z(x)\,\mathrm{d}x = \Theta_0(z), \quad 0 < z < \frac{1}{\varepsilon}. \tag{4.15}$$

Lemma 4.2. *If then functions h, g, Λ_0, Θ_0 satisfy the conditions in previous lemma, then problem (4.12)-(4.15) possesses a unique solution, denoted by $\phi = T_0(h, g, \Lambda_0, \Theta_0)$. Moreover,*

$$\|\phi\|_{H^2(\mathfrak{S})} \le C\,[\,\|h\|_{L^2(\mathfrak{S})} + \|g\|_{H^1(\mathfrak{S})} + \|\Lambda_0\|_{H^2(0,1/\varepsilon)} + \|\Theta_0\|_{H^2(0,1/\varepsilon)}\,].$$

Proof. From the argument in Lemma 4.1, it is sufficient to prove this result for the case $\Lambda_0 \equiv 0$, $\Theta_0 \equiv 0$ and $g \equiv 0$. For existence, we write again

$$h(x,z) = \sum_{k=0}^{\infty} h_k(x)\cos(\pi k \varepsilon z)$$

and consider the problem of finding $\phi_k \in H^1(\mathbb{R})$, and constants c_k, d_k such that

$$-k^2\pi^2\varepsilon^2\phi_k + \mathcal{L}_0(\phi_k) = h_k + c_k w_x + d_k Z \quad x \in \mathbb{R}$$

and

$$\int_{\mathbb{R}} \phi_k\, w_x\,\mathrm{d}x = 0, \quad \int_{\mathbb{R}} \phi_k\, Z\,\mathrm{d}x = 0,$$

where $\mathcal{L}_0(\phi_k) = \phi_{k,xx} - \phi_k + p w^{p-1}\phi_k$. Fredholm's alternative yields that this problem is solvable with the choices

$$c_k = -\frac{\int_{\mathbb{R}} h_k w_x\,\mathrm{d}x}{\int_{\mathbb{R}} w_x^2\,\mathrm{d}x}, \qquad d_k = -\frac{\int_{\mathbb{R}} h_k Z\,\mathrm{d}x}{\int_{\mathbb{R}} Z^2\,\mathrm{d}x}.$$

Observe in particular that

$$\sum_{k=0}^{\infty} |c_k|^2 \le C\varepsilon\|h\|^2_{L^2(\mathfrak{S})}, \quad \sum_{k=0}^{\infty} |d_k|^2 \le C\varepsilon\|h\|^2_{L^2(\mathfrak{S})}. \tag{4.16}$$

Finally define

$$\phi(x,z) = \sum_{k=0}^{\infty} \phi_k(x)\cos(\pi k\varepsilon z),$$

and correspondingly

$$c(\zeta)=\sum_{k=0}^{\infty} c_k \cos(\pi k\zeta)\,,\quad d(\zeta)=\sum_{k=0}^{\infty} d_k \cos(\pi k\zeta)\,.$$

Estimate (4.16) gives that $c(\varepsilon z)w_x$ and $d(\varepsilon z)Z$ have their $L^2(\mathfrak{S})$ norm controlled by that of h. The a priori estimates of the previous lemma tell us that the series for ϕ is convergent in $H^2(\mathfrak{S})$ and defines a unique solution for the problem with the desired bounds. □

As a special case of Lemma 4.2, we give a proof of Lemma 3.1.

Proof of Lemma 3.1. From the linear theory just developed in Lemma 4.2, the problem has a unique solution $\Phi\in H^2(\mathfrak{S})$ for some ρ. Careful checking the proof of Lemma 4.2 will give the estimate of ρ and Φ. In fact, we derive (3.3) by (4.16). On the other hand, uniqueness of the problem and evenness of the function g in the variable x imply that Φ is even in x for each z and $c(\varepsilon z)$ is identically zero.

We observe first that since

$$\int_{\mathbb{R}}\Phi(x,z)w_x\,\mathrm{d}x=0,\qquad \int_{\mathbb{R}}\Phi(x,z)Z\,\mathrm{d}x=0,$$

hence

$$\int_{-\infty}^{\infty}\Big[\,|\Phi_x(x,z)|^2+|\Phi(x,z)|^2-pw^{p-1}|\Phi(x,z)|^2\,\Big]\mathrm{d}x\geq\lambda_2\int_{-\infty}^{\infty}|\Phi(x,z)|^2\mathrm{d}x,\quad (4.17)$$

where $\lambda_2>0$ is the third eigenvalue of the operator

$$\mathcal{L}_0(\psi)=-\psi_{xx}+\psi-pw^{p-1}\psi\quad\text{ in }\mathbb{R}.$$

Consider function

$$H(z)=\int_{-\infty}^{\infty}|\Phi(x,z)|^2\,\mathrm{d}x.$$

From (4.17) it follows that

$$-H_{zz}+\lambda_2 H\leq 0$$

and from (3.4) we get that $|H_z(0)|\leq C$. Clearly we have also $H_z(1/\varepsilon)=0$ and thus by a comparison argument we get that

$$|H(z)|\leq Ce^{-\mu z},\quad \mu\leq\sqrt{\lambda_2}.$$

Using local elliptic estimates we then get

$$|\Phi(x,z)e^{\mu z}|\leq C\quad\text{ in }\mathfrak{S}.$$

From this, using the maximum principle we get (3.5). □

In order to apply the previous result to the resolution of the full problem (4.1)-(4.2), we define first the operator, for a fixed integer j

$$L^j(\phi)=\Delta\phi-\phi+pw_j^{p-1}\phi,$$

and consider the following problem

$$L^j(\phi)=h\,+\,c_j(\varepsilon z)\,w_{j,x}\,+\,d_j(\varepsilon z)\,Z_j\quad\text{in }\mathfrak{S},\qquad \frac{\partial\phi}{\partial\nu}=g\quad\text{on }\partial\mathfrak{S},\qquad (4.18)$$

$$\int_{\mathbb{R}}\phi(x,z)\,w_{j,x}(x)\,\mathrm{d}x=\Lambda_0(z),\quad 0<z<\frac{1}{\varepsilon},\qquad (4.19)$$

$$\int_{\mathbb{R}}\phi(x,z)\,Z_j(x)\,\mathrm{d}x=\Theta_0(z),\qquad 0<z<\frac{1}{\varepsilon}.\qquad (4.20)$$

We have

Lemma 4.3. *Problem (4.18)-(4.20) possesses a unique solution ϕ. Moreover,*

$$\|\phi\|_{H^2(\mathfrak{S})} \le C\,[\,\|h\|_{L^2(\mathfrak{S})} + \|\Lambda_0\|_{H^2(0,1/\varepsilon)} + \|\Theta_0\|_{H^2(0,1/\varepsilon)} + \|g\|_{H^1(\mathfrak{S})}\,].$$

Proof. We recall that $w_j = w(x - f_j(\varepsilon z))$ and $Z_j = Z(x - f_j(\varepsilon z))$. For a function $\xi(x,z)$ defined in $\mathfrak{S}$ we denote below

$$\tilde\xi(x,z) = \xi(x + f_j(\varepsilon z), z).$$

Direct computation gives that problem (4.18)-(4.20) is equivalent to

$$\begin{aligned}
\Delta\tilde\phi + B_1(\tilde\phi) - \tilde\phi + pw^{p-1}\tilde\phi &= \tilde h + c_j(\varepsilon z)\,w_x + d_j(\varepsilon z)\,Z \quad \text{in } \mathfrak{S},\\
\frac{\partial\tilde\phi}{\partial\nu} &= \tilde g + B_2(\tilde\phi) \quad \text{on } \partial\mathfrak{S},\\
\int_{\mathbb{R}} \tilde\phi(x,z)\,w_x(x)\,\mathrm{d}x &= \Lambda_0(z), \quad 0 < z < \frac{1}{\varepsilon},\\
\int_{\mathbb{R}} \tilde\phi(x,z)\,Z(x)\,\mathrm{d}x &= \Theta_0(z), \quad 0 < z < \frac{1}{\varepsilon},
\end{aligned}$$

where

$$\begin{aligned}
B_1(\tilde\phi) &= \varepsilon^2\Big(f_j'(\varepsilon z)\Big)^2\tilde\phi_{xx} - \varepsilon^2 f_j''(\varepsilon z)\tilde\phi_x - 2\varepsilon f'(\varepsilon z)\tilde\phi_{xz},\\
B_2(\tilde\phi) &= \varepsilon f_j'(\varepsilon z)\tilde\phi_x.
\end{aligned}$$

This problem is then equivalent to the fixed point linear problem

$$\tilde\phi = T_0(\tilde h + B_1(\tilde\phi),\, \tilde g + B_2(\tilde\phi),\, \Lambda_0,\, \Theta_0)$$

where T_0 is the linear operator defined by Lemma 4.2. The linear operators B_1 and B_2 are small in the sense that

$$\|B_1(\tilde\phi)\|_{L^2(\mathfrak{S})} + \|B_2(\tilde\phi)\|_{H^1(\mathfrak{S})} \le o(1)\,\|\tilde\phi\|_{H^2(\mathfrak{S})},$$

with $o(1) \to 0$ as $\varepsilon \to 0$. From this, unique solvability of the problem and the desired estimate immediately follow. □

Proof of Proposition 4.1. A first claim we make is that to prove the above result it suffices to consider the case $g = 0$, so that we will only need to find the operator $T(h, 0, \Lambda, \Theta)$. Indeed, let us consider the solution $\phi_0 = \phi_0(g)$ to the problem

$$\Delta\phi_0 - \phi_0 = 0 \quad \text{in } \mathfrak{S}, \quad \frac{\partial\phi_0}{\partial\nu} = g \quad \text{on } \partial\mathfrak{S}.$$

From standard elliptic theory, we find that

$$\|\phi_0\|_{H^2(\mathfrak{S})} \le C\,\|g\|_{H^1(\mathfrak{S})}.$$

On the other hand, we check directly that

$$\tilde\phi = \phi - \phi_0$$

satisfies a similar equation, but now for $g = 0$, with Λ_j and Θ_j replaced by $\tilde\Lambda_j$ and $\tilde\Theta_j$ respectively, where

$$\|\tilde\Lambda_j\|_{H^2(0,1/\varepsilon)} \le C\,[\,\|\Lambda_j\|_{H^2(0,1/\varepsilon)} + \|g\|_{H^1(\mathfrak{S})}\,],$$

$$\|\tilde\Theta_j\|_{H^2(0,1/\varepsilon)} \le C\,[\,\|\Theta_j\|_{H^2(0,1/\varepsilon)} + \|g\|_{H^1(\mathfrak{S})}\,],$$

and with h replaced by $\tilde h = \tilde h(h,g)$, a linear operator in its argument satisfying

$$\|\tilde h\|_{L^2(\mathfrak{S})} \le C\,[\,\|h\|_{L^2(\mathfrak{S})} + \|g\|_{H^1(\mathfrak{S})}\,]\,.$$

With the aid of this and the definition of $\tilde\phi$, the operator in Proposition 4.1 is thus built just from $T(\tilde h, 0, \tilde\Lambda, \tilde\Theta)$, as claimed.

We search for a solution of $\phi = T(h,0,\Lambda,\Theta)$ to problem (4.1)-(4.2) in the form

$$\phi = \sum_{j=1}^N \eta_j \bar\phi + \psi \tag{4.21}$$

where

$$\eta_j(x,z) = \eta_0\Big(\frac{x - f_j(\varepsilon z)}{R}\Big), \quad R = 2|\ln\varepsilon|,$$

and η_0 is smooth with $\eta_0(s) = 1$ for $|s| < 1/2$ and $= 0$ for $|s| > 1$. We will denote

$$\chi = 1 - \sum_{j=1}^N \eta_j.$$

It is readily checked that ϕ given by (4.21) solves problem (4.1)-(4.2) with $g = 0$ if the functions $\phi_j^* = \eta_j\bar\phi$ and ψ satisfy the following linear system of equations.

$$L^j(\phi_j^*) \;=\; \varpi(\phi_j^*) + c_j(\varepsilon z) w_{j,x} + d_j(\varepsilon z) Z_j \quad \text{in } \mathfrak{S}, \qquad \frac{\partial \phi_j^*}{\partial\nu} \;=\; 0 \quad \text{on } \partial\mathfrak{S}, \tag{4.22}$$

$$\int_{\mathbb R} \phi_j^*(x,z) w_{j,x}\,\mathrm{d}x \;=\; \tilde\Lambda_j, \quad \int_{\mathbb R} \phi_j^*(x,z) Z_j\,\mathrm{d}x \;=\; \tilde\Theta_j, \quad 0 < z < \frac{1}{\varepsilon}, \tag{4.23}$$

and

$$\begin{aligned}\Delta\psi - \chi\psi + \chi p w^{p-1}\psi \;&=\; \chi h + \sum_{j=1}^N (1-\eta_j) c_j(\varepsilon z) w_{j,x} + \sum_{j=1}^N (1-\eta_j) d_j(\varepsilon z) Z_j \\ &\quad - \sum_{j=1}^N \big[\, 2\nabla\eta_j\cdot\nabla\phi_j^* + \phi_j^*\Delta\eta_j\,\big],\end{aligned} \tag{4.24}$$

$$\frac{\partial\psi}{\partial\nu} \;=\; 0 \quad \text{on } \partial\mathfrak{S}, \tag{4.25}$$

where

$$\varpi(\phi_j^*) = \eta_j(h + \psi - pW^{p-1}\psi) - (pW^{p-1} - pw_j^{p-1})\eta_j\phi_j^*,$$

$$\tilde\Lambda_j = \Lambda_j + \int_{\mathbb R} (1-\eta_j)\phi_j w_{j,x}\,\mathrm{d}x - \sum_{k\neq j}\int_{\mathbb R} \eta_k\phi_k w_{j,x}\,\mathrm{d}x - \int_{\mathbb R}\psi w_{j,x},$$

$$\tilde\Theta_j = \Theta_j + \int_{\mathbb R} (1-\eta_j)\phi_j Z_j\,\mathrm{d}x - \sum_{k\neq j}\int_{\mathbb R} \eta_k\phi_k Z_j\,\mathrm{d}x - \int_{\mathbb R}\psi Z_j.$$

In order to solve this system we will set up a fixed point argument. To this end assume that function $\tilde\phi$ is given and define

$$\tilde\phi_j^* = \tilde\phi\,\eta_j, \quad \tilde\psi = \tilde\phi - \sum_{j=1}^N \tilde\phi_j^*.$$

First we replace ϕ_j^*, ψ by $\tilde{\phi}_j^*$, $\tilde{\psi}$ on the right hand sides of (4.22)-(4.23) and solve (4.22)–(4.23) for each $j = 1, \ldots, N$ using Lemma 4.3. We get the following estimate

$$\begin{aligned} \|\phi_j^*\|_{H^2(\mathfrak{S})} \quad \leq \quad & C\Big[\|h\|_{L^2(\mathfrak{S})} + \|\tilde{\psi}\|_{H^2(\mathfrak{S})} + \|\Lambda\|_{H^2(0,1/\varepsilon)} + \|\Theta\|_{H^2(0,1/\varepsilon)} \Big] \\ & + o(1) \sum_{j=1}^{N} \|\tilde{\phi}_j^*\|_{H^2(\mathfrak{S})}. \end{aligned} \tag{4.26}$$

Given $\tilde{\psi}$ we can now find functions $\phi_j^* = \phi_j^*(\tilde{\psi})$ which solve (4.22)–(4.23) by a fixed point argument. Next we observe that the norms $\|c_j(\varepsilon z) w_{j,x}\|_{L^2(\mathfrak{S})}$ and $\|d_j(\varepsilon z) Z_j\|_{L^2(\mathfrak{S})}$ are controlled by $\|h\|_{L^2(\mathfrak{S})}$ as it was pointed out in Lemma 4.2 (see (4.16) and the argument that follows). Therefore we can now solve (4.24)–(4.25) for ψ which in addition satisfies

$$\|\psi\|_{H^2(\mathfrak{S})} \leq C\|h\|_{L^2(\mathfrak{S})} + o(1) \sum_{j=1}^{N} \|\phi_j^*(\tilde{\psi})\|_{H^2(\mathfrak{S})}. \tag{4.27}$$

Combining this with (4.26) and applying a fixed point argument again we get finally a solution to (4.24)–(4.25). This ends the proof. □

5. **Solving the Nonlinear Intermediate Problem.** In this section we will solve problem (3.32)-(3.35). The linear problem associated to L_2 is

$$L_2(\phi) \; = \; h + \eta_\delta^\varepsilon \sum_{j=1}^{N} c_j(\varepsilon z)\, w_{j,x} + \eta_\delta^\varepsilon \sum_{j=1}^{N} d_j(\varepsilon z)\, Z_j \quad \text{in } \mathfrak{S}, \qquad \frac{\partial \phi}{\partial \nu} = g \quad \text{on } \partial\mathfrak{S}, \tag{5.1}$$

$$\int_{\mathbb{R}} \phi(x,z)\, w_{j,x}\, \mathrm{d}x = 0,\ 0 < z < \frac{1}{\varepsilon}\,,\ j = 1, \ldots, N, \tag{5.2}$$

$$\int_{\mathbb{R}} \phi(x,z)\, Z_j\, \mathrm{d}x = 0,\ 0 < z < \frac{1}{\varepsilon}\,,\ j = 1, \ldots, N. \tag{5.3}$$

From the choice of $\sigma = \varepsilon^{1/8}$, $\delta = \varepsilon^{1/6}$ and the definition of the operator $B_{x,z}$ we get

$$\|\eta_{6\delta}^\varepsilon B_{x,z}(\phi)\|_{L^2(\mathfrak{S})} \;\leq\; o(1)\|\phi\|_{H^2(\mathfrak{S})},$$

hence L_2 is just a small $H^2 \to L^2$-perturbation of the operator L treated in the previous section. Since the presence of the extra factor η_δ^ε can be dealt with easily because of the decay exponential of $w_{j,x}$ and Z_j as an immediate consequence of Proposition 4.1 we get the following.

Proposition 5.1. *There exists a constant $C > 0$, independent of ε and uniform with respect to $\mathbf{f}$ and e in (3.31) such that for all small ε problem (5.1)-(5.3) has a solution $\phi = T_{\mathbf{f},e}(h, g, 0, 0)$, which defines a linear operator in its argument and satisfies the estimate*

$$\|\phi\|_{H^2(\mathfrak{S})} \;\leq\; C\,[\,\|h\|_{L^2(\mathfrak{S})} + \|g\|_{H^1(\mathfrak{S})}\,]\,.$$

Our next goal is to solve the nonlinear problem (3.32)-(3.35). An important observation is that the terms $\tilde{E}_{110}$ and $\tilde{E}_{210}$ in the decomposition of $\tilde{E}$, have precisely the form $\eta_\delta^\varepsilon \sum_{j=1}^N d_j(\varepsilon z) Z_j$ and can be absorbed in that term. From the estimates in Lemma 3.2, $\tilde{E}_2 \equiv \tilde{E} - \tilde{E}_{110} - \tilde{E}_{210}$ is of order $O(\varepsilon^2 |\ln \varepsilon|^2)$, which is needed to apply contraction mapping theorem.

Proposition 5.2. *There exist numbers $D > 0$, $\gamma_0 > 0$ such that for all sufficiently small ε and all $\mathbf{f}$ and $\mathbf{e}$ in (3.31) problem (3.32)-(3.35) has a unique solution $\phi = \phi(\mathbf{f}, \mathbf{e})$ which satisfies*

$$\|\phi\|_{H^2(\mathfrak{S})} \le D\varepsilon^{\frac{3}{2}}|\ln\varepsilon|^2,$$
$$\|\phi\|_{L^\infty(|x|>\delta/\varepsilon)} + \|\nabla\phi\|_{L^\infty(|x|>\delta/\varepsilon)} \le e^{-\gamma_0\delta/\varepsilon}.$$

Besides ϕ depends continuously on $\mathbf{f}$ and $\mathbf{e}$.

Proof. Let $T_{\mathbf{f},\mathbf{e}}$ be the operator defined by Proposition 5.1. Let us denote

$$M(\phi) = -\eta^\varepsilon_{3\delta} b_\ell(x, z)\cdot\nabla_{x,z}\phi$$

Then, given $\mathbf{f}$ and $\mathbf{e}$ in (3.31), the equation (3.32)-(3.35) is equivalent to the fixed point problem for $\phi(\mathbf{f}, \mathbf{e})$:

$$\phi(\mathbf{f}, \mathbf{e}) = T_{\mathbf{f},\mathbf{e}}\Big(h,\, g,\, 0,\, 0\Big) \equiv \mathcal{A}(\phi, \mathbf{f}, \mathbf{e}) \tag{5.4}$$

with

$$h = -\eta^\varepsilon_\delta\tilde{E}_2(\mathbf{f}, \mathbf{e}) - \eta^\varepsilon_\delta N_2\big(\phi(\mathbf{f}, \mathbf{e})\big), \quad g = -\eta^\varepsilon_\delta\tilde{E}_b(\mathbf{f}, \mathbf{e}) + M\big(\phi(\mathbf{f}, \mathbf{e})\big).$$

In the sequel we will not emphasize the dependence on $\mathbf{f}$ and $\mathbf{e}$ whenever it is not necessary.

We will define now the region where contraction mapping principle applies. We consider the following closed, bounded subset of $H^2(\mathfrak{S})$:

$$\mathcal{B} = \left\{\phi \in H^2(\mathfrak{S}) \;\middle|\; \begin{array}{l} \|\phi\|_{H^2(\mathfrak{S})} \le D\varepsilon^{\frac{3}{2}}|\ln\varepsilon|^2, \\ \|\phi\|_{L^\infty(|x|>\delta/\varepsilon)} + \|\nabla\phi\|_{L^\infty(|x|>\delta/\varepsilon)} \le e^{-\gamma_0\delta/\varepsilon} \end{array}\right\}$$

and claim that there are constants $D, \gamma_0 > 0$ such that the map $\mathcal{A}$ defined in (5.4) is a contraction from $\mathcal{B}$ into itself, uniform with respect to $\mathbf{f}$ and $\mathbf{e}$. Given $\tilde{\phi} \in \mathcal{B}$ we denote $\phi = \mathcal{A}(\tilde{\phi}, \mathbf{f}, \mathbf{e})$ and then have the following estimates. Firstly, (3.27) and Lemma 3.2 imply that for $\tilde{\phi} \in \mathcal{B}$

$$\begin{aligned}\|\eta^\varepsilon_\delta\tilde{E}_2 + \eta^\varepsilon_\delta N_2(\tilde{\phi})\|_{L^2(\mathfrak{S})} \le\;& C_0\varepsilon^{3/2}|\ln\varepsilon|^2 + C\|\tilde{\phi}\|^2_{H^2(\mathfrak{S})} + e^{-\gamma\delta/\varepsilon} \\ &+C\varepsilon^{1/4}\Big[\|\tilde{\phi}\|_{L^\infty(|x|>\delta/\varepsilon)} + \|\nabla\tilde{\phi}\|_{L^\infty(|x|>\delta/\varepsilon)}\Big]\end{aligned} \tag{5.5}$$

with some $\gamma > 0$.
Secondly, using Lemma 3.2, and the fact that $|b_\ell(x, z)| \le C\varepsilon|x|$ we get for the H^2 extension of $-\eta^\varepsilon_\delta\tilde{E}_b + M(\tilde{\phi})$ (denoted by the same symbol)

$$\| -\eta^\varepsilon_\delta\tilde{E}_b + M(\tilde{\phi})\,\|_{H^2(\mathfrak{S})} \le C_1\varepsilon^{3/2}|\ln\varepsilon|^2 + C\delta\|\tilde{\phi}\|_{H^2(\mathfrak{S})}. \tag{5.6}$$

Finally, the exponential decay of the basic approximate solution W outside the region: $\{|x| > \delta_0/\varepsilon\}$ and the fact that $F'(W) = -1 + O(e^{-\gamma|x|})$ for some constant $\gamma > 0$ imply

$$\begin{aligned}&\|\mathcal{A}(\tilde{\phi}, \mathbf{f}, \mathbf{e})\|_{L^\infty(|x|>\delta/\varepsilon)} + \|\nabla\mathcal{A}(\tilde{\phi}, \mathbf{f}, \mathbf{e})\|_{L^\infty(|x|>\delta/\varepsilon)} \\ &\qquad\le Ce^{-\gamma\delta/\varepsilon} + C\delta\Big[\|\tilde{\phi}\|_{L^\infty(|x|>\delta/\varepsilon)} + \|\nabla\tilde{\phi}\|_{L^\infty(|x|>\delta/\varepsilon)}\Big].\end{aligned} \tag{5.7}$$

Since $\delta = \varepsilon^{1/6}$ from (5.5)–(5.7) we get that $\mathcal{A}$ indeed maps $\mathcal{B}$ into itself provided that D is chosen sufficiently large and γ_0 sufficiently small.

Let us analyze the Lipschitz dependence of the nonlinear operator involved in $\mathcal{A}$ for functions in $\mathcal{B}$, namely $N_2(\phi+\psi(\phi))$. For $\phi_1,\phi_2\in\mathcal{B}$ we have, using (3.27) and (3.28):

$$\begin{aligned}
&\Big\| \eta_\delta^\varepsilon N_2(\phi_1+\psi(\phi_1)) - \eta_\delta^\varepsilon N_2(\phi_2+\psi(\phi_2)) \Big\|_{L^2(\mathfrak{S})} \\
&\quad\le C\Big[\varepsilon^{3/2}|\ln\varepsilon|^2 + e^{-\gamma_0\delta/\varepsilon}\Big]\Bigg\{ \|\phi_1-\phi_2\|_{L^2(\mathfrak{S})} \\
&\qquad + \varepsilon^{1/4}\|\phi_1-\phi_2\|_{L^\infty(|x|>\delta/\varepsilon)} + \varepsilon^{1/4}\|\nabla(\phi_1-\phi_2)\|_{L^\infty(|x|>\delta/\varepsilon)}\Bigg\}
\end{aligned} \tag{5.8}$$

Using this one can show that $\mathcal{A}$ is a contraction map in $\mathcal{B}$ and thus show the existence of the fixed point.

A tedious but straightforward analysis of all terms involved in the differential operator and in the error yield that the operator $\mathcal{A}(\phi,\mathbf{f},\mathbf{e})$ is continuous with respect to $\mathbf{f}$ and $\mathbf{e}$. Indeed, indicating now the dependence on $\mathbf{f}$ and $\mathbf{e}$, let us make the following decomposition:

$$\begin{aligned}
&L_{2,\mathbf{f}_1,\mathbf{e}_1}(\,\phi(\mathbf{f}_1,\mathbf{e}_1)\,) - L_{2,\mathbf{f}_2,\mathbf{e}_2}(\,\phi(\mathbf{f}_2,\mathbf{e}_2)\,) \\
&\quad = L_{2,\mathbf{f}_1,\mathbf{e}_1}\Big[\phi(\mathbf{f}_1,\mathbf{e}_1)-\phi(\mathbf{f}_2,\mathbf{e}_2)\Big] + \Big[F'(W(\mathbf{f}_1,\mathbf{e}_1)) - F'(W(\mathbf{f}_2,\mathbf{e}_2))\Big]\phi(\mathbf{f}_2,\mathbf{e}_2),
\end{aligned}$$

$$\begin{aligned}
&\eta_\delta^\varepsilon\sum_{j=1}^N\Big[c_j(\varepsilon z;\mathbf{f}_1,\mathbf{e}_1)\,w_{j,x}(\mathbf{f}_1) - c_j(\varepsilon z;\mathbf{f}_2,\mathbf{e}_2)\,w_{j,x}(\mathbf{f}_2)\Big] \\
&\qquad = \eta_\delta^\varepsilon\sum_{j=1}^N\Big[c_j(\varepsilon z;\mathbf{f}_1,\mathbf{e}_1) - c_j(\varepsilon z;\mathbf{f}_2,\mathbf{e}_2)\Big]\,w_{j,x}(\mathbf{f}_1) \\
&\qquad\quad + \eta_\delta^\varepsilon\sum_{j=1}^N c_j(\varepsilon z;\mathbf{f}_2,\mathbf{e}_2)\Big[w_{j,x}(\mathbf{f}_1) - w_{j,x}(\mathbf{f}_2)\Big],
\end{aligned}$$

$$\begin{aligned}
&\eta_\delta^\varepsilon\sum_{j=1}^N\Big[d_j(\varepsilon z;\mathbf{f}_1,\mathbf{e}_1)\,Z_j(\mathbf{f}_1) - d_j(\varepsilon z;\mathbf{f}_2,\mathbf{e}_2)\,Z_j(\mathbf{f}_2)\Big] \\
&\qquad = \eta_\delta^\varepsilon\sum_{j=1}^N\Big[d_j(\varepsilon z;\mathbf{f}_1,\mathbf{e}_1) - d_j(\varepsilon z;\mathbf{f}_2,\mathbf{e}_2)\Big]\,Z_j(\mathbf{f}_1) \\
&\qquad\quad + \eta_\delta^\varepsilon\sum_{j=1}^N d_j(\varepsilon z;\mathbf{f}_2,\mathbf{e}_2)\Big[Z_j(\mathbf{f}_1) - Z_j(\mathbf{f}_2)\Big],
\end{aligned}$$

and finally, for each $j=1,\dots,N$

$$\begin{aligned}
&\int_{\mathbb{R}}[\,\phi(\mathbf{f}_1,\mathbf{e}_1)-\phi(\mathbf{f}_2,\mathbf{e}_2)\,]\,w_{j,x}(\mathbf{f}_1)\,\mathrm{d}x \\
&\qquad = -\int_{\mathbb{R}}\phi(\mathbf{f}_2,\mathbf{e}_2)\,[\,w_{j,x}(\mathbf{f}_1)-w_{j,x}(\mathbf{f}_2)\,]\,\mathrm{d}x, \\
&\int_{\mathbb{R}}[\,\phi(\mathbf{f}_1,\mathbf{e}_1)-\phi(\mathbf{f}_2,\mathbf{e}_2)\,]\,Z_j(\mathbf{f}_1)\,\mathrm{d}x \\
&\qquad = -\int_{\mathbb{R}}\phi(\mathbf{f}_2,\mathbf{e}_2)\,[\,Z_j(\mathbf{f}_1)-Z_j(\mathbf{f}_2)\,]\,\mathrm{d}x.
\end{aligned}$$

Using these decompositions one can estimate $\|\phi(\mathbf{f}_1, \mathbf{e}_1) - \phi(\mathbf{f}_2, \mathbf{e}_2)\|_{H^2(\mathfrak{S})}$ employing the theory developed in the previous section. Observe that this estimate does not depend on $c_j(\varepsilon z; \mathbf{f}_1) - c_j(\varepsilon z; \mathbf{f}_2)$ and $d_j(\varepsilon z; \mathbf{f}_1) - d_j(\varepsilon z; \mathbf{f}_2)$. Tedious but straightforward argument shows that in fact for fixed ε the fixed point of $\mathcal{A}$, $\phi(\mathbf{f}, \mathbf{e})$ is Lipschitz with respect to $\mathbf{f}$ and $\mathbf{e}$ and thus continuous with respect to $\mathbf{f}$ and $\mathbf{e}$. This ends the proof. □

Clearly Proposition 5.2 and the gluing procedure yield a solution to our original problem (2.5)–(2.6) if we can find $\mathbf{f}$ and $\mathbf{e}$ such that

$$\mathbf{c}(\mathbf{f}, \mathbf{e}) = 0, \tag{5.9}$$

$$\mathbf{d}(\mathbf{f}, \mathbf{e}) = 0. \tag{5.10}$$

As we will see this leads to a system of $2N$ nonlinear ODE's. We carry out this argument in the next section.

6. **Toda System.**

It is easy to see that the identities (5.9) and (5.10) are equivalent to the following system of equations

$$\int_{\infty}^{\infty} \big[\, L_2(\phi) \,+\, \eta_\delta^\varepsilon \tilde{E} \,+\, \eta_\delta^\varepsilon N_2(\phi) \,\big]\, w_{k,x}\, \mathrm{d}x \;=\; 0, \quad k = 1, ..., N, \tag{6.1}$$

$$\int_{\infty}^{\infty} \big[\, L_2(\phi) \,+\, \eta_\delta^\varepsilon \tilde{E} \,+\, \eta_\delta^\varepsilon N_2(\phi) \,\big]\, Z_k\, \mathrm{d}x \;=\; 0, \quad k = 1, ..., N. \tag{6.2}$$

6.1. **Estimates for parameter f.**

In this subsection we deal with the estimates of the terms in (6.1). Introduce the notations

$$\mathcal{S} = \{x \in \mathbb{R} : (x, z) \in \mathfrak{S}\}, \; \mathcal{S}_k = \{x \in \mathbb{R} : (x, z) \in A_k\},$$

and consider for each k, $(k = 1, \cdots, N)$, the following integrals

$$\begin{aligned} \int_{\mathcal{S}} \eta_\delta^\varepsilon\, \tilde{E}(x, z)\, w'(x - f_k(\varepsilon z))\, \mathrm{d}x &= \left\{ \int_{\mathcal{S}_k} + \int_{\mathcal{S}\setminus\mathcal{S}_k} \right\} \eta_\delta^\varepsilon\, \tilde{E}(x, z)\, w'(x - f_k(\varepsilon z))\, \mathrm{d}x \\ &\equiv \mathcal{E}_{k1}(\varepsilon z) \,+\, \mathcal{E}_{k2}(\varepsilon z). \end{aligned}$$

Note that in A_k, $\eta_\delta^\varepsilon \tilde{E}(x, z) = E_2(x, z)$, we begin with

$$\begin{aligned} \mathcal{E}_{k1}(\varepsilon z) = &-\varepsilon^2 \int_{\mathcal{S}_k} \sum_{j=1}^{N} \Big[\, f_j''\, w'(x - f_j) + (f_j')^2\, w''(x - f_j) \,\Big]\, w'(x - f_k)\, \mathrm{d}x \\ &+ \int_{\mathcal{S}_k} (B u_0)\, w'(x - f_k)\, \mathrm{d}x \,+\, \int_{\mathcal{S}_k} \Big[E_{02} - \sum_{j=1}^{N} F(w_j) \Big]\, w'(x - f_k)\, \mathrm{d}x \\ &+ \int_{\mathcal{S}_k} [\, E_{11} + E_{12} + E_{13}\,]\, w'(x - f_k)\, \mathrm{d}x \,+\, \int_{\mathcal{S}_k} \mathcal{N}(\phi^*)\, w'(x - f_k)\, \mathrm{d}x \\ &+ \int_{\mathcal{S}_k} [\, E_{21} + E_{22} + E_{23}\,]\, w'(x - f_k)\, \mathrm{d}x \,+\, \int_{\mathcal{S}_k} \mathcal{N}(\phi^{**})\, w'(x - f_k)\, \mathrm{d}x \\ &\equiv \sum_{i=1}^{7} I_i. \end{aligned}$$

Using the fact that $\int_{-\infty}^{\infty} w''(s)w'(s)\,ds = 0$ and the asymptotic formulas for w' in (2.2), we get

$$I_1 = -\varepsilon^2\gamma_1\, f_k''(\varepsilon z) + O(\varepsilon^3)\sum_{j=1}^{N} f_j''(\varepsilon z) + O(\varepsilon^3)\sum_{j=1}^{N}\Big(f_j'(\varepsilon z)\Big)^2,$$

where $\gamma_1 = \int_{-\infty}^{\infty}\big(w'(s)\big)^2\,ds$.
Now, from the definition of the operator $B_{x,z}$ in (2.8)

$$I_2(\varepsilon z) = \varepsilon^2\, b_\sigma(\varepsilon z) f_k' + O(\sigma^{-2}\varepsilon^3)\sum_{j=1}^{N}\Big[\, f_j' + (f_j')^2 + f_j''\,\Big],$$

where

$$b_\sigma(\varepsilon z) = \gamma_1 k_0 \eta_\sigma(\varepsilon z) \;+\; \gamma_1 k_1 \eta_\sigma(1-\varepsilon z). \tag{6.3}$$

As for I_3, using the expression of the error term E_{02} in (2.13), we find

$$\begin{aligned} E_{02} - \sum_{j=1}^{N} F(w_j) \;&=\; p(w_k)^{p-1}(u_0 - w_k) - \sum_{j\neq k}(w_j)^p \\ &\quad + \Big[\,\frac{1}{2}F''(w_k)(u_0-w_k)^2 + \max_{j\neq k} O(e^{-3|f_j - x|})\,\Big] \\ &=\; E_{020,k} \;+\; E_{021,k} \;+\; E_{022,k}. \end{aligned}$$

We will now write

$$\begin{aligned} &\int_{\mathcal{S}_k} E_{020,k}\, w'(x-f_k)\,\mathrm{d}x \\ &= p\int_{\mathcal{S}_k}\Big(w_k\Big)^{p-1}\Big[\,w_{k-1} + w_{k+1}\,\Big]\, w'(x-f_k)\,\mathrm{d}x \;+\; \varepsilon\max_{j\neq k} O(e^{-|f_j - f_k|}) \\ &= \gamma_2\Big[\,e^{-(f_k - f_{k-1})} - e^{-(f_{k+1}-f_k)}\,\Big] + \varepsilon^{\mu_1}\max_{j\neq k} O(e^{-|f_j-f_k|}), \end{aligned}$$

where μ_1 is a small positive constant depending only on p, and γ_2 is a positive constant given by the following expression

$$\gamma_2 = pC_p\int_0^{\infty} w^{p-1}(s)w'(s)e^{-s}ds - pC_p\int_0^{\infty} w^{p-1}(s)w'(s)e^{s}ds.$$

For k fixed let us consider the following integral

$$\frac{1}{2}\int_{\frac{f_{k-1}+f_k}{2}}^{f_k}\Big|\,F''(w_k)(u_0-w_k)^2 w'(x-f_k)\,\mathrm{d}x\Big| \le \varepsilon^{\mu_2}\max_{j\neq k} O(e^{-|f_j-f_k|}).$$

Similar estimates hold for

$$\frac{1}{2}\int_{f_k}^{\frac{f_{k+1}+f_k}{2}}\Big|\,F''(w_k)(u_0-w_k)^2 w'(x-f_k)\,\Big|\,\mathrm{d}x.$$

Therefore, in A_k, there exists a small positive constant μ_3 such that

$$I_3 \;=\; \gamma_2\Big[\,e^{-(f_k-f_{k-1})} - e^{-(f_{k+1}-f_k)}\,\Big] + \varepsilon^{\mu_3}\max_{j\neq k} O(e^{-|f_j-f_k|}).$$

Similarly,

$$I_4 \;=\; -\varepsilon^2\,\beta_1(z)\, f_k' \;+\; \varepsilon^{\mu_4}\max_{j\neq k} O(e^{-|f_j-f_k|}),$$

where

$$\beta_1(z) = 2\int_{\mathbb{R}} \big[\, k_0\,\Phi_{xz}(x,z) \,+\, k_1\Phi_{xz}(x, \frac{1}{\varepsilon} - z)\,\big] w^{'}(x)\,\mathrm{d}x.$$

In fact, we will use the smallness of $L^2(0,1)$-norm of the term involving $f_k^{'}(\varepsilon z)\Phi_{xz}\big(x - f_k(\varepsilon z), z\big)$ in I_4. Let

$$\kappa(\varepsilon z) = \varepsilon^2 \int_{\mathcal{S}_k} f_k^{'}(\varepsilon z)\Phi_{xz}\big(x - f_k(\varepsilon z), z\big)\, w'(x - f_k(\varepsilon z))\,\mathrm{d}x,$$

then

$$||\kappa||^2_{L^2(0,1)} \,\le\, C\varepsilon^4\,|\ln\varepsilon|^2 \int_{\mathcal{S}_k} \Big|\Phi_{xz}\big(x - f_k(\xi), \frac{\xi}{\varepsilon}\big)\Big|^2\,\mathrm{d}x\mathrm{d}\xi \,\le\, C\varepsilon^5\,|\ln\varepsilon|^2.$$

Using the facts that the function $w^{'}(x)$ is odd and the functions $\phi^*(x,z)$ and $Z(x)$ (in the definition of construction of the approximate solutions) are even, we can derive, for $i = 5, \cdots, 7$

$$||I_i||_{L^2(0,1)} \le C\varepsilon^{2+\mu_5},\ \text{for some small } \mu_4 > 0, \tag{6.4}$$

which end the estimates for all terms in $\mathcal{E}_{k1}(\varepsilon z)$.

To compute $\mathcal{E}_{k2}(\varepsilon z)$ we notice that for $(x,z) \in S_{\delta/\varepsilon} \setminus A_k$ we have

$$w'(x - f_k) = \max_{j\neq k} O(e^{-\frac{1}{2}|f_j - f_k|})$$

and thus we can estimate

$$\mathcal{E}_{k2}(\varepsilon z) = \varepsilon^{1/2} \max_{j\neq k} O(e^{-|f_j - f_k|}) + O(\varepsilon^{1/2})\sum_{i=1}^{7} I_i.$$

Gathering the above estimates, we get the following, for $k = 1, \ldots, N$

$$\begin{aligned} \int_{\mathcal{S}} \eta^\varepsilon_\delta\, \tilde{E}(x,z)\, w'(x - f_k(\varepsilon z))\,\mathrm{d}x \,&=\, -\varepsilon^2\gamma_1 f_k'' \,-\, \varepsilon^2\,\beta_2(z) f_k' \\ &+ \gamma_2\Big(e^{-(f_k - f_{k-1})} - e^{-(f_{k+1} - f_k)}\Big) + \mathcal{P}_k(\varepsilon z), \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} \mathcal{P}_k(\varepsilon z) \quad &= \quad O(\sigma^{-2}\varepsilon^3)\sum_{j=1}^{N}\big[\, f_j' + (f_j')^2 + f_j''\,\big] \,+\, \varepsilon^{\mu_6}\max_{j\neq k} O(e^{-|f_j - f_k|}), \\ \beta_2(z) \quad &= \quad \beta_1(z) \,-\, b_\sigma(\varepsilon z), \end{aligned}$$

with $\mu_6 \le \mu_i$, $i = 1, \cdots, 5$. Moreover f_k's need to satisfy boundary conditions (2.14). For further references we observe that

$$||\mathcal{P}_k||_{L^2(0,1)} \le C\varepsilon^{2+\mu_7},\ \text{for some } \mu_7 > 0,\ k = 1, \cdots, N. \tag{6.6}$$

The above estimate is possible thanks to the fact that we have chosen $\sigma = \varepsilon^{\frac{1}{8}}$.

Continuing with the terms involved in (6.1), using the quadratic nature of $N_2(\phi)$ and Proposition 5.2, we get for

$$\mathcal{Q}_k(\varepsilon z) = \int_{\mathcal{S}} \eta^\varepsilon_\delta\, N_2(\phi)\, w'(x - f_k(\varepsilon z))\,\mathrm{d}x$$

a similar estimate holds

$$||\mathcal{Q}_k||_{L^2(0,1)} \le C\varepsilon^{2+\mu_8},\quad k = 1, \cdots, N. \tag{6.7}$$

We point out that, by Proposition 5.2, $\mathcal{Q}_k$ is a continuous function of the parameters $\mathbf{f}$ and $\mathbf{e}$. The last term in (6.1) can be written as

$$\begin{aligned}\mathfrak{V}_k(\varepsilon z) &= \int_{\mathcal{S}} L_2(\phi)\, w'(x - f_k(\varepsilon z))\,\mathrm{d}x \\ &= \int_{\mathcal{S}} \phi_{zz}\, w'(x - f_k(\varepsilon z))\,\mathrm{d}x + \int_{\mathfrak{S}} B_{x,z}(\phi)\, w'(x - f_k(\varepsilon z))\,\mathrm{d}x \\ &\quad + \int_{\mathcal{S}} \phi\,\big[\, w'''(x - f_k(\varepsilon z)) + F^{'}(W)\, w'(x - f_k(\varepsilon z))\,\big]\,\mathrm{d}x.\end{aligned}$$

A similar estimate holds

$$||\mathfrak{V}_k||_{L^2(0,1)} \leq C\varepsilon^{2+\mu_9}, \quad k = 1, \cdots, N. \tag{6.8}$$

In fact, the proof is very straightforward. For example, using the orthogonality conditions we can get

$$\begin{aligned}\mathfrak{V}^1_k(\varepsilon z) &= \int_{\mathcal{S}} \phi_{zz}\, w'(x - f_k(\varepsilon z))\,\mathrm{d}x \\ &= \varepsilon^2 \int_{\mathcal{S}} \phi\,\big[\, f_k^{''} w''(x - f_k(\varepsilon z)) - (f_k^{'})^2 w'''(x - f_k(\varepsilon z))\,\big]\,\mathrm{d}x \\ &\quad + 2\varepsilon f_k^{'} \int_{\mathcal{S}} \phi_z\, w''(x - f_k(\varepsilon z))\,\mathrm{d}x\end{aligned}$$

The estimate for $\mathfrak{V}^1_k$ follows from Proposition 5.2. Moreover, it also depends continuously on the parameters $\mathbf{f}$ and $\mathbf{e}$.

We define for $\zeta = \varepsilon z$

$$\mathcal{N}_k(\xi, \mathbf{f}, \mathbf{f}^{'}, \mathbf{f}^{''}, \mathbf{e}, \mathbf{e}^{'}, \mathbf{e}^{''}) = \mathcal{P}_k + \mathcal{Q}_k + \mathfrak{V}_k.$$

From above discussion, we draw a conclusion as the following proposition

Proposition 6.1. *For $k = 1, \cdots, N$, there holds the following estimate*

$$\begin{aligned}&\int_{\infty}^{\infty} [L_2(\phi) \,+\, \eta^{\varepsilon}_{\delta} \tilde{E} \,+\, \eta^{\varepsilon}_{\delta} N_2(\phi)]\, w_{k,x}\,\mathrm{d}x \\ &\quad = -\varepsilon^2 \gamma_1 f_k'' \,-\, \varepsilon^2\, \beta_2(\frac{\zeta}{\varepsilon})\, f_k' \,+\, \gamma_2 \left(e^{-(f_k - f_{k-1})} - e^{-(f_{k+1} - f_k)} \right) + \mathcal{N}_k.\end{aligned}$$

Moreover, $\mathcal{N}_k$ can be decomposed in the following way

$$\mathcal{N}_k = \mathcal{N}_{k1}(\zeta, \boldsymbol{f}, \boldsymbol{f}^{'}, \boldsymbol{f}^{''}, \boldsymbol{e}, \boldsymbol{e}^{'}, \boldsymbol{e}^{''}) \,+\, \mathcal{N}_{k2}(\zeta, \boldsymbol{f}, \boldsymbol{f}^{'}, \boldsymbol{e}, \boldsymbol{e}^{'}).$$

where $\mathcal{N}_{k1}$ and $\mathcal{N}_{k2}$ are continuous of their arguments. Function $\mathcal{N}_{k1}$ satisfies the following properties for $k = 1, \cdots, N$

$$\begin{aligned}&\big|\big|\mathcal{N}_{k1}(\zeta, \boldsymbol{f}, \boldsymbol{f}^{'}, \boldsymbol{f}^{''}, \boldsymbol{e}, \boldsymbol{e}^{'}, \boldsymbol{e}^{''})\big|\big|_{L^2(0,1)} \,\leq\, C\varepsilon^{2+\mu_0}, \\ &\big|\big|\mathcal{N}_{k1}(\xi, \boldsymbol{f}, \boldsymbol{f}^{'}, \boldsymbol{f}^{''}, \boldsymbol{e}, \boldsymbol{e}^{'}, \boldsymbol{e}^{''}) - \mathcal{N}_{k1}(\zeta, \boldsymbol{f}_1, \boldsymbol{f}_1^{'}, \boldsymbol{f}_1^{''}, \boldsymbol{e}_1, \boldsymbol{e}_1^{'}, \boldsymbol{e}_1^{''})\big|\big|_{L^2(0,1)} \\ &\qquad \leq\, C\varepsilon^{2+\mu_0} |\ln\varepsilon|^q \left[\, ||\boldsymbol{f} - \boldsymbol{f}_1||_{H^2(0,1)} + ||\boldsymbol{e} - \boldsymbol{e}_1||_b \,\right].\end{aligned}$$

Function $\mathcal{N}_{k2}$ satisfies the following estimates for $k = 1, \cdots, N$

$$\big|\big|\mathcal{N}_{k2}(\zeta, \boldsymbol{f}, \boldsymbol{f}^{'}, \boldsymbol{e}, \boldsymbol{e}^{'})\big|\big|_{L^2(0,1)} \,\leq\, C\varepsilon^{2+\mu_0}.$$

We omit the proof of this proposition. In fact, careful examining of all terms will lead the decomposition of the operator $\mathcal{N}_k$ and the properties of its components $\mathcal{N}_{k1}$ and $\mathcal{N}_{k2}$. □

6.2. **Estimates for parameter e.** In this subsection we turn to the estimates of all terms in (6.2) and firstly consider for each k, $(k = 1, \dots, N)$, the following integrals

$$
\begin{aligned}
&\int_{\mathcal{S}} \eta_\delta^\varepsilon \, \tilde{E}(x,z)\, Z(x - f_k(\varepsilon z))\, \mathrm{d}x \\
&= \left\{ \int_{\mathcal{S}_k} + \int_{\mathcal{S}\setminus\mathcal{S}_k} \right\} \eta_\delta^\varepsilon \, \tilde{E}(x,z)\, Z(x - f_k(\varepsilon z))\, \mathrm{d}x \\
&\equiv \mathcal{F}_{k1}(\varepsilon z) + \mathcal{F}_{k2}(\varepsilon z).
\end{aligned}
$$

Note that in A_k, $\eta_\delta^\varepsilon \tilde{E}(x,z) = E_2(x,z)$, we begin with

$$
\begin{aligned}
&\mathcal{F}_{k1}(\varepsilon z) \\
&= -\varepsilon^2 \int_{\mathcal{S}_k} \sum_{j=1}^{N} \Big[f_j''\, w^{'}(x - f_j) + (f_j')^2\, w''(x - f_j) \Big]\, Z(x - f_k)\, \mathrm{d}x \\
&\quad + \int_{\mathcal{S}_k} (Bu_0)\, Z(x - f_k)\, \mathrm{d}x \; + \int_{\mathcal{S}_k} \Big[E_{02} - \sum_{j=1}^{N} F(w_j) \Big] Z(x - f_k)\, \mathrm{d}x \\
&\quad + \int_{\mathcal{S}_k} [\, E_{11} + E_{12} + E_{13} \,]\, Z(x - f_k)\, \mathrm{d}x \; + \int_{\mathcal{S}_k} \mathcal{N}(\phi^*)\, Z(x - f_k)\, \mathrm{d}x \\
&\quad + \int_{\mathcal{S}_k} [\, E_{21} + E_{22} + E_{23} \,]\, Z(x - f_k)\, \mathrm{d}x \; + \int_{\mathcal{S}_k} \mathcal{N}(\phi^{**})\, Z(x - f_k)\, \mathrm{d}x \\
&\equiv \sum_{i=1}^{7} J_i.
\end{aligned}
$$

Using the fact that $\int_{-\infty}^{\infty} w'(s) Z(s)\, ds = 0$ and the asymptotic formula for Z we get

$$
J_1 \;=\; \varepsilon^2 \gamma_3\, (f_k')^2 \;+\; O(\varepsilon^3) \sum_{j=1}^{N} \Big(f_j'(\varepsilon z) \Big)^2 \;+\; O(\varepsilon^3) \sum_{j=1}^{N} f_j''(\varepsilon z),
$$

where $\gamma_3 = \int_{\mathbb{R}} w''(s)\, Z(s) \mathrm{d}s$. By similar computation of I_2

$$
J_2(\varepsilon z) = \varepsilon^2 \gamma_3\, f_k'\, b_\sigma(\varepsilon z) + O(\sigma^{-2}\varepsilon^3) \sum_{j=1}^{N} \Big[\, f_j' + (f_j')^2 + f_j'' \,\Big],
$$

where $b_\sigma(\varepsilon z)$ is defined in (6.3) and γ_3 is a constant defined as above.

Similarly as in the computation of I_3, we get that there exists a small positive constant τ_1 with $\frac{1}{2} < \tau_1 < 1$ depending only on p such that

$$
J_3 \quad = \quad \gamma_4 \left[e^{\,-(f_k - f_{k-1})} + e^{\,-(f_{k+1} - f_k)} \right] + \varepsilon^{\tau_1} \max_{j \neq k} O(e^{-|f_j - f_k|})
$$

where

$$
\gamma_4 = pC_p \int_0^{\infty} w^{p-1}(s) Z(s) [\, e^{\,-s} + e^{\,s} \,]\, \mathrm{d}s.
$$

Since $w^{'}(x)$ is odd in the variable x and $Z(x)$ is even in the variable x

$$
\begin{aligned}
J_{41} &\equiv \int_{\mathcal{S}_k} E_{11}\, Z(x-f_k)\,\mathrm{d}x \\
&= \varepsilon\Big[k_0\rho(\varepsilon z)+k_1\rho(1-\varepsilon z)\Big]\int_{\mathcal{S}_k}\sum_{j=1}^{N} Z(x-f_j)\,Z(x-f_k)\,\mathrm{d}x \\
&\qquad + O(\varepsilon^3)|\ln\varepsilon|^q\int_{\mathcal{S}_k} e^{-(1-\varrho)|f_k-x|}\,Z(x-f_k)\,\mathrm{d}x \\
&= \varepsilon\,\alpha_1(\varepsilon z)\ +\ \varepsilon\max_{j\neq k} O(e^{-|f_j-f_k|}) \\
&\qquad + O(\varepsilon^3)|\ln\varepsilon|^q\int_{\mathcal{S}_k} e^{-(1-\varrho)|f_k-x|}\,Z(x-f_k)\,\mathrm{d}x,
\end{aligned}
$$

where

$$
\alpha_1(\varepsilon z)=k_0\rho(\varepsilon z)+k_1\rho(1-\varepsilon z). \tag{6.9}
$$

Since the functions $\phi^*(x,z)$ and $w(x)$ are even in the variable x

$$
\begin{aligned}
J_{42} &\equiv \int_{\mathcal{S}_k} E_{12}\, Z(x-f_k)\,\mathrm{d}x \\
&= \int_{\mathcal{S}_k} \varepsilon p\sum_{j=1}^{N}(\,u_0^{p-1}-w_j^{p-1}\,)\,\phi_j^*\,Z(x-f_k)\,\mathrm{d}x \\
&= \varepsilon\,\max_{j\neq k} O(e^{-(\frac{1-\nu}{2})|f_j-f_k|}),
\end{aligned}
$$

$$
\begin{aligned}
J_5 &= \int_{\mathcal{S}_k} \mathcal{N}(\phi^*)\,Z(x-f_k)\,\mathrm{d}x \\
&= \int_{\mathcal{S}_k} [\,F(u_0+\phi^*)-F(u_0)-F'(u_0)\phi^*\,]\ Z(x-f_k)\,\mathrm{d}x \\
&= \varepsilon^2\alpha_2(z)+\varepsilon^2\max_{j\neq k} O(e^{-|f_j-f_k|}),
\end{aligned}
$$

where

$$
\alpha_2(z)=\frac{p(p-1)}{2}\int_{\mathbb{R}} w^{p-2}(s)\Big[\,k_0\Phi(s,z)+k_1\Phi(s,\frac{1}{\varepsilon}-z)\,\Big]^2 Z(s)\,\mathrm{d}s.
$$

Since $Z^{'}(x)$ is odd in the variable x and $Z(x)$ is even in the variable x

$$\begin{aligned}
J_{61} &\equiv \int_{\mathcal{S}_k} E_{21}\, Z(x-f_k)\,\mathrm{d}x \\
&= \int_{\mathcal{S}_k} \sum_{j=1}^{N} \left[(\varepsilon^3 e_j^{''} + \varepsilon\lambda_0 e_j)\, Z_j \;+ \varepsilon^3 \sum_{j=1}^{N} (f_j^{'})^2\, e_j\, Z_j^{''} \right] Z(x-f_k)\,\mathrm{d}x \\
&\quad + \int_{\mathcal{S}_k} \sum_{j=1}^{N} \left(-2\varepsilon^3\, e_j^{'}\, f_j^{'}\, Z_j^{'} + \varepsilon^3\, e_j\, f_j^{''}\, Z_j^{'} \right) Z(x-f_k)\,\mathrm{d}x \\
&= \varepsilon^3 e_k^{''} + \varepsilon\lambda_0 e_k \;+ \sum_{j=1}^{N} (\varepsilon^{4+p} e_j^{''} + \varepsilon^{p+2} e_j) \\
&\quad + \varepsilon^{4+p} \sum_{j=1}^{N} (f_j^{'})^2 e_j \;+\; \varepsilon^{4+p} \sum_{j=1}^{N} (e_j f_j^{''} + 2 e_j^{'} f_j^{'}).
\end{aligned}$$

Since the functions $\phi_j^*(x,z)$ and $w(x)$ are even in the variable x

$$\begin{aligned}
J_{62} &\equiv \int_{\mathcal{S}_k} E_{22}\, Z(x-f_k)\,\mathrm{d}x \\
&= \int_{\mathcal{S}_k} \varepsilon p \sum_{j=1}^{N} \left[(u_0+\phi^*)^{p-1} - w_j^{p-1} \right] \phi_j^{**}\, Z(x-f_k)\,\mathrm{d}x \\
&= \varepsilon^2 \alpha_3(z)\, e_k + \varepsilon^{1/2} \max_{j\neq k} O(e^{-|f_j-f_k|}),
\end{aligned}$$

where

$$\alpha_3(z) = p(p-1) \int_{\mathbb{R}} w^{p-2}(s) \left[k_0\, \Phi(s,z) + k_1\, \Phi(s, \frac{1}{\varepsilon} - z) \right] Z^2(s)\,\mathrm{d}s.$$

Since ϕ^{**} is a term of order ε and from the assumption on e_k, we derive

$$\begin{aligned}
J_7 &= \int_{\mathcal{S}_k} \mathcal{N}(\phi^{**})\, Z(x-f_k)\,\mathrm{d}x \\
&= \int_{\mathcal{S}_k} [\, F(u_0+\phi^*+\phi^{**}) - F(u_0+\phi^*) - F'(u_0+\phi^*)\phi^{**}\,]\, Z(x-f_k)\,\mathrm{d}x \\
&= \frac{1}{2} p(p-1)\varepsilon^2 e_k^2 \int_{\mathbb{R}} w^{p-2}(s) Z^3(s) ds + O(\varepsilon^3) \sum_{k=1}^{N} e_k^3 \\
&= \varepsilon^2 \gamma_5\, e_k^2 + O(\varepsilon^3) \sum_{k=1}^{N} e_k^3,
\end{aligned}$$

where $\gamma_5 = \frac{1}{2} p(p-1) \int_{\mathbb{R}} w^{p-2}(s) Z^3(s)\,\mathrm{d}s$.

To compute $\mathcal{F}_{k2}(\varepsilon z)$ we notice that for $(x,z) \in S_{\delta/\varepsilon} \setminus A_k$

$$Z(x-f_k) = \max_{j\neq k} O(e^{-\frac{p}{2}|f_j-f_k|})$$

and thus we can estimate

$$\mathcal{F}_{k2}(\varepsilon z) = \varepsilon^{1/2} \max_{j\neq k} O(e^{-|f_j-f_k|}) + O(\varepsilon^{1/2}) \sum_{i=1}^{7} J_i.$$

Gathering the above estimates, we get the following, for $k = 1, \dots, N$

$$\begin{aligned}\int_{\mathcal{S}} \eta_\delta^\varepsilon\, \tilde{E}(x,z)\, Z(x - f_k(\varepsilon z))\,\mathrm{d}x \;&=\; \varepsilon^2\gamma_3\left(f_k'\right)^2 \;+\; \varepsilon^2\gamma_5\, e_k^2\\ &+\gamma_4\left[e^{-(f_k - f_{k-1})} + e^{-(f_{k+1}-f_k)}\right] + \varepsilon\alpha_1(z) + \varepsilon^2\alpha_2(z)\\ &+\varepsilon^3 e_k^{''} + \varepsilon^2\alpha_3(z)e_k + \varepsilon\lambda_0 e_k + \mathcal{R}_k(\varepsilon z),\end{aligned} \tag{6.10}$$

where

$$\begin{aligned}\mathcal{R}_k(\varepsilon z) = O(\sigma^{-2}\varepsilon^3)\sum_{j=1}^{N}[\,f_j' + (f_j')^2 + f_j''\,] + O(\varepsilon^3)\sum_{k=1}^{N} e_k^3\\ +\varepsilon^2\gamma_3\, f_k'\, b_\sigma(\varepsilon z) \;+\; \varepsilon^{\tau_1}\max_{j\neq k} O(e^{-|f_j - f_k|}).\end{aligned} \tag{6.11}$$

For further references we observe that

$$||\mathcal{R}_k||_{L^2(0,1)} \le C\varepsilon^{2+\tau_2},\ \text{ for some } \tau_2 > 0,\ \ k = 1,\cdots,N. \tag{6.12}$$

Continuing with the terms involved in (6.2), using the quadratic nature of $N_2(\phi)$ and Proposition 5.2, we get that for

$$\mathcal{H}_k(\varepsilon z) = \int_{\mathcal{S}} \eta_\delta^\varepsilon\, N_2(\phi)\, Z(x - f_k(\varepsilon z))\,\mathrm{d}x,$$

a similar estimate holds

$$||\mathcal{H}_k||_{L^2(0,1)} \le C\varepsilon^{2+\tau_3}, \quad k = 1,\cdots,N. \tag{6.13}$$

We point out that, by Proposition 5.2, $\mathcal{H}_k$ is a continuous function of the parameters $\mathbf{f}$ and $\mathbf{e}$.

The last term in (6.2) can be written as

$$\begin{aligned}\mathcal{U}_k(\varepsilon z) &= \int_{\mathcal{S}} L_2(\phi)\, Z(x - f_k(\varepsilon z))\,\mathrm{d}x\\ &= \int_{\mathcal{S}} \phi_{zz}\, Z(x - f_k(\varepsilon z))\,\mathrm{d}x + \int_{\mathcal{S}} B_{x,z}(\phi)\, Z(x - f_k(\varepsilon z))\,\mathrm{d}x\\ &\quad+ \int_{\mathcal{S}} \phi\left[\,Z''(x - f_k(\varepsilon z)) + F^{'}(W)\, Z(x - f_k(\varepsilon z))\,\right]\mathrm{d}x.\end{aligned}$$

A similar estimate holds

$$||\mathcal{U}_k||_{L^2(0,1)} \le C\varepsilon^{2+\tau_4}, \quad k = 1,\cdots,N. \tag{6.14}$$

In fact, the proof is very straightforward. For example, using the orthogonality conditions we can get

$$\begin{aligned}\mathcal{U}_k^1(\varepsilon z) &= \int_{\mathcal{S}} \phi_{zz}\, Z(x - f_k(\varepsilon z))\,\mathrm{d}x\\ &= \varepsilon^2\int_{\mathcal{S}} \phi\left[\,f_k^{''}\, Z'(x - f_k(\varepsilon z)) - (f_k^{'})^2 Z''(x - f_k(\varepsilon z))\,\right]\mathrm{d}x\\ &\quad+ 2\varepsilon f_k^{'}\int_{\mathcal{S}} \phi_z\, Z'(x - f_k(\varepsilon z))\,\mathrm{d}x.\end{aligned}$$

The estimate for $\mathfrak{U}_{k2}^1$ follows from Proposition 5.2. Moreover, it also depends continuously on the parameters $\mathbf{f}$ and $\mathbf{e}$.

We choose τ in (3.10) is a small constant such that $0 < \tau \le \min(\tau_i, \mu_0)$, $i = 1, \cdots, 4$ and define for $\zeta = \varepsilon z$

$$\mathcal{M}_k(\zeta, \mathbf{f}, \mathbf{f}', \mathbf{f}'', \mathbf{e}, \mathbf{e}', \mathbf{e}'') = \mathcal{R}_k + \mathcal{H}_k + \mathcal{U}_k + \varepsilon^2 \gamma_5\, e_k^2,$$

then we can conclude with the following proposition

Proposition 6.2. *For $k = 1, \cdots, N$ there hold the following estimates*

$$\begin{aligned}\int_{\infty}^{\infty} \left[L_2(\phi) \,+\, \eta_\delta^\varepsilon \tilde{E} \,+\, \eta_\delta^\varepsilon N_2(\phi) \right] Z_k \,\mathrm{d}x \;=\; \varepsilon^2 \gamma_3 \left(f_k' \right)^2 \\ + \gamma_4 \left[e^{-(f_k - f_{k-1})} + e^{-(f_{k+1} - f_k)} \right] \,+\, \varepsilon \alpha_1(\zeta) \,+\, \varepsilon^2 \alpha_2(\frac{\zeta}{\varepsilon}) \\ + \varepsilon^3 e_k'' \,+\, \varepsilon^2 \alpha_3(\frac{\zeta}{\varepsilon}) e_k \,+\, \varepsilon \lambda_0 e_k \,+\, \mathcal{M}_k.\end{aligned}$$

Moreover, the operator $\mathcal{M}_k$ can be decomposed in the following way

$$\mathcal{M}_k = \mathcal{M}_{k1}(\zeta, \boldsymbol{f}, \boldsymbol{f}', \boldsymbol{f}'', \boldsymbol{e}, \boldsymbol{e}', \boldsymbol{e}'') + \mathcal{M}_{k2}(\zeta, \boldsymbol{f}, \boldsymbol{f}', \boldsymbol{e}, \boldsymbol{e}'),$$

where $\mathcal{M}_{k1}$ and $\mathcal{M}_{k2}$ are continuous in their arguments. There exists a positive constant τ_0, $\tau_0 > \tau$, such that function $\mathcal{M}_{k1}$ satisfies the following properties for $k = 1, \cdots, N$

$$\big|\big|\mathcal{M}_{k1}(\zeta, \boldsymbol{f}, \boldsymbol{f}', \boldsymbol{f}'', \boldsymbol{e}, \boldsymbol{e}', \boldsymbol{e}'')\big|\big|_{L^2(0,1)} \le C\varepsilon^{2+\tau_0},$$

$$\begin{aligned}\big|\big|\mathcal{M}_{k1}(\zeta, \boldsymbol{f}, \boldsymbol{f}', \boldsymbol{f}'', \boldsymbol{e}, \boldsymbol{e}', \boldsymbol{e}'') - \mathcal{M}_{k1}(\zeta, \boldsymbol{f}_1, \boldsymbol{f}_1', \boldsymbol{f}_1'', \boldsymbol{e}_1, \boldsymbol{e}_1', \boldsymbol{e}_1'')\big|\big|_{L^2(0,1)} \\ \le C\varepsilon^{2+\tau_0} |\ln \varepsilon|^q \left[||\boldsymbol{f} - \boldsymbol{f}_1||_{H^2(0,1)} + ||\boldsymbol{e} - \boldsymbol{e}_1||_b \right],\end{aligned}$$

and function $\mathcal{M}_{k2}$ satisfies the following estimates for $k = 1, \cdots, N$

$$\big|\big|\mathcal{M}_{k2}(\zeta, \boldsymbol{f}, \boldsymbol{f}', \boldsymbol{e}, \boldsymbol{e}')\big|\big|_{L^2(0,1)} \le C\varepsilon^{2+\tau_0}.$$

□

7. **Location of the Concentrating Layers.** Setting

$$\theta = 2(\zeta - \frac{1}{2}),\ -1 < \theta < 1,$$

and defining the operators

$$\begin{aligned}\mathcal{L}_1^*(f_k) &\equiv \varepsilon^2\, \gamma\, f_k'' + \varepsilon^2 \sqrt{\gamma}\, \beta(\theta)\, f_k' \,-\, e^{-(f_k - f_{k-1})} + e^{-(f_{k+1} - f_k)}, \\ \mathcal{L}_2^*(e_k) &\equiv \varepsilon^2\, e_k'' \,+\, \varepsilon\, \alpha(\theta)\, e_k \,+\, \frac{\lambda_0}{4}\, e_k,\end{aligned}$$

from previous section, after obvious algebra, we have to deal with the following system with k running from 1 to N

$$\mathcal{L}_1^*(f_k) = \mathcal{N}_k, \tag{7.1}$$

$$\mathcal{L}_2^*(e_k) \,-\, \alpha_5(\theta) \,-\, \alpha_6(\theta) = \frac{1}{\varepsilon}\mathcal{M}_k, \tag{7.2}$$

$$f_k'(1) \,+\, K_+\, f_k(1) = 0, \tag{7.3}$$

$$f_k'(-1) \,+\, K_-\, f_k(-1) = 0, \tag{7.4}$$

$$e_k'(1) \,+\, K_+\, e_k(1) = 0, \tag{7.5}$$

$$e_k'(-1) \,+\, K_-\, e_k(-1) = 0, \tag{7.6}$$

where $K_- = \frac{1}{2}k_0$, $K_+ = \frac{1}{2}k_1$, $\gamma = 4\gamma_1/\gamma_2 > 0$, $f_0 = -\infty$, $f_{N+1} = \infty$ and the functions defined by

$$\begin{aligned}
\alpha_5(\theta) &= \alpha_1\Big(\frac{1-\theta}{2}\Big) + \varepsilon\alpha_2\Big(\frac{1-\theta}{2\varepsilon}\Big),\\
\alpha_6(\theta) &= \varepsilon\,\gamma_3\,\big[\,f_k'(\theta)\,\big]^2 + \frac{\gamma_4}{\varepsilon}\Big[e^{-(f_k-f_{k-1})} + e^{-(f_{k+1}-f_k)}\Big],\\
\alpha(\theta) &= C\int_{\mathbb{R}} w^{p-2}\,Z^2\left[k_0\,\Phi\Big(x,\frac{\theta+1}{2\varepsilon}\Big) + k_1\,\Phi\Big(x,\frac{1-\theta}{2\varepsilon}\Big)\right]\mathrm{d}x,\\
\beta(\theta) &= \frac{1}{\sqrt{\gamma_1\,\gamma_2}}\int_{\mathbb{R}}\left[k_0\,\Phi_{xz}\Big(x,\frac{\theta+1}{2\varepsilon}\Big) + k_1\,\Phi_{xz}\Big(x,\frac{1-\theta}{2\varepsilon}\Big)\right]w'(x)\,\mathrm{d}x\\
&\quad - \frac{k_0\sqrt{\gamma_1}}{2\sqrt{\gamma_2}}\,\eta_\sigma\Big(\frac{\theta+1}{2}\Big) - \frac{k_1\sqrt{\gamma_1}}{2\sqrt{\gamma_2}}\,\eta_\sigma\Big(\frac{1-\theta}{2}\Big).
\end{aligned}$$

Before solving the above system, we study two simpler problems in the following two subsections.

7.1. In this part, we consider the following Toda system for $k = 1, \cdots, N$

$$\mathcal{L}_1^*(f_k) = h_k, \tag{7.7}$$

$$f_k'(1) + K_+ f_k(1) = 0, \tag{7.8}$$

$$f_k'(-1) + K_- f_k(-1) = 0, \tag{7.9}$$

where $f_0 = -\infty$, $f_{N+1} = \infty$.

Proposition 7.1. *Assume that the following conditions hold:*

$$\begin{gathered}
K_- > 0, \quad K_+ < 0,\\
K_- - K_+ + 2K_-K_+ > 0,\\
\frac{K_- + K_+}{2K_-K_+} \in (-1,1),
\end{gathered} \tag{7.10}$$

and let functions h_k be such that

$$\|h_k\|_{L^2(-1,1)} \le C\varepsilon^{2+\mu},$$

with some $\mu > 0$. Then, for each sufficiently small ε there exists a unique solution $\mathbf{f} = \widetilde{T}_1(h_1, ..., h_N)$ to the system (7.7)-(7.9) which satisfies

$$\|f_k\|_{H^2(-1,1)} \le C|\ln\varepsilon|^2, \qquad k = 1, \ldots, N,$$

and for $k = 1, \ldots, N-1$

$$f_{k+1}(\theta) - f_k(\theta) > 2|\ln\varepsilon| - 4\ln|\ln\varepsilon|, \quad \sum_{k=1}^N f_k = \varepsilon^{\mu'}. \tag{7.11}$$

with some $0 < \mu' < \mu$. Moreover there exist $\lambda = c|\ln\varepsilon| + O(\ln|\ln\varepsilon|)$, $\theta_0 \in (0,1)$, which do not depend on $h_k, k = 1, \cdots, N$, such that we have the following representations, for $k = 1, \ldots, N-1$

$$f_k(\theta) = q_k\big(\lambda(\theta-\theta_0)\big) - (k-1)\ln\bar{\lambda}^2\varepsilon^2 + \phi_k(\theta),$$

and for $k = 1, \ldots, N-1$

$$f_{k+1}(\theta) - f_k(\theta) = q_{k+1}\big(\lambda(\theta-\theta_0)\big) - q_k\big(\lambda(\theta-\theta_0)\big) - \ln\bar{\lambda}^2\varepsilon^2 + \varphi_{k,k+1}(\theta).$$

Functions q_k are explicitly given as solutions of the Toda system

$$q_k'' - e^{(q_{k-1}-q_k)} + e^{(q_k-q_{k+1})} = 0 \quad \text{in } \mathbb{R}, \quad q_{k+1} - q_k \geq 0,$$

where we take $q_0 = -\infty$, $q_{N+1} = \infty$, and functions ϕ_k and $\varphi_{k,k+1}(\theta)$ satisfy

$$\|\phi_k\|_{H^2} \leq C\varepsilon^{\mu'}, \quad \|\varphi_{k,k+1}\|_{H^2} \leq C\varepsilon^{\mu'}.$$

Note that since $|\Gamma| = 1$ and $K_- = \frac{k_0}{2}$, $K_+ = \frac{k_1}{2}$, condition (7.10) is actually equivalent to (1.7). We mimic the proof of Theorem 6.1 in [9]. By setting

$$V = \sum_{j=1}^{N} f_j, \quad V_k = f_k - f_{k+1}, \ k = 1, \cdots, N-1,$$

$$H = \sum_{j=1}^{N} h_j/\varepsilon^2, \quad H_k = h_k - h_{k+1}, \ k = 1, \cdots, N-1,$$

problem (7.7)-(7.9) is equivalent to the following systems. One describes the location of the center of mass of N-concentration layers as

$$\gamma V'' + \sqrt{\gamma}\,\beta(\theta)\,V' = H, \tag{7.12}$$

$$V'(1) + K_+V(1) = 0, \quad V'(-1) + K_-V(-1) = 0, \tag{7.13}$$

and the other relate to the balance of mutual distance between N-concentration layers, for $k = 1, \cdots, N-1$,

$$\varepsilon^2\gamma V_k'' + \varepsilon^2\sqrt{\gamma}\,\beta(\theta)\,V_k' - e^{V_{k-1}} + 2e^{V_k} - e^{V_{k+1}} = H_k, \tag{7.14}$$

$$V_k'(1) + K_+V_k(1) = 0, \quad V_k'(-1) + K_-V_k(-1) = 0, \tag{7.15}$$

where $V_0 = V_N = \infty$.

The next proposition focuses on the solvability of (7.12)-(7.13) and gives a description the location of the center of mass for the N concentration layers, which will collapse to the line segment Γ because of the smallness of H.

Proposition 7.2. *Under the non-degenerate condition (1.6), if $H \in L^2(0,1)$ then there is a constant ε_0 for each $0 < \varepsilon < \varepsilon_0$ satisfying (1.11), the problem (7.12)-(7.13) has a unique solution $V \in H^2(0,1)$ which satisfies the following estimate*

$$||V||_\infty + ||V^{'}||_2 + ||V^{''}||_2 \leq C||H||_{L^2(0,1)}.$$

Proof. Under the non-degenerate condition (1.6), the existence part comes from the continuity method and a priori estimates, whose validity can be proved because the nontrivial term $\alpha_1(\frac{\theta}{\varepsilon})$ posses fast variable. Hence, we focus on the proof of the estimate by the method of Fourier expansion. There exists an orthonormal basis of $L^2(0,1)$ constituted by eigenfunctions $\{y_n\}$, associated to the eigenvalues $\{\zeta_n\}$, of the following eigenvalue problem

$$-y^{''}(\theta) = \zeta y(\theta), \ 0 < \theta < 1,$$

$$y^{'}(1) + k_1 y(1) = 0, \ \ y^{'}(0) + k_0 y(0) = 0.$$

The result in [20](on Page 9 and 10) shows that, as $n \to \infty$

$$\sqrt{\zeta_n} \ = n\pi + \frac{k_1 - k_0}{n\pi} + O(\frac{1}{n^3}). \tag{7.16}$$

It is easy to see that there exist a universal positive constant C such that

$$|\, y_n^{'}(\theta)\,| \leq C\,n,$$

for all $n \in \mathbb{N}$. We then expand

$$H(\theta) = \sum_{n=0}^{\infty} H_n y_n(\theta), \quad V(\theta) = \sum_{n=0}^{\infty} a_n y_n(\theta), \quad \beta(\theta) V^{'}(\theta) = \sum_{n=0}^{\infty} d_n y_n(\theta),$$

and carry out estimates of above Fourier coefficients d_n's. We only calculate two components of d_n in the sequel

$$\begin{aligned}
&\frac{k_0}{\sqrt{\gamma_1\,\gamma_2}}\Big|\int_{-1}^{1}\int_{\mathbb{R}} \Phi_{zx}(x, \frac{1+\theta}{\varepsilon})\, w_x(x)\, V^{'}(\theta)\, y_n(\theta)\, \mathrm{d}x\mathrm{d}\theta\Big| \\
&\leq C\Big[\int_{-1}^{1}\int_{\mathbb{R}} |V^{'}(\theta) w_x(x)|^2 \,\mathrm{d}x\mathrm{d}\theta\Big]^{\frac{1}{2}} \times \Big\{\int_{-1}^{1}\int_{\mathbb{R}} |\, \Phi_{zx}(x, \frac{1+\theta}{2\varepsilon})|^2 \,\mathrm{d}x\mathrm{d}\theta\Big\}^{\frac{1}{2}} \\
&\leq \sigma_1 ||V^{'}||_{L^2},
\end{aligned} \tag{7.17}$$

and

$$\begin{aligned}
&\frac{k_0\,\sqrt{\gamma_1}}{2\sqrt{\gamma_2}}\Big|\int_{-1}^{1} \eta_\sigma(\frac{1+\theta}{2})\, V^{'}(\theta)\, y_n(\theta)\, \mathrm{d}\theta\Big| \\
&\leq C\Big[\int_{-1}^{1} |V^{'}(\theta)|^2 \,\mathrm{d}\theta\Big]^{\frac{1}{2}} \times \Big\{\int_{-1}^{1} |\, \eta_\sigma(\frac{1+\theta}{2})|^2 \,\mathrm{d}\theta\Big\}^{\frac{1}{2}} \\
&\leq \sigma_1 ||V^{'}||_{L^2},
\end{aligned} \tag{7.18}$$

where σ_1 can be chosen small if the constant ε is sufficiently small. The estimates for other two components in d_n can be showed similarly.

From the equation

$$-\zeta_n a_n + d_n = h_n,$$

and estimates of the Fourier coefficients d_n's, we get

$$|a_n| \,\leq\, \Big|\frac{h_n}{\zeta_n}\Big| \,+\, \frac{\sigma_1}{\zeta_n}\,||V^{'}||_{L^2}.$$

Therefore, from asymptotic expression of ζ_n in (7.16) and the smallness of σ_1, we obtain

$$||V^{'}||^2_{L^2} \,+\, ||V||^2_{L^2} \,\leq\, C||H||^2_{L^2}.$$

Hence

$$\|V^{''}\|_{L^2} \,\leq\, C\,(\|h\|_{L^2} \,+\, \|V^{'}\|_{L^2} \,+\, \|V\|_{L^2}),$$

and the final result then follows. □

From now on, the subsection is devoted to the solvability of (7.14)–(7.15). Let us set

$$V_k = \tilde{V}_k + \ln(\varepsilon^2)$$

so that (7.14)–(7.15) becomes

$$\gamma\, \tilde{V}_k'' \,+\, \sqrt{\gamma}\,\beta(\theta)\, \tilde{V}_k' \,-\, e^{\tilde{V}_{k-1}} \,+\, 2e^{\tilde{V}_k} \,-\, e^{\tilde{V}_{k+1}} = \tilde{H}_k, \tag{7.19}$$

$$\tilde{V}_k'(-1) \,+\, K_-\, \tilde{V}_k(-1) = -K_-\, \ln(\varepsilon^2), \quad \tilde{V}_k'(1) \,+\, K_+\, \tilde{V}_k(1) = -K_+\, \ln(\varepsilon^2), \tag{7.20}$$

where $\tilde{H}_k \,=\, \varepsilon^{-2} H_k$. To solve the above problem we will take advantage of the fact that the equation (7.19) without the first order term considered on the whole real line has an explicit solution. To this end we introduce two parameters θ_0 and

$|\lambda| \gg 1$ and then define a change of variable $t = \lambda(\theta - \theta_0)$, and look for a solution to (7.19)–(7.20) in the form:

$$\tilde{V}_k(\theta) = \tilde{q}_k\Big(\lambda(\theta - \theta_0)\Big) + \ln \bar{\lambda}^2,$$

where for convenience we have set $\bar{\lambda} = \lambda\sqrt{\gamma}$. To solve for $k = 1, \cdots, N-1$

$$\begin{aligned} \tilde{q}_k'' + \tilde{\beta}(t)\,\tilde{q}_k' - e^{\tilde{q}_{k-1}} + 2e^{\tilde{q}_k} - e^{\tilde{q}_{k+1}} &= \bar{\lambda}^{-2}\tilde{H}_k, && (7.21)\\ \lambda\,\tilde{q}_k'(t^-) + K_-\,\tilde{q}_k(t^-) &= -K_-\ln(\bar{\lambda}^2\varepsilon^2), && (7.22)\\ \lambda\,\tilde{q}_k'(t^+) + K_+\,\tilde{q}_k(t^+) &= -K_+\ln(\bar{\lambda}^2\varepsilon^2), && (7.23) \end{aligned}$$

where $\tilde{\beta}(t) = \beta(\theta_0 + t/\lambda)/\bar{\lambda}$, $t^- = -\lambda(1+\theta_0)$, $t^+ = \lambda(1-\theta_0)$, we first solve (approximately) the homogeneous problem without the first order term to determine just two parameters θ_0 and λ and next set $\tilde{q}_k = q_k + \varphi_k$, where q_k satisfy the following equations

$$q_k'' - e^{q_{k-1}} + 2e^{q_k} - e^{q_{k+1}} = 0 \quad \text{in } \mathbb{R}, \qquad k = 1, \ldots, N-1$$

where $q_0 = q_N = \infty$. The procedure to find the corrections φ_k will be described in the following for $N = 2$. For general $N \geq 3$, the proof is similar and the reader is referred to the paper [9].

• **Cluster of two layers:** Since $N = 2$, system (7.21)–(7.23) can then be reduced to a single scalar equation

$$\begin{aligned} \tilde{q}'' + \tilde{\beta}(t)\,\tilde{q}' + 2e^{\tilde{q}} &= \bar{\lambda}^{-2}h, \quad t^- < t < t^+ && (7.24)\\ \lambda\,\tilde{q}'(t^-) + K_-\tilde{q}(t^-) &= -K_-\ln(\varepsilon^2\bar{\lambda}^2), && (7.25)\\ \lambda\,\tilde{q}'(t^+) + K_+\tilde{q}(t^+) &= -K_+\ln(\varepsilon^2\bar{\lambda}^2). && (7.26) \end{aligned}$$

The homogeneous version of the equation (7.24) without the first order term considered on $\mathbb{R}$ has an explicit solution

$$q_0(t) = \ln\left(\frac{1}{4\cosh^2(t/2)}\right).$$

It can be seen easily that $u_0(t) < 0$ and also that

$$q_0(t) = -|t| + O(e^{-|t|}), \qquad t \to \pm\infty.$$

We will now look for the first approximation of the solution of (7.24)–(7.26) in the form $q(t) = q_0(t)$ and then get the following system for the parameters θ_0, λ:

$$\begin{aligned} \lambda\,q_0'\big(-\lambda(1+\theta_0)\big) + K_-\,q_0\big(-\lambda(1+\theta_0)\big) &= -K_-\ln(\varepsilon^2\bar{\lambda}^2), && (7.27)\\ \lambda\,q_0'\big(\lambda(1-\theta_0)\big) + K_+\,q_0\big(\lambda(1-\theta_0)\big) &= -K_+\ln(\varepsilon^2\bar{\lambda}^2). && (7.28) \end{aligned}$$

This is in fact a nonlinear system for (θ_0, λ). Although it is in principle possible to find exact solution (θ_0, λ) to (7.27)–(7.28), we will not do it here. Instead, taking into account the asymptotic behavior of q_0, we will look for (θ_0, λ) that solve the following system

$$\begin{aligned} \lambda\big[\,1 - K_-(1+\theta_0)\big] &= -K_-\ln(\varepsilon^2\bar{\lambda}^2)\\ \lambda\big[-1 - K_+(1-\theta_0)\,\big] &= -K_+\ln(\varepsilon^2\bar{\lambda}^2) \end{aligned} \qquad (7.29)$$

which has a solution (θ_0, λ) such that $\lambda = O(|\ln\varepsilon|) > 0$, $\theta_0 \in (-1, 1)$ thanks to (7.10) and our assumption. In fact we have that:

$$\theta_0 = \frac{K_+ + K_-}{2K_+K_-}, \qquad \lambda = \frac{K_+\ln(\varepsilon^2\bar{\lambda}^2)}{1 + K_+(1-\theta_0)}. \qquad (7.30)$$

Notice that to find λ in (7.30) we have to solve a simple nonlinear equation. One can show that

$$\lambda = \frac{-2K_-K_+}{K_- - K_+ + 2K_-K_+}\ln\frac{1}{\varepsilon} + O(\ln\ln\frac{1}{\varepsilon}). \tag{7.31}$$

Denoting

$$\begin{aligned} g_+ &= \lambda\, q_0'(t) + K_+ q_0(t) + K_+\ln(\varepsilon^2\bar\lambda^2), \text{ for } t = \lambda(1-\theta_0),\\ g_- &= \lambda\, q_0'(t) + K_- q_0(t) + K_-\ln(\varepsilon^2\bar\lambda^2), \text{ for } t = -\lambda(1+\theta_0), \end{aligned}$$

we get from (7.31)

$$|g_\pm| \le C\varepsilon^\mu, \tag{7.32}$$

with some $\mu > 0$. We will seek an exact solution to (7.24)–(7.26) in the form

$$q(t) = q_0(t) + \varphi(t).$$

To find φ we will use a fixed point argument and thus we need to study the linearized version of (7.24)–(7.26) as

$$\phi'' + \tilde\beta(t)\,\phi' + 2e^{q_0}\phi = h, \tag{7.33}$$

$$\lambda\,\phi'(t^-) + K_-\phi(t^-) = g_-, \quad \lambda\,\phi'(t^+) + K_+\phi(t^+) = g_+, \tag{7.34}$$

with a given function $h \in L^2(t^-, t^+)$ and small constants $g_\pm$.

Before that, we need the solvability of a more simpler problem like,

$$\phi'' + 2e^{q_0}\phi = h,$$

with the same boundary condition as (7.34). Functions

$$\psi_1(t) = q_0'(t), \quad \psi_2(t) = tq_0'(t) + 2,$$

form the fundamental set for problem (7.33)–(7.34) without the first order term in the equation and the Wronskian is $W(\psi_1, \psi_2) = 1$. By variations of constants formula

$$\begin{aligned} \phi(t) &= -\psi_1(t)\int_{-\lambda(1+\theta_0)}^t \psi_2(s)h(s)\,\mathrm{d}s + \psi_2(t)\int_{-\lambda(1+\theta_0)}^t \psi_1(s)h(s)\,\mathrm{d}s\\ &\quad + c_1\psi_1(t) + c_2\psi_2(t). \end{aligned}$$

Functions ψ_1, ψ_2 satisfy the asymptotic formulas

$$\psi_1(t) = \mp 1 + O(e^{-|t|}), \qquad \psi_2(t) = -|t| + O(e^{-|t|}), \quad t \to \pm\infty$$

from which it follows easily

$$\|\phi\|_{L^2} \le C\bar\lambda^{3/2}\left[\,\|h\|_{L^2} + \bar\lambda^{-1}|c_1| + |c_2|\,\right]. \tag{7.35}$$

To determine constants c_1, c_2 we need to solve the system:

$$\begin{aligned} c_1\left[\lambda\psi_1' + K_-\psi_1\right] + c_2\left[\lambda\psi_2' + K_-\psi_2\right] &= g_-, &&\text{at } t = -\lambda(1+\theta_0),\\ c_1\left[\lambda\psi_1' + K_+\psi_1\right] + c_2\left[\lambda\psi_2' + K_+\psi_2\right] &= g_+ + \tilde g_+, &&\text{at } t = \lambda(1-\theta_0), \end{aligned}$$

where

$$\tilde g_+ = K_+\psi_1\int_{-\lambda(1+\theta_0)}^{\lambda(1-\theta_0)} \psi_2(s)h(s)\,\mathrm{d}s - K_+\psi_2\int_{-\lambda(1+\theta_0)}^{\lambda(1-\theta_0)} \psi_1(s)h(s)\,\mathrm{d}s.$$

This system has a unique solution for $\bar\lambda \gg 1$ since the matrix

$$\begin{pmatrix} K_- & 1 - K_-(1+\theta_0)\\ -K_+ & 1 - K_+(1-\theta_0) \end{pmatrix}$$

is nondegenerate thanks to the non-degeneracy conditions. In fact we find

$$\begin{aligned} |c_1| &\leq C\left[\,|g_-|+|g_+|+\bar\lambda^{3/2}\|h\|_{L^2}\,\right] \\ |c_2| &\leq C\bar\lambda^{-1}\left[\,|g_-|+|g_+|+\bar\lambda^{3/2}\|h\|_{L^2}\,\right]. \end{aligned}$$

From (7.35) we get

$$\|\phi\|_{L^2} \leq C\left[\bar\lambda^2\|h\|_{L^2}+\bar\lambda^{1/2}(|g_-|+|g_+|)\right]$$

By a straightforward argument we get a further estimate

$$\begin{aligned} \|\phi\|_{\bar\lambda} &\equiv \|\phi''\|_{L^2}+\|\phi'\|_{L^2}+\|\phi\|_{L^2} \\ &\leq C\left[\,\bar\lambda^2\|h\|_{L^2}+\bar\lambda^{1/2}(|g_-|+|g_+|)\,\right]. \end{aligned} \tag{7.36}$$

Hence, we can solve the following problem

$$\phi'' + 2e^{q_0}\phi = h,$$
$$\lambda\,\phi'(t^-) + K_-\phi(t^-) = g_-, \quad \lambda\,\phi'(t^+) + K_+\phi(t^+) = g_+,$$

and the estimate (7.36) holds.

Just as the computations in (7.17)-(7.18), it can be derived that

$$||\tilde\beta||_2 \leq \varepsilon^{\mu^{'}}, \text{ for } \mu^{'} > 0.$$

Given the last formula and (7.32) and assuming in addition that

$$\|h\|_{L^2} \leq \varepsilon^{\mu},$$

it is easy to solve (7.24)–(7.26) using a standard fixed point argument in the set of functions

$$\mathcal{X} = \left\{\phi \mid \|\phi\|_{\bar\lambda} < \varepsilon^{\mu''}\right\} \quad \text{with } \mu'' < \min(\mu, \mu^{'}).$$

After this is done one can go back to the original problem solving the following equation for f_1 and f_2

$$f_1 = \frac{1}{2}(V + V_1), \; f_2 = \frac{1}{2}(V - V_1).$$

It is easy to check the validity of Proposition 7.1 and we finish the proof for the case $N = 2$. □

7.2. We turn to the following linear problem

$$\mathcal{L}_2^*(e) = g(\theta), \; -1 < \theta < 1, \tag{7.37}$$

$$e^{'}(1) + \frac{1}{2}k_1 e(1) = 0, \quad e^{'}(-1) + \frac{1}{2}k_0 e(-1) = 0. \tag{7.38}$$

Proposition 7.3. *If $g \in L^2(0,1)$ then for ε satisfying (1.11) there is a unique solution $e = \widetilde{T}_2(g)$ in $H^2(0,1)$ to problem (7.37)-(7.38) which satisfies*

$$||e||_b \leq C\,\varepsilon^{-1}\,||g||_{L^2(0,1)}.$$

Moreover, if $g \in H^2(0,1)$ then

$$\varepsilon^2||e^{''}||_{L^2(0,1)} + ||e^{'}||_{L^2(0,1)} + ||e||_{L^\infty(0,1)} \leq C\,||g||_{H^2(0,1)}.$$

Proof. The proof is similar to that of Lemma 7.2 in [42].

There exists an orthonormal basis of $L^2(0,1)$ constituted by eigenfunctions $\{x_k\}$, associated to the eigenvalues $\{\xi_k\}$, of the following eigenvalue problem

$$-y''(\theta)-\xi y(\theta)=0,\quad 0<\theta<1,$$
$$y'(1)+\frac{1}{2}k_1y(1)=0,\quad y'(-1)+\frac{1}{2}k_0y(-1)=0.$$

The result in Page 9-10 of [20] shows that, as $k\to\infty$

$$\sqrt{\xi_k} = \frac{k\pi}{2}+\frac{k_1-k_0}{2k\pi}+O(\frac{1}{k^3}), \tag{7.39}$$
$$x_k = \cos\Big(\frac{k\pi}{2}(\theta+1)\Big)+O(\frac{1}{k}). \tag{7.40}$$

Let

$$g(\theta)=\sum_{k=0}^{\infty}g_kx_k(\theta),\quad e(\theta)=\sum_{k=0}^{\infty}a_kx_k(\theta),\quad \alpha(\theta)e(\theta)=\sum_{k=0}^{\infty}c_kx_k(\theta).$$

Now, we can use (7.39) and (7.40) to calculate the coefficients c_k

$$\begin{aligned} c_k &= \int_{-1}^{1}\alpha(\theta)\,e(\theta)\cos\Big(\frac{k\pi}{2}(\theta+1)\Big)\,d\theta+O(\frac{1}{k})\,||e||_{L^\infty}\\ &= \frac{-4}{(k\pi)^2}\int_{-1}^{1}\alpha(\theta)\,e(\theta)\Big[\cos\Big(\frac{k\pi}{2}(\theta+1)\Big)\Big]''\,d\theta+O(\frac{1}{k})\,||e||_{L^\infty}\\ &\quad+\frac{-4}{(k\pi)^2}\int_{-1}^{1}\Big[\alpha(\theta)\,e(\theta)\Big]''\cos\Big(\frac{k\pi}{2}(\theta+1)\Big)\,d\theta+O(\frac{1}{k})\,||e||_{L^\infty}. \end{aligned}$$

Using the formula of definition α, the estimates of $\Phi(x,z)$ in Lemma 3.1 and the equation for e, we can find

$$|c_k|\le\frac{C}{(k\pi)^2\varepsilon}\,||e||_{L^\infty}. \tag{7.41}$$

Using the equation

$$-\varepsilon^2\,\xi_k\,a_k+\varepsilon\,c_k+\frac{\lambda_0}{4}\,a_k=g_k$$

it is derived that

$$|a_k|\le C\Big[\frac{|g_k|}{|\lambda_0-4\varepsilon^2\xi_k|}+\frac{||e||_{L^\infty}}{(k\pi)^2|\,\lambda_0-4\varepsilon^2\xi_k\,|}\Big]. \tag{7.42}$$

From the gap condition (1.11) and (7.39), we get

$$|\lambda_0-4\varepsilon^2\xi_k|\ge C\varepsilon. \tag{7.43}$$

Using the asymptotic expression of ξ_k in (7.39) and the gap condition (1.11), it can be derived

$$\sum_{\frac{\sqrt{\lambda_0}}{2\varepsilon}\le\sqrt{\xi_k}\le\frac{3\sqrt{\lambda_0}}{2\varepsilon}}\frac{1}{(k\pi)^2|\,\lambda_0-4\varepsilon^2\xi_k\,|}\le C\varepsilon|\ln\varepsilon|. \tag{7.44}$$

Combining (7.42)-(7.44), some elementary analysis will give

$$\begin{aligned} ||e||_{L^\infty} &\le C\sum|a_k|\\ &\le C\varepsilon\,|\ln\varepsilon|\,||e||_{L^\infty}+\varepsilon^{-1}||g||_{L^2}. \end{aligned}$$

Hence,

$$||e||_{L^\infty} \le C\varepsilon^{-1}||g||_{L^2}. \tag{7.45}$$

Multiplying the equation (7.37) by e and integrating by parts, and then using (7.45), we can get

$$\varepsilon||e'||_{L^2} \le C\varepsilon^{-1}||g||_{L^2}. \tag{7.46}$$

The rest of the estimates can be derived easily. Under the gap condition (1.11), we can easily solve the following problem because of the good asymptotic expression of ξ_k in (7.39)

$$\varepsilon^2 e''(\theta) + \frac{\lambda_0}{4}e(\theta) = g(\theta),\ 0 < \theta < 1$$
$$e'(1) + \frac{1}{2}k_1 e(1) = 0, \quad e'(0) + \frac{1}{2}k_0 e(0) = 0.$$

The existence part comes from above estimates and the continuity method. □

We give a proof of the main theorem in the final part of this section.

Proof of Theorem 1.1: If $\hat{e}$ solves

$$\mathcal{L}_2^*(\hat{e}) = \alpha_5,\ -1 < \theta < 1$$
$$\hat{e}'(1) + K_+\,\hat{e}(1) = 0, \quad \hat{e}'(-1) + K_-\,\hat{e}(-1) = 0,$$

from the definition of α_5 and Lemma 3.1, we get

$$||\hat{e}||_{H^2(-1,1)} \le C\varepsilon^{1/2}.$$

Replacing e_k by $\hat{e} + e_k$, the system (7.1)-(7.6) keeps the same form except that the term α_5 disappear. Moreover, let $\tilde{e}_k$ solves

$$\mathcal{L}_2^*(\tilde{e}_k) = \alpha_{6k}(\theta),$$
$$\tilde{e}_k'(1) + K_+\,\tilde{e}_k(1) = 0, \quad \tilde{e}_k'(-1) + K_-\,\tilde{e}_k(-1) = 0,$$

where

$$\begin{aligned}\alpha_{6k}(\theta) \;&=\; \varepsilon\gamma_4\bar{\lambda}^2\, e^{q_k\big(\lambda(\theta-\theta_0)\big) - q_{k+1}\big(\lambda(\theta-\theta_0)\big)} + \varepsilon\gamma_3\bar{\lambda}^2\Big[q_k'(\lambda(\theta-\theta_0))\Big]^2 \\ &\quad + \varepsilon\gamma_4\bar{\lambda}^2\, e^{q_{k-1}\big(\lambda(\theta-\theta_0)\big) - q_k\big(\lambda(\theta-\theta_0)\big)},\end{aligned}$$

combining the fact that $q_k - q_{k-1} > 0$ and Proposition 7.3, it derives

$$||\tilde{e}_k||_{H^2(-1,1)} \le C\varepsilon^{\mu}.$$

Define

$$\mathcal{D} = \left\{ \mathbf{f}, \mathbf{e} \in H^2(\mathfrak{S}) \;\middle|\; \begin{array}{l} ||\mathbf{f}||_{H^2(-1,1)} \le D|\ln\varepsilon|^2, \\ ||\mathbf{e}||_b \le C\,\varepsilon^{\mu}. \end{array} \right\}$$

For $(\bar{\mathbf{f}}, \bar{\mathbf{e}}) \in \mathcal{D}$, we can set for $k = 1, \cdots, N$

$$\begin{aligned} h_k(\mathbf{f},\mathbf{e}) \;&\equiv\; \mathcal{N}_{k1}(\mathbf{f},\mathbf{f}',\mathbf{f}'',\mathbf{e},\mathbf{e}',\mathbf{e}'') + \mathcal{N}_{k2}(\bar{\mathbf{f}},\bar{\mathbf{f}}',\bar{\mathbf{e}},\bar{\mathbf{e}}'), \\ g_k(\mathbf{f},\mathbf{e}) \;&\equiv\; \frac{1}{\varepsilon}\mathcal{M}_{k1}(\mathbf{f},\mathbf{f}',\mathbf{f}'',\mathbf{e},\mathbf{e}',\mathbf{e}'') + \frac{1}{\varepsilon}\mathcal{M}_{k2}(\bar{\mathbf{f}},\bar{\mathbf{f}}',\bar{\mathbf{e}},\bar{\mathbf{e}}'). \end{aligned}$$

We now use Contraction Mapping Principle and Schauder Fixed Point Theorem to solve (7.1)-(7.6) with the right hand replacing by h_k and g_k. In fact, for any $\mathbf{f}^0$ and $\mathbf{e}^0$, we can use Proposition 7.1 to get

$$\mathbf{f}^1 = \widetilde{T}_1\big(\,h_1(\mathbf{f}^0,\mathbf{e}^0),\, \cdots,\, h_N(\mathbf{f}^0,\mathbf{e}^0)\,\big).$$

Moreover, for $k=1,\ldots,N$

$$f^1_k(\theta)=q_k\big(\,\lambda(\theta-\theta_0)\,\big)\;-\;(k-1)\ln\bar\lambda^2\varepsilon^2\;+\;\phi^1_k(\mathbf{f}^0,\mathbf{e}^0),$$

and for $k=1,\ldots,N-1$

$$f^1_{k+1}(\theta)-f^1_k(\theta)=q_{k+1}\big(\,\lambda(\theta-\theta_0)\,\big)-q_k\big(\,\lambda(\theta-\theta_0)\,\big)\;-\;\ln\bar\lambda^2\varepsilon^2\;+\;\varphi^1_{k,k+1}(\mathbf{f}^0,\mathbf{e}^0),$$

where we use $\phi^1_k(\mathbf{f}^0,\mathbf{e}^0)$ and $\varphi^1_{k,k+1}(\mathbf{f}^0,\mathbf{e}^0)$ to denote the dependence of these two functions on $\mathbf{f}^0$ and $\mathbf{e}^0$. Setting for $k=1,\cdots,N$

$$\begin{aligned}\mathcal{H}_k(\mathbf{f}^0,\mathbf{e}^0)\;=\;&\varepsilon\gamma_4\bar\lambda^2\,e^{q_k\big(\,\bar\lambda(\theta-\theta_0)\,\big)-q_{k+1}\big(\,\bar\lambda(\theta-\theta_0)\,\big)-t_1\varphi_{k,k+1}}\;+\;2\varepsilon\gamma_3\bar\lambda\,q_k^{'}(\bar\lambda(\theta-\theta_0))\,\phi_k^{'}\\&+\varepsilon\gamma_4\bar\lambda^2\,e^{q_{k-1}\big(\,\bar\lambda(\theta-\theta_0)\,\big)-q_k\big(\,\bar\lambda(\theta-\theta_0)\,\big)-t_2\varphi_{k-1,k}}\;+\;\varepsilon\gamma_3\Big[\phi_k^{'}\Big]^2\end{aligned}$$

where $t_1,t_2\in(0,1)$, then we can use Proposition 7.3 to set

$$\mathbf{e}^1\;=\;\widetilde{T}_2\Big(\mathcal{H}_1(\mathbf{f}^0,\mathbf{e}^0)+g_1(\mathbf{f}^0,\mathbf{e}^0),\;\cdots,\;\mathcal{H}_N(\mathbf{f}^0,\mathbf{e}^0)+g_N(\mathbf{f}^0,\mathbf{e}^0)\Big)\;+\;(\tilde e_1,...,\tilde e_N),$$

where

$$||e^1_k||_b\le C\varepsilon^\mu.$$

Whence, by the fact that $\mathcal{N}_{k1},\mathcal{M}_{k1}$ and $\mathcal{H}_k$ are contractions on $\mathcal{D}$, making use of the argument developed in Proposition 7.1 and 7.3 and the Contraction Mapping theorem, we find $\mathbf{f}$ and $\mathbf{e}$ for a fixed $\bar{\mathbf{f}}$ and $\bar{\mathbf{e}}$. In this way we define a mapping $\mathcal{Z}(\bar{\mathbf{f}},\bar{\mathbf{e}})=(\mathbf{f},\mathbf{e})$ and the solution of our problem is simply a fixed point of $\mathcal{Z}$. Continuity of $\mathcal{N}_{ki}$ and $\mathcal{M}_{ki}$, $i=1,2$, with respect to its parameters and a standard regularity arguments allows us to conclude that $\mathcal{Z}$ is compact as mapping from $H^1(-1,1)$ into itself. The Schauder Theorem applies to yield the existence of a fixed point of $\mathcal{Z}$ as required. This ends the proof of Theorem 1.1. □

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