Concentration on Lines for a Singularly Perturbed Neumann Problem in Two-Dimensional Domains

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Abstract

We consider the following singularly perturbed elliptic problem

$$\varepsilon^2 \triangle \tilde{u} - \tilde{u} + \tilde{u}^p = 0, \;\; \tilde{u} > 0 \quad \text{in } \Omega, \quad \frac{\partial \tilde{u}}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, ε is a small parameter, n denotes the outward normal of $\partial\Omega$ and the exponent $p > 1$. Let Γ be a straight line intersecting orthogonally with $\partial\Omega$ at exactly two points and satisfying a *non-degenerate condition.* We establish the existence of a solution u_ε concentrating along a curve near Γ, exponentially small in ε at any positive distance from the curve, provided ε is small and away from certain *critical numbers.* The concentrating curve will collapse to Γ as $\varepsilon \to 0$.

1 Introduction

We consider the following problem

$$\varepsilon^2 \triangle \tilde{u} - \tilde{u} + \tilde{u}^p = 0, \quad \tilde{u} > 0 \quad \text{in } \Omega \text{ and } \frac{\partial \tilde{u}}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, ε is a small parameter, n denotes the outward normal of $\partial\Omega$ and the exponent $p > 1$.

Problem (1.1) is known as stationary equation of Keller-Segel system in chemotaxis [20]. It can also be viewed as a limiting stationary equation of Gierer-Meinhardt system in biological pattern formation [10]. Problem (1.1) has been studied extensively in recent years. See [26] for backgrounds and references.

In the pioneering papers [20], [27]-[28], under the condition that p is subcritical, i.e., $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 < p < +\infty$ when $N = 2$, Lin, Ni and Takagi established the existence of a least-energy solution U_ε of (1.1) and showed that, for ε sufficiently small, U_ε has only one local maximum point $P_\varepsilon \in \partial\Omega$. Moreover, $H(P_\varepsilon) \to \max_{P \in \partial\Omega} H(P)$ as $\varepsilon \to 0$, where $H(\cdot)$ is the mean curvature of $\partial\Omega$. Such a solution is called boundary spike-layer.

Since then, many paper investigated further the solutions of (1.1) concentrating at one or multiple points of $\bar{\Omega}$. (These solutions are called spike-layers.) A general principle is that the location of interior spikes is determined by the distance function from the boundary. We refer the reader to the articles [2], [5], [7], [11]-[13], [31], and references therein. On the other hand, boundary spikes are related to the mean curvature of $\partial\Omega$. This aspect is discussed in the papers [3], [4], [6],[14],[19],[30],[32], and references therein. A good review of the subject up to 2004 can be found in [26].

The question of constructing higher-dimensional concentration sets has been investigated only in recent years. It has been conjectured in [26] that for any $1 \leq k \leq N-1$, problem (1.1) has a solution U_ε which concentrates on a k-dimensional subset of $\overline{\Omega}$. We mention some results that support such a conjecture.

In [24] and [25], Malchiodi and Montenegro proved that for $N \geq 2$, there exists a sequence of numbers $\varepsilon_k \to 0$ such that problem (1.1) has a solution U_{ε_k} which concentrates at the boundary $\partial\Omega$ (or any component of $\partial\Omega$). In [22, 23], Malchiodi showed the concentration phenomena for (1.1) along a closed non-degenerate geodesic of $\partial\Omega$ in three-dimensional smooth bounded domain Ω. Mahmoudi and Malchiodi in [21] proved a full general concentration of solutions along k-dimensional $(1 \leq k \leq N-1)$ non-degenerate minimal submanifolds of the boundary for $n \geq 3$ and $1 < p < \frac{N-k+2}{N-k-2}$.

In the above papers [21]-[25], the higher dimensional concentration set is *on* the boundary. A natural question is if there are solutions with high dimensional concentration set *inside* the domain. In this paper we consider problem (1.1) with solutions concentrating on a curve Γ_ε near a straight line Γ intersecting the boundary. More precisely, throughout the paper, we assume that $N = 2$. The curve $\Gamma \subset \Omega$ satisfies the following assumptions (see Figure 1): The curvature of Γ is zero and after translation and rotation, Γ is contained in the $\tilde{y}_1 = 0$ axis in the $(\tilde{y}_1, \tilde{y}_2)$ coordinates. Γ intersects $\partial\Omega$ at exactly two points, saying, γ_1, γ_0, and at these points $\Gamma \perp \partial\Omega$. The boundary $\partial\Omega$ can be represented as $\varphi_1(\tilde{y}_1)$ and $\varphi_0(\tilde{y}_1)$ near γ_1, γ_0 respectively. Moreover, after rescaling, we can always assume $|\Gamma| = 1$, i.e. $\varphi_1(0) - \varphi_0(0) = 1$. Let $-k_1$ and k_0 be the curvatures of the boundary

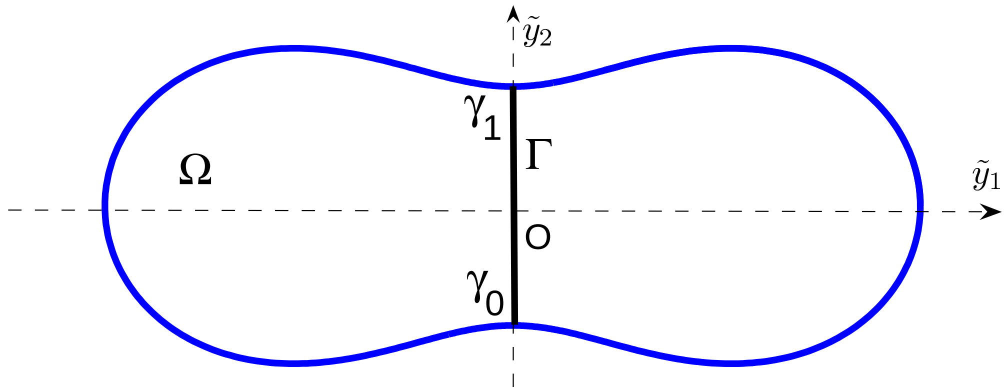

Figure 1:

$\partial\Omega$ at the points γ_1 and γ_0 respectively, where

$$k_1 = \varphi_1^{''}(0), \quad k_0 = \varphi_0^{''}(0). \tag{1.2}$$

We define a *geometric eigenvalue problem*

$$f^{''}(\theta) + \lambda\, f(\theta) = 0,\ 0 < \theta < 1, \quad f^{'}(1) + k_1 f(1) = 0, \quad f^{'}(0) + k_0 f(0) = 0. \tag{1.3}$$

We say that Γ is *non-degenerate* if (1.3) does not have a zero eigenvalue. This is equivalent to the following condition:

$$k_0 - k_1 + k_0 k_1 |\Gamma| \neq 0. \tag{1.4}$$

Before stating the main result, we introduce two functions w and Z. Let w be the unique (even) solution of

$$w^{''} - w + w^p = 0 \text{ and } w > 0 \text{ in } \mathbb{R},\ w^{'}(0) = 0,\ w(\pm\infty) = 0. \tag{1.5}$$

It is well known that the associated linearized eigenvalue problem,

$$h^{''} - h + p w^{p-1} h = \lambda h \text{ in } \mathbb{R},\ \int_{\mathbb{R}} h^2 = 1,\ h \in H^1(\mathbb{R}) \tag{1.6}$$

possesses a unique positive eigenvalue λ_0 with a unique even and positive eigenfunction Z.

Our main theorem can be stated as the following:

Theorem 1.1. *Assume that the line segment Γ satisfies the non-degenerate condition (1.4). Given a small constant $\tilde{c}$, there exists ε_0 such that for all $\varepsilon < \varepsilon_0$ satisfying the following gap condition*

$$\left|\lambda_0 - \frac{k^2\pi^2}{|\Gamma|^2}\,\varepsilon^2\right| \geq \tilde{c}\,\varepsilon, \quad \forall\, k \in \mathbb{N}, \tag{1.7}$$

problem (1.1) has a positive solution u_ε *concentrating along a curve* Γ_ε *near* Γ. *Near* Γ, u_ε *takes the form*

$$u_\varepsilon(\tilde{y}) \,=\, w\left(\frac{\tilde{y}_1}{\varepsilon}\right)\,(\,1\,+\,o(1)\,)\;. \tag{1.8}$$

Moreover, there exists some number c_0, *for* $\tilde{y}=(\tilde{y}_1,\tilde{y}_2)\in\Omega$, u_ε *satisfies globally,*

$$u_\varepsilon(\tilde{y})\le \exp\left[\,-c_0\;\varepsilon^{-1}\,dist(\tilde{y},\Gamma_\varepsilon)\,\right]$$

and the curve Γ_ε *will collapse to* Γ *as* $\varepsilon\to 0$.

Let us comment on related results, and the difficulties as well as main steps in proving Theorem 1.1.

The geometric eigenvalue problem (1.3) also appeared in the study of transition layer for the following Allen-Cahn equation

$$\varepsilon^2\Delta u+u-u^3=0 \text{ in } \Omega,\;\; \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega. \tag{1.9}$$

In an interesting paper [15], using $\Gamma-$convergence, Kohn and Sternberg constructed local minimizers to (1.9) with transition layer at straight line segment contained in Ω which locally minimizes length among all curves nearby with endpoints lying on $\partial\Omega$. (See Figure 1.) Later, M. Kowalczyk [16] extended the construction to non-minimizing line segments. More precisely, assuming that Γ satisfies (1.4), he constructed a solution u_ε whose zero set Γ_ε converges to Γ, for *all* ε sufficiently small. In [29], Pacard and Ritore constructed transition layer solutions to (1.9) near minimal submanifold on a closed Riemannian manifold.

To explain in a few words the difficulties we have encountered, let us assume for the moment that $\Omega=(-\infty,+\infty)\times[0,1]$ is an infinite strip. In terms of the stretched coordinates $(s,z)=\varepsilon^{-1}(\tilde{y}_1,\tilde{y}_2)$ the equation would look near the curve approximately like

$$v_{ss}+v_{zz}-v+v^p=0,\;(s,z)\in\mathfrak{S}:=\mathbb{R}\times(0,\frac{1}{\varepsilon}),\quad \frac{\partial v}{\partial z}=0 \text{ on } \partial\mathfrak{S}.$$

The effect of curvature and of the boundary conditions are here neglected. The linearization of this problem around the profile $w(s)$ becomes

$$\phi_{zz}+\phi_{ss}-\phi+pw^{p-1}\phi=0,\;(s,z)\in\mathfrak{S},\;\;\frac{\partial\phi}{\partial z}=0 \text{ on } \partial\mathfrak{S}.$$

Functions of the form

$$\phi^1=w_s(s)\cos(k\pi\varepsilon z),\quad \phi^2=Z(s)\cos(k\pi\varepsilon z),$$

are eigenfunctions associated to eigenvalues respectively $-k^2\varepsilon^2$ and $\lambda_0-k^2\varepsilon^2$. Many of these numbers are *small* and thus *"near non-invertibility"* of the linear operator occurs. These two effects, combined in principle orthogonally because of the L^2-orthogonality of Z and w_s, are actually coupled through the smaller order terms neglected.

In [1, 16, 17, 29], related singular perturbation problems, involving the Allen-Cahn equation (1.9), the translation effect ϕ^1 have been successfully treated through successive improvements of the approximation and fine spectral analysis of the actual linearized operator. The principle is simple: the better the approximation, higher the chances of a correct inversion of the linearized operator to obtain a contraction mapping formulation of the problem. In [24, 25] resonance phenomena similar to the "ϕ^2-effect" has been faced in the Neumann problem involving whole boundary concentration. In [21]-[22] this boundary concentration on a $k-$dimensional minimal surface of the boundary, involving both ϕ^1 and ϕ^2 effects, has been treated via arbitrary high order approximations.

In [8], M. del, Pino, M. Kowalczyk and the first author constructed the curve concentrations for nonlinear Schrödinger equation

$$\varepsilon^2 \Delta U - V(x)U + U^p = 0, \ U \in H^1(\mathbb{R}^2), \ U > 0. \tag{1.10}$$

There, they faced the coupled effect ϕ^1 and ϕ^2. They introduced a sort of *infinite Liapunov-Schmidt reduction method* which is close in spirit to that of finite dimensional Liapunov-Schmidt reduction method of Floer and Weinstein [9] and provides substantial simplification and flexibility to deal with larger noise and coupling of the non-invertibility of the linearized operator. Their idea is to solve first a natural projected problem where the linear operator is uniformly invertible, the resolution of the full problem becomes reduction to a nonlinear, nonlocal second order system of differential equations, which turns out to be directly solvable thanks to the assumptions made on the curve.

The main difficulty in our paper will come from the coupling of ϕ^1 and ϕ^2, and the boundary condition. In [8], the error term is of the order $O(\varepsilon^2)$, while here the error term is $O(\varepsilon)$ since the stretching of the boundary conditions gives $\frac{\partial \phi}{\partial z} + O(\varepsilon)$. However, the spectrum gap in (1.7) is also $O(\varepsilon)$ which creates additional difficulty. Worse than that, the spectrum gap caused by ϕ^2 and the boundary corrections are *strongly* coupled. We overcome these difficulties by first using successive improvements of the approximation and then perform the infinite-dimensional reduction to reduce the problem to two coupled nonlinear ODEs. The reduced ODEs involve coefficients of both fast and slow variables (see (7.8) and (7.13)). A careful analysis of Fourier modes is needed to ensure the invertibility.

The organization of the paper is as follows: in Section 2, we first study a linear problem in $\mathfrak{S}$ with both inhomogeneous right hand side and inhomogeneous boundary terms. Then we set up the problem in stretched variables (s, z) where $x = s - f(\varepsilon z)$, and introduce the first approximations involving two unknown functions $(f(\varepsilon z), e(\varepsilon z))$. In Section 3, a gluing procedure reduces the nonlinear problem to a projection problem on the infinite strip $\mathfrak{S}$, while in Section 4 and Section 5, we show that the projection problem has a unique solution for the pair of (f, e) in a chosen region. The final step is to adjust the parameters f and e which is equivalent to solving a nonlocal, nonlinear coupled second order system of differential equations for the pair (f, e) with boundary conditions. This is done in Section 6 and Section 7.

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2 Preliminaries and setting up the problem

In the sequel, w is the even function defined in (1.5). Z is the even eigenfunction defined in the eigenvalue problem (1.6). Throughout the paper, $\mathfrak{S}$ represents the strip $\{ (x, z) : x \in \mathbb{R}, 0 < z < \frac{1}{\varepsilon} \}$ in $\mathbb{R}^2$. $\partial_1\mathfrak{S}$ and $\partial_0\mathfrak{S}$ are two components of the boundary of $\mathfrak{S}$, i.e.

$$\partial_1\mathfrak{S} = \{ (x, z) : x \in \mathbb{R}, z = \frac{1}{\varepsilon} \}, \quad \partial_0\mathfrak{S} = \{ (x, z) : x \in \mathbb{R}, z = 0 \}.$$

2.1 A linear model problem

We first consider the following linear problem

$$\Delta\phi^0 - K\phi^0 + pw^{p-1}\phi^0 = 0 \quad \text{in } \mathfrak{S}, \tag{2.1}$$

$$\phi^0_z = G_1(x) \quad \text{on } \partial_1\mathfrak{S}, \tag{2.2}$$

$$\phi^0_z = G_0(x) \quad \text{on } \partial_0\mathfrak{S}, \tag{2.3}$$

$$\int_{\mathbb{R}} \phi^0(x, z) w_x(x) \mathrm{d}x = 0, \quad \int_{\mathbb{R}} \phi^0(x, z) Z(x) \mathrm{d}x = 0, \quad 0 < z < \frac{1}{\varepsilon}, \tag{2.4}$$

where $K > \lambda_0 + 1$ is a large positive constant. Suppose the following orthogonality conditions hold

$$\int_{\mathbb{R}} G_1(x) w_x(x) \mathrm{d}x = 0, \quad \int_{\mathbb{R}} G_0(x) w_x(x) \mathrm{d}x = 0, \tag{2.5}$$

$$\int_{\mathbb{R}} G_1(x) Z(x) \mathrm{d}x = 0, \quad \int_{\mathbb{R}} G_0(x) Z(x) \mathrm{d}x = 0. \tag{2.6}$$

Lemma 2.1. *If $G_1 \in L^2(\mathbb{R})$, $G_0 \in L^2(\mathbb{R})$ and the orthogonality conditions (2.5), (2.6) hold, then there is a unique solution ϕ^0 to the problem (2.1)-(2.4) for any large positive constant K. Moreover there is a constant $C > 0$, independent of ε, such that the solution to the problem (2.1)-(2.4) satisfies a priori estimate*

$$||\phi^0||_{H^2(\mathfrak{S})} \leq C\big(||G_1||_{L^2(\mathbb{R})} + ||G_0||_{L^2(\mathbb{R})}\big).$$

Proof. Since K is large, the proof of the existence and uniqueness of the solution and its estimate is standard. To show orthogonality in L^2, using the equations of $Z(x)$ and ϕ^0, for $\varphi(z) = \int_{\mathbb{R}} \phi^0(x,z)Z(x)\mathrm{d}x$, one finds

$$\varphi''(z) - (K - 1 - \lambda_0)\,\varphi(z) = 0, \quad 0 < z < \frac{1}{\varepsilon}, \quad \varphi'(0) = 0, \quad \varphi'(\frac{1}{\varepsilon}) = 0.$$

Choosing $K > \lambda_0 + 1$, we deduce that $\varphi(z) = \int_{\mathbb{R}} \phi^0(x,z)Z(x)\mathrm{d}x = 0, \forall\, z \in (0, \frac{1}{\varepsilon})$. Similarly we have $\int_{\mathbb{R}} \phi^0(x,z)w_x(x)\mathrm{d}x = 0, \forall\, z \in (0, \frac{1}{\varepsilon})$. □

A special case of Lemma 2.1 is the following problem: finding function $\hat{\phi} \in H^2(\mathfrak{S})$ such that

$$\Delta\hat{\phi} - \tilde{K}\hat{\phi} + pw^{p-1}\hat{\phi} = 0 \quad \text{in } \mathfrak{S}, \tag{2.7}$$

$$\hat{\phi}_z = G_1(x) \quad \text{on } \partial_1\mathfrak{S}, \tag{2.8}$$

$$\hat{\phi}_z = G_0(x) \quad \text{on } \partial_0\mathfrak{S}, \tag{2.9}$$

where $\tilde{K}$ is a large positive constant.

Lemma 2.2. *Suppose the functions $G_1(x), G_0(x)$ are even in x, then there exists a large positive constant $\tilde{K}$ such that the problem (2.7)-(2.9) has a unique solution $\hat{\phi}$, which is an even function in the variable x and satisfies*

$$||\hat{\phi}||_{H^2(\mathfrak{S})} \le C\big(||G_1||_{L^2(\mathbb{R})} + ||G_0||_{L^2(\mathbb{R})}\big).$$

Moreover, if $G_1(x), G_0(x)$ are exponentially decaying in x, then

$$\big|\hat{\phi}(x,z)\big| < Ce^{-\alpha|x|}, \tag{2.10}$$

where $\alpha > 0$ and the constant C does not depend on ε.

Proof. The existence of $\hat{\phi}$ follows from Lemma 2.1. By uniqueness and evenness of $G_1(x), G_0(x)$, $\hat{\phi}$ is even. By the exponentially decaying of $G_1(x), G_0(x)$, we also have (2.10). □

Next, we consider the following problem

$$\mathcal{L}_0(\widetilde{\phi}) \equiv \Delta\widetilde{\phi} - \widetilde{\phi} + pw^{p-1}\widetilde{\phi} = h \quad \text{in } \mathfrak{S}, \tag{2.11}$$

$$\widetilde{\phi}_z = G_1(x) \quad \text{on } \partial_1\mathfrak{S}, \tag{2.12}$$

$$\widetilde{\phi}_z = G_0(x) \quad \text{on } \partial_0\mathfrak{S}, \tag{2.13}$$

$$\int_{\mathbb{R}} \widetilde{\phi}(x,z)w_x(x)\mathrm{d}x = 0, \int_{\mathbb{R}} \widetilde{\phi}(x,z)Z(x)\mathrm{d}x = 0, \quad 0 < z < \frac{1}{\varepsilon}. \tag{2.14}$$

Lemma 2.3. *If $h \in L^2(\mathfrak{S})$, $G_1 \in L^2(\mathbb{R})$, $G_0 \in L^2(\mathbb{R})$ and the orthogonality conditions (2.5)-(2.6) hold, then for any solution $\widetilde{\phi}$ to problem (2.11)-(2.14) we have*

$$||\widetilde{\phi}||_{H^2(\mathfrak{S})} \le C\big[\ ||h||_{L^2(\mathfrak{S})} + ||G_1||_{L^2(\mathbb{R})} + ||G_0||_{L^2(\mathbb{R})}\ \big]$$

where the constant C does not depend on h, G_1, G_0 and ε. Furthermore, if $|h| + |G_0| + |G_1| \le Ce^{-\alpha|x|}$, then $|\phi| \le Ce^{-c\alpha|x|}$ for some $C, c > 0$.

Proof. Let $\phi^0(x,z)$ be defined in Lemma 2.1 and $\widetilde{\phi} = \phi^0 + \phi$. Then we have

$$\Delta\phi - \phi + pw^{p-1}\phi = h + (1-K)\phi^0 \quad \text{in } \mathfrak{S}, \tag{2.15}$$

$$\phi_z = 0 \quad \text{on } \partial_1\mathfrak{S}, \quad \phi_z = 0 \quad \text{on } \partial_0\mathfrak{S},$$

$$\int_{\mathbb{R}} \phi(x,z)w_x(x)\mathrm{d}x = 0, \quad \int_{\mathbb{R}} \phi(x,z)Z(x)\mathrm{d}x = 0, \quad 0 < z < \frac{1}{\varepsilon}.$$

Let us consider Fourier series decompositions for $h + (1-K)\phi^0$ and ϕ:

$$\phi(x,z) = \sum_{k=0}^{\infty} \phi_k(x)\cos(k\pi\varepsilon z),$$

$$h(x,z) + (1-K)\phi^0(x,z) = \sum_{k=0}^{\infty} h_k(x)\cos(k\pi\varepsilon z).$$

From the equation (2.15) we arrive at the following equations

$$-k^2\pi^2\varepsilon^2\phi_k + \phi_{k,xx} - \phi_k + pw^{p-1}\phi_k = h_k, \tag{2.16}$$

with the orthogonality condition

$$\int_{\mathbb{R}} \phi_k(x)w_x(x)\mathrm{d}x = 0, \qquad \int_{\mathbb{R}} \phi_k(x)Z(x)\mathrm{d}x = 0. \tag{2.17}$$

Let us consider the bilinear form in $H^1(\mathbb{R})$

$$B(\psi,\psi) = \int_{\mathbb{R}} [\ |\psi_x|^2 + |\psi|^2 - pw^{p-1}|\psi|^2\ ]\,\mathrm{d}x\ .$$

Since (2.17) holds uniformly in k we conclude that

$$C[\ \|\phi_k\|^2_{L^2(\mathbb{R})} + \|\phi_{k,x}\|^2_{L^2(\mathbb{R})}\ ] \le B(\phi_k,\phi_k)$$

for a constant $C > 0$ independent of k. Using this fact and equation (2.16) we arrive at

$$(1 + \pi^4k^4\varepsilon^4)\|\phi_k\|^2_{L^2(\mathbb{R})} + \|\phi_{k,x}\|^2_{L^2(\mathbb{R})} \le C\|h_k\|^2_{L^2(\mathbb{R})}. \tag{2.18}$$

Moreover, we see from (2.16) that ϕ_k satisfies an equation of the form

$$\phi_{k,xx} - \phi_k = \tilde{h}_k, \quad x \in \mathbb{R}$$

where $\|\tilde{h}_k\|_{L^2(\mathbb{R})} \leq C\|h_k\|_{L^2(\mathbb{R})}$. Hence it follows that

$$\|\phi_{k,xx}\|^2_{L^2(\mathbb{R})} \leq C\|h_k\|^2_{L^2(\mathbb{R})}. \tag{2.19}$$

Summing up estimates (2.18) and (2.19) in k, we conclude that

$$\|D^2\phi\|^2_{L^2(\mathfrak{S})} + \|D\phi\|^2_{L^2(\mathfrak{S})} + \|\phi\|^2_{L^2(\mathfrak{S})} \leq C\|h\|^2_{L^2(\mathfrak{S})}.$$

The final estimate follows from the estimates of ϕ and ϕ^0. □

A corollary of Lemma 2.3 is the following

Corollary 2.4. *Let $G_1 \in L^2(\mathbb{R})$, $G_0 \in L^2(\mathbb{R})$ satisfy the orthogonality conditions (2.5)-(2.6) and $h = 0$. Then problem (2.11)-(2.14) has a unique solution $\widetilde{\phi}$ such that*

$$||\widetilde{\phi}||_{H^2(\mathfrak{S})} \leq C\big[\ ||G_1||_{L^2(\mathbb{R})} + ||G_0||_{L^2(\mathbb{R})}\ \big]$$

where the constant C does not depend on G_1, G_0 and ε. Furthermore, if $|G_0| + |G_1| \leq Ce^{-\alpha|x|}$, then $|\phi| \leq Ce^{-c\alpha|x|}$ for some $C, c > 0$.

Proof. The proof of existence follows from the construction in Lemma 2.3. □

Finally in this subsection, we consider the following problem: given $h \in L^2(\mathfrak{S})$ and $\bar{G}_1,\ \bar{G}_0 \in L^2(\mathbb{R})$, finding functions $\hat{\phi} \in H^2(\mathfrak{S}),\ c, d \in L^2(0,1)$ and constants l_1, l_0, m_1, m_0 such that

$$\triangle\hat{\phi} - \hat{\phi} + pw^{p-1}\hat{\phi} = h + c(\varepsilon z)\,\chi(\varepsilon x)\,w_x + d(\varepsilon z)\,\chi(\varepsilon|x|)\,Z \quad \text{in } \mathfrak{S}, \tag{2.20}$$

$$\hat{\phi}_z = \bar{G}_1(x) - l_1\,\chi\,w_x - m_1\,\chi\,(\varepsilon|x|)Z \quad \text{on } \partial_1\mathfrak{S}, \tag{2.21}$$

$$\hat{\phi}_z = \bar{G}_0(x) - l_0\,\chi\,w_x - m_0\,\chi\,(\varepsilon|x|)Z \quad \text{on } \partial_0\mathfrak{S}, \tag{2.22}$$

$$\int_{\mathbb{R}} \hat{\phi}(x,z)w_x(x)\mathrm{d}x = 0, \quad \int_{\mathbb{R}} \hat{\phi}(x,z)Z(x)\mathrm{d}x = 0, \quad 0 < z < \frac{1}{\varepsilon} \tag{2.23}$$

where $\chi(t)$ a smooth cut-off function such that $\chi(t) = 1$ for $|t| \leq 10\delta$ and $\chi(t) = 0$ for $t \geq 20\delta$, and $\delta > 0$ is a small constant defined in Section 3.

Lemma 2.5. *There exist functions $c(\varepsilon z)$, $d(\varepsilon z)$ with respect to h and pairs of constants l_1, m_1 and l_0, m_0 with respect to $\bar{G}_1$ and $\bar{G}_0$ respectively such that the problem (2.20)-(2.23) has a unique solution $\hat{\phi} = T_1(h,\, \bar{G}_1,\, \bar{G}_0)$. Moreover,*

$$||\hat{\phi}||_{H^2(\mathfrak{S})} \leq C\,(||h||_{L^2(\mathfrak{S})} + ||\bar{G}_1||_{L^2(\mathbb{R})} + ||\bar{G}_0||_{L^2(\mathbb{R})}),$$

where the constant C does not depend on h, $\bar{G}_1$, $\bar{G}_0$ and ε.

Proof. Let

$$l_1 = \frac{\int_{\mathbb{R}} \bar{G}_1 w_x \mathrm{d}x}{\int_{\mathbb{R}} \chi w_x^2 \mathrm{d}x}, \quad m_1 = \frac{\int_{\mathbb{R}} \bar{G}_1 Z \mathrm{d}x}{\int_{\mathbb{R}} \chi Z^2 \mathrm{d}x},$$

$$l_0 = \frac{\int_{\mathbb{R}} \bar{G}_0 w_x \mathrm{d}x}{\int_{\mathbb{R}} \chi w_x^2 \mathrm{d}x}, \quad m_0 = \frac{\int_{\mathbb{R}} \bar{G}_0 Z \mathrm{d}x}{\int_{\mathbb{R}} \chi Z^2 \mathrm{d}x}.$$

Then the functions

$$G_1 = \bar{G_1} - l_1 \chi w_x - m_1 \chi Z, \quad G_0 = \bar{G_0} - l_0 \chi w_x - m_0 \chi Z,$$

satisfy the orthogonality conditions (2.5)-(2.6) and Lemma 2.1 is applicable. Let ϕ^0 be the function defined in Lemma 2.1 and set $\hat{\phi} = \phi^0 + \phi$. Then we can derive the following problem

$$\triangle \phi - \phi + p w^{p-1} \phi = h + (1-K)\phi^0 + c\, \chi\, w_x + d\, \chi\, Z \quad \text{in } \mathfrak{S},$$

$$\phi_z = 0 \quad \text{on } \partial_1 \mathfrak{S}, \quad \phi_z = 0 \quad \text{on } \partial_0 \mathfrak{S},$$

$$\int_{\mathbb{R}} \phi(x,z) w_x(x) \mathrm{d}x = 0, \quad \int_{\mathbb{R}} \phi(x,z) Z(x) \mathrm{d}x = 0, \quad 0 < z < \frac{1}{\varepsilon}.$$

To establish the existence, we assume that

$$h(x,z) = \sum_{k=0}^{\infty} h_k(x) \cos(k\pi\varepsilon z), \quad (1-K)\phi^0(x,z) = \sum_{k=0}^{\infty} \phi_k^0(x) \cos(k\pi\varepsilon z),$$

and consider the problem of finding $\phi_k \in H^1(\mathbb{R})$ and constants c_k, d_k such that

$$-k^2\pi^2\varepsilon^2 \phi_k + \phi_{k,xx} - \phi_k + p w^{p-1} \phi_k = h_k + c_k\, \chi\, w_x + d_k\, \chi\, Z + \phi_k^0. \tag{2.24}$$

Set

$$c_k = -\frac{\int_{\mathbb{R}} h_k w_x \mathrm{d}x}{\int_{\mathbb{R}} \chi w_x^2 \mathrm{d}x}, \quad d_k = -\frac{\int_{\mathbb{R}} h_k Z \mathrm{d}x}{\int_{\mathbb{R}} \chi Z^2 \mathrm{d}x}.$$

Then we can obtain the trivial identities

$$\int_{\mathbb{R}} (h_k + c_k\, \chi\, w_x + d_k\, \chi\, Z + \phi_k^0) w_x \mathrm{d}x = 0,$$

$$\int_{\mathbb{R}} (h_k + c_k\, \chi\, w_x + d_k\, \chi\, Z + \phi_k^0) Z \mathrm{d}x = 0.$$

Applying Fredholm's alternative to equation (2.24), we can find the existence and uniqueness of solutions ϕ_k for $k \geq 0$. Obviously,

$$\sum_{k=0}^{\infty} \left(|c_k|^2 + |d_k|^2\right) \leq C\varepsilon ||h||^2_{L^2(\mathfrak{S})}. \tag{2.25}$$

Set

$$\phi(x,z) = \sum_{k=0}^{\infty} \phi_k \cos(k\pi\varepsilon z), \quad c(\varepsilon z) = \sum_{k=0}^{\infty} c_k \cos(k\pi\varepsilon z), \quad d(\varepsilon z) = \sum_{k=0}^{\infty} d_k \cos(k\pi\varepsilon z).$$

Obviously, (2.25) shows that $c(\varepsilon z)w_x(x)$ and $d(\varepsilon z)Z(x)$ have their L^2-norms controlled by $||h||^2_{L^2}$. The a priori estimate of Lemma 2.3 gives that the series of ϕ is convergent in $H^2(\mathfrak{S})$ and define a unique solution ϕ with the properties

$$\int_{\mathbb{R}} \phi(x,z)w_x(x)\mathrm{d}x = 0, \quad \int_{\mathbb{R}} \phi(x,z)Z(x)\mathrm{d}x = 0.$$

Finally, the solution $\hat{\phi} = \phi^0 + \phi$ have the properties required in the Lemma. □

2.2 Setting up the problem

Now, we turn to the procedure of setting up the problem near Γ. By scaling $(y_1, y_2) = (\frac{\tilde{y}_1}{\varepsilon}, \frac{\tilde{y}_2}{\varepsilon})$, problem (1.1) becomes

$$\triangle u - u + u^p = 0 \text{ and } u > 0 \quad \text{in } \Omega_\varepsilon, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{2.26}$$

where $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$.

By our assumptions on Γ, there exists a small positive constant δ_0 such that we can introduce a local coordinate near $\frac{\Gamma}{\varepsilon}$,

$$(s,z) = \Big(y_1, \ \frac{y_2 - \varphi_0(\varepsilon y_1)/\varepsilon}{\varphi_1(\varepsilon y_1) - \varphi_0(\varepsilon y_1)} \Big), \ -\frac{\delta_0}{\varepsilon} < s < \frac{\delta_0}{\varepsilon}, \ 0 < z < \frac{1}{\varepsilon}. \tag{2.27}$$

The problem restricted to the region $\Omega'_\varepsilon = \{ \ (s,z) | -\frac{\delta_0}{\varepsilon} < s < \frac{\delta_0}{\varepsilon}, 0 < z < \frac{1}{\varepsilon} \ \}$ becomes

$$\begin{aligned} u_{ss} &+ \frac{1 + \big[(\varepsilon z - 1)\varphi'_0 - \varepsilon z \varphi'_1\big]^2}{(\varphi_1 - \varphi_0)^2} u_{zz} + \frac{2(\varepsilon z - 1)\varphi'_0 - 2\varepsilon z \varphi'_1}{(\varphi_1 - \varphi_0)} u_{sz} \\ &+ \varepsilon \frac{\big[(\varepsilon z - 1)\varphi''_0 - \varepsilon z \varphi''_1\big](\varphi_1 - \varphi_0) - 2\big[(\varepsilon z - 1)\varphi'_0 - \varepsilon z \varphi'_1\big](\varphi'_1 - \varphi'_0)}{(\varphi_1 - \varphi_0)^2} u_z \\ &\quad - u + u^p \ = \ 0 \quad \text{in } \Omega'_\varepsilon, \end{aligned} \tag{2.28}$$

$$\frac{\varphi'_1(\varphi_1 - \varphi_0)}{1 + (\varphi'_1)^2} u_s - u_z \ = \ 0, \quad -\frac{\delta_0}{\varepsilon} < s < \frac{\delta_0}{\varepsilon}, \ z = \frac{1}{\varepsilon}, \tag{2.29}$$

$$\frac{\varphi'_0(\varphi_1 - \varphi_0)}{1 + (\varphi'_0)^2} u_s - u_z \ = \ 0, \quad -\frac{\delta_0}{\varepsilon} < s < \frac{\delta_0}{\varepsilon}, \ z = 0. \tag{2.30}$$

Using the assumptions on Γ and Taylor's expansion, the problem near $\frac{\Gamma}{\varepsilon}$ can be rewritten as

$$u_{ss} + u_{zz} + B_1(u) - u + u^p \ = \ 0 \quad \text{in } \Omega'_\varepsilon, \tag{2.31}$$

$$k_1 \varepsilon s u_s + b_5 \varepsilon^2 s^2 u_s - u_z + D^1_0(u) \ = \ 0, \quad -\frac{\delta_0}{\varepsilon} < s < \frac{\delta_0}{\varepsilon}, \ z = \frac{1}{\varepsilon}, \tag{2.32}$$

$$k_0 \varepsilon s u_s + b_6 \varepsilon^2 s^2 u_s - u_z + D^0_0(u) \ = \ 0, \quad -\frac{\delta_0}{\varepsilon} < s < \frac{\delta_0}{\varepsilon}, \ z = 0, \tag{2.33}$$

where

$$
\begin{aligned}
B_1(u) &= \varepsilon^2 b_1 s^2 u_{zz} + \left[\, \varepsilon b_2 s + \varepsilon^2 a_1(z) s + \varepsilon^2 b_3 s^2 \,\right] u_{sz} \\
&\quad + \left[\, \varepsilon b_4 + \varepsilon^2 a_2(z) + \varepsilon^2 b_3 s \,\right] u_z + B_0(u), \qquad (2.34) \\
B_0(u) &= \left[\, \frac{1 + \left((\varepsilon z - 1)\varphi_0' - \varepsilon z \varphi_1'\right)^2}{(\varphi_1 - \varphi_0)^2} - 1 - \varepsilon^2 b_1 s^2 \,\right] u_{zz} \\
&\quad + \left[\, \frac{2(\varepsilon z - 1)\varphi_0' - 2\varepsilon z \varphi_1'}{(\varphi_1 - \varphi_0)} - b_2 \varepsilon s - a_1(z)\varepsilon^2 s - b_3 \varepsilon^2 s^2 \,\right] u_{sz} \\
&\quad + \left\{ \frac{[(\varepsilon z - 1)\varphi_0'' - \varepsilon z \varphi_1''](\varphi_1 - \varphi_0) - 2[(\varepsilon z - 1)\varphi_0' - \varepsilon z \varphi_1'](\varphi_1' - \varphi_0')}{(\varphi_1 - \varphi_0)^2} \right. \\
&\quad \left. - b_4 - a_2(z)\varepsilon - b_3 \varepsilon s \right\} \varepsilon u_z \\
&\equiv a_3(s,z) u_{zz} + a_4(s,z) u_{sz} + a_5(s,z) u_z, \qquad (2.35)
\end{aligned}
$$

$$
D_0^1(u) = \left[\frac{\varphi_1'(\varphi_1 - \varphi_0)}{1 + (\varphi_1')^2} - k_1 \varepsilon s - b_5 \varepsilon^2 s^2 \right] u_s, \qquad (2.36)
$$

$$
D_0^0(u) = \left[\frac{\varphi_0'(\varphi_1 - \varphi_0)}{1 + (\varphi_0')^2} - k_0 \varepsilon s - b_6 \varepsilon^2 s^2 \right] u_s, \qquad (2.37)
$$

$$
b_1 = k_0^2 - \frac{k_1 - k_0}{2}, \quad b_2 = -2k_0, \quad b_3 = \varphi_0'''(0), \quad b_4 = -k_0, \quad b_5 = \frac{1}{2}\varphi_1'''(0)
$$
$$
b_6 = \frac{1}{2}\varphi_0'''(0), \quad a_1(z) = 2(k_0 - k_1)z, \quad a_2(z) = (k_0 - k_1)z.
$$

Note that $B_0(u), D_0^1(u)$ and $D_0^0(u)$ are of size $O(\varepsilon^3)$.

We assume that the location of concentration of the solution is characterized by the curve $\Gamma_\varepsilon : s = f(\varepsilon z)$ in the (s,z) coordinates, where f satisfies the uniform constraint

$$
||f||_a = ||f||_{L^\infty(0,1)} + ||f'||_{L^\infty(0,1)} + ||f''||_{L^2(0,1)} \le \varepsilon^{\frac{1}{2}}. \qquad (2.38)
$$

We consider now a further change of variables as follows

$$
x = s - f(\varepsilon z), \quad z = z. \qquad (2.39)
$$

This gives the following problem on the infinite strip $\mathfrak{S}$

$$
S(u) \equiv u_{xx} + u_{zz} + B_3(u) - u + u^p = 0 \quad \text{in } \mathfrak{S}, \qquad (2.40)
$$

with boundary conditions

$$
\varepsilon(k_1 x + k_1 f + f')u_x + \varepsilon^2 b_5 (x+f)^2 u_x - u_z + D_0^1(u) = 0 \quad \text{on } \partial_1 \mathfrak{S}, \qquad (2.41)
$$

$$
\varepsilon(k_0 x + k_0 f + f')u_x + \varepsilon^2 b_6 (x+f)^2 u_x - u_z + D_0^0(u) = 0 \quad \text{on } \partial_0 \mathfrak{S}, \qquad (2.42)
$$

where

$$
\begin{aligned}
B_3(u) = & -\varepsilon 2f^{'}u_{zx} + \varepsilon b_2(x+f)u_{zx} + \varepsilon b_4 u_z \\
& + \varepsilon^2 (f^{'})^2 u_{xx} - \varepsilon^2 f^{''} u_x + \varepsilon^2 b_1 (x+f)^2 u_{zz} \\
& - \varepsilon^2 b_2 f^{'} (x+f) u_{xx} + \varepsilon^2 a_1(z)(x+f) u_{zx} + \varepsilon^2 b_3 (x+f)^2 u_{zx} \\
& - \varepsilon^2 b_4 f^{'} u_x + \varepsilon^2 a_2(z) u_z + \varepsilon^2 b_3 (x+f) u_z + B_2(u),
\end{aligned}
\tag{2.43}
$$

and

$$
\begin{aligned}
B_2(u) = & -\varepsilon^3 2b_1 (x+f)^2 f^{'} u_{xz} - \varepsilon^3 a_1(z)(x+f) f^{'} u_{xx} - \varepsilon^3 b_3 (x+f)^2 f^{'} u_{xx} \\
& - \varepsilon^3 a_2(z) f^{'} u_x - \varepsilon^3 b_3 (x+f) f^{'} u_x + \varepsilon^4 b_1 (x+f)^2 (f^{'})^2 u_{xx} \\
& - \varepsilon^4 b_1 (x+f)^2 f^{''} u_x + B_0(u).
\end{aligned}
\tag{2.44}
$$

Note that $B_2(u)$ is a term of size $O(\varepsilon^3)$. All the derivatives in $B_0(u)$, $D_0^1(u)$ and $D_0^0(u)$ are expressed in the variables (x, z).

We take $u_1 = w(x)$ as the first approximate solution of the problem in $\mathfrak{S}$. Then we compute

$$
S(w) = B_3(w) = \varepsilon^2 S_3 + \varepsilon^2 S_4 + B_2(w) \quad \text{in } \mathfrak{S}, \tag{2.45}
$$

where $S_3 = -f^{''} w_x - b_2 f^{'} x w_{xx} - b_4 f^{'} w_x$ is an odd function in the variable x, $S_4 = (f^{'})^2 w_{xx} - b_2 f^{'} f w_{xx}$ is an even function in the variable x. On the boundary, the errors become

$$
\varepsilon(k_1 x + k_1 f + f^{'}) w_x + \varepsilon^2 b_5 (x+f)^2 w_x + D_0^1(w) \quad \text{on } \partial_1 \mathfrak{S},
$$

$$
\varepsilon(k_0 x + k_0 f + f^{'}) w_x + \varepsilon^2 b_6 (x+f)^2 w_x + D_0^0(w) \quad \text{on } \partial_0 \mathfrak{S}.
$$

To cancel the term of first order of ε on the boundary (in the sense of projection against Z in L^2) and improve the approximation, we introduce a boundary layer term $\tilde{\phi}(x,z) = b(\varepsilon z) Z(x)$ with $b(z)$ satisfying

$$
\varepsilon^2 b^{''} + \lambda_0 b(z) = 0, \quad 0 < z < 1, \quad b^{'}(1) = c_1, \quad b^{'}(0) = c_0,
$$

where $c_1,\ c_0$ are constants such that

$$
c_1 = k_1 \int_{\mathbb{R}} x w_x Z \mathrm{d}x, \qquad c_0 = k_0 \int_{\mathbb{R}} x w_x Z \mathrm{d}x. \tag{2.46}
$$

It is easy to calculate directly that

$$
b(z) = \varepsilon \ \frac{c_0 \cos(\sqrt{\lambda_0}/\varepsilon) - c_1}{\sqrt{\lambda_0} \sin(\sqrt{\lambda_0}/\varepsilon)} \ \cos(\frac{\sqrt{\lambda_0}}{\varepsilon} \ z) + \varepsilon \ \frac{c_0}{\sqrt{\lambda_0}} \ \sin(\frac{\sqrt{\lambda_0}}{\varepsilon} \ z).
$$

Hence,

$$
\tilde{\phi}(x,z) = \varepsilon A(z) Z(x) \equiv \varepsilon \phi_{11}(x,z), \tag{2.47}
$$

where

$$A(z) = \frac{c_0 \cos(\sqrt{\lambda_0}/\varepsilon) - c_1}{\sqrt{\lambda_0}\sin(\sqrt{\lambda_0}/\varepsilon)} \cos(\sqrt{\lambda_0}\, z) \; + \; \frac{c_0}{\sqrt{\lambda_0}} \sin(\sqrt{\lambda_0}\, z). \tag{2.48}$$

By Corollary 2.4, there exists a unique solution (denoted by ϕ_{12}) of the following problem

$$\Delta\phi_{12} - \phi_{12} + pw^{p-1}\phi_{12} = 0 \quad \text{in } \mathfrak{S},$$

$$\frac{\partial\phi_{12}}{\partial z} = k_1 x w_x - c_1 Z \quad \text{on } \partial_1\mathfrak{S}, \quad \frac{\partial\phi_{12}}{\partial z} = k_0 x w_x - c_0 Z \quad \text{on } \partial_0\mathfrak{S}.$$

Moreover ϕ_{12} is even in x. We set

$$\phi_1 = \varepsilon\, \phi_{11}(x,z) + \varepsilon\, \chi_0(\varepsilon z)\, \phi_{12}(x,z). \tag{2.49}$$

where χ_0 is a smooth cut-off function such that $\chi_0(\eta) = 1$ if $|\eta| < \varepsilon^2$ or $|1-\eta| < \varepsilon^2$, $\chi_0(\eta) = 0$ if $2\varepsilon^2 < \eta < 1 - 2\varepsilon^2$. Note that $\phi_1(x,z)$ is size of $O(\varepsilon)$ under the gap condition (1.7).

Let $u_2 = w + \phi_1$ be the second approximate solution. We compute the new error

$$S(w+\phi_1) = S(w) + \varepsilon^2\, 2\,\chi_0^{'}\, \phi_{12,z} + \varepsilon^3 \chi_0^{''}\, \phi_{12} + B_3(\phi_1) + N_0(\phi_1) \quad \text{in } \mathfrak{S}, \tag{2.50}$$

where $N_0(\phi_1) = (w+\phi_1)^p - w^p - pw^{p-1}\phi_1$ and $S(w)$ is defined in (2.45).

On the boundary, the new errors become

$$\begin{aligned}
&\varepsilon(k_1 f + f^{'})w_x + \varepsilon^2 b_5 (x+f)^2 w_x + \varepsilon^2(k_1 x + k_1 f + f^{'})[A(\tfrac{1}{\varepsilon}) Z_x + \phi_{12,x}] \\
&\quad + \varepsilon^3 b_5 (x+f)^2 [A(\tfrac{1}{\varepsilon}) Z_x + \phi_{12,x}] + D_0^1(w+\phi_1) \quad \text{on } \partial_1\mathfrak{S}, \\
&\varepsilon(k_0 f + f^{'})w_x + \varepsilon^2 b_6 (x+f)^2 w_x + \varepsilon^2(k_0 x + k_0 f + f^{'})[A(0) Z_x + \phi_{12,x}] \\
&\quad + \varepsilon^3 b_6 (x+f)^2 [A(0) Z_x + \phi_{12,x}] + D_0^0(w+\phi_1) \quad \text{on } \partial_0\mathfrak{S}.
\end{aligned}$$

To improve the approximation for solution still keeping the term of ε^2, we need to introduce a new parameter e, in additional to f, and define the third approximate solution to the problem near $\frac{\Gamma}{\varepsilon}$ as

$$u_3 = w + \phi_1 + \varepsilon\, e(\varepsilon z) Z(x). \tag{2.51}$$

In all what follows, we shall assume the validity of the following uniform constraints on the parameter e

$$||e||_b \quad = \quad ||e||_{L^\infty(0,1)} + \varepsilon ||e^{'}||_{L^2(0,1)} + \varepsilon^2 ||e^{''}||_{L^2(0,1)} \le \varepsilon^{\frac{1}{2}}. \tag{2.52}$$

For simplicity, define

$$\digamma = \{\ (f,e)|\ \text{the functions } f \text{ and } e \text{ satisfy (2.38) and (2.52) respectively } \}. \tag{2.53}$$

To decompose the coupling of the parameters f and e on the boundary of $\mathfrak{S}$ (in the sense of projection against Z in L^2), by Lemma 2.2, we introduce a new term $\hat{\phi}$ (even in x) defined by the following problem

$$\begin{aligned} &\triangle\hat{\phi} - \tilde{K}\hat{\phi} + pw^{p-1}\hat{\phi} = 0 \quad \text{in } \mathfrak{S}, \\ &\hat{\phi}_z = 2b_5 f(1)\ xw_x + k_1 x[A(\frac{1}{\varepsilon})Z_x + \phi_{12,x}(\frac{1}{\varepsilon}, x)] \quad \text{on } \partial_1\mathfrak{S}, \\ &\hat{\phi}_z = 2b_6 f(0)\ xw_x + k_0 x[A(0)Z_x + \phi_{12,x}(0, x)] \quad \text{on } \partial_0\mathfrak{S}, \end{aligned}$$

where $\tilde{K}$ is a large positive constant. Define a boundary layer again

$$\phi_2 = \varepsilon^2\hat{\phi}. \tag{2.54}$$

ϕ_2 is an exponential decaying function which is order ε^2 and even in the variable x.

Finally our basic approximate solution to the problem near the interface $\frac{\Gamma}{\varepsilon}$ is

$$u_4 = w + \phi_1 + \varepsilon e(\varepsilon z)Z(x) + \phi_2. \tag{2.55}$$

We set up the full problem in the form $S(u_4 + \phi) = 0$ which can be expanded as follows

$$S(u_4 + \phi) = S(u_4) + \mathcal{L}_1(\phi) + B_3(\phi) + N_1(\phi) = 0 \quad \text{in } \mathfrak{S}, \tag{2.56}$$

with boundary condition

$$\begin{aligned} D_3^1(\phi) - \phi_z + D_0^1(u_4 + \phi) = g_1 \quad \text{on } \partial_1\mathfrak{S}, \\ D_3^0(\phi) - \phi_z + D_0^0(u_4 + \phi) = g_0 \quad \text{on } \partial_0\mathfrak{S}, \end{aligned} \tag{2.57}$$

where

$$\mathcal{L}_1(\phi) = \phi_{xx} + \phi_{zz} - \phi + pu_4^{p-1}\phi, \tag{2.58}$$

$$N_1(\phi) = (u_4 + \phi)^p - u_4^p - pu_4^{p-1}\phi, \tag{2.59}$$

$$D_3^1(\phi) = \varepsilon(k_1 x + k_1 f + f^{'})\phi_x + \varepsilon^2 b_5 (x + f)^2\phi_x, \tag{2.60}$$

$$D_3^0(\phi) = \varepsilon(k_0 x + k_0 f + f^{'})\phi_x + \varepsilon^2 b_6 (x + f)^2\phi_x, \tag{2.61}$$

$$\begin{aligned} g_1(x) = &-\varepsilon(k_1 f + f^{'})w_x - \varepsilon^2 b_5(x^2 + f^2)w_x + \varepsilon^2 e^{'} Z \\ &- \varepsilon^2(k_1 f + f^{'})[A(\frac{1}{\varepsilon})Z_x + \phi_{12,x}] - \varepsilon^2(k_1 x + k_1 f + f^{'})eZ_x \\ &- \varepsilon^3 b_5(x + f)^2[A(\frac{1}{\varepsilon})Z_x + \phi_{12,x}] - \varepsilon^3 b_5(x + f)^2 eZ_x \\ &- \varepsilon^3(k_1 x + k_1 f + f^{'})\hat{\phi}_x(x, \frac{1}{\varepsilon}) - \varepsilon^4 b_5(x + f)^2\hat{\phi}_x(x, \frac{1}{\varepsilon}), \end{aligned} \tag{2.62}$$

$$\begin{aligned} g_0(x) = &-\varepsilon(k_0 f + f^{'})w_x - \varepsilon^2 b_6(x^2 + f^2)w_x + \varepsilon^2 e^{'} Z \\ &- \varepsilon^2(k_0 f + f^{'})[A(0)Z_x + \phi_{12,x}] - \varepsilon^2(k_0 x + k_0 f + f^{'})eZ_x \\ &- \varepsilon^3 b_6(x + f)^2[A(0)Z_x + \phi_{12,x}] - \varepsilon^3 b_6(x + f)^2 eZ_x \\ &- \varepsilon^3(k_0 x + k_0 f + f^{'})\hat{\phi}_x(x, 0) - \varepsilon^4 b_6(x + f)^2\hat{\phi}_x(x, 0). \end{aligned} \tag{2.63}$$

The error of the approximation is

$$
\begin{aligned}
E_1 &= S(u_4) \\
&= S(w+\phi_1) + \varepsilon(\varepsilon^2 e^{''} Z + \lambda_0 e Z) + B_3(\varepsilon e Z) + \varepsilon^2(\tilde{K}-1)\hat{\phi} + \varepsilon^2 B_3(\hat{\phi}) \\
&\quad + (w+\phi_1+\varepsilon e Z + \varepsilon^2\hat{\phi})^p - (w+\phi_1)^p - p(w+\phi_1)^{p-1}\big[\ \varepsilon e Z + \varepsilon^2 \hat{\phi}\ \big] \\
&\quad + p\big[(w+\phi_1)^{p-1} - w^{p-1}\big]\big(\varepsilon e Z + \varepsilon^2\hat{\phi}\big),
\end{aligned}
\tag{2.64}
$$

where $S(w+\phi_1)$ is defined in (2.50) and the operator B_3 is given by (2.43). Moreover, we decompose

$$
E_1 = E_{11} + E_{12},\ g_0 = g_{01} + g_{02},\ g_1 = g_{11} + g_{12} \tag{2.65}
$$

with

$$
\begin{aligned}
E_{11} &= \varepsilon^3 e^{''} Z + \varepsilon\lambda_0 e Z \quad \text{and} \quad E_{12} = E_1 - E_{11}, \\
g_{01} &= -\varepsilon(k_0 f + f^{'})w_x + \varepsilon^2 e^{'} Z \quad \text{and} \quad g_{02} = g_0 - g_{01}, \\
g_{11} &= -\varepsilon(k_0 f + f^{'})w_x + \varepsilon^2 e^{'} Z \quad \text{and} \quad g_{12} = g_1 - g_{11}.
\end{aligned}
$$

For further reference, it is useful to estimate the $L^2(\mathfrak{S})$ norm of E_1. From the uniform bound of e in (2.52), it is easy to see that

$$
||E_{11}||_{L^2(\mathfrak{S})} \le C\varepsilon^{\frac{1}{2}}. \tag{2.66}
$$

Since ϕ_1 and $\varepsilon e Z$ are of size $O(\varepsilon)$, all terms in E_{12} carry ε^2 in front. We claim that

$$
||E_{12}||_{L^2(\mathfrak{S})} \le C\varepsilon^{\frac{3}{2}}. \tag{2.67}
$$

A rather delicate term in E_{12} is the one carrying $f^{''}$ since we only assume a uniform bound on $||f^{''}||_{L^2(0,1)}$. For example, we have a term $K_1 = \varepsilon^2 f^{''}$ in $S(w)$ which has bound like

$$
||K_1||_{L^2(\mathfrak{S})} \le C\varepsilon^2.
$$

Similarly, we have the following estimates

$$
||g_{02}||_{L^2(\mathbb{R})} + ||g_{12}||_{L^2(\mathbb{R})} \le C\varepsilon^{\frac{3}{2}}. \tag{2.68}
$$

Since

$$
|N_0(\phi_1)| = |(w+\phi_1)^p - w^p - p w^{p-1}\phi_1| = |p(w+t\phi_1)^{p-2}\phi_1^2|,
$$

we obtain

$$
||N_0(\phi_1)||_{L^2(\mathfrak{S})} \le C\varepsilon^{\frac{3}{2}}.
$$

Other terms can be estimated in the similar way. Moreover, for the Lipschitz dependence of the term of error E_{12} on the parameter f and e for the norm defined in (2.38) and (2.52), we have the validity of the estimate

$$
||E_{12}(f_1,e_1) - E_{12}(f_2,e_2)||_{L^2(\mathfrak{S})} \le C\varepsilon^{3/2}\big[\ ||f_1-f_2||_a + ||e_1-e_2||_b\ \big]. \tag{2.69}
$$

Similarly we obtain

$$\begin{aligned}||g_{02}(f_1,e_1)-g_{02}(f_2,e_2)||_{L^2(\mathbb{R})} \quad &+ \quad ||g_{12}(f_1,e_1)-g_{12}(f_2,e_2)||_{L^2(\mathbb{R})} \\ &\leq \quad C\varepsilon^{3/2}\big[|||f_1-f_2||_a+||e_1-e_2||_b\,\big]. \qquad (2.70)\end{aligned}$$

3 The gluing procedure

In this section, we use a gluing technique (as in [8]) to reduce the problem in $\frac{1}{\varepsilon}\Omega$ to the infinite strip $\mathfrak{S}$.

Let $\delta<\delta_0/100$ be a fixed number, where δ_0 is a constant defined in (2.27). We consider a smooth cut-off function $\eta_\delta(t)$ where $t\in\mathbb{R}_+$ such that $\eta_\delta(t)=1$ for $0\leq t\leq\delta$ and $\eta(t)=0$ for $t>2\delta$. Set $\eta_\delta^\varepsilon(s)=\eta_\delta(\varepsilon|s|)$, where s is the normal coordinate to $\frac{\Gamma}{\varepsilon}$. Let $u_4(s,z)$ denote the approximate solution constructed near the curve $\frac{\Gamma}{\varepsilon}$ in the coordinates (s,z), which was introduced in (2.27). We define our first global approximation to be simply

$$W=\eta_{3\delta}^\varepsilon(s)u_4. \tag{3.1}$$

In the coordinates (y_1,y_2) introduced in (2.26), W is a function defined on Ω_ε which is extended globally as 0 beyond the $6\delta/\varepsilon$-neighborhood of $\frac{\Gamma}{\varepsilon}$.

For $u=W+\hat{\phi}$ where $\hat{\phi}$ globally defined in Ω_ε, denote

$$S(u)=\triangle_y u-u+u^p \quad \text{in } \Omega_\varepsilon.$$

Then u satisfies (2.26) if and only if

$$\widetilde{\mathcal{L}}(\hat{\phi})=-\widetilde{E}-\widetilde{N}(\hat{\phi}) \quad \text{in } \Omega_\varepsilon, \tag{3.2}$$

with boundary condition

$$\frac{\partial\hat{\phi}}{\partial n}+\frac{\partial W}{\partial n}=0 \quad \text{on } \partial\Omega_\varepsilon, \tag{3.3}$$

where

$$\widetilde{E}=S(W),\ \widetilde{\mathcal{L}}(\hat{\phi})=\triangle_y\hat{\phi}-\hat{\phi}+pW^{p-1}\hat{\phi},\ \widetilde{N}(\hat{\phi})=(W+\hat{\phi})^p-W^p-pW^{p-1}\hat{\phi}.$$

We further separate $\hat{\phi}$ in the following form

$$\hat{\phi}=\eta_{3\delta}^\varepsilon\phi+\psi$$

where, in the coordinates (x,z) of the form (2.39), we assume that ϕ is defined in the whole strip $\mathfrak{S}$. Obviously, (3.2)-(3.3) is equivalent to the following problem

$$\eta_{3\delta}^\varepsilon\Big(\triangle_y\phi-\phi+pW^{p-1}\phi\Big)=\eta_\delta^\varepsilon\Big[-\widetilde{N}(\eta_{3\delta}^\varepsilon\phi+\psi)-\widetilde{E}-pW^{p-1}\psi\Big], \tag{3.4}$$

$$\Delta_y\psi - \psi + (1-\eta^\varepsilon_\delta)pW^{p-1}\psi = -\varepsilon^2(\Delta_y\eta^\varepsilon_{3\delta})\phi - 2\varepsilon(\nabla_y\eta^\varepsilon_{3\delta})(\nabla_y\phi) \\ - (1-\eta^\varepsilon_\delta)\widetilde{N}(\eta^\varepsilon_\delta\phi+\psi) - (1-\eta^\varepsilon_\delta)\widetilde{E}. \tag{3.5}$$

On the boundary, we get

$$\eta^\varepsilon_{3\delta}\frac{\partial\phi}{\partial n} + \eta^\varepsilon_\delta\frac{\partial W}{\partial n} = 0, \tag{3.6}$$

$$\frac{\partial\psi}{\partial n} + (1-\eta^\varepsilon_\delta)\frac{\partial W}{\partial n} + \varepsilon\frac{\partial\eta^\varepsilon_{3\delta}}{\partial n}\phi = 0. \tag{3.7}$$

The key observation is that, after solving (3.5) and (3.7), the problem can be transformed to the following nonlinear problem involving the parameter ψ

$$\widetilde{\mathcal{L}}(\phi) = \eta^\varepsilon_\delta\Big[\, -\widetilde{N}(\phi+\psi) - \widetilde{E} - pW^{p-1}\psi\,\Big] \quad \text{in } \mathfrak{S}, \tag{3.8}$$

$$\frac{\partial\phi}{\partial n} + \eta^\varepsilon_\delta\frac{\partial W}{\partial n} = 0 \quad \text{on } \partial\mathfrak{S}. \tag{3.9}$$

Notice that the operator $\widetilde{\mathcal{L}}$ in Ω_ε may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of $\frac{\Gamma}{\varepsilon}$ in the strip $\mathfrak{S}$ and the operator $\frac{\partial}{\partial n}$ may be taken as any compatible extension outside the $6\delta/\varepsilon$-neighborhood of $\frac{\Gamma}{\varepsilon}$ on the boundary $\partial\mathfrak{S}$.

First, we solve, given a small ϕ, problem (3.5) and (3.7) for ψ. Assume now that ϕ satisfies the following decay property

$$\big|\nabla\phi(y)\big| + \big|\phi(y)\big| \le e^{-\gamma/\varepsilon} \quad \text{if } \ |s| > \delta/\varepsilon, \tag{3.10}$$

for certain constant $\gamma > 0$. The solvability can be done in the following way: let us observe that W is exponentially small for $|s| > \delta/\varepsilon$, where s is the normal coordinate to $\frac{\Gamma}{\varepsilon}$. Then the problem

$$\Delta\psi - \big[1-(1-\eta^\varepsilon_\delta)pW^{p-1}\big]\psi = h \quad \text{in } \Omega_\varepsilon,$$
$$\frac{\partial\psi}{\partial n} = -(1-\eta^\varepsilon_\delta)\frac{\partial W}{\partial n} - \varepsilon\frac{\partial\eta^\varepsilon_{3\delta}}{\partial n}\phi \quad \text{on } \Omega_\varepsilon,$$

has a unique bounded solution ψ whenever $||h||_\infty \le +\infty$. Moreover,

$$||\psi||_\infty \le C||h||_\infty.$$

Since $\widetilde{N}$ is power-like with power greater than one, a direct application of contraction mapping principle yields that (3.5) and (3.7) has a unique (small) solution $\psi = \psi(\phi)$ with

$$||\psi(\phi)||_{L^\infty} \le C\varepsilon\big[\ ||\phi||_{L^\infty(|s|>\delta/\varepsilon)} + ||\nabla\phi||_{L^\infty(|s|>\delta/\varepsilon)} + e^{-\delta/\varepsilon}\ \big], \tag{3.11}$$

where $|s| > \delta/\varepsilon$ denotes the complement in Ω_ε of δ/ε-neighborhood of $\frac{\Gamma}{\varepsilon}$. Moreover, the nonlinear operator ψ satisfies a Lipschitz condition of the form

$$||\psi(\phi_1) - \psi(\phi_2)||_{L^\infty} \le C\varepsilon\big[\ ||\phi_1-\phi_2||_{L^\infty(|s|>\delta/\varepsilon)} + ||\nabla\phi_1 - \nabla\phi_2||_{L^\infty(|s|>\delta/\varepsilon)}\ \big]. \tag{3.12}$$

Therefore, from the above discussion, the full problem has been reduced to solving the following (nonlocal) problem in the infinite strip $\mathfrak{S}$

$$\mathcal{L}_2(\phi) = \eta_\delta^\varepsilon \Big[-\widetilde{N}(\phi + \psi(\phi)) - \widetilde{E} - pW^{p-1}\psi(\phi)\Big] \quad \text{in } \mathfrak{S}, \tag{3.13}$$

$$\mathcal{B}(\phi) + \eta_\delta^\varepsilon \frac{\partial W}{\partial n} = 0 \quad \text{on the boundary of the strip } \mathfrak{S}, \tag{3.14}$$

for $\phi \in H^2(\mathfrak{S})$ satisfying condition (3.10). Here $\mathcal{L}_2$ denotes a linear operator that coincides with $\widetilde{\mathcal{L}}$ on the region $|s| < 8\delta/\varepsilon$, $\mathcal{B}$ denotes the outward normal derivatives of $\mathfrak{S}$ that coincides with outward normal $\frac{\partial}{\partial n}$ of Ω_ε on the region $|s| < 8\delta/\varepsilon$.

The definitions of these operators can be showed as follows. The operator $\widetilde{\mathcal{L}}$ for $|s| < 8\delta/\varepsilon$ is given in coordinates (x, z) by formula (2.56). We extend it for functions ϕ defined in the strip $\mathfrak{S}$ in terms of (x, z) as the following

$$\mathcal{L}_2(\phi) = \mathcal{L}_1(\phi) + \chi(\varepsilon|x|)B_3(\phi) \quad \text{in } \mathfrak{S}, \tag{3.15}$$

where $\chi(r)$ is a smooth cut-off function which equals 1 for $0 \le r < 10\delta$ and vanishes identically for $r > 20\delta$, $\mathcal{L}_1$ and B_3 are the operators defined in (2.58) and (2.43). Similarly, the boundary conditions can be written as

$$\begin{aligned} &\chi(\varepsilon|x|)D_3^1(\phi) - \phi_z + \chi(\varepsilon|x|)D_0^1(W + \phi) = \chi(\varepsilon|x|)\ g_1 \quad \text{on } \partial_1\mathfrak{S}, \\ &\chi(\varepsilon|x|)D_3^0(\phi) - \phi_z + \chi(\varepsilon|x|)D_0^0(W + \phi) = \chi(\varepsilon|x|)\ g_0 \quad \text{on } \partial_0\mathfrak{S}, \end{aligned} \tag{3.16}$$

where the operators D_3^1 and D_3^0 are defined in (2.60)-(2.61) and the operators D_0^1, D_0^0 are defined in (2.36)-(2.37).

Rather than solving problem (3.13)-(3.14), we deal with the following projection problem: for each pair of parameters f and e in $\mathcal{F}$, finding functions $\phi \in H^2(\mathfrak{S})$, $c, d \in L^2(0, 1)$ and constants $l_1,\ l_0,\ m_1,\ m_0$ such that

$$\mathcal{L}_2(\phi) = -\chi E_1 - \chi N_2(\phi) + c(\varepsilon z)\ \chi w_x + d(\varepsilon z)\ \chi Z \quad \text{in } \mathfrak{S}, \tag{3.17}$$

$$\chi D_3^1(\phi) - \phi_z + \chi D_0^1(W + \phi) = \chi g_1 + l_1\ \chi w_x + m_1\ \chi Z \quad \text{on } \partial_1\mathfrak{S}, \tag{3.18}$$

$$\chi D_3^0(\phi) - \phi_z + \chi D_0^0(W + \phi) = \chi g_0 + l_0\ \chi w_x + m_0\ \chi Z \quad \text{on } \partial_0\mathfrak{S}, \tag{3.19}$$

$$\int_{\mathbb{R}} \phi(x, z)w_x(x)\mathrm{d}x = \int_{\mathbb{R}} \phi(x, z)Z(x)\mathrm{d}x = 0, \quad 0 < z < \frac{1}{\varepsilon}. \tag{3.20}$$

where $N_2(\phi) = \widetilde{N}(\phi + \psi(\phi)) + pW^{p-1}\psi(\phi)$. In Proposition 5.1, we will prove that this problem has a unique solution ϕ whose norm is controlled by the L^2-norm, not of the errors E_1, g_0, g_1, but rather of their components E_{12}, g_{02}, g_{12}. Moreover, ϕ will satisfies (3.10).

After this has been done, our task is to adjust the parameters f and e such that the functions c and d are identically zero, and the constants $l_1,\ l_0,\ m_1,\ m_0$ are zero too. It is equivalent to solving a nonlocal, nonlinear coupled second order system of differential equations for the pair (f, e) with boundary conditions. In Section 7, we will prove this system is solvable in $\mathcal{F}$.

4 The invertibility of $\mathcal{L}_2$

Let $\mathcal{L}_2$ be the operator defined in $H^2(\mathfrak{S})$ by (3.15) and g_1, g_0 be the functions in (2.62)-(2.63). Note that the function $\chi(\varepsilon|x|)$ is even in the definition of $\mathcal{L}_2$. In this section, We study the following linear problem: for given $h \in L^2(\mathfrak{S}), g_0, g_1 \in L^2(\mathbb{R})$, finding functions $\phi \in H^2(\mathfrak{S}),\ c, d \in L^2(0,1)$ and constants $l_1,\ l_0,\ m_1,\ m_0$ such that

$$\mathcal{L}_2(\phi) = h + c(\varepsilon z)\,\chi w_x + d(\varepsilon z)\,\chi Z \quad \text{in } \mathfrak{S}, \tag{4.1}$$

$$\chi D_3^1(\phi) - \phi_z + \chi D_0^1(W+\phi) = \chi g_1 + l_1\ \chi w_x + m_1\ \chi Z \quad \text{on } \partial_1\mathfrak{S}, \tag{4.2}$$

$$\chi D_3^0(\phi) - \phi_z + \chi D_0^0(W+\phi) = \chi g_0 + l_0\ \chi w_x + m_0\ \chi Z \quad \text{on } \partial_0\mathfrak{S}, \tag{4.3}$$

$$\int_{\mathbb{R}} \phi(x,z) w_x(x) \mathrm{d}x = 0, \qquad \int_{\mathbb{R}} \phi(x,z) Z(x) \mathrm{d}x = 0, \quad 0 < z < \frac{1}{\varepsilon}. \tag{4.4}$$

Proposition 4.1. *If δ in the definition of $\mathcal{L}_2$ is chosen small enough and $h \in L^2(\mathfrak{S})$, then there exists a constant $C > 0$, independent of ε, such that for all small ε, the problem (4.1)-(4.4) has a unique solution $\phi = T_2(h)$ which satisfies*

$$||\phi||_{H^2(\mathfrak{S})} \le C(||h||_{L^2(\mathfrak{S})} + ||g_{02}||_{L^2(\mathbb{R})} + ||g_{12}||_{L^2(\mathbb{R})}) \tag{4.5}$$

Moreover, if h, g_0, g_1 have compact supports contained in $|x| \le 20\delta/\varepsilon$, then

$$\big|\phi(x,z)\big| + \big|\nabla\phi(x,z)\big| \le ||\phi||_{L^\infty}\ e^{-2\delta/\varepsilon} \quad \textit{for } |x| > 40\delta/\varepsilon. \tag{4.6}$$

Proof. Note that χg_{11} and χg_{01} can be absorbed by $-l_1\ \chi w_x - m_1\ \chi Z$ and $-l_0\ \chi w_x - m_0\ \chi Z$, the problem can be written as

$$\begin{aligned} \triangle\phi - \phi + pw^{p-1}\phi = &- p(W^{p-1} - w^{p-1})\phi - \chi B_3(\phi) + h \\ &+ c(\varepsilon z)\ \chi w_x + d(\varepsilon z)\ \chi Z \quad \text{in } \mathfrak{S}, \end{aligned}$$

$$\phi_z = -\chi g_{12} + \chi D_3^1(\phi) + \chi D_0^1(W+\phi) - l_1\ \chi w_x - m_1\ \chi Z \quad \text{on } \partial_1\mathfrak{S},$$

$$\phi_z = -\chi g_{02} + \chi D_3^0(\phi) + \chi D_0^0(W+\phi) - l_0\ \chi w_x - m_0\ \chi Z \quad \text{on } \partial_0\mathfrak{S},$$

$$\int_{\mathbb{R}} \phi(x,z) w_x(x) \mathrm{d}x = 0, \qquad \int_{\mathbb{R}} \phi(x,z) Z(x) \mathrm{d}x = 0, \quad 0 < z < \frac{1}{\varepsilon}.$$

Let

$$\varphi = T_1\Big(h - p(W^{p-1} - w^{p-1})\phi - \chi B_3(\phi),\ \ \bar{G}_1,\ \ \bar{G}_0\Big)$$

where

$$\bar{G}_1(\phi) = -\chi g_{12} + \chi D_3^1(\phi) + \chi D_0^1(W+\phi),$$

$$\bar{G}_0(\phi) = -\chi g_{02} + \chi D_3^0(\phi) + \chi D_0^0(W+\phi),$$

and T_1 is the bounded operator defined by Lemma 2.5.

The key point is that the operator

$$B_4(\phi) = -\chi B_3(\phi) - p(W^{p-1} - w^{p-1})\phi$$

is small in the sense that

$$||B_4(\phi)||_{L^2(\mathfrak{S})} \le C\delta ||\phi||_{H^2(\mathfrak{S})}.$$

Similar results hold for $\bar{G}_0(\phi)$ and $\bar{G}_1(\phi)$. Hence, the results can be derived by the invertibility conclusion of Lemma 2.5 if we choose δ sufficiently small.

Since χ is supported on $|x| < 20\delta/\varepsilon$, then ϕ satisfies for $|x| > 20\delta/\varepsilon$ a problem of the form

$$\phi_{zz} + \phi_{xx} - (1 + o(1))\phi = 0 \quad |x| > 20\delta/\varepsilon, 0 < z < \frac{1}{\varepsilon},$$
$$\phi_z = 0, \quad z = \frac{1}{\varepsilon},$$
$$\phi_z = 0, \quad z = 0,$$

Hence, the validity of formula (4.6) can be showed easily.

□

5 Solving the nonlinear projection problem

In this section, we will solve (3.17)-(3.20) in $\mathfrak{S}$. A first elementary, but crucial observation is the following: The term

$$E_{11} = \varepsilon^3 e^{''} Z + \varepsilon\lambda_0 e Z$$

in the decomposition of E_1, has precisely the form $d(\varepsilon z)Z$ and can be absorbed in that term $\chi d(\varepsilon z)Z$. Then, the equivalent equation of (3.17) is

$$\mathcal{L}_2(\phi) = \chi E_{12} + \chi N_2(\phi) + c(\varepsilon z)\ \chi w_x + d(\varepsilon z)\ \chi Z$$

Similarly we can also absorb the $O(\varepsilon)$ in g_0 and g_1.

Let T_2 be the bounded operator defined by Proposition 4.1. Then the problem (3.17)-(3.20) is equivalent to the following fixed point problem

$$\phi = T_2\big(\ \chi E_{12} + \chi N_2(\phi)\ \big) \equiv \mathcal{A}(\phi). \tag{5.1}$$

We collect some useful facts to find the domain of the operator $\mathcal{A}$ such that $\mathcal{A}$ becomes a contraction mapping.

The big difference between E_{11} and E_{12} is their sizes. From (2.66) and (2.67)

$$||E_{12}||_{L^2(\mathfrak{S})} \le c_*\ \varepsilon^{3/2}, \tag{5.2}$$

while E_{11} is only of size $O(\varepsilon^{1/2})$. Similarly, we have

$$||g_{02}||_{L^2(\mathbb{R})} + ||g_{12}||_{L^2(\mathbb{R})} \le c_* \ \varepsilon^{3/2}. \tag{5.3}$$

The operator T_2 has a useful property: assume $\hat{h}$ has a support contained in $|x| \le 20\delta/\varepsilon$, then $\phi = T_2(\hat{h})$ satisfies the estimate

$$\big|\phi(x,z)\big| + \big|\nabla\phi(x,z)\big| \le ||\phi||_{L^\infty} \ e^{-2\delta/\varepsilon} \quad \text{for } |x| > 40\delta/\varepsilon. \tag{5.4}$$

Recall that the operator $\psi(\phi)$ satisfies, as seen directly from its definition

$$||\psi(\phi)||_{L^\infty} \le C\varepsilon\Big[\ \Big|\Big| \ |\phi| + |\nabla\phi| \ \Big|\Big|_{L^\infty(|x|>20\delta/\varepsilon)} + e^{-\delta/\varepsilon} \ \Big], \tag{5.5}$$

and a Lipschitz condition of the form

$$||\psi(\phi_1) - \psi(\phi_2)||_{L^\infty} \le C\varepsilon\Big[\ \Big|\Big| \ |\phi_1 - \phi_2| + |\nabla(\phi_1 - \phi_2)| \ \Big|\Big|_{L^\infty(|x|>20\delta/\varepsilon)} \ \Big]. \tag{5.6}$$

Now, the facts above will allow us to construct a region where contraction mapping principle applies and then solve the problem (3.17)-(3.20). Consider the following closed, bounded subset

$$\mathfrak{D} = \left\{ \phi \in H^2(\mathfrak{S}) \ \middle| \ \begin{array}{l} ||\phi||_{H^2(\mathfrak{S})} \le \tau\varepsilon^{3/2}, \\ \Big|\Big| \ |\phi| + |\nabla\phi| \ \Big|\Big|_{L^\infty(|x|>40\delta/\varepsilon)} \le \ ||\phi||_{H^2(\mathfrak{S})} \ e^{-\delta/\varepsilon}. \end{array} \right\} \tag{5.7}$$

We claim that if the constant τ is sufficiently large, then the map $\mathcal{A}$ defined in (5.1) is a contraction form $\mathfrak{D}$ into itself. Let us analyze the Lipschitz character of the nonlinear operator involved in $\mathcal{A}$ for functions in $\mathfrak{D}$

$$\begin{aligned} \chi N_2(\phi) \quad &= \quad \chi N_1(\phi + \psi(\phi)) + \chi p W^{p-1}\psi(\phi) \\ &\equiv \quad \bar{N}_2(\phi) + \chi p W^{p-1}\psi(\phi). \end{aligned} \tag{5.8}$$

Note that $N_1(\varphi) = p\big[(W + t\varphi)^{p-1} - W^{p-1}\big]\varphi$ for some $t \in (0,1)$. From here it follows that

$$|N_1(\varphi)| \le C(|\varphi|^p + |\varphi|^2).$$

Denoting $S_\delta = \mathfrak{S} \cap \big\{|x| < 10\delta/\varepsilon\big\}$, we have that for $\phi \in \mathfrak{D}$

$$||\bar{N}_2(\phi)||_{L^2(\mathfrak{S})} \le C\big[\ ||\phi||^p_{L^{2p}(\mathfrak{S})} + ||\phi||^2_{L^4(\mathfrak{S})} + ||\psi(\phi)||^p_{L^{2p}(S_\delta)} + ||\psi(\phi)||^2_{L^4(S_\delta)} \ \big].$$

Using Sobolev's embedding, we derive

$$||\phi||^p_{L^{2p}(\mathfrak{S})} + ||\phi||^2_{L^4(\mathfrak{S})} \le C\big(\ ||\phi||^p_{H^2(\mathfrak{S})} + ||\phi||^2_{H^2(\mathfrak{S})} \ \big).$$

Using estimates (5.4), the facts that $\phi \in \mathfrak{D}$, (5.3), that of the area of S_δ is of order $O(\delta/\varepsilon)$ and Sobolev's embedding, we get

$$||\psi(\phi)||^p_{L^{2p}(S_\delta)} + ||\psi(\phi)||^2_{L^4(S_\delta)} \le Ce^{-\delta/4\varepsilon}\Big[\ 1 + ||\phi||^p_{H^2(\mathfrak{S})} + ||\phi||^2_{H^2(\mathfrak{S})} \ \Big].$$

Hence, from the properties of W and $\psi(\phi)$ we obtain

$$||\chi N_2(\phi)||_{L^2(\mathfrak{S})} \le C(\varepsilon^{3/2}\tau^p + \varepsilon^3\tau^2). \tag{5.9}$$

As for Lipschitz condition, we find after a direct calculations

$$||N_1(\varphi_1) - N_1(\varphi_2)||_{L^2(\mathfrak{S})} \le C\Big[||\varphi_1||^{p-1}_{L^{2p}(\mathfrak{S})} + ||\varphi_1||_{L^4(\mathfrak{S})} + ||\varphi_2||^{p-1}_{L^{2p}(\mathfrak{S})} + ||\varphi_2||_{L^4(\mathfrak{S})}\Big]$$
$$\times \big(||\varphi_1 - \varphi_2||_{L^{2p}(\mathfrak{S})} + ||\varphi_1 - \varphi_2||_{L^4(\mathfrak{S})}\big).$$

Hence,

$$\begin{aligned} ||\bar{N}_2(\phi_1) - \bar{N}_2(\phi_2)||_{L^2(\mathfrak{S})} &\le ||N_1(\phi_1 + \psi(\phi_1)) - N_1(\phi_2 + \psi(\phi_1))||_{L^2(S_\delta)} \\ &\quad + ||N_1(\phi_2 + \psi(\phi_1)) - N_1(\phi_2 + \psi(\phi_2))||_{L^2(S_\delta)} \\ &\le \upsilon\Big(||\phi_1 - \phi_2||_{L^4(S_\delta)} + ||\phi_1 - \phi_2||_{L^{2p}(S_\delta)}\Big) \\ &\quad + \upsilon\Big(||\psi(\phi_1) - \psi(\phi_2)||_{L^4(S_\delta)} + ||\psi(\phi_1) - \psi(\phi_2)||_{L^{2p}(S_\delta)}\Big), \end{aligned}$$

where $\upsilon = \upsilon_1 + \upsilon_2$ with

$$\upsilon_l = ||\phi_l||^{p-1}_{L^{2p}(S_\delta)} + ||\psi(\phi_l)||^{p-1}_{L^{2p}(S_\delta)} + ||\phi_l||_{L^4(S_\delta)} + ||\psi(\phi_l)||_{L^4(S_\delta)}.$$

Arguing as above and using the Lipschitz dependence of ψ on ϕ, it can be derived

$$||\chi N_2(\phi_1) - \chi N_2(\phi_2)||_{L^2(\mathfrak{S})} \le C\big(\varepsilon^{\frac{3}{2}(p-1)}\tau^{p-1} + \varepsilon^{\frac{3}{2}}\tau\big)||\phi_1 - \phi_2||_{H^2(\mathfrak{S})}. \tag{5.10}$$

Now, we can find the solution of (5.1) in the sequel. Let $\phi \in \mathfrak{D}$ and $\nu = \mathcal{A}(\phi)$, then from (5.2)-(5.3) and (5.9)

$$||\nu||_{H^2(\mathfrak{S})} \le ||T_2||\Big[\ c_*\varepsilon^{3/2} + C\tau^p\varepsilon^{3p/2} + C\tau^2\varepsilon^3\ \Big].$$

Choosing any number $\tau > C_*||T_2||$, we get that for small ε

$$||\nu||_{H^2(\mathfrak{S})} \le \tau\varepsilon^{3/2}.$$

From (5.4)

$$\Big|\Big||\nu| + |\nabla\nu|\Big|\Big|_{L^\infty(|x|>40\delta/\varepsilon)} \le ||\nu||_\infty\ e^{\frac{-2\delta}{\varepsilon}} \le ||\nu||_{H^2(\mathfrak{S})}\ e^{\frac{-\delta}{\varepsilon}}.$$

Therefore, $\nu \in \mathfrak{D}$. $\mathcal{A}$ is clearly a contraction thanks to (5.10) and we can conclude that (5.1) has a unique solution in $\mathfrak{D}$.

The error E_{12} and the operator T_2 itself carry the functions f and e as parameters. For future reference, we should consider their Lipschitz dependence on these parameters. (2.69) is just the formula about the Lipschitz dependence of error E_{12} on these two parameters. The other task can be realized by careful and direct computations of all terms involved in the differential operator which will show this dependence is indeed Lipschitz with respect to the H^2-norm (for all ε).

Within the operator, consider for instance the following term involving f''

$$Q_f(\phi) = \varepsilon^2 f'' \phi_x.$$

Then we have

$$||Q_f(\phi)||^2_{L^2(\mathfrak{S})} \le \varepsilon^3 \int_0^1 \left|f''(\theta)\right|^2 d\theta \Big(\sup_z \int_{\mathbb{R}} |\phi_x(x,z)|^2 \mathrm{d}x\Big).$$

Let $\mu(z) = \int_{\mathbb{R}} |\phi_x(x,z)|^2 \mathrm{d}x$. Then

$$\begin{aligned}\sup_z \mu(z) &\le \varepsilon \int_{\mathbb{R}} |\phi_x|^2 \mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}} |\phi_x||\phi_{xz}|\mathrm{d}x \\ &\le \varepsilon \int_{\mathbb{R}} |\phi_x|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} |\phi_x|^2 \mathrm{d}x + \frac{4}{\varepsilon^2}\int_{\mathbb{R}} |\phi_{xz}|^2 \mathrm{d}x\end{aligned} \tag{5.11}$$

and we can obtain

$$\mu(z) \le C\varepsilon^{-2} ||\phi||^2_{H^2(\mathfrak{S})}.$$

Therefore,

$$||Q_f(\phi)||^2_{L^2(\mathfrak{S})} \le C\varepsilon ||f||_a.$$

Similar estimates can be applied to other terms in the operator involving f''.

For the linear operator T_2, we have the following Lipschitz dependence

$$||T_2(f_1) - T_2(f_2)|| \le C\varepsilon ||f_1 - f_2||_a.$$

Moreover, the operator N_2 also has Lipschitz dependence on (f, e). It is easily checked that for $\phi \in \mathfrak{D}$ we have, with obvious notation

$$||\chi N_{2,(f_1,e_1)}(\phi) - \chi N_{2,(f_2,e_2)}(\phi)||_{L^2(\mathfrak{S})} \le C\varepsilon^{5/2}\Big[\ ||f_1 - f_2||_a + ||e_1 - e_2||_b\ \Big].$$

Hence, from the fixed point characterization we get that

$$||\phi(f_1, e_1) - \phi(f_2, e_2)||_{H^2(\mathfrak{S})} \le C\varepsilon^{3/2}[\ ||f_1 - f_2||_a + ||e_1 - e_2||_b\ ]. \tag{5.12}$$

Proposition 5.1. *There is a number $\tau > 0$ such that for all ε small enough satisfying (1.7) and all parameters (f, e) in $\mathcal{F}$, problem (3.17)-(3.20) has a unique solution $\phi = \phi(f, e)$ which satisfies*

$$||\phi||_{H^2(\mathfrak{S})} \le \tau\varepsilon^{3/2},$$

$$\Big|\Big|\ |\phi| + |\nabla\phi|\ \Big|\Big|_{L^\infty(|x| > 40\delta/\varepsilon)} \le ||\phi||_{H^2(\mathfrak{S})} e^{-\delta/\varepsilon}.$$

Moreover, ϕ depends Lipschitz-continuously on the parameters f and e in the sense of the estimate (5.12).

□

6 Estimates of the projection against w_x and Z

As we mentioned in Section 3, in the next part of the paper, we will set up equations for the parameters f and e which are equivalent to making $c(\varepsilon z)$, $d(\varepsilon z)$, l_1, l_0, m_1, m_0 zero in the system (3.17)-(3.20). These equations are obtained by simply integrating the equations (3.17)-(3.19) (only in x) against w_x and Z respectively. Using the equations of w and Z in section 1 and formula (3.20) and the fact that χ is an even function in the variable x, it is easy to derive the following equations

$$\int_{\mathbb{R}}\Big[\ \chi E_1+\chi N_2(\phi)+\chi B_3(\phi)+p\big(W^{p-1}-w^{p-1}\big)\phi\ \Big]w_x\mathrm{d}x \;=\; 0, \tag{6.1}$$

$$\int_{\mathbb{R}}\Big[\ \chi E_1+\chi N_2(\phi)+\chi B_3(\phi)+p\big(W^{p-1}-w^{p-1}\big)\phi\ \Big]\ Z\mathrm{d}x \;=\; 0, \tag{6.2}$$

$$\int_{\mathbb{R}}\Big[\ \chi g_1-\chi D_3^1\big(\phi(x,1/\varepsilon)\big)+\chi D_0^1\big(\phi(x,1/\varepsilon)\big)\ \Big]w_x\mathrm{d}x \;=\; 0, \tag{6.3}$$

$$\int_{\mathbb{R}}\Big[\ \chi g_0-\chi D_3^0\big(\phi(x,0)\big)+\chi D_0^0\big(\phi(x,0)\big)\ \Big]w_x\mathrm{d}x \;=\; 0, \tag{6.4}$$

$$\int_{\mathbb{R}}\Big[\ \chi g_1-\chi D_3^1\big(\phi(x,1/\varepsilon)\big)+\chi D_0^1\big(\phi(x,1/\varepsilon)\big)\ \Big]\ Z\mathrm{d}x \;=\; 0, \tag{6.5}$$

$$\int_{\mathbb{R}}\Big[\ \chi g_0-\chi D_3^0\big(\phi(x,0)\big)+\chi D_0^0\big(\phi(x,0)\big)\ \Big]\ Z\mathrm{d}x \;=\; 0. \tag{6.6}$$

It is therefore of crucial importance to carry out computations of the estimates of the terms $\displaystyle\int_{\mathbb{R}} E_1 w_x\mathrm{d}x$ and $\displaystyle\int_{\mathbb{R}} E_1 Z\mathrm{d}x$ and, similarly, some other terms involving ϕ.

6.1 Estimates for projections of the error

We carry out some estimates for the terms $\int_{\mathbb{R}} E_1 w_x\mathrm{d}x$ and $\int_{\mathbb{R}} E_1 Z\mathrm{d}x$ in this section. For the pair (f,e) satisfying (2.38)and (2.52), denote by $b_{1\varepsilon}$ and $b_{2\varepsilon}$, generic, uniformly bounded continuous functions of the form

$$b_{l\varepsilon}=b_{l\varepsilon}\Big(\ z,\ f(\varepsilon z),\ e(\varepsilon z),\ f^{'}(\varepsilon z),\ \varepsilon e^{'}(\varepsilon z)\ \Big),\qquad l=1,2$$

where $b_{1\varepsilon}$ is uniformly Lipschitz in its four last arguments. Let $\alpha_l(z)$, $l=1,2,3$, be smooth bounded functions depending only on z.

First, multiplying (2.64) by w_x and integrating over the variable x, using the decomposition of E_1 (2.65) and the fact that the functions Z and $\hat{\phi}$ are both even in x while w_x is an odd function in x, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}} E_1 w_x \mathrm{d}x \\
&= \int_{\mathbb{R}} E_{12} w_x \mathrm{d}x \\
&= \int_{\mathbb{R}} S(w+\phi_1) w_x \mathrm{d}x + \int_{\mathbb{R}} \left[\, \varepsilon B_3(eZ) + \varepsilon^2 B_3(\hat{\phi})\, \right] w_x \mathrm{d}x \\
&\quad + \varepsilon p \int_{\mathbb{R}} \Big[\, (w+\phi_1)^{p-1} - w^{p-1}\, \Big] (\, eZ + \varepsilon \hat{\phi}\,) w_x \mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \Big[\, (w+\phi_1+\varepsilon e Z + \varepsilon^2 \hat{\phi})^p - (w+\phi_1)^p - p(w+\phi_1)^{p-1}(\varepsilon e Z + \varepsilon^2 \hat{\phi})\, \Big] w_x \mathrm{d}x \\
&\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

We calculate these terms as follows. From (2.50) and the fact that ϕ_{12} is even in the variable x, I_1 can be rewritten as

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}} S(w+\phi_1) w_x \mathrm{d}x \\
&= \int_{\mathbb{R}} S(w) w_x \mathrm{d}x + \int_{\mathbb{R}} B_3(\phi_1) w_x \mathrm{d}x + \int_{\mathbb{R}} \Big[(w+\phi_1)^p - w^p - p w^{p-1}\phi_1\Big] w_x \mathrm{d}x \\
&\equiv I_{11} + I_{12} + I_{13}.
\end{aligned}
$$

From the formula (2.45), using $b_4 - \frac{1}{2} b_2 = 0$, we get

$$
\begin{aligned}
I_{11} &= \varepsilon^2 \int_{\mathbb{R}} S_3 w_x \mathrm{d}x + \int_{\mathbb{R}} B_2(w) w_x \mathrm{d}x \\
&= -\varepsilon^2 f^{''} \int_{\mathbb{R}} w_x^2 \mathrm{d}x - \varepsilon^2 b_2 f^{'} \int_{\mathbb{R}} x w_{xx} w_x \mathrm{d}x - \varepsilon^2 b_4 f^{'} \int_{\mathbb{R}} w_x^2 \mathrm{d}x \\
&\quad - \varepsilon^4 b_1 f^{''} \int_{\mathbb{R}} x^2 w_x^2 \mathrm{d}x - \varepsilon^4 b_1 f^{''} f^2 \int_{\mathbb{R}} w_x^2 \mathrm{d}x - \varepsilon^2 f^{''} \int_{\mathbb{R}} a_3(x+f,z) w_x^2 \mathrm{d}x + \varepsilon^3 b_{1\varepsilon} \\
&= \varepsilon^2 \delta_1 f^{''} + \varepsilon^4 b_{2\varepsilon} f^{''} + \varepsilon^3 b_{1\varepsilon}
\end{aligned}
$$

where $\delta_1 = -\int_{\mathbb{R}} w_x^2 \mathrm{d}x$.

From the definitions of A and b_2, we can estimate the component $\int_{\mathbb{R}} B_3(\varepsilon\phi_{11}) w_x \mathrm{d}x$ in I_{12}

$$
\begin{aligned}
\int_{\mathbb{R}} B_3(\varepsilon\phi_{11}) w_x \mathrm{d}x &= -\varepsilon^2 2 A^{'} f^{'} \int_{\mathbb{R}} Z_x w_x \mathrm{d}x + \varepsilon^2 b_2 A^{'} f \int_{\mathbb{R}} Z_x w_x \mathrm{d}x - \varepsilon^3 A f^{''} \int_{\mathbb{R}} Z_x w_x \mathrm{d}x \\
&\quad - \varepsilon^5 b_1 A f^{''} \int_{\mathbb{R}} x^2 Z_x w_x \mathrm{d}x - \varepsilon^5 b_1 A f^2 f^{''} \int_{\mathbb{R}} Z_x w_x \mathrm{d}x \\
&\quad - \varepsilon^3 f^{''} A \int_{\mathbb{R}} a_3(x+f,z) Z_x w_x \mathrm{d}x + \varepsilon^3 b_{2\varepsilon} \\
&= \varepsilon^2 \delta_1 (\bar{\alpha}_1(z) f^{'} + \bar{\alpha}_2(z) f) + \varepsilon^3 \Big[b_{2\varepsilon} f^{''} + b_{2\varepsilon} f^{'} + b_{2\varepsilon} \Big]
\end{aligned}
$$

where

$$
\bar{\alpha}_1(z) = -2\, \frac{c_1 - c_0 \cos(\sqrt{\lambda_0}/\varepsilon)}{\delta_1 \sin(\sqrt{\lambda_0}/\varepsilon)}\, \sin(\sqrt{\lambda_0}\, z) + 2\, \frac{c_0}{\delta_1}\, \cos(\sqrt{\lambda_0}\, z) \tag{6.7}
$$

$$
\bar{\alpha}_2(z) = k_0 \bar{\alpha}_1(z). \tag{6.8}
$$

A similar computation gives

$$\int_{\mathbb{R}} B_3(\varepsilon\,\chi_0(\varepsilon\,z)\,\phi_{12})w_x\mathrm{d}x = \varepsilon^2\delta_1(\tilde{\alpha}_1(z)f^{'} + \tilde{\alpha}_2(z)f) + \varepsilon^3\Big[b_{2\varepsilon}f^{''} + b_{2\varepsilon}f^{'} + b_{2\varepsilon}\Big]$$

where

$$\begin{aligned} \tilde{\alpha}_1(z) &= -\frac{2}{\delta_1}\int_{\mathbb{R}}\chi_0(\varepsilon\,z)\,\phi_{12,zx}(x,z)w_x\mathrm{d}x, \\ \tilde{\alpha}_2(z) &= k_0\,\tilde{\alpha}_1(z). \end{aligned}$$

Hence,

$$I_{12} = \varepsilon^2\delta_1(\alpha_1(z)f^{'} + \alpha_2(z)f) + \varepsilon^3\Big[b_{2\varepsilon}f^{''} + b_{2\varepsilon}f^{'} + b_{2\varepsilon}\Big], \tag{6.9}$$

where

$$\alpha_1(z) = \bar{\alpha}_1(z) + \tilde{\alpha}_1(z), \quad \alpha_2(z) = \bar{\alpha}_2(z) + \tilde{\alpha}_2(z). \tag{6.10}$$

Since ϕ_1 is of size $O(\varepsilon)$ and even in the variable x, it follows

$$\begin{aligned} I_{13} &= \frac{1}{2}p(p-1)\int_{\mathbb{R}} w^{p-2}(\phi_1)^2 w_x\mathrm{d}x + \varepsilon^3 b_{1\varepsilon} \\ &= \varepsilon^3 b_{1\varepsilon}. \end{aligned}$$

In the term of I_2, since the terms in $\varepsilon^2 B_3(\hat{\phi})$ are of order $O(\varepsilon^3)$, we only need to find those parts in $B_3(eZ)$ which are odd in x

$$\begin{aligned} I_2 = &-2\varepsilon^3 f^{'}e^{'}\int_{\mathbb{R}} Z_x w_x\mathrm{d}x + \varepsilon^3 b_2 f e^{'}\int_{\mathbb{R}} Z_x w_x\mathrm{d}x - \varepsilon^3 f^{''}e\int_{\mathbb{R}} Z_x w_x\mathrm{d}x \\ &+ 2\varepsilon^5 b_1 f e^{''}\int_{\mathbb{R}} xZw_x\mathrm{d}x - \varepsilon^3 b_2 f' e\int_{\mathbb{R}} xZ_{xx}w_x\mathrm{d}x - \varepsilon^3 b_4 f^{'}e\int_{\mathbb{R}} Z_x w_x\mathrm{d}x \\ &- \varepsilon^5 b_1 f^{''}e\int_{\mathbb{R}} x^2 Z_x w_x\mathrm{d}x - \varepsilon^5 b_1 f^2 f^{''}e\int_{\mathbb{R}} Z_x w_x\mathrm{d}x \\ &- \varepsilon^2 f^{''}\int_{\mathbb{R}} a_3(x+f,z)w_x^2\mathrm{d}x + \varepsilon^3 b_{2\varepsilon} \\ = &\,\varepsilon^3 b_{1\varepsilon}^1 e + \varepsilon^3 b_{1\varepsilon}^2 e^{'} + \varepsilon^5 b_{1\varepsilon}e^{''} + \varepsilon^3 b_{1\varepsilon}^3 f^{''} + \varepsilon^3 b_{2\varepsilon}. \end{aligned}$$

Obviously, since ϕ_1 is a term of size $O(\varepsilon)$ and even in the variable x

$$\begin{aligned} &I_3 + I_4 \\ &= p(p-1)\Big[\varepsilon e\int_{\mathbb{R}} w^{p-2}\phi_1 Z w_x\mathrm{d}x + \frac{\varepsilon^2}{2}\int_{\mathbb{R}}(w+\phi_1)^{p-2}e^2 Z^2 w_x\mathrm{d}x\Big] + \varepsilon^3 b_{1\varepsilon} \\ &= \varepsilon^3 b_{1\varepsilon}. \end{aligned}$$

Therefore, we conclude that

$$\int_{\mathbb{R}} E_1 w_x\mathrm{d}x = \varepsilon^2\delta_1\Big[f^{''} + \alpha_1(z)f^{'} + \alpha_2(z)f\Big] + \varepsilon^3 b_{1\varepsilon}\Big[e + e^{'} + \varepsilon^2 e^{''}\Big] + \varepsilon^3 b_{2\varepsilon}f^{''} + \varepsilon^3 b_{2\varepsilon}. \tag{6.11}$$

Next, multiplying (2.64) by Z and integrating over the variable x and using the decomposition of E_1 (2.65), we get

$$\int_{\mathbb{R}} E_1 Z \mathrm{d}x = \int_{\mathbb{R}} E_{11} Z \mathrm{d}x + \int_{\mathbb{R}} E_{12} Z \mathrm{d}x$$

where

$$\int_{\mathbb{R}} E_{11} Z \mathrm{d}x = \varepsilon(\varepsilon^2 e^{''} + \lambda_0 e) \int_{\mathbb{R}} Z^2 \mathrm{d}x = \varepsilon^3 e^{''} + \varepsilon \lambda_0 e,$$

$$\begin{aligned}
&\int_{\mathbb{R}} E_{12} Z \mathrm{d}x \\
&= \int_{\mathbb{R}} S(w+\phi_1) Z \mathrm{d}x + \int_{\mathbb{R}} \left[\ \varepsilon B_3(eZ) + \varepsilon^2 B_3(\hat{\phi})\ \right] Z \mathrm{d}x \\
&\quad + \varepsilon^2 \int_{\mathbb{R}} (\tilde{K}-1)\hat{\phi} Z \mathrm{d}x + \varepsilon p \int_{\mathbb{R}} \Big[(w+\phi_1)^{p-1} - w^{p-1}\Big] \big(\ eZ + \varepsilon\hat{\phi}\ \big) Z \mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \Big[(w+\phi_1+\varepsilon e Z + \varepsilon^2\hat{\phi})^p - (w+\phi_1)^p - p(w+\phi_1)^{p-1}\big(\ \varepsilon e Z + \varepsilon^2 \hat{\phi}\ \big)\Big] Z \mathrm{d}x \\
&\equiv J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

The detailed computation for these terms are listed in the following: The formula (2.50) gives

$$\begin{aligned}
J_1 &= \int_{\mathbb{R}} S(w+\phi_1) Z \mathrm{d}x \\
&= \int_{\mathbb{R}} S(w) Z \mathrm{d}x + \varepsilon^2 \int_{\mathbb{R}} [2\,\chi_0^{'}(\varepsilon z)\,\phi_{12,z} + \varepsilon \chi_0^{''}(\varepsilon z)\,\phi_{12,z}] Z \mathrm{d}x + \int_{\mathbb{R}} B_3(\phi_1) Z \mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \Big[(w+\phi_1) - w^p - w^{p-1}\phi_1\Big] Z \mathrm{d}x \\
&= J_{11} + J_{12} + J_{13} + J_{14}.
\end{aligned}$$

We deal with the components of J_1 in the sequel. The expression (2.45) and the properties of its components of $S(w)$ give

$$\begin{aligned}
J_{11} &= \varepsilon^2 \int_{\mathbb{R}} S_4 Z \mathrm{d}x + \int_{\mathbb{R}} B_2(w) Z \mathrm{d}x \\
&= \varepsilon^2 \int_{\mathbb{R}} (f^{'})^2 w_{xx} Z \mathrm{d}x - \varepsilon^2 \int_{\mathbb{R}} b_2 f f^{'} w_{xx} Z \mathrm{d}x \\
&\quad - \varepsilon^4 2 b_1 f f^{''} \int_{\mathbb{R}} x w_x Z \mathrm{d}x - \varepsilon^2 f^{''} \int_{\mathbb{R}} a_3(x+f, z) w_x Z \mathrm{d}x + \varepsilon^3 b_{1\varepsilon} \\
&= \varepsilon^3 b_{1\varepsilon} + \varepsilon^4 b_{2\varepsilon} f^{''}.
\end{aligned}$$

For further reference, it is obvious to make the following notation

$$J_{12} = \varepsilon^2 \alpha_{41}(z) + \varepsilon^3 b_{1\varepsilon},$$

where $\alpha_{41}(z) = 2\,\chi_0^{'}(\varepsilon z) \int_{\mathbb{R}} \phi_{12,z}(x,z) Z \mathrm{d}x$.

Using the facts $b_4 = \frac{1}{2} b_2$, $a_2(z) = \frac{1}{2} a_1(z)$ and the relation $\int_{\mathbb{R}} x Z_x Z \mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}} Z^2 \mathrm{d}x$, we cancel the terms of order $O(\varepsilon^2)$ in the component $\int_{\mathbb{R}} B_3(\varepsilon\phi_{11})Z\mathrm{d}x$ in J_{12} and obtain

$$\begin{aligned}\int_{\mathbb{R}} B_3(\varepsilon\phi_{11})Z\mathrm{d}x &= \varepsilon^2 b_2 A^{'}\int_{\mathbb{R}} xZ_xZ\mathrm{d}x + \varepsilon^2 b_4 A^{'}\int_{\mathbb{R}} Z^2\mathrm{d}x \\ &+ \varepsilon^3 a_1(z)A^{'}\int_{\mathbb{R}} Z_xZ\mathrm{d}x + \varepsilon^3 a_2(z)A^{'}\int_{\mathbb{R}} Z^2\mathrm{d}x \\ &+ \varepsilon^5 2b_1 Af^{''}\int_{\mathbb{R}} xZ_xZ\mathrm{d}x - \varepsilon^3 Af^{''}\int_{\mathbb{R}} a_3(x+f,z)Z_xZ\mathrm{d}x + \varepsilon^3 b_{2\varepsilon} \\ &= \varepsilon^3 b_{1\varepsilon} + \varepsilon^5 b_{2\varepsilon} f^{''}.\end{aligned}$$

Similar computation will give

$$\int_{\mathbb{R}} B_3(\varepsilon\,\chi_0(\varepsilon z)\,\phi_{12})Z\mathrm{d}x = \varepsilon^2\alpha_{42}(z) + \varepsilon^3 b_{1\varepsilon} + \varepsilon^5 b_{2\varepsilon} f^{''}.$$

where

$$\alpha_{42}(z) = b_2\,\chi_0(\varepsilon z)\int_{\mathbb{R}} \phi_{12,zx}(x,z)xZ\mathrm{d}x + b_4\,\chi_0(\varepsilon z)\int_{\mathbb{R}} \phi_{12,z}(x,z)Z\mathrm{d}x.$$

Hence,

$$J_{12} = \varepsilon^2\alpha_{42}(z) + \varepsilon^3 b_{1\varepsilon} + \varepsilon^5 b_{2\varepsilon} f^{''}.$$

From the formula (2.47)

$$\begin{aligned}J_{13} &= \frac{1}{2}p(p-1)\int_{\mathbb{R}} w^{p-2}\phi_1^2 Z\mathrm{d}x + \varepsilon^3 b_{1\varepsilon} \\ &= \frac{\varepsilon^2}{2}p(p-1)\int_{\mathbb{R}} w^{p-2}[A(z)Z(x) + \chi_0(\varepsilon z)\phi_{12}(x,z)]^2 Z(x)\mathrm{d}x + \varepsilon^3 b_{1\varepsilon} \\ &\equiv \varepsilon^2\alpha_{43}(z) + \varepsilon^3 b_{1\varepsilon}\end{aligned}$$

In the term of J_2, since the term $\varepsilon^2 B_3(\hat{\phi})$ is of order $O(\varepsilon^3)$, we shall find those parts in $B_3(eZ)$ which are even functions in x:

$$\begin{aligned}J_2 &= \varepsilon\int_{\mathbb{R}} B_3(eZ)Z\mathrm{d}x + \varepsilon^3 b_{1\varepsilon} \\ &= \varepsilon^3 b_2 e^{'}\int_{\mathbb{R}} xZ_xZ\mathrm{d}x + \varepsilon^3 b_4 e^{'}\int_{\mathbb{R}} Z^2\mathrm{d}x - \varepsilon^5 2b_1 ff^{''}e\int_{\mathbb{R}} xZ_xZ\mathrm{d}x \\ &- \varepsilon^3 f^{''}e\int_{\mathbb{R}} a_3(x+f,z)Z_xZ\mathrm{d}x + \varepsilon\int_{\mathbb{R}} B_2(eZ)Z^2\mathrm{d}x + \varepsilon^4 b_{2\varepsilon} + \varepsilon^3 b_{1\varepsilon} \\ &= \varepsilon^3 b_{1\varepsilon} + \varepsilon^4 b_{1\varepsilon} e^{''} + \varepsilon^5 b_{2\varepsilon} f^{''}.\end{aligned}$$

where we use $\int_{\mathbb{R}} xZ_xZ\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}} Z^2\mathrm{d}x$ and $b_4 - \frac{1}{2}b_2 = 0$.

We denote

$$\begin{aligned}J_3 &= \varepsilon^2\int_{\mathbb{R}} (\tilde{K}-1)\hat{\phi}(x,z)Z\mathrm{d}x \\ &\equiv \varepsilon^2\alpha_{44}(z).\end{aligned}$$

Since ϕ_1 is of size $O(\varepsilon)$, from the assumption on e, then we get

$$
\begin{aligned}
J_4 + J_5 &= \varepsilon p(p-1)e\int_{\mathbb{R}} w^{p-2}\phi_1 Z^2 \mathrm{d}x \;+\; \frac{p(p-1)}{2}\,\varepsilon^2\int_{\mathbb{R}}(w+\phi_1)^{p-2}e^2 Z^3\mathrm{d}x \;+\; \varepsilon^3 b_{1\varepsilon}\\
&= \varepsilon^2 p(p-1)\Big\{e\int_{\mathbb{R}} w^{p-2}[AZ+\chi_0\,\phi_{12}]Z^2\mathrm{d}x \;+\; \frac{1}{2}e^2\int_{\mathbb{R}} w^{p-2}Z^3\mathrm{d}x\Big\} \;+\; \varepsilon^3 b_{1\varepsilon}\\
&= \varepsilon^2\alpha_3(z)e \;+\; \varepsilon^3 b_{2\varepsilon}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_3(z) &= \bar{\alpha}_3(z)+\tilde{\alpha}_3(z),\\
\bar{\alpha}_3(z) &= p(p-1)A\int_{\mathbb{R}} w^{p-2}Z^3\mathrm{d}x,\\
\tilde{\alpha}_3(z) &= p(p-1)\chi_0(\varepsilon z)\int_{\mathbb{R}} w^{p-2}\phi_{12}Z^2\mathrm{d}x.
\end{aligned}
\tag{6.12}
$$

Therefore, we conclude that

$$
\int_{\mathbb{R}} E_1 Z\mathrm{d}x = \varepsilon^3 e^{''} + \varepsilon\lambda_0 e + \varepsilon^2\alpha_3(z)e + \varepsilon^2\alpha_4(z) + \varepsilon^4 b_{1\varepsilon}e^{''} + \varepsilon^4 b_{2\varepsilon}f^{''} + \varepsilon^3 b_{1\varepsilon}. \tag{6.13}
$$

where

$$
\alpha_4(z) = \alpha_{41}(z)+\alpha_{42}(z)+\alpha_{43}(z)+\alpha_{44}(z). \tag{6.14}
$$

6.2 Projection of terms involving ϕ

We will estimate the terms that involve ϕ in (6.1)-(6.2) integrated against the functions w_x and Z in the variable x. Concerning w_x, we denote by $\Lambda(\phi)$ the sum of these terms, which can be decomposed as $\Lambda(\phi)=\sum_{i=1}^{3}\Lambda_i(\phi)$.

Let $\Lambda_1(\varepsilon z) = \int_{\mathbb{R}}\chi B_3(\phi)w_x\mathrm{d}x$. We make the following observation: all terms in $B_3(\phi)$ carry ε and involve powers of x times derivatives of $1,\ 2$ orders of ϕ. The conclusion is that since w_x has exponential decay then

$$
\int_0^1 \big|\Lambda_1(\theta)\big|^2 d\theta \le C\varepsilon^3||\phi||^2_{H^2(\mathfrak{S})}
$$

Hence,

$$
||\Lambda_1||_{L^2(0,1)} \le C\varepsilon^3. \tag{6.15}
$$

We shall analyze the properties of the operator Λ_1 acting on the pair (f,e) in $H^2(0,1)$. We single out two less regular terms which are operators depending Lipschitz continuously on (f,e). The one whose coefficient depends on $f^{''}$ explicitly has the form

$$
\begin{aligned}
\Lambda_{1*} &= -\varepsilon^2 f^{''}\int_{\mathbb{R}}\phi_x\Big[\frac{1+\big((\varepsilon z-1)\varphi_0^{'}-\varepsilon z\varphi_1^{'}\big)^2}{(\varphi_1-\varphi_0)^2} - 1 - \varepsilon^2 b_1(x+f)^2\Big]w_x\mathrm{d}x\\
&= \varepsilon^2 f^{''}\int_{\mathbb{R}}\phi\Big\{\Big[\frac{1+\big((\varepsilon z-1)\varphi_0^{'}-\varepsilon z\varphi_1^{'}\big)^2}{(\varphi_1-\varphi_0)^2} - 1 - \varepsilon^2 b_1(x+f)^2\Big]w_x\Big\}_x\mathrm{d}x.
\end{aligned}
$$

Since ϕ has Lipschitz dependence on (f,e) in the form (5.12), from Sobolev's embedding, we derive that

$$||\phi(f_1,e_1)-\phi(f_2,e_2)||_{L^\infty(\mathfrak{S})}\le C\varepsilon^{3/2}\big[||f_1-f_2||_a+||e_1-e_2||_b\big].$$

Hence,

$$||\Lambda_{1*}(f_1,e_1)-\Lambda_{1*}(f_2,e_2)||_{L^2(0,1)}\le C\varepsilon^{3+\frac{1}{2}}\big[||f_1-f_2||_a+||e_1-e_2||_b\big]. \tag{6.16}$$

The another one comes from second derivative of ϕ in z

$$\Lambda_{1**}=\int_{\mathbb{R}}\phi_{zz}\Big[\frac{1+\big((\varepsilon z-1)\varphi_0'-\varepsilon z\varphi_1'\big)^2}{(\varphi_1-\varphi_0)^2}-1-\varepsilon^2 b_1(x+f)^2\Big]w_x\mathrm{d}x.$$

Then

$$||\Lambda_{1**}(f_1,e_1)-\Lambda_{1**}(f_2,e_2)||_{L^2(0,1)}\le C\varepsilon^{3}\big[||f_1-f_2||_a+||e_1-e_2||_b\big]. \tag{6.17}$$

For fixed ε, the remainder $\Lambda_1-\Lambda_{1*}-\Lambda_{1**}$ actually defines a compact operator of the pair (f,e) from $H^2(0,1)$ into $L^2(0,1)$. This is a consequence of the fact that weak convergence in $H^2(\mathfrak{S})$ implies local strong convergence in $H^1(\mathfrak{S})$, and the same is the case for $H^2(0,1)$ and $C^1[0,1]$. If f_j and e_j are bounded sequences in $H^2(0,1)$, then clearly the functions $\phi(f_j,e_j)$ constitute a bounded sequence in $H^2(\mathfrak{S})$. In the above remainder we can integrate by parts once in x if necessary. Averaging against w_x which decays exponentially localizes the situation and the desired result follows.

We also observe that $\Lambda_2(\varepsilon z)=\int_{\mathbb{R}}\chi N_2(\phi)w_x\mathrm{d}x$ can be estimated similarly. Using the definition of $N_2(\phi)$ and the exponential decay of w_x we get

$$||\Lambda_2||_{L^2(0,1)}\le C\varepsilon^{\frac{1}{2}}||\phi||_{H^2(\mathfrak{S})}\le C\varepsilon^3.$$

Let us consider $\Lambda_3(\varepsilon z)=\int_{\mathbb{R}}p\big[W^{p-1}-w^{p-1}\big]\phi w_x\mathrm{d}x$. Since $W=w+\phi_1+\varepsilon eZ+\varepsilon^2\hat{\phi}_2$ and $\phi_1+\varepsilon eZ$ can be estimated as

$$\varepsilon\big|eZ\big|+\big|\phi_1(x,z)\big|\le C\varepsilon e^{-|x|},$$

we obtain that for some $\sigma>0$ the following uniform bound holds

$$\big|(W^{p-1}-w^{p-1})w_x\big|\le C\varepsilon e^{-\sigma|x|}.$$

From here we find that

$$||\Lambda_3||_{L^2(0,1)}\le C\varepsilon^{\frac{3}{2}}||\phi||_{H^2(\mathfrak{S})}\le C\varepsilon^3.$$

These two terms Λ_2 and Λ_3 also define compact operators similarly as before. We observe that exactly the same estimates can be carried out in the terms obtained from integration against Z.

6.3 Projection of errors on the boundary

In this subsection we compute the projection of error on the boundary. The main errors on the boundary integrated against w_x and Z in the variable x can be calculated as the following:

$$\begin{aligned}\int_{\mathbb{R}} g_1 w_x \mathrm{d}x &= \varepsilon\delta_1\Big[\ k_1 f(1) + f^{'}(1)\ \Big] \ +\ \varepsilon^2\delta_2 b_5 +\ \varepsilon^2\delta_1 b_5 f^2(1) \\ &\quad +\ \varepsilon^2\Big[\ k_1 f(1) + f^{'}(1)\ \Big]\{\ \delta_3 A(1) + \int_{\mathbb{R}} \phi_{12,x}(\frac{1}{\varepsilon}, x) w_x \mathrm{d}x\ \} \\ &\quad +\ \varepsilon^2\delta_3\Big[\ k_1 f(1)\ +\ f^{'}(1)\ \Big] e(1)\ +\ O(\varepsilon^3),\end{aligned}$$

where $\delta_2 = -\int_{\mathbb{R}} x^2 w_x^2 \mathrm{d}x$ and $\delta_3 = -\int_{\mathbb{R}} Z_x w_x \mathrm{d}x$.

Similarly,

$$\begin{aligned}\int_{\mathbb{R}} g_0 w_x \mathrm{d}x &= \varepsilon\delta_1\Big[\ k_0 f(0) + f^{'}(0)\ \Big] \ +\ \varepsilon^2\delta_2 b_6 +\ \varepsilon^2\delta_1 b_6 f^2(0) \\ &\quad +\ \varepsilon^2\Big[\ k_0 f(0) + f^{'}(0)\ \Big]\{\ \delta_3 A(0) + \int_{\mathbb{R}} \phi_{12,x}(0, x) w_x \mathrm{d}x\ \} \\ &\quad +\ \varepsilon^2\delta_3\Big[\ k_0 f(0)\ +\ f^{'}(0)\ \Big] e(0)\ +\ O(\varepsilon^3).\end{aligned}$$

Using the definitions of c_1 and c_0 in (2.46) to cancel the terms of order $O(\varepsilon)$ and using the following formulas

$$\int_{\mathbb{R}} Z^2 \mathrm{d}x = -2\int_{\mathbb{R}} x Z_x Z \mathrm{d}x = 1,$$

we get the following two estimates

$$\begin{aligned}\int_{\mathbb{R}} g_1 Z \mathrm{d}x &= -\ \varepsilon^2\Big[\ e^{'}(1) + \frac{1}{2}k_1 e(1)\ \Big]\ -\ \varepsilon^3 b_5 f(1)\{\ A(\frac{1}{\varepsilon}) + + \int_{\mathbb{R}} \phi_{12,x}(\frac{1}{\varepsilon}, x) w_x \mathrm{d}x\ \} \\ &\quad -\ \varepsilon^3 b_5 f(1) e(1)\ -\ \varepsilon^3 k_1 \int_{\mathbb{R}} x\hat{\phi}_x(x, \frac{1}{\varepsilon}) Z \mathrm{d}x\ +\ O(\varepsilon^4),\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}} g_0 Z \mathrm{d}x &= -\ \varepsilon^2\Big[\ e^{'}(0) + \frac{1}{2}k_0 e(0)\ \Big]\ -\ \varepsilon^3 b_6 f(0)\{\ A(0) + \int_{\mathbb{R}} \phi_{12,x}(0, x) w_x \mathrm{d}x\ \} \\ &\quad -\ \varepsilon^3 b_6 f(0) e(0)\ -\ \varepsilon^3 k_0 \int_{\mathbb{R}} x\hat{\phi}_x(x, 0) Z \mathrm{d}x\ +\ O(\varepsilon^4).\end{aligned}$$

Higher order errors can be proceeded as follows:

$$\begin{aligned}\int_{\mathbb{R}} D_3^1(\phi(x, \frac{1}{\varepsilon})) w_x \mathrm{d}x &= \varepsilon\int_{\mathbb{R}}\Big[\ k_1 x + k_1 f(1) + f^{'}(1)\ \Big]\phi_x(x, \frac{1}{\varepsilon}) w_x \mathrm{d}x \\ &\quad + \varepsilon^2 b_5 \int_{\mathbb{R}}\Big[\ x + f(1)\ \Big]^2 \phi_x(x, \frac{1}{\varepsilon}) w_x \mathrm{d}x \\ &= O(\varepsilon^{\frac{5}{2}}),\end{aligned} \tag{6.18}$$

$$\begin{aligned}\int_{\mathbb{R}} D_3^0(\phi(x, 0)) w_x \mathrm{d}x &= \varepsilon\int_{\mathbb{R}}\Big[\ k_0 x + k_0 f(0) + f^{'}(0)\ \Big]\phi_x(x, 0) w_x \mathrm{d}x \\ &\quad + \varepsilon^2 b_6 \int_{\mathbb{R}}\Big[\ x + f(0)\ \Big]^2 \phi_x(x, 0) w_x \mathrm{d}x \\ &= O(\varepsilon^{\frac{5}{2}}),\end{aligned} \tag{6.19}$$

$$
\begin{aligned}
\int_{\mathbb{R}} D_3^1\big(\phi(x,\frac{1}{\varepsilon})\big)Z\mathrm{d}x &= \varepsilon\int_{\mathbb{R}}\Big[\ k_1x + k_1f(1) + f^{'}(1)\ \Big]\phi_x(x,\frac{1}{\varepsilon})Z\mathrm{d}x \\
&\quad + \varepsilon^2 b_5\int_{\mathbb{R}}\Big[\ x + f(1)\ \Big]^2\phi_x(x,\frac{1}{\varepsilon})Z\mathrm{d}x \\
&= O(\varepsilon^{\frac{5}{2}}),
\end{aligned}
\tag{6.20}
$$

$$
\begin{aligned}
\int_{\mathbb{R}} D_3^0\big(\phi(x,0)\big)Z\mathrm{d}x &= \varepsilon\int_{\mathbb{R}}\Big[\ k_0x + k_0f(0) + f^{'}(0)\ \Big]\phi_x(x,0)Z\mathrm{d}x \\
&\quad + \varepsilon^2 b_6\int_{\mathbb{R}}\Big[\ x + f(0)\ \Big]^2\phi_x(x,0)Z\mathrm{d}x \\
&= O(\varepsilon^{\frac{5}{2}}).
\end{aligned}
\tag{6.21}
$$

The other terms $D_0^1(\phi)$ and $D_0^0(\phi)$ on the boundary integrated against w_x and Z in the variable x are of size of order $O(\varepsilon^3)$.

7 The system for (f, e): proof of the theorem

Using the estimates in previous section, we find the following nonlinear, nonlocal system of differential equations for the parameters (f,e) in the variable $\theta = \varepsilon z$

$$
\mathcal{L}_1^*(f) \equiv f^{''}(\theta) + \alpha_1(\frac{\theta}{\varepsilon})f^{'}(\theta) + \alpha_2(\frac{\theta}{\varepsilon})f(\theta) = \varepsilon M_{1\varepsilon}, \quad 0<\theta<1, \tag{7.1}
$$

$$
\mathcal{L}_2^*(e) \equiv \varepsilon^2 e^{''}(\theta) + \varepsilon\alpha_3(\frac{\theta}{\varepsilon})e(\theta) + \lambda_0 e(\theta) = \varepsilon\alpha_4(z) + \varepsilon^2 M_{2\varepsilon}, \quad 0<\theta<1, \tag{7.2}
$$

with the boundary conditions

$$
f^{'}(1) + k_1 f(1) + M_1^1(f,e) = 0, \tag{7.3}
$$

$$
f^{'}(0) + k_0 f(0) + M_0^1(f,e) = 0, \tag{7.4}
$$

$$
e^{'}(1) + \frac{1}{2}k_1 e(1) + M_1^0(f,e) = 0, \tag{7.5}
$$

$$
e^{'}(0) + \frac{1}{2}k_0 e(0) + M_0^0(f,e) = 0, \tag{7.6}
$$

where $\alpha_1(z)$ and $\alpha_2(z)$ are smooth functions defined in (6.10), $\alpha_3(z), \alpha_4(z)$ are smooth functions defined in (6.12) and (6.14) respectively. M_j^i are some terms of order $O(\varepsilon^{\frac{1}{2}})$. The operators $M_{1\varepsilon}$ and $M_{2\varepsilon}$ can be decomposed in the following form

$$
M_{l\varepsilon}(f,e) = A_{l\varepsilon}(f,e) + K_{l\varepsilon}(f,e), \quad l = 1,2
$$

where $K_{l\varepsilon}$ is uniformly bounded in $L^2(0,1)$ for (f,e) in $\mathcal{F}$ and is also compact. The operator $A_{l\varepsilon}$ is Lipschitz in this region, see (6.16)-(6.17),

$$
||A_{l\varepsilon}(f_1,e_1) - A_{l\varepsilon}(f_2,e_2)||_{L^2(0,1)} \le C\big[\ ||f_1-f_2||_a + ||e_1-e_2||_b\ \big]. \tag{7.7}
$$

Before solving (7.1)-(7.6), some basic facts about the invertibility of corresponding linear operators are derived. Firstly, we consider the following problem

$$\begin{aligned} f''(\theta) + \alpha_1(\frac{\theta}{\varepsilon})f'(\theta) + \alpha_2(\frac{\theta}{\varepsilon})f(\theta) = h(\theta), \ 0 < \theta < 1, \\ f'(1) + k_1 f(1) = 0, \\ f'(0) + k_0 f(0) = 0. \end{aligned} \tag{7.8}$$

Lemma 7.1. *Under the non-degenerate condition (1.4), if* $h \in L^2(0,1)$ *then there is a constant* ε_0 *for each* $0 < \varepsilon < \varepsilon_0$ *satisfying (1.7), the problem (7.8) has a unique solution* $f \in H^2(0,1)$ *which satisfies* $||f||_a \leq C||h||_{L^2(0,1)}$.

Proof. Under the non-degenerate condition (1.4), the existence part comes from the a priori estimate and the continuity method. Hence, we focus on the proof of the estimate. Note that if the non-degenerate condition (1.4) holds, then $\alpha_1(\frac{\theta}{\varepsilon}) \neq 0$ in the sense of the equality of two functions. There exists an orthonormal basis of $L^2(0,1)$ constituted by eigenfunctions $\{y_n\}$, associated to the eigenvalues $\{\zeta_n\}$, of the following eigenvalue problem

$$\begin{aligned} -y''(\theta) = \zeta y(\theta), \ 0 < \theta < 1, \\ y'(1) + k_1 y(1) = 0, \ \ y'(0) + k_0 y(0) = 0. \end{aligned}$$

The result in [18](on Page 9 and 10) shows that, as $n \to \infty$

$$\sqrt{\zeta_n} = n\pi + \frac{k_1 - k_0}{n\pi} + O(\frac{1}{n^3}). \tag{7.9}$$

It is easy to see that there exist a positive constant C such that $|y_n'(\theta)| \leq Cn$ for all $n \in \mathbb{N}$. We then expand

$$\begin{aligned} h(\theta) &= \sum_{n=0}^{\infty} h_n y_n(\theta), & f(\theta) &= \sum_{n=0}^{\infty} a_n y_n(\theta), \\ \bar{\alpha}_1(\frac{\theta}{\varepsilon})f'(\theta) &= \sum_{n=0}^{\infty} \bar{d}_n y_n(\theta), & \bar{\alpha}_2(\frac{\theta}{\varepsilon})f(\theta) &= \sum_{n=0}^{\infty} \bar{c}_n y_n(\theta), \\ \tilde{\alpha}_1(\frac{\theta}{\varepsilon})f'(\theta) &= \sum_{n=0}^{\infty} \tilde{d}_n y_n(\theta), & \tilde{\alpha}_2(\frac{\theta}{\varepsilon})f(\theta) &= \sum_{n=0}^{\infty} \tilde{c}_n y_n(\theta). \\ \alpha_1(\frac{\theta}{\varepsilon})f'(\theta) &= \sum_{n=0}^{\infty} d_n y_n(\theta), & \alpha_2(\frac{\theta}{\varepsilon})f(\theta) &= \sum_{n=0}^{\infty} c_n y_n(\theta). \end{aligned}$$

Now, we estimate the above Fourier coefficients. Let

$$\Phi_{n,1}(\theta) = \int_0^\theta \sin(\frac{\sqrt{\lambda_0}}{\varepsilon} s) y_n(s) ds, \qquad \Phi_{n,2}(\theta) = \int_0^\theta \cos(\frac{\sqrt{\lambda_0}}{\varepsilon} s) y_n(s) ds.$$

Using the following formula, the equation for y_n and integrating by parts two times

$$\Phi_{n,1}(\theta) = -\frac{\varepsilon^2}{\lambda_0} \int_0^\theta \Big[\sin(\frac{\sqrt{\lambda_0}}{\varepsilon} s) \Big]'' y_n(s) ds,$$

we can get

$$|\Phi_{n,1}(\theta)| \leq \frac{C\varepsilon(\varepsilon n+1)}{|\lambda_0 - \varepsilon^2\zeta_n|}. \tag{7.10}$$

Similarly, it can be derived

$$|\Phi_{n,2}(\theta)| \leq \frac{C\varepsilon(\varepsilon n+1)}{|\lambda_0 - \varepsilon^2\zeta_n|}. \tag{7.11}$$

From the gap condition (1.7) and the expression of $\bar{\alpha}_2(z)$ in formula (6.8)

$$\bar{\alpha}_2(\frac{\theta}{\varepsilon}) = \bar{\kappa}_a \sin(\frac{\sqrt{\lambda_0}}{\varepsilon}\ \theta) + \bar{\kappa}_b \cos(\frac{\sqrt{\lambda_0}}{\varepsilon}\ \theta),$$

for parameters $\bar{\kappa}_a, \bar{\kappa}_b$ depending on ε, which are also bonded by a universal constant independent of ε. Hence,

$$\bar{c}_n = \bar{\kappa}_a \int_0^1 \Phi'_{n,1}(\theta) f(\theta) d\theta + \bar{\kappa}_b \int_0^1 \Phi'_{n,2}(\theta) f(\theta) d\theta.$$

Integrating once by parts and using (7.10)-(7.11), we obtain

$$|\bar{c}_n| \leq \frac{C\ \varepsilon(\varepsilon n+1)}{|\lambda_0 - \varepsilon^2\zeta_n|} \Big\{ \ ||f'||_{L^2} \ + \ ||f||_{L^2} \ \Big\}.$$

Similar approach will implies

$$|\bar{d}_n| \leq \frac{C\ \varepsilon(\varepsilon n+1)}{|\lambda_0 - \varepsilon^2\zeta_n|} \Big\{ \ ||f'||_{L^2} \ + \ ||f||_{L^2} \ + \ ||h||_{L^2} \ \Big\}.$$

The estimate of $\tilde{d}_n$ can be showed as

$$\begin{aligned}
|\tilde{d}_n| &= \frac{2}{\delta_1} \Big| \int_0^1 \int_{\mathbb{R}} y_n(\theta) f'(\theta)\, \chi_0(\theta)\, \phi_{12,zx}(x, \frac{\theta}{\varepsilon}) w_x(x) \mathrm{d}x \mathrm{d}\theta \Big| \\
&\leq \Big[\int_0^1 \int_{\mathbb{R}} |f'(\theta) w_x(x)|^2 \mathrm{d}x \mathrm{d}\theta \Big]^{\frac{1}{2}} \times \Big\{ \int_0^1 \int_{\mathbb{R}} |\, \chi_0(\theta)\, \phi_{12,zx}(x, \frac{\theta}{\varepsilon})|^2 \mathrm{d}x \mathrm{d}\theta \Big\}^{\frac{1}{2}} \\
&\leq \sigma_1 ||f'||_{L^2},
\end{aligned}$$

where σ_1 can be chosen small if ε is sufficiently small. Similar estimate hold for $\tilde{c}_n$

$$|\tilde{c}_n| \leq \sigma_1 ||f||_{L^2}.$$

From the equations

$$\begin{aligned}
-\zeta_n a_n + c_n + d_n &= h_n, \\
\bar{c}_n + \tilde{c}_n &= c_n, \\
\bar{d}_n + \tilde{d}_n &= d_n,
\end{aligned}$$

and estimates of the Fourier coefficients $\bar{c}_n, \bar{d}_n, \tilde{c}_n, \tilde{d}_n$, we get

$$|a_n| \leq \Big| \frac{h_n}{\zeta_n} \Big| \ + \ \frac{C\ \varepsilon(\varepsilon n+1)}{|\zeta_n(\lambda_0 - \varepsilon^2\zeta_n)|} \times \Big\{ \ ||f'||_{L^2} + ||f||_{L^2} + ||h||_{L^2} \ \Big\} \ + \ \frac{\sigma_1}{\zeta_n} \times \Big\{ \ ||f'||_{L^2} + ||f||_{L^2} \Big\}.$$

Therefore, from asymptotic expression of ζ_n in (7.9) and the smallness of σ_1, we get

$$\begin{aligned}&||f^{'}||_{L^2}^2 + ||f||_{L^2}^2 \\ &\qquad \leq C||h||_{L^2}^2 \ + \ C_* \sum_n \frac{n^2 \ \varepsilon^2(\varepsilon n+1)^2}{|\zeta_n|^2(\lambda_0-\varepsilon^2\zeta_n)^2} \times \Big\{ \ ||f^{'}||_{L^2}^2 + ||f||_{L^2}^2 + ||h||_{L^2}^2 \ \Big\}.\end{aligned}$$

where the positive constant C_* does not depend on ε. From the asymptotic expression of ζ_n, there exists a positive constant ε_0 such that, for all positive $\varepsilon < \varepsilon_0$, there holds

$$\sum_{2\varepsilon^2\zeta_n>\lambda_0} \frac{n^2}{|\zeta_n|^2} \ \leq \ \frac{1}{1000C_*} \times \Big\{ \ \max{(1,\lambda_0)} \ \Big\}^{-1}.$$

For all positive $\varepsilon < \varepsilon_0$ satisfying (1.7), elementary analysis will imply the following estimates

$$\sum_{2\varepsilon^2\zeta_n\geq 3\lambda_0} \frac{n^2 \ \varepsilon^2(\varepsilon n+1)^2}{|\zeta_n|^2(\lambda_0-\varepsilon^2\zeta_n)^2} \leq \frac{1}{50C_*},$$

$$\sum_{\lambda_0<2\varepsilon^2\zeta_n<3\lambda_0} \frac{n^2 \ \varepsilon^2(\varepsilon n+1)^2}{|\zeta_n|^2(\lambda_0-\varepsilon^2\zeta_n)^2} \leq \frac{1}{50C_*}$$

$$\sum_{2\varepsilon^2\zeta_n\leq\lambda_0} \frac{n^2 \ \varepsilon^2(\varepsilon n+1)^2}{|\zeta_n|^2(\lambda_0-\varepsilon^2\zeta_n)^2} \leq C\varepsilon.$$

Since

$$\|f^{''}\|_{L^2} \leq C(\|h\|_{L^2} + \|f^{'}\|_{L^2} + \|f\|_{L^2}), \tag{7.12}$$

the final result then follows. □

Secondly, we consider the following problem

$$\begin{aligned}\varepsilon^2 e^{''}(\theta) + \varepsilon\alpha_3(\frac{\theta}{\varepsilon})e + \lambda_0 e(\theta) = g(\theta), \ 0<\theta<1 \\ e^{'}(1) + \frac{1}{2}k_1 e(1) = 0, \\ e^{'}(0) + \frac{1}{2}k_0 e(0) = 0.\end{aligned} \tag{7.13}$$

Lemma 7.2. *If $g \in L^2(0,1)$ then for all small ε satisfying (1.7) there is a unique solution $e \in H^2(0,1)$ to problem (7.13) which satisfies $||e||_b \leq C \ \varepsilon^{-1} \ ||g||_{L^2(0,1)}$. Moreover, if $g \in H^2(0,1)$ then*

$$\varepsilon^2||e^{''}||_{L^2(0,1)} + ||e^{'}||_{L^2(0,1)} + ||e||_{L^\infty(0,1)} \leq C||g||_{H^2(0,1)}.$$

Proof. The proof is similar to Lemma ??. Let $\{x_{\}}$ be an orthonormal eigenfunction basis of $L^2(0,1)$, associated to the eigenvalues $\{\xi_k\}$, of the following eigenvalue problem

$$\begin{aligned}&-y^{''}(\theta) = \xi y(\theta), \ 0<\theta<1, \\ &\quad y^{'}(1) + \frac{1}{2}k_1 y(1) = 0, \ \ y^{'}(0) + \frac{1}{2}k_0 y(0) = 0.\end{aligned}$$

By [18], as $k \to \infty$

$$\begin{aligned} \sqrt{\xi_k} &= k\pi + \frac{k_1 - k_0}{k\pi} + O(\frac{1}{k^3}). \\ x_k &= \cos(k\pi\theta) - \frac{k_1 - k_0}{k\pi}\theta \sin(k\pi\theta) - \frac{k_0}{k}\sin(k\pi\theta) + O(\frac{1}{k^2}). \end{aligned} \tag{7.14}$$

We expand

$$g(\theta) = \sum_{k=0}^{\infty} g_k x_k(\theta), \quad e(\theta) = \sum_{k=0}^{\infty} a_k x_k(\theta),$$

$$\bar{\alpha}_3(\frac{\theta}{\varepsilon})e(\theta) = \sum_{k=0}^{\infty} \bar{c}_k x_k(\theta), \quad \tilde{\alpha}_3(\frac{\theta}{\varepsilon})e(\theta) = \sum_{k=0}^{\infty} \tilde{c}_k x_k(\theta), \quad c_k = \bar{c}_k + \tilde{c}_k.$$

Now, we can use (7.14) to calculate the coefficients $\bar{c}_k$

$$\begin{aligned} \bar{c}_k &= \int_0^1 \bar{\alpha}_3(\frac{\theta}{\varepsilon})\, e(\theta)\, x_k(\theta)\mathrm{d}\theta \\ &= \tilde{\kappa}_a \int_0^1 \cos(\frac{\sqrt{\lambda_0}\,\theta}{\varepsilon})\cos(2k\pi\theta)\, e(\theta)\mathrm{d}\theta + \tilde{\kappa}_b \int_0^1 \sin(\frac{\sqrt{\lambda_0}\,\theta}{\varepsilon})\cos(2k\pi\theta)\, e(\theta)\mathrm{d}\theta + O(\frac{1}{k})||e||_{L^\infty} \\ &= \frac{1}{2}\tilde{\kappa}_b \int_0^1 \sin[(\frac{\sqrt{\lambda_0}}{\varepsilon} - 2k\pi)\theta]\, e(\theta)\mathrm{d}\theta + \frac{1}{2}\tilde{\kappa}_b \int_0^1 \sin[(\frac{\sqrt{\lambda_0}}{\varepsilon} + 2k\pi)\theta]\, e(\theta)\mathrm{d}\theta \\ &\quad + \frac{1}{2}\tilde{\kappa}_a \int_0^1 \cos[(\frac{\sqrt{\lambda_0}}{\varepsilon} - 2k\pi)\theta]\, e(\theta)\mathrm{d}\theta + \frac{1}{2}\tilde{\kappa}_a \int_0^1 \cos[(\frac{\sqrt{\lambda_0}}{\varepsilon} + 2k\pi)\theta]\, e(\theta)\mathrm{d}\theta + O(\frac{1}{k})||e||_{L^\infty} \\ &\equiv \sum_{i=1}^{4} \bar{c}_{k,i} + O(\frac{1}{k})||e||_{L^\infty}. \end{aligned}$$

Using the following formula, the equation for e and integrating by parts twice, we have

$$\bar{c}_{k,1} = -\frac{1}{2}\tilde{\kappa}_b \Big[\frac{\sqrt{\lambda_0}}{\varepsilon} - 2k\pi \Big]^{-2} \int_0^1 \Big[\sin(\frac{\sqrt{\lambda_0}}{\varepsilon} - 2k\pi)\theta \Big]'' e(\theta))ds,$$

and hence

$$|\bar{c}_{k,1}| \le C \frac{1}{2k\pi\varepsilon\, |k\pi - \frac{\sqrt{\lambda_0}}{\varepsilon}|} ||e||_{L^\infty}.$$

Similar estimates hold for other terms as well:

$$|\bar{c}_{k,4}|,\ |\bar{c}_{k,2}| \le C \frac{1}{2k\pi\varepsilon\, |k\pi + \frac{\sqrt{\lambda_0}}{\varepsilon}|} ||e||_{L^\infty}, \quad |\bar{c}_{k,3}| \le C \frac{1}{2k\pi\varepsilon\, |k\pi - \frac{\sqrt{\lambda_0}}{\varepsilon}|} ||e||_{L^\infty}.$$

Adding the last four estimates, we obtain

$$|\bar{c}_k| \le C \Big\{ \frac{1}{2k\pi\varepsilon\, |k\pi - \frac{\sqrt{\lambda_0}}{\varepsilon}|} + \frac{1}{k} \Big\} ||e||_{L^\infty}.$$

Similar to the estimate of $\tilde{d}_n$ in Lemma 7.1, we can get

$$|\tilde{c}_k| \le \varepsilon ||e||_{L^\infty}.$$

Using the equation

$$-\varepsilon^2\xi_k a_k + \varepsilon c_k + \lambda_0 a_k = g_k$$

we get

$$\begin{aligned}
|a_k| &\le \frac{\varepsilon|c_k| + |g_k|}{|\lambda_0 - \varepsilon^2\xi_k|} \\
&\le \frac{|g_k|}{|\lambda_0 - \varepsilon^2\xi_k|} + \frac{\varepsilon||e||_{L^\infty}}{k|\lambda_0 - \varepsilon^2\xi_k|} + \frac{\varepsilon||e||_{L^\infty}}{2k\pi\varepsilon\,|k\pi - \frac{\sqrt{\lambda_0}}{\varepsilon}| \cdot |\lambda_0 - \varepsilon^2\xi_k|} + \frac{\varepsilon^2||e||_{L^\infty}}{|\lambda_0 - \varepsilon^2\xi_k|} \\
&\le \frac{|g_k|}{|\lambda_0 - \varepsilon^2\xi_k|} + \frac{||e||_{L^\infty}}{k|\frac{\sqrt{\lambda_0}}{\varepsilon} - \sqrt{\xi_k}\,|} + \frac{||e||_{L^\infty}}{2k\pi\varepsilon\,|k\pi - \frac{\sqrt{\lambda_0}}{\varepsilon}| \cdot |\frac{\sqrt{\lambda_0}}{\varepsilon} - \sqrt{\xi_k}\,|} + \frac{\varepsilon^2||e||_{L^\infty}}{|\lambda_0 - \varepsilon^2\xi_k|}.
\end{aligned}$$

From the gap condition (1.7), we have

$$|\lambda_0 - \varepsilon^2\xi_k| \ge C\varepsilon. \tag{7.15}$$

Using the asymptotic expression of ξ_k in (7.14) and the gap condition (1.7), we derive

$$\sum_{\frac{\sqrt{\lambda_0}}{2\varepsilon} \le \sqrt{\xi_k} \le \frac{2\sqrt{\lambda_0}}{2\varepsilon}} \frac{1}{k\,|\frac{\sqrt{\lambda_0}}{\varepsilon} - \sqrt{\xi_k}\,|} \le C\varepsilon|\ln\varepsilon|, \tag{7.16}$$

$$\sum_{\frac{\sqrt{\lambda_0}}{2\varepsilon} \le \sqrt{\xi_k} \le \frac{3\sqrt{\lambda_0}}{2\varepsilon}} \frac{1}{2k\pi\varepsilon\,|k\pi - \frac{\sqrt{\lambda_0}}{\varepsilon}| \cdot |\frac{\sqrt{\lambda_0}}{\varepsilon} - \sqrt{\xi_k}\,|} \le C\varepsilon|\ln\varepsilon|. \tag{7.17}$$

Combining (7.15)-(7.17), some elementary calculus gives

$$\begin{aligned}
||e||_{L^\infty} &\le C\sum |a_k| \\
&\le C\varepsilon\,|\ln\varepsilon| \cdot ||e||_{L^\infty} + \varepsilon^{-1}||g||_{L^2}.
\end{aligned}$$

Hence,

$$||e||_{L^\infty} \le C\varepsilon^{-1}||g||_{L^2}. \tag{7.18}$$

Multiplying the first equation in (7.13) by e and integrating by parts, and then using (7.18), we can get

$$\varepsilon||e'||_{L^2} \le C\varepsilon^{-1}||g||_{L^2}. \tag{7.19}$$

The rest part of the estimates can be derived easily.

Under the gap condition (1.7) and (7.14), we can solve the following problem

$$\begin{aligned}
&\varepsilon^2 e''(\theta) + \lambda_0 e(\theta) = g(\theta),\ 0 < \theta < 1 \\
&e'(1) + \frac{1}{2}k_1 e(1) = 0, \\
&e'(0) + \frac{1}{2}k_0 e(0) = 0.
\end{aligned}$$

The existence part comes from the a priori estimate and the continuity method. □

Thirdly, we consider the following system

$$\begin{aligned} \mathcal{L}(f,e) \equiv (\ \mathcal{L}_1^*(f),\ \mathcal{L}_2^*(e)\ ) &= (\ h(\theta),\ g(\theta)\ ), \quad 0<\theta<1 \\ f^{'}(1)+k_1 f(1) &= \Gamma_1^1, \\ f^{'}(0)+k_0 f(0) &= \Gamma_0^1, \\ e^{'}(1)+\frac{1}{2}k_1 e(1) &= \Gamma_1^0, \\ e^{'}(0)+\frac{1}{2}k_0 e(0) &= \Gamma_0^0, \end{aligned} \tag{7.20}$$

where $\Gamma_j^i, i,j=0,1$ are some constants.

Lemma 7.3. *Under the non-degenerate condition (1.4), if* $h,\ g \in L^2(0,1)$ *then there exists* $\varepsilon_0>0$ *such that for all* $0<\varepsilon<\varepsilon_0$ *satisfying (1.7) there is a unique solution* (f,e) *in* $H^2(0,1)$ *to problem (7.20) which satisfies*

$$||f||_a + ||e||_b \le C[\ ||h||_{L^2(0,1)} + \varepsilon^{-1}||g||_{L^2(0,1)} + \Sigma_{i,j=0}^1 |\Gamma_j^i|\ ].$$

Proof. Under the non-degenerate condition (1.4) and the gap condition (1.7), there exist f_0 and e_0 satisfying

$$\begin{aligned} f_0^{''} &= 0, \\ \varepsilon^2 e_0^{''} + \lambda_0 e_0 &= 0, \\ f_0^{'}(1)+k_1 f_0(1) &= \Gamma_1^1, \\ f_0^{'}(0)+k_0 f_0(0) &= \Gamma_0^1, \\ e_0^{'}(1)+\frac{1}{2}k_1 e_0(1) &= \Gamma_1^0, \\ e_0^{'}(0)+\frac{1}{2}k_0 e_0(0) &= \Gamma_0^0. \end{aligned}$$

Setting $f=\tilde{f}+f_0,\ e=\tilde{e}+e_0$ to the system (7.20), the final conclusion can be derived from Lemma 7.1 and Lemma 7.2. □

Proof of Theorem 1.1: Let $\hat{e}$ solves

$$\begin{aligned} &\mathcal{L}_2^*(\hat{e}) = \varepsilon\alpha_2,\ 0<\theta<1 \\ &\hat{e}^{'}(1)+\frac{1}{2}k_1\hat{e}(1)=0, \\ &\hat{e}^{'}(0)+\frac{1}{2}k_0\hat{e}(0)=0. \end{aligned}$$

By Lemma 7.2, we get

$$||\hat{e}||_b \le C\varepsilon.$$

Setting $e = \hat{e} + \tilde{e}$, the system (7.1)-(7.6) keeps the same form except that the term $\varepsilon\alpha_2$ disappear. Let $(\tilde{f}, \tilde{e}) \in \mathcal{F}$, where $\mathcal{F}$ is defined in (2.53), and define

$$\begin{aligned} (\ h(f,e),\ g(f,e)\ ) &= \Big(\ \varepsilon A_{1\varepsilon}(f,e) + \varepsilon K_{1\varepsilon}(\tilde{f},\tilde{e}),\ \varepsilon^2 A_{2\varepsilon}(f,e) + \varepsilon^2 K_{2\varepsilon}(\tilde{f},\tilde{e})\ \Big), \\ \Gamma^i_j &= M^i_j(\tilde{f},\tilde{e}), \quad i,j = 0,1. \end{aligned}$$

From (7.7), $A_{1\varepsilon}$ and $A_{2\varepsilon}$ are contraction mappings of its arguments in $\mathcal{F}$. By Banach Contraction Mapping theorem and Lemma 9.3, we can solve the nonlinear problem

$$\mathcal{L}(f,e) \equiv \Big(\mathcal{L}^*_1(f), \mathcal{L}^*_2(e)\Big) = (h,g),$$

with the boundary conditions defined in (7.20) on the region $\mathcal{F}$. Hence, we can define a new operator $\mathcal{Z}$ from $\mathcal{F}$ into $\mathcal{F}$ by $\mathcal{Z}(\tilde{f},\tilde{e}) = (f,e)$. Finding a solution to the problem (7.1)-(7.6) is equivalent to locating a fixed point of $\mathcal{Z}$. Schauder's fixed point theorem applies to finish the proof of its existence. Hence, by Proposition 5.1 and the lines followed, we complete the existence part of Theorem 1.1. Other properties of u_ε in Theorem 1.1 can be showed easily. □

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