

NONRADIAL SYMMETRIC BOUND STATES FOR A SYSTEM OF COUPLED SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider bound state solutions of the coupled elliptic system

$$\begin{aligned} \Delta u - u + u^3 + \beta v^2 u &= 0 \text{ in } \mathbb{R}^N, \\ \Delta v - v + v^3 + \beta u^2 v &= 0 \text{ in } \mathbb{R}^N, \\ u > 0, v > 0, u, v &\in \mathbb{H}^1(\mathbb{R}^N) \end{aligned}$$

where $N = 2, 3$. It is known ([13]) that when $\beta < 0$, there are no ground states, i.e., no least energy solutions. We show that, for certain finite subgroups of $O(N)$ acting on $\mathbb{H}^1(\mathbb{R}^N)$, least energy solutions can be found within the associated subspaces of symmetric functions. For $\beta \leq -1$ these solutions are nonradial. From this we deduce, for every $\beta \leq -1$, the existence of infinitely many nonradial bound states of the system.

1. INTRODUCTION

In this paper, we study solitary wave solutions of time-dependent coupled nonlinear Schrödinger equations given by

$$(1.1) \quad \begin{cases} -i \frac{\partial}{\partial t} \Phi_1 = \Delta \Phi_1 + \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1 \text{ for } y \in \mathbb{R}^N, t > 0, \\ -i \frac{\partial}{\partial t} \Phi_2 = \Delta \Phi_2 + \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2 \text{ for } y \in \mathbb{R}^N, t > 0, \\ \Phi_j = \Phi_j(y, t) \in \mathbb{C}, \quad j = 1, 2, \\ \Phi_j(y, t) \rightarrow 0 \text{ as } |y| \rightarrow +\infty, t > 0, j = 1, 2 \end{cases}$$

where μ_1, μ_2 are positive constants, $n \leq 3$, and β is a coupling constant.

The system (1.1) arises in applications of many physical problems, especially in the study of incoherent solitons in nonlinear optics. We refer to [19, 20] for experimental results, and [1, 6, 10, 11, 12] for a comprehensive list of references. Physically, the solution Φ_j denotes the j -th component of the beam in Kerr-like photorefractive media. The positive constant μ_j is for self-focusing in the j -th component of the beam. The coupling constant β is the *interaction* between the first and the second component of the beam. As $\beta > 0$, the interaction is attractive, while the interaction is repulsive if $\beta < 0$.

Problem (1.1) also arises in the Hartree-Fock theory for a double condensate i.e. a binary mixture of Bose-Einstein condensates in two different hyperfine states [1]

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and $|2\rangle$ ([8]). Physically, Φ_1 and Φ_2 are the corresponding condensate amplitudes, μ_j and β are the intraspecies and interspecies scattering lengths. The sign of the scattering length β determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive. When $\beta < 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive ([24]). In contrast, when $\beta > 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are attractive. For atoms of the single state $|j\rangle$, when $\mu_j > 0$, the interactions of the single state $|j\rangle$ are attractive. To obtain solitary wave solutions of the system (1.1), we set $\Phi_1(x, t) = e^{i\lambda_j t} u(x)$, $\Phi_2(x, t) = e^{i\lambda_j t} v(x)$, and we may transform the system (1.1) to a coupled elliptic system given by

$$(1.2) \quad \begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta v^2 u = 0 & \text{in } \mathbb{R}^N, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^N. \end{cases}$$

An important class of solutions are *bound states*, namely, solution (u, v) satisfying (1.2) and the following conditions

$$(1.3) \quad u, v > 0 \text{ in } \mathbb{R}^N, \quad u(y), v(y) \rightarrow 0 \text{ as } |y| \rightarrow +\infty.$$

In [2, 3, 4, 18, 22], the existence of bound states when $\beta > 0$ is proved. Notice that in this case, all solutions of (1.2), (1.3) are radially symmetric up to translation (see [25]), and thus one may restrict to the class of *radially symmetric* functions. When $\beta < 0$, this is no longer true: a result in [16] says that when

$$(1.4) \quad N = 2, \quad \min\left(\sqrt{\frac{\lambda_1}{\lambda_2}}, \sqrt{\frac{\lambda_2}{\lambda_1}}\right) < \sin \frac{\pi}{k} \quad \text{for some } k \geq 2,$$

then, for $\beta < 0$, $|\beta|$ sufficiently small, there are positive solutions to (1.2) with one component concentrating at the center, and the other component concentrating around a regular k -polygon.

The main purpose of the present paper is to study the existence of *nonradial* solutions in the case where $\beta < 0$, and $\lambda_1 = \lambda_2, \mu_1 = \mu_2$. Note that in this case (1.4) fails, so that the result of [16] does not apply. Without loss of generality, we may assume that $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. Namely we consider the following system of elliptic equations

$$(1.5) \quad \begin{cases} \Delta u - u + u^3 + \beta v^2 u = 0 & \text{in } \mathbb{R}^N, \\ \Delta v - v + v^3 + \beta u^2 v = 0 & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \quad u(y), v(y) \rightarrow 0 \text{ as } |y| \rightarrow +\infty. \end{cases}$$

Solutions of (1.5) are critical points of the energy functional $E : (\mathbb{H}^1(\mathbb{R}^N))^2 \rightarrow \mathbb{R}$ defined by

$$E[u, v] = \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{1}{4} \int_{\mathbb{R}^N} (u^4 + v^4) - \frac{\beta}{2} \int_{\mathbb{R}^N} u^2 v^2,$$

where $\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx$ for $u \in \mathbb{H}^1(\mathbb{R}^N)$. All nontrivial solutions of (1.5) belong to the Nehari set

$$\mathbf{N} = \left\{ (u, v) \in (\mathbb{H}^1(\mathbb{R}^N))^2 : u, v \geq 0, u, v \not\equiv 0, \right. \\ \left. \|u\|^2 = \int_{\mathbb{R}^N} (u^4 + \beta u^2 v^2), \|v\|^2 = \int_{\mathbb{R}^N} (v^4 + \beta u^2 v^2) \right\}$$

A solution (\bar{u}, \bar{v}) of (1.5) is called a *ground state* if $E(\bar{u}, \bar{v}) = c_0$, where

$$(1.6) \quad c_0 = \inf_{(u,v) \in \mathbf{N}} E[u, v].$$

In particular, $E(\bar{u}, \bar{v}) \leq E(u, v)$ for any nontrivial solution (u, v) of (1.5). Concerning the existence of ground states, it was proved in [14] that c_0 is attained for $\beta > 0$ small, whereas c_0 is not attained for any $\beta < 0$. To explain this phenomenon, it is worth pointing out that E fails to satisfy the Palais-Smale condition since the embedding $\mathbb{H}^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$ is not compact. Moreover, for $\beta < 0$ the interaction of the two species is repulsive. Therefore a spatial separation of u and v in \mathbb{R}^N is observed for $(u, v) \in \mathbf{N}$ with energy close to c_0 . In fact, the repulsion of u and v seems to be closely related to the repulsion of positive and negative bumps in the study of sign changing solutions of the single equation $-\Delta u + u = u^3$ in \mathbb{R}^N , see e.g. [26].

For $\beta > -1$, (1.5) admits the scalar solutions

$$(1.7) \quad (u, v) = \frac{1}{\sqrt{1+\beta}}(w_0, w_0),$$

(and its translations), where $w_0 \in \mathbb{H}^1(\mathbb{R}^N)$ is the unique solution of the scalar elliptic problem

$$(1.8) \quad \begin{cases} -\Delta w + w = w^3, & w > 0 \text{ in } \mathbb{R}^N, \\ w(0) = \max_{y \in \mathbb{R}^N} w(y), & w \in \mathbb{H}^1(\mathbb{R}^N), \end{cases}$$

cf. [7, 9]. As remarked above, these solutions are not ground states for $-1 < \beta < 0$. For $\beta \leq -1$, (1.5) does not admit any solutions with $u = v$. Indeed, for $\beta \leq -1$, it is evident that

$$(1.9) \quad u \neq v \quad \text{for every } (u, v) \in \mathbf{N}.$$

In the present paper we prove, for any $\beta < 0$, the existence of ground states within spaces of functions invariant under the action of a finite subgroup \mathcal{G} of $O(N)$. In these spaces, E still fails to satisfy the Palais-Smale condition, but we recover compactness of energy minimizing sequences by balancing the self-attraction of the single species with the repulsion of different species and by applying concentration compactness arguments. To state our main results, we recall some notation for

a (nontrivial) finite subgroup $\mathcal{G} \leq O(N)$. We set $\mathcal{G}x = \{Ax : A \in \mathcal{G}\} \subset \mathbb{R}^N$ and $\mathcal{G}^x := \{A \in \mathcal{G} : Ax = x\} \subset \mathcal{G}$ for $x \in \mathbb{R}^N$, and we denote by $|\mathcal{G}x|$ resp. $|\mathcal{G}^x|$ the number of elements in $\mathcal{G}x$, \mathcal{G}^x , respectively. Moreover, we set $\text{Fix}(\mathcal{G}) = \{x \in \mathbb{R}^N : \mathcal{G}x = \{x\}\}$ which is a subspace of \mathbb{R}^N , and we write $V_{\mathcal{G}} = \text{Fix}(\mathcal{G})^\perp$ for the orthogonal complement of $\text{Fix}(\mathcal{G})$ in \mathbb{R}^N . Finally, we set $l(\mathcal{G}) = \min\{|\mathcal{G}y| : y \in V_{\mathcal{G}} \setminus \{0\}\}$. If $u \in \mathbb{H}^1(\mathbb{R}^N)$, we say that u is \mathcal{G} -symmetric if $u(Ax) = u(x)$ for every $A \in \mathcal{G}$, and we put

$$H_{\mathcal{G}} = \{u \in \mathbb{H}^1(\mathbb{R}^N) : u \text{ is } \mathcal{G}\text{-symmetric}\}$$

We introduce the following definition.

Definition 1.1. *Let $B \in O(N)$, and let $\mathcal{G} \leq O(N)$ be a finite subgroup of $O(N)$. We call the pair (B, \mathcal{G}) admissible if*

- (a) *B is contained in the normalizer of \mathcal{G} , and $B^2 \in \mathcal{G}$;*
- (b) *$Bx = x$ for every $x \in \text{Fix}(\mathcal{G})$.*
- (c) *there exists $x_0 \in V_{\mathcal{G}} \setminus \{0\}$ with*
 - (c1) $|\mathcal{G}x_0| = l(\mathcal{G})$,
 - (c2) $\min_{A \in \mathcal{G} \setminus \mathcal{G}^{x_0}} |x_0 - Ax_0| < 2 \min_{A \in \mathcal{G}} |x_0 - BAx_0|$.

Condition (c2) in particular implies that $B \notin \mathcal{G}$. Condition (a) ensures that $\mathcal{G}_B = \mathcal{G} \cup B\mathcal{G}$ is a subgroup of $O(N)$, and that an action $*$ of \mathcal{G}_B on $(\mathbb{H}^1(\mathbb{R}^N))^2$ is well defined by $A * (u, v) := (u \circ A^{-1}, v \circ A^{-1})$ for $A \in \mathcal{G}$ and $B * (u, v) := (v \circ B^{-1}, u \circ B^{-1})$. The $*$ -invariant elements of $(\mathbb{H}^1(\mathbb{R}^N))^2$ are precisely of the form $(u, u \circ B)$ with $u \in H_{\mathcal{G}}$. We define

$$\mathbf{N}(B, \mathcal{G}) := \{u \in H_{\mathcal{G}} : (u, u \circ B) \in \mathbf{N}\}$$

and

$$(1.10) \quad c(B, \mathcal{G}) = \inf_{u \in \mathbf{N}(B, \mathcal{G})} E(u, u \circ B).$$

Now we state our main result.

Theorem 1.2. *Let $N = 2$ or $N = 3$, let (B, \mathcal{G}) be an admissible pair, and let $\beta < 0$. Then:*

(a) *$\mathbf{N}(B, \mathcal{G})$ is nonempty and $c(B, \mathcal{G})$ is attained. Moreover, every minimizer $u \in \mathbf{N}(B, \mathcal{G})$ for (1.10) gives rise to a \mathcal{G} -symmetric solution $(u, u \circ B)$ of (1.5) with $u > 0$ everywhere on \mathbb{R}^N .*

(b) *If $|\beta| < 1$ is small, then $c(B, \mathcal{G}) = \frac{1}{2(1+\beta)} \|w_0\|^2$, and this value is attained only at the solutions (1.7) and its translations.*

From part (a) and (1.9), we directly deduce the following.

Corollary 1.3. *Under the assumptions of Theorem 1.2, for $\beta \leq -1$, there exists a \mathcal{G} -symmetric solution $(u, u \circ B)$ of (1.5) with $u \neq u \circ B$. Hence u is not \mathcal{G}_B -symmetric and therefore nonradial.*

We briefly comment on Definition 1.1. Part (a) of this Definition is clearly related to the action $*$ defined above. Part (b) ensures that $\mathbf{N}(B, \mathcal{G})$ and the reduced energy $u \mapsto E(u, u \circ B)$ are invariant under translations of the form $u \mapsto u(\cdot - y)$ for $y \in \text{Fix}(G)$. Part (c) will be crucial for estimating the value of $c(B, \mathcal{G})$ and thus for finding a relatively compact energy minimizing sequence in $\mathbf{N}(B, \mathcal{G})$. A classification of admissible pairs (B, \mathcal{G}) in arbitrary dimension seems out of reach. In dimensions $N \leq 3$ the finite subgroups of $O(N)$ and its properties are well known (see e.g. [5]), and therefore we can determine all admissible pairs. In combination with Corollary 1.3, the following list highlights the rich structure of the solution set of (1.5) for $\beta \leq -1$.

Example 1.4. (i) *Polygonal symmetry in \mathbb{R}^2* : Let $N = 2$, fix $k \in \mathbb{N}$ and let $B_k \in O(2)$ denote the (counter-clockwise) rotation by $\theta_k = \frac{\pi}{2k}$, i.e.,

$$B_k(x) = (x_1 \cos \theta_k - x_2 \sin \theta_k, x_1 \sin \theta_k + x_2 \cos \theta_k) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

We set $\mathcal{G}_k = \{\text{Id}, B_k^2, B_k^4, \dots, B_k^{2k-2}\}$. Then the admissibility condition (a) is clearly satisfied for the pair (B_k, \mathcal{G}_k) . Note also that $\text{Fix}(\mathcal{G}_k) = \{0\}$. Moreover, for every $x \in \mathbb{R}^2 \setminus \{0\}$ we have $|\mathcal{G}_k x| = l(\mathcal{G}_k) = k$ and

$$\min_{\substack{A \in \mathcal{G}_k \\ A \neq \text{Id}}} |x - Ax| = |x - B_k^2 x| = 2 \sin \theta_k < 4 \sin\left(\frac{\theta_k}{2}\right) = 2|x - B_k x| = 2 \min_{A \in \mathcal{G}_k} |x - B_k A x|.$$

Hence the pair (B_k, \mathcal{G}_k) is admissible.

(ii) *Polygonal symmetry in \mathbb{R}^3* : Let $N = 3$, fix $k \in \mathbb{N}$ and let $B_k \in O(3)$ denote the rotation of (x_1, x_2) by $\theta_k = \frac{\pi}{2k}$, i.e.,

$$B_k(x) = (x_1 \cos \theta_k - x_2 \sin \theta_k, x_1 \sin \theta_k + x_2 \cos \theta_k, 0) \quad \text{for } x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

With this choice of B_k we may define \mathcal{G}_k as in (i), and again admissibility condition (a) is satisfied for the pair (B_k, \mathcal{G}_k) . In contrast to (i) we now have a nontrivial space of fixed points $\text{Fix}(\mathcal{G}_k) = \{(0, 0, \xi) : \xi \in \mathbb{R}\}$. Nevertheless, for every $x \in V_{\mathcal{G}_k} \setminus \{0\}$ we still have $|\mathcal{G}_k x| = l(\mathcal{G}_k) = k$ and

$$\min_{\substack{A \in \mathcal{G}_k \\ A \neq \text{Id}}} |x - Ax| = |x - B_k^2 x| = 2 \sin \theta_k < 4 \sin\left(\frac{\theta_k}{2}\right) = 2|x - B_k x| = 2 \min_{A \in \mathcal{G}_k} |x - B_k A x|.$$

Hence the pair (B_k, \mathcal{G}_k) is admissible.

(iii) *Tetrahedral symmetry in \mathbb{R}^3* : Let $N = 3$, and consider the group $\mathcal{G} \leq O(3)$ generated by the coordinate permutations $(x_1, x_2, x_3) \mapsto (x_{\pi_1}, x_{\pi_2}, x_{\pi_3})$ and $F \in O(3)$ defined by $F(x) = (x_1, -x_2, -x_3)$. Then $|\mathcal{G}| = 12$. Let $B \in O(3)$ be defined by $B(x) = -x$. Then $B^2 = \text{Id} \in \mathcal{G}$, and since B commutes with permutations and with F , the admissibility condition (a) is satisfied for the pair (B, \mathcal{G}) . We also note that $\text{Fix}(\mathcal{G}) = \{0\}$. For $x_0 = (1, 1, 1)$ we have

$$\mathcal{G}x_0 = \{(1, 1, 1), (-1, -1, 1), (1, -1, -1), (-1, 1, -1)\},$$

so that $|\mathcal{G}x_0| = 4 = l(\mathcal{G})$. Moreover, since

$$B\mathcal{G}x_0 = \{(-1, -1, -1), (1, 1, -1), (-1, 1, 1), (1, -1, 1)\},$$

we have

$$\min_{A \in \mathcal{G} \setminus \mathcal{G}x_0} |x_0 - Ax_0| = \sqrt{2} < 2 = 2 \min_{A \in \mathcal{G}} |x_0 - BAx_0|.$$

Hence the pair (B, \mathcal{G}) is admissible. Note that the group \mathcal{G} leaves the tetrahedron with vertices $(1, 1, 1)$, $(-1, -1, 1)$, $(1, -1, -1)$ and $(-1, 1, -1)$ fixed.

By choosing $k_j = 2^j$, $j \in \mathbb{N}$ in Examples 1.4 (i) and (ii) above, Corollary 1.3 implies the existence of bound states (u_j, v_j) of (1.5) which are G_{k_j} -symmetric but not $G_{k_{j+1}}$ -symmetric. In particular, (u_j, v_j) , $j \in \mathbb{N}$ are pairwise different nonradial solutions. Thus we conclude our last main result.

Corollary 1.5. *For $N = 2, 3$ and $\beta \leq -1$, the system (1.5) admits infinitely many nonradial bound states.*

The paper is organized as follows. In Section 2 we recall known facts and collect preliminary results. In Section 3 we prove Theorem 1.2(a), and Section 4 contains the proof of Theorem 1.2(b).

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2. PRELIMINARIES

Throughout the remainder of this paper, we assume that $\beta \leq 0$. We fix some notation. As usual, we endow the Hilbert space $\mathbb{H}^1(\mathbb{R}^N)$ with the scalar product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv) dx, \quad u, v \in \mathbb{H}^1(\mathbb{R}^N),$$

and we set $\|u\|^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx$ as before. Moreover, for $1 \leq p \leq \infty$ and $u \in L^p(\mathbb{R}^N)$ we denote by $|u|_p$ the usual L^p -norm of u . It is well known (see [7]) that the (unique) solution w_0 of the scalar problem (1.8) is a radial and radially decreasing function which minimizes the Sobolev quotient of the embedding $\mathbb{H}^1(\mathbb{R}^N) \hookrightarrow L^4(\mathbb{R}^N)$, i.e.,

$$(2.11) \quad \|w_0\| = \frac{\|w_0\|^2}{|w_0|_4^2} = \min_{u \in \mathbb{H}^1(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^2}{|u|_4^2}.$$

We recall the following asymptotic estimates for w_0 , see e.g. [9, 17]:

$$(2.12) \quad \left\{ \begin{array}{l} w_0(y) = a_N |y|^{-\frac{N-1}{2}} e^{-|y|} (1 + o(1)) \\ w_0'(|y|) = -a_N |y|^{-\frac{N-1}{2}} e^{-|y|} (1 + o(1)) \end{array} \right\} \quad \text{as } |y| \rightarrow \infty.$$

Here $a_N > 0$ is a constant depending only on the dimension N . Similarly as in [14, Lemma 2.6] we deduce some integral estimates.

Lemma 2.1. *As $y \rightarrow \infty$,*

$$(2.13) \quad \frac{1}{w_0(y)} \int_{\mathbb{R}^N} w_0^3(x) w_0(x-y) dx \rightarrow b_N > 0.$$

where $b_N = a_N \int_{\mathbb{R}^N} w_0^3 dx$. Moreover, for $0 < \delta < 2$,

$$(2.14) \quad \frac{1}{w_0(\delta y)} \int_{\mathbb{R}^N} w_0^2(x) w_0^2(x-y) dx \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

Proof. By (2.12),

$$(2.15) \quad \frac{w_0(x-y)}{w_0(y)} \rightarrow a_N \quad \text{as } |y| \rightarrow \infty \quad \text{for every } x \in \mathbb{R}^N.$$

Moreover, there is $c > 1$ such that

$$(2.16) \quad c^{-1} \min\{1, |y|^{-\frac{N-1}{2}}\} e^{-|y|} \leq w_0(y) \leq c \min\{1, |y|^{-\frac{N-1}{2}}\} e^{-|y|}$$

for every $y \in \mathbb{R}^N$. Let $|y| \geq 1$, and put $\bar{c} = c^5 2^{\frac{N-1}{2}}$. If $|x| \geq \frac{|y|}{2}$, then

$$w_0^3(x) \frac{w_0(x-y)}{w_0(y)} \leq c^5 \left(\frac{|y|}{|x|} \right)^{\frac{N-1}{2}} e^{-3|x|-|x-y|+|y|} \leq \bar{c} e^{-3|x|-|x-y|+|y|} \leq \bar{c} e^{-2|x|},$$

and for $|x| \leq \frac{|y|}{2}$ we also have

$$w_0^3(x) \frac{w_0(x-y)}{w_0(y)} \leq c^5 \left(\frac{|y|}{|x-y|} \right)^{\frac{N-1}{2}} e^{-3|x|-|x-y|+|y|} \leq \bar{c} e^{-2|x|}.$$

Consequently,

$$w_0^3(x) \frac{w_0(x-y)}{w_0(y)} \leq \bar{c} e^{-2|x|} \quad \text{for } |y| \geq 1 \text{ and every } x.$$

Hence, by (2.15) and Lebesgue's Theorem,

$$\lim_{|y| \rightarrow \infty} \frac{1}{w_0(y)} \int_{\mathbb{R}^N} w_0^3(x) w_0(x-y) dx = a_N \int_{\mathbb{R}^N} w_0^3(x) dx = b_N.$$

Next we consider (2.14), and we may assume that $\delta \geq 1$. Using (2.16) we estimate, for $|y| \geq 1$,

$$\begin{aligned} \frac{w_0^2(x) w_0^2(x-y)}{w_0(\delta y)} &\leq c^5 (\delta |y|)^{\frac{N-1}{2}} e^{-2|x|-2|x-y|+\delta|y|} \leq c^5 (\delta |y|)^{\frac{N-1}{2}} e^{-2|x|-\frac{2+\delta}{2}|x-y|+\delta|y|} \\ &\leq c^5 (\delta |y|)^{\frac{N-1}{2}} e^{-\frac{2-\delta}{2}(|x|+|y|)} = f_\delta(y) e^{-\frac{2-\delta}{2}|x|}. \end{aligned}$$

where $f_\delta(y) := c^5 (\delta |y|)^{\frac{N-1}{2}} e^{-\frac{2-\delta}{2}|y|} \rightarrow 0$ as $|y| \rightarrow \infty$. Hence

$$\frac{1}{w_0(\delta y)} \int_{\mathbb{R}^N} w_0^2(x) w_0^2(x-y) dx \leq f_\delta(y) \int_{\mathbb{R}^N} e^{-\frac{2-\delta}{2}|x|} dx \rightarrow 0$$

as $|y| \rightarrow \infty$, as claimed. \square

Next, we fix an admissible pair (B, \mathcal{G}) in the sense of Definition 1.1. We consider the reduced energy functional

$$E_{\mathcal{G}} \in C^2(H_{\mathcal{G}}, \mathbb{R}), \quad E_{\mathcal{G}}(u) = \frac{1}{2} \|u\|^2 - \frac{1}{4} |u|_4^4 - \beta Q(u),$$

where the C^2 -functional $Q : \mathbb{H}^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ is defined by

$$Q(u) = \frac{1}{4} \int_{\mathbb{R}^N} u^2(x) u^2(Bx) dx = \frac{1}{4} |u \cdot (u \circ B)|_2^2.$$

Lemma 2.2. *For $u \in H_{\mathcal{G}}$, $v \in H^1(\mathbb{R}^N)$ we have*

$$\langle \nabla Q(u), v \rangle = \int_{\mathbb{R}^N} u^2(Bx) u(x) v(x) dx$$

Proof. For $u, v \in H^1(\mathbb{R}^N)$ we find

$$\begin{aligned} \langle \nabla Q(u), v \rangle &= \frac{1}{2} \int_{\mathbb{R}^N} \left(u^2(x) u(Bx) v(Bx) + u^2(Bx) u(x) v(x) \right) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \left(u^2(B^{-1}x) + u^2(Bx) \right) u(x) v(x) dx \end{aligned}$$

For $u \in H_{\mathcal{G}}$ we have $u \circ B = u \circ B^{-1}$, since $B^2 \in \mathcal{G}$ and $u \circ A = u$ for every $A \in \mathcal{G}$.

We thus conclude that

$$(2.17) \quad \langle \nabla Q(u), v \rangle = \int_{\mathbb{R}^N} u^2(Bx) u(x) v(x) dx \quad \text{for } u \in H_{\mathcal{G}}, v \in \mathbb{H}^1(\mathbb{R}^N).$$

\square

Corollary 2.3. *If $u \in H_{\mathcal{G}}$ is a nontrivial and nonnegative critical point of $E_{\mathcal{G}}$, then $(u, u \circ B)$ is a solution of (1.5).*

Proof. For $v \in \mathbb{H}^1(\mathbb{R}^N)$ we have by Lemma 2.2:

$$\begin{aligned} 0 &= \langle \nabla E_{\mathcal{G}}(u), v \rangle = \langle u, v \rangle - \int_{\mathbb{R}^N} u^3 v \, dx - \beta \langle \nabla Q(u), v \rangle \\ &= \langle u, v \rangle - \int_{\mathbb{R}^N} u^3 v \, dx - \beta \int_{\mathbb{R}^N} (u \circ B)^2 uv \, dx \end{aligned}$$

Thus u is a weak solution of the equation $-\Delta u + u - u^3 = \beta(u \circ B)^2 u$. By standard elliptic regularity, u is in fact a classical solution. Moreover, since $u \geq 0$ and $u \not\equiv 0$, it follows from the strong maximum principle that $u > 0$ in \mathbb{R}^N . Now $u \circ B$ solves

$$-\Delta(u \circ B) + (u \circ B) - (u \circ B)^3 = \beta(u \circ B)^2(u \circ B) = \beta u^2(u \circ B),$$

since $B^2 \in \mathcal{G}$. Hence $(u, u \circ B)$ is a classical solution of (1.5). \square

Next we put

$$\mathbf{N}_{\mathcal{G}} = \{u \in H_{\mathcal{G}} : u \neq 0, E'_{\mathcal{G}}(u)u = 0\} = \{u \in H_{\mathcal{G}} : u \neq 0, \|u\|^2 = |u|_4^4 + \beta|u \cdot (u \circ B)|_2^2\},$$

where the second equality follows from Lemma 2.2. We note that $\mathbf{N}(B, \mathcal{G}) = \{u \in \mathbf{N}_{\mathcal{G}} : u \geq 0\}$. We need the following lemma.

Lemma 2.4.

- (i) $|u|_4^2 \geq \|u\| \geq \kappa$ for some constant $\kappa > 0$ (independent of $\beta \leq 0$) and every $u \in \mathbf{N}_{\mathcal{G}}$.
- (ii) $\mathbf{N}_{\mathcal{G}} \subset H_{\mathcal{G}}$ is a closed C^1 -manifold.
- (iii) $E_{\mathcal{G}}(u) = \frac{\|u\|^2}{4}$ for $u \in \mathbf{N}_{\mathcal{G}}$.
- (iv) If $u \in H_{\mathcal{G}} \setminus \{0\}$ satisfies $|u|_4^4 > |\beta| |u \cdot (u \circ B)|_2^2$, then $\sqrt{t(u)}u \in \mathbf{N}_{\mathcal{G}}$ for

$$t(u) = \frac{\|u\|^2}{|u|_4^4 + \beta|u \cdot (u \circ B)|_2^2} > 0.$$

Proof. (i) By definition and Sobolev embeddings, we have $\|u\|^2 \leq |u|_4^4 \leq \kappa_0 \|u\|^4$ for some $\kappa_0 > 0$ and $u \in \mathbf{N}_{\mathcal{G}}$, so that $|u|_4^2 \geq \|u\| \geq \kappa$ for $\kappa = \sqrt{\kappa_0^{-1}}$.

(ii) By (i), $\mathbf{N}_{\mathcal{G}}$ is closed in $H_{\mathcal{G}}$. Moreover, $\mathbf{N}_{\mathcal{G}}$ is the zero set of the functional

$$(2.18) \quad F \in C^1(H_{\mathcal{G}}, \mathbb{R}), \quad F(u) = \|u\|^2 - |u|_4^4 - 4\beta Q(u).$$

Since for $u \in \mathbf{N}_{\mathcal{G}}$ we have

$$(2.19) \quad F'(u)u = 2\|u\|^2 - 4(|u|_4^4 + \beta|u \cdot (u \circ B)|_2^2) = -2\|u\|^2 \neq 0,$$

$\mathbf{N}_{\mathcal{G}}$ is a C^1 -submanifold of $H_{\mathcal{G}}$.

(iii) For $u \in \mathbf{N}_{\mathcal{G}}$ we have

$$E_{\mathcal{G}}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}(|u|_4^4 + \beta|u \cdot (u \circ B)|_2^2) = \frac{1}{4}\|u\|^2.$$

(iv) This also follows by direct computation. \square

3. EXISTENCE OF MINIMIZERS

In this section we prove part (a) of Theorem 1.2, which is an immediate consequence of the following proposition. Here we set

$$(3.20) \quad \tilde{c} := \inf_{u \in \mathbf{N}_{\mathcal{G}}} E_{\mathcal{G}}(u).$$

Proposition 3.1. (i) *The value \tilde{c} is attained.*

(ii) *$\tilde{c} = c(B, \mathcal{G})$, and if $u \in \mathbf{N}_{\mathcal{G}}$ is a minimizer for (3.20), then either $(u, u \circ B)$ or $(-u, -u \circ B)$ is a solution of (1.5). In particular, either $u \in \mathbf{N}(B, \mathcal{G})$ or $-u \in \mathbf{N}(B, \mathcal{G})$.*

The remainder of this section is devoted to the proof of Proposition 3.1. The proof consists of two steps; first we obtain an estimate for the value of \tilde{c} in terms of $\|w_0\|$, and then we analyze minimizing sequences for (3.20) via concentration compactness arguments. The strict inequality in the following estimate is crucial.

Proposition 3.2. *We have $\tilde{c} < \frac{k}{4}\|w_0\|^2$, where $k = l(\mathcal{G}) = |\mathcal{G}x_0|$ and x_0 is given by Definition 1.1.*

Proof. Let $A_1 = \text{Id} \in O(N)$, and let $A_2, \dots, A_k \subset \mathcal{G} \setminus \mathcal{G}^{x_0}$ be such that $\mathcal{G}x_0 = \{A_1x_0, \dots, A_kx_0\}$. We put

$$\mu = \min_{j \neq 1} |x_0 - A_jx_0| = \min_{i \neq j} |A_ix_0 - A_jx_0| > 0$$

and

$$\nu = \min_j |x_0 - BA_jx_0| = \min_{i,j} |A_ix_0 - A_jBx_0|,$$

so that $\mu < 2\nu$ by Definition 1.1(c2). For $r > 0$ and $j = 1, \dots, k$ we set $w_r^j = w_0(\cdot - rA_jx_0)$, and we consider $U_r = \sum_{j=1}^k w_r^j \in H_{\mathcal{G}}$. As $r \rightarrow \infty$, (2.13) implies that

$$d_r := \sum_{i \neq j} \int_{\mathbb{R}^N} (w_r^i)^3 w_r^j dx = (b_N + o(1)) \sum_{i \neq j} w_0(r[A_ix_0 - A_jx_0]),$$

hence

$$(b_N + o(1))(\mu r)^{-\frac{N-1}{2}} e^{-\mu r} \leq d_r \leq \frac{k(k-1)}{2} (b_N + o(1))(\mu r)^{-\frac{N-1}{2}} e^{-\mu r}.$$

Moreover, (2.14) yields for $1 \leq i, j \leq k$ and $\delta = \frac{\mu}{\nu} < 2$ the estimate

$$\begin{aligned} \int_{\mathbb{R}^N} (w_r^i(x))^2 (w_r^j(Bx))^2 dx &= o(w_0(\delta r[A_i x_0 - A_j B x_0])) \\ (3.21) \qquad \qquad \qquad &= o((\delta \nu r)^{-\frac{N-1}{2}} e^{-\delta \nu r}) = o(d_r) \end{aligned}$$

as $r \rightarrow \infty$. We also have

$$\begin{aligned} \|U_r\|^2 &= k\|w_0\|^2 + \sum_{i \neq j} \int_{\mathbb{R}^N} (\nabla w_r^i \nabla w_r^j + w_r^i w_r^j) dx = k\|w_0\|^2 + \sum_{i \neq j} \int_{\mathbb{R}^N} (w_r^i)^3 w_r^j dx \\ (3.22) \qquad \qquad \qquad &= k\|w_0\|^2 + d_r, \end{aligned}$$

and

$$\begin{aligned} |U_r|_4^4 &= \int_{\mathbb{R}^N} \left(\sum_{j=1}^k w_r^j \right)^4 dx \geq \sum_{j=1}^k \int_{\mathbb{R}^N} (w_r^j)^4 dx + 4 \sum_{i \neq j} \int_{\mathbb{R}^N} (w_r^i)^3 w_r^j dx \\ (3.23) \qquad \qquad \qquad &= k|w_0|_4^4 + 4d_r = k\|w_0\|^2 + 4d_r. \end{aligned}$$

Furthermore we estimate

$$\begin{aligned} \int_{\mathbb{R}^N} U_r^2(x) U_r^2(Bx) dx &= \int_{\mathbb{R}^N} \left(\sum_{i,j} w_r^i(x) w_r^j(x) \right) \left(\sum_{i,j} w_r^i(Bx) w_r^j(Bx) \right) dx \\ &\leq \frac{1}{4} \int_{\mathbb{R}^N} \left(\sum_{i,j} [(w_r^i)^2(x) + (w_r^j)^2(x)] \right) \left(\sum_{i,j} [(w_r^i)^2(Bx) + (w_r^j)^2(Bx)] \right) dx \\ (3.24) \qquad \qquad \qquad &\leq k^2 \sum_{i,j} \int_{\mathbb{R}^N} (w_r^i)^2(x) (w_r^j)^2(Bx) dx = o(d_r) \end{aligned}$$

by (3.21). Let

$$t_r := t(U_r) = \frac{\|U_r\|^2}{|U_r|_4^4 + \beta |U_r(U_r \circ B)|_2^2},$$

so that $\sqrt{t_r} U_r \in \mathbf{N}_{\mathcal{G}}$ by Lemma 2.4(iv). Combining (3.22), (3.23) and (3.24), we obtain

$$\begin{aligned} E_{\mathcal{G}}(\sqrt{t_r} U_r) &= \frac{1}{4} \|\sqrt{t_r} U_r\|^2 = \frac{1}{4} \cdot \frac{\|U_r\|^4}{|U_r|_4^4 + \beta |U_r(U_r \circ B)|_2^2} \\ &\leq \frac{1}{4} \cdot \frac{(k\|w_0\|^2 + d_r)^2}{k\|w_0\|^2 + 4d_r + o(d_r)} = \frac{k}{4} \|w_0\|^2 \cdot \frac{k\|w_0\|^2 + 2d_r + o(d_r)}{k\|w_0\|^2 + 4d_r + o(d_r)}, \end{aligned}$$

so that $\tilde{c} \leq E_{\mathcal{G}}(\sqrt{t_r} U_r) < \frac{k}{4} \|w_0\|^2$ for r large. \square

Lemma 3.3. *There exists a sequence $(u_n)_n \subset \mathbf{N}_{\mathcal{G}}$ with $E_{\mathcal{G}}(u_n) \rightarrow \tilde{c}$ and $E'_{\mathcal{G}}(u_n) \rightarrow 0$ in $H_{\mathcal{G}}^*$.*

Proof. Since $\mathbf{N}_{\mathcal{G}}$ is a C^1 -manifold, we may invoke Ekeland's variational principle (see e.g. [23]) to deduce the existence of a sequence $(u_n)_n \subset \mathbf{N}_{\mathcal{G}}$ such that $E_{\mathcal{G}}(u_n) \rightarrow \tilde{c}$ and

$$(3.25) \quad o(1) = (E_{\mathcal{G}}|_{\mathbf{N}_{\mathcal{G}}})'(u_n) = E'_{\mathcal{G}}(u_n) - \lambda_n F'(u_n) \quad \text{in } H_{\mathcal{G}}^*$$

for a sequence $(\lambda_n)_n \subset \mathbb{R}$, where F is defined in (2.18). Since $u_n \in \mathbf{N}_{\mathcal{G}}$, (2.19) and (3.25) imply that

$$(3.26) \quad o(1)\|u_n\| = \lambda_n F'(u_n)(u_n) = -2\lambda_n \|u_n\|^2,$$

and therefore $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ by Lemma 2.4(i). Thus (3.25) yields $E'_{\mathcal{G}}(u_n) \rightarrow 0$ as $n \rightarrow \infty$, as claimed. \square

Proof of Proposition 3.1 (completed). (a) Let $(u_n)_n \subset \mathbf{N}_{\mathcal{G}}$ be a sequence as provided by Lemma 3.3, and let $y_n \in \mathbb{R}^N$, $n \in \mathbb{N}$ satisfy

$$\int_{B_1(y_n)} u_n^4 dx = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} u_n^4 dx.$$

Since $\mathcal{N}_{\mathcal{G}}$ and $E_{\mathcal{G}}$ are invariant and $\nabla E_{\mathcal{G}}$ is equivariant under translations $u \mapsto u(\cdot + y)$ with $y \in \text{Fix}(\mathcal{G})$ (cf. Definition 1.1(b)), we may assume that $y_n \in V_{\mathcal{G}} = \text{Fix}(\mathcal{G})^\perp$ for every n . We recall that u_n is bounded in $\mathbb{H}^1(\mathbb{R}^N)$ and $|u_n|_4^2 \geq \kappa > 0$ for every n by Lemma 2.4, so a result of Lions [17, Lemma I.1] implies that

$$(3.27) \quad \liminf_{n \rightarrow \infty} \int_{B_1(y_n)} u_n^4 dx > 0.$$

We claim that

$$(3.28) \quad (y_n)_n \text{ is bounded.}$$

Suppose this is false. Then we may pass to a subsequence with $|y_n| \rightarrow \infty$ and $\frac{y_n}{|y_n|} \rightarrow y \in V_{\mathcal{G}} \setminus \{0\}$. Since $k := l(\mathcal{G}) \leq |\mathcal{G}y|$, there are $A_1, \dots, A_k \in \mathcal{G}$ such that

$$(3.29) \quad \text{the points } A_j y, j = 1, \dots, k, \text{ are pairwise different.}$$

Let $\hat{u}_n = u_n(\cdot + y_n)$. Up to a subsequence, $\hat{u}_n \rightharpoonup \hat{u} \in \mathbb{H}^1(\mathbb{R}^N)$ weakly, where $\hat{u} \neq 0$ by (3.27). Since $\nabla E_{\mathcal{G}}(u_n) \rightarrow 0$ in $\mathbb{H}^1(\mathbb{R}^N)$,

$$\begin{aligned} o(1) &= \langle \nabla E_{\mathcal{G}}(u_n), \hat{u}(\cdot - y_n) \rangle \\ &= \langle u_n, \hat{u}(\cdot - y_n) \rangle - \int_{\mathbb{R}^N} u_n^3 \hat{u}(\cdot - y_n) dx - \beta \langle Q(u_n), \hat{u}(\cdot - y_n) \rangle \\ &= \langle \hat{u}_n, \hat{u} \rangle - \int_{\mathbb{R}^N} \hat{u}_n^3 \hat{u} dx - \beta \int_{\mathbb{R}^N} u_n^2(Bx) u_n(x) \hat{u}(x - y_n) dx \\ &= \langle \hat{u}_n, \hat{u} \rangle - \int_{\mathbb{R}^N} \hat{u}_n^3 \hat{u} dx + |\beta| \int_{\mathbb{R}^N} u_n^2(Bx + y_n) \hat{u}_n(x) \hat{u}(x) dx \\ &\geq \|\hat{u}\|^2 - |\hat{u}|_4^4 + |\beta| \int_{\mathbb{R}^N} u_n^2(Bx + y_n) (\hat{u}_n(x) - \hat{u}(x)) \hat{u}(x) dx + o(1), \end{aligned}$$

while

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} u_n^2(Bx + y_n) (\hat{u}_n(x) - \hat{u}(x)) \hat{u}(x) dx \right| \\ &\leq \left(\int_{\mathbb{R}^N} u_n^4(Bx + y_n) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} (\hat{u}_n(x) - \hat{u}(x))^2 \hat{u}^2(x) dx \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We therefore conclude that $0 < \|\hat{u}\|^2 \leq |\hat{u}|_4^4$, and thus

$$\|\hat{u}\|^2 \geq \frac{\|\hat{u}\|^4}{|\hat{u}|_4^4} \geq \frac{\|w_0\|^4}{|w_0|_4^4} = \|w_0\|^2$$

by (2.11). Using Proposition 3.2, we may choose $R = R(\varepsilon) > 0$ such that

$$(3.30) \quad \int_{B_R(0)} (|\nabla \hat{u}|^2 + \hat{u}^2) dx > \frac{4\tilde{c}}{k}.$$

By (3.29), the balls $B_R(A_j y_n)$, $j = 1, \dots, k$ are disjoint for n large. Therefore

$$\begin{aligned} E_{\mathcal{G}}(u_n) &= \frac{1}{4} \|u_n\|^2 \geq \frac{1}{4} \sum_{j=1}^k \int_{B_R(A_j y_n)} (|\nabla u_n|^2 + u_n^2) dx \\ &\geq \frac{k}{4} \int_{B_R(y_n)} (|\nabla u_n|^2 + u_n^2) dx = \frac{k}{4} \int_{B_R(0)} (|\nabla \hat{u}_n|^2 + \hat{u}_n^2) dx. \end{aligned}$$

Since $\hat{u}_n \rightharpoonup \hat{u}$ weakly, (3.30) yields

$$\liminf_{n \rightarrow \infty} E_{\mathcal{G}}(u_n) \geq \frac{k}{4} \int_{B_R(0)} (|\nabla \hat{u}|^2 + \hat{u}^2) dx > \tilde{c},$$

which contradicts the fact that $(u_n)_n$ is a minimizing sequence for (3.20).

Thus (3.28) holds. Consequently, we may pass to a subsequence such that $u_n \rightharpoonup u$ weakly in $H_{\mathcal{G}}$, where $u \in H_{\mathcal{G}} \setminus \{0\}$. Since $E'_{\mathcal{G}} : H_{\mathcal{G}} \rightarrow H_{\mathcal{G}}^*$ is weak-to-weak continuous, we conclude that u is a critical point of $E_{\mathcal{G}}$, so that $u \in \mathbf{N}_{\mathcal{G}}$. Moreover,

$$\tilde{c} = \lim_{n \rightarrow \infty} E_{\mathcal{G}}(u_n) = \frac{1}{4} \lim_{n \rightarrow \infty} \|u_n\|^2 \geq \frac{1}{4} \|u\|^2 = E_{\mathcal{G}}(u),$$

so that u is a minimizer of (3.30). Hence \tilde{c} is attained, and the proof of (a) is finished.

(b) If $u \in \mathbf{N}_{\mathcal{G}}$ is a minimizer for (3.20), then

$$(3.31) \quad 0 = (E_{\mathcal{G}}|_{\mathbf{N}_{\mathcal{G}}})'(u) = E'_{\mathcal{G}}(u) - \lambda F'(u)$$

for some $\lambda \in \mathbb{R}$, since $\mathbf{N}_{\mathcal{G}} = F^{-1}(0)$ is a C^1 -manifold. Hence

$$0 = E'_{\mathcal{G}}(u)u - \lambda F'(u)u = -\lambda F'(u)u = -2\lambda \|u\|^2$$

by (2.19), which yields $\lambda = 0$ and therefore $E'_{\mathcal{G}}(u) = 0$. We consider $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\} \in H_{\mathcal{G}}$. Then

$$\begin{aligned} 0 &= E'_{\mathcal{G}}(u)u^{\pm} = \|u^{\pm}\|^2 - |u^{\pm}|_4^4 - \beta Q'(u)u^{\pm} \\ &= \|u^{\pm}\|^2 - |u^{\pm}|_4^4 + |\beta| \int_{\mathbb{R}^N} u^2(Bx)u(x)u^{\pm}(x) dx \\ &\geq \|u^{\pm}\|^2 - |u^{\pm}|_4^4 + |\beta| \int_{\mathbb{R}^N} [u^{\pm}(Bx)]^2 [u^{\pm}(x)]^2 dx \end{aligned}$$

Hence, if $0 < \|u^+\| < \|u\|$, then $t(u^+) \leq 1$ (cf. Lemma 2.4(iv)) and

$$E_{\mathcal{G}}(\sqrt{t(u^+)}u^+) \leq \frac{1}{4}\|u^+\|^2 < \frac{1}{4}\|u\|^2 = E_{\mathcal{G}}(u),$$

contradicting the assumption that u is a minimizer for (3.20). Similarly, $0 < \|u^-\| < \|u\|$ leads to a contradiction. We therefore conclude that u does not change sign. By Lemma 2.3, either $(u, u \circ B)$ or $(-u, -u \circ B)$ is a solution of (1.5). The proof is finished. \square

4. PROOF OF (B) OF THEOREM 1.2

Here we prove part (b) of Theorem 1.2. For this we consider $\beta_n < 0$, $n \in \mathbb{N}$ with $\beta_n \rightarrow 0$ and a sequence of corresponding minimizers $(u_n, u_n \circ B)_n$ of (1.10). We only need to show that $u_n = u_n \circ B$ for large n , because then the uniqueness result in [7] for solutions of $-\Delta u + u = (1 + \beta_n)u^3$ implies that $u_n = u_n \circ B = \frac{1}{\sqrt{1+\beta_n}}w_0$ up to translation in $\text{Fix}(\mathcal{G})$. So we assume by contradiction that, for a subsequence,

$$(4.1) \quad u_n \neq u_n \circ B \quad \text{for every } n.$$

The minimization property and (2.11) imply that

$$\begin{aligned} \frac{1}{4}\|u_n\|^2 &= \inf \left\{ \frac{\|u\|^2}{4} : u \in \mathbb{H}^1(\mathbb{R}^N) \setminus \{0\} : \|u\|^2 = |u|_4^4 + \beta_n |u \cdot (u \circ B)|_2^2 \right\} \\ &= \inf \left\{ \frac{\|u\|^2}{4} : u \in \mathbb{H}^1(\mathbb{R}^N) \setminus \{0\} : \|u\|^2 = |u|_4^4 \right\} + o(1) \\ (4.2) \quad &= \frac{1}{4}\|w_0\|^2 + o(1). \end{aligned}$$

Hence $(u_n)_n$ is bounded in $\mathbb{H}^1(\mathbb{R}^N)$, and $|u_n|_4^2 \geq \kappa > 0$ by Lemma 2.4(i). Similarly as in the proof of Proposition 3.1, we may assume that

$$(4.3) \quad \int_{B_1(y_n)} u_n^4 dx = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} u_n^4 dx \geq c > 0$$

for points $y_n \in V_G$, $n \in \mathbb{N}$ and a constant $c > 0$. Setting $\hat{u}_n = u(\cdot + y_n)$, we have $\hat{u}_n \rightharpoonup \hat{u} \neq 0$ (after passing to a subsequence), where \hat{u} is a solution of the scalar problem

$$(4.4) \quad -\Delta \hat{u} + \hat{u} = \hat{u}^3, \quad u \in \mathbb{H}^1(\mathbb{R}^N), \quad u > 0,$$

so that \hat{u} equals w_0 up to translation. By (4.2) we thus have $\|\hat{u}\| = \|w_0\| = \lim_{n \rightarrow \infty} \|u_n\| = \lim_{n \rightarrow \infty} \|\hat{u}_n\|$, hence $\hat{u}_n \rightarrow \hat{u}$ strongly in $H^1(\mathbb{R}^N)$. Since $u_n \in H_G$,

$$\hat{u}_n(x) = u_n(x + y_n) = u_n(Ax + Ay_n) = \hat{u}_n(Ax + (Ay_n - y_n)) \quad \text{for } A \in \mathcal{G}, x \in \mathbb{R}^N,$$

so that the relative compactness of $(\hat{u}_n)_n$ in $\mathbb{H}^1(\mathbb{R}^N)$ implies the boundedness of the sequence $(Ay_n - y_n)_n \subset \mathbb{R}^N$ for every $A \in \mathcal{G}$. Recalling that $(y_n)_n \subset V_G = \text{Fix}(G)^\perp$, we conclude that $(y_n)_n$ is bounded. Since y_n is bounded, we infer that $u_n \rightarrow u$ in H_G , where $u \in H_G$ is a nontrivial solution of (4.4). This then implies that $u = w_0(\cdot - z_0)$ for some $z_0 \in \text{Fix}(\mathcal{G})$. Since $\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} w_0^4 dx$ is attained precisely at $y = 0$, we deduce from (4.3) that $z_0 = 0$, so that $u_n \rightarrow w_0$ in H_G . Combining this information with elliptic estimates similarly as in [26, Sec. 2], we find that

$$(4.5) \quad u_n \rightarrow w_0 \quad \text{uniformly on } \mathbb{R}^N.$$

We set $\varphi_n = u_n - u_n \circ B \in \mathcal{H}_G$ and note that φ_n satisfies

$$(4.6) \quad \Delta \varphi_n - \varphi_n + 3w_0^2 \varphi_n + c_n(x) \varphi_n = 0,$$

where

$$(4.7) \quad c_n = u_n^2 + (u_n \circ B)^2 + u_n(u_n \circ B) - 3w_0^2 - \beta_n u_n(u_n \circ B) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

uniformly in \mathbb{R}^N . By (4.1), we may choose $x_n \in \mathbb{R}^N$ such that $\varphi_n(x_n) = \max_{x \in \mathbb{R}^3} |\varphi_n(x)| > 0$. Using (4.6) and (4.7), we deduce that $(x_n)_n \subset \mathbb{R}^N$ is a bounded sequence. We consider $\hat{\varphi}_n = \frac{\varphi_n}{|\varphi_n(x_n)|}$ which satisfies

$$(4.8) \quad \Delta \hat{\varphi}_n - \hat{\varphi}_n + 3w_0^2 \hat{\varphi}_n + c_n(x) \hat{\varphi}_n = 0, \quad \hat{\varphi}_n(x_n) = 1.$$

Using elliptic estimates we derive that, for a subsequence, $x_n \rightarrow x_0 \in \mathbb{R}^N$ and $\hat{\varphi}_n \rightarrow \hat{\varphi}_0$ in $C_{loc}^1(\mathbb{R}^N)$ as $n \rightarrow \infty$, where $\hat{\varphi}_0 \in H_G \cap C^2(\mathbb{R}^N)$ is a solution of $\Delta \hat{\varphi}_0 - \hat{\varphi}_0 + 3w_0^2 \hat{\varphi}_0 = 0$ with $\hat{\varphi}_0(x_0) = 1$. It follows from Appendix C of [21] that

$$(4.9) \quad \hat{\varphi}_0 = \frac{\partial w_0}{\partial \tau} \quad \text{for some vector } \tau \in \mathbb{R}^N.$$

Since w_0 is radial, the \mathcal{G} -symmetry of $\frac{\partial w_0}{\partial \tau} = \hat{\varphi}_0 \in H_{\mathcal{G}}$ implies that $\tau \in \text{Fix}(\mathcal{G})$. But then $B\tau = \tau = B^{-1}\tau$ by Definition 1.1(b), and therefore

$$\hat{\varphi}_0(Bx) = \frac{\partial w_0}{\partial \tau}(Bx) = \frac{\partial w_0}{\partial \tau}(x) = \hat{\varphi}_0(x) \quad \text{for all } x \in \mathbb{R}^N.$$

On the other hand, by definition we have $\varphi_n \circ B = -\varphi_n$ and therefore $\hat{\varphi}_0 \circ B = -\hat{\varphi}_0$. We conclude that $\hat{\varphi}_0 \equiv 0$, contradicting $\hat{\varphi}_0(x_0) = 1$. We conclude that $u_n = u_n \circ B$ for n large, as required. The proof of Theorem 1.2(b) is finished.

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