MULTIPLE INTERIOR PEAK SOLUTIONS FOR SOME SINGULARLY PERTURBED NEUMANN PROBLEMS

CHANGFENG GUI AND JUNCHENG WEI

ABSTRACT. We consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 & \text{in} \quad \Omega \\ u > 0 \ \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in R^N, $\varepsilon > 0$ is a small parameter and f is a superlinear, subcritical nonlinearity. It is known that this equation possesses boundary spike solutions that concentrate, as ε approaches zero, at a critical point of the mean curvature function $H(P), P \in \partial\Omega$. It is also proved that this equation has single interior spike solutions at a local maximum point of the distance function $d(P, \partial\Omega), P \in \Omega$.

In this paper, we prove the existence of interior $K - peak$ ($K \geq 2$) solutions at the local maximum points of the following function

$$\varphi(P_1, P_2, ..., P_K) = \min_{i,k,l=1,\ldots,K; k \neq l} (d(P_i, \partial\Omega), \frac{1}{2}|P_k - P_l|)$$

We first use Liapunov-Schmidt reduction method to reduce the problem to a finite dimensional problem. Then we use a maximizing procedure to obtain multiple interior spikes. The function $\varphi(P_1, ..., P_K)$ appears naturally in the asymptotic expansion of the energy functional.

1. Introduction

The aim of this paper is to construct a family of multiple interior peak solutions to the following singularly perturbed elliptic problem

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 \ \text{ in } \Omega, \\ u > 0 \ \text{ in } \ \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, Ω is a bounded smooth domain in R^N, $\varepsilon > 0$ is a constant, the exponent p satisfies $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$ and $1 < p < \infty$ for $N = 2$ and $\nu(x)$ denotes the unit outward normal at $x \in \partial\Omega$.

1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J40.
Key words and phrases. Multiple Interior Spikes, Nonlinear Elliptic Equations.

Equation (1.1) is known as the stationary equation of the Keller-Segal system in chemotaxis. It can also be seen as the limiting stationary equation of the so-called Gierer-Meinhardt system in biological pattern formation, see [40] for more details.

In the pioneering papers of [22], [26] and [27], Lin, Ni and Takagi established the existence of least-energy solutions and showed that for ε sufficiently small the least-energy solution has only one local maximum point P_ε and $P_\varepsilon \in \partial\Omega$. Moreover, $H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\varepsilon \to 0$, where $H(P)$ is the mean curvature of P at $\partial\Omega$. In [28] , Ni and Takagi constructed boundary spike solutions for axially symmetric domains. The second author in [40] studied the general domain case and showed that for single boundary spike solutions, the boundary spike must approach a critical point of the mean curvature; on the other hand, for any nondegenerate critical point of $H(P)$, one can construct boundary spike solutions with spike approaching that point. The first author in [14] constructed multiple boundary spike layer solutions at multiple local maximum poi nts of $H(P)$. Later the second author and Winter in [43] constructed multiple boundary spike layer solutions at multiple nondegenerate critical points of $H(P)$. Similar results are also obtained by Y. Y. Li in [24] independently. When $p = \frac{N+2}{N-2}$, similar results for the boundary spike layer solutions have been obtained by [2], [3], [4], [15], [25], [44] etc.

In all the above papers, only *boundary* spike layer solutions are obtained and studied. It remains a question whether or not *interior* spike layer solutions exist for problem (1.1). It was proved in [39] that under very restrictive geometric conditions, one can construct single interior spike solutions.

In [41], the second author obtained the first result in constructing *single interior* spike solutions by using the distance function, $d(P, \partial\Omega)$. More precisely, it was proved in [41] that for any smooth domain Ω, there always exists a single interior spike solution wh ich concentrates at the most centered part of the domain, namely, the points which attains the maximum of the distance function. (We note that the distance function has already appeared in the study of the corresponding Dirichlet problem in [29], [37].) We also like to point out that formal asymptotic analysis for single interior peak solution is done in [45], and some results on multiple interior peak solutions are obtained in [20] under very complicated conditions.

In this paper, we study the existence of multiple interior peak solutions by using the geometry of the domain. Moreover, we are able to deal with more general nonlinearities.

More precisely, we consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega. \end{cases} \tag{1.2}$$

We will assume that $f : R^+ \to R$ is of class $C^{1+\sigma}$ and satisfies the following conditions

(f1) $f(u) \equiv 0$ for $u \leq 0$ and $f(u) \to +\infty$ as $u \to \infty$.

(f2) $f(0) = 0, f'(0) = 0$ and

$$f(u) = O(|u|^{p_1}), f'(u) = O(|u|^{p_2 - 1}) \text{ as } |u| \to \infty$$

for some $1 < p_1, p_2$ and there exists $1 < p_3$ such that

$$|f_u(u + \phi) - f_u(u)| \leq \begin{cases} C|\phi|^{p_3 - 1} \text{ if } p_3 > 2 \\ C(|\phi| + |\phi|^{p_3 - 1}) \quad \text{if } p_3 \leq 2 \end{cases}$$

(f3) The following equation

$$\begin{cases} \triangle w - w + f(w) = 0 \quad \text{in } R^N \\ w > 0, w(0) = \max_{z \in R^N} \; w(z) \\ w \to 0 \quad \text{at } \infty \end{cases} \tag{1.3}$$

has a unique solution $w(y)$ (by the results of [16], w is radial, i.e., $w = w(r)$ and $w' < 0$ for $r = |y| \neq 0$) and w is nondegenerate. Namely the operator

$$L := \triangle - 1 + f'(w) \tag{1.4}$$

is invertible in the space $H_r^2(R^N) := \{u = u(|y|) \in H^2(R^N)\}$

Some important examples of f are the following.

Example 1 (chemotaxis and pattern formation)

Let $f(u) = u^p$ where $1 < p < (\frac{N+2}{N-2})_+ (= \infty$ if $N = 2; = \frac{N+2}{N-2}$ if $N > 2)$. It is easy to see that f satisfies (f1), (f2) and (f3). This problem ar ises from the Keller-Segal model in chemotaxis and Gierer-Meinhardt system in pattern formation (see [26], [27] and the references therein).

Example 2 (population dynamics and chemical reaction theory)

Let $g(u) = u(u-a)(1-u)$ and $f(u) = g(u) - (a+1)u$, where $0 < a < \frac{1}{2}$. This is a famous model from population dynamics and chemical reaction theory (see [5], [10], [35]). By the result of [17], f satisfies (f1)-(f3).

Other nonlinearities satisfying $(f1), (f2)$ and $(f3)$ can be found in [11].

In what follows, we state precisely our assumption on the domain.

For any $\mathbf{P} = (P_1, ..., P_K) \in \Omega^K = \Omega \times \Omega \times ... \times \Omega$, we introduce the following function

$$\varphi(P_1, P_2, ..., P_K) = \min_{i,k,l=1,...,K; k \neq l} (d(P_i, \partial\Omega), \frac{1}{2}|P_k - P_l|)$$

We assume that a subset Λ of Ω^K satisfies

$$\max_{(P_1,...,P_K) \in \Lambda} \varphi(P_1, ..., P_K) > \max_{(P_1,...,P_K) \in \partial\Lambda} \varphi(P_1, ..., P_K) \tag{1.5}$$

We emphasize that such a set Λ always exists . For example, we can take $\Lambda = \Omega^K$. We also observe that any such Λ can be modified so that for $\mathbf{P} = (P_1, ..., P_K) \in \Lambda$ we have

$$\min_{I=1,...,K} d(P_i, \partial\Omega) > \delta > 0, \qquad \min_{k,l=1,...,K; k \neq l} |P_k - P_l| > 2\delta > 0 \tag{1.6}$$

for some sufficiently small $\delta > 0$.

Next we discuss some other examples of Λ for some special domains. If $d(P, \partial\Omega)$ has K strict local maximum points $P_1, ..., P_K$ in Ω such that $\min_{i \neq j,} |P_i - P_j| > 2 \max_{i=1,...,K} d(P_i, \partial\Omega)$, we can choose Λ such that (1.5) holds with $\max_{(P_1,...,P_K) \in \Lambda} \varphi(P_1, ..., P_K)$ achieved at $\mathbf{P} = (P_1, ..., P_k)$. When $\Omega = B_R(0)$ and $K = 2$, one can take $P_1 = (R/2, 0, ..., 0), P_2 = (-R/2, ..., 0)$ and $\Lambda = \{(X_1, X_2) : R/2 - \delta < |X_i| < R/2 + \delta, i = 1, 2, |X_1 - X_2| > \delta\}$ with δ small. Then (1.5) holds and $\max_{(P_1,P_2) \in \Lambda} \varphi(P_1, P_2) = R/2$ is achieved at $\mathbf{P} = (P_1, P_2)$.

We now state our main result in this paper.

Theorem 1.1. *Assume that condition (1.5) is satisfied. Let f satisfy assumptions (f1)-(f3). Then for ε sufficiently small problem (1.2) has a solution u_ε which possesses exactly K local maximum points $Q_1^\varepsilon, ..., Q_K^\varepsilon$ with $\mathbf{Q}^\varepsilon =$*

$(Q_1^\varepsilon,...,Q_K^\varepsilon)\in\Lambda$. Moreover $\varphi(Q_1^\varepsilon,...,Q_K^\varepsilon)\to\max_{(Q_1,...,Q_K)\in\Lambda}\varphi(Q_1,...,Q_K)$ as $\varepsilon\to 0$. Furthermore, we have

$$u_\varepsilon(x)\le aexp(-\frac{b\min_{i=1,...,K}(|x-Q_i^\varepsilon|)}{\varepsilon}) \tag{1.7}$$

for certain positive constants a, b.

More detailed asymptotic behavior of u_ε can be found in the proof of Theorem 1.1.

We have the following interesting corollary.

Corollary 1.2. *For any smooth and bounded domain and any fixed positive integer $K\in Z$, there always exists interior K-peaked solutions which concentrates at the maximum point of the function $\varphi(P_1,...,P_K)$.*

Remark 1.3. *: It can be shown that the maximum of $\varphi(P_1,...,P_K)$ in Ω^K is attained at some point $(Q_1,...,Q_K)$ with $d(Q_i,\partial\Omega)=\max\varphi(P_1,...,P_K)$ for some i. In other words, the distance between each $Q_i's$ is always larger than or equal to the smallest $d(Q_i,\partial\Omega)$. If we connect the maximum point of $\varphi(P_1,...,P_K)$ with the ball packing problem and call the set of the centers of K equal balls packed in Ω with the largest radius a K packing center, then the K interior peaks of the above solution converges to a K packing center. After the paper was completed, we were told by Professor P. Bates that he and Fusco [6] had made similar connection between the locations of spikes and the ball packing problem in their study of Cahn-Hilliard equation.*

Theorem 1.1 is the first result in proving the existence of multiple interior spike solutions for problem (1.2). Note that for the corresponding Dirichlet problem, multiple interior spike solutions have been constructed in [9], [8], etc.

To introduce the main idea of the proof of Theorem 1.1, we need to give some necessary notations and definitions first.

Let w be the unique solution of (1.3). It is known (see [16]) that w is radially symmetric, decreasing and

$$\lim_{|y|\to\infty} w(y)e^{|y|}|y|^{\frac{N-1}{2}}=c_0>0$$

Associated with problem (1.2) is the following energy functional

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2|\nabla u|^2 + u^2) - \int_\Omega F(u)$$

where $F(u) = \int_0^u f(s)ds$ and $u \in H^1(\Omega)$.

For any smooth bounded domain U, we set $P_U w$ to be the unique solution of

$$\begin{cases} \Delta u - u + f(w) = 0 \text{ in } U, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } U. \end{cases} \tag{1.8}$$

For $P \in \Omega$, we set

$$\Omega_\varepsilon = \{y : \varepsilon y \in \Omega\}, \quad \Omega_{\varepsilon,P} = \{y : \varepsilon y + P \in \Omega\},$$

Fix $\mathbf{P} = (P_1, P_2, ..., P_K) \in \Lambda$, we set

$$Pw_i(y) = P_{\Omega_{\varepsilon,P_i}} w, w_i = w(y - \frac{P_i}{\varepsilon}), \quad y \in \Omega_\varepsilon$$

$$u = \sum_{i=1}^K P_{\Omega_{\varepsilon,P_i}} w + \Phi_{\varepsilon,\mathbf{P}} \in H^2(\Omega_\varepsilon)$$

$$\mathcal{K}_{\varepsilon,\mathbf{P}} = span\{\frac{\partial P_{\Omega_{\varepsilon,P_i}} w}{\partial P_{i,j}}, i = 1, ..., K, j = 1, ..., N\} \subset H^2(\Omega_\varepsilon)$$

$$\mathcal{C}_{\varepsilon,\mathbf{P}} = span\ \{\frac{\partial P_{\Omega_{\varepsilon,P_i}} w}{\partial P_{i,j}}, i = 1, ..., K, j = 1, ..., N\} \subset L^2(\Omega_\varepsilon)$$

We first solve $\Phi_{\varepsilon,\mathbf{P}}$ in $\mathcal{K}_{\varepsilon,\mathbf{P}}$ by using Liapunov-Schmidt reduction method. We show that $\Phi_{\varepsilon,\mathbf{P}}$ is C^1 in $\mathbf{P}$. After that, we define a new function

$$M_\varepsilon(\mathbf{P}) = J_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) \tag{1.9}$$

We maximize $M_\varepsilon(\mathbf{P})$ over Λ. We show that the resulting solution has the properties of Theorem 1.1.

The paper is organized as follows. Notation, preliminaries and some useful estimates are explained in Section 2. Section 3 contains the setup of our problem and we solve (1.2) up to approximate kernel and cokernel, respectively. We set up a maximization problem in Section 4. Finally we show that the solution to the maximization problem is indeed a solution of (1.2) and satisfies all properties of Theorem 1.1.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. $\delta > 0$ is a very small number.

Acknowledgement. The research of the first author is supported by a NSERC research grant of Canada. The research of the second author is supported by an Earmarked Grant from RGC of Hong Kong.

2. Technical Analysis

In this section we introduce a projection and derive some useful estimates.

Let w be the unique solution of (1.3).

Let

$$I(w) = \frac{1}{2}\int_{R^N} (|\nabla w|^2 + w^2) - \frac{1}{p+1}\int_{R^N} w^{p+1}$$

be the energy of w.

Recall that $P_{\Omega_{\varepsilon,P}} w$ is the unique solution of

$$\begin{cases} \Delta v - v + f(w) = 0 \text{ in } \Omega_{\varepsilon,P}, \\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega_{\varepsilon,P} \end{cases} \tag{2.1}$$

where $\Omega_{\varepsilon,P} := \{y | \varepsilon y + P \in \Omega\}$

Set

$$\varphi_{\varepsilon,P}(x) = w(\frac{|x-P|}{\varepsilon}) - P_{\Omega_{\varepsilon,P}} w(y), \ \varepsilon y + P = x.$$

Then $\varphi_{\varepsilon,P}(x)$ satisfies

$$\begin{cases} \varepsilon^2 \triangle v - v = 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu} w(\frac{|x-P|}{\varepsilon}) & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

It is immediately seen that on $\partial\Omega$

$$\begin{aligned} \frac{\partial}{\partial \nu} w(\frac{|x-P|}{\varepsilon}) &= \frac{1}{\varepsilon} w'(\frac{|x-P|}{\varepsilon}) \frac{< x-P, \nu >}{|x-P|} \\ = -\frac{1}{\varepsilon}\Big(|x-P|^{-(N-1)/2} \cdot \varepsilon^{+\frac{N-1}{2}} e^{-\frac{|x-P|}{\varepsilon}} &(1 + O(\varepsilon)) \Big) \frac{< x-P, \nu >}{|x-P|} \\ &= -\varepsilon^{\frac{N-3}{2}} e^{-\frac{|x-P|}{\varepsilon}} (1 + O(\varepsilon)) \frac{< x-P, \nu >}{|x-P|^{\frac{N+1}{2}}}. \end{aligned} \tag{2.3}$$

To analyze $P_{\Omega_{\varepsilon,P}}w$, we introduce another linear problem. Let $P^D_{\Omega_{\varepsilon,P}}w$ be the unique solution of

$$\begin{cases} \varepsilon^2\triangle v - v + f(w) = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Set

$$\varphi^D_{\varepsilon,P} = w - P^D_{\Omega_{\varepsilon,P}}w, \quad \psi^D_{\varepsilon,P}(x) = -\varepsilon\log\varphi^D_{\varepsilon,P}(x).$$

Then $\psi^D_{\varepsilon,P}$ satisfies

$$\begin{cases} \varepsilon\triangle v - |\nabla v|^2 + 1 = 0 & \text{in } \Omega, \\ v = -\varepsilon\log(w(\frac{|x-P|}{\varepsilon})) & \text{on } \partial\Omega. \end{cases}$$

Note that for $x \in \partial\Omega$

$$\psi^D_{\varepsilon,P}(x) = -\varepsilon\log\left((\frac{|x-P|}{\varepsilon})^{-\frac{N-1}{2}}e^{-\frac{|x-P|}{\varepsilon}}(1+O(\varepsilon))\right)$$
$$= |x-P| + \frac{N-1}{2}\varepsilon\log(\frac{|x-P|}{\varepsilon}) + O(\varepsilon^2)$$

By the results of Section 4 of [29] and Section 3 in [38], we have

Lemma 2.1. (1) $\dfrac{\partial\psi^D_{\varepsilon,P}}{\partial\nu} = -(1+O(\varepsilon))\dfrac{< x-P,\nu >}{|x-P|}$,
(2) $\psi^D_{\varepsilon,P}(x) \longrightarrow \psi_0(x) = \inf_{z\in\partial\Omega}(|z-x| + |z-P|)$ *as* $\varepsilon \to 0$ *uniformly in* $\bar{\Omega}$. *In particular* $\psi_0(P) = 2d(P,\partial\Omega)$.

Let us now compare $\varphi_{\varepsilon,P}(x)$ and $\varphi^D_{\varepsilon,P}(x)$. To this end, we introduce another function. Let U_ε be the solution for the following problem

$$\begin{cases} \varepsilon^2\Delta U_\varepsilon - U_\varepsilon = 0 \text{ in } \Omega, \\ U_\varepsilon = 1 \text{ on } \partial\Omega \end{cases}$$

Set

$$\Psi_\varepsilon = -\varepsilon\log(U_\varepsilon)$$

Then by Theorem 1 of [12], we have

$$\Psi_\varepsilon(x) = d(x,\partial\Omega) + O(\varepsilon), \frac{\partial\Psi_\varepsilon}{\partial\nu} = -1 + O(\varepsilon)$$

and

$$|U_\varepsilon(x)| \le Ce^{-\frac{d(x,\partial\Omega)}{\varepsilon}}.$$

Moreover, for any $\sigma_0 > 0$ we have

$$\frac{U_\varepsilon(\varepsilon y + P)}{U_\varepsilon(P)} \le Ce^{(1+\sigma_0)|y|} \tag{2.4}$$

This leads to the following

Lemma 2.2. *There exist $\eta_0, \alpha_0 > 0, \varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, we have*

$$-(1+\eta_0\varepsilon)\varphi^D_{\varepsilon,P} - Ce^{-\frac{1}{\varepsilon}(1+\alpha_0)d(P,\partial\Omega)}U_\varepsilon < \varphi_{\varepsilon,P} < -(1-\eta_0\varepsilon)\varphi^D_{\varepsilon,P} + Ce^{-\frac{1}{\varepsilon}(1+\alpha_0)d(P,\partial\Omega)}U_\varepsilon$$

Proof: We first assume that Ω is convex with respect to P. Namely, there is a constant $c_0 > 0$ such that

$$\langle x - P, \nu_x\rangle \ge c_0 > 0$$

for all $x \in \partial\Omega$, where ν_x is the unit normal at $x \in \partial\Omega$. Then on $\partial\Omega$, we have

$$\begin{aligned}
\frac{\partial\varphi^D_{\varepsilon,P}}{\partial\nu} &= e^{-\frac{\psi^D_{\varepsilon,P}(x)}{\varepsilon}}(-\frac{1}{\varepsilon})\frac{\partial\psi^D_{\varepsilon,P}(x)}{\partial\nu} \\
&= -\frac{1}{\varepsilon}(w)\frac{\partial\psi^D_{\varepsilon,P}(x)}{\partial\nu} \\
&= \frac{1}{\varepsilon}w(1+O(\varepsilon))\frac{< x - P, \nu >}{|x-P|} \\
&= -(1+O(\varepsilon))\frac{\partial\varphi_{\varepsilon,P}}{\partial\nu}.
\end{aligned}$$

Note that since Ω is convex with respect to P, we have $\frac{\partial\varphi^D_{\varepsilon,P}}{\partial\nu} < 0$, hence by comparison principles

$$-(1+\eta_0\varepsilon)\varphi^D_{\varepsilon,P} \le \varphi_{\varepsilon,P} \le -(1-\eta_0\varepsilon)\varphi^D_{\varepsilon,P}.$$

Now for any bounded smooth Ω, we can choose a constant $R = (1 + 2\alpha_0)d(P,\partial\Omega)$ for some $\alpha_0 > 0$ such that $\Omega_1 := B_R(P) \cap \Omega$ is strictly convex with respect to P, i.e.

$$< x - P, \nu_x > \ge c_0 > 0, \quad x \in \partial\Omega_1.$$

Then on $\partial\Omega_1 \cap \partial\Omega = \Gamma_1$, we have

$$\frac{\partial\varphi_{\varepsilon,P}}{\partial\nu} \le -(1+O(\varepsilon))\frac{\partial\varphi^D_{\varepsilon,P}}{\partial\nu}$$

On $\partial\Omega\backslash\Gamma_1$, we have

$$\left|\frac{\partial\varphi^D_{\varepsilon,P}}{\partial\nu}\right| \le Ce^{-(1+2\alpha_0)\frac{1}{\varepsilon}d(P,\partial\Omega)}$$

$$\frac{\partial \varphi_{\varepsilon,P}}{\partial \nu} \leq Ce^{-(1+2\alpha_0)\frac{1}{\varepsilon}d(P,\partial\Omega)} \leq Ce^{-(1+\alpha_0)\frac{1}{\varepsilon}d(P,\partial\Omega)}\frac{\partial U_\varepsilon}{\partial \nu}$$

By comparison principle, we get the inequality.

Lemma 2.2 is thus proved. □

By Lemma 2.2, we have that

$$\psi_\varepsilon(P) := -\varepsilon \log\left(-\varphi_{\varepsilon,P}(P)\right) \to 2d(P,\partial\Omega)$$

since

$$\varphi_{\varepsilon,P}(P) = (-1+O(\varepsilon))\varphi^D_{\varepsilon,P}(P) + O(e^{-(2+\alpha_0)\frac{1}{\varepsilon}d(P,\partial\Omega)}).$$

Let

$$V_{\varepsilon,P}(y) = \frac{1}{\varphi_{\varepsilon,P}(P)}\cdot\varphi_{\varepsilon,P}(\varepsilon y + P).$$

Then $V_{\varepsilon,P}(0) = 1$ (hence $V_{\varepsilon,P}(y) > 0$ by Harnack inequality) and we have

Lemma 2.3. *For every sequence $\varepsilon_k \to 0$, there is a subsequence $\varepsilon_{k\ell} \to 0$ such that $V_{\varepsilon_{k\ell},P} \to \bar{V}$ uniformly on every compact set of R^N, where $\bar{V}$ is a positive solution of*

$$\begin{cases} \triangle u - u = 0 \ \text{in } R^N, \\ u > 0 \ \text{in } R^N \ \text{and } u(0) = 1. \end{cases} \tag{2.5}$$

Moreover for any $c_1 > 0$, $\sup_{z\in\Omega_{\varepsilon_{k\ell},P}} e^{-(1+c_1)|z|}\left|V_{\varepsilon_{k\ell},P}(z) - \bar{V}\right| \to 0$ as $\varepsilon_{k\ell} \to 0$.

Proof: The proof is similar to that of Lemma 4.4 (ii) in [29]. □

Next we state some useful lemmas about the interactions of two w's.

Lemma 2.4. *Let $f \in C(R^N)\cap L^\infty(R^N), g \in C(R^N)$ be radially symmetric and satisfy for some $\alpha \geq 0, \beta \geq 0, \gamma_0 \in R$*

$$f(x)exp(\ \alpha|x|)|x|^\beta \to \gamma_0 \ \text{as } |x| \to \infty$$

$$\int_{R^N} |g(x)|exp(\ \alpha|x|)(1+|x|^\beta) < \infty$$

Then

$$exp(\alpha|y|)|y|^\beta \int_{R^N} g(x+y)f(x)dx \to \gamma_0 \int_{R^N} g(x)exp(\ -\alpha x_1)dx \ \text{as } |y| \to \infty.$$

For the proof, see [7].

We then have the following estimates.

Lemma 2.5. $\frac{1}{w(\frac{|P_1-P_2|}{\varepsilon})}\int_{R^N} f(w_1)w_2 \to \gamma > 0$ *where*

$$\gamma = \int_{R^N} f(w(y))e^{-y_1}dy \tag{2.6}$$

The next lemmma is the key result in this section.

Lemma 2.6. *For any* $\mathbf{P} = (P_1, ..., P_K) \in \Lambda$ *and* ε *sufficiently small*

$$J_\varepsilon(\sum_{i=1}^K Pw_i) = \varepsilon^N[KI(w) - \frac{1}{2}(\gamma + o(1))\sum_{i=1}^K (e^{-\frac{1}{\varepsilon}\psi_\varepsilon(P_i)}$$

$$-(\gamma + o(1))\sum_{i,l=1,i\neq l}^K w(\frac{|P_i - P_l|}{\varepsilon})], \tag{2.7}$$

where γ *is defined by (2.6).*

Proof:

We shall prove the case when $K = 2$. The other cases are similar.

By (1.6), we have that $d(P_i, \partial\Omega) > \delta > 0, i = 1, 2, |P_1 - P_2| > 2\delta > 0$.

Note that by Lemma 2.3 and similar arguments as in the proof of Lemma 5.1 of [29] we have

$$\varepsilon^2\int_\Omega |\nabla P_{\Omega_{\varepsilon,P}}w|^2 + \int_\Omega |P_{\Omega_{\varepsilon,P}}w|^2 = \varepsilon^N\int_{\Omega_{\varepsilon,P}} f(w)P_{\Omega_{\varepsilon,P}}w$$

$$= \varepsilon^N(\int_{\Omega_{\varepsilon,P}} f(w)w + \int_{\Omega_{\varepsilon,P}} f(w)(P_{\Omega_{\varepsilon,P}}w - w))$$

$$= \varepsilon^N(\int_{R^N} f(w)w - \varphi_{\varepsilon,P}(P)\int_{\Omega_{\varepsilon,P}} f(w)V_{\varepsilon,P} + o(\varphi_{\varepsilon,P}(P)))$$

$$= \varepsilon^N(\int_{R^N} f(w)w - \varphi_{\varepsilon,P}(P)\gamma + o(\varphi_{\varepsilon,P}(P)))$$

since

$$\gamma = \int_{R^N} f(w)\tilde{V} = \int_{R^N} f(w)e^{-y_1}$$

for any solution $\tilde{V}$ of (2.5) (see Lemma 4.7 in [29]).

Similarly we have

$$\int_\Omega F(P_{\Omega_{\varepsilon,P}}w) = \varepsilon^N(\int_{R^N} F(w) - \varphi_{\varepsilon,P}(P)\gamma + o(\varphi_{\varepsilon,P}(P)))$$

By Lemmas 2.1, 2.2 and 2.5, we have

$$\int_\Omega f(w_1)Pw_2 = \varepsilon^N(\gamma + o(1))w(\frac{|P_1 - P_2|}{\varepsilon}) + \varepsilon^N o(\varphi_{\varepsilon,P_2}(P_2)).$$

$$\int_\Omega f(Pw_1)Pw_2 = \varepsilon^N(\gamma + o(1))w(\frac{|P_1 - P_2|}{\varepsilon}) + o(\sum_{i=1}^2 \varphi_{\varepsilon,P_i}(P_i)).$$

Let $\delta > 0$ be a sufficiently small number. We then have

$$\int_\Omega F(Pw_1+Pw_2) = \int_{\Omega_1} F(Pw_1+Pw_2)+\int_{\Omega_2} F(Pw_1+Pw_2)+\int_{\Omega_3} F(Pw_1+Pw_2)$$

$$= I_1 + I_2 + I_3$$

where $I_i, i = 1, 2, 3$ are defined at the last equality and

$$\Omega_1 = \{|x-P_1| \le \frac{1-\delta}{2}|P_1-P_2|\}, \Omega_2 = \{|x-P_2| \le \frac{1-\delta}{2}|P_1-P_2|\}, \Omega_3 = \Omega\backslash(\Omega_1\cup\Omega_2)$$

For I_3, we have

$$|I_3| \le C\int_{\Omega_3}(w_1 + w_2)^{2+\sigma} = O(e^{-(1+\sigma)\frac{1}{\varepsilon}|P_1-P_2|})$$

To estimate I_1, we first observe that

$$\int_{\Omega_1} |F(Pw_1 + Pw_2) - F(Pw_1) - f(Pw_1)Pw_2| \le C\int_{\Omega_1} |Pw_1|^{p_3-\sigma}|Pw_2|^{1+\sigma}$$

(for $0 < \sigma < \min(1, (p_3 - 1)/2)$)

$$\le O(e^{-(1+\sigma)\frac{|P_1-P_2|}{\varepsilon}})$$

by Lemma 2.4 since $p_3 - \sigma > 1 + \sigma$.

Therefore, we have

$$\begin{aligned} I_1 =& \int_{\Omega_1}(F(Pw_1) + f(Pw_1)Pw_2) + O(e^{-(1+\sigma)\frac{1}{\varepsilon}|P_1-P_2|}) \\ =& \int_\Omega F(Pw_1) + \int_{\Omega_1} f(w_1)w_2 + O(e^{-(1+\sigma)\frac{1}{\varepsilon}|P_1-P_2|}) \\ =&\varepsilon^N[\int_{R^N} F(w) - \gamma\varphi_{\varepsilon,P_1}(P_1) + \int_{\Omega_1} f(w_1)w_2 + O(e^{-(1+\sigma)\frac{1}{\varepsilon}|P_1-P_2|}) \\ &+o(\varepsilon^N\sum_{i=1}^2 \varphi_{\varepsilon,P_i}(P_i)). \end{aligned}$$

Similarly,

$$I_2 = \varepsilon^N [\int_{R^N} F(w) - \gamma \varphi_{\varepsilon,P_2}(P_2) + \int_{\Omega_2} f(w_2) w_1$$
$$+ O(e^{-(1+\sigma)\frac{1}{\varepsilon}|P_1 - P_2|}) + o(\sum_{i=1}^{2} \varphi_{\varepsilon,P_i}(P_i))]$$

Hence

$$\varepsilon^{-N} J_\varepsilon(\sum_{i=1}^{K} Pw_i) = \int_{\Omega_\varepsilon} [\frac{1}{2}(\sum_{i=1}^{2}(|\nabla Pw_i|^2 + (Pw_i)^2)) + \nabla Pw_1 \nabla Pw_2 + Pw_1 Pw_2]$$
$$- \int_{\Omega_\varepsilon} (F(Pw_1 + Pw_2))$$
$$= \int_{\Omega_\varepsilon} [\frac{1}{2}(\sum_{i=1}^{2}(|\nabla Pw_i|^2 + (Pw_i)^2))] + \int_{\Omega_\varepsilon} f(w_1) Pw_2$$
$$- \int_{\Omega_\varepsilon} F(Pw_1 + Pw_2)$$
$$= 2I(w) - \frac{1}{2}(\gamma + o(1)) \sum_{i=1}^{2} \varphi_{\varepsilon,P_i}(P_i) + \int_{\Omega_\varepsilon} f(w_1) Pw_2$$
$$- \int_{\Omega_1} f(w_1) w_2 - \int_{\Omega_2} f(w_2) w_1 + o(w(\frac{|P_1 - P_2|}{\varepsilon}))$$
$$= 2I(w) - \frac{1}{2}(\gamma + o(1)) \sum_{i=1}^{2} \varphi_{\varepsilon,P_i}(P_i) - (\gamma + o(1)) w(\frac{|P_1 - P_2|}{\varepsilon})$$

since

$$\int_{\Omega_\varepsilon} f(w_1) w_2 = (\gamma + o(1)) w(\frac{|P_1 - P_2|}{\varepsilon}),$$

$$\int_{\Omega_1} f(w_1) w_2 = (\gamma + o(1)) w(\frac{|P_1 - P_2|}{\varepsilon}),$$

and

$$\int_{\Omega_3} f(w_2) w_1 = (\gamma + o(1)) w(\frac{|P_1 - P_2|}{\varepsilon})$$

3. Liapunov-Schmidt Reduction

In this section, we solve problem (1.2) in approriate kernel and cokernel. We first introduce some notations.

Let $H^2_N(\Omega_\varepsilon)$ be the Hilbert space defined by

$$H^2_N(\Omega_\varepsilon)=\left\{u\in H^2(\Omega_\varepsilon)\,\middle|\,\frac{\partial u}{\partial\nu_\varepsilon}=0 \text{ on } \partial\Omega_\varepsilon\right\}.$$

Define

$$S_\varepsilon(u)=\Delta u-u+f(u)$$

for $u\in H^2_N(\Omega_\varepsilon)$. Then solving equation (1.2) is equivalent to

$$S_\varepsilon(u)=0, u\in H^2_N(\Omega_\varepsilon).$$

Fix $\mathbf{P}=(P_1,...,P_K)\in\Lambda$. To study (1.2) we first consider the linearized operator

$$\tilde{L}_\varepsilon: u(z)\mapsto \Delta u(z)-u(z)+f'\left(\sum_{i=1}^K P_{\Omega_{\varepsilon,P_i}}w\right)u(z),$$
$$H^2_N(\Omega_\varepsilon)\to L^2(\Omega_\varepsilon).$$

It is easy to see (integration by parts) that the cokernel of $\tilde{L}_\varepsilon$ coincides with its kernel. Choose approximate cokernel and kernel as

$$\mathcal{C}_{\varepsilon,\mathbf{P}}=\mathcal{K}_{\varepsilon,\mathbf{P}}$$
$$=\text{span}\left\{\frac{\partial P_{\Omega_{\varepsilon,P_i}}w}{\partial P_{i,j}}\,\middle|\, i=1,\ldots,K,\, j=1,\ldots,N\right\}.$$

Let $\pi_{\varepsilon,P_1,\ldots,P_M}$ denote the projection in $L^2(\Omega_\varepsilon)$ onto $\mathcal{C}^\perp_{\varepsilon,P_1,\ldots,P_M}$. Our goal in this section is to show that the equation

$$\pi_{\varepsilon,\mathbf{P}}\circ S_\varepsilon(\sum_{i=1}^K P_{\Omega_{\varepsilon,P_i}}w+\Phi_{\varepsilon,P_1,\ldots,P_K})=0$$

has a unique solution $\Phi_{\varepsilon,\mathbf{P}}\in\mathcal{K}^\perp_{\varepsilon,\mathbf{P}}$ if ε is small enough. Moreover $\Phi_{\varepsilon,\mathbf{P}}$ is C^1 in $\mathbf{P}=(P_1,...,P_K)$.

As a preparation in the following two propositions we show invertibility of the corresponding linearized operator.

Proposition 3.1. *Let $L_{\varepsilon,\mathbf{P}}=\pi_{\varepsilon,\mathbf{P}}\circ\tilde{L}_\varepsilon$. There exist positive constants $\overline{\varepsilon},\lambda$ such that for all $\varepsilon\in(0,\overline{\varepsilon})$ and $\mathbf{P}=(P_1,...,P_K)\in\Lambda$*

$$\|L_{\varepsilon,\mathbf{P}}\Phi\|_{L^2(\Omega_\varepsilon)}\geq\lambda\|\Phi\|_{H^2(\Omega_\varepsilon)} \tag{3.1}$$

for all $\Phi\in\mathcal{K}^\perp_{\varepsilon,\mathbf{P}}$.

Proposition 3.2. *For all $\varepsilon \in (0, \bar{\varepsilon})$ and $\mathbf{P} = (P_1, ..., P_K) \in \Lambda$ the map*

$$L_{\varepsilon,\mathbf{P}} = \pi_{\varepsilon,\mathbf{P}} \circ \tilde{L}_\varepsilon : \mathcal{K}^\perp_{\varepsilon,\mathbf{P}} \to \mathcal{C}^\perp_{\varepsilon,\mathbf{P}}$$

is surjective.

Proof of Proposition 3.1: We will follow the method used in [13], [30], [31], and [42]. Suppose that (3.1) is false. Then there exist sequences $\{\varepsilon_k\}$, $\{P_i^k\}$, and $\{\Phi_k\}$ $(i = 1, 2, \ldots, K,\ k = 1, 2, \ldots)$ with $\mathbf{P}^k = (P_1^k, ..., P_K^k) \in \Lambda$, $\Phi_k \in \mathcal{K}^\perp_{\varepsilon_k,\mathbf{P}^k}$ such that

$$\|L_{\varepsilon_k,\mathbf{P}^k}\Phi_k\|_{L^2} \to 0, \tag{3.2}$$

$$\|\Phi_k\|_{H^2} = 1, \qquad k = 1,\, 2,\, \ldots\,. \tag{3.3}$$

We omit the argument Ω_{ε_k} where this can be done without causing confusion.

As before, we set

$$Pw_{i,k}(y) = P_{\Omega_{\varepsilon_k,P_i^k}} w(y - P_i^k/\varepsilon), \quad y \in \Omega_{\varepsilon_k}.$$

For $j = 1, 2, \ldots, N$, we denote

$$e_{ij,k} = \frac{\partial}{\partial P_{i,j}^k} Pw_{i,k} \Big/ \left\| \frac{\partial}{\partial P_{i,j}^k} Pw_{i,k} \right\|_{L^2}.$$

Note that

$$< e_{i_1j_1,k}, e_{i_2j_2,k} >= \delta_{i_1i_2}\delta_{j_1j_2} + O(\varepsilon_k) \qquad \text{as } k \to \infty$$

since $\frac{\partial Pw_i}{\partial P_{i,j}} - \frac{\partial w_i}{\partial P_{i,j}}$ is exponentially small, where $\delta_{i_1i_2}$ is the Kronecker symbol. Furthermore because of (3.2),

$$\|L_{\varepsilon_k,\mathbf{P}^k}\Phi_k\|^2_{L^2} - \sum_{i=1}^K \sum_{j=1}^N \left(\int_{\Omega_{\varepsilon_k}} L_{\varepsilon_k,\mathbf{P}^k}\Phi_k e_{ij,k} \right)^2 \to 0 \tag{3.4}$$

as $k \to \infty$.

We introduce new sequences $\{\varphi_{i,k}\}$ by

$$\varphi_{i,k}(y) = \chi(\varepsilon y - P_i^k)\Phi_k(y), \quad y \in \Omega_{\varepsilon_k} \tag{3.5}$$

where $\chi(z)$ is a cut-off function such that $\chi(z) = 1$ for $|z| \le \delta$ and $\chi(z) = 0$ for $|z| > 2\delta$ where δ is small (actually we choose δ as in (1.6)).

It follows from (3.3) and the smoothness of χ that

$$\|\varphi_{i,k}(\cdot + P_i^k/\varepsilon_k)\|_{H^2(R^N)} \le C$$

for all k sufficiently large. Therefore there exists a subsequence, again denoted by $\{\varphi_{i,k}\}$ which converges weakly in $H^2(R^N)$ to a limit $\varphi_{i,\infty}$ as $k \to \infty$. We are now going to show that $\varphi_{i,\infty} \equiv 0$. As a first step we deduce

$$\int_{R^N} \varphi_{i,\infty} \frac{\partial w}{\partial y_j} = 0, \qquad j = 1, \dots, N. \tag{3.6}$$

This statement is shown as follows.

$$\int_{R^N} \varphi_{i,k}(y + P_i^k/\varepsilon_k) \frac{\partial w}{\partial y_j} dy$$
$$= \int_{\Omega_{\varepsilon_k, P_i^k}} \chi(\varepsilon y) \Phi_k(y + P_i^k) \frac{\partial P w_{i,k}}{\partial P_{i,j}^k} dy + o(1)$$
$$= \int_{\Omega_{\varepsilon_k}} (\chi(\varepsilon y - P_i^k) - 1) \Phi_k \frac{\partial P w_{i,k}}{\partial P_{i,j}^k} dy + o(1)$$
$$= o(1)$$

Here we have used the facts that $\Phi_k \in \mathcal{K}_{\varepsilon_k, \mathbf{P}^k}$ and $w(\frac{|x - P_i^k|}{\varepsilon})$ is exponentially decaying outside $B_\delta(P_i^k)$. This implies (3.6).

Let $\mathcal{K}_0$ and $\mathcal{C}_0$ be the kernel and cokernel, respectively, of the linear operator $S_0'(w)$ which is the Fréchet derivative at w of

$$S_0(v) = \Delta v - v + f(v),$$
$$S_0 : H^2(R^N) \to L^2(R^N),$$

Note that

$$\mathcal{K}_0 = \mathcal{C}_0 = \text{span} \left\{ \frac{\partial w}{\partial y_j} \,|j = 1, \dots, N \right\}.$$

Equation (3.6) implies that $\varphi_{i,\infty} \in \mathcal{K}_0^\perp$. By the exponential decay of w and by (3.2) we have after possibly taking a further subsequence that

$$\Delta \varphi_{i,\infty} - \varphi_{i,\infty} + f'(w) \varphi_{i,\infty} = 0,$$

i.e. $\varphi_{i,\infty} \in \mathcal{K}_0$. Therefore $\varphi_{i,\infty} = 0$.

Hence

$$\varphi_{i,k} \rightharpoonup 0 \qquad \text{weakly in } H^2(R^N) \tag{3.7}$$

as $k \to \infty$. By the definition of $\varphi_{i,k}$ we get $\Phi_k \rightharpoonup 0$ in H^2 and

$$\|f'(\sum_{i=1}^K P_{\Omega_{\varepsilon_k, P_i^k}} w) \Phi_k\|_{L^2} \to 0 \qquad \text{as } k \to \infty.$$

Furthermore,

$$\|(\Delta - 1)\Phi_k\|_{L^2} \to 0 \qquad \text{as } k \to \infty.$$

Since

$$\int_{\Omega_{\varepsilon_k}} |\nabla \Phi_k|^2 + \Phi_k^2 = \int_{\Omega_{\varepsilon_k}} [(1-\Delta)\Phi_k]\Phi_k$$
$$\leq C\|(\Delta - 1)\Phi_k\|_{L^2}$$

we have that

$$\|\Phi_k\|_{H^1} \to 0 \qquad \text{as } k \to \infty.$$

In summary:

$$\|\Delta\Phi_k\|_{L^2} \to 0 \text{ and } \|\Phi_k\|_{H^1} \to 0. \tag{3.8}$$

From (3.8) and the following elliptic regularity estimate (for a proof see Appendix B in [42])

$$\|\Phi_k\|_{H^2} \leq C(\|\Delta\Phi_k\|_{L^2} + \|\Phi_k\|_{H^1}) \tag{3.9}$$

for $\Phi_k \in H^2_N$ we imply that

$$\|\Phi_k\|_{H^2} \to 0 \qquad \text{as } k \to \infty.$$

This contradicts the assumption

$$\|\Phi_k\|_{H^2} = 1$$

and the proof of Proposition 3.1 is completed. □

Proof of Proposition 3.2: We define a linear operator T from $L^2(\Omega_\varepsilon)$ to itself by

$$T = \pi_{\varepsilon,\mathbf{P}} \circ \tilde{L} \circ \pi_{\varepsilon,\mathbf{P}}$$

It's domain of definition is $H^2_N(\Omega_\varepsilon)$. By the theory of elliptic equations and by integration by parts it is easy to see that T is a (unbounded) self-adjoint operator on $L^2(\Omega_\varepsilon)$ and a closed operator. The L^2 estimates of elliptic equations imply that the range of T is closed in $L^2(\Omega_\varepsilon)$. Then by the Closed Range Theorem ([46], page 205), we know that the range of T is the orthogonal complement of its kernel which is, by Proposition 3.1, $\mathcal{K}_{\varepsilon,\mathbf{P}}$. This leads to Proposition 3.2. □

We are now in a position to solve the equation

$$\pi_{\varepsilon,\mathbf{P}} \circ S_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) = 0. \tag{3.10}$$

Note that

$$S_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,P_1,\ldots,P_K}) = \tilde{L}_\varepsilon(\Phi_{\varepsilon,\mathbf{P}}) + N_{\varepsilon,\mathbf{P}} + M_{\varepsilon,\mathbf{P}} \tag{3.11}$$

where

$$N_{\varepsilon,\mathbf{P}} = f(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) - f(\sum_{i=1}^K Pw_i) - f'(\sum_{i=1}^K Pw_i)\Phi_{\varepsilon,\mathbf{P}}$$

and

$$M_{\varepsilon,\mathbf{P}} = f(\sum_{i=1}^K Pw_i) - \sum_{i=1}^K f(w_i).$$

Before we move on, we need the following error estimates.

Lemma 3.3. *For ε sufficiently small, we have*

$$|N_{\varepsilon,\mathbf{P}}| \le C(|\Phi_{\varepsilon,\mathbf{P}}|^{1+\sigma} + |\Phi_{\varepsilon,\mathbf{P}}|^{p_1}) \tag{3.12}$$

$$\|M_{\varepsilon,\mathbf{P}}\|_{L^2(\Omega_\varepsilon)} \le Ce^{-\frac{1+\sigma}{2}\frac{1}{\varepsilon}\varphi(P_1,\ldots,P_K)} \tag{3.13}$$

Proof: It is easy to derive (3.12) from the mean value theorem.

To prove (3.13), we divide the domain into $(K+1)$ parts: let $\Omega = \cup_{i=1}^{K+1}\Omega_i$ where

$$\Omega_i = \{|x - P_i| \le \frac{1}{2}\min_{k\neq l}|P_k - P_l|\}, \quad i = 1, ..., K,$$

$$\Omega_{K+1} = \Omega \setminus \cup_{i=1}^K \Omega_i.$$

We now estimate $M_{\varepsilon,\mathbf{P}}$ in each domain.

In Ω_{K+1}, we have

$$|M_{\varepsilon,P_1,\ldots,P_K}| \le (w_1 + ... + w_K)^{1+\sigma}$$
$$\le O(e^{-\frac{1+\sigma}{2}\frac{1}{\varepsilon}\min_{k\neq l}|P_k - P_l|})$$

Hence

$$\|M_{\varepsilon,\mathbf{P}}\|_{L^2((\Omega_{K+1})_\varepsilon)} \le O(e^{-\frac{1+\sigma}{2}\frac{1}{\varepsilon}\varphi(P_1,\ldots,P_K)})$$

In $\Omega_i, i = 1, ..., K$, we have

$$|M_{\varepsilon,\mathbf{P}}| \le \sum_{j\neq i}\Big(|f'(w_i)w_j| + |f'(w_i)(Pw_j - w_j)|\Big)\,|$$

$$+O(\sum_{j\neq i}(|Pw_j|^{1+\sigma}+|w_j|^{1+\sigma}))+O(|Pw_i-w_i|^{1+\sigma})$$

Hence using Lemmas 2.1, 2.2 and the exponential decay of w, we have

$$\|M_{\varepsilon,\mathbf{P}}\|_{L^2}\leq O(e^{-\frac{1+\sigma}{2}\frac{1}{\varepsilon}\varphi(P_1,\ldots,P_K)})$$

□

Next we solve (3.12). Since $L_{\varepsilon,\mathbf{P}}|_{\mathcal{K}^\perp_{\varepsilon,\mathbf{P}}}$ is invertible (call the inverse $L^{-1}_{\varepsilon,\mathbf{P}}$) we can rewrite

$$\begin{aligned}\Phi &= -(L^{-1}_{\varepsilon,\mathbf{P}}\circ\pi_{\varepsilon,\mathbf{P}})(M_{\varepsilon,\mathbf{P}})\\ &\quad -(L^{-1}_{\varepsilon,\mathbf{P}}\circ\pi_{\varepsilon,\mathbf{P}})N_{\varepsilon,\mathbf{P}}(\Phi)\\ &\equiv G_{\varepsilon,\mathbf{P}}(\Phi)\end{aligned}\tag{3.14}$$

where the operator $G_{\varepsilon,\mathbf{P}}$ is defined by the last equation for $\Phi\in H^2_N(\Omega_\varepsilon)$. We are going to show that the operator $G_{\varepsilon,\mathbf{P}}$ is a contraction on

$$B_{\varepsilon,\delta}\equiv\{\Phi\in H^2_N(\Omega_\varepsilon)|\|\Phi\|_{H^2(\Omega_\varepsilon)}<\delta\}$$

if δ is small enough. We have

$$\begin{aligned}\|G_{\varepsilon,\mathbf{P}}(\Phi)\|_{H^2(\Omega_\varepsilon)}&\leq\lambda^{-1}(\|\pi_{\varepsilon,\mathbf{P}}\circ N_{\varepsilon,\mathbf{P}}(\Phi)\|_{L^2(\Omega_\varepsilon)}\\ &\quad+\|\pi_{\varepsilon,\mathbf{P}}\circ(M_{\varepsilon,\mathbf{P}})\|_{L^2(\Omega_\varepsilon)})\\ &\leq\lambda^{-1}C(c(\delta)\delta+\delta_\varepsilon)\end{aligned}$$

where $\lambda>0$ is independent of $\delta>0$, $\delta_\varepsilon=e^{-\frac{1+\sigma}{2}\frac{1}{\varepsilon}\varphi(P_1,\ldots,P_K)}$ and $c(\delta)\to0$ as $\delta\to0$. Similarly we show

$$\|G_{\varepsilon,\mathbf{P}}(\Phi)-G_{\varepsilon,\mathbf{P}}(\Phi')\|_{H^2(\Omega_\varepsilon)}\leq\lambda^{-1}c(\delta)\|\Phi-\Phi'\|_{H^2(\Omega_\varepsilon)}$$

where $c(\delta)\to0$ as $\delta\to0$. Therefore $G_{\varepsilon,\mathbf{P}}$ is a contraction on B_δ. The existence of a fixed point $\Phi_{\varepsilon,\mathbf{P}}$ now follows from the Contraction Mapping Principle and $\Phi_{\varepsilon,\mathbf{P}}$ is a solution of (3.14).

Because of

$$\begin{aligned}\|\Phi_{\varepsilon,\mathbf{P}}\|_{H^2(\Omega_{\varepsilon,\mathbf{P}})}&\leq\lambda^{-1}(\|N_{\varepsilon,\mathbf{P}}(\Phi_{\varepsilon,\mathbf{P}})\|_{L^2(\Omega_\varepsilon)}\\ &\quad+\|M_{\varepsilon,P_1,\ldots,P_K}\|_{L^2})\\ &\leq\lambda^{-1}(C\delta_\varepsilon+c(\delta)\|\Phi_{\varepsilon,\mathbf{P}}\|_{H^2(\Omega_\varepsilon)})\end{aligned}$$

we have

$$(1-\lambda^{-1}c(\delta))\|\Phi_{\varepsilon,\mathbf{P}}\|_{H^2}\leq C\delta_\varepsilon.$$

We have proved

Lemma 3.4. *There exists $\bar{\varepsilon} > 0$ such that for any $0 < \varepsilon < \bar{\varepsilon}$ and $\mathbf{P} \in \Lambda$ there exists a unique $\Phi_{\varepsilon,\mathbf{P}} \in \mathcal{K}^{\perp}_{\varepsilon,\mathbf{P}}$ satisfying $S_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) \in \mathcal{C}^{\perp}_{\varepsilon,\mathbf{P}}$ and*

$$\|\Phi_{\varepsilon,\mathbf{P}}\|_{H^2(\Omega_\varepsilon)} \leq C\delta_\varepsilon. \tag{3.15}$$

where $\delta_\varepsilon = e^{-\frac{1+\sigma}{2}\frac{1}{\varepsilon}\varphi(P_1,\ldots,P_K)}$.

Finally we show that $\Phi_{\varepsilon,\mathbf{P}}$ is actually smooth in $\mathbf{P}$

Lemma 3.5. *Let $\Phi_{\varepsilon,\mathbf{P}}$ be defined by Lemma 3.4. Then $\Phi_{\varepsilon,\mathbf{P}} \in C^1$ in $\mathbf{P}$.*

Proof:

Recall that $\Phi_{\varepsilon,\mathbf{P}}$ is a solution of the equation

$$\pi_{\varepsilon,\mathbf{P}} \circ S_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) = 0 \tag{3.16}$$

such that

$$\Phi_{\varepsilon,\mathbf{P}} \in \mathcal{K}^{\perp}_{\varepsilon,\mathbf{P}}. \tag{3.17}$$

Note that by differentiating equation (3.16) twice we easily conclude that the functions Pw_i and $\partial^2 Pw_i/(\partial P_{i,j}\partial P_{i,k})$ are C^1 in $\mathbf{P}$. This implies that the projection $\pi_{\varepsilon,\mathbf{P}}$ is C^1 in $\mathbf{P}$. Applying $\partial/\partial P_{i,j}$ to (3.16) gives

$$\pi_{\varepsilon,\mathbf{P}} \circ DS_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) \left(\sum_{i=1}^K \frac{\partial Pw_i}{\partial P_{i,j}} + \frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}}\right)$$

$$+\frac{\partial \pi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}} \circ S_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) = 0. \tag{3.18}$$

where

$$DS_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) = \Delta - 1 + f'(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}).$$

We decompose $\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}}$ into two parts:

$$\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}} = \left(\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_1 + \left(\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_2$$

where $\left(\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_1 \in \mathcal{K}_{\varepsilon,\mathbf{P}}$ and $\left(\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_2 \in \mathcal{K}^{\perp}_{\varepsilon,\mathbf{P}}$.

We can easily see that $\left(\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}}\right)_1$ is continuous in $\mathbf{P}$ since

$$\int_{\Omega_\varepsilon} \Phi_{\varepsilon,\mathbf{P}} \frac{\partial P w_k}{\partial P_{k,l}} = 0, \quad k, l = 1, ..., N$$

and hence

$$\int_{\Omega_\varepsilon} \frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial \tau_{P_{i,j}}} \frac{\partial P w_k}{\partial P_{k,l}} + \int_{\Omega_\varepsilon} \Phi_{\varepsilon,\mathbf{P}} \frac{\partial^2 P w_k}{\partial \tau_{P_{i,j}} \partial P_{k,l}} = 0$$

where i, j, k, l are indice from 1 to K.

Now we can write equation (3.23) as

$$\pi_{\varepsilon,\mathbf{P}} \circ DS_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) \left((\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}})_2 \right)$$

$$+\pi_{\varepsilon,\mathbf{P}} \circ DS_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) \left(\sum_{i=1}^K \frac{\partial P w_i}{\partial P_{i,j}} + (\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}})_1 \right)$$

$$+\frac{\partial \pi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}} \circ S_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}}) = 0. \tag{3.19}$$

As in the proof of Propositions 3.1 and 3.2 we can show that the operator

$$\pi_{\varepsilon,\mathbf{P}} \circ DS_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}})$$

is invertible from $\mathcal{K}_{\varepsilon,\mathbf{P}}^\perp$ to $\mathcal{C}_{\varepsilon,\mathbf{P}}^\perp$. Then we can take inverse of $\pi_{\varepsilon,\mathbf{P}} \circ DS_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,\mathbf{P}})$ in above equation, and the inverse is continuous in $\mathbf{P}$.

Since $\frac{\partial P w_i}{\partial \tau_{P_{i,j}}}, (\frac{\partial \Phi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}})_1 \in \mathcal{K}_{\varepsilon,\mathbf{P}}$ are continuous in $\mathbf{P} \in \Lambda$ and so is $\frac{\partial \pi_{\varepsilon,\mathbf{P}}}{\partial P_{i,j}}$, we conclude that $(\partial \Phi_{\varepsilon,\mathbf{P}}/(\partial P_{i,j}))_2$ is also continuous in $\mathbf{P}$. This is the same as the C^1 dependence of $\Phi_{\varepsilon,\mathbf{P}}$ in $\mathbf{P}$. The proof is finished. □

4. The reduced problem: A Maximization Procedure

In this section, we study a maximization problem.

Fix $\mathbf{P} \in \overline{\Lambda}$. Let $\Phi_{\varepsilon,\mathbf{P}}$ be the solution given by Lemma 3.4. We define a new functional

$$M_\varepsilon(\mathbf{P}) = J_\varepsilon(\sum_{i=1}^K Pw_i + \Phi_{\varepsilon,P_1,...,P_K}) : \overline{\Lambda} \to R \tag{4.1}$$

We shall prove

Proposition 4.1. *For ε small, the following maximization problem*

$$\max\{M_\varepsilon(\mathbf{P}) : \mathbf{P} \in \overline{\Lambda}\} \tag{4.2}$$

has a solution $\mathbf{P}^\varepsilon \in \Lambda$.

Proof: Since $J_\varepsilon(\sum_{i=1}^K P_{\Omega_{\varepsilon,P_i}} w + \Phi_{\varepsilon,\mathbf{P}})$ is continuous in $\mathbf{P}$, the maximization problem has a solution. Let $M_\varepsilon(\mathbf{P}^\varepsilon)$ be the maximum where $\mathbf{P}^\varepsilon \in \overline{\Lambda}$.

We claim that $\mathbf{P}^\varepsilon \in \Lambda$.

In fact for any $\mathbf{P} \in \overline{\Lambda}$, we have

$$M_\varepsilon(\mathbf{P}) = J_\varepsilon(\sum_{i=1}^K Pw_i) + g_{\varepsilon,\mathbf{P}}(\Phi_{\varepsilon,\mathbf{P}}) + O(\|\Phi_{\varepsilon,\mathbf{P}}\|^2_{H^2})$$

where

$$\begin{aligned} & g_{\varepsilon,\mathbf{P}}(\Phi_{\varepsilon,\mathbf{P}}) \\ = & \int_{\Omega_\varepsilon} \sum_{i=1}^K (\nabla Pw_i \nabla \Phi_{\varepsilon,\mathbf{P}} + Pw_i \Phi_{\varepsilon,\mathbf{P}}) - \int_{\Omega_\varepsilon} f(\sum_{i=1}^K Pw_i)\Phi_{\varepsilon,\mathbf{P}} \\ = & \int_{\Omega_\varepsilon} [\sum_{i=1}^K f(w_i) - f(\sum_{i=1}^K Pw_i)]\Phi_{\varepsilon,\mathbf{P}} \\ \leq & \|\sum_{i=1}^K f(w_i) - f(\sum_{i=1}^K Pw_i)\|_{L^2} \|\Phi_{\varepsilon,\mathbf{P}}\|_{L^2} \\ \leq & O(e^{-(1+\sigma)\frac{1}{\varepsilon}\varphi(P_1,...,P_K)}) \end{aligned}$$

By Lemma 2.6 and Lemma 3.4, we have

$$M_\varepsilon(\mathbf{P}) = \varepsilon^N[KI(w) - \frac{1}{2}(\gamma + o(1))(\sum_{i=1}^K e^{-\frac{1}{\varepsilon}\psi_\varepsilon(P_i)}) - (\gamma + o(1)) \sum_{k \neq l} w(\frac{|P_k - P_l|}{\varepsilon})]$$

Since $M_\varepsilon(\mathbf{P}^\varepsilon)$ is the maximum, we have

$$\frac{1}{2}\sum_{i=1}^K e^{-\frac{1}{\varepsilon}\psi_\varepsilon(P_i^\varepsilon)} + \sum_{k\neq l} w(\frac{|P_k^\varepsilon - P_l^\varepsilon|}{\varepsilon}) \leq \frac{1}{2}\sum_{i=1}^K e^{-\frac{1}{\varepsilon}\psi_\varepsilon(P_i)} + \sum_{k \neq l} w(\frac{|P_k - P_l|}{\varepsilon}) + o(1)$$

for any $\mathbf{P} = (P_1, ..., P_K) \in \overline{\Lambda}$. This implies that

$$\varphi(P_1^\varepsilon, ..., P_K^\varepsilon) \geq \max_{\mathbf{P} \in \overline{\Lambda}} \varphi(P_1, ..., P_K) - \delta$$

for any $\delta > 0$.

So $\varphi(P_1^\varepsilon, ..., P_K^\varepsilon) \to \max_{\mathbf{P}\in\Lambda} \varphi(P_1, ..., P_K)$ as $\varepsilon \to 0$. By condition (1.5), we conclude $\mathbf{P}^\varepsilon \in \Lambda$. This completes the proof of Proposition 4.1.

5. Proof of Theorem 1.1

In this section section, we apply results in Section 3 and Section 4 to prove Theorem 1.1 and Corollary 1.2.

Proofs of Theorem 1.1 and Corollary 1.2

By Lemma 3.4 and Lemma 3.5, there exists ε_0 such that for $\varepsilon < \varepsilon_0$ we have a C^1 map which, to any $\mathbf{P} \in \overline{\Lambda}$, associates $\Phi_{\varepsilon,P_1,\ldots,P_K} \in \mathcal{K}^{\perp}_{\varepsilon,\mathbf{P}}$ su ch that

$$S_\varepsilon(\sum_{i=1}^{K} Pw_i + \Phi_{\varepsilon,P_1,\ldots,P_K}) = \sum_{k=1,\ldots,K;l=1,\ldots,N} \alpha_{kl} \frac{\partial Pw_k}{\partial P_{k,l}} \tag{5.1}$$

for some constants $\alpha_{kl} \in R^{KN}$.

By Proposition 4.1, we have $\mathbf{P}^\varepsilon \in \Lambda$, achieving the maximum of the maximization problem in Proposition 4.1. Let $u_\varepsilon = \sum_{i=1}^{K} P_{\Omega_{\varepsilon,P_i^\varepsilon}} w + \Phi_{\varepsilon,P_1^\varepsilon,\ldots,P_K^\varepsilon}$. Then we have

$$D_{P_{i,j}}|_{P_i=P_i^\varepsilon} M_\varepsilon(\mathbf{P}^\varepsilon) = 0, i = 1, \ldots, K, j = 1, \ldots, N$$

Hence we have

$$\int_{\Omega_\varepsilon} [\nabla u_\varepsilon \nabla \frac{\partial(\sum_{i=1}^{K} Pw_i + \Phi_{\varepsilon,P_1,\ldots,P_K})}{\partial P_{i,j}}|_{P_i=P_i^\varepsilon} + u_\varepsilon \frac{\partial(\sum_{i=1}^{K} Pw_i + \Phi_{\varepsilon,P_1,\ldots,P_K})}{\partial P_{i,j}}|_{P_i=P_i^\varepsilon}$$

$$-f(u_\varepsilon)\frac{\partial(\sum_{i=1}^{K} Pw_i + \Phi_{\varepsilon,P_1,\ldots,P_K})}{\partial P_{i,j}}|_{P_i=P_i^\varepsilon}] = 0$$

Hence we have

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \frac{\partial(Pw_i + \Phi_{\varepsilon,P_1,\ldots,P_K})}{\partial P_{i,j}}|_{P_i=P_i^\varepsilon}$$

$$+u_\varepsilon \frac{\partial(Pw_i + \phi_{\varepsilon,P_1,\ldots,P_K})}{\partial P_{i,j}}|_{P_i=P_i^\varepsilon} - f(u_\varepsilon)\frac{\partial(Pw_i + \phi_{\varepsilon,P_1,\ldots,P_K})}{\partial P_{i,j}}|_{P_i=P_i^\varepsilon} = 0$$

for $i = 1, \ldots, K$ and $j = 1, \ldots, N$.

Therefore we have

$$\sum_{k=1,\ldots,K;l=1,\ldots,N} \alpha_{kl} \int_{\Omega_\varepsilon} \frac{\partial Pw_k}{\partial P_{k,l}} \frac{\partial(Pw_i + \Phi_{\varepsilon,P_1,\ldots,P_K})}{\partial P_{i,j}} = 0 \tag{5.2}$$

Since $\Phi_{\varepsilon,P_1,\ldots,P_K} \in \mathcal{K}_{\varepsilon,\mathbf{P}}$, we have that

$$\int_{\Omega_\varepsilon} \frac{\partial Pw_k}{\partial P_{k,l}} \frac{\partial \Phi_{\varepsilon,P_1,\ldots,P_K}}{\partial P_{i,j}} = -\int_{\Omega_\varepsilon} \frac{\partial^2 Pw_i}{\partial P_{k,l}\partial P_{i,j}} \Phi_{\varepsilon,P_1,\ldots,P_K}$$

$$= \|\frac{\partial^2 Pw_i}{\partial P_{k,l}\partial P_{i,j}}\|_{L^2} \|\Phi_{\varepsilon,P_1,\ldots,P_K}\|_{L^2}$$

$$= O(e^{-\frac{1+\sigma}{2}\frac{1}{\varepsilon}\varphi(P_1^\varepsilon,\ldots,P_K^\varepsilon)})$$

Thus equation (5.2) becomes a system of homogeneous equations for α_{kl} and the matrix of the system is nonsingular since it is diagonally dominant. So $\alpha_{kl} \equiv 0, k = 1, \ldots, K, l = 1, \ldots N$.

Hence $u_\varepsilon = \sum_{i=1}^K P_{\Omega_{\varepsilon,P_i^\varepsilon}} w + \Phi_{\varepsilon,P_1^\varepsilon,\ldots,P_K^\varepsilon}$ is a solution of (1.2).

By our construction, it is easy to see that by Maximum Principle $u_\varepsilon > 0$ in Ω. Moreover $\varepsilon^N J_\varepsilon(u_\varepsilon) \to KI(w)$ and u_ε has only K local maximum points $Q_1^\varepsilon, \ldots, Q_K^\varepsilon$. By the structure of u_ε we see that (up to a permutation) $Q_i^\varepsilon - P_i^\varepsilon = o(1)$. Hence $\varphi(Q_1^\varepsilon, \ldots, Q_K^\varepsilon) \to \max_{\mathbf{P}\in\Lambda} \varphi(P_1, \ldots, P_K)$. This proves Theorem 1.1.

Finally, Corollary 1.2 can be easily proved by taking $\Lambda = \{(x_1, \ldots, x_K) \in \Omega^K : d(x_i, \partial\Omega) > \delta, 1 = 1, \ldots, K, |x_k - x_l| > \delta, k \neq l\}$ where $\delta > 0$ is small.

References

[1] Agmon, S.,*Lectures on Elliptic Boundary Value Problems,* Van Nostrand, Princeton, 1965.

[2] Adimurthi, G. Mancinni and S.L. Yadava, The role of mean curvature in a semilinear Neumann problem involving the critical Sobolev exponent, *Comm. P.D.E.* 20 (1995), 591-631.

[3] Adimurthi, F. Pacella and S.L. Yadava, Interaction between the geometry of the boundary and positive solutions of a semilinear Neumann problem with critical nonlinearity. *J. Funct. Anal.* (1993), 31 8-350.

[4] Adimurthi, F. Pacella and S.L. Yadava, Characterization of concentration points and L^∞-estimates for solutions involving the critical Sobolev exponent, *Diff. Int. Eqn.* 8(1) 1995, 41-68.

[5] D.G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. in Math.*, 1978, Vol. **30**, pp. 33-76.

[6] P. Bates and G. Fusco, Equilibria with many nuclei for the Cahn-Hilliard equation, preprint.

[7] A. Bahri and Y. Li, On a min-max procedure for the existence of a positive solution for certain scalar field equations in R^N, *Revista Mat. Iberoa.* 6(1990), 1-15.

[8] D. Cao, N. Dancer, E. Noussair and S. Yan, On the existence and profile of multi-peaked solutions to singularly perturbed semilinear Dirichlet problem, *Discrete and Continuous Dynamical Systems*, Vol. 2, No 2, 1996.

[9] M. Del Pino, P. Felmer and J. Wei, Multiple-peak solutions for some singular perturbation problems, *Cal. Var. and P.D.E.*, to appear.

[10] J. Jang, On spike solutions of singularly perturbed semilinear Dirichlet problems, *J. Diff. Eqns* 114(1994), 370-395.

[11] Dancer, E.N., A note on asymptotic uniqueness for some nonlinearities which change sign, *Rocky Mountain Math. J.*, to appear.

[12] W.H. Fleming and P.E. Souganidis, Asymptotic series and the method of vanishing viscosity, *Indiana Univ. Math. J.* 35 (2) 1986, 425-447.

[13] Floer, A. and Weinstein, A., Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.*, 1986, Vol. **69**, pp. 397-408.

[14] Gui, C., Multi-peak solutions for a semilinear Neumann problem, *Duke Math. J.* 1996, to appear.

[15] Gui, C. and Ghoussoub N., Multi-peak Solutions for a Semilinear Neumann Problem Invovling the Critical Sobolev Exponent, preprint.

[16] Gidas, B., Ni, W.-M., and Nirenberg, L., Symmetry of positive solutions of nonlinear elliptic equations in R^n, *Mathematical Analysis and Applications*, Part A, Adv. Math. Suppl. Studies Vol. **7A**, pp. 369-402, Academic Press, New York, 1981.

[17] R. Gardner and L.A. Peletier, The set of positive solutions of semilinear equations in large balls, *Proc. Roy. Soc. Edinburgh* 104 A (198 6), 53-72.

[18] Gilbarg, D. and Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order,* 2nd edition, Springer, Berlin, 1983.

[19] Helffer, B. and Sjöstrand, J., Multiple wells in the semi-classical limit I, *Comm. PDE*, 1984, Vol. **9**, pp. 337-408.

[20] Kowalczyk, M., Multiple spike layers in the shadow Gierer-Meihardt system: existence of equilibria and approximate invariant manifold, preprint.

[21] M. K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n, *Arch. Rational Mech. Anal.* 105 (1989), 243-266.

[22] Lin, C., Ni, W.-M., and Takagi, I., Large amplitude stationary solutions to a chemotaxis systems, *J. Diff. Eqns.*, 1988, Vol. **72**, pp. 1-27.

[23] Lions, J.L. and Magenes, E., *Non-Homogeneous Boundary Value Problems and Applications*, Vol **I**, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1972.

[24] Li, Y. Y., *On a singularly perturbed equation with Neumann boundary condition,* Comm. in PDE., to appear.

[25] Ni, W.-M., Pan, X., and Takagi, I., Singular behavior of least-energy solutions of a semilinear Neumann problem involving critical Sobolev exponents, *Duke Math. J.*, 1992, Vol. **67**, pp. 1-20.

[26] Ni, W.-M. and Takagi, I., On the shape of least energy solution to a semilinear Neumann problem, *Comm. Pure Appl. Math.*, 1991, Vol. **41**, pp. 819-851.

[27] Ni, W.-M. and Takagi, I., Locating the peaks of least energy solutions to a semilinear Neumann problem, *Duke Math. J.*, 1993, Vol. **70**, pp. 247-281.

[28] W.-M. Ni and I. Takagi, Point-condensation generated be a r eaction-diffusion system in axially symmetric domains, *Japan J. Industrial Appl . Math.* 12(1995), 327-365.

[29] Ni, W.-M. and Wei, J., On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, *Comm. Pure Appl. Math.*, 1995, Vol. **48**, pp. 731-768.

[30] Oh, Y.G., Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_a$, 1988, *Comm. PDE*, Vol. **13(12)**, pp. 1499-1519.

[31] Oh, Y.G., On positive multi-lump bound states of nonlinear Schrödinger equations under multiple-well potentials, 1990, *Comm. Math. Phys.*, Vol. **131**, pp. 223-253.

[32] X.B. Pan, Condensation of least-energy solutions of a semilinear Neumann problem, *J. Partial Diff. Eqns.* 8 (1995), 1-36.

[33] X.B. Pan, Condensation of least-energy solutions: the effect of boundary conditions, *Nonlinear Analysis, TMA* 24 (1995), 195-222.

[34] X.B. Pan, Further study on the effect of boundary conditions, *J. Diff. Eqns.* 117(1995), 446-468.

[35] J. Smoller and A. Wasserman, Global bifurcation of steady-state solutions, *J. Diff. Eqns* 39 (1981), 269-290.

[36] Ward, M., An asymptotic analysis of localized solutions for some reaction-diffusion models in multidimensional domains, *Studies in Appl. Math,* 97 (1996), 103-126.

[37] J. Wei, On the construction of single-peaked solutions of a singularly perturbed semilinear Dirichlet problem, *J. Diff. Eqns.* 129 (1996), 315-333.

[38] J.Wei, On the effect of the geometry of the domain in a singularly perturbed Dirichlet problem, *Diff. Int. Eqns.*, to appear.

[39] J. Wei, On the interior spike layer solutions of a singularly perturbed Neumann problem, *Tohoku Math. J.* 50(1998), 159-178.

[40] Wei, J., On the boundary spike layer solutions of singularly perturbed semilinear Neumann problem, *J. Diff. Eqns.*, 1997, Vol. **134**.

[41] Wei, J., On the construction of single interior peak solutions for a singularly perturbed Neumann problem, submitted.

[42] Wei, J. and Winter, M., Stationary solutions for the Cahn-Hilliard equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 15(1998), 459-492.

[43] Wei, J. and Winter, M., Multiple Boundary Spike Solutions for a Wide Class of Singular Perturbation Problems, *J. London Math. Soc.*, to appear.

[44] Z.-Q. Wang, On the existence of multiple single-peaked solutions for a semilinear Neumann problem, *Arch. Rational Mech. Anal.* 120(1992),375-399.

[45] M. Ward, An asymptotic analysis of localized solutions for some reaction-diffusion models in multidimensional domains, *Stud. Appl. Math.,* 97 (1996), 103-126.

[46] K. Yosida, *Functional Analysis,* fifth edition, Springer, Berlin, Heidelberg, New York, 1978.

[47] Zeidler, E., *Nonlinear Functional Analysis and its Applications I, Fixed-Point Theorems,* Springer, New York, Berlin, Heidelberg, Tokyo, 1986.

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada

E-mail address: cgui@math.ubc.ca

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk