On the equation $\Delta u + K(x)u^{\frac{n+2}{n-2}\pm\epsilon^2} = 0$ in R^n

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Abstract

In this paper we apply the Lyapunov-Schmidt reduction method, as developed by Floer and Weinstein, to construct solutions to semilinear elliptic equations with nearly critical exponents in R^n. The solutions will have fast decay at infinity and will blow up as the exponent tends to critical exponent.

1 Introduction and Main Result

Of concern is the existence of solutions of

$$\begin{cases} \Delta u + K(x) u^q = 0, \\ u > 0, \quad x \in R^n, \end{cases} \tag{1.1}$$

where $n \geq 3$, $q > 1$ and $K(x)$ is smooth.

In this paper, we shall prove the existence of decaying solutions of (1.1) when q is "near" critical, i.e. $q = \frac{n+2}{n-2} \pm \epsilon^2$ with $\epsilon > 0$ small, under a local condition on K and a reasonable growth restriction on K at infinity.

Equation (1.1) arises in various areas including Riemannian geometry ($q = \frac{n+2}{n-2}$), and astrophysics (when $K \equiv 1$, it is called Lane-Emden-Fowler equation; when $K = 1/(1+|x|^2)$, it is called Matukuma equation). In the last decade, equation (1.1) has been studied extensively by many authors. In 1982, Ni [N] proved that if $|K(x_1, x_2)| \leq C(1 + |x_1|)^l$ for some $l < -2$, where $x = (x_1, x_2) \in R^3 \times R^{n-3}$, then (1.1) has infinitely many bounded solutions, which are also bounded away from zero. Since then, various existence results when K decays like or faster than $C|x|^{-2}$ at infinity have been obtained by Gui [G1, G2], Kawano [K], Kawano, Satsuma and Yotsutani [KSY], Naito[Na], Naito and Usami [NU], etc.. When K is nonpositive, Ni [N] and F.-H. Lin [Ln] proved that if $|K|$ decays like or slower than $C|x|^{-2}$

at infinity, then (1.1) has no solution (see K.-S. Cheng and J.-T. Lin [CLn] for refinements). It has been shown in [N] that if K grows like or faster than $C|x|^{(n-2)(q-1)-2}$ at infinity, then (1.1) has no solution, either. Thus when either $|K|$ decays fast, or grows fast, or K is nonpositive, we have fairly clean results about (1.1), as far as the existence is concerned.

In the more interesting range of K, i.e. when K is nonnegative and decays slower than $C|x|^{-2}$ but grows slower than $C|x|^{(n-2)(q-1)-2}$ at infinity, our understanding of (1.1) is not as complete. Ding and Ni [DN] showed that if K is nonincreasing along every ray starting at the origin and K satisfies a certain symmetry condition, then equation (1.1) with $q \geq \frac{n+2}{n-2}$ has infinitely many solutions. They also showed that if K is radial and radially increasing, and if K is a slight perturbation (near the origin) of the constant function 1, then (1.1) with $q = \frac{n+2}{n-2}$ has no radial solution. Combining this with their existence results, we see that (1.1) is rather sensitive to even the slightest perturbation of K, at least when $q = \frac{n+2}{n-2}$. For radial K, there are many other works, of which we only mention a few: Bianchi and Egnell [BE], Cheng and Chern [CC], Johnson, Pan and Yi [JPY], Li and Ni [LN], Lin and Lin [LL], Nous! sair and Swanson [NS], X. Pan [P], W.Rother [R], and Yanagida and Yotsutani [YY1,YY2]. In particular, [YY1,YY2] contain rather general existence and classification results, especially some interesting results concerning the Ding-Ni phenomena for the critical case $q = \frac{n+2}{n-2}$. On the other hand, for nonradial K, there are comparatively fewer results. C. Gui [G1,G2] studied existence of (1.1) for slow decaying K and large q, while J. Wei [We] obtained some existence results for K decaying only in a two-dimensional subspace and $q \geq \frac{n+2}{n-2}$. Y. Y. Li [L] showed that there are infinitely many solutions of (1.1) with finite energy for $n = 3$ and $q = 5$ (critical) assuming, among other things, that K is periodic in one variable. If

$K(x)$ is well defined on S^n (via sterograhic projection) and $q = \frac{n+2}{n-2}$, equation (1.1) has been studied in for example [BC], [CY] and [ES] and the references therein.

In this paper, we shall prove the following existence result:

Theorem 1.1 *Let $K(x)$ be a nonnegative $C^{2+\alpha}$ function ($0 < \alpha < 1$) satisfying $K(x) \leq C(1 + |x|)^m$ for all $x \in R^n$, where C is some positive constant and $m < 2$. Suppose also that K has a nondegenerate critical point x_0 such that $K(x_0) > 0$, $\Delta K(x_0) < 0$ when $q < n^* = \frac{n+2}{n-2}$ and $\Delta K(x_0) > 0$ when $q > n^*$. Then for $n > \frac{12-2m}{2-m}$ and q near critical, i.e. for $q = n^* \pm \epsilon^2$, $\epsilon > 0$ small, equation (1.1) has a positive solution u_ϵ with*

$$u_\epsilon(x) \leq C|x|^{2-n}, \quad |x| \geq 1 \tag{1.2}$$

where C is a constant independent of ϵ. Furthermore, u_ϵ concentrates and blows up at x_0 as ϵ shrinks to zero.

(The sense in which u_ϵ concentrates and blows up will be made clear in Section 4. See, in particular, (4.5).)

Remarks 1. Recall from [N] that if $K(x)$ grows like or faster than $C|x|^2$ at infinity, equation (1.1) with $q = \frac{n+2}{n-2}$ has no positive solution. So the growth restriction on K at infinity in Theorem 1.1 is a reasonable one.

2. Note that if x_0 is a nondegenerate local maximum (minimum) point of K, then $\Delta K(x_0)$ is negative (positive) and hence Theorem 1.1 applies. Of course, Theorem 1.1 applies even if x_0 is a saddle point. Observe also that if $m = 0$, i.e. if K is bounded, then the requirement on n is that $n > 6$. If $m > 0$, n is required to be larger. In all cases, our solution u_ϵ decays fast at infinity.

3. Theorem 1.1 allows for *arbitrary* behavior of K(except for the reasonable growth restriction) outside a neighborhood of x_0. Furthermore, by the

proof of Theorem 1.1, we actually do not need $K \in C^{2+\alpha}_{loc}(R^n)$, $K \in C^{\alpha}_{loc}(R^n)$ and $K \in C^{2+\alpha}$ in a neighborhood of x_0 are sufficient.

It is interesting to compare Theorem 1.1 to the following nonexistence result of Li and Ni [LN].

Theorem A. *Suppose that K decays faster than $C|x|^{-2}$ at infinity and that*

$$L(x) = \left[n - \frac{(n-2)(q+1)}{2}\right] K(x) + x \cdot \nabla K(x)$$

never changes sign in R^n. Then (1.1) has no solutions which decay at infinity.

As remarked in [LN], when $q < n^*$, the assumptions stated in Theorem A are never satisfied, except when $K \equiv 0$. When $q > n^*$, if K has a critical point x_0 as stated in Theorem 1.1, then it is easy to verify directly that $L(x)$ must change sign, provided q is close to n^*. This may not be the case if $\Delta K(x_0) < 0$ or if $\Delta K(x_0) < 0$ but q is not close to n^*. Theorem A also implies that Theorem 1.1 generally fails to be true for $q = n^*$ since our conditions on K at x_0 alone are certainly not enough for $L(x) \equiv x \cdot \nabla K(x)$ to change sign.

To illustrate the idea behind Theorem 1.1, we may assume without loss of generality that x_0 in the statement of Theorem 1.1 is the origin, and that $K(x_0) = 1$. We first observe that if we rescale u in (1.1) as $\epsilon^{\frac{2}{q-2}} u(\epsilon x + z)$ (where z is fixed in R^n), then (1.1) is transformed into an equivalent form

$$\begin{cases} \Delta u + K(\epsilon x + z) u^q = 0, \\ u > 0, x \in R^n. \end{cases} \tag{1.3}$$

(The reason for introducing z will be seen in the future.)

If $q = n^* \pm \epsilon^2$ (as assumed in Theorem 1.1), then, letting both ϵ and z go to zero, we see that the limiting equation of (1.3) is:

$$\begin{cases} \Delta u + u^{n^*} = 0, \\ u > 0, x \in R^n. \end{cases} \tag{1.4}$$

It is well-known (see [CGS] or [CLi]) that all solutions of (1.4) are of the form (up to translations)

$$U_\lambda(x) = \lambda^{\frac{n-2}{2}} U(\lambda x), \tag{1.5}$$

where $\lambda > 0$ and $U(x) = (1 + \frac{|x|^2}{n(n-2)})^{\frac{2-n}{2}}$. One then tends to guess (perhaps very cautiously) that when ϵ and $|z|$ are small, equation (1.3) should have a solution u close to U_λ for some $\lambda > 0$. If this is indeed the case, (1.1) must have a solution u_ϵ close to $\epsilon^{-\frac{2}{q-1}} U_\lambda(\frac{x-z}{\epsilon})$ for some $\lambda > 0$.

If we take $z = o(\epsilon)$ and if we can bound λ between two fixed positive constants as $\epsilon \to 0$, then we see that u_ϵ should concentrate and blow up at x_0. This is precisely what is claimed in Theorem 1.1.

Rescaling u in (1.3) as $\lambda^{-\frac{2}{q-1}} u(\lambda^{-1} x)$, $\lambda > 0$, we see that equation (1.3) is equivalent to:

$$\begin{cases} \Delta u + K(\frac{\epsilon x}{\lambda} + z) u^q = 0, \\ u > 0, \quad x \in R^n. \end{cases} \tag{1.6}$$

Since equation (1.3) is expected to have a solution close to U_λ for some λ (bounded between two positive constants), equation (1.6) should have a solution close to

$$\lambda^{-\frac{2}{q-1}} U_\lambda(\lambda^{-1} x) = \lambda^{\frac{n-2}{2} - \frac{2}{q-1}} U(x) \approx U(x). \tag{1.7}$$

It is our goal to find a solution u of (1.6) of the form $u = U + \phi$, where ϕ is "small", $z = o(\epsilon)$, λ is bounded by two positive constants and ϵ is small. To this end, we apply the Lyapunov-Schmidt reduction method, as developed by Floer and Weinstein in [FW], where they studied the semi-classical solutions of the nonlinear Schrödinger equation

$$\frac{h^2}{2} \Delta u - (V(x) - E) u + u^p = 0 \text{ in } R^n, \tag{1.8}$$

for h small. (The same method was used in [O1] and in [O2].) At the starting point of the Lyapunov-Schmidt procedure, we need to study the linearized operator (obtained by linearizing equation (1.6) at U)

$$L \equiv \Delta + n^* U^{n^*-1}. \tag{1.9}$$

Of importance here are the kernel of L and the Fredholm property of L, in some appropriate setting. For L to have the Fredholm property, we have to work in weighted Sobolev spaces, in which the general linear theory is well developed by Nirenberg and Walker [LW], Cantor[C1,C2], Lockhart [Lo] and McOwen [Mc].

In this paper, we shall prove in detail Theorem 1.1 in case $q = n^* - \epsilon^2$. The case $q = n^* + \epsilon^2$ can be handled by slightly modifying the arguments in Sections 3 and 4 . So from now on, we assume $q = n^* - \epsilon^2$.

Throughout the remainder of this paper, we assume, without loss of generality, that K satisfies the hypotheses of Theorem 1.1, that $x_0 = 0$ and $K(0) = 1$.

This paper is organized as follows. In Section 2, we recall some properties of weighted Sobolev spaces and study the kernel of the linearized operator L and its Fredholm property. Section 3 contains some error estimates and the reduction to finite dimensions. Finally, in Section 4, we apply a degree-theoretic argument to solve the reduced problem and we also explain why u_ϵ concentrates and blows up as $\epsilon \to 0$, thereby completing the proof.

2 Weighted Sobolev Spaces and the Linearized Operator

For $1 < p < \infty$, a nonnegative integer l and a real number β, the weighted Sobolev space $W^p_{l,\beta}$ is defined to be the completion of $C^\infty_0(R^n)$ under the norm:

$$||u||_{p,l,\beta} = \sum_{|\alpha|=0}^{l} || < x >^{\beta+|\alpha|} \partial^\alpha u||_{L^p(R^n)},$$

where $< x >= (1+|x|^2)^{\frac{1}{2}}$. If $\beta = 0 = l$, then $W^p_{l,\beta}$ is just the usual L^p space. When $l = 0$, we write $W^p_{0,\beta}$ as L^p_β. It is easy to see that the conjugate of L^p_β is $L^{p'}_{-\beta}$ under the action $< f, u >= \int_{R^n} f(x)u(x)dx$, $f \in L^{p'}_{-\beta}$, $u \in L^p_\beta$, where p' denotes, as always, the conjugate of p: $\frac{1}{p} + \frac{1}{p'} = 1$.

The following known results will be useful in this paper.

Proposition 2.1 *(1). $W^p_{l,\beta}$ can be continously embedded into L^q_γ, provided $\frac{l}{n} - \frac{1}{p} + \frac{1}{q} \geq 0$, $\infty > q \geq p > 1$ and $\gamma \leq \beta + \frac{n}{p} - \frac{n}{q}$.*

(2). Suppose $l > \frac{n}{p}$ and $\frac{n}{p} + \beta > \delta$. If $f \in W^p_{l,\beta}$, then $f \in C^{l-\frac{n}{p}}_{loc}(R^n)$ and $|f(x)| < x >^\delta \leq C(\delta, p, l, \beta, n)||f||_{p,l,\beta}$.

(3). The integral operator

$$Tu(x) = \int_{R^n} \frac{u(y)}{|x-y|^{n-2}} dy$$

is a bounded operator from L^p_β to $L^p_{\beta-2}$, provided $2 - \frac{n}{p} < \beta < \frac{n}{p'}$.

Proof: (1) is a special case of [Lo], Theorem 2.14. (2) follows from the proof of [C2], Theorem 5.4 (the Hölder continuity of f follows from the fact that $W^p_{l,\beta} \hookrightarrow W^{l,p}_{loc}(R^n)$ (the usual Sobolev space) and the embedding theorem). (3) follows from [NW], Lemma 2.1, and is also proved in Corollay 1, [Mc].$\square$

We now turn to the linear operator $L = \Delta + n^* U^{n^*-1} : L^p_\beta \to L^p_{\beta+2}$ with domain $Dom(L) = W^p_{2,\beta}$. The conjugate L^* of L is a linear mapping from $L^{p'}_{-(\beta+2)} \to L^{p'}_{-\beta}$ with domain $Dom(L^*)$, which is the space of all fuctions $u \in L^{p'}_{-(\beta+2)}$ with the property that $\exists C > 0$, such that $| < u, Lv > | = |\int_{R^n} uLvdx| \leq C||v||_{L^p_\beta}$, for all $v \in W^p_{2,\beta}$. Clearly, $W^{p'}_{2,-(\beta+2)} \subset Dom(L^*)$ and $L^*v = (\Delta + n^* U^{n^*-1})v$ for $v \in W^{p'}_{2,-(\beta+2)}$. On the other hand, for any $u \in Dom(L^*)$, we have $(\Delta + n^* U^{n^*-1})u \in L^{p'}_{-\beta}$. By Theorem 3.1 in [NW], u must belong to $W^{p'}_{2,-(\beta+2)}$. (Note that the condition that u belongs to the usual Sobolev space $W^{2,p'}$ can be removed from [NW] in our present situation.) Thus $Dom(L^*) = W^{p'}_{2,-(\beta+2)}$.

Proposition 2.2 *If $-\frac{n}{p} < \beta < \frac{n}{p'} - 2$, then $Ker(L) = X = Ker(L^*)$, where $X =$ span $\{\frac{\partial U}{\partial x_1}, ..., \frac{\partial U}{\partial x_n}, x \cdot \nabla U + \frac{n-2}{2}U\}$.*

Proof: We shall only prove $Ker(L) = X$. The proof of $Ker(L^*) = X$ is similar.

Since $\beta < \frac{n}{p'} - 2$, by some simple calculations, we see that $X \subset Ker(L)$. On the other hand, we claim that the dimension of $Ker(L) \leq n+1$. Then the desired conclusion follows immediately. The proof of this claim is identical to that of [W], Lemma 4.2, except that now, for any $\varphi \in Ker(L)$, the argument leading to the fact that φ and $\nabla\varphi$ decay at infinity is different(also see an earlier paper [NT] which concerns a different operator but involves the same idea). Therefore, we shall only prove the decay of φ and $\nabla\varphi$ here.

Observe that once we know that φ decays, the decay of $\nabla\varphi$ follows from the interior L^p estimates and embedding theorems. Observe also that if $p > \frac{n}{2}$, then by Proposition 2.1, $\varphi \to 0$ as $|x| \to \infty$. So in the rest of this proof, we may assume $p \leq \frac{n}{2}$.

We shall first prove that $\varphi \in L^r(R^n)$ for some $r > 1$.

If $p = \frac{n}{2}$, then by (1) of Proposition 2.1, $\phi \in L^r$ for $r \geq max(\frac{n}{\beta+2}, p)$, so in the remainder of the proof of the assertion "$\phi \in L^r$", we assume $p < \frac{n}{2}$.

Notice that for any $u \in W^p_{2,\beta}$, we have the representation formula:

$$u(x) = \int_{R^n} \Gamma(x-y)\Delta u(y)dy, \quad x \in R^n,$$

where Γ is the fundamental solution of Δ in R^n. This follows from an argument involving cut-off functions, as in [NW], p. 278. Here we use the assumption that $\beta > -\frac{n}{p}$.

For any $\varphi \in Ker(L)$, we have

$$\Delta\varphi + n^* U^{n^*-1}\varphi = 0 \ \text{ in } R^n \tag{2.1}$$

and $\varphi \in C^\infty(R^n)$ by the elliptic regularity theory. Thus

$$\varphi(x) = \int_{R^n} \Gamma(x-y)f(y)dy, \quad x \in R^n, \tag{2.2}$$

where $f = -n^* U^{n^*-1}\varphi$. Observe that $f \in L^p_{4+\beta}$. Now if $4 + \beta \geq \frac{n}{p'}$, then of course $f \in L^p_0$. In this case, by (2.2) and the Hardy-Littlewood-Sobolev inequality, $\varphi \in L^r$ where $\frac{1}{r} = \frac{1}{p} - \frac{2}{n}$ (recall that we have assumed $p < \frac{n}{2}$, so $r > 1$). On the other hand, if $4 + \beta < \frac{n}{p'}$ (recall that $-\frac{n}{p} < \beta$), then by (2.2) and Proposition 2.1, $\varphi \in L^p_{\beta+2}$, which in turn implies that $\Delta\varphi = f \in L^p_{6+\beta}$. If $6 + \beta \geq \frac{n}{p'}$, the Hardy-Littlewood-Sobolev inequality again implies $\varphi \in L^r, \frac{1}{r} = \frac{1}{p} - \frac{2}{n}$. Thus $\varphi \in L^p_{4+\beta}$ and hence $\Delta\varphi \in L^p_{8+\beta}$. Repeating this process finitely many times, we eventually have $\varphi \in L^r$ for some $r > 1$.

Applying the one-sided Harnack inequality ([GT], Theorem 8.17) to (2.1), we have

$$|\varphi|(Q) \leq C(n,p)||\varphi||_{L^r(B_1(Q))},$$

where $B_1(Q)$ is the unit ball centered at $Q \in R^n$. Letting $|Q| \to \infty$ and recalling that $\varphi \in L^r$, we have $\varphi(Q) \to 0$ as $|Q| \to \infty$. The proof of Proposition 2.2 is complete. □

By [Lo], Corollary 5.7, L is Fredholm if $-\frac{n}{p} < \beta < \frac{n}{p'} - 2$. So in our case, $Range(L)$ is closed and equal to

$$W \equiv (KerL^*)^{\perp} = X^{\perp} = \{u \in L^p_{\beta+2} | < u, v >= \int_{R^n} u(x)v(x)dx = 0, v \in X\}.$$

Observe that when $2 - \frac{n}{p} < \beta < \frac{n}{p'} - 2$, we can decompose $W^p_{2,\beta}$ as

$$\begin{array}{l} W^p_{2,\beta} = X \bigoplus Y, \text{ where} \\ Y = \{u \in W^p_{2,\beta} | \int_{R^n} uvdx = 0, \text{for all } v \in X\} \end{array} \tag{2.3}$$

(The condition "$\beta > 2 - \frac{n}{p}$" implies that $uv \in L^1$ for $u \in L^p_{\beta}, v \in X$.)

Y is a Banach space equipped with $W^p_{2,\beta}$ norm and $W = Range(L)$ is also a Banach space equipped with $L^p_{\beta+2}$ norm. By the Closed Graph Theorem, we have

Proposition 2.3 *Suppose $n > 4$ and $2 - \frac{n}{p} < \beta < \frac{n}{p'} - 2$. Then*

$$L^{-1} : W \to Y$$

exists and is bounded.

Finally, let us make a remark about the decomposition of $L^p_{2+\beta}$. If $\beta < \frac{n}{p'} - 4$, it is easy to see that $X \subset L^p_{2+\beta}$. Recall that when $-\frac{n}{p} < \beta < \frac{n}{p'} - 2, W = Range(L) = (KerL^*)^{\perp} = X^{\perp}$. So if $-\frac{n}{p} < \beta < \frac{n}{p'} - 4$, $L^p_{2+\beta}$ can be decomposed as

$$L^p_{2+\beta} = X \bigoplus W. \tag{2.4}$$

3 Reduction to finite dimensions

As discussed in Section 1, to prove Theorem 1.1, we need only show that for small $\epsilon > 0$, equation (1.6) has a solution u of the form $u = U + \phi$, where ϕ is "small", $|z| = o(\epsilon)$ and λ is bounded by two positive constants independent of ϵ.

Let $S_\epsilon(u) = \Delta u + K(\frac{\epsilon x}{\lambda} + z)u_+^q$, where $u_+ = \max(0, u)$. Then equation (1.6) is equivalent to

$$S_\epsilon(u) = 0,\ u_+ \not\equiv 0,\ x \in R^n. \tag{3.1}$$

For if u satisfies (3.1), then by the Maximum Principle, $u > 0$ in R^n and hence (1.6) is satisfied.

Observe that

$$\begin{aligned} S_\epsilon(U+\phi) &= \Delta(U+\phi) + K(\frac{\epsilon x}{\lambda} + z)(U+\phi)_+^q, \\ &= L\phi + N_\epsilon^1(\phi) + N_\epsilon^2(\phi) + M_\epsilon^1 + M_\epsilon^2, \end{aligned}$$

where

$$\begin{aligned} L\phi &= \Delta\phi + n^* U^{n^*-1}\phi, \\ N_\epsilon^1(\phi) &= K(\frac{\epsilon x}{\lambda} + z)((U+\phi)_+^q - U^q - n^* U^{n^*-1}\phi), \\ N_\epsilon^2(\phi) &= (K(\frac{\epsilon x}{\lambda} + z) - 1)(n^* U^{n^*-1}\phi), \\ M_\epsilon^1 &= (K(\frac{\epsilon x}{\lambda} + z) - 1)U^{n^*}, \\ M_\epsilon^2 &= K(\frac{\epsilon x}{\lambda} + z)(U^q - U^{n^*}). \end{aligned}$$

So (3.1) is equivalent to

$$L\phi + N_\epsilon^1(\phi) + N_\epsilon^2(\phi) + M_\epsilon^1 + M_\epsilon^2 = 0, \quad (U+\varphi)_+ \not\equiv 0. \tag{3.2}$$

We shall solve (3.2) in the $W_{2,\beta}^p$ setting. In this section, we solve (3.2) modulo $X(= Ker(L))$ for $\phi \in Y$, and by doing so, we reduce our problem to a finite dimensional one.

Recall that when $2 - \frac{n}{p} < \beta < \frac{n}{p'} - 4$, we can decompose both $W^p_{2,\beta}$ and $L^p_{2+\beta}$ as in (2.3) and (2.4) (it is here we require $n > 6$). Let Q be the projection of $L^p_{2+\beta}$ onto X. Then $P = I - Q$ is the projection of $L^p_{2+\beta}$ onto W.

Let $F_\epsilon(\phi) = -L^{-1}P(N^1_\epsilon(\phi) + N^2_\epsilon(\phi) + M^1_\epsilon + M^2_\epsilon)$, $\phi \in W^p_{2,\beta}$. Then solving (3.2) modulo X is equivalent to finding a fixed point of $F_\epsilon(\phi)$.

We shall always assume that $|z| \leq 1$ and $\lambda \in [a, b]$, where a and b are positive constants to be specified in the next section.

The following technical lemmas guarantee that if $\phi \in W^p_{2,\beta}$, then $N^i_\epsilon(\phi)$ and M^i_ϵ $(i = 1, 2)$ are in $L^p_{2+\beta}$ for suitable p and β . We shall delay their proofs until the end of this section..

Lemma 3.1 *Suppose $n > 6, p > \frac{2n}{n+2}$ and $\frac{(2+m)(n-2)}{4} - \frac{n}{p} < \beta$. Then there exist a small constant $\epsilon_1 = \epsilon_1(p, n, \beta, m, a, b)$ and a constant C independent of ϵ, z and λ, such that for $0 \leq \epsilon \leq \epsilon_1$ and $\phi \in W^p_{2,\beta}$,*

$$||N^1_\epsilon(\phi)||_{p,0,2+\beta} \leq C(\epsilon^2||\phi||_{p,2,\beta} + ||\phi||^{n^*-\epsilon_1^2}_{p,2,\beta} + ||\phi||^{n^*}_{p,2,\beta}). \tag{3.3}$$

Lemma 3.2 *For every $p > 1$ and $\beta \in R^1$, there exists a constant C, independent of ϵ, z and λ, such that for $\phi \in W^p_{2,\beta}$,*

$$||N^2_\epsilon(\phi)||_{p,0,2+\beta} \leq C(\epsilon^2 + |z|^2)||\phi||_{p,2,\beta}. \tag{3.4}$$

Lemma 3.3 *For every $p > 1, \beta \in R^1$ satifying $\beta < \frac{n}{p'} - 2$, there exist a small constant $\epsilon_2 = \epsilon_2(p, n, \beta, m, a, b)$ and a constant C independent of ϵ, z and λ, such that for $0 \leq \epsilon \leq \epsilon_2$,*

$$||M^1_\epsilon||_{p,0,2+\beta} \leq C(\epsilon^2 + |z|^2), \tag{3.5}$$

$$||M^2_\epsilon||_{p,0,2+\beta} \leq C\epsilon^2. \tag{3.6}$$

Combining Lemmas 3.1-3.3 and recalling (2.4) and Proposition 2.3, we have the following:

Proposition 3.4 *Suppose* $p > \frac{2n}{n+2}$ *and* $\frac{(2+m)(n-2)}{4} - \frac{n}{p} < \beta < \frac{n}{p'} - 4$ *(this requires that* $n > \frac{12-2m}{2-m}$*). Then there exist a small constant* $\epsilon_3 = \epsilon_3(p, n, \beta, m, a, b)$ *and a constant* C *independent of* ϵ, z *and* λ*, such that for* $0 \leq \epsilon \leq \epsilon_3$ *and* $\phi \in W^p_{2,\beta}$*,*

$$\begin{aligned} ||F_\epsilon(\phi)||_{p,2,\beta} \;\leq\;& C\{||\phi||^{n^*}_{p,2,\beta} + ||\phi||^{n^*-\epsilon_3^2}_{p,2,\beta} \\ & + (\epsilon^2 + |z|^2)||\phi||_{p,2,\beta} + (\epsilon^2 + |z|^2)\}. \end{aligned} \tag{3.7}$$

Now let B_δ be the ball in Y with center 0 and radius δ. Take z such that $|z| \leq \epsilon$. If $\frac{(2+m)(n-2)}{4} - \frac{n}{p} < \beta < \frac{n}{p'} - 4$, then for $0 \leq \epsilon \leq \epsilon_3$ and $\phi \in B_\delta$, we have $F_\epsilon(\phi) \in Y$ and

$$||F_\epsilon(\phi)||_{p,2,\beta} \leq C((\delta)^{n^*} + (\delta)^{n^*-\epsilon_3^2} + (\epsilon_3)^2) \leq \delta,$$

provided δ and ϵ_3 are chosen small enough. Thus, under the above conditions, F_ϵ is a mapping from B_δ to B_δ.

Next, we want to show that F_ϵ is a contraction mapping. To this end, we need the following estimates, whose proofs will also be postponed until the end of this section.

Proposition 3.5 *Suppose* $n > \frac{12-2m}{2-m}, p > \frac{2n}{n+2}$ *and* $\frac{(2+m)(n-2)}{4} - \frac{n}{p} < \beta < \frac{n}{p'} - 4$ *. Then there exist a small constant* $\epsilon_4 = \epsilon_4(p, n, \beta, m, a, b)$ *and a constant* C *independent of* ϵ, z *and* λ*, such that for* $0 \leq \epsilon \leq \epsilon_4$ *and* $\phi \in W^p_{2,\beta}$*,*

$$\begin{aligned} ||F_\epsilon(\phi_1) - F_\epsilon(\phi_2)||_{p,2,\beta} \;\leq\;& C\{||\phi_1 - \phi_2||^{n^*}_{p,2,\beta} + ||\phi_1 - \phi_2||^{n^*-\epsilon_4^2}_{p,2,\beta} \\ & + ||\phi_1 - \phi_2||_{p,2,\beta}(\epsilon^2 + |z|^2 + \\ & ||\phi_2||^{n^*-1-\epsilon_4^2}_{p,2,\beta} + ||\phi_2||^{n^*-1}_{p,2,\beta})\}. \end{aligned} \tag{3.8}$$

From this result, we see that $F_\epsilon : B_\delta \to B_\delta$ is a contraction mapping for $\lambda \in [a,b], |z| \le \epsilon$ and $0 \le \epsilon \le \epsilon_0$, provided that ϵ_0 and δ are chosen sufficiently small (of course $\epsilon_0 \le min(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$). By the Contraction Mapping Principle, F_ϵ has a unique fixed point $\phi_{\epsilon,z,\lambda}$ on $B_\delta \subset Y$. Obviously, $\phi_{\epsilon,z,\lambda}$ solves

$$L\phi + P(N_\epsilon^1(\phi) + N_\epsilon^2(\phi) + M_\epsilon^1 + M_\epsilon^2) = 0. \tag{3.9}$$

Hence it is a solution of (3.2) modulo X.

We now show that $\phi_{\epsilon,z,\lambda}$ is "small ". By Proposition 3.4, we have (recall that we take z such that $|z| \le \epsilon$),

$$\begin{aligned} ||\phi_{\epsilon,z,\lambda}||_{p,2,\beta} &= ||F_\epsilon(\phi_{\epsilon,z,\lambda})||_{p,2,\beta} \\ &\le C(||\phi_{\epsilon,z,\lambda}||_{p,2,\beta}^{n^*} + ||\phi_{\epsilon,z,\lambda}||_{p,2,\beta}^{n^*-\epsilon_0^2} \\ &\quad + \epsilon^2||\phi||_{p,2,\beta} + \epsilon^2), \end{aligned}$$

for $0 \le \epsilon \le \epsilon_0$. Taking ϵ_0 and δ small, we obtain

$$||\phi_{\epsilon,z,\lambda}||_{p,2,\beta} \le C\epsilon^2 \quad \text{for } 0 \le \epsilon \le \epsilon_0. \tag{3.10}$$

Summarizing what we have obtained in this section, we have:

Proposition 3.6 *Suppose that the hypotheses in Proposition 3.5 hold. Then there exists a small constant* $\epsilon_0 = \epsilon_0(p,n,\beta,m,a,b)$*, such that for every triple* $(\epsilon, z, \lambda) \in [0,\epsilon_0] \times B_\epsilon(0) \times [a,b]$*, (3.9) has a unique solution* $\phi_{\epsilon,z,\lambda} \in Y$*. Furthermore (3.10) holds.*

In the next section, we will show that if z and λ are chosen suitably, $\phi_{\epsilon,z,\lambda}$ is actually a solution of (3.2).

We devote the remainder of this section to the proofs of Lemmas 3.1-3.3 and Proposition 3.5.

Proof of Lemma 3.1: Note that

$$\begin{aligned}
N^1_\epsilon(\phi) &= K(\frac{\epsilon x}{\lambda}+z)((U+\phi)^q_+ - U^q - n^* U^{n^*-1}\phi) \\
&= K(\frac{\epsilon x}{\lambda}+z)\{[(U+\phi)^{n^*-\epsilon^2}_+ - U^{n^*-\epsilon^2} - (n^*-\epsilon^2)U^{n^*-1-\epsilon^2}\phi] \\
&\quad + n^*(U^{n^*-1-\epsilon^2} - U^{n^*-1})\phi - \epsilon^2 U^{n^*-1-\epsilon^2}\phi\} \\
&= K(\frac{\epsilon x}{\lambda}+z)\{I_1+I_2+I_3\},
\end{aligned}$$

where I_1, I_2 and I_3 are defined at the last equality.

Take a small $\epsilon_1 > 0$, to be specified later. Then for $0 \le \epsilon \le \epsilon_1$,

$$\begin{aligned}
|K(\frac{\epsilon x}{\lambda}+z)I_3| &\le C\epsilon^2(1+|\frac{\epsilon x}{\lambda}+z|)^m U^{n^*-1-\epsilon_1^2}|\phi| \\
&\le C\epsilon^2 < x >^{m-4+\epsilon_1^2(n-2)} |\phi|; \\
|K(\frac{\epsilon x}{\lambda}+z)I_2| &\le C\epsilon^2 < x >^m U^{n^*-1-\epsilon_1^2}|\log U||\phi| \\
&\le C\epsilon^2 < x >^{m-4+2\epsilon_1^2(n-2)} |\phi|.
\end{aligned}$$

Thus

$$\begin{aligned}
||K(\frac{\epsilon x}{\lambda}+z)(I_2+I_3)||_{p,0,2+\beta} &\le C\epsilon^2(\int_{R^n}(< x >^{m-4+2\epsilon_1^2(n-2)+2+\beta} |\phi|)^p dx)^{\frac{1}{p}} \\
&\le C\epsilon^2||\phi||_{p,0,\beta} \\
&\le C\epsilon^2||\phi||_{p,2,\beta}, \qquad (3.11)
\end{aligned}$$

provided $0 \le \epsilon \le \epsilon_1$, with ϵ_1 taken so small that $m-2+2\epsilon_1^2(n-2) \le 0$.

To estimate I_1, observe that since $n > 6$, $1 < n^* < 2$. So there exist a small $\epsilon_1 > 0$ and a constant C independent of ϵ, such that for $0 \le \epsilon \le \epsilon_1$, $\xi \in R^1$,

$$|(1+\xi)^{n^*-\epsilon^2}_+ - 1 - (n^*-\epsilon^2)\xi| \le C|\xi|^{n^*-\epsilon^2}. \qquad (3.12)$$

¿From this it follows that for $0 \le \epsilon \le \epsilon_1$,

$$\begin{aligned}
|K(\frac{\epsilon x}{\lambda}+z)I_1| &\le C < x >^m |\phi|^{n^*-\epsilon^2} \\
&\le C < x >^m (|\phi|^{n^*-\epsilon_1^2} + |\phi|^{n^*}).
\end{aligned}$$

Hence for $0 \leq \epsilon \leq \epsilon_1$,

$$
\begin{aligned}
||K(\frac{\epsilon x}{\lambda}+z)I_1||_{p,0,2+\beta} &\leq C\{(\int_{R^n}(<x>^{2+m+\beta}|\phi|^{n^*-\epsilon_1^2})^p dx)^{\frac{1}{p}} \\
&\quad +(\int_{R^n}(<x>^{2+m+\beta}|\phi|^{n^*})^p dx)^{\frac{1}{p}}\} \\
&= C\{||\phi||^{n^*-\epsilon_1^2}_{p(n^*-\epsilon_1^2),0,(2+m+\beta)/(n^*-\epsilon_1^2)}+||\phi||^{n^*}_{pn^*,0,(2+m+\beta)/n^*}\}.
\end{aligned}
$$

So

$$
||K(\frac{\epsilon x}{\lambda}+z)I_1||_{p,0,2+\beta} \leq C(||\phi||^{n^*-\epsilon_1^2}_{p,2,\beta}+||\phi||^{n^*}_{p,2,\beta}), \tag{3.13}
$$

provided the conditions for embedding (see Proposition 2.1) are satisfied: $\frac{2}{n}-\frac{1}{p}+\frac{1}{pn^*} \geq 0$, $(2+m+\beta)/(n^*-\epsilon_1^2) \leq \beta+\frac{n}{p}-n/p(n^*-\epsilon_1^2)$. It is easy to see that these conditions are satisfied if $p > \frac{2n}{n+2}$, $\frac{(2+m)(n-2)}{4}-\frac{n}{p} < \beta$ and if ϵ_1 is taken sufficiently small.

Combining (3.11) and (3.13), we reach the desired conclusion. □

Proof of Lemma 3.2:

Recall that the C^2 function K satisfies:$K(0)=1, \nabla K(0)=0$ and $0 \leq K(x) \leq C(1+|x|)^m$ with $m<2$. Then it is easy to see that

$$
\begin{aligned}
|K(\frac{\epsilon x}{\lambda}+z)-1| &\leq C|\frac{\epsilon x}{\lambda}+z|^2, \\
&\leq C(\epsilon^2|x|^2+|z|^2).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
||N^2_\epsilon(\phi)||_{p,0,2+\beta} &\leq C(\int_{R^n}((\epsilon^2|x|^2+|z|^2)<x>^{-4}<x>^{2+\beta}|\phi|)^p dx)^{\frac{1}{p}} \\
&\leq C(\epsilon^2+|z|^2)||\phi||_{p,0,\beta} \\
&\leq C(\epsilon^2+|z|^2)||\phi||_{p,2,\beta}. \quad \square
\end{aligned}
$$

Proof of Lemma 3.3: The proof of (3.5) is similar to that of Lemma 3.2 and hence we omit the details.

To show (3.6), we observe that for $0 \le \epsilon \le \epsilon_2$,

$$\begin{aligned} M_\epsilon^2 &= |K(\frac{\epsilon x}{\lambda}+z)(U^{n^*-\epsilon^2}-U^{n^*})| \\ &\le C(1+\epsilon|x|+|z|)^m \epsilon^2 U^{n^*-\epsilon^2}|\log U| \\ &\le C\epsilon^2 <x>^2 <x>^{-(n+2)+2\epsilon^2(n-2)} . \end{aligned}$$

So for $0 \le \epsilon \le \epsilon_2$, we have

$$\begin{aligned} ||M_\epsilon^2||_{p,0,2+\beta} &\le C\epsilon^2 (\int_{R^n} (<x>^{2+\beta-n+2\epsilon^2(n-2)})^p dx)^{\frac{1}{p}} \\ &\le C\epsilon^2, \end{aligned}$$

provided ϵ_2 is taken so small that

$$(2+\beta-n+2\epsilon_2^2(n-2))p+n<0.$$

But this is possible since we assumed $\beta < \frac{n}{p'}-2$. $\square$

Proof of Lemma 3.5: By Proposition 2.3 we have

$$\begin{aligned} ||F_\epsilon(\phi_1)-F_\epsilon(\phi_2)||_{p,2,\beta} &\le C(||N_\epsilon^1(\phi_1)-N_\epsilon^1(\phi_2)||_{p,0,\beta}+||N_\epsilon^2(\phi_1)-N_\epsilon^2(\phi_2)||_{p,0,\beta}) \\ &= I_4+I_5, \end{aligned}$$

where I_4 and I_5 are defined at the last equality.

Since $N_\epsilon^2(\phi)$ is linear in ϕ, then by Lemma 3.2,

$$I_5 \le C(\epsilon^2+|z|^2)||\phi_1-\phi_2||_{p,2,\beta}. \tag{3.14}$$

To estimate I_4, we observe that

$$\begin{aligned} &|N_\epsilon^1(\phi_1)-N_\epsilon^1(\phi_2)| \le \\ &K(\frac{\epsilon x}{\lambda}+z)|(U+\phi_1)_+^{n^*-\epsilon^2}-(U+\phi_2)_+^{n^*-\epsilon^2}-(n^*-\epsilon^2)(U+\phi_2)_+^{n^*-1-\epsilon^2}(\phi_1-\phi_2)| \\ &+K(\frac{\epsilon x}{\lambda}+z)(n^*-\epsilon^2)|(U+\phi_2)_+^{n^*-1-\epsilon^2}-U^{n^*-1-\epsilon^2}||\phi_1-\phi_2| \\ &+K(\frac{\epsilon x}{\lambda}+z)|(n^*-\epsilon^2)U^{n^*-1-\epsilon^2}-n^*U^{n^*-1}||\phi_1-\phi_2|. \end{aligned}$$

Since $1 < n^* < 2$, there exist a small constant ϵ_4 and a constant C independent of ϵ, such that for $0 \le \epsilon \le \epsilon_4$, ξ and $\eta \in R^1$,

$$|(1+\xi)_+^{n^*-\epsilon^2} - (1+\eta)_+^{n^*-\epsilon^2} - (n^*-\epsilon^2)(1+\eta)_+^{n^*-1-\epsilon^2}(\xi-\eta)| \le C|\xi-\eta|^{n^*-\epsilon^2}, \tag{3.15}$$

$$|(1+\xi)_+^{n^*-\epsilon^2-1} - 1| \le C|\xi|^{n^*-\epsilon^2-1}. \tag{3.16}$$

Thus for $0 \le \epsilon \le \epsilon_4$,

$$\begin{aligned}
& |N_\epsilon^1(\phi_1) - N_\epsilon^1(\phi_2)| \\
\le\ & C < x >^m (|\phi_1-\phi_2|^{n^*-\epsilon^2} + |\phi_2|^{n^*-\epsilon^2-1}||\phi_1-\phi_2| \\
& +\epsilon^2 U^{n^*-1-\epsilon^2}|\log U||\phi_1-\phi_2|) \\
\le\ & C < x >^m \{|\phi_1-\phi_2|^{n^*-\epsilon_4^2} + |\phi_1-\phi_2|^{n^*} \\
& +|\phi_1-\phi_2|(|\phi_2|^{n^*-\epsilon_4^2-1} + |\phi_2|^{n^*-1} + \epsilon^2 < x >^{-4+2\epsilon_4^2(n-2)})\}.
\end{aligned} \tag{3.17}$$

Observe that the $L^p_{2+\beta}$ norm of $< x >^m |\phi_1-\phi_2|^{n^*-\epsilon_4^2}$ is equal to

$$||\phi_1-\phi_2||^{n^*-\epsilon_4^2}_{p(n^*-\epsilon_4^2),0,(2+m+\beta)/(n^*-\epsilon_4^2)} \le C||\phi_1-\phi_2||^{n^*-\epsilon_4^2}_{p,2,\beta},$$

if we take ϵ_4 small and then use Proposition 2.1 (see the proof of Lemma 3.1 for the same situation).

Observe also that the $L^p_{2+\beta}$ norm of $< x >^m |\phi_1-\phi_2||\phi_2|^{n^*-\epsilon_4^2-1}$ is equal to

$$\begin{aligned}
& (\int_{R^n}(< x >^{2+m+\beta} |\phi_1-\phi_2||\phi_2|^{n^*-1-\epsilon_4^2})^p dx)^{\frac{1}{p}} \\
\le\ & (\int_{R^n} < x >^{p(2+m+\beta)} |\phi_2|^{p(n^*-\epsilon_4^2)}dx)^{(n^*-\epsilon_4^2-1)/p(n^*-\epsilon_4^2)} \\
& \times (\int_{R^n}(< x >^{p(2+m+\beta)} |\phi_1-\phi_2|^{p(n^*-\epsilon_4^2)}dx)^{1/(p(n^*-\epsilon_4^2))} \\
=\ & ||\phi_2||^{n^*-\epsilon_4^2-1}_{p(n^*-\epsilon_4^2),0,(2+m+\beta)/(n^*-\epsilon_4^2)}||\phi_1-\phi_2||_{p(n^*-\epsilon_4^2),0,(2+m+\beta)/(n^*-\epsilon_4^2)} \\
\le\ & C||\phi_2||^{n^*-1-\epsilon_4^2}_{p,2,\beta}||\phi_1-\phi_2||_{p,2,\beta},
\end{aligned}$$

where at the first step, we use Hölder's inequality, and at the last step we use Proposition 2.1.

Estimating each term on the right side of (3.17) in the same fashion, we have

$$
\begin{aligned}
I_2 &= ||N_\epsilon^2(\phi_1) - N_\epsilon^2(\phi_2)||_{p,0,2+\beta} \\
&\leq C\{\sum_{j=0}^{1} ||\phi_1 - \phi_2||_{p,2,\beta}^{n^*-j\epsilon_4^2} + ||\phi_1 - \phi_2||_{p,2,\beta}(\epsilon^2 + |z|^2) \\
&\quad + (\sum_{j=0}^{1} ||\phi_2||_{p,2,\beta}^{n^*-1-j\epsilon_4^2})||\phi_1 - \phi_2||_{p,2,\beta}\}.
\end{aligned}
$$

Combining this with (3.14), we are done. □

4 Reduced Problem

In this section, we shall prove that the fixed point $\phi_{\epsilon,z,\lambda}$ in Proposition 3.6 is indeed a solution of (3.2) for suitable z and λ. To this end, it suffices to show

$$
< S_\epsilon(U + \phi_{\epsilon,z,\lambda}), \Psi >= 0, \tag{4.1}
$$

for all $\Psi \in X =$ span $\{\frac{\partial U}{\partial x_1}, ..., \frac{\partial U}{\partial x_n}, x \cdot \nabla U + \frac{n-2}{2}U\}$.

Throughout the remainder of this paper, we will assume that the conditions in Proposition 3.5 hold, that $0 < \epsilon < \epsilon_0$ and $\lambda \in [a, b]$, where ϵ_0 is given as in Proposition 3.6.

Take $z = \epsilon^\gamma \eta$, where η is in the closed unit ball $B_1(0)$ in R^n, and γ is a constant (to be specified later) greater than 1 . Define a vector field

$$
V_\epsilon = (V_\epsilon^1, V_\epsilon^2, ..., V_\epsilon^{n+1}) : \Omega \equiv B_1(0) \times [a, b] \to R^{n+1}
$$

by the following:

$$V_\epsilon^j(\eta,\lambda) := \frac{1}{\epsilon^{1+\gamma}}\left\{\int_{R^n} S_\epsilon(U+\phi_{\epsilon,z,\lambda})\frac{\partial U}{\partial x_j}dx\right\}, j=1,...,n,$$

$$V_\epsilon^{n+1}(\eta,\lambda) := \frac{1}{\epsilon^2}\left\{\int_{R^n} S_\epsilon(U+\phi_{\epsilon,z,\lambda})(x\nabla U+\frac{n-2}{2}U)dx\right\},$$

where $(\eta,\lambda)\in\Omega$.

Proposition 4.1 *(1). For each fixed ϵ, V_ϵ is continuous on Ω.*

(2). Take the constant γ such that $1<\gamma<min(2(n^*-\epsilon_0^2),1+\alpha)$ (α is the Hölder exponent of D^2K). Then V_ϵ converges to V_0 uniformly on Ω as $\epsilon\to 0$, where

$$V_0^j(\eta,\lambda) = -\frac{n-2}{2n\lambda}\int_{R^n} U^{n^*+1}(x)dx\sum_{l=1}^n K_{jl}(0)\eta_l, j=1,...,n,$$

$$V_0^{n+1}(\eta,\lambda) = -\frac{n-2}{2n}(\frac{\Delta K(0)}{\lambda^2}\int_{R^n} x_1^2U^{n^*+1}(x)dx+\frac{n-2}{2}\int_{R^n} U^{n^*+1}(x)dx)$$

(where $K_{jl}(0)$ stands for $\frac{\partial^2 K}{\partial x_j\partial x_l}(0)$).

We delay the proof of this result but use it to prove Theorem 1.1 now.

Since $\Delta K(0)<0$, there exists a unique $\lambda_0>0$ such that $V_0^{n+1}(\eta,\lambda_0)=0$ for $\eta\in B_1(0)$ (λ_0 depends only on n). Now choose $a=\frac{\lambda_0}{2}, b=2\lambda_0$. Since 0 is a nondegenerate critical point of K, i.e. $det(K_{ij}(0))\neq 0$, V_0 never vanishes on the boundary of Ω and $deg(V_0,\Omega,0)\neq 0$. By Proposition 4.1, the continuous vector field V_ϵ is close to, and hence, homotopic to V_0 for each small $\epsilon>0$. Thus $deg(V_\epsilon,\Omega,0)=deg(V_0,\Omega,0)\neq 0$ and consequently V_ϵ has a zero point $(\eta(\epsilon),\lambda(\epsilon))\in B_1(0)\times[\frac{\lambda_0}{2},2\lambda_0]$. Now (4.1) holds for $z=z(\epsilon)=\epsilon^\gamma\eta(\epsilon),\lambda=\lambda(\epsilon)$ and small $\epsilon>0$. Therefore $v_\epsilon=U+\phi_\epsilon$, where ϕ_ϵ stands for $\phi_{\epsilon,z_\epsilon,\lambda_\epsilon}$, is a weak and hence positive classical solution of (3.1) and of (1.6) (the fact that $(v_\epsilon)_+ \not\equiv 0$ follows from (3.10)). Going backwards from (1.6) to (1.1), we see that

$$u_\epsilon(x) = (\frac{\lambda(\epsilon)}{\epsilon})^{\frac{2}{q-1}} v_\epsilon(\frac{\lambda(\epsilon)(x - z(\epsilon))}{\epsilon}) \tag{4.2}$$

is a solution of (1.1).

Next we discuss the decay rate and asymptotic behavior of u_ϵ. It is clear that for fixed p, $W^p_{2,\beta}$ and $L^p_{2+\beta}$ get smaller as β gets larger. So for fixed $p > \frac{2n}{n+2}$, ϕ_ϵ (which is the fixed point of F_ϵ on $B_\delta \subset Y \subset W^p_{2,\beta}$) is independent of β, where β satisfies the condition in Proposition 3.5, i.e.

$$\frac{(2+m)(n-2)}{4} < \beta + \frac{n}{p} < n - 4. \tag{4.3}$$

Now choose p so large that $2 > \frac{n}{p}$. Then by (4.3), (2) of Proposition 2.1 and (3.10), we have for $l < n-4$,

$$\begin{aligned} |\phi_\epsilon(x)| &\le C < x >^{-l} ||\phi_\epsilon||_{p,2,\beta} \\ &\le C\epsilon^2 < x >^{-l}, x \in R^n, \end{aligned} \tag{4.4}$$

where C is a constant, dependent on l but independent of ϵ. This and (4.2) imply that $u_\epsilon(0) \to \infty$ as $\epsilon \to 0$ and that

$$\begin{aligned} 0 < u_\epsilon(x) &\le C(\frac{\lambda(\epsilon)}{\epsilon})^{\frac{2}{q-1}} U(\frac{\lambda(\epsilon)(x-z(\epsilon))}{\epsilon}) + C(\frac{\lambda(\epsilon)}{\epsilon})^{\frac{2}{q-1}-l} \epsilon^2 |x - z(\epsilon)|^{-l} \\ &\le C(\frac{\lambda(\epsilon)}{\epsilon})^{\frac{2}{q-1}-(n-2)} |x - z(\epsilon)|^{-(n-2)} + C(\frac{\lambda(\epsilon)}{\epsilon})^{\frac{2}{q-1}-(l+2)} |x - z(\epsilon)|^{-l} \\ &\le C\epsilon^{l+2-\frac{n-2}{2}} (|x - z(\epsilon)|^{-(n-2)} + |x - z(\epsilon)|^{-l}) \end{aligned}$$

(recall $\lambda(\epsilon) \le 2\lambda_0(n)$).

Hence for any fixed $\delta > 0$ and $l < n-4$, we have

$$0 < u_\epsilon(x) \le C\epsilon^{l+2-\frac{n-2}{2}} |x|^{-l}, \text{for } |x| \ge \delta, \tag{4.5}$$

where C is a constant, independent of small ϵ but dependent on l and δ. If l is close to $n-4$, then $l+2-\frac{n-2}{2}$ is close to $\frac{n-2}{2}>0$. So (4.5) implies that u_ϵ decays uniformly outside any fixed neighborhood of the origin.

We have thus established the conclusion in Theorem 1.1 concerning u_ϵ 's blow-up and concentration. Next, we shall prove (1.2), i.e. u_ϵ's decay rate. Assume now that $|x|\geq 1$. In the following, C always denotes a constant independent of ϵ but dependent on p,β,l and n.

By (4.4) and (4.2), we have that

$$\begin{aligned}
\int_{|y|\leq 1} K(y)u_\epsilon^q(y)dy &\leq C\int_{|y|\leq 1} u_\epsilon^q(y)dy \\
&\leq C\int_{|y|\leq 1} (\frac{\lambda(\epsilon)}{\epsilon})^{\frac{2q}{q-1}} v_\epsilon^q(\frac{\lambda(\epsilon)(y-z(\epsilon))}{\epsilon})dy \\
&= C\lambda(\epsilon)^{\frac{2q}{q-1}-n}\epsilon^{n-\frac{2q}{q-1}} \int_{|\frac{\epsilon x}{\lambda(\epsilon)}+z(\epsilon)|\leq 1} v_\epsilon^q(x)dx \\
&\leq C\epsilon^{n-\frac{2q}{q-1}} \int_{|x|\leq\frac{4\lambda_0}{\epsilon}} v_\epsilon^q(x)dx \\
&\leq C\epsilon^{n-\frac{2q}{q-1}}(\int_{|x|\leq\frac{4\lambda_0}{\epsilon}} U^q(x)dx + \int_{|x|\leq\frac{4\lambda_0}{\epsilon}} |\phi_\epsilon(x)|^q dx) \\
&\leq C\epsilon^{n-\frac{2q}{q-1}}(C+\epsilon^{lq-n+2q}).
\end{aligned}$$

Hence, we have

$$\int_{|y|\leq 1} K(y)u_\epsilon^q(y)dy \leq C\int_{|y|\leq 1} u_\epsilon^q(y)dy \leq C. \tag{4.6}$$

Observe that

$$\begin{aligned}
u_\epsilon(x) &= c_n\int_{R^n}\frac{K(y)u_\epsilon^q(y)}{|x-y|^{n-2}}dy \\
&\leq C(\int_{|y-x|\leq\frac{|x|}{2}} + \int_{\frac{|x|}{2}\leq|y-x|\leq 2|x|} + \int_{2|x|\leq|y-x|})\frac{K(y)u_\epsilon^q(y)}{|x-y|^{n-2}}dy \\
&= I_6+I_7+I_8,
\end{aligned}$$

where I_6, I_7 and I_8 are defined by the last equality.

For I_6, we note that in $\Omega_1 \equiv \left\{y \in R^n || y-x| \leq \frac{|x|}{2}\right\}$, $\frac{1}{2} \leq \frac{|x|}{2} \leq |y| \leq \frac{3|x|}{2}$. Thus by (4.5),

$$\begin{aligned}
I_6 &\leq \int_{|y-x| \leq \frac{|x|}{2}} \frac{C < y >^m < y >^{-ql}}{|x-y|^{n-2}} dy \\
&= C \int_{|y-x| \leq \frac{|x|}{2}} \frac{< y >^{m-ql}}{|x-y|^{n-2}} dy \\
&\leq C|x|^{m-ql} \int_0^{\frac{|x|}{2}} \frac{1}{r^{n-2}} r^{n-1} dr \\
&= C|x|^{m-ql+2}.
\end{aligned}$$

I_8 may be estimated similarly. For in $\Omega_3 \equiv \{y \in R^n | 2|x| \leq |y-x|\}$, $1 \leq |x| \leq \frac{|y-x|}{2} \leq |y| \leq \frac{3|y-x|}{2}$. Hence,

$$\begin{aligned}
I_8 &\leq C \int_{\Omega_3} \frac{< y >^{m-ql}}{|x-y|^{n-2}} dy \\
&\leq C \int_{\Omega_3} \frac{< x-y >^{m-ql}}{|x-y|^{n-2}} dy \\
&\leq C \int_{2|x|}^{\infty} \frac{r^{n-1}}{r^{n-2} r^{ql-m}} dr \\
&= C|x|^{m-ql+2}.
\end{aligned}$$

where we choose l close to $n-4$ and ϵ small enough such that $m-ql+2 < 0$.

To estimate I_7, let $\Omega_2 \equiv \left\{y \in R^n | \frac{|x|}{2} \leq |y-x| \leq 2|x|\right\}$, and observe that by (4.6),

$$\begin{aligned}
I_7 &\leq \frac{C}{|x|^{n-2}} \int_{\Omega_2} K(y) u_\epsilon^q(y) dy \\
&\leq \frac{C}{|x|^{n-2}} \left(\int_{|y| \leq 1} K(y) u_\epsilon^q(y) dy + \int_{1 \leq |y| \leq 3|x|} K(y) u_\epsilon^q(y) dy \right) \\
&\leq \frac{C}{|x|^{n-2}} \left(C + \int_1^{3|x|} \frac{dy}{|y|^{ql-m}} \right)
\end{aligned}$$

$$\leq \begin{cases} C|x|^{2-n}, & \text{if } n-1+m-ql<-1. \\ C|x|^{2-n}(\log|x|+1), & \text{if } n-1+m-ql=-1. \\ C|x|^{2-n}(C+|x|^{n+m-ql}), & \text{if } n-1+m-ql>-1. \end{cases}$$

Now we see that

$$u_\epsilon(x) \leq \begin{cases} C|x|^{2-n}, & \text{if } n+m-ql<0. \\ C|x|^{2-n}(\log|x|+1), & \text{if } n+m-ql=0. \\ C|x|^{m-ql+2}, & \text{if } n+m-ql>0. \end{cases} \tag{4.7}$$

But since $n+m-ql>0$, it follows that

$$u_\epsilon(x) \leq C|x|^{m-ql+2}, \quad |x| \geq 1. \tag{4.8}$$

This decay result is an improvement of (4.5), since $m-ql+2<-l$ for l close to $n-4$ and ϵ small (recall that $n>(12-2m)/(2-m)$). Moreover, we can keep on improving (4.8) until we reach (1.2) as follows.

Let $l_0=-(m-ql+2)$. Using (4.8) and repeating the arguments leading to (4.7), we have that (4.7) with l replaced by l_0 holds. If $n+m-ql_0<0$, we are done; If $n+m-ql_0=0$, by increasing l a little we reduce this case to the previous one and hence we are also done; If $n+m-ql_0>0$, then (4.8) with l replaced by l_0 holds. In this case we let $l_1=-(m-ql_0+2)$ and repeat the above process with l_0 replaced by l_1, which leads to either (1.2) or (4.8) with l replaced by l_1. If the latter occurs, we iterate the above process again and again, and each time we are led to (4.8) with l replaced by

$$l_k=-(m-ql_{k-1}+2), \quad k=1,2,..., \tag{4.9}$$

until we reach $n+m-ql_k \leq 0$ (and hence (1.2)). On the other hand, it is easy to see that $l<l_0<l_1<...<l_k \to \infty$ as $k \to \infty$. Thus after finitely many iterations, we eventually have (1.2).

The proof of Theorem 1.1 is complete.

Proof of Proposition 4.1:

(1) Since ϵ is fixed, in this part of the proof we can always assume that $\epsilon = 1$ (which causes no loss of generality), and we thus write $\phi_{\epsilon,z,\lambda}$ as $\phi_{z,\lambda}$, $F_\epsilon(\phi)$ (defined in Section 3) as $F_{z,\lambda}(\phi)$, V_ϵ as V. Now $z=\eta$, and hence our goal is to show the continuity of $V(z,\lambda)$ on Ω.

We claim that $\phi_{z,\lambda}$ is continuous in (z,λ) with respect to the $W^p_{2,\beta}$ norm. To prove this, observe that since $\phi_{z,\lambda}$ is the fixed point of the contraction mapping $F_{z,\lambda}$ on the ball $B_\delta \subset Y$ (δ is independent of $(z,\lambda)\in\Omega$), we have, for some positive constant $\overline{c} < 1$,

$$\begin{aligned}
& ||\phi_{z_1,\lambda_1} - \phi_{z_2,\lambda_2}||_{p,2,\beta} \\
\leq\ & ||F_{z_1,\lambda_1}(\phi_{z_1,\lambda_1}) - F_{z_1,\lambda_1}(\phi_{z_2,\lambda_2})||_{p,2,\beta} + ||F_{z_1,\lambda_1}(\phi_{z_2,\lambda_2}) - F_{z_2,\lambda_2}(\phi_{z_2,\lambda_2})||_{p,2,\beta} \\
\leq\ & \overline{c}||\phi_{z_1,\lambda_1} - \phi_{z_2,\lambda_2}||_{p,2,\beta} + ||F_{z_1,\lambda_1}(\phi_{z_2,\lambda_2}) - F_{z_2,\lambda_2}(\phi_{z_2,\lambda_2})||_{p,2,\beta}.
\end{aligned}$$

Thus to prove the claim, we need only verify that for any fixed $\phi \in B_\delta$, we have

$$||F_{z_1,\lambda_1}(\phi) - F_{z_2,\lambda_2}(\phi)||_{p,2,\beta} \to 0 \text{ as } (z_1,\lambda_1) \to (z_2,\lambda_2). \tag{4.10}$$

Observe that

$$\begin{aligned}
& ||F_{z_1,\lambda_1}(\phi) - F_{z_2,\lambda_2}(\phi)||_{p,2,\beta} \\
=\ & ||L^{-1}P\left[(K(\frac{x}{\lambda_1}+z_1) - K(\frac{x}{\lambda_2}+z_2))(U+\phi)^q_+\right]||_{p,2,\beta} \\
\leq\ & C||(K(\frac{x}{\lambda_1}+z_1) - K(\frac{x}{\lambda_2}+z_2))(U+\phi)^q_+||_{p,o,2+\beta}
\end{aligned} \tag{4.11}$$

and that

$$|(K(\frac{x}{\lambda_1}+z_1) - K(\frac{x}{\lambda_2}+z_2))(U+\phi)^q_+| \leq C < x >^m (U+\phi)^q_+, \tag{4.12}$$

whose $L^p_{2+\beta}$ norm is equal to

$$||(U+\phi)_+||^q_{qp,0,(m+2+\beta)/q} \leq C||(U+\phi)_+||^q_{p,2,\beta} < \infty.$$

At the last step, we used Proposition 2.1, just as we did in the derivation of (3.13).

Now (4.10) follows from (4.11), (4.12) and the Lebesgue Dominated Convergence Theorem. Hence we have shown the continuity of $\phi_{z,\lambda}$ in (z,λ) w.r.t. $W^p_{2,\beta}$ norm.

To prove the continuity of V on Ω, we need only prove the continuity of $S(U+\phi_{z,\lambda}) \equiv \Delta(U+\phi_{z,\lambda}) + K(\frac{x}{\lambda}+z)(U+\phi_{z,\lambda})^q_+$ in $(z,\lambda) \in \Omega$ w.r.t. $L^p_{2+\beta}$ norm.

To this end, we deduce that

$$\begin{aligned}
& ||S(U+\phi_{z_1,\lambda_1}) - S(U+\phi_{z_2,\lambda_2})||_{p,0,2+\beta} \\
\leq\ & ||\Delta(\phi_{z_1,\lambda_1} - \phi_{z_2,\lambda_2})||_{p,0,2+\beta} \\
& + ||K(\frac{x}{\lambda_1}+z_1)\left[(U+\phi_{z_1,\lambda_1})^q_+ - (U+\phi_{z_2,\lambda_2})^q_+\right]||_{p,0,2+\beta} \\
& + ||(K(\frac{x}{\lambda_1}+z_1) - K(\frac{x}{\lambda_2}+z_2))(U+\phi_{z_2,\lambda_2})^q_+||_{p,0,2+\beta} \\
=\ & I_9 + I_{10} + I_{11},
\end{aligned}$$

where I_9, I_{10} and I_{11} are defined at the last equality.

But

$$I_9 \leq C||\phi_{z_1,\lambda_1} - \phi_{z_2,\lambda_2}||_{p,2,\beta} \to 0$$

as $(z_1,\lambda_1) \to (z_2,\lambda_2)$. Aslo as shown in the proof of (4.10), $I_{11} \to 0$ when $(z_1,\lambda_1) \to (z_2,\lambda_2)$. I_{10} can be handled similarly as in the proof of Proposition 3.5, and by combining the estimates there with the continuity of $\phi_{z,\lambda}$ in (z,λ), we see that $I_{10} \to 0$ as $(z_1,\lambda_1) \to (z_2,\lambda_2)$.

(1) of Proposition 4.1 is proved.

(2) Since $\frac{\partial U}{\partial x_j} \in X = Ker(L^*)$, we have

$$
\begin{aligned}
& \int_{R^n} S_\epsilon(U + \phi_{\epsilon,z,\lambda}) \frac{\partial U}{\partial x_j} dx \\
= & \int_{R^n} \left[L\phi_{\epsilon,z,\lambda} + N_\epsilon^1(\phi_{\epsilon,z,\lambda}) + N_\epsilon^2(\phi_{\epsilon,z,\lambda}) + M_\epsilon^1 + M_\epsilon^2 \right] \frac{\partial U}{\partial x_j} dx \\
= & < N_\epsilon^1(\phi_{\epsilon,z,\lambda}), \frac{\partial U}{\partial x_j} > + < N_\epsilon^2(\phi_{\epsilon,z,\lambda}), \frac{\partial U}{\partial x_j} > + < M_\epsilon^1, \frac{\partial U}{\partial x_j} > + < M_\epsilon^2, \frac{\partial U}{\partial x_j} > \\
= & II_1 + II_2 + II_3 + II_4,
\end{aligned}
$$

where II_1, II_2, II_3 and II_4 are defined at the last equality.

By Lemma 3.1, Lemma 3.2 and (3.10), we have

$$
\begin{aligned}
|II_1| & \leq ||N_\epsilon^1(\phi_{\epsilon,z,\lambda})||_{p,0,2+\beta} || \frac{\partial U}{\partial x_j} ||_{p,0,-(2+\beta)} \\
& \leq C(\epsilon^4 + \epsilon^{2(n^* - \epsilon_0^2)}) \qquad (4.13)
\end{aligned}
$$

and

$$
|II_2| \leq C\epsilon^2(\epsilon^2 + |z|^2) \leq C\epsilon^4. \qquad (4.14)
$$

Since 0 is a critical point of $K \in C_{loc}^{2+\alpha}(R^n)(0 < \alpha < 1)$ and $0 \leq K(x) \leq C(1 + |x|)^m$ with $m < 2$, we have

$$
K(x) = 1 + \frac{1}{2} \sum_{i,l=1}^{n} K_{il}(0) x_i x_l + O(|x|^{2+\alpha}), \quad x \in R^n \qquad (4.15)
$$

where $|O(|x|^{2+\alpha})| \leq C|x|^{2+\alpha}$ for some constant C and all $x \in R^n$. By this, together with the fact that $\frac{\partial U}{\partial x_j}$ is odd in x_j and even in the other variables, we obtain

$$
II_3 = \int_{R^n} (K(\frac{\epsilon x}{\lambda} + z) - 1) U^{n^*} \frac{\partial U}{\partial x_j} dx
$$

$$
\begin{aligned}
&= \int_{R^n}\left[\frac{1}{2}\sum_{i,l=1}^{n} K_{il}(0)(\frac{\epsilon x_i}{\lambda}+z_i)(\frac{\epsilon x_l}{\lambda}+z_l)+O(|\frac{\epsilon x}{\lambda}+z|^{2+\alpha})\right]U^{n^*}\frac{\partial U}{\partial x_j}dx \\
&= \frac{\epsilon}{\lambda}\int_{R^n}\sum_{l=1}^{n} K_{jl}(0)z_l U^{n^*}x_j\frac{\partial U}{\partial x_j}dx+O(\epsilon^{2+\alpha}+|z|^{2+\alpha}) \\
&= \frac{\epsilon}{(n^*+1)\lambda}\int_{R^n} x_j\frac{\partial U^{n^*+1}}{\partial x_j}dx\sum_{l=1}^{n} K_{jl}(0)z_l+O(\epsilon^{2+\alpha}+|z|^{2+\alpha}) \\
&= -\frac{\epsilon}{(n^*+1)\lambda}\int_{R^n} U^{n^*+1}dx\sum_{l=1}^{n} K_{jl}(0)z_l+O(\epsilon^{2+\alpha}+|z|^{2+\alpha}) \\
&\quad (\text{ integrating by parts. })
\end{aligned}
$$

Hence

$$
II_3=-\frac{\epsilon^{1+\gamma}}{(n^*+1)\lambda}\int_{R^n} U^{n^*+1}dx\sum_{l=1}^{n} K_{jl}(0)\eta_l+O(\epsilon^{2+\alpha}). \tag{4.16}
$$

Since $\frac{\partial U}{\partial x_j}$ is odd in x_j, we see that

$$
\begin{aligned}
II_4 &= \int_{R^n} K(\frac{\epsilon x}{\lambda}+z)(U^{n^*-\epsilon^2}-U^{n^*})\frac{\partial U}{\partial x_j}dx \\
&= \int_{R^n}(K(\frac{\epsilon x}{\lambda}+z)-1)(U^{n^*-\epsilon^2}-U^{n^*})\frac{\partial U}{\partial x_j}dx.
\end{aligned}
$$

So by (4.15), the mean value theorem and the fact that $U\le 1$, we deduce that

$$
\begin{aligned}
|II_4| &\le C\int_{R^n}(\epsilon^2|x|^2+|z|^2)\epsilon^2 U^{n^*-\epsilon_0^2}|\log U||\frac{\partial U}{\partial x_j}|dx \\
&\le C\epsilon^4\int_{R^n}|x|^2 U^{n^*-\epsilon_0^2}|\log U||\frac{\partial U}{\partial x_j}|dx \\
&\quad +C\epsilon^2|z|^2\int_{R^n} U^{n^*-\epsilon_0^2}|\log U||\frac{\partial U}{\partial x_j}|dx \\
&\le C(\epsilon^4+\epsilon^2|z|^2).
\end{aligned}
$$

Combining this with (4.13), (4.14) and (4.16), we see that V_ϵ^j converges to V_0^j uniformly on Ω as $\epsilon\to 0, j=1,...,n$.

It remains to show that V_ϵ^{n+1} converges uniformly to V_0^{n+1}. We shall denote $x \cdot \nabla U + \frac{n-2}{2} U$ by Ψ.

As before, we have

$$\begin{aligned}
& \int_{R^n} S_\epsilon(U + \phi_{\epsilon,z,\lambda}) \Psi dx \\
= \; & < N_\epsilon^1(\phi_{\epsilon,z,\lambda}), \Psi > + < N_\epsilon^2(\phi_{\epsilon,z,\lambda}), \Psi > + < M_\epsilon^1, \Psi > + < M_\epsilon^2, \Psi > \\
= \; & II_5 + II_6 + II_7 + II_8,
\end{aligned}$$

where II_5, II_6, II_7 and II_8 are defined at the last equality, and that

$$|II_5| \le C(\epsilon^4 + \epsilon^{2(n^* - \epsilon_0^2)}) \tag{4.17}$$

$$|II_6| \le C\epsilon^2(\epsilon^2 + |z|^2) \le C\epsilon^4. \tag{4.18}$$

For II_7, by using (4.15) and the symmetry property of Ψ, we deduce that

$$\begin{aligned}
II_7 &= \int_{R^n} (K(\frac{\epsilon x}{\lambda} + z) - 1) U^{n^*} \Psi dx \\
&= \int_{R^n} \left[\frac{1}{2} \sum_{i,l=1}^n K_{il}(0)(\frac{\epsilon x_i}{\lambda} + z_i)(\frac{\epsilon x_l}{\lambda} + z_l) + O(|\frac{\epsilon x}{\lambda} + z)|^{2+\alpha}) \right] U^{n^*} \Psi dx \\
&= \frac{1}{2} \int_{R^n} \sum_{i=1}^n K_{ii}(0) \frac{\epsilon^2}{\lambda^2} x_i^2 U^{n^*} \Psi dx \\
&\quad + \frac{1}{2} \int_{R^n} \sum_{i,l=1}^n K_{il}(0) z_i z_l U^{n^*} \Psi dx + O(\epsilon^{2+\alpha} + |z|^{2+\alpha}) \\
&= J_1 + J_2 + O(\epsilon^{2+\alpha} + |z|^{2+\alpha}),
\end{aligned}$$

where J_1, J_2 are defined at the last equality.

To compute J_2, we first observe that $\Psi = \frac{\partial U_\lambda}{\partial \lambda}|_{\lambda=1}$, where $U_\lambda(x) = \lambda^{\frac{n-2}{2}} U(\lambda x)$. Hence

$$\begin{aligned}
\int_{R^n} U^{n^*} \Psi dx &= \int_{R^n} U_\lambda^{n^*} \frac{\partial U_\lambda}{\partial \lambda} dx|_{\lambda=1} = \frac{1}{n^*+1} \frac{\partial}{\partial \lambda} \int_{R^n} U_\lambda^{n^*+1} dx|_{\lambda=1} \\
&= \frac{1}{n^*+1} \frac{\partial}{\partial \lambda} \int_{R^n} U^{n^*+1} dx|_{\lambda=1} = 0
\end{aligned}$$

and thus $J_2 = 0$. To compute J_1, we apply integration by parts on

$$\begin{aligned}
\int_{R^n} x_1^2 U^{n^*} \Psi dx &= \int_{R^n} x_1^2 U^{n^*} (x\nabla U + \frac{n-2}{2} U) dx \\
&= \frac{n-2}{2} \int_{R^n} x_1^2 U^{n^*+1} dx - \sum_{k=1}^{n} \int_{R^n} \frac{U^{n^*+1}}{n^*+1} \frac{\partial}{\partial x_k} (x_1^2 x_k) dx \\
&= \frac{2-n}{2} \int_{R^n} x_1^2 U^{n^*+1} dx.
\end{aligned}$$

Thus

$$II_7 = \frac{(2-n)\epsilon^2}{2n\lambda^2} \Delta K(0) \int_{R^n} x_1^2 U^{n^*+1} dx. \tag{4.19}$$

Write II_8 as

$$\begin{aligned}
II_8 &= \int_{R^n} (K(\frac{\epsilon x}{\lambda} + z) - 1)(U^{n^*-\epsilon^2} - U^{n^*}) \Psi dx + \int_{R^n} (U^{n^*-\epsilon^2} - U^{n^*}) \Psi dx \\
&= J_3 + J_4,
\end{aligned}$$

where J_3 and J_4 are defined at the last equality.

J_3 can be estimated just as we did for II_4:

$$|J_3| \le C\epsilon^2(\epsilon^2 + |z|^2) \le C\epsilon^4. \tag{4.20}$$

By Taylor's theorem, we have that for some $0 < t < 1$,

$$\begin{aligned}
J_4 &= \int_{R^n} \left[-\epsilon^2 U^{n^*} \log U + \frac{1}{2} \epsilon^4 U^{n^*-t\epsilon^2} (\log U)^2 \right] \Psi dx \\
&= -\epsilon^2 \int_{R^n} U^{n^*} \log U \Psi dx + O(\epsilon^4) \\
&= -\epsilon^2 \int_{R^n} U_\lambda^{n^*} \log U_\lambda \frac{\partial U_\lambda}{\partial \lambda} dx|_{\lambda=1} + O(\epsilon^4) \\
&= -\frac{\epsilon^2}{n^*+1} \int_{R^n} \frac{\partial}{\partial \lambda} (U_\lambda^{n^*+1} \log U_\lambda) dx|_{\lambda=1} + \frac{\epsilon^2}{n^*+1} \int_{R^n} U_\lambda^{n^*} \frac{\partial U_\lambda}{\partial \lambda} dx|_{\lambda=1} + O(\epsilon^4) \\
&= -\frac{\epsilon^2}{n^*+1} \int_{R^n} \frac{\partial}{\partial \lambda} (U_\lambda^{n^*+1} \log U_\lambda) dx|_{\lambda=1} + 0 + O(\epsilon^4).
\end{aligned}$$

Observe that

$$\begin{aligned}\int_{R^n} U_\lambda^{n^*+1}\log U_\lambda dx &= \int_{R^n} \lambda^n U^{n^*+1}(\lambda x)(\frac{n-2}{2}\log\lambda + \log U(\lambda x))dx \\ &= \frac{n-2}{2}\log\lambda \int_{R^n} U^{n^*+1}(y)dy + \int_{R^n} U^{n^*+1}(y)\log U(y)dy.\end{aligned}$$

So we see that

$$\begin{aligned}J_4 &= -\frac{\epsilon^2}{n^*+1}\frac{n-2}{2}\int_{R^n} U^{n^*+1}(y)dy + O(\epsilon^4) \\ &= -\frac{\epsilon^2(n-2)^2}{4n}\int_{R^n} U^{n^*+1}(y)dy + O(\epsilon^4).\end{aligned}$$

This and (4.19) imply

$$II_8 = -\frac{\epsilon^2(n-2)^2}{4n}\int_{R^n} U^{n^*+1}(y)dy + O(\epsilon^4 + \epsilon^2|z|^2). \tag{4.21}$$

Now (4.16)-(4.21) imply the desired uniform convergence of V_ϵ^{n+1} to V_0^{n+1} on Ω as $\epsilon \to 0$. □

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