VORTEX RINGS PINNING FOR THE GROSS-PITAEVSKII EQUATION IN THREE DIMENSIONAL SPACE

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Abstract. We construct stationary solutions possessing two vortex rings to the nonlinear Schrödinger type problem

$$iu_t = -\varepsilon^2 \triangle u + \Big(V + |u|^2\Big)u,$$

where the unknown function u is defined as $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, ε is a small positive parameter and V is a real smooth potential. These two vortex rings will be pinned at a fixed site by the potential V. They lie in the same plane and have neighboring interaction in the normal direction, or in two parallel planes with interaction in the binormal direction, in such a way that the neighboring interaction will be balanced by the effect of the potential.

1. Introduction

1.1. **Background.** In the present paper, we consider the existence of solutions with vortex rings to the nonlinear Schrödinger type problem

$$iu_t = -\varepsilon^2 \triangle u + \Big(V + |u|^2\Big)u, \tag{1.1}$$

where the unknown function u is defined as $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$, $\triangle$ is the Laplace operator in $\mathbb{R}^3$, ε is a small positive parameter and V is a real smooth potential. The equation (1.1) called Gross-Pitaevskii equation[69] is a well-known mathematical model to describe Bose-Einstein condensates.

Quantized vortices have gained major interest in the past few years due to the experimental realization of Bose-Einstein condensates (cf.[9], [32]). Vortices in Bose-Einstein condensates are quantized, and their size, origin, and significance are quite different from those in normal fluids since they exemplify superfluid properties (cf.[33], [10], [11]). In addition to the simpler two-dimensional point vortices, two types of individual topological defects in three-dimensional Bose-Einstein condensates have focused the attention of the scientific community in recent years: vortex lines[81, 78, 41] and vortex rings. Quantized vortex rings with cores have been proved to exist when charged particles are accelerated through superfluid helium[71]. The achievements of quantized vortices in a trapped Bose-Einstein condensate [85], [65], [64] have suggested the possibility of producing vortex rings with ultracold atoms. The existence and dynamics of vortex rings in a trapped Bose-Einstein condensates have been studied by several authors [8], [46], [47], [35], [73], [40], [75], [45]. Vortex ring and their two-dimensional analogy(vortex-antivortex pair) have played an important role in the study of complex quantized structures such as superfluid turbulence and so attracted much attention [11], [10], [53], [44]. The reader can refer to the review papers [36], [38], [11] for more details on quantized vortices in physical works.

In the present paper, we are concerned with the construction of vortices by rigorous mathematical method. For the steady state, (1.1) becomes the problem

$$-\varepsilon^2 \triangle u + \Big(V(y) + |u|^2\Big)u = 0, \tag{1.2}$$

where the unknown function u is defined as $u : \mathbb{R}^3 \to \mathbb{C}$, ε is a small positive parameter and V is a smooth potential. The study of the problem (1.2) in the homogeneous case, i.e. $V \equiv -1$, on bounded domain with suitable boundary condition started from [14] by F. Bethuel, H. Brezis, F.

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Helein in 1994, see also the book by K. Hoffmann and Q. Tang[43]. Since then, there are many references on the existence, asymptotic behavior and dynamical behavior of solutions. We refer to the books [2] and [76] for references and backgrounds. Regarding the construction of solutions, we mention two works which are relevant to present one. F. Pacard and T. Riviere derived a non-variational method to construct solutions with coexisting degrees of +1 and -1 in [67]. The proof is based on an analysis of the linearized operator around an approximation. M. del Pino, M. Kowalczyk and M. Musso [27] derived a reduction method for general existence for vortex solutions under Neumann (or Dirichlet)boundary conditions. The reader can refer to [54]-[56], [58], [79], [86], [23]-[24], [48]-[51], [80] and the references therein.

Traveling wave solutions are known to play an important role in the full dynamics of (1.1). More precisely, when $V \equiv -1$, these are solutions in the form

$$u(y,t) \,=\, \tilde{u}\big(y_1,\, y_2,\, y_3 - \varepsilon\, C\, t\big).$$

Then by a suitable rescaling, $\tilde{u}$ is a solution of the nonlinear elliptic problem

$$-\,i\,C\,\frac{\partial \tilde{u}}{\partial y_3} \,=\, -\triangle \tilde{u} \,+\, \Big(|\tilde{u}|^2 - 1\Big)\tilde{u}. \tag{1.3}$$

In two dimensional plane, F. Bethuel and J. Saut constructed a traveling wave with two vortices of degree ± 1 in [18]. In higher dimension, by minimizing the energy, F. Bethuel, G. Orlandi and D. Smets constructed solutions with a vortex ring[17]. See [22] for another proof by Mountain Pass Lemma and the extension of results in [16]. The reader can refer to the review paper [15] by F. Bethuel, P. Gravejat and J. Saut and the references therein. For a similar existence result of vortex rings for Shrödinger map equation, F. Lin and J. Wei [60] gave a new proof by using a reduction method.

1.2. **The pinning phenomena.** Before stating our assumptions and main result, we review some references on pinning phenomena of vortices.

We start the review by going through the pinning phenomena in superconductors which are described by the well known Ginzburg-Landau model and most relevant to the topics for Gross-Pitaevskii equations. When a superconductor of type II is placed in an external magnetic field, the field penetrates the superconductor in thin tubes of magnetic flux called magnetic vortices. This will cause the dissipation of energy due to creeping or flow of magnetic vortices[82]. In the application of superconductors, it is of importance to pin vortices at fixed locations, i.e. prevent their motion. Various mechanisms have been advances by physicists, engineers and mathematicians, such as introducing impurities into the superconducting material sample, changing the thickness of the superconducting material sample, so as to derive various variants of the original Ginzburg-Landau mode of superconductivity.

We first mention the results for the modified Ginzburg-Landau equations of a superconductor with impurities

$$\begin{aligned} -\triangle_A \psi \,+\, \lambda(|\psi|^2 - 1)\psi \,+\, W(x)\psi \,&=\, 0,\\ \nabla \times \nabla \times A \,+\, \mathrm{Im}(\bar{\psi}\, \nabla_A\, \psi) \,&=\, 0, \end{aligned} \tag{1.4}$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is a potential of impurities, $\nabla_A = \nabla - iA$ is the covariant gradient and $\triangle_A = \nabla_A \cdot \nabla_A$. For a vector field A, $\nabla \times A = \partial_1 A_2 - \partial_2 A_1$. Numerical evidence that fundamental magnetic vortices(degrees of ± 1) of the same degree are attracted to maxima of $W(x)$ can be found in works by Chapman, Du and Gunzburger[20], Du, Gunzburger and Peterson[34]. Ting and Gustafson[42] have shown dynamic stability/instability of single pinned fundamental vortices. Pakylak, Ting and Wei show the pinning phenomena of multi-vortices in [68]. Sigal and Strauss[77] have derived the effective dynamics of the magnetic vortex in a local potential. For a small positive number ϵ and some $q > 0$, by defining

- Strength of external potential W: $W \in H^1(\mathbb{R}^2)$ with $\|W\|_{H^1} \leq \epsilon^q$

- Smallness of W and derivatives of W: $\sup_{x\in\mathbb{R}^2} |\partial_x^\alpha W(x)| \leq \epsilon^q$ for $0 \leq |\alpha| \leq 1$,

Ting[83] has studied the effective dynamics of multi-vortices in an external potential for the strength of external potential for $0 < q < 1$ and $q > 1$ (strong and weak external potentials).

As an extreme of impurities, the presence of point defect or normal inclusion in some disjoint, smooth connected regions contained in the superconductor sample will also cause the pinning phenomena. Let $\mathcal{D} \subset \mathbb{R}^2$ be a smooth simply connected domain. For functions $\psi \in H^1(\mathcal{D};\mathbb{C})$, $A \in H^1(\mathcal{D};\mathbb{R}^2)$, N. Andre, P. Bauman and D. Phillips considered the minimizers of the energy[1]

$$E_\varepsilon(\psi, A) \equiv \int_{\mathcal{D}} \left\{ \frac{1}{2}\big|(\nabla - iA)\psi\big|^2 + \frac{1}{4\varepsilon^2}\big(|\psi|^2 - a(x)\big)^2 + \frac{1}{2}\big(\nabla \times A - h_{ex}\mathbf{e}_3\big)^2 \right\} \mathrm{d}x. \tag{1.5}$$

The domain $\mathcal{D}$ represents the cross-section of an infinite cylindrical body with $\mathbf{e}_3$ as its generator. The body is subjected to an applied magnetic field, $h_{ex}\mathbf{e}_3$ where $h_{ex} \geq 0$ is constant. If the smooth function a is nonnegative and is allowed to vanish at finite many points, the local minimizers exhibit vortex pinning at the zeros of a. Later on, for functions $\psi \in H^1(\mathcal{D};\mathbb{C})$, $A \in H^1(\mathcal{D};\mathbb{R}^2)$, S. Alama and L. Bronsard consider the minimizers of the energy[5]

$$E_\varepsilon(\psi, A) \equiv \int_{\mathcal{D}} \left\{ \frac{1}{2}\big|(\nabla - iA)\psi\big|^2 + \frac{1}{4\varepsilon^2}\Big[\big(|\psi|^2 - a(x)\big)^2 - (a^-)^2\Big] + \frac{1}{2}\big(h - h_{ex}\big)^2 \right\} \mathrm{d}x, \tag{1.6}$$

where $h = \mathrm{curl}A$ and h_{ex} is a constant applied field. They assume that $a \in C^2(\mathcal{D})$; $\{x \in \bar{\mathcal{D}} : a(x) \leq 0\} = \overline{\cup_{m=1}^n \omega_m}$ with finitely many smooth, simply connected domains $\omega_m \subset\subset \mathcal{D}$; $\nabla a(x) \neq 0$ for all $x \in \partial w_m$, $m = 1, \ldots, n$. For bounded applied fields(independent of ε), they showed that the normal regions acted as "giant vortices" acquiring large vorticity for large (fixed) applied field h_{ex}. Note that these configurations cannot have any vortices in the sense of zeros of ψ in $\Omega = \mathcal{D} - \cup_{m=1}^n \omega_m$, nevertheless they do exhibit vorticity around the holes ω_m due to the nontrivial topology of the domain Ω. For $h_{ex} = O(|\log\varepsilon|)$, the pinning effect of the holes eventually breaks down and free vortices begin to appear in the superconducting region $a(x) > 0$, at a point set which is determined by solving an elliptic boundary-valued problems. The reader can refer to [6] and [3].

Work has also been done on non-magnetic vortices ($A = 0$) with pinning (see[7], [13]). For example, in the model for the variance of the thickness of the superconducting material sample considered by[7], a weight function $p(x)$ is introduced into the energy

$$\mathcal{E}_\lambda = \frac{1}{2}\int_\Omega \Big[p|\bigtriangledown u|^2 + \lambda(1 - |u|^2)^2 \Big], \tag{1.7}$$

with a bounded domain and $\lambda \to \infty$. They show that non-magnetic vortices are localized near minima of $p(x)$ in the first part of [7]. In the second part, they also analyzed the 'interaction energy' between vortices approaching the same limit site by deriving estimates of the mutual distances between these vortices. In fact, they showed that the mutual distance between vortices(approaching the same limit site) is of order $O(1/\sqrt{|\log\lambda|})$. See also the paper by Lin and Du[57].

In 2006, experimentalists succeeded in creating a rotating optical lattice potential with square geometry, which they applied to a Bose-Einstein condensates with a vortex lattice[84]. They observed the pinning of vortices at the potential minima for sufficient optical strength and confirmed the theoretical prediction by Reijnders and Duine[72]. See also the papers[3], [66] for pinning phenomena of vortices in single and multicomponent Bose-Einstein condensates.

Note that the above results we mentioned are two-dimensional cases and the location of the vortices (in the sense of zero of the order parameter) was determined by the properties of the potential. However, in the present paper and [61], we consider the vortex rings in three-dimensional space. The location of vortex rings will be also affected by their shape factors(the curvatures). This will be explained in the sequel.

To the leading order, the vortex lines in the Ginzburg-Landau theory move in the binormal direction with curvature-dependent velocity[70]. Moreover, the motion of vortex lines in quantum mechanics are essentially determined by fours factors[19]: the shape of vortex line, the shape of the back ground condensate wave function, the interaction between vortex lines and possible external forces. Note that vortex rings can also be considered as a special case of vortex lines. By formal asymptotic expansion, A. Svidzinsky and A. Fetter[81] gave a complete description of qualitative features of dynamics of a single vortex line in a trapped Bose-Einstein in the Thomas-Fermi limit. To be specific, we shall consider a trapping potential $V(y) = m(\omega_\perp^2 r^2 + \omega_z^2 z^2)/2$ in the cylindrical coordinates (r, θ, z), with aspect ration defined by $\lambda = \omega_z/\omega_\perp$. The density profile of the condensate is given by $\rho(y) = \rho_0(1 - r^2/R_\perp^2 - z^2/R_z^2)$ in Thomas-Fermi limit, where $R_\perp = \sqrt{2\mu/m\omega_\perp^2}$ and $R_z = \sqrt{2\mu/m\omega_z^2}$ are, respectively, the radial and axial Thomas-Fermi radii of the trapped Bose-Einstein condensates; μ is the chemical potential and $\rho_0 = \mu m/4\pi\hbar^2 a$ is the central particle density. Thus the velocity of a vortex line at y in nonrotating trap is given by(cf. (38) in [81])

$$\mathcal{V} = \Lambda(\xi, k)\left(\frac{T \times \bigtriangledown V(y)}{\mu\rho(y)/\rho_0} + kB\right) \tag{1.8}$$

where T and B are tangent vector and binormal of the vortex line. In the above, $\Lambda(\xi, k) = (-\hbar/2m)\log\left(\xi\sqrt{R_\perp^{-2} + k^2/8}\right)$ and k is the curvature of the vortex line. For more details, the reader can refer to [81] and the references therein. Recently, T. Lin, J. Wei and J. Yang[61] construct solutions with a single stationary(and also a traveling) vortex ring for (1.1) with inhomogeneous trap potential by the finite dimensional reduction method in PDE theory.

It is worth to mention that the authors[81], [61] studied the motion of a single vortex ring under the effects of the three factors except the interaction between vortex rings. More precisely, from (1.8) we see that the shape parameter(the curvature), the wave function and the gradient of the potential will determine the limit site of the stationary vortex lines, i.e. the potential will pin the vortex lines. We will call the role of the gradient of the potential as **the effect of first order of the potential**. Hence in the present paper we want to study the role of the factor of interaction between vortex rings by adding one more vortex ring. We will find that the interaction between vortex rings will be balanced by the second derivative of the potential, see Remark 3.1 and Remark 3.2. We will call the role of the second derivatives of the potential as **the effect of second order of the potential**.

We are now interested in showing the existence of stationary state of (1.1) possessing vortex rings with neighboring interaction, which will be pinned by the trap potential. In other words, we are looking for solutions to problem (1.1) in the form[63]

$$u(y, t) \,=\, e^{i\,\nu\,t}\,\tilde{u}(y_1,\, y_2,\, y_3),$$

which also has vortex rings. Here ν is a constant to be determined latter (cf. (1.21)). Then $\tilde{u}$ is a solution of the nonlinear elliptic problem

$$-\varepsilon^2 \bigtriangleup \tilde{u} \,+\, \Big(V(y) + \nu + |\tilde{u}|^2\Big)\tilde{u} \,=\, 0, \quad \tilde{u} \in H^1(\mathbb{R}^3). \tag{1.9}$$

We first consider the case for two rings lying in the same plane in such a way that they have vortex-vortex interaction in the normal direction, which will be balanced by the effect of second order of the trap potential. Then we construct two parallel vortex-vortex rings whose neighboring interaction in the binormal direction will be also balanced by the effect of second order of the trap potential. We call the first kind of interaction as **type I** and the second as **type II**.

1.3. **Assumptions and results.** We assume that the real function V in (1.1) has the following properties **(A1)-(A4)**.

(A1): *V is a symmetric function with the form*

$$V(y_1, y_2, y_3) = V\big(r, y_3\big) = V\big(r, -y_3\big) \quad \textit{with } r = \sqrt{y_1^2 + y_2^2}\,.$$

(A2): *There is a number r_0 such that the following solvability condition holds*

$$\frac{\partial V}{\partial r}\Big|_{(r_0,0)} - \frac{a}{r_0} = 0. \tag{1.10}$$

Here a is a positive constant defined by (cf. (7.11))

$$a \equiv \frac{1}{\pi}\int_{\mathbb{R}^2} \rho(|s|)\rho'(|s|)\frac{1}{|s|}\,\mathrm{d}s > 0, \tag{1.11}$$

where ρ is defined by (2.1). We also assume that r_0 is non-degenerate in the sense that

$$\frac{\partial^2 V}{\partial r^2}\Big|_{(r_0,0)} + \frac{a}{r_0^2} \neq 0. \tag{1.12}$$

Moreover, the following holds

$$F_1 \equiv \frac{\partial^2 V}{\partial r^2}\Big|_{(r_0,0)} \neq 0 \quad or \quad F_2 \equiv \frac{\partial^2 V}{\partial y_3^2}\Big|_{(r_0,0)} \neq 0. \tag{1.13}$$

(A3): *There exists a number r_2 with $r_2 - r_0 = \tau_0$ such that*

$$-1 + \Big[V(r,y_3) - V(r_0,0)\Big] < 0, \quad if \sqrt{r^2+y_3^2} \in \big(0,\, r_2\big), \tag{1.14}$$

and also the following conditions

$$-1 + \Big(V(r,y_3) - V(r_0,0)\Big) = 0, \quad V'(r,y_3) > 0, \quad V''(r,y_3) > 0, \tag{1.15}$$

hold along the circle $\sqrt{r^2+y_3^2} = r_2$. Here τ_0 is a universal positive constant independent of ε and the derivatives were taken with respect to the out normal of the circle $\sqrt{r^2+y_3^2} = r_2$.

As a consequence of (1.14) and (1.15) there exist positive constants c_1, c_2, τ_1, τ_2 and τ_3 with $\tau_1,\, \tau_2 < 1/100$ such that

$$-1 + \Big[V(r,y_3) - V(r_0,0)\Big] \leq -c_1, \quad \text{if } \sqrt{r^2+y_3^2} \in \big(0,\, r_2 - \tau_1\big), \tag{1.16}$$

$$-1 + \Big[V(r,y_3) - V(r_0,0)\Big] \geq c_2, \quad \text{if } \sqrt{r^2+y_3^2} \in (r_2+\tau_2, r_2+\tau_3), \tag{1.17}$$

Hence, we finally assume that

(A4): *Outside the ball of radius $r_2 + \tau_2$, the potential V satisfies*

$$-1 + \Big[V(r,y_3) - V(r_0,0)\Big] \geq c_2, \quad if \sqrt{r^2+y_3^2} \in (r_2+\tau_2, +\infty). \tag{1.18}$$

Some words are in order to explain the physical and mathematical motivations of the assumptions in (**A1**)-(**A4**).

Remark 1.1.

- *We will need the symmetries in (**A1**) to transfer the problem (1.9) into a two-dimensional case in subsection 3.1, in such a way that we can apply the mathematical method from* [60]. *Moreover, we will use these symmetries to determine the locations of vortex rings, see Remark 1.3.*
- *The assumptions in* **(A2)** *will determine the dynamics of vortex rings with neighboring interactions, see Remark 1.3 and Remark 1.6. For mathematical explanations of conditions (1.12) and (1.13), see Remarks 3.1 and 3.2.*
- *We will determined the density function(i.e. the absolute value $|u|$ of a solution u) with decay by the classical Thomas-Fermi approach in outer region of vortices. So we impose the conditions in* **(A3)**, *see Remark 1.5 and Remark 1.2.*

- *It is also worth to mention that we assume that V satisfy (1.18) outside the ball of radius $r_2+\tau_2$. This is due to that facts that it is a vortexless region and we do not care the effect of the potential V there. Moreover, the assumption in (1.18) will be helpful for dealing with the problem in mathematical aspect and then determining the density function with decay at infinity, see part 5 of the proof of Lemma 6.1.*

□

Remark 1.2. *A typical form of V in physical model is the harmonic type, see [81]*

$$V(y_1,y_2,y_3)\,=\,|y_1|^2+|y_2|^2+|y_3|^2.$$

*It is easy to check that the harmonic potential satisfies the assumptions (**A1**)-(**A4**). In recent experiments in which a laser beam is superimposed upon the magnetic trap holding the atoms, the trapping potential W is of a type[74]*

$$W(r,y_3)\,=\,r^2\,+\,y_3^2\,+\,b_2\,e^{-b_1 r^2},\quad r^2\,=\,y_1^2\,+\,y_2^2, \tag{1.19}$$

*where b_1 are b_2 are two positive constants. This potential W satisfies (**A1**). Trivial computations give that*

$$\frac{\partial W}{\partial r}\,=\,2r\,-\,2b_1\,b_2\,r\,e^{-b_1 r^2},\qquad \frac{\partial^2 W}{\partial y_3^2}\,=\,2,$$
$$\frac{\partial^2 W}{\partial r^2}\,=\,2\,-\,2b_1\,b_2\,e^{-b_1 r^2}\,+\,4b_1^2\,b_2 r^2\,e^{-b_1 r^2}.$$

By solving the equation

$$2\,-\,2b_1\,b_2\,e^{-b_1 r^2}\,=\,\frac{a}{r^2},$$

we can find r_0 satisfies (1.10) and also (1.12) because of the relation

$$\frac{\partial^2 W}{\partial r^2}\,-\,\frac{a}{r^2}\,=\,2\,-\,2b_1\,b_2\,e^{-b_1 r^2}\,-\,\frac{a}{r^2}\,+\,4b_1^2\,b_2 r^2\,e^{-b_1 r^2}.$$

If b_1 and b_2 are small enough, $\frac{\partial W}{\partial r}>0$ and $\frac{\partial^2 W}{\partial r^2}>0$.

*However, one can check that W does not satisfy the assumptions in (**A3**) and (**A4**). In fact the relation*

$$-1\,+\,\Big[W(r,y_3)-W(r_0,0)\Big]\,=\,-1\,-\,r_0^2\,-\,b_2\,e^{-b_1 r_0^2}\,+\,r^2\,+\,y_3^2\,+\,b_2\,e^{-b_1 r^2}<0$$

*hold in a region $\check{D}$, which is not a ball. In the present work, we focus on the pinning phenomena and do not care the profile of the order parameter u far from the vortex region. As we have stated in Remark 1.1, we will use the classical Thomas-Fermi approach in outer region of vortices to find the order parameter u, which will bring singularity and be improved by a correction term around the corner of $\partial\check{D}$, see Remark 1.5. For the convenience of arguments of dealing with the problem in a small neighborhood of $\partial\check{D}$, we consider the potential satisfying the assumptions in (**A3**) and (**A4**). In fact, we can modify the function W and get V in the form*

$$V(r,y_3)\,=\,r^2\,+\,y_3^2\,+\,\eta(r,y_3)\,b_2\,e^{-b_1 r^2},\quad r^2\,=\,y_1^2\,+\,y_2^2, \tag{1.20}$$

where η is a smooth cut-off function such that $\eta(r,y_3)=1$ for $\sqrt{r^2+y_3^2}\,\leq\,r_0+\tau$ for some positive constant τ and $\eta(r,y_2)=0$ for $\sqrt{r^2+y_3^2}\,\geq\,r_0+2\tau$. Moreover, we require that

$$\eta\big(r,y_3\big)\,=\,\eta\big(r,-y_3\big).$$

Now, we have

$$-1\,+\,\Big[V(r,y_3)-V(r_0,0)\Big]\,=\,-1\,-\,r_0^2\,-\,b_2\,e^{-b_1 r_0^2}\,+\,r^2\,+\,y_3^2\,+\,\eta(r,y_3)\,b_2\,e^{-b_1 r^2}.$$

*Careful computations will give that V satisfy the assumptions in (**A3**) and (**A4**) if we choose τ small enough. For a general potential, $\check{D}$ may be a smooth bounded domain without symmetries. It* *is an interesting problem to study the Thomas-Fermi approximation and its improvement around $\partial\check{D}$, which deserves an independent long paper. The reader can refer to a recent paper [52] and the references therein for a complete discussion.* □

By setting

$$V(r_0,0)+\nu \;=\; -1 \quad \text{and} \quad \tilde V(r,y_3)\;=\;V(r,y_3)\;-\;V(r_0,0) \tag{1.21}$$

in (1.9), we shall consider the following problem

$$-\varepsilon^2\triangle\tilde u+\Big(-1+\tilde V(r,y_3)+|\tilde u|^2\Big)\tilde u=0,\quad \tilde u\in H^1(\mathbb{R}^3). \tag{1.22}$$

By the setting in (1.21), we can consider the equation as a perturbation of the classical Ginzburg-Landau equation in a neighborhood of $(r_0,0)$ in the (r,y_3) coordinates and then construct vortex rings, see Remark 1.5. In the above, the new potential $\tilde V$ possesses the properties:

$$\frac{\partial\tilde V}{\partial y_3}\Big|_{(r,0)}=0,\qquad \frac{\partial\tilde V}{\partial r}\Big|_{(0,y_3)}=0,\qquad \tilde V(r_0,0)=0,\qquad \frac{\partial\tilde V}{\partial r}\Big|_{(r_0,0)}-\frac{a}{\tilde r_0}=0, \tag{1.23}$$

and also

$$-1\;+\;\tilde V(r,y_3)=0,$$

along the circle $\sqrt{r^2+y_3^2}=r_2$.

The main object of the present paper is to construct a solution to problem (1.22) with a pair of vortex rings approaching the circle $(r_0,0)$ in the (r,y_3) coordinates. In other words, we will construct a solution to (1.9) with two vortex rings, characterized by the curves

$$\begin{aligned}\sqrt{y_1^2+y_2^2}\;&=\;f_1,\quad y_3\;=\;d_1,\\ \sqrt{y_1^2+y_2^2}\;&=\;f_2\quad y_3\;=\;d_2,\end{aligned} \tag{1.24}$$

where d_1, d_2, f_1 and f_2 are four parameters to be determined in the reduction procedure. If $\digamma_1\neq 0$ the locations of the neighboring vortex rings satisfy

$$\textbf{Type I:}\quad d_1\;=\;d_2=0,\quad f_1+f_2\;=\;2r_0+O(\varepsilon),\quad f_1-f_2\;=\;O\big(1/\sqrt{|\log\varepsilon|}\,\big). \tag{1.25}$$

If $\digamma_2\neq 0$ the locations of the neighboring vortex rings satisfy

$$\textbf{Type II:}\quad d_1\;+\;d_2\;=\;0,\quad d_1\;-\;d_2\;=\;O\big(1/\sqrt{|\log\varepsilon|}\,\big),\quad f_1\;=\;f_2\;=\;r_0+O(\varepsilon). \tag{1.26}$$

Remark 1.3. *As we have stated before, there are some works on the dynamics of vortex lines with the action of trapped potential, base on formal expansion, see [81] and the references therein. In our case, we will set two vortex rings very close in the sense that their distance is of order $O\big(1/\sqrt{|\log\varepsilon|}\,\big)$. So they are pinned in the same limit site, see Remark 1.6. By using the assumption (**A1**) and the formulas in (1.24)-(1.26) for the locations of vortex rings, the curvatures k of vortex rings obey $k\sim 1/r_0$, while in a neighborhood of the vortex rings $\nabla V\sim\frac{\partial V}{\partial r}\Big|_{(r_0,0)}N$ with the normal vector N of the vortex rings. We want the stationary vortex ring to be trapped by the potential V, so we impose the condition (1.10) to make the vortex curvature and the trap potential compensate each other in the first order. This is the case of zero velocity in (1.8).* □

Before stating the main results, we introduce the notations

$$\ell_1=\Big[(|y'|-f_1)^2+(y_3-d_1)^2\Big]^{1/2},\qquad \ell_2=\Big[(|y'|-f_2)^2+(y_3-d_2)^2\Big]^{1/2},$$

$$\ell=\sqrt{y_1^2+y_2^2+y_3^2}\,,\qquad \delta_\varepsilon=\varepsilon\frac{\partial\tilde V}{\partial\ell}\Big|_{(r_2,0)}>0,$$

and $\varphi_{01}(y_1,y_2,y_3)=\varphi_{01}(r,y_3)$, $\varphi_{02}(y_1,y_2,y_3)=\varphi_{02}(r,y_3)$ are the angle arguments of the vectors $(r-f_1, y_3-d_1)$ and $(r-f_2, y_3-d_2)$ in the (r,y_3) plane. It is well known that $\rho(\ell)e^{i\varphi_{01}}$ is a standard vortex (of degree $+1$) solution around (f_1,d_1) where $\rho(z)$ is the unique solution of the problem

$$\rho''+\frac{1}{z}\rho'-\frac{1}{z^2}\rho+(1-|\rho|^2)\rho=0 \quad \text{for } z\in(0,+\infty), \quad \rho(0)=0, \quad \rho(+\infty)=1. \tag{1.27}$$

Let q be the unique solution to the following problem

$$q''-q\big(\ell+q^2\big)=0 \quad \text{on } \mathbb{R}, \qquad q(\ell)\to 0 \text{ as } \ell\to+\infty, \qquad q(\ell)\to+\infty \text{ as } \ell\to-\infty. \tag{1.28}$$

The functions ρ and q will be described in more detail in section 2. Since we will describe two types, called type I and II as before, of interactions of neighboring vortex rings, by recalling the condition (1.13), we will choose a parameter j for type I in the form

$$j=\begin{cases} 1, & \text{if } \digamma_1<0,\\ 2, & \text{if } \digamma_1>0,\end{cases} \qquad \text{with} \quad \digamma_1=\frac{\partial^2 V}{\partial r^2}\Big|_{(r_0,0)}, \tag{1.29}$$

while for type II by

$$j=\begin{cases} 1, & \text{if } \digamma_2<0,\\ 2, & \text{if } \digamma_2>0,\end{cases} \qquad \text{with} \quad \digamma_2=\frac{\partial^2 V}{\partial y_3^2}\Big|_{(r_0,0)}. \tag{1.30}$$

This choice of the parameter j will be explained in Remark 1.6. The main result reads:

Theorem 1.4. *For ε sufficiently small, there exists an axially symmetric solution to problem (1.22) in the form $u=u(|y'|,y_3)\in C^\infty(\mathbb{R}^3,\mathbb{C})$ with a pair of vortex rings. More precisely, the solution u possesses the following asymptotic profile*

$$u(y_1,y_2,y_3)=\begin{cases} \rho\Big(\frac{\ell_1}{\varepsilon}\Big)\rho\Big(\frac{\ell_2}{\varepsilon}\Big)\,e^{i\big(\varphi_{01}+(-1)^j\varphi_{02}\big)}(1+o(1)), & y\in\mathcal{D}_2=\{\ell<\varepsilon^{1-\lambda_1}\},\\[2ex] \sqrt{1-\tilde V(r,y_3)}\,e^{i(\varphi_{01}+(-1)^j\varphi_{02})}(1+o(1)), & y\in\mathcal{D}_1=\{\ell<r_2-\varepsilon^{1-\lambda_2}\}\setminus\mathcal{D}_2,\\[2ex] \delta_\varepsilon^{1/3}q\Big(\delta_\varepsilon^{1/3}\frac{\ell-r_2}{\varepsilon}\Big)\,e^{i\big(\varphi_{01}+(-1)^j\varphi_{02}\big)}(1+o(1)), & y\in\mathcal{D}_3=\{\ell>r_2-\varepsilon^{1-\lambda_2}\},\end{cases}$$

where λ_1 and λ_2 are two positive constants with $\lambda_1,\lambda_2<1/3$. The locations of these two vortex rings satisfy (1.25) or (1.26). □

To explain the result, we give several remarks. The reader can refer to Subsections 4.1 and 4.2 for more details on the asymptotic behavior of the solution.

Remark 1.5. *In the vortex core region $\mathcal{D}_2$, we consider the problem (1.22) as a perturbation of the homogeneous case of (1.2), i.e. the case of $V\equiv-1$ in (1.2). Hence we can set two vortex rings with profile in the form*

$$\rho\Big(\frac{\ell_1}{\varepsilon}\Big)e^{i\varphi_{01}}\rho\Big(\frac{\ell_2}{\varepsilon}\Big)\,e^{i(-1)^j\varphi_{02}}. \tag{1.31}$$

These two vortex rings have interaction, see Remark 1.6. In the region $\mathcal{D}_1$, we neglect the kinetic energy by the classical Thomas-Fermi approach and determine the density function by solving the equation

$$-1+\tilde V(r,y_3)+|u|^2=0$$

due to the assumption **(A3)**. *This was justified by G. Baym and C. Pethick in [12]. The reader can refer to the monograph [69] for more discussions. However, this approach does not describe properly the decay of the wave function near the outer edge of the cloud. In other words, if we substitute the approximation $\sqrt{1-\tilde V(r,y_3)}\,e^{i(\varphi_{01}+(-1)^j\varphi_{02})}$ into the kinetic part, the derivatives of the function $\sqrt{1-\tilde V}$ will bring singularity around the edge $\sqrt{r^2+y_3^2}=r_2$, see the formula (4.39) and the discussion in subsection 4.1. For the correction of the Thomas-Fermi approximation, there are also some formal expansions in physical works such as [62] and [37]. Here we use q in Lemma*

2.4 to describe the profile beyond the Thomas-Fermi approximation. The function q is a solution of a type of Painlevé equation. The reader can refer to [2], [4] *and* [52]. □

Remark 1.6. *If $\mathsf{F}_1 < 0$ or $\mathsf{F}_2 < 0$, then V is repulsive hump-shaped around r_0. To achieve the balance of the interaction of vortex-vortex and the effect of second order of the potential we choose $j = 1$ in (1.31) in such a way that we put two attractive vortex rings (the vortex ring of degree $+1$ and its anti-pair of degree -1). On the other hand, for the case of $\mathsf{F}_1 > 0$ or $\mathsf{F}_2 > 0$, we will choose $j = 2$ to get two repulsive vortex rings of the same degrees. For mathematical explanations of conditions (1.12) and (1.13), see Remarks 3.1 and 3.2.*

Note that the distance between neighboring vortex rings is of order $O\big(1/\sqrt{|\log\varepsilon|}\,\big)$, which implies that they are pinned at the same limit site as ε tends to zero. Thus the interaction of neighboring vortex rings is strong enough to make it 'comparable' to the effect of second order of the trap potential. This quantity is determined by solving an algebraic system (7.16) or (7.19), which was derived by the finite dimensional method in Section 7, see also Remarks 3.1 an 3.2. It is worth to mention that this phenomenon also appears in pinning of two dimensional vortices. The reader can refer to, for example, [7]. *In addition, for the foliation of multiple phase transition layers of the Allen-Cahn equation (real valued)*

$$\varepsilon^2 \triangle u + u - u^3 = 0$$

on a smooth bounded domain(with homogeneous Neumann boundary condition) or a compact s-mooth Riemannian manifolds, the authors also used the infinite dimensional reduction method[28] *to derive a system of nonlinear PDEs (Toda system*[29] *or Jacobi-Toda system*[31] *respectively) to describe the neighboring interaction of multiple phase transition layers with mutual distance of order $O(\varepsilon|\log\varepsilon|)$. The reader can refer to the review paper*[30] *for more references on Jacoi-Toda system.*

□

Remark 1.7. *In Theorem 1.4, the solutions we have constructed satisfy*

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) < +\infty. \tag{1.32}$$

Thus the asymptotic behavior of the solutions is quite different from those constructed with constant trapping potential ([17]*). The reason for this is clear: because of the trapping potential, there exists a vortexless solution satisfying (1.32). Outside the vortex our solutions behaves like this vortexless solution. A major difficulty is the matching of vortex solution with vortexless solution.* □

The remaining part of the present paper is devoted to the complete proof of Theorem 1.4. We will use the finite dimensional reduction method in the sense that by the reduction procedure we transfer the PDE problem into a problem(such as an algebraic problem) which can be solved on a finite dimensional abstract space. The finite dimensional reduction procedure has been used in many other problems. See [25], [39], [59] and the references therein. M. del Pino, M. Kowalczyk and M. Musso [27] were the first to use this procedure to study Ginzburg-Landau equation in a bounded domain. F. Lin and the first author adopted this approach to the Schrödinger map equation in [60]. It is worth to mention that an extension of the finite dimensional reduction method was introduced in the work [28], which was called the infinite dimensional reduction method by transferring the PDE to a new problem(such as another ODE or PDE) which can be solved on an infinite dimensional abstract space. The main steps of finite dimensional reduction method will be given in subsection 3.2.

The organization is as follows: in section 2, we give some preliminary results. After transferring the problem (1.22) to a two dimensional case (3.3)-(3.4), and providing a collection of notations in subsection 3.1, we sketch the outline of strategy of the proof in subsection 3.2. In section 4, we construct an approximate solution and estimate its error. As in the standard reduction method, sections 5-7 are devoted to solving a nonlinear projected problem (5.50) for given parameters f_1, f_2, d_1 and d_2 and then solving a system involving f_1, f_2, d_1 and d_2 to get a real solution of problem (3.3)-(3.4).

2. Preliminaries

By (ℓ,φ) designating the usual polar coordinates $s_1 = \ell\cos\varphi$, $s_2 = \ell\sin\varphi$, we introduce the standard vortex block solution

$$U_0(s_1, s_2) = \rho(\ell)e^{i\varphi}, \tag{2.1}$$

with degree +1 in the whole plane, where $\rho(\ell)$ is the unique solution of the problem

$$\rho'' + \frac{1}{\ell}\rho' - \frac{1}{\ell^2}\rho + (1-|\rho|^2)\rho = 0 \quad \text{for } \ell \in (0,+\infty), \quad \rho(0) = 0, \quad \rho(+\infty) = 1. \tag{2.2}$$

It is easy to check that

$$\triangle U_0 \,+\, (1-|U_0|^2)U_0 \,=\, 0. \tag{2.3}$$

The properties of the function ρ are stated in the following lemma.

Lemma 2.1. *There hold the following properties:*
(1) $\rho(0) = 0, \quad \rho'(0) > 0, \quad 0 < \rho(\ell) < 1, \quad \rho'(\ell) > 0$ *for all* $\ell > 0$,
(2) $\rho(\ell) = 1 - \frac{1}{2\ell^2} + O(\frac{1}{\ell^4})$ *for large* ℓ,
(3) $\rho(\ell) = k\ell - \frac{k}{8}\ell^3 + O(\ell^5)$ *for* ℓ *close to* 0.

Proof. The proof can be found in Theorem 1.1 in [21]. □

We introduce the bilinear form associated to problem (2.3)

$$\mathcal{B}(\phi,\phi) = \int_{\mathbb{R}^2} |\bigtriangledown \phi|^2 \,-\, \int_{\mathbb{R}^2}(1-\rho^2)|\phi|^2 \,+\, 2\int_{\mathbb{R}^2} |\mathrm{Re}(\bar{U}_0\phi)|^2, \tag{2.4}$$

defined in the natural space $\mathcal{H}$ of all locally-H^1 functions with

$$||\phi||_{\mathcal{H}} = \int_{\mathbb{R}^2} |\bigtriangledown \phi|^2 \,-\, \int_{\mathbb{R}^2}(1-\rho^2)|\phi|^2 \,+\, 2\int_{\mathbb{R}^2} |\mathrm{Re}(\bar{U}_0\phi)|^2 < +\infty. \tag{2.5}$$

Let us consider, for a given ϕ, its associated ψ defined by the relation

$$\phi = iU_0\psi. \tag{2.6}$$

Then we decompose ψ in the form

$$\psi = \psi_0(\ell) + \sum_{m\geq 1}\Big[\psi_m^1 + \psi_m^2\Big], \tag{2.7}$$

where we have denoted

$$\begin{aligned}
\psi_0 &= \psi_{01}(\ell) + i\psi_{02}(\ell),\\
\psi_m^1 &= \psi_{m1}^1(\ell)\cos(m\vartheta) + i\psi_{m2}^1(\ell)\sin(m\vartheta),\\
\psi_m^2 &= \psi_{m1}^2(\ell)\sin(m\vartheta) + i\psi_{m2}^2(\ell)\cos(m\vartheta).
\end{aligned}$$

This bilinear form is non-negative, see the first section of [26] and the references therein. The nondegeneracy of U_0 is contained in the following lemma, whose proof can be found in Lemma A.1 in the appendix of [27].

Lemma 2.2. *There exists a constant* $C > 0$ *such that if* $\phi \in \mathcal{H}$ *decomposes like in (2.6)-(2.7) with* $\psi_0 \equiv 0$, *and satisfies the orthogonality conditions*

$$\mathrm{Re}\int_{B(0,1/2)} \bar{\phi}\,\frac{\partial U_0}{\partial s_l} = 0, \quad l = 1, 2,$$

then there holds

$$\mathcal{B}(\phi,\phi) \geq C\int_{\mathbb{R}^2} \frac{|\phi|^2}{1+\ell^2}.$$

□

The linear operator L_0 corresponding to the bilinear form $\mathcal{B}$ can be defined by

$$L_0(\phi) = \Big(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2}\Big)\phi + (1-|\rho|^2)\phi - 2\mathrm{Re}\big(\bar{U}_0\phi\big)U_0. \tag{2.8}$$

The nondegeneracy of U_0 can also be stated as the following lemma, whose proof can be found in Theorem 1 of [26]. The method relied on the decompositions in (2.6)-(2.7).

Lemma 2.3. *Suppose that $L_0[\phi] = 0$ with $\phi \in \mathcal{H}$, then*

$$\phi = c_1\frac{\partial U_0}{\partial s_1} + c_2\frac{\partial U_0}{\partial s_2}, \tag{2.9}$$

for some real constants c_1, c_2. □

To construct approximate solutions in Section 4, we also prepare the following lemma.

Lemma 2.4. *There exists a unique solution q to the following problem*

$$q'' - q\big(\ell + q^2\big) = 0 \quad \text{on } \mathbb{R}, \qquad q(\ell) \to 0 \text{ as } \ell \to +\infty, \qquad q(\ell) \to +\infty \text{ as } \ell \to -\infty.$$

Moreover, q has the properties

$$q(\ell) > 0 \quad \text{for all } \ell \in \mathbb{R}, \qquad q'(\ell) < 0 \quad \text{for all } \ell > 0,$$

$$q(\ell) \sim \exp\left(-\ell^{3/2}\right) \quad \text{as } \ell \to +\infty, \qquad q(\ell) \sim \sqrt{-\ell} \quad \text{as } \ell \to -\infty.$$

□

The proof was given in Lemma 2.4 in [61].

3. The symmetric formulation of the problem and Outline of the proof

By using its symmetry, we will first transfer the problem (1.22) to a two dimensional case in (3.3)-(3.4) and then give an outline of the proof for Theorem 1.4. For the convenience of readers, we also provide a collection of notations in subsection 3.1.

3.1. The symmetric formulation of the problem.

Making rescaling $y = \varepsilon\hat{y}$, problem (1.22) takes the form

$$-\triangle\tilde{u} + \Big(-1 + \tilde{V}(\varepsilon\hat{y}) + |\tilde{u}|^2\Big)\tilde{u} = 0. \tag{3.1}$$

Introduce new coordinates $(\hat{r}, \theta, \hat{y}_3) \in (0, +\infty) \times (0, 2\pi] \times \mathbb{R}$ in the form

$$\hat{y}_1 = \hat{r}\cos\theta, \quad \hat{y}_2 = \hat{r}\sin\theta, \quad \hat{y}_3 = \hat{y}_3.$$

Then problem (3.1) takes the form

$$-\Big(\frac{\partial^2}{\partial\hat{r}^2} + \frac{\partial^2}{\partial\hat{y}_3^2} + \frac{1}{\hat{r}^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{\hat{r}}\frac{\partial}{\partial\hat{r}}\Big)\tilde{u} + \Big(-1 + \tilde{V}(\varepsilon\hat{r}, \varepsilon\hat{y}_3) + |\tilde{u}|^2\Big)\tilde{u} = 0. \tag{3.2}$$

In the present paper, we want to construct a solution with vortex rings, which does not depend on the variable θ. Hence, we consider a two-dimensional problem, for $(x_1, x_2) \in \mathbb{R}^2$

$$S[u] \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)u \; + \; \Big(1 - \tilde{V}(\varepsilon|x_1|, \varepsilon x_2) - |u|^2\Big)u = 0, \tag{3.3}$$

with Neumann boundary condition

$$\frac{\partial u}{\partial x_1}(0, x_2) = 0, \quad |u| \to 0 \text{ as } |x| \to +\infty. \tag{3.4}$$

For the convenience of readers, a collection of notations is provided.

Notations: *For further convenience, we have used x_1, x_2 to denote $\hat{r}, \hat{y}_3$ in the above equations, and also*

$$x = (x_1, x_2) \in \mathbb{R}^2, \quad \hat{\ell} = |x|, \quad \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}. \tag{3.5}$$

In this rescaled coordinates, we write

$$\hat{d}_1 = d_1/\varepsilon, \quad \hat{d}_2 = d_2/\varepsilon, \quad \hat{f}_1 = f_1/\varepsilon, \quad \hat{f}_2 = f_2/\varepsilon, \tag{3.6}$$

$$\hat{r}_2 = r_2/\varepsilon, \qquad \hat{r}_0 = r_0/\varepsilon, \tag{3.7}$$

where the constants f_1, f_2, d_1, d_2, r_2 and r_0 are defined in (1.24), (1.15) and (1.10). By the notations, see Figures 1 and 2

$$\vec{e}_1 = (\hat{f}_1, \hat{d}_1), \quad \vec{e}_2 = (\hat{f}_2, \hat{d}_2), \quad \vec{e}_3 = (\hat{f}_3, \hat{d}_3), \quad \vec{e}_4 = (\hat{f}_4, \hat{d}_4), \tag{3.8}$$

where $\hat{f}_3$, $\hat{f}_4$, $\hat{d}_3$ and $\hat{d}_4$ are given in (3.15) and (3.16), we also introduce the local translated variable

$$s = x - \vec{e}_1 \quad \text{or} \quad z = x - \vec{e}_2, \tag{3.9}$$

in a small neighborhood of the vortices. We will use these notations without any further statement in the sequel.

For any given (x_1, x_2) in $\mathbb{R}^2$, let $\varphi_{01}(x_1, x_2)$, $\varphi_{02}(x_1, x_2)$, $\varphi_{03}(x_1, x_2)$ and $\varphi_{04}(x_1, x_2)$ be respectively the angle arguments of the vectors $(x_1 - \hat{f}_1, x_2 - \hat{d}_1)$, $(x_1 - \hat{f}_2, x_2 - \hat{d}_2)$, $(x_1 - \hat{f}_3, x_2 - \hat{d}_3)$ and $(x_1 - \hat{d}_4, x_2 - \hat{d}_4)$ in the (x_1, x_2) plane , see Figures 1 and 2. We also let

$$\begin{aligned} \hat{\ell}_1(x_1, x_2) &= \sqrt{(x_1 - \hat{f}_1)^2 + (x_2 - \hat{d}_1)^2}\,, \quad \hat{\ell}_2(x_1, x_2) = \sqrt{(x_1 - \hat{f}_2)^2 + (x_2 - \hat{d}_2)^2}\,, \\ \hat{\ell}_3(x_1, x_2) &= \sqrt{(x_1 - \hat{f}_3)^2 + (x_2 - \hat{d}_3)^2}\,, \quad \hat{\ell}_4(x_1, x_2) = \sqrt{(x_1 - \hat{f}_4)^2 + (x_2 - \hat{d}_4)^2}\,, \end{aligned} \tag{3.10}$$

be the distance functions between the point (x_1, x_2) and the of vortices locating at the points $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ and $\vec{e}_4$.

To construct the approximate solution, we will first decompose the whole space into D_1, D_2 and D_3, see (4.24). Then we make further decompositions in (5.9) and (5.10) such that

$$D_1 = D_{1,1} \cup D_{1,2}, \quad D_2 = \cup_{m=1}^{6} D_{2,m}, \quad D_3 = D_{3,1} \cup D_{3,2}.$$

The reader can refer to Figure 1 and Figure 2.

Finally, we decompose the operator in (3.3) as

$$S[u] \equiv S_0[u] + S_1[u], \tag{3.11}$$

with the explicit form

$$S_0[u] \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)u, \quad S_1[u] \equiv \Big(1 - \tilde{V}(\varepsilon|x_1|, \varepsilon x_2) - |u|^2\Big)u. \tag{3.12}$$

For later use, we give the formula

$$S_0[fg] = S_0[f] + S_0[g] + 2\nabla f \cdot \nabla g \tag{3.13}$$

for any given smooth functions f and g. □

To handle the influence of the potential, we here look for vortex ring solutions vanishing as $|x|$ approaching $+\infty$. As we stated in (1.24), we assume that the two vortex rings are characterized by the curve, in the original coordinates $\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$

$$\begin{aligned} \sqrt{\hat{y}_1^2 + \hat{y}_2^2} &= \hat{f}_1, \quad \hat{y}_3 = \hat{d}_1, \\ \sqrt{\hat{y}_1^2 + \hat{y}_2^2} &= \hat{f}_2, \quad \hat{y}_3 = \hat{d}_2. \end{aligned} \tag{3.14}$$

In other words, in the two dimensional situation with (x_1, x_2) coordinates, we will construct two vortices at $\vec{e}_1$ and $\vec{e}_2$ and its anti-pairs at $\vec{e}_4$ and $\vec{e}_3$, see Figures 1 and 2. For the construction of

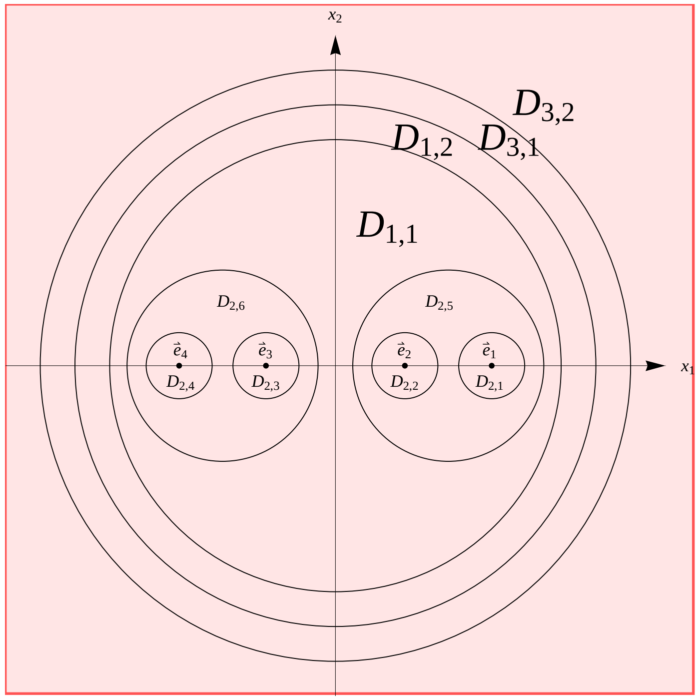

FIGURE 1. Decomposition of domain for type I

solutions possessing vortex rings with interaction of type I in Theorem (1.4), we assume that the parameters satisfy the constraints

$$\begin{aligned} &\hat{f}_1 + \hat{f}_4 = 0, \quad \hat{f}_2 + \hat{f}_3 = 0, \quad \hat{d}_1 = \hat{d}_2 = \hat{d}_3 = \hat{d}_4 = 0, \\ &\hat{f}_1 + \hat{f}_2 \;=\; 2\hat{r}_0 + O(1), \quad \hat{f}_1 - \hat{f}_2 \;=\; O\big(\,\varepsilon^{-1}\,|\log\varepsilon|^{-1/2}\,\big), \end{aligned} \tag{3.15}$$

while for vortex rings with interaction of type II by

$$\begin{aligned} &\hat{f}_1 + \hat{f}_4 = 0, \quad \hat{f}_2 + \hat{f}_3 = 0, \quad \hat{f}_1 = \hat{f}_2, \quad \hat{f}_1 + \hat{f}_2 \;=\; 2\hat{r}_0 + O(1), \\ &\hat{d}_1 + \hat{d}_2 = 0, \quad \hat{d}_3 + \hat{d}_4 = 0, \quad \hat{d}_1 = \hat{d}_4, \quad \hat{d}_1 - \hat{d}_2 \;=\; O\big(\,\varepsilon^{-1}\,|\log\varepsilon|^{-1/2}\,\big). \end{aligned} \tag{3.16}$$

These parameters will be determined by the reduction procedure in section 7, see also Remark 3.1 and Remark 3.2.

3.2. **Outline of the Proof.** In order to prove Theorem 1.4, we will use the finite dimensional reduction method to find a solution to (3.3)-(3.4). Here are the main steps of the finite dimensional reduction method.

Step 1: Construction of approximate solutions

To construct a solution to (3.3)-(3.4) and prove the result in Theorem 1.4, the first step is to construct an approximate solution, denoted by u_2 in (4.58), possessing a pair of neighboring vortices locating at $\vec{e}_1$ and $\vec{e}_2$ and their antipairs at $\vec{e}_4$ and $\vec{e}_3$.

The heuristic method is to find suitable approximations in different regions and then patch them together, see Remark 1.5. So we decompose the whole plane into different regions D_1, D_2, D_3

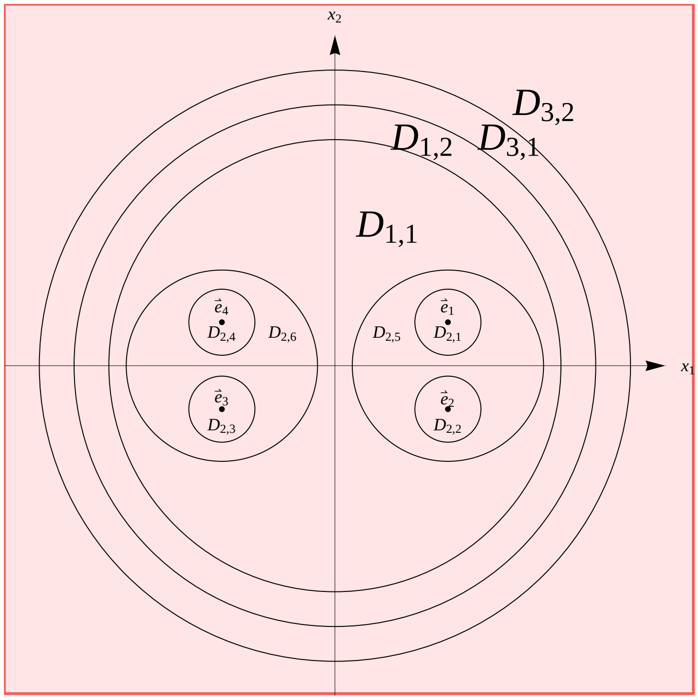

FIGURE 2. Decomposition of domain for type II

as in (4.24), see Figures 1 and 2. Note that the components of D_2 center at $(-\hat{r}_0, 0)$ or $(\hat{r}_0, 0)$. The first approximation u_1 to a solution has a profile of a pair of standard vortices in D_2, which possess vortex-vortex interaction and centers $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_4$, $\vec{e}_3$, see (4.26). Then in D_1 we set u_1 by Thomas-Fermi approximation in (4.29) and then make a trivial extension to the region D_3 at the moment. As we have stated in Remark 1.5, we shall find an improvement beyond the Thomas-Fermi approximation. This will be explained in the following.

Now there are two types of singularities caused by the phase term of standard vortices and the Thomas-Fermi approximation, which will be described in subsection 4.1. In fact, in the region D_2, to cancel the first singular term $\frac{1}{x_1}\frac{\varphi_0^j}{\partial x_1}$ caused by the phase term φ_0^j in (4.28) we here add one more correction term φ_1^j in (4.44) to the pase component as the works [60] and [61]. On the other hand, for the second singularity caused by the Thomas-Fermi approximation, by some type of rescaling in (4.53), in D_3 we use q(a solution of Painlevé II equation) in Lemma 2.4 as a bridge when $|x|$ crossing the circle of radius $\hat{r}_2$ and then reduce the norm of the approximate solution to zero as $|x|$ tends to ∞. The improvement procedure will be done in subsection 4.2

Finally we get the approximate solution u_2 in (4.58), which has the symmetry

$$u_2(x_1, x_2) = \overline{u_2(x_1, -x_2)}, \qquad u_2(x_1, x_2) = u_2(-x_1, x_2). \tag{3.17}$$

Note that u_2 is a function depending on the parameters $\hat{f}_1, ..., \hat{f}_4$, $\hat{d}_1, ..., \hat{d}_4$ with constraints in (3.15) or (3.16). The subsection 4.3 is devoted to the deriving of the estimation of the error $S[u_2]$ in suitable weighted norms. The reader can refer to the papers [27] and [60].

Step 2: Finding a perturbation

We intend to look for a solution of (3.3)-(3.4) by adding a perturbation term to u_2 where the perturbation term is small in suitable norms. More precisely, for the perturbation $\psi = \psi_1 + i\psi_2$

with symmetry (5.5), we take the solution u in the form (cf. (5.4))

$$u = \Big[\chi\big(v_2 + iv_2\psi\big) + (1-\chi)(v_1+v_2)e^{i\psi}\Big] \,+\, \Big[v_3 + i\eta_3 e^{i\varphi}\psi\Big],$$

where v_1, v_2, v_3 are local forms of the approximate solution u_2, see (5.3). This perturbation method near the vortices was introduced in [27].

For given parameters $\hat{f}_1, ..., \hat{f}_4$, $\hat{d}_1, ..., \hat{d}_4$ with constraints in (3.15) or (3.16) and ε small, instead of considering the problem (3.3)-(3.4), we look for a ψ and Lagrange multipliers c_1, c_2 to satisfy the projected problem

$$\begin{cases} \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)u + \Big(1 - \tilde{V}(\varepsilon|x_1|, \varepsilon x_2) - |u|^2\Big)u \,=\, c_1\Lambda_1 \,+\, c_2\Lambda_2, \\ \mathrm{Re}\big(\int_{\mathbb{R}^2} \bar{\phi}\,\Lambda_1\,\mathrm{d}x\big) = 0, \quad \mathrm{Re}\big(\int_{\mathbb{R}^2} \bar{\phi}\,\Lambda_2\,\mathrm{d}x\big) = 0 \quad \text{with } \phi = iv_2\psi, \end{cases} \tag{3.18}$$

where Λ_1 and Λ_2 are defined in (5.48) or (5.49), which constitute the kernel of the linearized problem at u_2 because of Lemma 2.3. After writing problem (3.18) into a problem (with a linear part and a nonlinear part) in the form of the perturbation term ψ, we can find the perturbation term ψ through *a priori estimates* and contraction mapping theorem.

The procedure will be done in the following way. It is well known that the establishment of *a priori estimates* relies heavily on the properties of the corresponding linear operators. To get explicit information of the linearized problem at the approximate solution u_2, we then also divide further D_2, D_1 and D_3 into small parts in (5.9) and (5.10), see Figures 1 and 2. In section 5, we then formulate the problem in suitable local forms (with linear parts and a nonlinear parts) in different regions.

The key points that we shall mention are the roles of local forms of the linear operators for further deriving of the *a priori estimates* in Lemma 6.1. In D_1, the linear operators have approximate forms, (cf. (5.14) and (5.18))

$$\tilde{L}_1(\psi_1) \,\equiv\, \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 \,+\, \frac{2}{\beta_1}\,\nabla\beta_1\cdot\nabla\psi_1,$$

$$\bar{L}_1(\psi_2) \,\equiv\, \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 \,-\, 2|u_2|^2\psi_2 \,+\, \frac{2}{\beta_1}\,\nabla\beta_1\cdot\nabla\psi_2.$$

The type of the linear operator $\tilde{L}_1$ was handled in [60], while $\bar{L}_1$ is a good operator since $|u_2|$ stays away from 0 in D_1 by the assumption (**A3**), see (5.12) and (5.16). In the vortex core regions $D_{2,1}$, $D_{2,2}$, $D_{2,3}$ and $D_{2,4}$, we use a type of symmetry (3.17) to deal with the kernel of the linear operator related to the standard vortex. In $D_{3,1}$, the lowest approximations of the linear operators are, (cf. (5.42))

$$L_{31*}(\psi_1) = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_1 \,-\, \big(\lambda + q^2(\lambda)\big)\psi_1\,,$$

$$L_{31**}(\psi_2) = \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_2 \,-\, \big(\lambda + 3q^2(\lambda)\big)\psi_2\,.$$

By Lemma 2.4, the facts that $L_{31*}(q) = 0$ and $L_{31**}(-q') = 0$ with $-q' > 0$ and $q > 0$ on $\mathbb{R}$ will give the application of maximum principle. The linear operators in the region $D_{3,2}$ can be approximated by a good linear operator of the form, (cf.(5.47))

$$L_{32*}[\tilde{\psi}] \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\tilde{\psi} \,+\, \big(1 - \tilde{V}\big)\tilde{\psi},$$

with $\big(1 - \tilde{V}\big) \le -c_2 < 0$ by the assumption (**A4**). For more details, the reader can refer to proof of Lemma 6.1.

After deriving the linear resolution theory by Lemmas 6.1 and 6.2, we can solve the nonlinear projected problem (3.18), (i.e. 5.50) in section 6.

Step 3: Adjusting the parameters

Note that the perturbation term ψ and the Lagrange multipliers c_1 and c_1 are functions of the parameters $\hat{f}_1,\dots,\hat{f}_4$, $\hat{d}_1,\dots,\hat{d}_4$. To get a real solution to (3.3)-(3.4), we shall choose suitable parameters $\hat{f}_1,\dots,\hat{f}_4$, $\hat{d}_1,\dots,\hat{d}_4$ with constraints (3.15) or (3.16) such that c_1 and c_2 are identically zero. It is equivalent to solve a reduced algebraic system for the Lagrange multipliers

$$c_1(\hat{f}_1,\dots,\hat{f}_4,\hat{d}_1,\dots,\hat{d}_4)=0,\quad c_2(\hat{f}_1,\dots,\hat{f}_4,\hat{d}_1,\dots,\hat{d}_4)=0. \tag{3.19}$$

In fact, multiplying the first equation in (3.18) by Λ_1 or Λ_2 and then integrating on $\mathbb{R}^2$, we can derive the equations in (3.19). This will be done in section 7.

In other words, we achieve the balance between the vortex-vortex interaction and the effect of the trap potential by adjusting the locations of the vortex rings. Recall that

$$\hat{d}_1=d_1/\varepsilon,\quad \hat{d}_2=d_2/\varepsilon,\quad \hat{f}_1=f_1/\varepsilon,\quad \hat{f}_2=f_2/\varepsilon.$$

By using the symmetries in (3.15) or (3.16), we only need to solve some algebraic equations to find the pair (f_1,f_2) or (f_1,d_1). More precisely, for the interaction of the type I in (1.25) we will solve the following algebraic equations, (cf. (7.16))

$$\begin{aligned} c_1(f_1,f_2) \,=\, & 2\,\varepsilon\pi\left[\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}\log\frac{1}{\varepsilon}-\frac{a}{f_1}\log\frac{f_1}{\varepsilon}\right] \\ & +\,2\,\varepsilon\pi\left[\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}(f_1-r_0)\,\log\frac{1}{\varepsilon}-(-1)^j\frac{4a}{f_1-f_2}\right]+\,M_{1,1}(f_1,f_2), \end{aligned}\tag{3.20}$$

$$\begin{aligned} c_2(f_1,f_2) \,=\, & 2\,\varepsilon\pi\left[\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}\log\frac{1}{\varepsilon}-\frac{a}{f_2}\log\frac{f_2}{\varepsilon}\right] \\ & +\,2\,\varepsilon\pi\left[\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}(f_2-r_0)\,\log\frac{1}{\varepsilon}+(-1)^j\frac{4a}{f_1-f_2}\right]+\,M_{1,2}(f_1,f_2), \end{aligned}\tag{3.21}$$

where $M_{1,1}$ and $M_{1,2}$ of order $O(\varepsilon)$ are continuous functions of the parameters f_1 and f_2. By recalling the solvability condition (1.10) and the non-degeneracy condition (1.12) as well as the choice of j in (1.29), we can find a zero of $\big(c_1(f_1,f_2),\,c_2(f_1,f_2)\big)$ at (f_1,f_2) with constraints (1.25) due to the help of the simple mean-value theorem.

Remark 3.1. *In the type I case, see Figure 1, the two rings lie in the same plane in such a way that they have vortex-vortex interaction in the normal direction, which will be balanced by the effect of second order of the trap potential. By the symmetries in (3.15), we here only need to determine f_1 and f_2. Note that the terms in the first brackets of (3.20) and (3.21) will determine the limit locations of these two rings due to the relation in (1.10). This implies that f_1 and f_2 have the same leading order in their asymptotic behaviors. The mutual distance(i.e. $|f_2-f_1|$) between vortex rings will determine the interaction of these two vortex rings. Hence, the terms in the second brackets of (3.20) and (3.21) will show the balance of the interaction of neighboring vortex rings and the effects of second derivatives of the potential $\tilde{V}$(i.e. V). Moreover, f_1-f_2 is of order $O\big(1/\sqrt{|\log\varepsilon|}\,\big)$.* □

One the other hand, for the interaction of the type II in (1.26) we then solve the following system, (cf. (7.19))

$$c_1(f_1,d_1)\,=2\,\varepsilon\pi\left[\frac{\partial^2\tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}d_1\,\log\frac{1}{\varepsilon}-(-1)^j\frac{8a}{d_1}\right]+\,M_{2,1}(f_1,d_1), \tag{3.22}$$

$$\begin{aligned} c_2(f_1,d_1) &= 2\,\varepsilon\pi\left[\frac{\partial\tilde V}{\partial r}\Big|_{(r_0,0)}\log\frac{1}{\varepsilon}-\frac{a}{f_1}\log\frac{f_1}{\varepsilon}\right] \\ &\quad + 2\pi\varepsilon\,\frac{\partial^2\tilde V}{\partial r^2}\Big|_{(r_0,0)}\big(f_1-r_0\big)\log\frac{1}{\varepsilon}+M_{2,2}(f_1,d_1), \end{aligned} \tag{3.23}$$

where $M_{2,1}$ and $M_{2,2}$ of order $O(\varepsilon)$ are continuous functions of the parameters f_1 and d_1. By recalling the solvability condition (1.10) and the non-degeneracy conditions (1.12) as well as the choice of j in (1.30), we can find (f_1, d_1) such that the constraints in (1.26) are fulfilled and also

$$\Big(c_1\big(f_1,d_1\big),\, c_2\big(f_1,d_1\big)\Big)=0,$$

with the help of the simple mean-value theorem.

Remark 3.2. *In the type II case, see Figure 2, we construct two parallel vortex-vortex rings whose neighboring interaction in the binormal direction will be also balanced by the effect of second order of the trap potential. By the symmetries in (3.15), we here only need to determine f_1 and d_1. Note that the terms in the bracket of (3.23) will determine the limit locations(the leading order of f_1) of these two rings due to the relation in (1.10), while the terms in the bracket of (3.22) will show the balance of the interaction of neighboring vortex rings and the effect of second order of the potential $\tilde V$(i.e. V). The mutual distance(i.e. $|2d_1|$) of these two vortex rings is also of order $O\big(1/\sqrt{|\log\varepsilon|}\,\big)$.* □

4. The approximate solution

The main object of this section will focus on the construction a good approximate solution, say u_2, in a suitable form and then estimate the error $S[u_2]$.

4.1. First approximate solution.

In this subsection, we only consider the case for $x_1 > 0$ because of the symmetry of the problem. By the constraints in (3.15) and (3.16), the vortices will be attracted to $(\hat r_0, 0)$ and $(-\hat r_0, 0)$. Hence, we decompose the plane into different regions D_1, D_2 and D_3 in the following form, see Figures 1 and 2

$$\begin{aligned} D_2 &\equiv \left\{(x_1,x_2): \sqrt{(x_1+\hat r_0)^2+x_2^2}<\varepsilon^{-\lambda_1}\ \text{or}\ \sqrt{(x_1-\hat r_0)^2+x_2^2}<\varepsilon^{-\lambda_1}\right\}, \\ D_1 &\equiv \Big\{(x_1,x_2): |x|<\hat r_2-\varepsilon^{-\lambda_2}\Big\}\setminus D_2, \qquad D_3=\Big\{(x_1,x_2): |x|>\hat r_2-\varepsilon^{-\lambda_2}\Big\}. \end{aligned} \tag{4.24}$$

where λ_1 and λ_2 are positive constants with $\lambda_1,\, \lambda_2 < 1/3$.

Recall the notations defined in subsection 3.1. By using the definition of the standard vortex of degree $+1$ in (2.1), in the small neighborhoods of $\vec e_1$, $\vec e_2$, $\vec e_3$ and $\vec e_4$, we locally define the vortices by

$$\begin{aligned} w_1=\rho(\hat\ell_1)e^{i\varphi_{01}}, &\quad w_2=\rho(\hat\ell_2)e^{i(-1)^j\varphi_{02}}, \\ w_3=\rho(\hat\ell_3)e^{-i(-1)^j\varphi_{03}}, &\quad w_4=\rho(\hat\ell_4)e^{-i\varphi_{04}}. \end{aligned} \tag{4.25}$$

Here we recall the choice of the parameter j in (1.29)-(1.30) as the following: to show the result in (1.25), in the sequel we will choose j as

$$j=\begin{cases} 1, & \text{if } \digamma_1<0,\\ 2, & \text{if } \digamma_1>0,\end{cases}$$

where $\digamma_1$ is given in (1.13). To show the result in (1.26), we will choose j as

$$j=\begin{cases} 1, & \text{if } \digamma_2<0,\\ 2, & \text{if } \digamma_2>0,\end{cases}$$

where F_2 is given in (1.13). It is easy to show that the functions w_k, $k = 1, 2, 3, 4$, satisfy the equation (2.3). The **first approximate solution** can be roughly defined as follows, see Remark 1.5:

(1) If $(x_1, x_2) \in D_2$, we choose u_1 by

$$u_1(x_1, x_2) \,=\, U_2(x_1, x_2) \,\equiv\, w_1 w_2 w_3 w_4. \tag{4.26}$$

It can be expressed by

$$U_2 \,=\, \rho(\hat{\ell}_1)\rho(\hat{\ell}_2)\rho(\hat{\ell}_3)\rho(\hat{\ell}_4)e^{i\varphi_0^j}, \tag{4.27}$$

where the phase term φ_0^j is defined by

$$\varphi_0^j \,=\, \varphi_{01} \,-\, \varphi_{04} \,+\, (-1)^j\big(\varphi_{02} \,-\, \varphi_{03}\big). \tag{4.28}$$

In other words, if $j = 1$, u_1 possesses a pair of neighboring vortices of degrees ± 1 in the neighborhood of $(\hat{r}_0, 0)$ as well as their anti-pairs in the neighborhood of $(-\hat{r}_0, 0)$. On the other hand, in the case $j = 2$, u_1 possesses a pair of neighboring vortices of the same degrees in the neighborhood of $(\hat{r}_0, 0)$ as well as their anti-pairs in the neighborhood of $(-\hat{r}_0, 0)$.

(2) If $(x_1, x_2) \in D_1$, we write

$$u_1(x_1, x_2) \,=\, U_1(x_1, x_2) \,\equiv\, \sqrt{1 - \tilde{V}(\varepsilon|x_1|, \varepsilon x_2)}\, e^{i\varphi_0^j}\,. \tag{4.29}$$

The choice of u_1 is well defined due to the assumption (**A3**). Here we use the standard Thomas-Fermi approximation, see [69].

(3) As we are looking for solutions vanishing at infinity, we heuristically define $u_1 \,=\, \tilde{U}_3 \,\equiv\, 0$ for $(x_1, x_2) \in D_3$. Note that this extension is not good, we will make an improvement in next subsection.

For further improvement of the approximation, it is crucial to evaluate the error of this approximation by substituting the approximate solution into the equation (3.3), which will be carried out in the sequel. Obviously, there hold the trivial formulas

$$\begin{aligned} \nabla_{x_1,x_2} w_1 \,&=\, \frac{\rho'(\hat{\ell}_1)}{\hat{\ell}_1}\big(x_1 - \hat{f}_1,\, x_2 - \hat{d}_1\big), \qquad \nabla_{x_1,x_2} w_2 \,=\, \frac{\rho'(\hat{\ell}_2)}{\hat{\ell}_2}\big(x_1 - \hat{f}_2,\, x_2 - \hat{d}_2\big), \\ \nabla_{x_1,x_2} w_4 \,&=\, \frac{\rho'(\hat{\ell}_4)}{\hat{\ell}_4}\big(x_1 - \hat{f}_3,\, x_2 - \hat{d}_3\big), \qquad \nabla_{x_1,x_2} w_3 \,=\, \frac{\rho'(\hat{\ell}_3)}{\hat{\ell}_3}\big(x_1 - \hat{f}_4,\, x_2 - \hat{d}_4\big), \end{aligned} \tag{4.30}$$

and also

$$\begin{aligned} \frac{\partial \varphi_0^j}{\partial x_1} \,&=\, -\,\frac{x_2 - \hat{d}_1}{(\hat{\ell}_1)^2} + \frac{x_2 - \hat{d}_4}{(\hat{\ell}_4)^2} \,+\, (-1)^j\Big(-\,\frac{x_2 - \hat{d}_2}{(\hat{\ell}_2)^2} + \frac{x_2 - \hat{d}_3}{(\hat{\ell}_3)^2}\Big), \\ \frac{\partial \varphi_0^j}{\partial x_2} \,&=\, \frac{x_1 - \hat{f}_1}{(\hat{\ell}_1)^2} - \frac{x_1 - \hat{f}_4}{(\hat{\ell}_4)^2} \,+\, (-1)^j\Big(\frac{x_1 - \hat{f}_2}{(\hat{\ell}_2)^2} - \frac{x_1 - \hat{f}_3}{(\hat{\ell}_3)^2}\Big). \end{aligned} \tag{4.31}$$

As we have stated, we work directly in the half space $\mathbb{R}^2_+ \,=\, \{(x_1, x_2) \,:\, x_1 \,>\, 0\}$ in the sequel because of the symmetry of the problem.

First, we estimate the error near the vortex rings. Note that for $x_1 > 0$, the errors between 1 and the functions w_4 and w_3 are $(\hat{\ell}_4)^{-2}$ and $(\hat{\ell}_3)^{-2}$, which are of order ε^2, we may ignore w_4 and w_3 in the computations below. Note that

$$\begin{aligned} S_0[U_2] \,=\,& w_2 w_3 w_4 \triangle w_1 \,+\, w_1 w_3 w_4 \triangle w_2 \,+\, 2 w_3 w_4 \nabla w_1 \cdot \nabla w_2 \\ &+\, \frac{1}{x_1}\frac{\partial w_1}{\partial x_1} w_2 w_3 w_4 \,+\, \frac{1}{x_1}\frac{\partial w_2}{\partial x_1} w_1 w_3 w_4 \,+\, O(\varepsilon^2). \end{aligned} \tag{4.32}$$

The main terms in (4.32) will be estimated in the following. There holds

$$
\begin{aligned}
\frac{1}{x_1}\frac{\partial w_1}{\partial x_1}w_2w_3w_4 \,+\, \frac{1}{x_1}\frac{\partial w_2}{\partial x_1}w_1w_3w_4 \,&= \frac{1}{x_1}\frac{\partial \rho(\hat{\ell}_1)}{\partial x_1}\frac{U_2}{\rho(\hat{\ell}_1)} \,+\, \frac{1}{x_1}\frac{\partial \rho(\hat{\ell}_2)}{\partial x_1}\frac{U_2}{\rho(\hat{\ell}_2)} \,+\, iU_2\frac{1}{x_1}\frac{\partial \varphi_0^j}{\partial x_1} \,+\, O(\varepsilon^2)\\
&= \frac{x_1-\hat{f}_1}{x_1\hat{\ell}_1}\rho'(\hat{\ell}_1)\frac{U_2}{w(\hat{\ell}_1)} \,+\, \frac{x_1-\hat{f}_2}{x_1\hat{\ell}_2}\rho'(\hat{\ell}_2)\frac{U_2}{w(\hat{\ell}_2)}\\
&\quad +\, iU_2\frac{1}{x_1}\frac{\partial \varphi_0^j}{\partial x_1} \,+\, O(\varepsilon^2).
\end{aligned}
$$

Using (3.15) or (3.16), we denote the last term in the above formula by

$$
F_{21} \,\equiv\, iU_2\frac{1}{x_1}\frac{\partial \varphi_0^j}{\partial x_1} \,=\, -iU_2\frac{x_2-\hat{d}_1}{x_1(\hat{\ell}_1)^2} \,+\, iU_2\frac{x_2-\hat{d}_4}{x_1(\hat{\ell}_4)^2} \,+\, (-1)^j iU_2\Big(-\frac{x_2-\hat{d}_2}{x_1(\hat{\ell}_2)^2} + \frac{x_2-\hat{d}_3}{x_1(\hat{\ell}_3)^2}\Big). \tag{4.33}
$$

By using the formulas in (4.30) and (4.31)

$$
\begin{aligned}
2w_3w_4\nabla w_1\cdot\nabla w_2 \,&= 2U_2\left(\frac{\nabla\rho_1}{\rho_1} + i\nabla\varphi_{01}\right)\cdot\left(\frac{\nabla\rho_2}{\rho_2} + i(-1)^j\nabla\varphi_{02}\right)\\
&= 2U_2\frac{(x_1-\hat{f}_1)(x_1-\hat{f}_2)+(x_2-\hat{d}_1)(x_2-\hat{d}_2)}{\hat{\ell}_1\hat{\ell}_2}\,\frac{\rho'(\hat{\ell}_1)}{\rho(\hat{\ell}_1)}\,\frac{\rho'(\hat{\ell}_2)}{\rho(\hat{\ell}_2)}\\
&\quad -\, 2(-1)^jU_2\frac{(x_2-\hat{d}_1)(x_2-\hat{d}_2)+(x_1-\hat{f}_1)(x_1-\hat{f}_2)}{(\hat{\ell}_1)^2(\hat{\ell}_2)^2}\\
&\quad +\, 2i(-1)^jU_2\frac{-(x_2-\hat{d}_2)(x_1-\hat{f}_1)+(x_1-\hat{f}_2)(x_2-\hat{d}_1)}{\hat{\ell}_1(\hat{\ell}_2)^2}\,\frac{\rho'(\hat{\ell}_1)}{\rho(\hat{\ell}_1)}\\
&\quad +\, 2iU_2\frac{-(x_2-\hat{d}_1)(x_1-\hat{f}_2)+(x_1-\hat{f}_1)(x_2-\hat{d}_2)}{\hat{\ell}_2(\hat{\ell}_1)^2}\,\frac{\rho'(\hat{\ell}_2)}{\rho(\hat{\ell}_2)}.
\end{aligned}
$$

It is worth mentioning that the above formula will play an important role in the interaction between the neighboring two vortex rings. Recall the notation r_0 in (1.10) and that

$$
\frac{\partial\tilde{V}}{\partial y_3}\Big|_{(r,0)} = 0.
$$

In a small neighborhood of the point $(r_0, 0) = (\varepsilon\hat{r}_0, 0)$, by Taylor expansion we also write $\tilde{V}(\varepsilon|x_1|, \varepsilon x_2)$ in the form

$$
\tilde{V}(\varepsilon|x_1|,\varepsilon x_2) \,=\, \varepsilon\,\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}(x_1-\hat{r}_0) \,+\, \frac{\varepsilon^2}{2}\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}(x_1-\hat{r}_0)^2 \,+\, \frac{\varepsilon^2}{2}\frac{\partial^2\tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}x_2^2 \,+\, \varepsilon^3\,O(\hat{\ell}_1^3+\hat{\ell}_2^3),
$$

where we have used the assumption (1.23). It is easy to derive that

$$
\begin{aligned}
S_1[U_2] \,&=\, \Big(1-\tilde{V}-|U_2|^2\Big)U_2\\
&=\, \Big(1-|w_1|^2\Big)U_2 \,+\, \Big(1-|w_2|^2\Big)U_2 \,+\, \Big(-1-|U_2|^2+|w_1|^2+|w_2|^2\Big)U_2\\
&\quad -\, \varepsilon U_2\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}(x_1-\hat{r}_0) \,-\, \frac{\varepsilon^2}{2}U_2\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}(x_1-\hat{r}_0)^2\\
&\quad -\, \varepsilon^2\frac{\partial^2\tilde{V}}{\partial r\partial y_3}\Big|_{(r_0,0)}(x_1-\hat{r}_0)x_2 \,-\, \frac{\varepsilon^2}{2}U_2\frac{\partial^2\tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}x_2^2 \,+\, \varepsilon^3O(\hat{\ell}_1^3+\hat{\ell}_2^3)U_2.
\end{aligned}
$$

By adding all terms together and then using the equation (2.3), we express the error, near the vortex rings,

$$
S[U_2] \,=\, S_0[U_2] \,+\, S_0[U_2],
$$

in the form

$$
\begin{aligned}
S[U_2] \,=\,& U_2\,\frac{x_1-\hat f_1}{x_1\hat\ell_1}\,\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)}\;+\;U_2\,\frac{x_1-\hat f_2}{x_1\hat\ell_2}\,\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)}\\
&+\;2U_2\,\frac{(x_1-\hat f_1)(x_1-\hat f_2)+(x_2-\hat d_1)(x_2-\hat d_2)}{\hat\ell_1\hat\ell_2}\;\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)}\,\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)}\\
&-\;2(-1)^jU_2\,\frac{(x_2-\hat d_1)(x_2-\hat d_2)+(x_1-\hat f_1)(x_1-\hat f_2)}{(\hat\ell_1)^2(\hat\ell_2)^2}\\
&+\;2i(-1)^jU_2\,\frac{-(x_2-\hat d_2)(x_1-\hat f_1)+(x_1-\hat f_2)(x_2-\hat d_1)}{\hat\ell_1(\hat\ell_2)^2}\;\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)}\\
&+\;2iU_2\,\frac{-(x_2-\hat d_1)(x_1-\hat f_2)+(x_1-\hat f_1)(x_2-\hat d_2)}{\hat\ell_2(\hat\ell_1)^2}\;\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)}.\\
&-\;U_2\Bigg[\varepsilon\frac{\partial\tilde V}{\partial r}\Big|_{(r_0,0)}(x_1-\hat r_0)\;+\;\frac{\varepsilon^2}{2}\,\frac{\partial^2\tilde V}{\partial r^2}\Big|_{(r_0,0)}(x_1-\hat r_0)^2\\
&\qquad\quad+\;\frac{\varepsilon^2}{2}\,\frac{\partial^2\tilde V}{\partial y_3^2}\Big|_{(r_0,0)}x_2^2+\varepsilon^3O(\hat\ell_1^3+\hat\ell_2^3)\Bigg]\;+\;F_{21}\\
\equiv\,& F_{21}\;+\;F_{22}.
\end{aligned}
\tag{4.34}
$$

Since F_{21} is a singular term defined in (4.33), we will introduce a further correction to improve the approximation. This will be done in subsection 4.2. On the other hand, by careful checking, we find that F_{22} is a term defined on the region D_2 with properties, for $\ell_1>3$ and $\ell_2>3$

$$
\Big|\mathrm{Re}\,\frac{F_{22}}{-iU_2}\Big|\;\le\;\frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_1)^3}\;+\;\frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_2)^3}\;+\;\frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_3)^3}\;+\;\frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_4)^3},
\tag{4.35}
$$

$$
\Big|\mathrm{Im}\,\frac{F_{22}}{-iU_2}\Big|\;\le\;\frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_1)^{1+\sigma}}\;+\;\frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_2)^{1+\sigma}}\;+\;\frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_3)^{1+\sigma}}\;+\;\frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_4)^{1+\sigma}},
\tag{4.36}
$$

and

$$
\Big\|\frac{F_{22}}{-iU_2}\Big\|_{L^p\left(\{\hat\ell_1<3\}\cup\{\hat\ell_2<3\cup\{\hat\ell_3<3\cup\{\hat\ell_4<3\}\right)}\;\le\;C\varepsilon|\log\varepsilon|,
\tag{4.37}
$$

where σ and p are some universal constants with

$$
\frac{2\lambda_1}{2-\lambda_1}<\sigma<1.
\tag{4.38}
$$

Second, we compute the error for U_1. There holds

$$
\begin{aligned}
\frac{\partial}{\partial x_1}\sqrt{1-\tilde V(\varepsilon|x_1|,\varepsilon x_2)}\;&=\;-\;\frac{\varepsilon}{2\sqrt{1-\tilde V}}\,\frac{\partial\tilde V}{\partial r},\\
\frac{\partial^2}{\partial x_1^2}\sqrt{1-\tilde V(\varepsilon|x_1|,\varepsilon x_2)}\;&=\;-\;\frac{\varepsilon^2}{4}\big(1-\tilde V\big)^{-3/2}\Big|\frac{\partial\tilde V}{\partial r}\Big|^2\;-\;\frac{\varepsilon^2}{2\sqrt{1-\tilde V}}\,\frac{\partial^2\tilde V}{\partial r^2}.
\end{aligned}
\tag{4.39}
$$

It is easy to check that the error of U_1 is

$$
S[U_1]\;=\;S_0[U_1].
$$

Note that

$$\begin{aligned} S_0[U_1] &= S_0\Big[\sqrt{1-\tilde V}\Big]e^{i\varphi_0^j} + 2ie^{i\varphi_0^j}\nabla\sqrt{1-\tilde V}\cdot\nabla\varphi_0^j - \sqrt{1-\tilde V}e^{i\varphi_0^j}\big|\nabla\varphi_0^j\big|^2 \\ &\quad + iS_0[\varphi_0^j]\sqrt{1-\tilde V}e^{i\varphi_0^j}. \end{aligned}$$

Hence the error is

$$\begin{aligned} S[U_1] &= -\frac{1}{4}\varepsilon^2\big|\tilde\nabla\tilde V\big|^2\frac{U_1}{\big(1-\tilde V\big)^2} - \frac{1}{2}\varepsilon^2\tilde\triangle\tilde V\frac{U_1}{1-\tilde V} - i\frac{U_1}{1-\tilde V}\nabla\tilde V\cdot\nabla\varphi_0^j - U_1\big|\nabla\varphi_0^j\big|^2 \\ &\quad - \frac{1}{2}\varepsilon U_1\frac{1}{x_1}\frac{\partial\tilde V}{\partial r}\frac{1}{1-\tilde V} + iU_1S_0[\varphi_0^j] \\ &\equiv F_{12} + F_{11}, \end{aligned} \tag{4.40}$$

where we have denoted

$$\tilde\nabla\tilde V = \Big(\frac{\partial\tilde V}{\partial r},\frac{\partial\tilde V}{\partial y_3}\Big), \quad \tilde\triangle\tilde V = \frac{\partial^2\tilde V}{\partial r^2} + \frac{\partial^2\tilde V}{\partial y_3^2},$$

and also

$$F_{11} = -i\frac{U_1}{1-V}\nabla\tilde V\cdot\nabla\varphi_0^j + iU_1S_0[\varphi_0^j]. \tag{4.41}$$

Note that F_{11} is not a singular term in the region D_1. On the other hand, out of the region D_1, if $|x|$ is close to $\hat r_2$, then $\big(1-\tilde V\big)^{-1}$ brings singularity. So we need another correction term to improve the approximation in a neighborhood of the curve $|x| = \hat r_2$. On the region D_1, we will evaluate the terms in F_{12}. The condition $\frac{\partial\tilde V}{\partial r}\Big|_{(0,y_3)} = 0$ implies that

$$\frac{1}{x_1}\frac{\partial\tilde V}{\partial r} = O(\varepsilon).$$

Whence, we find that F_{12} is a term defined on the region D_1 with properties,

$$\left|\mathrm{Re}\,\frac{F_{12}}{-iU_1}\right| \le \frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_1)^3} + \frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_2)^3} + \frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_3)^3} + \frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_4)^3} \tag{4.42}$$

$$\left|\mathrm{Im}\,\frac{F_{12}}{-iU_1}\right| \le \frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_1)^{1+\sigma}} + \frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_2)^{1+\sigma}} + \frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_3)^{1+\sigma}} + \frac{O(\varepsilon^{1-2\sigma})}{(1+\hat\ell_4)^{1+\sigma}}. \tag{4.43}$$

In the above estimates, we need $2\lambda_2 < \sigma < 1$.

4.2. **Further improvement of approximation.** To handle the singular term F_{21}, as the argument in [60], we here introduce a further correction $\varphi_1^j(x_1,x_2)$ to the phase term in the form

$$\varphi_1^j = \varphi_s^j + \varphi_r^j. \tag{4.44}$$

By setting the smooth cut-off function

$$\eta(s) = \begin{cases} 1, & |s| \le 1/10, \\ 0, & |s| \ge 1/5, \end{cases} \tag{4.45}$$

the singular part φ_s^j is defined as

$$\begin{aligned} \varphi_s^j &= \Big(\eta\big(\varepsilon^{\lambda_1}\hat\ell_1\big) + \eta\big(\varepsilon^{\lambda_1}\hat\ell_4\big)\Big)\left[\frac{x_2-\hat d_1}{4\hat f_1}\log(\hat\ell_1)^2 + \frac{x_2-\hat d_4}{4\hat f_4}\log(\hat\ell_4)^2\right] \\ &\quad + (-1)^j\Big(\eta\big(\varepsilon^{\lambda_1}\hat\ell_2\big) + \eta\big(\varepsilon^{\lambda_1}\hat\ell_3\big)\Big)\Big[\frac{x_2-\hat d_2}{4\hat f_2}\log(\hat\ell_2)^2 + \frac{x_2-\hat d_3}{4\hat f_3}\log(\hat\ell_3)^2\Big]. \end{aligned} \tag{4.46}$$

For later use, we compute:

$$
\begin{aligned}
\frac{\partial \varphi_s^j}{\partial x_1} = {} & \varepsilon^{\lambda_1}\Big(\eta'\big(\varepsilon^{\lambda_1}\hat{\ell}_1\big)\frac{x_1-\hat{f}_1}{\hat{\ell}_1}+\eta'\big(\varepsilon^{\lambda_4}\hat{\ell}_1\big)\frac{x_1-\hat{f}_4}{\hat{\ell}_4}\Big)\left[\frac{x_2-\hat{d}_1}{4\hat{f}_1}\,\log(\hat{\ell}_1)^2+\frac{x_2-\hat{d}_4}{4\hat{f}_4}\,\log(\hat{\ell}_4)^2\right] \\
& + (-1)^j\varepsilon^{\lambda_1}\Big(\eta'\big(\varepsilon^{\lambda_1}\hat{\ell}_2\big)\frac{x_1-\hat{f}_2}{\hat{\ell}_2}+\eta'\big(\varepsilon^{\lambda_1}\hat{\ell}_3\big)\frac{x_1-\hat{f}_3}{\hat{\ell}_3}\Big)\left[\frac{x_2-\hat{d}_2}{4\hat{f}_2}\,\log(\hat{\ell}_2)^2+\frac{x_2-\hat{d}_3}{4\hat{f}_3}\,\log(\hat{\ell}_3)^2\right] \\
& + \Big(\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_1\big)+\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_4\big)\Big)\left[\frac{(x_2-\hat{d}_1)(x_1-\hat{f}_1)}{2\hat{f}_1\hat{\ell}_1^2}+\frac{(x_2-\hat{d}_4)(x_1-\hat{f}_4)}{2\hat{f}_4\hat{\ell}_4^2}\right] \\
& + (-1)^j\Big(\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_2\big)+\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_3\big)\Big)\Big[\frac{(x_2-\hat{d}_2)(x_1-\hat{f}_2)}{2\hat{f}_2\hat{\ell}_2^2}+\frac{(x_2-\hat{d}_3)(x_1-\hat{f}_3)}{2\hat{f}_3\hat{\ell}_3^2}\Big],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_s^j}{\partial x_2} = {} & \varepsilon^{\lambda_1}\Big(\eta'\big(\varepsilon^{\lambda_1}\hat{\ell}_1\big)\frac{x_2-\hat{d}_1}{\hat{\ell}_1}+\eta'\big(\varepsilon^{\lambda_1}\hat{\ell}_1\big)\frac{x_2-\hat{d}_4}{\hat{\ell}_4}\Big)\left[\frac{x_2-\hat{d}_1}{4\hat{f}_1}\,\log(\hat{\ell}_1)^2+\frac{x_2-\hat{d}_4}{4\hat{f}_4}\,\log(\hat{\ell}_4)^2\right] \\
& + (-1)^j\varepsilon^{\lambda_1}\Big(\eta'\big(\varepsilon^{\lambda_1}\hat{\ell}_2\big)\frac{x_2-\hat{d}_2}{\hat{\ell}_2}+\eta'\big(\varepsilon^{\lambda_1}\hat{\ell}_3\big)\frac{x_2-\hat{d}_3}{\hat{\ell}_3}\Big)\left[\frac{x_2-\hat{d}_2}{4\hat{f}_2}\,\log(\hat{\ell}_2)^2+\frac{x_2-\hat{d}_3}{4\hat{f}_3}\,\log(\hat{\ell}_3)^2\right] \\
& + \Big(\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_1\big)+\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_4\big)\Big)\left[\frac{1}{4\hat{f}_1}\log(\hat{\ell}_1)^2+\frac{1}{4\hat{f}_4}\log(\hat{\ell}_4)^2\right] \\
& + (-1)^j\Big(\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_2\big)+\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_3\big)\Big)\left[\frac{1}{4\hat{f}_2}\log(\hat{\ell}_2)^2+\frac{1}{4\hat{f}_3}\log(\hat{\ell}_3)^2\right] \\
& + \Big(\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_1\big)+\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_4\big)\Big)\left[\frac{(x_2-\hat{d}_1)^2}{2\hat{f}_1\hat{\ell}_1^2}+\frac{(x_2-\hat{d}_4)^2}{2\hat{f}_4\hat{\ell}_4^2}\right] \\
& + (-1)^j\Big(\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_2\big)+\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_3\big)\Big)\left[\frac{(x_2-\hat{d}_2)^2}{2\hat{f}_2\hat{\ell}_2^2}+\frac{(x_2-\hat{d}_3)^2}{2\hat{f}_3\hat{\ell}_3^2}\right].
\end{aligned}
$$

Hence, by recalling (3.15) and (3.16), we obtain

$$
\begin{aligned}
\triangledown\varphi_s^j = {} & \Big(\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_1\big)+\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_4\big)\Big)\Big[\frac{1}{2\hat{f}_1}(0,\log\hat{f}_1)+\frac{1}{2\hat{f}_4}(0,\log\hat{f}_4)+O(\varepsilon\log\hat{\ell}_1)+O(\varepsilon\log\hat{\ell}_4)\Big] \\
& + (-1)^j\Big(\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_2\big)+\eta\big(\varepsilon^{\lambda_1}\hat{\ell}_3\big)\Big)\Big[\frac{1}{2\hat{f}_2}(0,\log\hat{f}_2)+\frac{1}{2\hat{\ell}_3}(0,\log\hat{f}_3) \\
& \qquad + O(\varepsilon\log\hat{\ell}_2)+O(\varepsilon\log\hat{\ell}_3)\Big].
\end{aligned}
\tag{4.47}
$$

Note that the function φ_s^j is continuous but $\triangledown\varphi_s^j$ is not. The singularity of φ_s^j comes from its derivatives, which will play an important role in the final reduction procedure.

Remark 4.1. The reader can also refer to formula (16) in [81] for the formal derivation of general improvement of the phase term. □

On the other hand, we choose the regular part φ_r^j by solving the equation

$$
S_0[\varphi_r^j] + \eta_1\,\frac{1}{1-\tilde{V}}\nabla\tilde{V}\cdot\nabla\varphi_r^j = -S_0[\varphi_0^j+\varphi_s^j] - \eta_1\,\frac{1}{1-\tilde{V}}\nabla\tilde{V}\cdot\nabla\big(\varphi_0^j+\varphi_s^j\big). \tag{4.48}
$$

In the above, the cut-off function η_1 is defined in (4.57). It can be done as follows. For (x_1,x_2) in the region

$$
\mathfrak{D} \equiv B_{\tilde{\tau}}(\vec{e}_1)\cup B_{\tilde{\tau}}(\vec{e}_2)\cup B_{\tilde{\tau}}(\vec{e}_3)\cup B_{\tilde{\tau}}(\vec{e}_4),
$$

where for any j, $B_{\tilde{\tau}}(\vec{e}_j)$ is a ball of radius $\tilde{\tau} = \varepsilon^{-\lambda_1}/5$ with center $\vec{e}_j$, trivial computation gives that,

$$S_0[\varphi_0^j + \varphi_s^j] \,+\, \eta_1 \,\frac{1}{1-\tilde{V}} \nabla \tilde{V} \cdot \nabla\big(\varphi_0^j + \varphi_s^j\big) \,=\, O(\varepsilon^2).$$

For $(x_1, x_2) \in \mathcal{D}^c$, the error is also $O(\varepsilon^2)$. In fact, for $(x_1, x_2) \in \mathcal{D}^c$, we have $\varphi_s^j = 0$ and then

$$S_0[\varphi_0^j + \varphi_s^j] \,+\, \eta_1 \,\frac{1}{1-\tilde{V}} \nabla \tilde{V} \cdot \nabla\big(\varphi_0^j + \varphi_s^j\big) \,=\, \eta_1 \, F_{11} \,+\, F_{21}.$$

Going back to the original variables (r, y_3) and setting $\hat{\varphi}(r, y_3) = \varphi_r^j(r/\varepsilon, y_3/\varepsilon)$, we see that

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial y_3^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big)\hat{\varphi} \,\le\, C\Big(1 + r^2 + y_3^2\Big)^{-3/2}.$$

Whence, by solving problem (4.48), we can choose φ_r^j such that there holds

$$\hat{\varphi} = O\Big(\frac{1}{\sqrt{1 + r^2 + y_3^2}}\Big).$$

Moreover the term φ_r^j is C^1-smooth. As a consequence, we have chosen the phase correction term φ_1^j such that

$$S_0[\varphi_1^j + \varphi_0^j] \,+\, \eta_1 \,\frac{1}{1-\tilde{V}} \nabla \tilde{V} \cdot \nabla\big(\varphi_1^j + \varphi_0^j\big) \,=\, 0. \tag{4.49}$$

Now we shall deal with the singularity as x approaching the circle $|x| = \hat{r}_2$. By the assumption (**A3**), there exists a small positive ε_0 such that for $0 < \varepsilon < \varepsilon_0$

$$\delta_\varepsilon \,=\, \varepsilon \frac{\partial \tilde{V}}{\partial \ell}\Big|_{(r_2, 0)} > 0, \tag{4.50}$$

where $\ell = \sqrt{r^2 + y_3^2}$. Then for (r, y_3) with foot point (p_1, p_2) on the circle of radius r_2, there holds

$$\begin{aligned} 1 \,-\, \tilde{V}(r, y_3) \,&=\, 1 \,-\, \tilde{V}(p_1, p_2) \,-\, \frac{\partial \tilde{V}}{\partial \ell}\Big|_{(p_1, p_2)} \varepsilon(\hat{\ell} - \hat{r}_2) + O\big(\varepsilon^2 (\hat{\ell} - \hat{r}_2)^2\big) \\ &=\, -\delta_\varepsilon(\hat{\ell} - \hat{r}_2) + O\big(\varepsilon^2 (\hat{\ell} - \hat{r}_2)^2\big). \end{aligned} \tag{4.51}$$

Let q be the unique solution given by Lemma 2.4. Now we define $\tilde{q}(\lambda) = \delta_\varepsilon^{1/3} q\big(\delta_\varepsilon^{1/3}\lambda\big)$. Then it is easy to check that

$$\tilde{q}_{\lambda\lambda} \,-\, \tilde{q}\big(\delta_\varepsilon \lambda + \tilde{q}^2\big) \,=\, 0. \tag{4.52}$$

In other words, if we choose

$$\hat{q}(x_1, x_2) = \delta_\varepsilon^{1/3} q\Big(\delta_\varepsilon^{1/3}(\hat{\ell} - \hat{r}_2)\Big) \tag{4.53}$$

with $\hat{\ell} = \sqrt{x_1^2 + x_2^2}$, then $\hat{q}$ satisfies

$$\begin{aligned} &\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\hat{q} \,+\, \Big(1 - \tilde{V}(\varepsilon|x_1|, \varepsilon x_2)\Big)\hat{q} \,-\, \hat{q}^{\,3} \\ &\qquad = \, \delta_\varepsilon^{2/3} \, \frac{1}{\hat{\ell}} \, q'\Big(\delta_\varepsilon^{1/3}(\hat{\ell} - \hat{r}_2)\Big) \,+\, O\big(\varepsilon^2 (\hat{\ell} - \hat{r}_2)^2\big)\hat{q}. \end{aligned} \tag{4.54}$$

This implies that we can use

$$U_3(x_1, x_2) = \hat{q}(x_1, x_2) e^{i\varphi_0^j}, \tag{4.55}$$

as an approximation near the circle $\ell = \hat{r}_2$.

By defining smooth cut-off functions as follows

$$\tilde{\eta}_2(s) = \begin{cases} 1, & |s| \le 1, \\ 0, & |s| \ge 2, \end{cases} \qquad \tilde{\eta}_3(s) = \begin{cases} 1, & s \ge -1, \\ 0, & s \le -2, \end{cases} \tag{4.56}$$

we choose the cut-off functions by

$$\begin{aligned}\eta_2\big(x_1,x_2\big) \;&=\; \tilde\eta_2\big(\,\varepsilon^{\lambda_1}|x-(\hat r_0,0)|\,\big)\;+\;\tilde\eta_2\big(\,\varepsilon^{\lambda_1}|x+(\hat r_0,0)|\,\big)\\ \eta_3\big(x_1,x_2\big)\;&=\;\tilde\eta_3\big(\varepsilon^{\lambda_2}(\hat\ell-\hat r_2)\big),\\ \eta_1\big(x_1,x_2\big)\;&=\;1-\eta_2-\eta_3.\end{aligned}\tag{4.57}$$

We then choose the **final approximate solution** to (3.3)-(3.4) by, for $(x_1,x_2)\in\mathbb{R}^2$,

$$\begin{aligned}u_2(x_1,x_2)\;=\;&\sqrt{1-\tilde V(\varepsilon|x_1|,\varepsilon x_2)}\,\eta_1\,e^{i\varphi}\;+\;\rho(\hat\ell_1)\rho(\hat\ell_2)\rho(\hat\ell_4)\rho(\hat\ell_3)\,\eta_2\,e^{i\varphi}\\ &+\;\hat q(x_1,x_2)\,\eta_3\,e^{i\varphi},\end{aligned}\tag{4.58}$$

where the new phase term

$$\varphi=\varphi_0^j+\varphi_1^j.\tag{4.59}$$

By recalling the definition of U_1,U_2,U_3 in (4.29), (4.26) and (4.55), we also write the approximation as

$$u_2\;=U_1\,\eta_1\,e^{\,i\varphi_1^j}\;+\;U_2\,\eta_2\,e^{\,i\varphi_1^j}\;+\;U_3\,\eta_3\,e^{\,i\varphi_1^j}.\tag{4.60}$$

By using (3.15) or (3.16), it is easy to check that u_2 has the symmetry

$$u_2(x_1,x_2)=\overline{u_2(x_1,-x_2)},\qquad u_2(x_1,x_2)=u_2(-x_1,x_2).\tag{4.61}$$

Moreover, there holds

$$\frac{\partial u_2}{\partial x_1}(0,x_2)=0.\tag{4.62}$$

4.3. **Estimates of the error.** As we have stated, we work directly in the half space $\mathbb{R}^2_+=\{(x_1,x_2):\,x_1>0\}$ in the sequel because of the symmetry of the problem. Recalling the definitions of the operators in (3.12), let us start to compute the error:

$$\begin{aligned}\mathbb{E}\;=\;S[u_2]\;=\;&S[U_1]\eta_1e^{i\varphi_1^j}\;+\;U_1S_0[\eta_1e^{i\varphi_1^j}]\;+\;2\nabla U_1\cdot\nabla\Big(\eta_1e^{i\varphi_1^j}\Big)\\ &+\;S[U_2]\eta_2e^{i\varphi_1^j}\;+\;U_2S_0[\eta_2e^{i\varphi_1^j}]\;+\;2\nabla U_2\cdot\nabla\Big(\eta_2e^{i\varphi_1^j}\Big)\\ &+\;S[U_3]\eta_3e^{i\varphi_1^j}\;+\;U_3S_0[\eta_3e^{i\varphi_1^j}]\;+\;2\nabla U_3\cdot\nabla\Big(\eta_3e^{i\varphi_1^j}\Big)\;+\;\mathbb{N},\end{aligned}\tag{4.63}$$

where the nonlinear term $\mathbb{N}$ is defined by

$$\mathbb{N}=\eta_1|U_1|^2U_1e^{i\varphi_1^j}\;+\;\eta_2|U_2|^2U_2e^{i\varphi_1^j}\;+\;\eta_3|U_3|^2U_3e^{i\varphi_1^j}\;-\;|u_2|^2u_2.\tag{4.64}$$

The main components in the above formula can be estimated as follows.

We first consider the error in the region D_1(cf. (4.24)). Recall the estimate of $S[U_1]$ in (4.40). There holds

$$\begin{aligned}\eta_1e^{i\varphi_1^j}S[U_1]\;+\;U_1S_0[\eta_1e^{i\varphi_1^j}]\;=\;&\eta_1\,e^{i\varphi_1^j}\,F_{12}\;+\;\eta_1\,e^{i\varphi_1^j}\,F_{11}\;+\;i\,\eta_1\,U_1\,e^{i\varphi_1^j}\,S_0[\varphi_1^j]\;+\;U_1\,S_0[\eta_1]\,e^{i\varphi_1^j}\\ &+\;2iU_1e^{i\varphi_1^j}\nabla\eta_1\cdot\nabla\varphi_1^j\;-\;\eta_1\,U_1\,e^{i\varphi_1^j}\big|\nabla\varphi_1^j\big|^2.\end{aligned}$$

Using the formula (4.47), we obtain

$$\begin{aligned}2\nabla U_1\cdot\nabla\Big(\eta_1e^{i\varphi_1^j}\Big)\;=\;&2i\,\eta_1\,e^{i\varphi_1^j}\nabla U_1\cdot\nabla\varphi_1^j\;+\;2\,e^{i\varphi_1^j}\nabla U_1\cdot\nabla\eta_1\\ =\;&i\,\eta_1\,U_1\,e^{i\varphi_1^j}\,\frac{1}{1-\tilde V}\nabla\tilde V\cdot\nabla\varphi_1^j\;-\;2\,\eta_1\,U_1\,e^{i\varphi_1^j}\,\nabla\varphi_0^j\cdot\nabla\varphi_1^j\\ &+\;e^{i\varphi_1^j}\,\frac{U_1}{1-\tilde V}\nabla\tilde V\cdot\nabla\eta_1\;+\;2i\,e^{i\varphi_1^j}U_1\,\nabla\varphi_0^j\cdot\nabla\eta_1.\end{aligned}$$

From the relation (4.49), the term F_{11} defined in (4.41) has been canceled. In the region D_1, the error is estimated by

$$\mathbb{E}_1 \equiv \mathbb{E} = \eta_1 e^{i\varphi_1^j} F_{12} + U_1 S_0[\eta_1] e^{i\varphi_1^j} + 2iU_1 e^{i\varphi_1^j}\nabla\eta_1\cdot\nabla\varphi_1^j - \eta_1 U_1 e^{i\varphi_1^j}\big|\nabla\varphi_1^j\big|^2$$
$$- 2\eta_1 U_1 e^{i\varphi_1^j}\nabla\varphi_0^j\cdot\nabla\varphi_1^j + e^{i\varphi_1^j}\frac{U_1}{1-\tilde V}\nabla\tilde V\cdot\nabla\eta_1 + 2i e^{i\varphi_1^j}U_1\nabla\varphi_0^j\cdot\nabla\eta_1 + \mathbb{N}.$$

Note that, in this region, $|\bigtriangledown\varphi_0^j| = O(\varepsilon)$ and $|\bigtriangledown\varphi_1^j| = O(\varepsilon)$.

Using the equation (4.49), the singular term F_{21} in $S[U_2]$ in is canceled and we then get

$$S[U_2]\,\eta_2 e^{i\varphi_1^j} + U_2 S_0[\eta_2 e^{i\varphi_1^j}] = F_{22}\,\eta_2 e^{i\varphi_1^j} + U_2 S_0[\eta_2] e^{i\varphi_1^j} + 2iU_2\nabla\eta_2\cdot\nabla\varphi_1^j - U_2\eta_2 e^{i\varphi_1^j}\big|\nabla\varphi_1^j\big|^2.$$

Whence, there holds

$$S[U_2]\,\eta_2 e^{i\varphi_1^j} + U_2 S_0[\eta_2 e^{i\varphi_1^j}] = F_{22}\,\eta_2 e^{i\varphi_1^j} + \varepsilon^2 O(|\hat\ell_1|^2 + |\hat\ell_2|^2).$$

The formula in (4.31) and (4.47) imply that

$$F_{23} \equiv 2\nabla U_2\cdot\nabla\Big(\eta_2 e^{i\varphi_1^j}\Big)$$
$$= \eta_2 U_2 e^{i\varphi_1^j}\frac{1}{\hat f_1}\log\hat f_1\Big[\, i\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)}\frac{x_2-\hat d_1}{\hat\ell_1} + i\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)}\frac{x_2-\hat d_2}{\hat\ell_2} - \frac{x_1-\hat f_1}{\hat\ell_1^2} - (-1)^j\frac{x_1-\hat f_2}{\hat\ell_2^2}\,\Big]$$
$$+ \eta_2 U_2 e^{i\varphi_1^j}\frac{1}{\hat f_2}\log\hat f_2\Big[\, i\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)}\frac{x_2-\hat d_1}{\hat\ell_1} + i\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)}\frac{x_2-\hat d_2}{\hat\ell_2} - \frac{x_1-\hat f_1}{\hat\ell_1^2} - \frac{x_1-\hat f_2}{\hat\ell_2^2}\,\Big]$$
$$+ O(\varepsilon\log\hat\ell_1 + \varepsilon\log\hat\ell_2).$$

It is worth mentioning that, in the vortex-core region D_2(cf. (4.24)), we estimate the error by

$$\mathbb{E}_2 \equiv \mathbb{E} = \eta_2 e^{i\varphi_1^j}F_{22} + F_{23} + O(\varepsilon\log\hat\ell_1 + \varepsilon\log\hat\ell_2) + \mathbb{N}, \tag{4.65}$$

where F_{22} is defined in (4.34). The singularity of the term F_{23} will play an important role in the final reduction step.

In the region D_3(cf. (4.24)), we first compute the error of U_3

$$S[U_3] = S_0[U_3] + S_1[U_3],$$

where

$$S_0[U_3] = \delta_\varepsilon\, q''\Big(\delta_\varepsilon^{1/3}(\hat\ell - \hat r_2)\Big) e^{i\varphi_0^j} + 2\delta_\varepsilon^{2/3}\frac{1}{\hat\ell}\, q'\Big(\delta_\varepsilon^{1/3}(\hat\ell-\hat r_2)\Big) e^{i\varphi_0^j}$$
$$+ 2\nabla\hat q\cdot\nabla e^{i\varphi_0^j} - \hat q\, e^{i\varphi_0^j}\big|\nabla\varphi_0^j\big|^2 + iS_0[\varphi_0^j]\,\hat q\, e^{i\varphi_0^j}.$$

We also write $S_1[U_3]$ of the form

$$S_1[U_3] = \delta_\varepsilon\Bigg[-\delta_\varepsilon^{1/3}(\hat\ell-\hat r_2) - q^2\Big(\delta_\varepsilon^{1/3}(\hat\ell-\hat r_2)\Big)\Bigg] q\Big(\delta_\varepsilon^{1/3}(\hat\ell-\hat r_2)\Big) e^{i\varphi_0^j}$$
$$+ \Big[(1-\tilde V) + \delta_\varepsilon(\hat\ell - \hat r_2)\Big]\hat q e^{i\varphi_0^j}.$$

The equation of q in Lemma 2.4 implies that there holds

$$S[U_3] = 2\delta_\varepsilon^{2/3}\frac{1}{\hat\ell}\, q'\Big(\delta_\varepsilon^{1/3}(\hat\ell-\hat r_2)\Big) e^{i\varphi_0^j} + 2\nabla\hat q\cdot\nabla e^{i\varphi_0^j} - \hat q\, e^{i\varphi_0^j}\big|\nabla\varphi_0^j\big|^2$$
$$+ iS_0[\varphi_0^j]\,\hat q\, e^{i\varphi_0^j} + \Big[(1-\tilde V) + \delta_\varepsilon(\hat\ell-\hat r_2)\Big]\hat q e^{i\varphi_0^j},$$
$$\equiv F_{31} + F_{32},$$

where the term F_{31} is defined by

$$F_{31} \equiv iS_0[\varphi_0^j]\,\hat q\, e^{i\varphi_0^j} = i\,U_3\, S_0[\varphi_0^j].$$

The other two terms can be estimated as

$$\begin{aligned} U_3 S_0[\eta_3 e^{i\varphi_1^j}] \,+\, 2\nabla U_3\cdot\nabla\Big(\eta_3 e^{i\varphi_1^j}\Big) \;&=\; iU_3\,\eta_3 e^{i\varphi_1^j}\,S_0[\varphi_1^j] \,-\, U_3\,\eta_3 e^{i\varphi_1^j}\,|\varphi_1^j|^2 \\ &\quad+\, U_3\,e^{i\varphi_1^j}\,S_0[\eta_3] \,+\, 2iU_3\,e^{i\varphi_1^j}\nabla\eta_3\cdot\nabla\varphi_1^j \\ &\quad+\, 2e^{i\varphi_1^j}\nabla U_3\cdot\nabla\eta_3 \,+\, 2i\,\eta_3\,e^{i\varphi_1^j}\nabla U_3\cdot\nabla\varphi_1^j \\ &\equiv\; \eta_3 e^{i\varphi_1^j}\,F_{3,3} \,+\, F_{3,4}, \end{aligned}$$

where we have denoted

$$F_{3,3} \;=\; iU_3\,S_0[\varphi_1^j]. \tag{4.66}$$

By using the relation (4.49), $F_{3,1}$ and $F_{3,3}$ will be canceled. Hence, in the region D_3, the error takes the form

$$\mathbb{E}_3 \;\equiv\; \mathbb{E} \;=\; \eta_3 e^{i\varphi_1^j}\,F_{3,2} \,+\, F_{3,4} \,+\, \mathbb{N}.$$

To get a real solution to (3.3)-(3.4), we will find a perturbation term ψ to u_2 in the form of (5.4), which will lead to different local forms of the problem, and so the error $\mathbb{E}$, see section 5. Hence, we define

$$\tilde{\mathbb{E}} \;=\; \frac{\mathbb{E}_1}{-i\eta_1\,U_1\,e^{i\varphi_1^j}} \,+\, \frac{\mathbb{E}_2}{-i\eta_2\,U_2\,e^{i\varphi_1^j}} \,+\, \mathbb{E}_3. \tag{4.67}$$

For a function $h = h_1 + ih_2$ with real functions h_1, h_2, define a norm of the form

$$\begin{aligned} ||h||_{**} \;\equiv\; &\sum_{m=1}^{4}||iu_2 h||_{L^p(\hat\ell_m<3)} \,+\, \sum_{m=1}^{4}\Big[\,||\hat\ell_m^{2+\sigma}h_1||_{L^\infty(\tilde D)} \,+\, ||\hat\ell_m^{1+\sigma}h_2||_{L^\infty(\tilde D)}\,\Big] \\ &+\, \sum_{m=1}^{4}\Big[\,||\hat\ell_m^{2+\sigma}h_1||_{L^\infty(D_{1,2})} \,+\, ||\varepsilon^{\lambda_2-1}\hat\ell_m^{\sigma}h_2||_{L^\infty(D_{1,2})}\Big] \,+\, \sum_{m=1}^{2}||h_m||_{L^p(D_3)}\,, \end{aligned} \tag{4.68}$$

where we have denoted

$$\tilde D = D_2\cup D_{1,1}\setminus\Big\{\,\hat\ell_1<3 \text{ or } \hat\ell_2<3 \text{ or } \hat\ell_3<3 \text{ or } \hat\ell_4<3\,\Big\}. \tag{4.69}$$

As a conclusion, we have the following lemma.

Lemma 4.2. *There hold, for $x\in\tilde D\cup D_{1,2}$,*

$$\Big|\mathrm{Re}\frac{\mathbb{E}}{-iu_2}\Big| \;\le\; \frac{C\varepsilon^{1-2\sigma}}{(1+\hat\ell_1)^3} + \frac{C\varepsilon^{1-2\sigma}}{(1+\hat\ell_2)^3} + \frac{C\varepsilon^{1-2\sigma}}{(1+\hat\ell_3)^3} + \frac{C\varepsilon^{1-2\sigma}}{(1+\hat\ell_4)^3},$$

$$\Big|\mathrm{Im}\frac{\mathbb{E}}{-iu_2}\Big| \;\le\; \frac{C\varepsilon^{1-2\sigma}}{(1+\hat\ell_1)^{1+\sigma}} + \frac{C\varepsilon^{1-2\sigma}}{(1+\hat\ell_2)^{1+\sigma}} + \frac{C\varepsilon^{1-2\sigma}}{(1+\hat\ell_3)^{1+\sigma}} + \frac{C\varepsilon^{1-2\sigma}}{(1+\hat\ell_4)^{1+\sigma}};$$

and also

$$\big\|\mathbb{E}\big\|_{L^p\left(\{\hat\ell_1<3\}\cup\{\hat\ell_2<3\}\cup\{\hat\ell_3<3\}\cup\{\hat\ell_4<3\}\right)} \;\le\; C\varepsilon|\log\varepsilon|,$$

where $\sigma\in(0,1)$ is a constant satisfying (4.38) and $2\lambda_2<\sigma<1$. As a consequence, there also holds

$$\big\|\tilde{\mathbb{E}}\big\|_{**}\le C\varepsilon^{1-2\sigma}.$$

□

5. Local Setting-up of the Problem

We look for a solution $u = u(x_1, x_2)$ to problem (3.3)-(3.4) in the form of small perturbation of u_2, with additional symmetry:

$$u(x_1, x_2) = \bar{u}(x_1, -x_2). \tag{5.1}$$

Let $\tilde{\chi} : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function defined by

$$\tilde{\chi}(s) = \begin{cases} 1, & s \leq 1, \\ 0, & s \geq 2. \end{cases} \tag{5.2}$$

Recalling (4.56)-(4.60) and setting the components of the approximation u_2 as

$$v_1(x_1, x_2) \;=\; \eta_1 \, U_1 \, e^{i\varphi_1^j}, \quad v_2(x_1, x_2) \;=\; \eta_2 \, U_2 \, e^{i\varphi_1^j}, \quad v_3(x_1, x_2) \;=\; \eta_3 \, U_3 \, e^{i\varphi_1^j}, \tag{5.3}$$

we want to choose the ansatz of the form

$$u \;= \Big[\chi \big(v_2 + i v_2 \psi \big) + \big(1 - \chi \big)(v_1 + v_2) e^{i\psi} \Big] \;+\; \Big[v_3 + i \eta_3 e^{i\varphi} \psi \Big], \tag{5.4}$$

where $\chi(x_1, x_2) \;=\; \tilde{\chi}(\hat{\ell}_1) + \tilde{\chi}(\hat{\ell}_2) + \tilde{\chi}(\hat{\ell}_3) + \tilde{\chi}(\hat{\ell}_4)$. The above nonlinear decomposition of the perturbation was first introduced in [27], see also [60].

To find the perturbation terms, the main object of this section is to write the equation for the perturbation as a linear one with a right hand side given by a lower order nonlinear term. Note that we shall carefully derive the equations due to different forms of local setting of the perturbation term ψ. The symmetry imposed on u can be transmitted to the symmetry on the perturbation terms

$$\psi(x_1, -x_2) = -\overline{\psi(x_1, x_2)}, \quad \psi(x_1, x_2) = \psi(-x_1, x_2). \tag{5.5}$$

This type of symmetry will play an important role in our further arguments. Let us observe that

$$\begin{aligned} u &= \Big[(v_1 + v_2) + i(v_1 + v_2)\psi + \big(1 - \chi\big)(v_1 + v_2)\big(e^{i\psi} - 1 - i\psi\big) \Big] \;+\; \Big(v_3 + i\eta_3 e^{i\varphi}\psi \Big) \\ &= u_2 \;+\; i(v_1 + v_2)\psi \;+\; i\eta_3 e^{i\varphi}\psi \;+\; \Gamma \end{aligned}$$

where we have denoted

$$\Gamma = \big(1 - \chi\big)(v_1 + v_2)\big(e^{i\psi} - 1 - i\psi\big). \tag{5.6}$$

A direct computation shows that

$$\begin{aligned} |u|^2 \;&=\; |u_2|^2 \;+\; 2\mathrm{Re}\big(\bar{u}_2 i (v_1 + v_2)\psi\big) \;+\; 2\eta_3 \mathrm{Re}(\bar{u}_2 i e^{i\varphi}\psi) \\ &\quad +\; 2\mathrm{Re}(\bar{u}_2 \Gamma) \;+\; \big| i(v_1 + v_2)\psi + i\eta_3 e^{i\varphi}\psi + \Gamma \big|^2. \end{aligned}$$

Then the nonlinear term in (3.3) can be expressed by

$$\begin{aligned} |u|^2 u \;&= |u_2|^2 u_2 \;+\; i|u_2|^2 (v_1 + v_2)\psi \;+\; i|u_2|^2 \eta_3 e^{i\varphi}\psi \;+\; |u_2|^2 \Gamma \\ &\quad +\; 2\mathrm{Re}\big(\bar{u}_2 i(v_1 + v_2)\psi\big) u_2 \;+\; 2\eta_3 \mathrm{Re}(\bar{u}_2 i e^{i\varphi}\psi) u_2 \;+\; \mathfrak{N}, \end{aligned}$$

where $\mathfrak{N}$ is defined by

$$\begin{aligned} \mathfrak{N} \equiv& \Big[2\mathrm{Re}\big(\bar{u}_2 i(v_1 + v_2)\psi\big) + 2\eta_3 \mathrm{Re}\big(\bar{u}_2 i e^{i\varphi}\psi\big) \Big] \times \Big(i(v_1 + v_2)\psi + i\eta_3 e^{i\varphi}\psi + \Gamma \Big) \\ &+ \Big[2\mathrm{Re}(\bar{u}_2 \Gamma) + \big| i(v_1 + v_2)\psi + i\eta_3 e^{i\varphi}\psi + \Gamma \big|^2 \Big] \times \Big(u_2 + i(v_1 + v_2)\psi + i\eta_3 e^{i\varphi}\psi + \Gamma \Big). \end{aligned} \tag{5.7}$$

Whence u is a solution to problem (3.3)-(3.4) if and only if

$$
\begin{aligned}
& i(v_1+v_2)S_0[\psi] \,+\, 2i\,\nabla(v_1+v_2)\cdot\nabla\psi \,+\, i\Big[1-\tilde{V}-|u_2|^2\Big](v_1+v_2)\psi \\
& + iS_0[v_1+v_2]\psi \,-\, 2\mathrm{Re}\big(\bar{u}_2 i(v_1+v_2)\psi\big)u_2 \\
& + i\eta_3 e^{i\varphi}S_0[\psi] \,+\, 2i\,\nabla\Big[\eta_3 e^{i\varphi}\Big]\cdot\nabla\psi \,+\, i\eta_3 e^{i\varphi}\Big[1-\tilde{V}-|u_2|^2\Big]\psi \\
& + iS_0\Big[\eta_3 e^{i\varphi}\Big]\psi \,-\, 2\eta_3\mathrm{Re}\big(\bar{u}_2 i e^{i\varphi}\psi\big)u_2 \,=\, -\,\mathbb{E}\,+\,N,
\end{aligned}
\tag{5.8}
$$

where the error term $\mathbb{E}$ is defined in (4.63) and N is the nonlinear operator defined by

$$
N \,=\, -\,S_0\big[\Gamma\big] \,-\, \big(1-\tilde{V}-|u_2|^2\big)\Gamma \,-\, \mathfrak{N}.
$$

We shall explicitly write the equation in suitable local forms and then analyze the property of the corresponding linear operators, which will be done in the following.

By recalling the notations in subsection 3.1, and also D_1, D_2, D_3 in (4.24), we divide further D_2 into small parts, see Figure 1 and Figure 2

$$
\begin{aligned}
& D_{2,1} \,\equiv\, \big\{(x_1,x_2):\hat{\ell}_1<1\big\}, \qquad D_{2,4} \,\equiv\, \big\{(x_1,x_2):\hat{\ell}_4<1\big\}, \\
& D_{2,2} \,\equiv\, \big\{(x_1,x_2):\hat{\ell}_2<1\big\}, \qquad D_{2,3} \,\equiv\, \big\{(x_1,x_2):\hat{\ell}_3<1\big\}, \\
& D_{2,5} \,\equiv\, \left\{(x_1,x_2):\sqrt{(x_1-\hat{r}_0)^2+x_2^2}\,<\,\varepsilon^{-\lambda_1}\right\}\setminus\Big(D_{2,1}\cup D_{2,2}\Big), \\
& D_{2,6} \,\equiv\, \left\{(x_1,x_2):\sqrt{(x_1+\hat{r}_0)^2+x_2^2}\,<\,\varepsilon^{-\lambda_1}\right\}\setminus\Big(D_{2,4}\cup D_{2,3}\Big).
\end{aligned}
\tag{5.9}
$$

We also divide D_1 and D_3 into two components

$$
\begin{aligned}
& D_{1,1} \,\equiv\, \Big\{\,(x_1,x_2):|x|<\hat{r}_2-\frac{\tau_1}{\varepsilon}\,\Big\}\setminus D_2, \\
& D_{1,2} \,\equiv\, \Big\{(x_1,x_2):\hat{r}_2-\frac{\tau_1}{\varepsilon}<|x|<\hat{r}_2-\varepsilon^{-\lambda_2}\,\Big\}, \\
& D_{3,1} \,\equiv\, \Big\{\,(x_1,x_2):\hat{r}_2-\varepsilon^{-\lambda_2}<|x|<\hat{r}_2+\frac{\tau_2}{\varepsilon}\,\Big\}, \\
& D_{3,2} \,\equiv\, \Big\{(x_1,x_2):|x|>\hat{r}_2+\frac{\tau_2}{\varepsilon}\,\Big\}.
\end{aligned}
\tag{5.10}
$$

Here τ_1 and τ_2 are given in the assumptions (**A3**) and (**A4**).

In the region D_1 far from the vortex core region, directly from the form of the ansatz $u=u_2e^{i\psi}$ with the approximation as

$$
u_2(x_1,x_2) \,=\, \sqrt{1-\tilde{V}}\,\eta_1\,e^{i(\varphi_0^j+\varphi_1^j)} \,+\, \rho(\hat{\ell}_1)\rho(\hat{\ell}_2)\rho(\hat{\ell}_3)\rho(\hat{\ell}_4)\,\eta_2\,e^{i(\varphi_0^j+\varphi_1^j)} \,+\, \eta_3\,\hat{q}\,e^{i(\varphi_0^j+\varphi_1^j)}
$$

we see that the equation takes the simple form

$$
\begin{aligned}
L_1(\psi) \,&\equiv\, \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi+2\frac{\nabla u_2}{u_2}\cdot\nabla\psi-2i|u_2|^2\psi_2 \\
&=\, E_1\,-\,i(\nabla\psi)^2\,+\,i|u_2|^2\big(1-e^{-2\psi_2}+2\psi_2\big),
\end{aligned}
$$

where $E_1=i\,\mathbb{E}/u_2$. We intend next to describe in more accurate form the equation above. Let us also write

$$
u_2=e^{i\varphi}\beta_1 \quad \text{with } \beta_1=\sqrt{1-\tilde{V}}\,\eta_1\,+\,\rho(\hat{\ell}_1)\rho(\hat{\ell}_2)\rho(\hat{\ell}_3)\rho(\hat{\ell}_4)\,\eta_2\,+\,\hat{q}\,\eta_3.
$$

In the region $D_{1,1}$, there holds

$$
u_2 \,=\, \beta_1 e^{i\varphi}=\sqrt{1-\tilde{V}}\,e^{i\varphi}\,\eta_1\,+\,\rho(\hat{\ell}_1)\rho(\hat{\ell}_2)\rho(\hat{\ell}_3)\rho(\hat{\ell}_4)\,e^{i\varphi}\,\eta_2.
\tag{5.11}
$$

For $x \in D_{1,1}$, by the definitions of the cut-off functions η_1 and η_2 in (4.57), we have $\eta_1 = 1 - \eta_2$. Moreover, there exists a positive constant c_5 such that, for $x \in D_{1,1}$,

$$\rho(\hat{\ell}_1)\rho(\hat{\ell}_2)\rho(\hat{\ell}_3)\rho(\hat{\ell}_4) \geq \sqrt{c_5}.$$

Hence, by using the assumption (**A**3), we have

$$|u_2|^2 \geq \min\{1 - \tilde{V}, c_5\} \geq \min\{c_1, c_5\} \equiv c_6 > 0. \tag{5.12}$$

Direct computation also gives that

$$\begin{aligned} 2\frac{\nabla u_2}{u_2} \cdot \nabla\psi &= \frac{2}{\beta_1} \nabla \beta_1 \cdot \nabla\psi_1 - 2 \nabla \varphi \cdot \nabla\psi_2 + i\frac{2}{\beta_1} \nabla \beta_1 \cdot \nabla\psi_2 + 2i \nabla \varphi \cdot \nabla\psi_1 \\ &= (A_1, 0) \cdot \nabla\psi_1 - (A_2, B_2) \cdot \nabla\psi_2 + i(A_1, 0) \cdot \nabla\psi_2 + i(A_2, B_2) \cdot \nabla\psi_1, \end{aligned}$$

where $A_1 = O(\varepsilon|\log\varepsilon|)$, $A_2 = O(\varepsilon)$, $B_2 = O(\varepsilon)$. The equations become

$$\tilde{L}_{1,1}(\psi_1) = \tilde{E}_{1,1} + \tilde{N}_{1,1}, \qquad \bar{L}_{1,1}(\psi_2) = \bar{E}_{1,1} + \bar{N}_{1,1}. \tag{5.13}$$

In the above, we have denoted the linear operators by

$$\begin{aligned} \tilde{L}_{1,1}(\psi_1) &\equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 + \frac{2}{\beta_1} \nabla \beta_1 \cdot \nabla\psi_1, \\ \bar{L}_{1,1}(\psi_2) &\equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 - 2|u_2|^2\psi_2 + \frac{2}{\beta_1} \nabla \beta_1 \cdot \nabla\psi_2. \end{aligned} \tag{5.14}$$

The nonlinear operators are

$$\begin{aligned} \tilde{N}_{1,1} &= -2 \nabla \varphi \cdot \nabla\psi_2 + 2 \nabla \psi_1 \cdot \nabla\psi_2, \\ \bar{N}_{1,1} &= 2 \nabla \varphi \cdot \nabla\psi_1 + |u_2|^2\big(1 - e^{-2\psi_2} + 2\psi_2\big) + |\nabla \psi_1|^2 - |\nabla \psi_2|^2. \end{aligned}$$

For $x \in D_{1,2}$, there holds,

$$u_2 = \beta_1 e^{i\varphi} = \sqrt{1 - \tilde{V}}\, e^{i\varphi}\, \eta_1 + \hat{q}\, e^{i\varphi}\, \eta_3. \tag{5.15}$$

For $x \in D_{1,2}$, by the definitions of the cut-off functions η_1 and η_3 in (4.57), we have $\eta_1 = 1 - \eta_3$. Moreover, by recalling the definition of $\hat{q}$ in (4.53) and the asymptotic behavior of q in Lemma 2.4,

$$\hat{q}(x_1, x_2) = \delta_\varepsilon^{1/3} q\Big(\delta_\varepsilon^{1/3}(\hat{\ell} - \hat{r}_2)\Big) \geq C\varepsilon^{(1-\lambda_2)/2} > 0.$$

Hence, by using the assumption (**A**3), we have

$$|u_2|^2 \geq c_7\, \varepsilon^{1-\lambda_2}, \quad x \in D_{1,2} > 0. \tag{5.16}$$

The equations become

$$\tilde{L}_{1,2}(\psi_1) = \tilde{E}_{1,2} + \tilde{N}_{1,2}, \qquad \bar{L}_{1,2}(\psi_2) = \bar{E}_{1,2} + \bar{N}_{1,2}. \tag{5.17}$$

In the above, we have denoted the linear operators by

$$\begin{aligned} \tilde{L}_{1,2}(\psi_1) &\equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 + \frac{2}{\beta_1} \nabla \beta_1 \cdot \nabla\psi_1, \\ \bar{L}_{1,2}(\psi_2) &\equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 - 2|u_2|^2\psi_2 + \frac{2}{\beta_1} \nabla \beta_1 \cdot \nabla\psi_2. \end{aligned} \tag{5.18}$$

The nonlinear operators are

$$\begin{aligned} \tilde{N}_{1,2} &= -2 \nabla \varphi \cdot \nabla\psi_2 + 2 \nabla \psi_1 \cdot \nabla\psi_2, \\ \bar{N}_{1,2} &= 2 \nabla \varphi \cdot \nabla\psi_1 + |u_2|^2\big(1 - e^{-2\psi_2} + 2\psi_2\big) + |\nabla \psi_1|^2 - |\nabla \psi_2|^2. \end{aligned}$$

Consider the linearization of the problem on the vortex-core region $\cup_{i=1}^4 D_{2,i}$.

$$\beta_{2,i} = \frac{U_2}{w_i}, \quad i = 1, 2, 3, 4, \tag{5.19}$$

where U_2 and w_i are defined in (4.26) and (4.25). Here we only argue in the region $D_{2,1} = \big\{(x_1, x_2) : \hat{\ell}_1 < 1\}$. It is more convenient to do this in the translated variable $(s_1, s_2) = (x_1 - \hat{f}_1, x_2 - \hat{d}_1)$. Now the term ψ is small, however possibly unbounded near the vortex. Whence, in the sequel, by setting

$$\tilde{\phi} = iv_2\psi \quad \text{with } \psi = \psi_1 + i\psi_2, \tag{5.20}$$

we shall require that $\tilde{\phi}$ is bounded (and smooth) near the vortices. We shall write the equation in term of a type of the function $\tilde{\phi}$ for $\hat{\ell} < \delta/\varepsilon$. In the region $D_{2,1}$, let us write u_2, i.e. v_2, as the form

$$v_2 = \beta_{2,1} U_0 \quad \text{ with } \beta_{2,1} = \rho(\hat{\ell}_2)\rho(\hat{\ell}_3)\rho(\hat{\ell}_4)e^{i(\varphi_0^j - \varphi_{01}) + i\varphi_1^j}, \tag{5.21}$$

where U_0, φ_0^j, φ_{01} and φ_1^j are defined in (2.1), (4.27), (3.10) and (4.44). We define the function

$$\phi(s) = iU_0\psi \quad \text{for } |s| < \delta/\varepsilon, \tag{5.22}$$

namely

$$\tilde{\phi} = \beta_{2,1}\phi. \tag{5.23}$$

Hence, in the translated variable, the ansatz becomes in this region

$$u_2 \;=\; \beta_{2,1}(s)U_0 \;+\; \beta_{2,1}(s)\phi \;+\; (1-\chi)\beta_{2,1}(s)U_0\Big(e^{\phi/U_0} - 1 - \frac{\phi}{U_0}\Big). \tag{5.24}$$

We also call $\Gamma_{2,1} = (1-\chi)U_0\Big(e^{\phi/U_0} - 1 - \frac{\phi}{U_0}\Big)$. The support of this function is contained in set $|s| > 1$. In this vortex-core region, the problem, written in (s_1, s_2) coordinates, can be stated as

$$L_{2,1}(\phi) = E_{2,1} + N_{2,1}. \tag{5.25}$$

Let us consider the linear operator defined in the following way: for ϕ and ψ linked through formula (5.22) we set

$$\begin{aligned} L_{2,1}(\phi) \;=& L_0(\phi) \;+\; \frac{1}{s_1 + \hat{f}_1}\frac{\partial}{\partial s_1}\phi \;+\; 2\big(1 - |\beta_{2,1}|^2\big)\mathrm{Re}\big(\bar{U}_0\phi\big)U_0 \\ &+ \left[\varepsilon\frac{\partial \tilde{V}}{\partial r}\Big|_{(r_0+\vartheta s_1, 0)} + 1 - |\beta_{2,1}|^2\right]\phi \;+\; 2\frac{\nabla\beta_{2,1}}{\beta_{2,1}}\cdot\nabla\phi \;+\; \chi\frac{E_{2,1}}{U_0}\phi, \end{aligned} \tag{5.26}$$

where ϑ is a small constant. Here we also have defined L_0 as

$$L_0(\phi) = \Big(\frac{\partial^2}{\partial s_1^2} + \frac{\partial^2}{\partial s_2^2}\Big)\phi + (1 - |\rho|^2)\phi - 2\mathrm{Re}\big(\bar{U}_0\phi\big)U_0.$$

Here, by writing the error $\mathbb{E}$ in the translated variable s, the error $E_{2,1}$ is given by

$$E_{2,1} = \mathbb{E}/\beta_{2,1}. \tag{5.27}$$

Observe that, in the region $D_{2,1}$, the error $E_{2,1}$ takes the expression

$$\begin{aligned}
E_{2,1} \,=\, & \rho(\hat\ell_1)\, e^{i\varphi_{01}} \Bigg[\frac{x_1-\hat f_1}{x_1\hat\ell_1}\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)} \,+\, \frac{x_1-\hat f_2}{x_1\hat\ell_2}\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)} \,-\, \varepsilon\frac{\partial\tilde V}{\partial r}\Big|_{(r_0,0)}(x_1-\hat r_0) \\
& \qquad -\, \frac{\varepsilon^2}{2}\frac{\partial^2\tilde V}{\partial r^2}\Big|_{(r_0,0)}(x_1-\hat r_0)^2 \,-\, \frac{\varepsilon^2}{2}\frac{\partial^2\tilde V}{\partial y_3^2}\Big|_{(r_0,0)} x_2^2 \Bigg] \\
& +\, 2\rho(\hat\ell_1)\, e^{i\varphi_{01}} \frac{(x_1-\hat f_1)(x_1-\hat f_2)+(x_2-\hat d_1)(x_2-\hat d_2)}{\hat\ell_1\hat\ell_2}\,\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)}\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)} \\
& -\, 2(-1)^j\rho(\hat\ell_1)\, e^{i\varphi_{01}} \frac{(x_2-\hat d_1)(x_2-\hat d_2)+(x_1-\hat f_1)(x_1-\hat f_2)}{(\hat\ell_1)^2(\hat\ell_2)^2} \\
& +\, 2i(-1)^j\rho(\hat\ell_1)\, e^{i\varphi_{01}} \frac{-(x_2-\hat d_2)(x_1-\hat f_1)+(x_1-\hat f_2)(x_2-\hat d_1)}{\hat\ell_1(\hat\ell_2)^2}\,\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)} \\
& +\, 2i\rho(\hat\ell_1)\, e^{i\varphi_{01}} \frac{-(x_2-\hat d_1)(x_1-\hat f_2)+(x_1-\hat f_1)(x_2-\hat d_2)}{\hat\ell_2(\hat\ell_1)^2}\,\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)} \\
& +\, \rho(\hat\ell_1)e^{i\varphi_{01}}\frac{1}{\hat f_1}\log\hat f_1\Big[\, i\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)}\frac{x_2-\hat d_1}{\hat\ell_1} + i\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)}\frac{x_2-\hat d_2}{\hat\ell_2} - \frac{x_1-\hat f_1}{\hat\ell_1^2} - (-1)^j\frac{x_1-\hat f_2}{\hat\ell_2^2}\,\Big] \\
& +\, \rho(\hat\ell_1)e^{i\varphi_{01}}\frac{1}{\hat f_2}\log\hat f_2\Big[\, i\frac{\rho'(\hat\ell_1)}{\rho(\hat\ell_1)}\frac{x_2-\hat d_1}{\hat\ell_1} + i\frac{\rho'(\hat\ell_2)}{\rho(\hat\ell_2)}\frac{x_2-\hat d_2}{\hat\ell_2} - \frac{x_1-\hat f_1}{\hat\ell_1^2} - \frac{x_1-\hat f_2}{\hat\ell_2^2}\,\Big] \\
& +\, O(\varepsilon\log\hat\ell_1+\varepsilon\log\hat\ell_2),
\end{aligned} \tag{5.28}$$

while the nonlinear term is given by

$$\begin{aligned}
N_{2,1}(\phi) = & -\,\frac{\triangle(\beta_{2,1}\,\Gamma_{2,1})}{\beta_{2,1}} \,+\, \Big(1-\tilde V-|U_0|^2\Big)\Gamma_{2,1} \,-\, 2|\beta_{2,1}|^2\,\mathrm{Re}(\bar U_0\phi)\big(\phi+\Gamma_{2,1}\big) \\
& -\,\Big(2|\beta_{2,1}|^2\mathrm{Re}\big(\bar U_0\Gamma_{2,1}\big)+|\beta_{2,1}|^2\big|\phi+\Gamma_{2,1}\big|^2\Big)\big(U_0+\phi+\Gamma_{2,1}\big) \,+\, (\chi-1)\frac{E_{2,1}}{U_0}\phi.
\end{aligned} \tag{5.29}$$

Taking into account to the explicit form of the function $\beta_{2,1}$ we get

$$\triangledown\beta_{2,1} = O(\varepsilon), \quad \triangle\beta_{2,1} = O(\varepsilon^2), \quad |\beta_{2,1}| \sim 1+O(\varepsilon^2), \tag{5.30}$$

provided that $|s|<\delta/\varepsilon$. With this in mind, we see that the linear operator is a small perturbation of L_0.

In the region $D_{2,5}$ far from the vortex core, directly from the form of the ansatz $u=\big(1-\chi\big)u_2e^{i\psi}$, we see that, for $\hat\ell_1>2$, the equation takes the simple form

$$\begin{aligned}
L_{2,5}(\psi) \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi + 2\frac{\triangledown u_2}{u_2}\cdot\triangledown\psi - 2i|u_2|^2\psi_2 \\
=\, & E_{2,5} \,-\, i(\triangledown\psi)^2 \,+\, i|u_2|^2\big(1-e^{-2\psi_2}+2\psi_2\big),
\end{aligned} \tag{5.31}$$

where $E_{2,5}=i\mathbb{E}/u_2$. We intend next to describe in more accurate form the equation above. As before, let us also write

$$u_2=\beta U_0 \quad \text{ with } \beta = \,\rho(\hat\ell_2)e^{-i\varphi_0^{j-}+i\varphi_1^j}. \tag{5.32}$$

For $\hat\ell_1<\varepsilon^{-\lambda_1}$, there are two real functions A and B such that

$$\beta = e^{iA+B}, \tag{5.33}$$

furthermore, a direct computation shows that, in this region, there holds

$$\triangledown A = O(\varepsilon), \qquad \triangle A = O(\varepsilon^2), \qquad \triangledown B = O(\varepsilon^3), \qquad \triangle B = O(\varepsilon^4). \tag{5.34}$$

The equations become

$$\tilde{L}_{2,5}(\psi_1) = \tilde{E}_{2,5} + \tilde{N}_{2,5}, \qquad \bar{L}_{2,5}(\psi_2) = \bar{E}_{2,5} + \bar{N}_{2,5}. \tag{5.35}$$

In the above, we have denoted the linear operators by

$$\tilde{L}_{2,5}(\psi_1) \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 + \Big(\bigtriangledown B + \frac{\rho'(\hat{\ell}_1)}{\rho(\hat{\ell}_1)}\frac{s}{\hat{\ell}_1}\Big)\cdot\bigtriangledown\psi_1,$$

$$\bar{L}_{2,5}(\psi_2) \equiv \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 - 2|u_2|^2\psi_2 + 2\Big(\bigtriangledown B + \frac{\rho'(\hat{\ell}_1)}{\rho(\hat{\ell}_1)}\frac{s}{\hat{\ell}_1}\Big)\cdot\bigtriangledown\psi_2,$$

where have used $s = (x_1 - \hat{d}_1, x_2)$. The nonlinear operators are

$$\tilde{N}_{2,5} = -2(\bigtriangledown A + \bigtriangledown\varphi_{01})\cdot\bigtriangledown\psi_2 + 2\bigtriangledown\psi_1\bigtriangledown\psi_2,$$

$$\bar{N}_{2,5} = -2(\bigtriangledown A + \bigtriangledown\varphi_{01})\cdot\bigtriangledown\psi_1 + |u_2|^2\big(1 - e^{-2\psi_2} + 2\psi_2\big) + |\bigtriangledown\psi_1|^2 - |\bigtriangledown\psi_2|^2.$$

In the region $D_{3,1}$, the approximation takes the form

$$u_2 = \hat{q}\, e^{i\varphi}.$$

We write the ansatz as

$$u = u_2 + ie^{i\varphi}\psi + \Gamma_{3,1}, \tag{5.36}$$

where $\Gamma_{3,1}$ is defined as

$$\Gamma_{3,1} = i\big(\eta_3 - 1\big)e^{i\varphi}\psi. \tag{5.37}$$

The equation becomes

$$\begin{aligned} L_{3,1}[\psi] &\equiv S_0[\psi] + 2i\bigtriangledown\varphi\cdot\bigtriangledown\psi - |\bigtriangledown\varphi|^2\psi + iS_0[\varphi]\psi \\ &\quad + \Big(1 - \tilde{V} - |u_2|^2\Big)\psi + 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2 \\ &= E_{3,1} + N_{3,1}, \end{aligned}$$

where $E_{3,1} = ie^{-i\varphi}\mathbb{E}$. The nonlinear operator is defined by

$$\begin{aligned} N_{3,1}(\psi) =& \, ie^{-i\varphi}\Big[\triangle\Gamma_{3,1} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Gamma_{3,1} + \big(1 - \tilde{V} - |u_2|^2\big)\Gamma_{3,1}\Big] \\ & - ie^{-i\varphi}\Big[2\mathrm{Re}(\bar{u}_2\Gamma_{3,1}) - |ie^{i\varphi}\psi + \Gamma_{3,1}|^2\Big](u_2 + ie^{i\varphi}\psi + \Gamma_{3,1}) \\ & - 2ie^{-i\varphi}\mathrm{Re}(\bar{u}_2 ie^{i\varphi}\psi)(ie^{i\varphi}\psi + \Gamma_{3,1}). \end{aligned}$$

More precisely, in the region $D_{3,1}$, the linear operator $L_{3,1}$ is defined as

$$\begin{aligned} L_{3,1}[\psi] =& \, S_0[\psi] - \big(\delta_\varepsilon(\hat{\ell} - \hat{r}_2) + \hat{q}^2\big)\psi + 2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2 \\ & + \Big[1 - \tilde{V} + \delta_\varepsilon(\hat{\ell} - \hat{r}_2)\Big]\psi + 2i\bigtriangledown\varphi\cdot\bigtriangledown\psi + S_0[\varphi]\psi - |\bigtriangledown\varphi|^2\psi. \end{aligned}$$

where we have used the definition of $\hat{q}$ in (4.53). We shall analyze other terms in the linear operator $L_{3,1}$. It is obvious that

$$2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2 ie^{i\varphi}\psi\big)u_2 = -2i\,\hat{q}^2\,\psi_2. \tag{5.38}$$

Whence we decompose the equation in the form

$$\begin{aligned} \tilde{L}_{3,1}[\psi_1] \equiv& \Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)\psi_1 - \big(\delta_\varepsilon(\hat{\ell} - \hat{r}_2) + \hat{q}^2\big)\psi_1 + \Big[1 - \tilde{V} + \delta_\varepsilon(\hat{\ell} - \hat{r}_2)\Big]\psi_1 \\ & + \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_1 - 2\bigtriangledown\varphi\cdot\bigtriangledown\psi_2 + S_0[\varphi]\psi_1 - |\bigtriangledown\varphi|^2\psi_1 \\ =& \, \tilde{E}_{3,1} + \tilde{N}_{3,1}, \end{aligned} \tag{5.39}$$

$$\begin{aligned}\bar{L}_{3,1}[\psi_2] \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 \,-\, \big(\delta_\varepsilon(\hat{\ell}-\hat{r}_2)+3\hat{q}^2\big)\psi_2 \,+\, \Big[1-\tilde{V}+\delta_\varepsilon(\hat{\ell}-\hat{r}_2)\Big]\psi_2 \\ & +\, \frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 \,+\, 2\bigtriangledown\varphi\cdot\bigtriangledown\psi_1 \,+\, S_0[\varphi]\psi_2 \,-\, |\bigtriangledown\varphi|^2\psi_2 \\ =\, & \bar{E}_{3,1}+\bar{N}_{3,1}.\end{aligned} \tag{5.40}$$

If $\hat{r}_2-\varepsilon^{-\lambda_2}<|x|<\hat{r}_2+\tau_2/\varepsilon$, by using (4.50) and the assumption (**A3**), we then have

$$\Xi_{3,1}\,\equiv\,1-\tilde{V}+\delta_\varepsilon(\hat{\ell}-\hat{r}_2)\,=\,-\frac{\varepsilon^2}{2}\frac{\partial^2\tilde{V}}{\partial\ell^2}(\hat{\ell}-\hat{r}_2)^2\,+\,O\big(\varepsilon^3(\hat{\ell}-\hat{r}_2)^3\big)<0. \tag{5.41}$$

This is due to the facts that $\frac{\partial^2\tilde{V}}{\partial\ell^2}>0$ along the circle $\sqrt{r^2+y_3^2}=r_2$ and that we can choose τ_2 small enough. The other terms with φ_0^j are also lower order terms. Whence the linear operators $\tilde{L}_{3,1}$ and $\bar{L}_{3,1}$ are small perturbations of the following linear operators

$$\begin{aligned}L_{31*}[\psi_1]\equiv\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1\,-\,\big(\delta_\varepsilon(\hat{\ell}-\hat{r}_2)+\hat{q}^2\big)\psi_1,\\ L_{31**}[\psi_2]\equiv\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2\,-\,\big(\delta_\varepsilon(\hat{\ell}-\hat{r}_2)+3\hat{q}^2\big)\psi_2.\end{aligned} \tag{5.42}$$

In the region $D_{3,2}$ the approximation takes the form

$$u_2\,=\,\hat{q}(x_1,x_2)e^{i\varphi},$$

and the ansatz is

$$u\,=\,u_2\,+\,ie^{i\varphi}\psi.$$

The equation becomes

$$\begin{aligned}L_{3,2}[\psi]\,\equiv\, & S_0[\psi]\,+\,\Big(1-\tilde{V}\Big)\psi\,-\,|u_2|^2\psi\,+\,2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2ie^{i\varphi}\psi\big)u_2\\ & -\,|\bigtriangledown\varphi|^2\psi\,+\,iS_0[\varphi]\psi\,+\,2i\bigtriangledown\varphi\cdot\bigtriangledown\psi\\ =\, & E_{3,2}\,+\,N_{3,2},\end{aligned} \tag{5.43}$$

where $E_{3,2}=ie^{-i\varphi}\mathbb{E}$. The nonlinear operator is defined by

$$N_{3,2}(\psi)\,=\,-\,ie^{-i\varphi}(u_2+ie^{i\varphi}\psi)|\psi|^2\,+\,2i\mathrm{Re}(\bar{u}_2ie^{i\varphi}\psi)\psi.$$

More precisely, for other term, we have

$$-|u_2|^2\psi\,+\,2ie^{-i\varphi}\mathrm{Re}\big(\bar{u}_2ie^{i\varphi}\psi\big)u_2\,=\,-\hat{q}^2\,\psi_1\,-\,3i\hat{q}^2\,\psi_2.$$

The equation can be decomposed in the form

$$\begin{aligned}\tilde{L}_{3,2}[\psi_1]\,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1+\Big(1-\tilde{V}\Big)\psi_1-\hat{q}\,\psi_1\\ & +\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2-|\bigtriangledown\varphi|^2\psi_1+iS_0[\varphi]\psi_1-2\bigtriangledown\varphi\cdot\bigtriangledown\psi_2\\ =\, & \tilde{E}_{3,2}\,+\,\tilde{N}_{3,2},\end{aligned} \tag{5.44}$$

$$\begin{aligned}\bar{L}_{3,2}[\psi_2]\,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2+\Big(1-\tilde{V}\Big)\psi_2-\hat{q}\,\psi_2\\ & +\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2-|\bigtriangledown\varphi|^2\psi_2+iS_0[\varphi]\psi_2+2\bigtriangledown\varphi\cdot\bigtriangledown\psi_1\\ =\, & \bar{E}_{3,2}\,+\,\bar{N}_{3,2}.\end{aligned} \tag{5.45}$$

The assumption (**A4**) implies that, for any sufficiently small ε there holds

$$\Xi_{3,2} \ = \ 1-\tilde{V} < -c_2 \quad \text{for } |x| > \hat{r}_2 + \tau_2/\varepsilon. \tag{5.46}$$

The other terms with φ_0^j are lower order terms. From the asymptotic properties of q in Lemma 2.4, $\hat{q}\,\psi_2$ and $\hat{q}\,\psi_1$ are also lower order term. Whence the linear operators $\tilde{L}_{3,2}$ and $\bar{L}_{3,2}$ are small perturbations of the following linear operator

$$L_{32*}[\tilde{\psi}] \equiv \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\tilde{\psi} \ + \ \big(1-\tilde{V}\big)\tilde{\psi}. \tag{5.47}$$

Let $\tilde{\chi}$ be the cut-off function defined in (5.2) and fix a large positive constant $R>0$. By recalling the definition of $\beta_{2,i}$'s in (5.19), for the case of interaction of type I in (1.25) we now define

$$\begin{aligned} \Lambda_1 \ &\equiv \ \frac{\partial u_2}{\partial \hat{f}_1}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_1|/R\big)}{\beta_{2,1}} \ + \ \frac{\partial u_2}{\partial \hat{f}_4}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_4|/R\big)}{\beta_{2,4}}\,, \\ \Lambda_2 \ &\equiv \ \frac{\partial u_2}{\partial \hat{f}_2}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_2|/R\big)}{\beta_{2,2}} \ + \ \frac{\partial u_2}{\partial \hat{f}_3}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_3|/R\big)}{\beta_{2,3}}\,. \end{aligned} \tag{5.48}$$

Similarly, for the case of interaction of type II in (1.26) we choose

$$\begin{aligned} \Lambda_1 \ \equiv\ &\frac{\partial u_2}{\partial \hat{f}_1}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_1|/R\big)}{\beta_{2,1}} \ + \ \frac{\partial u_2}{\partial \hat{f}_4}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_4|/R\big)}{\beta_{2,4}} \\ &+\ \frac{\partial u_2}{\partial \hat{f}_2}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_2|/R\big)}{\beta_{2,2}} \ + \ \frac{\partial u_2}{\partial \hat{f}_3}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_3|/R\big)}{\beta_{2,3}}\,, \\ \Lambda_2 \ \equiv\ &\frac{\partial u_2}{\partial \hat{d}_1}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_1|/R\big)}{\beta_{2,1}} \ + \ \frac{\partial u_2}{\partial \hat{d}_4}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_4|/R\big)}{\beta_{2,4}} \\ &+\ \frac{\partial u_2}{\partial \hat{d}_2}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_2|/R\big)}{\beta_{2,2}} \ + \ \frac{\partial u_2}{\partial \hat{d}_3}\cdot\frac{\tilde{\chi}\big(|x-\vec{e}_3|/R\big)}{\beta_{2,3}}\,. \end{aligned} \tag{5.49}$$

In summary, for any given parameters $\hat{f}_1,\dots,\hat{f}_4$ and $\hat{d}_1,\dots,\hat{d}_4$ with requirements in (3.15) or (3.16), we want to solve the projected equation for ψ satisfying the symmetry (5.5)

$$\mathcal{L}(\psi)+\mathcal{N}(\psi)=\mathcal{E}+c_1\Lambda_1+c_2\Lambda_2, \qquad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}\Lambda_1 \ = \ 0, \qquad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}\Lambda_2 \ = \ 0, \tag{5.50}$$

where have denoted

$$\mathcal{L}(\psi)=L_1(\phi) \quad \text{in } D_1, \qquad \mathcal{L}(\psi)=L_{2,m}(\psi) \quad \text{in } D_{2,m} \text{ for } m=1,2,3,4,5,6$$

$$\mathcal{L}(\psi)=L_{3,1}(\psi) \quad \text{in } D_{3,1}, \qquad \mathcal{L}(\psi)=L_{3,2}(\psi) \quad \text{in } D_{3,2},$$

with the relation

$$\phi=iu_2\psi \quad \text{in } D_2. \tag{5.51}$$

As we have stated, the nonlinear operator $\mathcal{N}$ and the error term $\mathcal{E}$ also have suitable local forms in different regions.

6. The resolution of the Projected Nonlinear Problem

6.1. **The Linear resolution theory.** The main object is to consider the resolution of the linear part of the projected problem in previous section, which was stated in Lemma 6.2.

For that purpose, we shall firs get a priori estimates expressed in suitable norms. By recalling the norm $\|\cdot\|_{**}$ defined in (4.68), for fixed small positive numbers $0<\sigma<1$, $0<\gamma<1$, we define

$$\begin{aligned}||\psi||_* \equiv & \sum_{m=1}^{4}||\phi||_{W^{2,p}(\hat\ell_m<3)} + \sum_{m=1}^{4}\Big[||\hat\ell_m^{\sigma}\psi_1||_{L^\infty(\tilde D\cup D_{1,2})}+||\hat\ell_m^{1+\sigma}\bigtriangledown\psi_1||_{L^\infty(\tilde D\cup D_{1,2})}\Big]\\ &+\sum_{m=1}^{4}\Big[||\hat\ell_m^{1+\sigma}\psi_2||_{L^\infty(\tilde D)}+||\hat\ell_m^{2+\sigma}\bigtriangledown\psi_2||_{L^\infty(\tilde D)}\Big]\\ &+\sum_{m=1}^{4}\Big[||\hat\ell_m^{\sigma}\psi_2||_{L^\infty(D_{1,2})}+||\hat\ell_m^{1+\sigma}\bigtriangledown\psi_2||_{L^\infty(D_{1,2})}\Big] + ||\psi||_{W^{2,p}(D_3)},\end{aligned}$$

where we have used the relation $\phi = iu_2\psi$ and the region $\tilde D$ is defined in (4.69). We then consider the following problem: for given h, finding ψ with the symmetry in (5.5)

$$\mathcal{L}(\psi)=h \quad \text{in } \mathbb{R}^2,\quad \mathrm{Re}\int_{\mathbb{R}^2}\bar\phi\Lambda_1 = \mathrm{Re}\int_{\mathbb{R}^2}\bar\phi\Lambda_2 0 \qquad \text{with } \phi=iu_2\psi. \tag{6.1}$$

Lemma 6.1. *There exists a constant C, depending on σ only, such that for all ε sufficiently small, and any solution of (6.1), we have the estimate*

$$||\psi||_* \le C\,||h||_{**}\,.$$

Proof. We prove the result by contradiction. Suppose that there is a sequence of $\varepsilon=\varepsilon_n$, functions ψ^n, h_n which satisfy (6.1) with

$$||\psi^n||_*=1,\quad ||h_n||_{**}=o(1).$$

Before any further argument, by the symmetry assumption (5.5) for $\psi=\psi_1+i\psi_2$, we have

$$\psi_1(x_1,-x_2)=-\psi_1(x_1,x_2),\quad \psi_1(-x_1,x_2)=\psi_1(x_1,x_2),$$
$$\psi_2(x_1,-x_2)=\psi_2(x_1,x_2),\quad \psi_2(-x_1,x_2)=\psi_2(x_1,x_2).$$

We may just need to consider the problem in $\mathbb{R}^2_+=\{(x_1,x_2): x_1>0\}$. Then we have

$$\mathrm{Re}\int_{\mathbb{R}^2}\bar\phi_n\Lambda_1 = 2\mathrm{Re}\int_{\mathbb{R}^2_+}\bar\phi_n\Lambda_1 = 0,$$
$$\mathrm{Re}\int_{\mathbb{R}^2}\bar\phi_n\Lambda_2 = 2\mathrm{Re}\int_{\mathbb{R}^2_+}\bar\phi_n\Lambda_2 = 0,$$

for any $\phi_n=iu_2\psi^n$. To get good estimate and then derive a contradiction, we will use suitable local forms of the linear operator $\mathcal{L}$ in different regions, which were stated in previous section. Hence we divide the proof into five parts.

Part 1. In the outer part D_1, we use the following barrier function

$$\mathcal{B}(x) = \mathcal{B}_1(x) + \mathcal{B}_2(x),$$

where

$$\mathcal{B}_1(x) = |\hat\ell_1|^\varrho|x_2|^\gamma + |\hat\ell_2|^\varrho|x_2|^\gamma + |\hat\ell_3|^\varrho|x_2|^\gamma + |\hat\ell_4|^\varrho|x_2|^\gamma,\qquad \mathcal{B}_2(x)=C_1(1+|x|^2)^{-\sigma/2},$$

where $\varrho+\gamma=-\sigma$, $0<\sigma<\gamma<1$, and C_1 is a large number depending on σ,ϱ,γ only. Trivial computations derive that

$$\triangle\mathcal{B}_1 \le -C\big(|\hat\ell_1|^2+|\hat\ell_2|^2+|\hat\ell_3|^2+|\hat\ell_4|^2\big)^{-1-\sigma/2},$$
$$\triangle\mathcal{B}_2+\frac{1}{x_1}\frac{\partial\mathcal{B}_2}{\partial x_1} \le -CC_1(1+|x|^2)^{-1-\sigma/2}.$$

On the other hand,

$$\frac{1}{x_1}\frac{\partial \mathcal{B}_1}{\partial x_1}\ \le\ \frac{|x_2|^\gamma}{x_1}\Big[\,|\hat\ell_1|^{\varrho-2}(x_1-\hat f_1)\ +\ |\hat\ell_2|^{\varrho-2}(x_1-\hat f_2)\ +\ |\hat\ell_3|^{\varrho-2}(x_1-\hat f_3)\ +\ |\hat\ell_4|^{\varrho-2}(x_1-\hat f_4)\Big].$$

Thus for x in the region

$$\check D\ \equiv\ \Big\{\ \big|x-(\hat r_0,0)\big|<c_\sigma\hat r_0\ \text{ or }\ \big|x-(-\hat r_0,0)\big|<c_\sigma\hat r_0\ \Big\},$$

where c_σ is small, by using the constraints in (3.15) or (3.16) we have

$$\frac{1}{x_1}\frac{\partial \mathcal{B}_1}{\partial x_1}\ \le\ Cc_\sigma\Big[\,|\hat\ell_1|^2+|\hat\ell_2|^2+|\hat\ell_3|^2+|\hat\ell_4|^2\,\Big]^{-1-\sigma/2}.$$

For $x\in\check D^c$, we have

$$\frac{1}{x_1}\frac{\partial \mathcal{B}_1}{\partial x_1}\le C(1+|x|^2)^{-1-\sigma/2}.$$

By choosing C_1 large, we have

$$\triangle\mathcal{B}+\frac{1}{x_1}\frac{\partial \mathcal{B}}{\partial x_1}\ \le\ -C\Big(\,|\hat\ell_1|^2+|\hat\ell_2|^2+|\hat\ell_3|^2+|\hat\ell_4|^2\,\Big)^{-1-\sigma/2}.$$

For the details of the above computations, the reader can refer to the proof of Lemma 7.2 in [60]. Now we can use this barrier function to derive the estimates for ψ on D_1. In fact, in D_1, there holds

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1\ +\ \frac{2}{\beta_1}\triangledown\beta_1\cdot\triangledown\psi_1\ =\ h_1.$$

By comparison principle on the set D_1, we obtain

$$|\psi_1|\ \le\ C\mathcal{B}\big(||h||_{**}+o(1)\big),\quad \forall\, x\in D_1.$$

On the other hand, in the region D_1 the equation for ψ_2 is

$$\Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2\ -\ 2|u_2|^2\psi_2\ +\ \frac{2}{\beta_1}\triangledown\beta_1\cdot\nabla\psi_2\ =\ h_2.$$

For $x\in D_{1,1}$, from (5.12), there holds $|u_2|^2>c_6>0$. By standard elliptic estimates we have

$$||\psi_2||_{L^\infty(D_{1,1})}\ \le\ C||\psi_2||_{L^\infty(\hat\ell_i=R)}(1+||\psi||_*)||h||_{**}(1+\hat\ell_1+\hat\ell_2+\hat\ell_3+\hat\ell_4)^{-1-\sigma},$$

$$\|\triangledown\psi_2\|_{L^\infty(D_{1,1})}\ \le\ C||\psi_2||_{L^\infty(\hat\ell_i=R)}(1+||\psi||_*)||h||_{**}(1+\hat\ell_1+\hat\ell_1+\hat\ell_3+\hat\ell_4)^{-2-\sigma}.$$

By the computations in the above, for $x\in D_{1,2}$ we have

$$\triangle\mathcal{B}\ +\ \frac{1}{x_1}\frac{\partial \mathcal{B}}{\partial x_1}\ -\ 2|u_2|^2\mathcal{B}\ \le\ -C\varepsilon^{1-\lambda_2}\big(|\hat\ell_1|^2+|\hat\ell_2|^2+|\hat\ell_3|^2+|\hat\ell_4|^2\big)^{-\sigma/2},$$

where we have used $|u_2|^2>c_7\,\varepsilon^{1-\lambda_2}$ for $x\in D_{1,2}$, (cf. (5.16)). By comparison principle on the set $D_{1,2}$, we obtain

$$|\psi_1|\ \le\ C\mathcal{B}\big(||h||_{**}+o(1)\big),\quad \forall\, x\in D_{1,2}.$$

Part 2. We here only derive the estimates in the vortex-core region $D_{2,1}$ near $\vec e_1$, see Figure 1 or Figure 2. Since $||h||_{**}=o(1),\psi^n\to\psi^0$, which satisfies

$$L_{2,1}(\psi^0)=0,\quad ||\psi^0||_*\le 1.$$

Whence, we get $L_0(\phi_0)=0$ with the operator L_0 defined in (2.8). By the nondegeneracy in Lemma 2.3, we have

$$\phi_0\ =\ \tilde c_1\frac{\partial U_0}{\partial s_1}\ +\ \tilde c_2\frac{\partial U_0}{\partial s_2}.$$

Observe that ϕ_0 inherits the symmetries of ϕ and hence

$$\phi_0(x_1,x_2) \,=\, \overline{\phi_0(x_1,-x_2)}, \quad \phi_0(x_1,x_2) \,=\, \phi_0(-x_1,x_2).$$

In the case of the interaction of type I, see (3.15), the symmetry of $\phi_0(x_1,x_2) = \phi_0(-x_1,x_2)$ is not preserved under the translation $s = x - \vec{e}_1$. Obviously, the term $\frac{\partial U_0}{\partial s_2}$ does not enjoy the symmetry $\phi_0(x_1,x_2) = \overline{\phi_0(x_1,-x_2)}$. This implies that $\phi_0 = \tilde{c}_1\frac{\partial U_0}{\partial s_1}$. On the other hand, taking a limit of the orthogonality condition

$$\mathrm{Re}\int_{\mathbb{R}^2_+}\bar{\phi}_n\Lambda_1 \,=\, 0,$$

we obtain

$$\mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}_0\frac{U_0}{\partial s_1} = 0,$$

and moreover $\tilde{c}_1 = 0$ and $\phi_0 = 0$.

In the case of the interaction of type II, see (3.16), the symmetries of

$$\phi_0(x_1,x_2) \,=\, \phi_0(-x_1,x_2), \quad \phi_0(x_1,x_2) \,=\, \overline{\phi_0(x_1,-x_2)},$$

are not preserved under the translation $s = x - \vec{e}_1$. Hence we can not use the symmetry to cancel the kernel of the operator L_0 in the coordinates (s_1,s_2). On the other hand, taking a limit of the orthogonality condition

$$\mathrm{Re}\int_{\mathbb{R}^2_+}\bar{\phi}_n\Lambda_1 \,=\, \mathrm{Re}\int_{\mathbb{R}^2_+}\bar{\phi}_n\Lambda_2 \,=\, 0,$$

we obtain

$$\mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}_0\frac{U_0}{\partial s_1} \,=\, \mathrm{Re}\int_{\mathbb{R}^2}\bar{\phi}_0\frac{U_0}{\partial s_2} \,=\, 0,$$

and moreover $\tilde{c}_1 = \tilde{c}_2 = 0$ and $\phi_0 = 0$.

Hence, for any fixed $R > 0$, there holds for $\phi = \phi_1 + i\phi_2$

$$||\phi_1||_{L^\infty(\hat{\ell}_1<R)} + ||\phi_2||_{L^\infty(\hat{\ell}_1<R)} + ||\bigtriangledown\phi_1||_{L^\infty(\hat{\ell}_1<R)} + ||\bigtriangledown\phi_2||_{L^\infty(\hat{\ell}_1<R)} \,=\, o(1).$$

Part 3. In the outer part $D_{2,6}\cup D_{2,5}$, we use the same barrier function $\mathcal{B}$ as Part 1. In the region $D_{2,6}\cup D_{2,5}$, we have

$$\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_1 \,+\, \Big(\bigtriangledown B + \frac{\rho'(\hat{\ell}_1)}{\rho(\hat{\ell}_1)}\frac{s}{\hat{\ell}_1}\Big)\cdot\bigtriangledown\psi_1 \,=\, h_1,$$

where have used $s = (x_1 - \hat{d}_1, x_2)$. By comparison principle on the set $D_{2,6}\cup D_{2,5}$, we obtain

$$|\psi_1| \le C\mathcal{B}\big(||h||_{**} + o(1)\big), \quad \forall x\in D_{2,6}\cup D_{2,5}.$$

On the other hand, the equation for ψ_2 is

$$\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{x_1}\frac{\partial}{\partial x_1}\Big)\psi_2 \,-\, 2|u_2|^2\psi_2 \,+\, 2\Big(\bigtriangledown B + \frac{\rho'(\hat{\ell}_1)}{\rho(\hat{\ell}_1)}\frac{s}{\hat{\ell}_1}\Big)\cdot\bigtriangledown\psi_2 \,=\, h_2.$$

For $x\in D_{2,6}\cup D_{2,5}$, there holds $|u_2|\sim 1$. By standard elliptic estimates we have

$$||\psi_2||_{L^\infty(\hat{\ell}_i>4)} \,\le\, C||\psi_2||_{L^\infty(\hat{\ell}_i=4)}(1+||\psi||_*)||h||_{**}(1+\hat{\ell}_1+\hat{\ell}_2+\hat{\ell}_3+\hat{\ell}_4)^{-1-\sigma},$$

$$|\bigtriangledown\psi_2| \,\le\, C||\psi_2||_{L^\infty(\hat{\ell}_i=R)}(1+||\psi||_*)||h||_{**}(1+\hat{\ell}_1+\hat{\ell}_2+\hat{\ell}_3+\hat{\ell}_4)^{-2-\sigma}.$$

Part 4. In the region $D_{3,1}$, we have

$$\begin{aligned}L_{3,1}[\psi_1] \equiv & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1 \,-\, \big(\delta_\varepsilon(\hat\ell-\hat r_2)+\hat q^2\big)\psi_1 \,+\, \Big[1-\tilde V+\delta_\varepsilon(\hat\ell-\hat r_2)\Big]\psi_1\\ &+\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_1 \,-\, 2\bigtriangledown\varphi_0^j\cdot\bigtriangledown\psi_2 \,+\, S_0[\varphi_0^j]\psi_1 \,-\, |\bigtriangledown\varphi_0^j|^2\psi_1\\ =&\, h_1,\end{aligned}$$

$$\begin{aligned}L_{3,1}[\psi_2] \equiv & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 \,-\, \big(\delta_\varepsilon(\hat\ell-\hat r_2)+3\hat q^2\big)\psi_2 \,+\, \Big[1-\tilde V+\delta_\varepsilon(\hat\ell-\hat r_2)\Big]\psi_2\\ &+\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 \,+\, 2\bigtriangledown\varphi_0^j\cdot\bigtriangledown\psi_1 \,+\, S_0[\varphi_0^j]\psi_2 \,-\, |\bigtriangledown\varphi_0^j|^2\psi_2\\ =&\, h_2.\end{aligned}$$

By defining a new translated variable $\lambda=\delta_\varepsilon^{1/3}(\hat\ell-\hat r_2)$, the linear operators L_{31*} and L_{31**} in (5.42) become

$$L_{31*}(\psi_{1*}) = \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_{1*} \,-\, \big(\lambda+q^2(\lambda)\big)\psi_{1*}\,,$$
$$L_{31**}(\psi_{2**}) = \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_{2**} \,-\, \big(\lambda+3q^2(\lambda)\big)\psi_{2**}\,.$$

From Lemma 2.4, $-q'(\lambda)>0$ for all $\lambda\in\mathbb{R}$, and $L_{31**}(-q')=0$. We apply the maximum principle to $-\psi_2/q'$ and then obtain

$$|\psi_2|\leq C|q'|(||h||_{**}+o(1)),\quad \forall\, x\in D_{3,1}.$$

On the other hand, $q(\lambda)>0$ for all $\lambda\in\mathbb{R}$, and $L_{31*}(q)=0$. We apply the maximum principle to ψ_1/q and then obtain

$$|\psi_1|\leq Cq(||h||_{**}+o(1)),\quad \forall\, x\in D_{3,1}.$$

Part 5. In $D_{3,2}$, we consider the problem

$$\begin{aligned}L_{3,2}[\psi_1] \,\equiv\, & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_1 \,+\, \big(1-\tilde V\big)\psi_1 \,-\, \hat q\,\psi_1\\ &+\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 \,-\, |\bigtriangledown\varphi_0^j|^2\psi_1 \,+\, iS_0[\varphi_0^j]\psi_1 \,-\, 2\bigtriangledown\varphi_0^j\cdot\bigtriangledown\psi_2\\ =&\, h_1,\end{aligned}$$

$$\begin{aligned}L_{3,2}[\psi_2] \equiv & \Big(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\Big)\psi_2 \,+\, \big(1-\tilde V\big)\psi_2 \,-\, \hat q(|x_1|)\psi_2\\ &+\,\frac{1}{x_1}\frac{\partial}{\partial x_1}\psi_2 \,-\, |\bigtriangledown\varphi_0^j|^2\psi_2 \,+\, iS_0[\varphi_0^j]\psi_2 \,+\, 2\bigtriangledown\varphi_0^j\cdot\bigtriangledown\psi_1\\ =&\, h_2.\end{aligned}$$

By using the properties of $\Xi_{3,2}$ in (5.46), i.e.

$$\Sigma_{3,2}=\big(1-\tilde V\big)<-c_2\quad\text{in } D_{3,2}.$$

we have

$$||\psi||_{W^{2,p}(D_{3,2})}\,\leq\, C||h||_{**}.$$

Combining all the estimates in the above, we obtain that $||\psi||_*=o(1)$, which is a contradiction. □

We now consider the following linear projected problem: finding ψ with the symmetry in (5.5)

$$\mathcal{L}[\psi] \,=\, h \,+\, c_1\Lambda_1 \,+\, c_2\Lambda_2, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda_1 \,=\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda_2 \,=\, 0 \quad \text{ with } \phi = iu_2\psi. \tag{6.2}$$

Lemma 6.2. *There exists a constant C, depending on σ only, such that for all ε sufficiently small, the following holds: if $||h||_{**} < +\infty$, there exists a unique solution $(\psi, c_1, c_2) \equiv \mathcal{T}(h)$ to (6.2). Furthermore, there holds*

$$||\psi||_* \,\le\, C\,||h||_{**}.$$

Proof. The proof is similar to that of Proposition 4.1 in [27]. Instead of solving (6.2) in $\mathbb{R}^2$, we solve it in a bounded domain first:

$$\mathcal{L}[\psi] \,=\, h \,+\, c_1\Lambda_1 \,+\, c_2\Lambda_2, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda_1 \,=\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda_2 \,=\, 0 \quad \text{ with } \phi = iu_2\psi,$$
$$\phi = 0 \quad \text{ on } \partial B_M(0), \quad \psi \text{ satisfies the symmetry (5.5)}.$$

where $M > 10\hat{r}_2$. By the standard proof of a priori estimates, we also obtain the following estimates for any solution ψ_M of above problem

$$||\psi||_* \le C||h||_{**}.$$

By working with the Sobole space $H_0^1(B_M(0))$, the existence will follow by Fredholm alternatives. Now letting $M \to +\infty$, we obtain a solution with the required properties. □

6.2. **Projected Nonlinear Problem.** We then consider the following problem: finding ψ with the symmetry in (5.5)

$$\mathcal{L}[\psi] \,+\, \mathcal{N}[\psi] \,=\, \mathcal{E} \,+\, c_1\Lambda_1 \,+\, c_2\Lambda_2, \quad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda_1 \,=\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda_2 \,=\, 0 \quad \text{ with } \phi = iu_2\psi. \tag{6.3}$$

Proposition 6.3. *There exists a constant C, depending on σ only, such that for all ε sufficiently small, there exists a unique solution (ψ, c_1, c_2) to (6.3), and*

$$||\psi||_* \le C||h||_{**}.$$

Furthermore, ψ is continuous in the parameters $\hat{f}_1, \ldots, \hat{f}_4$ and $\hat{d}_1, \ldots, \hat{d}_4$.

Proof. Using of the operator defined by Lemma 6.2, for ε small and the parameters $\hat{f}_1, \ldots, \hat{f}_4$, $\hat{d}_1, \ldots, \hat{d}_4$ with constraints in (3.15) or (3.16), we can write problem (6.3) as

$$\psi = \mathcal{T}\big(-\mathcal{N}[\psi] + \mathcal{E}\big) \equiv \mathcal{G}(\psi).$$

Using Lemma 4.2, we see that

$$||\mathcal{E}||_{**} \le C\varepsilon^{1-2\sigma}.$$

Let

$$\psi \in \mathbb{B} = \big\{\, ||\psi||_* < C\varepsilon^{1-2\sigma} \,\big\},$$

then we have, using the explicit form of $\mathcal{N}(\psi)$ in section 5

$$||\mathcal{N}(\psi)||_{**} \le C\varepsilon.$$

Whence, there holds

$$||\mathcal{G}(\psi)||_{**} \le C\Big(||\mathcal{N}(\psi)||_{**} + ||\mathcal{E}||_{**}\Big) \le C\varepsilon^{1-2\sigma}.$$

Similarly, we can also show that, for any $\check{\psi}$, $\hat{\psi} \in \mathbb{B}$

$$||\mathcal{G}(\check{\psi}) - \mathcal{G}(\hat{\psi})||_{**} \le o(1)||\check{\psi} - \hat{\psi}||_{**}\,.$$

By contraction mapping theorem, we confirm the result of the Lemma. □

7. Reduction procedure

To find a real solution to problem (3.3)-(3.4) and fulfill the proof of Theorem 1.4, in this section, we will solve a reduced problem and find suitable parameters $\hat{f}_1, \dots, \hat{f}_4$ and $\hat{d}_1, \dots, \hat{d}_4$ with properties in (3.15) or (3.16) such that the constants c_1 and c_2 in (5.50) are identical zero for any sufficiently small ε.

This can be done in the following way. In previous section, for any given parameters $\hat{f}_1, \dots, \hat{f}_4$ and $\hat{d}_1, \dots, \hat{d}_4$ with properties in (3.15) or (3.16), we have deduced the existence of ψ with the symmetry (5.5) to the projected problem

$$\mathcal{L}(\psi) + \mathcal{N}(\psi) = \mathcal{E} + c_1\Lambda_1 + c_2\Lambda_2, \qquad \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda_1 \,=\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\phi}\Lambda_2 \,=\, 0, \tag{7.1}$$

with the relation

$$\phi = iu_2\psi \quad \text{in } D_2.$$

Multiplying the first equation in (7.1) by $\bar{\Lambda}_1$, Λ_2 and integrating, we obtain

$$c_1\,\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_1\Lambda_1 \,=\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_1\mathcal{L}(\psi) \,-\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_1\mathcal{N}(\psi) \,-\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_1\mathcal{E}, \tag{7.2}$$

$$c_2\,\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_2\Lambda_1 \,=\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_2\mathcal{L}(\psi) \,-\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_2\mathcal{N}(\psi) \,-\, \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_2\mathcal{E}. \tag{7.3}$$

Hence we can derive the equations of the type

$$c_1(\hat{f}_1, \dots, \hat{f}_4,\, \hat{d}_1, \dots, \hat{d}_4) = 0, \quad c_1(\hat{f}_1, \dots, \hat{f}_4,\, \hat{d}_1, \dots, \hat{d}_4) = 0, \tag{7.4}$$

by computing the integrals of the right hand sides of the above formulas (7.2)-(7.3), which can be solved due to the assumptions in (**A3**). This procedure will be carried out in the next two subsections.

Recall the translated variables in (3.9) in the form

$$s \,=\, x \,-\, \vec{e}_1, \quad z \,=\, x \,-\, \vec{e}_2.$$

We use the local form of $\mathcal{E}$ in (5.28) and then write

$$\mathcal{E} \,=\, \Sigma_1 \,+\, \Sigma_2 \,+\, \Sigma_3 \,+\, \Sigma_4 \,+\, O(\varepsilon\log|s| + \varepsilon\log|z|),$$

where we have denoted

$$\begin{aligned}\Sigma_1 \,&=\eta_2\,\Theta\bigg[\,\frac{s_1}{(s_1+\hat{f}_1)|s|}\,\frac{\rho'(|s|)}{\rho(|s|)} \,+\, \frac{z_1}{(z_1+\hat{f}_2)|z|}\,\frac{\rho'(|z|)}{\rho(|z|)}\,\bigg],\\ \Sigma_2 \,&=\, -\,\eta_2\,\Theta\left[\,\varepsilon\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}(x_1-\hat{r}_0) \,+\, \frac{\varepsilon^2}{2}\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}(x_1-\hat{r}_0)^2 \,+\, \frac{\varepsilon^2}{2}\frac{\partial^2\tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}x_2^2\,\right],\end{aligned} \tag{7.5}$$

and also

$$\begin{aligned}\Sigma_3 \,=&\, 2\Theta\frac{s_2z_2+s_1z_1}{|s||z|}\,\frac{\rho'(|s|)}{\rho(|s|)}\frac{\rho'(|z|)}{\rho(|z|)} \,-\, 2(-1)^j\Theta\frac{s_2z_2+s_1z_1}{|s|^2|z|^2}\\ &+\, 2i(-1)^j\Theta\frac{s_2z_1-z_2s_1}{|s||z|^2}\,\frac{\rho'(|s|)}{\rho(|s|)} \,+\, 2i\Theta\frac{z_2s_1-s_2z_1}{|z||s|^2}\,\frac{\rho'(|z|)}{\rho(|z|)}.\end{aligned} \tag{7.6}$$

The last term in the error is

$$\begin{aligned}\Sigma_4 \,=&\, \eta_2\,\Theta\,\frac{1}{\hat{f}_1}\log\hat{f}_1\bigg[\,i\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_2}{|s|} + i\frac{\rho'(|z|)}{\rho(|z|)}\frac{z_2}{|z|} - \frac{s_1}{|s|^2} - (-1)^j\frac{s_1}{|z|^2}\,\bigg]\\ &+\, \eta_2\,\Theta\,\frac{1}{\hat{f}_2}\log\hat{f}_2\bigg[\,i\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_2}{|s|} + i\frac{\rho'(|z|)}{\rho(|z|)}\frac{z_2}{|z|} - \frac{s_1}{|s|^2} - \frac{z_1}{|z|^2}\,\bigg].\end{aligned} \tag{7.7}$$

In the above formulas, there hold the relations

$$z_2 = s_2 - (\hat{f}_2 - \hat{f}_1), \quad \text{and} \quad z_1 = s_1 - (\hat{f}_2 - \hat{f}_1).$$

7.1. **Case 1: Interaction of Type I.** To show the interaction of neighboring vortex rings of Type I in (1.25), by recalling (3.15), we here only need to determine $\hat{f}_1$ and $\hat{f}_2$. The main object of this section is to derive the system of equations with variables $\hat{f}_1$ and $\hat{f}_2$ and then solve it.

Recall the definitions Λ_1 and Λ_2 in (5.48). By the symmetries of the terms $\mathcal{E}$, Λ_1, Λ_2, we begin with the computations of

$$\text{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_1 \mathcal{E} \,=\, 2\text{Re}\int_{\mathbb{R}^2_+} \bar{\Lambda}_1 \mathcal{E}, \qquad \text{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_2 \mathcal{E} \,=\, 2\text{Re}\int_{\mathbb{R}^2_+} \bar{\Lambda}_2 \mathcal{E},$$

where $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$. The constraints in (3.15) imply that, in the region $\mathbb{R}^2_+$, the terms Λ_1 and Λ_2 have their supports contained in different regions as follows

$$\{(x_1, x_2) : |x - \vec{e}_1| < 2R\}, \qquad \{(x_1, x_2) : |x - \vec{e}_2| < 2R\}.$$

Hence we recall the translated variables in (3.9) in the form

$$s \,=\, x \,-\, \vec{e}_1, \quad z \,=\, x \,-\, \vec{e}_2.$$

We will find that it is convenient to compute

$$\text{Re}\int_{\mathbb{R}^2_+} \bar{\Lambda}_1 \mathcal{E}, \quad \text{Re}\int_{\mathbb{R}^2_+} \bar{\Lambda}_2 \mathcal{E}$$

on the variables (s_1, s_2) or (z_1, z_2). We can also deal with other terms similarly.

We first consider the estimates in the equation (7.2). Since Λ_1 has its support contained in a neighborhood of $\vec{e}_1$, we will choose the variables (s_1, s_2). Note that, in the region $\mathbb{R}^2_+$, there hold

$$\frac{\partial u_2}{\partial \hat{f}_1} \,=\, u_2\Big[-\frac{\rho'(\hat{\ell}_1)}{\rho(\hat{\ell}_1)} \frac{x_1 - \hat{f}_1}{\hat{\ell}_1} \,+\, i\,\frac{x_2}{(\hat{\ell}_1)^2}\Big] \,+\, O(\varepsilon^2) u_2,$$

which implies that

$$\Lambda_1 \,=\, \tilde{\chi}\big(|s|/R\big)\bigg[\, -\, \frac{\rho'(|s|)}{\rho(|s|)}\frac{s_1}{|s|} \,+\, i\,\frac{s_2}{|s|^2}\bigg]\Theta \,+\, O(\varepsilon^2),$$

where we have denoted

$$\Theta = \rho(|s|)\rho(|z|)e^{i(\varphi_{02}+\varphi_{01})}, \tag{7.8}$$

with φ_{01} and φ_{02} defined in (3.10).

In the case of interaction of neighboring vortex rings of type I, there hold the relations

$$z_2 = s_2, \quad \text{and} \quad z_1 = s_1 - (\hat{f}_2 - \hat{f}_1),$$

due to the constraints in (3.15). By defining $\vec{e}_T = (f_1 - f_2, 0)$, we have $|z| = |s - \vec{e}_T|$. We then obtain

$$\begin{aligned} 2\text{Re}\int_{\mathbb{R}^2_+} \bar{\Lambda}_1 \Sigma_1 \mathrm{d}x \,=\, &-\, 2\int_{\{|s|<R\}} \rho(|s|)\rho(|s-\vec{e}_T|)\rho'(|s|)\rho'(|s-\vec{e}_T|)\frac{s_1^2 - s_1(\hat{f}_2 - \hat{f}_1)}{(s_1 + \hat{f}_1)|s||s - \vec{e}_T|}\,\mathrm{d}s \\ &-\, 2\int_{\{|s|<R\}} \Big[\rho(|s-\vec{e}_T|)\Big]^2\Big[\rho'(|s|)\Big]^2\frac{s_1^2}{(s_1+\hat{f}_1)|s|^2}\,\mathrm{d}s \,+\, O(\varepsilon) \\ =\, &O(\varepsilon), \end{aligned}$$

In the above, we have used the fact that $|z| = |s - \vec{e}_T| = O\big(\varepsilon^{-1}\,|\log\varepsilon|^{-1/2}\big)$ for $|s| < R$ due the constraints in (3.15). For the convenience of notations, we will still the keep the notation $|z|$ in the computations of the formulas (7.9)-(7.13) by remembering its asymptotic behavior.

$$
\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}^2_+}\bar{\Lambda}_1\Sigma_2\mathrm{d}x \;=&\; 2\varepsilon\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}\int_{\{|s|<R\}}\rho(|s|)\rho'(|s|)\frac{s_1\big(s_1+(\hat{f}_1-\hat{r}_0)\big)}{|s|}\,\mathrm{d}s\\
&+\,\varepsilon^2\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}\int_{\{|s|<R\}}\rho(|s|)\rho'(|s|)\frac{s_1\big(s_1+(\hat{f}_1-\hat{r}_0)\big)^2}{|s|}\,\mathrm{d}s\\
&+\,\varepsilon^2\frac{\partial^2\tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}\int_{\{|s|<R\}}\rho(|s|)\rho'(|s|)\frac{s_1s_2^2}{|s|}\,\mathrm{d}s\,+\,O(\varepsilon)\\
=&\; 2\pi\varepsilon\,|\log\varepsilon|\,\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}\,+\,2\pi\varepsilon^2\,|\log\varepsilon|\,\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}(\hat{f}_1-\hat{r}_0)\,+\,O(\varepsilon),
\end{aligned}
\tag{7.9}
$$

$$
\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}^2_+}\bar{\Lambda}_1\Sigma_4\mathrm{d}x \;=&\; \frac{2}{\hat{f}_1}\log\hat{f}_1\int_{\{|s|<R\}}\rho(|s|)\rho'(|s|)\Big[\,(-1)^j\frac{s_1z_1}{|s||z|^2}+\frac{s_2z_2}{|s|^2|z|}\,\Big]\,\mathrm{d}s\\
&-\,\frac{2}{\hat{f}_1}\log\hat{f}_1\int_{\{|s|<R\}}\rho(|s|)\rho'(|s|)\frac{1}{|s|}\,\mathrm{d}s\,+\,O(\varepsilon)\\
=&\;-\,2\frac{a\,\pi}{\hat{f}_1}\log\hat{f}_1\,+\,O(\varepsilon).
\end{aligned}
\tag{7.10}
$$

where

$$
a=\frac{1}{\pi}\int_{\mathbb{R}^2}\rho(|s|)\rho'(|s|)\frac{1}{|s|}\,\mathrm{d}s>0. \tag{7.11}
$$

We write the term Σ_3 in the form

$$
\begin{aligned}
\Sigma_3 \;=&\; 2\Theta\left[\frac{|s|}{|z|}-\frac{s_1(\hat{f}_2-\hat{f}_1)}{|s||z|}\right]\frac{\rho'(|s|)}{\rho(|s|)}\frac{\rho'(|z|)}{\rho(|z|)}\,+\,2(-1)^j\Theta\left[\frac{1}{|z|^2}-\frac{s_1(\hat{f}_2-\hat{f}_1)}{|s|^2|z|^2}\right]\\
&-\,2i(-1)^j\Theta\frac{s_2(\hat{f}_2-\hat{f}_1)}{|s||z|^2}\,\frac{\rho'(|s|)}{\rho(|s|)}\,+\,2i\Theta\frac{s_2(\hat{f}_2-f_1)}{|z||s|^2}\,\frac{\rho'(|z|)}{\rho(|z|)}.
\end{aligned}
\tag{7.12}
$$

The above term play an important role in describing the interaction of neighboring vortex rings.

$$
\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}^2_+}\bar{\Lambda}_1\Sigma_3\mathrm{d}x \;=&\; -\,4\int_{\{|s|<R\}}\left[\frac{|s|}{|z|}-\frac{s_1(\hat{f}_2-\hat{f}_1)}{|s||z|}\right]\frac{\rho'(|s|)}{\rho(|s|)}\frac{\rho'(|z|)}{\rho(|z|)}\,\Theta\bar{\Theta}\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_1}{|s|}\,\mathrm{d}s\\
&-\,4(-1)^j\int_{\{|s|<R\}}\left[\frac{1}{|z|^2}-\frac{s_1(\hat{f}_2-\hat{f}_1)}{|s|^2|z|^2}\right]\Theta\bar{\Theta}\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_1}{|s|}\,\mathrm{d}s\\
&+\,4(-1)^j\int_{\{|s|<R\}}\Theta\bar{\Theta}\frac{s_2(\hat{f}_2-\hat{f}_1)}{|s||z|^2}\,\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_2}{|s|^2}\,\mathrm{d}s\\
&-\,4\int_{\{|s|<R\}}\Theta\bar{\Theta}\frac{s_2(\hat{f}_2-\hat{f}_1)}{|z||s|^2}\,\frac{\rho'(|z|)}{\rho(|z|)}\frac{s_2}{|s|^2}\,\mathrm{d}s\,+\,O(\varepsilon)\\
=&\; 4\int_{\{|s|<R\}}\frac{(\hat{f}_2-\hat{f}_1)}{|z|}\,\big|\rho'(|s|)\big|^2\rho'(|z|)\rho(|z|)\,\mathrm{d}s\\
&+\,4(-1)^j\int_{\{|s|<R\}}\frac{(\hat{f}_2-\hat{f}_1)}{|s||z|^2}\,\big|\rho(|z|)\big|^2\rho'(|s|)\rho(|s|)\,\mathrm{d}s\,+\,O(\varepsilon)\\
=&\; (-1)^j\,\frac{8a\pi}{\hat{f}_2-\hat{f}_1}+O(\varepsilon),
\end{aligned}
\tag{7.13}
$$

Hence, there holds

$$
\begin{aligned}
I_1 \equiv \mathrm{Re}\int_{\{|s|<R\}} \bar{\Lambda}_1 \mathcal{E} \mathrm{d}x &= 2\pi\varepsilon \frac{\partial \tilde{V}}{\partial r}\Big|_{(r_0,0)} \log\frac{1}{\varepsilon} - 2\,a\,\pi \frac{1}{\hat{f}_1}\log \hat{f}_1 \\
&\quad + 2\pi\varepsilon^2 |\log\varepsilon| \frac{\partial^2 \tilde{V}}{\partial r^2}\Big|_{(r_0,0)} (\hat{f}_1 - \hat{r}_0) + (-1)^j \frac{8a\pi}{\hat{f}_2 - \hat{f}_1} + O(\varepsilon).
\end{aligned} \tag{7.14}
$$

Using Proposition 6.3, and the expression in (5.29), we deduce that

$$
\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_1 \mathcal{N}(\psi) = \mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_1 N_2(\psi) = O(\varepsilon).
$$

On the other hand, integration by parts, we have

$$
\mathrm{Re}\int_{\mathbb{R}^2} \bar{\Lambda}_1 \mathcal{L}(\psi) = \mathrm{Re}\int_{\mathbb{R}^2} \bar{\psi} \mathcal{L}(\bar{\Lambda}_1) = O(\varepsilon).
$$

Similarly, we can deal with the estimate in the equation (7.3). Hence, we only consider the main components. Note that we can also write Σ_3 in the form

$$
\begin{aligned}
\Sigma_3 &= 2\Theta \frac{s_2 z_2 + s_1 z_1}{|s||z|} \frac{\rho'(|s|)}{\rho(|s|)} \frac{\rho'(|z|)}{\rho(|z|)} - 2(-1)^j \Theta \frac{s_2 z_2 + s_1 z_1}{|s|^2 |z|^2} \\
&\quad + 2i(-1)^j \Theta \frac{s_2 z_1 - z_2 s_1}{|s||z|^2} \frac{\rho'(|s|)}{\rho(|s|)} + 2i\Theta \frac{z_2 s_1 - s_2 z_1}{|z||s|^2} \frac{\rho'(|z|)}{\rho(|z|)} \\
&= 2\Theta \left[\frac{|z|}{|s|} - \frac{z_1(\hat{f}_1 - \hat{f}_2)}{|s||z|} \right] \frac{\rho'(|s|)}{\rho(|s|)} \frac{\rho'(|z|)}{\rho(|z|)} + 2(-1)^j \Theta \left[\frac{1}{|s|^2} - \frac{z_1(\hat{f}_1 - \hat{f}_2)}{|s|^2|z|^2} \right] \\
&\quad - 2i(-1)^j \Theta \frac{z_2(\hat{f}_2 - \hat{f}_1)}{|s||z|^2} \frac{\rho'(|s|)}{\rho(|s|)} + 2i\Theta \frac{z_2(\hat{f}_2 - f_1)}{|z||s|^2} \frac{\rho'(|z|)}{\rho(|z|)}.
\end{aligned} \tag{7.15}
$$

Since Λ_2 has its support contained in the neighborhood of $\vec{e}_2$, we here choose the variables (z_1, z_2). Note that, in the region $\mathbb{R}^2_+$, there holds

$$
\frac{\partial u_2}{\partial \hat{f}_2} = u_2 \Big[-\frac{\rho'(\hat{\ell}_2)}{\rho(\hat{\ell}_2)} \frac{x_1 - \hat{f}_2}{\hat{\ell}_2} + i\,\frac{x_2}{(\hat{\ell}_2)^2} \Big] + O(\varepsilon^2) u_2,
$$

which implies that

$$
\Lambda_2 = \tilde{\chi}\big(|z|/R\big) \left[-\frac{\rho'(|z|)}{\rho(|z|)} \frac{z_1}{|z|} + i\,\frac{z_2}{|z|^2} \right] \Theta + O(\varepsilon^2).
$$

Hence the interaction of neighboring vortex rings can be estimated by

$$
\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}^2_+} \bar{\Lambda}_2\Sigma_3\,\mathrm{d}x \,=\, & -4\int_{\{|z|<R\}}\left[\frac{|z|}{|s|}-\frac{z_1(\hat{f}_1-\hat{f}_2)}{|s||z|}\right]\frac{\rho'(|s|)}{\rho(|s|)}\frac{\rho'(|z|)}{\rho(|z|)}\,\Theta\bar{\Theta}\frac{\rho'(|z|)}{\rho(|z|)}\frac{z_1}{|z|}\,\mathrm{d}z\\
& -4(-1)^j\int_{\{|z|<R\}}\left[\frac{1}{|s|^2}-\frac{z_1(\hat{f}_1-\hat{f}_2)}{|s|^2|z|^2}\right]\Theta\bar{\Theta}\frac{\rho'(|z|)}{\rho(|z|)}\frac{z_1}{|z|}\,\mathrm{d}z\\
& +4(-1)^j\int_{\{|z|<R\}}\Theta\bar{\Theta}\frac{z_2(\hat{f}_2-\hat{f}_1)}{|s||z|^2}\,\frac{\rho'(|s|)}{\rho(|s|)}\frac{z_2}{|z|^2}\,\mathrm{d}z\\
& -4\int_{\{|z|<R\}}\Theta\bar{\Theta}\frac{z_2(\hat{f}_2-\hat{f}_1)}{|z||s|^2}\,\frac{\rho'(|z|)}{\rho(|z|)}\frac{z_2}{|z|^2}\,\mathrm{d}z\\
=\, & 4\int_{\{|z|<R\}}\frac{(\hat{f}_1-\hat{f}_2)}{|s|}\,\big|\rho'(|z|)\big|^2\rho'(|s|)\rho(|s|)\,\mathrm{d}z\\
& \quad +4(-1)^j\int_{\{|z|<R\}}\frac{(\hat{f}_1-\hat{f}_2)}{|s||z|^2}\,\big|\rho(|s|)\big|^2\rho'(|z|)\rho(|z|)\,\mathrm{d}z\,+\,O(\varepsilon)\\
=\, & (-1)^j\frac{8a\pi}{\hat{f}_1-\hat{f}_2}\,+\,O(\varepsilon),
\end{aligned}
$$

In the above, we have use fact that $|s|=|z-\vec{e}_T|=O\big(\varepsilon^{-1}\,|\log\varepsilon|^{-1/2}\big)$ for $|z|<R$. Other terms in (7.3) can be estimated similarly as we have done before.

Combining all estimates together and rescaling back to the original parameters by the relations in (3.6), we obtain the following equations

$$
\begin{aligned}
c_1(f_1,f_2) \,=\, & 2\,\varepsilon\pi\left[\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}\log\frac{1}{\varepsilon}-\frac{a}{f_1}\log\frac{f_1}{\varepsilon}\right]\\
& +2\,\varepsilon\pi\left[\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}(f_1-r_0)\,\log\frac{1}{\varepsilon}-(-1)^j\frac{4a}{f_1-f_2}\right]\,+\,M_{1,1}(f_1,f_2),\\
c_2(f_1,f_2) \,=\, & 2\,\varepsilon\pi\left[\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}\log\frac{1}{\varepsilon}-\frac{a}{f_2}\log\frac{f_2}{\varepsilon}\right]\\
& +2\,\varepsilon\pi\left[\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}(f_2-r_0)\,\log\frac{1}{\varepsilon}+(-1)^j\frac{4a}{f_1-f_2}\right]\,+\,M_{1,2}(f_1,f_2),
\end{aligned}
\tag{7.16}
$$

where $M_{1,1}$ and $M_{1,2}$ are continuous functions of the parameters f_1 and f_2 of order $O(\varepsilon)$. Now we recall the condition (1.13) and that we have chosen j in (1.29) in the form

$$
j=\begin{cases}1, & \text{if } \digamma_1<0,\\ 2, & \text{if } \digamma_1>0,\end{cases}\qquad \text{with}\quad \digamma_1=\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}.
$$

By the solvability condition (1.10) and the non-degeneracy condition (1.12), we can find (f_1,f_2) such that the constraints in (1.25) are fulfilled and also

$$
\Big(c_1\big(f_1,f_2\big),\,c_2\big(f_1,f_2\big)\Big)=0,
$$

with the help of the simple mean-value theorem.

7.2. **Case 2: Interaction of Type II.** To show the interaction of neighboring vortex rings of Type II in (1.26), by recalling the requirements in (3.16) we shall find $\hat{f}_1$ and $\hat{d}_1$. The main object of this part is to compute the integrals in (7.2)-(7.3), which will become a system of equations with variables $\hat{f}_1$ and $\hat{d}_1$. This system can be solved due the assumptions in (**A3**).

Recall the definitions of Λ_1 and Λ_2 in (5.49). By the symmetries of the terms $\mathcal{E}$, Λ_1, Λ_2, we begin with the computations of

$$\mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}_1\mathcal{E} \,=\, 4\,\mathrm{Re}\int_{\mathbb{R}^2_{++}}\bar{\Lambda}_1\mathcal{E}, \qquad \mathrm{Re}\int_{\mathbb{R}^2}\bar{\Lambda}_2\mathcal{E} \,=\, 4\,\mathrm{Re}\int_{\mathbb{R}^2_{++}}\bar{\Lambda}_2\mathcal{E},$$

where we have denoted

$$\mathbb{R}^2_{++} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\, x_2 > 0\}.$$

The constraints in (3.16) imply that, in the region $\mathbb{R}^2_{++}$, the term Λ_1 and Λ_2 have their supports contained in the same region as follows

$$\{(x_1, x_2) : |x - \vec{e}_1| < 2R\}.$$

Hence we recall the translated variables in (3.9) in the form

$$s \,=\, x \,-\, \vec{e}_1, \quad z \,=\, x \,-\, \vec{e}_2.$$

It is convenient to compute

$$\mathrm{Re}\int_{\mathbb{R}^2_{++}}\bar{\Lambda}_1\mathcal{E}, \qquad \mathrm{Re}\int_{\mathbb{R}^2_{++}}\bar{\Lambda}_2\mathcal{E}$$

on the variables (s_1, s_2). We can also deal with other terms similarly.

Note that, in the region $\mathbb{R}^2_{++}$, there holds

$$\frac{\partial u_2}{\partial \hat{f}_1} \,=\, u_2\Big[-\frac{\rho'(\hat{\ell}_1)}{\rho(\hat{\ell}_1)}\frac{x_1 - \hat{f}_1}{\hat{\ell}_1} \,+\, i\,\frac{x_2 - \hat{d}_1}{(\hat{\ell}_1)^2}\Big] \,+\, O(\varepsilon^2)u_2,$$

$$\frac{\partial u_2}{\partial \hat{d}_1} \,=\, u_2\Big[-\frac{\rho'(\hat{\ell}_1)}{\rho(\hat{\ell}_1)}\frac{x_2 - \hat{d}_1}{\hat{\ell}_1} \,-\, i\,\frac{x_1 - \hat{f}_1}{(\hat{\ell}_1)^2}\Big] \,+\, O(\varepsilon^2)u_2,$$

which implies that

$$\Lambda_1 \,=\, \chi\big(|s|/R\big)\left[\,-\,\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_1}{|s|} \,+\, i\,\frac{s_2}{|s|^2}\right]\Theta \,+\, O(\varepsilon^2),$$

$$\Lambda_2 \,=\, \chi\big(|s|/R\big)\left[\,-\,\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_2}{|s|} \,-\, i\,\frac{s_1}{|s|^2}\right]\Theta \,+\, O(\varepsilon^2),$$

where Θ is given in (7.8). In the case of interaction of neighboring vortex rings of type II, there also hold the relations

$$z_1 = s_1 \quad \text{and} \quad z_2 = s_2 - (\hat{d}_2 - \hat{d}_1).$$

due to the constraints in (3.16). In the coordinates (s_1, s_2), we will keep the notation $|z|$ with

$$|z| = |s - (0, d_1 - d_2)|,$$

which has the asymptotic behavior $|z| = O\big(\varepsilon^{-1}\,|\log\varepsilon|^{-1/2}\big)$ for $|s| < R$.

We first estimate the terms in (7.3). We then obtain

$$
\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}^2_{++}} \bar{\Lambda}_2\Sigma_1 \mathrm{d}x \,=\, & -\,2\int_{\{|s|<R\}} \Big[\rho(|z|)\Big]^2\Big[\rho'(|s|)\Big]^2 \frac{s_1s_2}{(s_1+\hat{f}_1)|s|^2}\,\mathrm{d}s \\
& -\,2\int_{\{|s|<R\}} \rho(|s|)\rho(|z|)\Big[\rho'(|s|)\Big]^2 \frac{s_1s_2}{(s_1+\hat{f}_2)|z||s|}\,\mathrm{d}s \,+\, O(\varepsilon) \\
=\, & O(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}^2_{++}} \bar{\Lambda}_2\Sigma_2 \mathrm{d}x \,=\, & 2\varepsilon\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}\int_{\{|s|<R\}} \rho(|s|)\rho'(|s|)\frac{s_2\big(s_1+(\hat{f}_1-\hat{r}_0)\big)}{|s|}\,\mathrm{d}s \\
& +\,\varepsilon^2\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}\int_{\{|s|<R\}} \rho(|s|)\rho'(|s|)\frac{s_2\big(s_1+(\hat{f}_1-\hat{r}_0)\big)^2}{|s|}\,\mathrm{d}s \\
& +\,\varepsilon^2\frac{\partial^2\tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}\int_{\{|s|<R\}} \rho(|s|)\rho'(|s|)\frac{s_2(s_2+\hat{d}_1)^2}{|s|}\,\mathrm{d}s \,+\, O(\varepsilon) \\
=\, & 2\pi\varepsilon^2\,|\log\varepsilon|\,\frac{\partial^2\tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}\,\hat{d}_1 \,+\, O(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}^2_{++}} \bar{\Lambda}_2\Sigma_4 \mathrm{d}x \,=\, & \frac{2}{\hat{f}_1}\log\hat{f}_1\int_{\{|s|<R\}} \rho(|s|)\rho'(|s|)\bigg[(-1)^j\frac{s_2s_1}{|s||z|^2}+\frac{s_1z_2}{|s|^2|z|}\bigg]\,\mathrm{d}s \\
& -\,\frac{4}{\hat{f}_1}\log\hat{f}_1\int_{\{|s|<R\}} \rho(|s|)\rho'(|s|)\frac{s_1s_2}{|s|^3}\,\mathrm{d}s \,+\, O(\varepsilon) \\
=\, & O(\varepsilon).
\end{aligned}
$$

We write the component Σ_3 in the form

$$
\begin{aligned}
\Sigma_3 \,=\, & 2\Theta\frac{s_2z_2+s_1z_1}{|s||z|}\,\frac{\rho'(|s|)}{\rho(|s|)}\frac{\rho'(|z|)}{\rho(|z|)} \,-\, 2(-1)^j\Theta\frac{s_2z_2+s_1z_1}{|s|^2|z|^2} \\
& +\,2i(-1)^j\Theta\frac{s_2z_1-z_2s_1}{|s||z|^2}\,\frac{\rho'(|s|)}{\rho(|s|)} \,+\, 2i\Theta\frac{z_2s_1-s_2z_1}{|z||s|^2}\,\frac{\rho'(|z|)}{\rho(|z|)} \\
=\, & 2\Theta\bigg[\frac{|s|}{|z|}-\frac{s_2(\hat{d}_2-\hat{d}_1)}{|s||z|}\bigg]\frac{\rho'(|s|)}{\rho(|s|)}\frac{\rho'(|z|)}{\rho(|z|)} \,+\, 2(-1)^j\Theta\bigg[\frac{1}{|z|^2}-\frac{s_2(\hat{d}_2-\hat{d}_1)}{|s|^2|z|^2}\bigg] \\
& +\,2i(-1)^j\Theta\frac{s_1(\hat{d}_2-\hat{d}_1)}{|s||z|^2}\,\frac{\rho'(|s|)}{\rho(|s|)} \,-\, 2i\Theta\frac{s_1(\hat{d}_2-\hat{d}_1)}{|z||s|^2}\,\frac{\rho'(|z|)}{\rho(|z|)}.
\end{aligned}
\tag{7.17}
$$

The above term play an important role in describing the interaction of neighboring vortex rings

$$
\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}^2_{++}} \bar{\Lambda}_2\Sigma_3 \mathrm{d}x \,=\, & -4\int_{\{|s|<R\}}\left[\frac{|s|}{|z|}-\frac{s_2(\hat{d}_2-\hat{d}_1)}{|s||z|}\right]\frac{\rho'(|s|)}{\rho(|s|)}\frac{\rho'(|z|)}{\rho(|z|)}\,\Theta\bar{\Theta}\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_2}{|s|}\,\mathrm{d}s\\
& -4(-1)^j\int_{\{|s|<R\}}\left[\frac{1}{|z|^2}-\frac{s_2(\hat{d}_2-\hat{d}_1)}{|s|^2|z|^2}\right]\Theta\bar{\Theta}\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_2}{|s|}\,\mathrm{d}s\\
& -4(-1)^j\int_{\{|s|<R\}}\Theta\bar{\Theta}\frac{s_1(\hat{d}_2-\hat{d}_1)}{|s||z|^2}\,\frac{\rho'(|s|)}{\rho(|s|)}\frac{s_1}{|s|^2}\,\mathrm{d}s\\
& +4\int_{\{|s|<R\}}\Theta\bar{\Theta}\frac{s_1(\hat{d}_2-\hat{d}_1)}{|z||s|^2}\,\frac{\rho'(|z|)}{\rho(|z|)}\frac{s_1}{|s|^2}\,\mathrm{d}s\\
=\, & 4\int_{\{|s|<R\}}\frac{(\hat{d}_2-\hat{d}_1)}{|z|}\,\big|\rho'(|s|)\big|^2\rho'(|z|)\rho(|z|)\,\mathrm{d}s\\
& +4(-1)^j\int_{\{|s|<R\}}\frac{(\hat{d}_2-\hat{d}_1)}{|s||z|^2}\,\big|\rho(|z|)\big|^2\rho'(|s|)\rho(|s|)\,\mathrm{d}s\,+\,O(\varepsilon)\\
=\, & (-1)^j\frac{8a\pi}{\hat{d}_2-\hat{d}_1}+O(\varepsilon),
\end{aligned}
$$

Hence, there holds

$$
I_2\,\equiv\,\mathrm{Re}\int_{\{|s|<R\}}\bar{\Lambda}_1\mathcal{E}\mathrm{d}x\,=2\pi\varepsilon^2\,|\log\varepsilon|\,\frac{\partial^2\tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}\hat{d}_1\,+\,(-1)^j\frac{8a\pi}{\hat{d}_2-\hat{d}_1}\,+\,O(\varepsilon). \tag{7.18}
$$

Using Proposition 6.3, and the expression in (5.29), we deduce that

$$
\mathrm{Re}\int_{\{|s|<R\}}\bar{\Lambda}_1\mathcal{N}(\psi)\,=\,\mathrm{Re}\int_{\{|s|<R\}}\bar{\Lambda}_1N_2(\psi)\,=\,O(\varepsilon).
$$

On the other hand, integration by parts, we have

$$
\mathrm{Re}\int_{\{|s|<R\}}\bar{\Lambda}_1\mathcal{L}(\psi)\,=\,\mathrm{Re}\int_{\{|s|<R\}}\bar{\psi}\mathcal{L}(\bar{\Lambda}_1)\,=\,O(\varepsilon).
$$

Similarly, we can deal with the estimates in the equation (7.2). Hence, we only consider the main components here.

$$
\begin{aligned}
2\mathrm{Re}\int_{\mathbb{R}^2_{++}}\bar{\Lambda}_1\Sigma_2\mathrm{d}x\,=\, & 2\varepsilon\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}\int_{\{|s|<R\}}\rho(|s|)\rho'(|s|)\frac{s_1\big(s_1+(\hat{f}_1-\hat{r}_0)\big)}{|s|}\,\mathrm{d}s\\
& +\,\varepsilon^2\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}\int_{\{|s|<R\}}\rho(|s|)\rho'(|s|)\frac{s_1\big(s_1+(\hat{f}_1-\hat{r}_0)\big)^2}{|s|}\,\mathrm{d}s\\
& +\,\varepsilon^2\frac{\partial^2\tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}\int_{\{|s|<R\}}\rho(|s|)\rho'(|s|)\frac{s_1(s_2+\hat{d}_1)^2}{|s|}\,\mathrm{d}s\,+\,O(\varepsilon)\\
=\, & 2\pi\varepsilon\,|\log\varepsilon|\,\frac{\partial\tilde{V}}{\partial r}\Big|_{(r_0,0)}\,+\,2\pi\varepsilon^2\,|\log\varepsilon|\,\frac{\partial^2\tilde{V}}{\partial r^2}\Big|_{(r_0,0)}\big(\hat{f}_1-\hat{r}_0\big)\,+\,O(\varepsilon),
\end{aligned}
$$

Combining all estimates together and rescaling back to the original parameters, we obtain the following equation

$$
\begin{aligned}
c_1(f_1, d_1) &= 2\,\varepsilon\pi \left[\frac{\partial^2 \tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)} d_1 \log \frac{1}{\varepsilon} - (-1)^j \frac{2a}{d_1} \right] + M_{2,1}(f_1, d_1), \\
c_2(f_1, d_1) &= 2\,\varepsilon\pi \left[\frac{\partial \tilde{V}}{\partial r}\Big|_{(r_0,0)} \log \frac{1}{\varepsilon} - \frac{a}{f_1} \log \frac{f_1}{\varepsilon} \right] \\
&\quad + 2\pi\varepsilon \frac{\partial^2 \tilde{V}}{\partial r^2}\Big|_{(r_0,0)} \big(f_1 - r_0\big) \log \frac{1}{\varepsilon} + M_{2,2}(f_1, d_1),
\end{aligned}
\tag{7.19}
$$

where $M_{2,1}$ and $M_{2,2}$ are continuous functions of the parameters f_1 and d_1 of order $O(\varepsilon)$. Now we recall the condition (1.13) and that we have chosen the parameter j in (1.30) as the following

$$
j = \begin{cases} 1, & \text{if } \mathcal{F}_2 < 0, \\ 2, & \text{if } \mathcal{F}_2 > 0, \end{cases} \qquad \text{with} \quad \mathcal{F}_2 = \frac{\partial^2 \tilde{V}}{\partial y_3^2}\Big|_{(r_0,0)}.
$$

By the solvability condition (1.10) and the non-degeneracy conditions (1.12), we can find (f_1, d_1) such that the constraints in (1.26) are fulfilled and also

$$
\Big(c_1\big(f_1, d_1\big),\, c_2\big(f_1, d_1\big) \Big) = 0,
$$

with the help of the simple mean-value theorem.

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