Uniqueness and nondegeneracy of sign-changing radial solutions to an almost critical elliptic problem

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April 11, 2016

We study uniqueness of sign-changing radial solutions for the following semi-linear elliptic equation

$$\Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N),$$

where $1 < p < \frac{N+2}{N-2}$, $N \geq 3$. It is well-known that this equation has a unique positive radial solution. The existence of sign-changing radial solutions with exactly k nodes is also known. However the uniqueness of such solutions is open. In this paper, we show that such sign-changing radial solution is unique when p is close to $\frac{N+2}{N-2}$. Moreover, those solutions are non-degenerate, i.e., the kernel of the linearized operator is exactly N-dimensional.

2010 Mathematics Subject Classification. Primary:35B25, Secondary: 35J61, 35Q55.

1 Introduction and main results

In this paper we establish the uniqueness and nondegeneracy of sign-changing radially symmetric solutions to the following semi-linear elliptic equation

$$\Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N) \tag{1.1}$$

where $1 < p < 2^* - 1$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent for the embedding of $H^1(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N), N \geq 3$. More precisely, for any $k \in \mathbb{N}$, we will prove that the following ODE problem

$$\begin{cases} u'' + \frac{N-1}{r}u' - u + |u|^{p-1}u = 0, \quad r \in (0, \infty), \quad N \geq 3, \\ u'(0) = 0, \quad \lim\limits_{r \to \infty} u(r) = 0, \end{cases} \tag{1.2}$$

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has a unique solution $u \in C^2[0,\infty)$ such that $u(0) > 0$ and u has exactly k zeros. Moreover, this unique solution is non-degenerate in the space of H^1 functions.

Equation (1.1) arises in various models in physics, mathematical physics and biology. In particular, the study of standing waves for the nonlinear Klein-Gordon or Schrödinger equations can be reduced to (1.1). The uniqueness and nondegeneracy of standing waves play essential role in the study of soliton dynamics or blow up for nonlinear Schrödinger equations (NLS). We refer to the papers of Berestycki and Lions [4], [5] for mathematical foundations of (1.1), Rapahael [34], Merle and Raphael [26], Nakanishi and Schlag [31] for backgrounds on dynamics and blow-ups of NLS.

The classical work of Gidas, Ni and Nirenberg [14] states that all positive solutions of (1.1) are radially symmetric around some point. The uniqueness of positive radial solutions to (1.1) has been extensively studied during the last thirty years. It was initiated by Coffman [7] with $p = 3$ and $N = 3$, and then improved by McLeod and Serrin [23] to $1 < p \leq \frac{N}{N-2}$, and finally extended by Kwong [21] to full range of exponents $1 < p < \frac{N+2}{N-2}$ by shooting method. Since then there have been many extensions and refinements, see for example the works [6], [33], [35] and references therein. An essential tool in studying (1.1) is the shooting method, i.e., one studies the behavior of solutions $u(r,\alpha)$ to the initial value problem

$$\begin{cases} u'' + \frac{N-1}{r}u' - u + u^p = 0, \quad r \in (0,\infty), \quad N \geq 3, \\ u(0) = \alpha, \quad u'(0) = 0, \end{cases} \tag{1.3}$$

for $\alpha \in (0,\infty)$ and obtains series of comparison results between two solutions to (1.3) with different initial values. One feature of this approach is that it can be extended to the m-Laplacian operator and more general nonlinearities, see [35] for example. However, it seems very hard to apply it to sign-changing solutions if one does not understand the complicated intersection between two solutions to (1.1) in the second nodal domain.

For sign-changing radial solutions, the existence results have been established by Coffman [8] and McLeod, Troy and Weissler [24] using ODE shooting techniques and a scaling argument. There are also other approaches including variational methods (Bartsch and William [3], Struwe [36]) and heat flow (Conti-Verzini-Terracinni [10]). But for the uniqueness of sign-changing solutions, to our knowledge, there are few works. In [9], using Coffman's approach, Cortazar, Garcia-Huidobro and Yarur study the uniqueness of sign-changing radial solution to

$$\Delta u + f(u) = 0, \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

under some convexity and sublinear growth conditions of $f(u)$. In the canonical case of $f(u) = |u|^{p-1}u - |u|^{q-1}u$, the condition on p and q is that:

$$p \geq 1, \ 0 < q < p, \quad \text{and } p + q \leq \frac{2}{N-2}. \tag{1.5}$$

The result we present here is a contribution to this matter which covers the $q = 1$ case and superlinear case. We shall employ a different method–the Liapunov-Schmidt reduction–to prove our result.

Up to now, the Lyapunov-Schmidt reduction method, which reduces an infinite-dimensional problem to a finite-dimensional one, has been widely and successfully used in constructing various solutions, see for example [38], [28], [15], [30]. It can also be used to establish uniqueness of finitely many spike solutions. For the uniqueness problem, the second author [38] applied this method and established the uniqueness and non-degeneracy of boundary spike solutions for the following singularly perturbed Neumann boundary problem:

$$\begin{cases} \epsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\epsilon > 0$ is a small parameter, Ω is a smooth bounded domain in $\mathbb{R}^N$, and p is subcritical. The main idea is to reduce the problem in $H^2(\Omega)$ into a finite-dimensional problem on the space of spikes and then compute the number of critical points for a finite-dimensional problem. The same idea has also been used successfully by Grossi [17] in computing the number of single-peak solutions of the nonlinear Schrödinger equation

$$\begin{cases} -\epsilon^2 \Delta u + V(x) u = u^p \quad \text{in } \mathbb{R}^N; \\ u > 0, \end{cases} \tag{1.7}$$

for a suitable class of potentials V and critical point P. For critical exponent problems, M. Grossi [18] used reduction approach to obtain uniqueness of solutions for slightly subcritical problem

$$\Delta u + u^{\frac{N+2}{N-2}-\epsilon} = 0, \ u > 0 \text{ in } \Omega, \ u \in H_0^1(\Omega)$$

when Ω is a symmetric convex domain.

The purpose of this paper is to establish the uniqueness of sign-changing radial solutions to (1.1) by the Liapunov-Schmidt reduction. After setting $p = \frac{N+2}{N-2} - \epsilon$, then problem (1.1) will become a singularly perturbed one and then we can use the idea in [38] and [11] to establish the uniqueness and non-degeneracy of sign-changing solution to (1.1) for sufficient small $\epsilon > 0$.

Our first result concerns the uniqueness of sign-changing radial solution:

Theorem 1.1. *For any positive integer k, there exists a positive constant ϵ_0 such that for $p \in \left(\frac{N+2}{N-2} - \epsilon_0, \frac{N+2}{N-2}\right)$, there exists an unique sign-changing radial solution to (1.1) with $u(0) > 0$ and exactly k zeros.*

Remark. *Using the same idea, we can give a new proof on the uniqueness of positive solution to the equation (1.1) with an almost critical power.*

Our second result concerns the eigenvalue estimates associated with the linearized operator at u_ϵ, the solutions obtained in Theorem 1.1:

$$\overline{L}_\epsilon \equiv \Delta - 1 + p|u_\epsilon|^{p-1}. \tag{1.8}$$

We have the following non-degeneracy result:

Theorem 1.2. *There exists a number $\epsilon_0 > 0$ such that for $p \in (\frac{N+2}{N-2} - \epsilon_0, \frac{N+2}{N-2})$, u_ϵ is non-degenerate, i.e., if ϕ satisfies*

$$\Delta\phi - \phi + p|u|^{p-1}\phi = 0, \qquad |\phi| \le 1, \ \text{in } \mathbb{R}^N \tag{1.9}$$

then

$$\phi \in \operatorname{span}\left\{ \frac{\partial u_\epsilon}{\partial x_1}, \cdots, \frac{\partial u_\epsilon}{\partial x_N} \right\}. \tag{1.10}$$

The organization of the paper is as follows. In Section 2 we give some preliminary analysis. In Section 3 a finite dimensional reduction procedure is given. In Section 4 we show the existence and uniqueness. Finally in Section 5 we give the small eigenvalue estimate and complete the proof of Theorem 1.2.

Throughout this paper we denote various generic constants by C. We use $O(B), o(B)$ to mean $|O(B)| \le C|B|, o(B)/|B| \to 0$ as $|B| \to 0$, respectively.

Acknowledgments: The research of J. Wei is partially supported by NSERC of Canada.

2 The asymptotic behavior of the solutions

In this section, we will give some preliminary analysis. First by Pohozaev's non-existence result, equation (1.1) only has trivial solution $u = 0$ when $p \ge 2^* - 1$, see [32]. So if u_ϵ is a sign-changing solution to (1.1) with $p = (N+2)/(N-2) - \epsilon, \epsilon > 0$, then u_ϵ must become unbounded as $\epsilon \to 0$. Moreover, by the result of Felmer, Quaas, Tang and Yu [12],

$$u_\epsilon(0) \to \infty \quad \text{and} \quad R_\epsilon \to 0 \tag{2.1}$$

as $\epsilon \to 0$, where R_ϵ is the first zero point of u_ϵ. Without loss of generality, in this paper, we assume that $u_\epsilon(0) > 0$ and we will consider sign-changing once radial solutions to (1.1). The proofs can be easily modified to deal with sign-changing solutions with more than one nodes.

The key estimate we shall obtain first is the relation between $u_\epsilon(0)$ and the first radius R_ϵ. To this end, we take the so-called Emden-Fowler transformation to u_ϵ as in [11]. Let

$$v_\epsilon(t) = r^\alpha u_\epsilon(r), \quad r = e^t, \ \alpha = \frac{2}{p-1}.$$

Then v_ϵ satisfies

$$v'' - \beta v' - (\gamma + e^{2t})v + |v|^{p-1}v = 0, \quad t \in (-\infty, \infty), \tag{2.2}$$

where

$$p = \frac{N+2}{N-2} - \epsilon, \ \beta = \frac{(N-2)^2\epsilon}{4-(N-2)\epsilon} \quad \text{and} \quad \gamma = \frac{(N-2)^2}{4} - \frac{\beta^2}{4}. \tag{2.3}$$

Recall that the corresponding energy functional of equation (1.1) is

$$\widetilde{E}_\epsilon(u) = \frac{1}{2}\int_0^\infty \left(|u'|^2 + |u|^2\right) r^{N-1}\, dr - \frac{1}{p+1}\int_0^\infty |u|^{p+1} r^{N-1}\, dr,$$

and by the Emden-Fowler transformation,

$$\int_0^\infty |u'|^2 r^{N-1}\, dr = \int_{-\infty}^\infty \Big[|v'|^2 + \gamma |v|^2\Big] e^{-\beta t}\, dt;$$

$$\int_0^\infty |u|^2 r^{N-1}\, dr = \int_{-\infty}^\infty e^{2t}|v|^2 e^{-\beta t}\, dt;$$

$$\int_0^\infty |u|^{p+1} r^{N-1}\,dr = \int_{-\infty}^\infty |v|^{p+1} e^{-\beta t}\,dt.$$

Thus the corresponding energy functional for equation (2.2) is

$$E_\epsilon(v) = \frac{1}{2}\int_{-\infty}^\infty \Big[|v'|^2 + (\gamma + e^{2t})|v|^2\Big] e^{-\beta t}\,dt - \frac{1}{p+1}\int_{-\infty}^\infty |v|^{p+1} e^{-\beta t}\,dt, \tag{2.4}$$

and $u(r) \in H^1(\mathbb{R}^N)$ if and only if $v(t) \in H$, where H is the Hilbert space defined by

$$H \equiv \{v \in H^1(\mathbb{R}) | \int_{-\infty}^\infty \Big[|v'|^2 + (\gamma + e^{2t})|v|^2\Big] e^{-\beta t}\,dt < \infty\} \tag{2.5}$$

with the inner product

$$(v, w)_\epsilon = \int_{-\infty}^\infty \Big[v'w' + (\gamma + e^{2t})vw\Big] e^{-\beta t}\,dt. \tag{2.6}$$

Similarly, we define the weighted L^2-product as follows:

$$\langle v, w\rangle_\epsilon = \int_{-\infty}^\infty vwe^{-\beta t}\,dt. \tag{2.7}$$

To get the asymptotic behaviour of the solutions, by standard blow-up analysis, we first have the following Lemma:

Lemma 2.1. *Let v_ϵ be a solution of (2.2). Then there exists a positive constant $C = C(N)$ such that*

$$\|v_\epsilon\|_\infty \le C. \tag{2.8}$$

Since the uniqueness of positive solutions is known for u in ball and annulus, so is it for v and we have the following a priori estimate of energy of v_ϵ:

Lemma 2.2. *Let v_ϵ be a solution of (2.2). Then there exists a small positive constant δ such that*

$$E_\epsilon(v_\epsilon) < 2E_\epsilon(w_0) + \delta < 3E_\epsilon(w_0), \tag{2.9}$$

where w_0 is the unique positive solution of the following problem

$$\begin{cases} w'' - \frac{(N-2)^2}{4} w + w^{2^*-1} = 0, w > 0 \quad \text{in } \mathbb{R}; \\ w(0) = \max_{t\in\mathbb{R}} w(t), \quad w(t) \to 0, \text{ as } |t| \to \infty. \end{cases} \tag{2.10}$$

Using the above a priori estimate of energy, we can follow the argument of [27] to prove the following asymptotic behavior of v_ϵ:

Lemma 2.3. *Suppose v_ϵ is a sign-changing once solution of (2.2), then v_ϵ has exactly one local maximum point t_1 and one local minimum point t_2 in $(-\infty, \infty)$, provided that ϵ is sufficiently small. Moreover,*

$$v_\epsilon(t) = w_0(t - t_1) - w_0(t - t_2) + o(1) \tag{2.11}$$

and

$$t_1 < t_2, \quad t_1 \to -\infty, \quad t_2 \to -\infty, \quad |t_2 - t_1| \to \infty, \quad as\ \epsilon \to 0 \tag{2.12}$$

where w_0 is the unique positive solution to equation (2.10) and $o(1) \to 0$ as $\epsilon \to 0$.

Proof. First we show that the local maximum point must go to $-\infty$ as $\epsilon \to 0$. Suppose not, there exists a sequence of local maximum points t_ϵ of v_ϵ such that $t_\epsilon \to t_0$. By the estimate of energy of v_ϵ we get $v_\epsilon(t + t_\epsilon) \to v_0$ in C^2_{loc}, where v_0 satisfies

$$\begin{aligned} &v'' - (\gamma_0 + e^{2t})v + v^{2^\star - 1} = 0, \ v \geq 0 \text{ in } \mathbb{R}, \\ &v(t) \to 0, \ \text{as } |t| \to \infty, \end{aligned} \tag{2.13}$$

where $\gamma_0 = \frac{(N-2)^2}{4}$. But by Pohozave's identity, $v_0 \equiv 0$. This contradicts with $v_0(0) \geq \gamma_0^{1/(2^\star - 1)} > 0$.

Next we show that the distance of local maximum point and zero point of v_ϵ goes to ∞. Suppose not, using the same notation above, there exists $d \in \mathbb{R}$ such that, $v_\epsilon(t + t_\epsilon) \to v_0$ in $C^2_{loc}((-\infty, d))$, where v_0 satisfies

$$\begin{aligned} v'' - (\gamma_0 + e^{2t})v + v^{2^\star - 1} = 0, \ v \geq 0 \text{ in } (-\infty, d), \\ v(d) = 0, \quad v(t) \to 0, \ \text{as } |t| \to \infty. \end{aligned} \tag{2.14}$$

This is also a contradiction to the Pohozave's identity.

Now we show that there only exists one local maximum point. Suppose not, there are at least are two local maximum points t_1 and t_2. We first show that $|t_1 - t_2| \to \infty$. Suppose not, $|t_1 - t_2|$ is bounded, then using the same notations, $v_\epsilon(t + t_1) \to v_0$ in $C^2_{loc}(\mathbb{R})$, where v_0 satisfies (2.13). Moreover since $v'_\epsilon(0) = 0$, $v'_0(0) = 0$ and then applying Lemma 4.2 in [27] and the arguement right after the proof of Lemma 4.2, we get a contradiction. Thus $|t_1 - t_2| \to \infty$. Then we have a lower bound of the energy functional $E_\epsilon(v_\epsilon) > 2E_\epsilon(w_0) + C_1 > 2E_\epsilon(w_0) + \delta$ for some $C_1 > 0$ independent of ϵ small, which contradicts with Lemma 2.2.

For the negative part, we can get the similar result and complete the proof. □

Now we set

$$S_\epsilon[v] = v'' - \beta v' - (\gamma + e^{2t})v + |v|^{p-1}v. \tag{2.15}$$

To get more accurate information on asymptotic behaviour, we introduce the function w be the unique positive solution of

$$\begin{cases} w'' - \frac{(N-2)^2}{4} w + w^p = 0 \quad \text{in } \mathbb{R}; \\ w(0) = \max_{t \in \mathbb{R}} w(t), \quad w(t) \to 0, \ \text{as } |t| \to \infty. \end{cases} \tag{2.16}$$

It is standard that

$$\begin{cases} w(t) = A_{\epsilon,N} e^{-(N-2)t/2} + O\big(e^{-p(N-2)t/2}\big), \quad t \geq 0; \\ \\ w'(t) = -\frac{N-2}{2} A_{\epsilon,N} e^{-(N-2)t/2} + O\big(e^{-p(N-2)t/2}\big), \quad t \geq 0, \end{cases} \tag{2.17}$$

where $A_{\epsilon,N} > 0$ is a constant depending only on ϵ and N. Actually the function $w(t)$ can be written explicitly and has the following form

$$w(t) = \gamma_0^{\frac{1}{p-1}} (\frac{p+1}{2})^{\frac{1}{p-1}} \Big[\cosh\big(\frac{p-1}{2}\gamma_0^{1/2} t\big)\Big]^{-\frac{2}{p-1}}.$$

Testing (2.17) with w and w' and integrating by parts, one arrives at the following identity:

$$\int_{\mathbb{R}} |w'|^2 \, dt = (\frac{1}{2} - \frac{1}{p+1}) \int_{\mathbb{R}} w^{p+1} \, dt = \gamma_0 (\frac{p-1}{p+3}) \int_{\mathbb{R}} w^2 \, dt. \tag{2.18}$$

Note that $w \notin H$ when $N = 3, 4$. For t_1, t_2 obtained in Lemma 2.3, we set w_{j,t_j} to be the unique solution of

$$v'' - (\gamma_0 + e^{2s})v + w_{t_j}^p = 0, \quad \text{where } w_{t_j}(s) = w(s - t_j),\ j = 1, 2 \tag{2.19}$$

in the Hilbert space H. The existence and uniqueness of w_{j,t_j} are derived from the Riesz's representation theorem.

Using the ODE analysis, we can obtain the following asymptotic expansion of w_{j,t_j}, $j = 1, 2$ for whose proof we postpone to Appendix A:

Lemma 2.4. *For ϵ sufficient small,*

$$w_{j,t_j} = w_{t_j} + \phi_{j,t_j} + O(e^{2t_j}), \tag{2.20}$$

where for $N = 3$,

$$\phi_{j,t_j}(s) = -e^{t_j/2} A_{\epsilon,3}\, e^{-s/2}\big(1 - e^{-e^s}\big); \tag{2.21}$$

for $N = 4$,

$$\phi_{j,t_j}(s) = -e^{t_j} A_{\epsilon,4}\, e^{-s}\big[1 - \rho_0(\frac{1}{4}e^{2s})\big], \tag{2.22}$$

and

$$\rho_0(r) = 2\sqrt{r} K_1(2\sqrt{r}),$$

where $K_1(z)$ is the modified Bessel function of second kind and satisfies

$$z^2 K_1''(z) + z K_1'(z) - (z^2 + 1) K_1(z) = 0;$$

for $N = 5$,

$$\phi_{j,t_j}(s) = -e^{3t_j/2} A_{\epsilon,5}\, e^{-3s/2}\Big[1 - (1+e^s)e^{-e^s}\Big]; \tag{2.23}$$

for $N = 6$,

$$\phi_{j,t_j} = -e^{2t_j} A_{\epsilon,6} e^{-2s}\big[1 - u_0(\frac{1}{16^2}e^{4s})\big],$$

where

$$u_0(r) = 8\sqrt{r} K_2(4r^{1/4}),$$

where $K_2(z)$ is the modified Bessel function of second kind and satisfies

$$z^2 K_2''(z) + zK_2'(z) - (z^2+4)K_2(z) = 0;$$

for $N > 6$, $\phi_{j,t_j} = 0$.

Remark. *By the maximum principle, we have the following useful estimates:*

$$0 < w_{j,t_j} < w_{t_j}, \quad -w_{t_j} < \phi_{j,t_j} < 0, \tag{2.24}$$

and

$$|w'_{t_j}| \le c_1 w_{t_j} \le c_2, \ |w'_{j,t_j}| \le c_1 w_{j,t_j} \le c_2, \tag{2.25}$$

where c_1, c_2 are two positive constants independent of ϵ small.

From the above Lemma 2.4 and (2.12), we see that $w_{j,t_j} = w_{t_j} + o(1) = w_{0,t_j} + o(1)$ in all the cases for $j = 1, 2$. Thus by (2.11),

$$v_\epsilon(t) = w_{\epsilon,\mathbf{t}} + o(1), \tag{2.26}$$

where

$$w_{\epsilon,\mathbf{t}}(t) = w_{1,t_1}(t) - w_{2,t_2}(t). \tag{2.27}$$

Before studying the properties of $w_{\epsilon,\mathbf{t}}$, we need some preliminary Lemmas. The first one is a useful inequality:

Lemma 2.5. *For $x \ge 0$, $y \ge 0$,*

$$|x^p - y^p| \le \begin{cases} |x-y|^p, & \text{if } 0 < p < 1, \\ p|x-y|(x^{p-1} + y^{p-1}), & \text{if } 1 \le p < \infty. \end{cases} \tag{2.28}$$

The following Lemma is proved in Proposition 1.2 of [2].

Lemma 2.6. *Let $f \in C(R) \cap L^\infty(R)$, $g \in C(R)$ be even and satisfy for some $\alpha \geq 0$, $\beta \geq 0$, $\gamma_0 \in R$,*

$$f(x)exp(\alpha|x|)|x|^\beta \to \gamma_0 \text{ as } |x| \to \infty,$$

$$\int_R |g(x)|exp(\alpha|x|)(1+|x|^\beta)dx < \infty.$$

Then

$$exp(\alpha|y|)|y|^\beta \int_R g(x+y)f(x)dx \to \gamma_0 \int_R g(x)exp(-\alpha x_1)dx \text{ as } |y| \to \infty.$$

Next we state a useful Lemma about the interactions of two w's:

Lemma 2.7. *For $|r-s| \gg 1$ and $\eta > \theta > 0$, there hold*

$$w^\eta(t-r)w^\theta(t-s) = O(w^\theta(|r-s|)); \tag{2.29}$$

$$\int_{-\infty}^{\infty} w^\eta(t-r)w^\theta(t-s)\,dt = \big(1+o(1)\big)w^\theta(|r-s|)\int_{-\infty}^{\infty} w^\eta(t)e^{\theta\sqrt{\gamma_0}t}\,dt, \tag{2.30}$$

where $o(1) \to 0$ as $|t-s| \to \infty$.

Proof. The conclusion follows from (2.17) and the Lebesgue's Dominated Convergence Theorem, see for example [22].

□

Now we have the following error estimates:

Lemma 2.8. *For ϵ sufficiently small and t_1, t_2 satisfy (2.12), there is a constant C independent of ϵ, t_1 and t_2 such that*

$$\|S_\epsilon[w_{\epsilon,\boldsymbol{t}}]\|_\infty + \int_{-\infty}^{\infty}\Big|S_\epsilon[w_{\epsilon,\boldsymbol{t}}]\Big|e^{-\beta t}\,dt \leq C\big[\beta + e^{\tau t_2} + e^{-\tau|t_1-t_2|/2}\big] \quad \text{for } N=3;$$

$$\|S_\epsilon[w_{\epsilon,\boldsymbol{t}}]\|_\infty + \int_{-\infty}^{\infty}\Big|S_\epsilon[w_{\epsilon,\boldsymbol{t}}]\Big|e^{-\beta t}\,dt \leq C\big[\beta + t_2^\tau e^{2\tau t_2} + e^{-\tau|t_1-t_2|}\big] \quad \text{for } N=4;$$

$$\|S_\epsilon[w_{\epsilon,\boldsymbol{t}}]\|_\infty + \int_{-\infty}^{\infty}\Big|S_\epsilon[w_{\epsilon,\boldsymbol{t}}]\Big|e^{-\beta t}\,dt \leq C\big[\beta + e^{2\tau t_2} + e^{-\tau(N-2)|t_1-t_2|/2}\big] \quad \text{for } N \geq 5,$$

where τ is a constant satisfying $\frac{1}{2} < \tau < \frac{\min\{p,2\}}{2}$.

Proof. By the equation satisfied by w_{j,t_j}, we have

$$S_\epsilon[w_{\epsilon,\mathbf{t}}] = -\beta w'_{\epsilon,\mathbf{t}} - (\gamma-\gamma_0)w_{\epsilon,\mathbf{t}} + |w_{\epsilon,\mathbf{t}}|^{p-1}w_{\epsilon,\mathbf{t}} - w_{t_1}^p + w_{t_2}^p. \tag{2.31}$$

From the exponential decay of w_j, (2.24), (2.25) and $\gamma-\gamma_0=-\frac{\beta^2}{4}$, we deduce that

$$|\beta w'_{\epsilon,\mathbf{t}}|\le C\beta(w_{t_1}+w_{t_2}),$$

and

$$|(\gamma-\gamma_0)w_{\epsilon,\mathbf{t}}|\le C\beta^2(w_{t_1}+w_{t_2}).$$

Next, we divide $(-\infty,\infty)$ into 2 intervals I_1,I_2 defined by

$$I_1=(-\infty,\frac{t_1+t_2}{2}),\quad I_2=[\frac{t_1+t_2}{2},\infty).$$

Then on I_i, $i=1,2$, we have $w_{t_j}\le w_{t_i}$ and then $w_{j,t_j}\le w_{i,t_i}$ by the maximum principle for $i\neq j$. So on I_1 we use inequality (2.28) to get

$$\begin{aligned}\Big||w_{\epsilon,\mathbf{t}}|^{p-1}w_{\epsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\Big| &\le Cw_{t_1}^{p-1}w_{t_2}+Cw_{t_1}^{p-1}\phi_{1,t_1}\\ &\le Cw_{t_1}^{p-\tau}w_{t_2}^{\tau}+Cw_{t_1}^{p-\tau}\phi_{1,t_1}^{\tau},\end{aligned}$$

for any $\tau\in(0,1]$.

Similarly on I_2 the following inequality holds,

$$\begin{aligned}\Big||w_{\epsilon,\mathbf{t}}|^{p-1}w_{\epsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\Big| &\le Cw_{t_2}^{p-1}w_{t_1}+Cw_{t_2}^{p-1}\phi_{2,t_2}\\ &\le Cw_{t_2}^{p-\tau}w_{t_1}^{\tau}+Cw_{t_2}^{p-\tau}\phi_{2,t_2}^{\tau},\end{aligned}$$

for any $\tau\in(0,1]$.

By the above inequalities and using Lemma 2.6, the desired result follows.

□

In order to obtain the a priori estimate of t_1,t_2 and compute the energy expansion $E_\epsilon[w_{\epsilon,\mathbf{t}}]$, we need to estimate

$$\|v_\epsilon-w_{\epsilon,\mathbf{t}}\|_\infty\quad\text{and } \|v_\epsilon-w_{\epsilon,\mathbf{t}}\|_H.$$

Lemma 2.9. *For ϵ sufficiently small, there is a constant C independent of ϵ such that*

$$v_\epsilon=w_{\epsilon,\mathbf{t}}+\phi_\epsilon, \tag{2.32}$$

where

$$\begin{cases}\|\phi_\epsilon\|_\infty+\|\phi_\epsilon\|_H\le C\big[\beta+e^{\tau t_2}+e^{-\tau|t_1-t_2|/2}\big] & \text{for } N=3;\\ \|\phi_\epsilon\|_\infty+\|\phi_\epsilon\|_H\le C\big[\beta+t_2^{\tau}e^{2\tau t_2}+e^{-\tau|t_1-t_2|}\big] & \text{for } N=4;\\ \|\phi_\epsilon\|_\infty+\|\phi_\epsilon\|_H\le C\big[\beta+e^{2\tau t_2}+e^{-\tau(N-2)|t_1-t_2|/2}\big] & \text{for } N\ge 5,\end{cases} \tag{2.33}$$

where τ satisfies $\frac{1}{2}<\tau<\frac{\min\{p,2\}}{2}$.

Proof. We may follow the arguments given in the proof of Lemma 2.4 in [19]. First by the properties of w_{j,t_j}'s we can choose proper t_j's such that the maximum points r_ϵ and minimum points s_ϵ of v_ϵ are also the ones of $w_{\epsilon,\mathbf{t}}$, respectively. Let $v_\epsilon = w_{\epsilon,\mathbf{t}} + \phi_\epsilon$, then $\phi_\epsilon \to 0$ and satisfies

$$\phi'' - \beta\phi' - (\gamma + e^{2t})\phi + p|w_{\epsilon,\mathbf{t}}|^{p-1}\phi + S_\epsilon[w_{\epsilon,\mathbf{t}}] + N_\epsilon[\phi] = 0,$$

where

$$N_\epsilon[\phi] = |w_{\epsilon,\mathbf{t}} + \phi|^{p-1}(w_{\epsilon,\mathbf{t}} + \phi) - |w_{\epsilon,\mathbf{t}}|^{p-1}w_{\epsilon,\mathbf{t}} - p|w_{\epsilon,\mathbf{t}}|^{p-1}\phi.$$

Now we prove the estimates for ϕ_ϵ by contradiction. Denote the right hand side order term of (2.33) by K_ϵ and suppose that

$$\|\phi_\epsilon\|_\infty / K_\epsilon \to \infty.$$

Let $\widetilde{\phi}_\epsilon = \phi_\epsilon / \|\phi_\epsilon\|_\infty$, then $\widetilde{\phi}_\epsilon$ satisfies

$$\widetilde{\phi}'' - \beta\widetilde{\phi}' - (\gamma + e^{2t})\widetilde{\phi} + p\big|w_{\epsilon,\mathbf{t}}\big|^{p-1}\widetilde{\phi} + \frac{S_\epsilon[w_{\epsilon,\mathbf{t}}]}{\|\phi_\epsilon\|_\infty} + \frac{N_\epsilon[\phi_\epsilon]}{\|\phi_\epsilon\|_\infty} = 0. \tag{2.34}$$

Note that

$$\Big|\frac{S_\epsilon[w_{\epsilon,\mathbf{t}}]}{\|\phi_\epsilon\|_\infty}\Big| \le CK_\epsilon/\|\phi_\epsilon\|_\infty, \quad \Big|\frac{N_\epsilon[\phi_\epsilon]}{\|\phi_\epsilon\|_\infty}\Big| \le C\|\phi_\epsilon\|_\infty^{\min\{p-1,1\}}. \tag{2.35}$$

Let t_ϵ be such that $\widetilde{\phi}_\epsilon(t_\epsilon) = \|\widetilde{\phi}_\epsilon\|_\infty = 1$ (the same proof applies if $\widetilde{\phi}_\epsilon(t_\epsilon) = -1$). Then by (2.34), (2.35) and the Maximum Principle, we have $|t_\epsilon - t_1| \le C$ or $|t_\epsilon - t_2| \le C$. Thus $|t_\epsilon - r_\epsilon| \le C$ or $|t_\epsilon - s_\epsilon| \le C$. Without loss of generality, we assume that $|t_\epsilon - r_\epsilon| \le C$. Then by the usual elliptic regular theory, we may take a subsequence $\widetilde{\phi}_\epsilon(t + r_\epsilon) \to \widetilde{\phi}_0(t)$ as $\epsilon \to 0$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ since $|r_\epsilon - t_1| \to 0$, where $\widetilde{\phi}_0$ satisfies

$$\widetilde{\phi}_0'' - \frac{(N-2)^2}{4}\widetilde{\phi}_0 + \frac{N+2}{N-2}w_0^{4/(N-2)}\widetilde{\phi}_0 = 0, \text{ and } \widetilde{\phi}_0'(0) = 0,$$

which implies $\widetilde{\phi}_0 \equiv 0$. This contradicts to the fact that $1 = \widetilde{\phi}_\epsilon(t_\epsilon) \to \widetilde{\phi}_0(t_0)$ for some t_0. Therefore we complete the proof.

□

The following is the basic technical estimate in this paper which gives the a priori estimates for t_1 and t_2:

Lemma 2.10. *For ϵ sufficient small we have for $N = 3$,*

$$\begin{cases} t_1 = \log a + 2\log b + 3\log\beta; \\ t_2 = \log a + \log\beta, \end{cases} \tag{2.36}$$

where a, b are constants and

$$a \to a_{0,3}, \;\; b \to b_{0,3}.$$

For $N=4$,

$$\begin{cases} t_1 - t_2 = \log b + \log\beta; \\ -2t_2 e^{2t_2} = a\beta, \end{cases} \tag{2.37}$$

where a, b are constants and

$$a \to a_{0,4}, \ \ b \to b_{0,4}.$$

For $N \geq 5$,

$$\begin{cases} t_1 = \frac{1}{2}\log a + \frac{2}{N-2}\log b + \frac{N+2}{2(N-2)}\log\beta; \\ t_2 = \frac{1}{2}\log a + \frac{1}{2}\log\beta, \end{cases} \tag{2.38}$$

where a, b are constants and

$$a \to a_{0,N}, \ \ b \to b_{0,N}.$$

Here $a_{0,N}, b_{0,N}$ are positive constants.

Proof. Here we only give the proof for $N=3$, for $N>3$, the proof is similar. From $S_\epsilon[v_\epsilon]=0$ and $v_\epsilon = w_{\epsilon,\mathbf{t}} + \phi_\epsilon$ we deduce

$$L_{\epsilon,\mathbf{t}}[\phi] + S_\epsilon[w_{\epsilon,\mathbf{t}}] + N_\epsilon[\phi] = 0, \tag{2.39}$$

where

$$L_{\epsilon,\mathbf{t}}[\phi] = \phi'' - \beta\phi' - (\gamma + e^{2t})\phi + p|w_{\epsilon,\mathbf{t}}|^{p-1}\phi,$$

and

$$N_\epsilon[\phi] = |w_{\epsilon,\mathbf{t}} + \phi|^{p-1}(w_{\epsilon,\mathbf{t}} + \phi) - |w_{\epsilon,\mathbf{t}}|^{p-1}w_{\epsilon,\mathbf{t}} - p|w_{\epsilon,\mathbf{t}}|^{p-1}\phi.$$

Multiplying (2.39) by w'_{1,t_1} and integrating over $\mathbb{R}$, we obtain

$$\int_{-\infty}^{\infty} L_{\epsilon,\mathbf{t}}[\phi]w'_{1,t_1}\,dt + \int_{-\infty}^{\infty} S_\epsilon[w_{\epsilon,\mathbf{t}}]w'_{1,t_1} + \int_{-\infty}^{\infty} N_\epsilon[\phi]w'_{1,t_1}\,dt = 0.$$

Integrating by parts and using Lemma 2.9 we have

$$\begin{aligned} \int_{-\infty}^{\infty} L_{\epsilon,\mathbf{t}}[\phi]w'_{1,t_1}\,dt &= \int_{-\infty}^{\infty}\Big[w'''_{1,t_1} + \beta w''_{1,t_1} - (\gamma + e^{2t})w'_{1,t_1} + p|w_{\epsilon,\mathbf{t}}|^{p-1}w'_{1,t_1}\Big]\phi\,dt \\ &= \int_{-\infty}^{\infty}\beta\big[(\gamma_0 + e^{2t})w_{1,t_1} - w_{t_1}^p\big]\phi\,dt - (\gamma-\gamma_0)\int_{-\infty}^{\infty} w'_{1,t_1}\phi\,dt \\ &\quad + 2\int_{-\infty}^{\infty} e^{2t}w_{1,t_1}\phi\,dt + p\int_{-\infty}^{\infty}\Big[|w_{\epsilon,\mathbf{t}}|^{p-1}w'_{1,t_1} - w_{t_1}^{p-1}w'_{t_1}\Big]\phi\,dt \\ &= o(\big[\beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2}\big]). \end{aligned}$$

Similarly we can obtain

$$\int_{-\infty}^{\infty} L_{\epsilon,\mathbf{t}}[\phi] w'_{2,t_2}\, dt = o\big(\big[\beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2}\big]\big).$$

For the nonlinearity term using (2.28) we get

$$\begin{aligned}\Big|N_\epsilon[\phi]\Big| &= \Big||w_{\epsilon,\mathbf{t}} + \phi|^{p-1}(w_{\epsilon,\mathbf{t}} + \phi) \\ &\quad - |w_{\epsilon,\mathbf{t}}|^{p-1} w_{\epsilon,\mathbf{t}} - p|w_{\epsilon,\mathbf{t}}|^{p-1}\phi\Big| \\ &\leq C|\phi|^{\min\{p,2\}}.\end{aligned}$$

So using the exponential decay of w and taking $\tau > \max\{\frac{1}{2}, \frac{1}{p}\}$ we deduce

$$\int_{-\infty}^{\infty} N_\epsilon[\phi] w'_{1,t_1}\, dt = O(\|\phi\|_\infty^2) = o\big(\big[\beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2}\big]\big).$$

Similarly we can obtain

$$\int_{-\infty}^{\infty} N_\epsilon[\phi] w'_{2,t_2}\, dt = o\big(\big[\beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2}\big]\big).$$

To estimate $\int_{-\infty}^{\infty} S_\epsilon[w_{\epsilon,\mathbf{t}}] w'_{1,t_1}\, dt$, we write

$$\begin{aligned}\int_{-\infty}^{\infty} S_\epsilon[w_{\epsilon,\mathbf{t}}] w'_{1,t_1}\, dt &= \int_{-\infty}^{\infty} \Big[- \beta w'_{\epsilon,\mathbf{t}} - (\gamma - \gamma_0) w_{\epsilon,\mathbf{t}} + |w_{\epsilon,\mathbf{t}}|^{p-1} w_{\epsilon,\mathbf{t}} - w_{t_1}^p + w_{t_2}^p \Big] w'_{1,t_1}\, dt \\ &= E_1 + E_2 + E_3,\end{aligned}$$

where

$$\begin{aligned}E_1 &= -\beta \int_{-\infty}^{\infty} w'_{\epsilon,\mathbf{t}} w'_{1,t_1}\, dt; \\ E_2 &= -(\gamma - \gamma_0) \int_{-\infty}^{\infty} w_{\epsilon,\mathbf{t}} w'_{1,t_1}\, dt; \\ E_3 &= \int_{-\infty}^{\infty} \Big[|w_{\epsilon,\mathbf{t}}|^{p-1} w_{\epsilon,\mathbf{t}} - w_{t_1}^p + w_{t_2}^p\Big] w'_{1,t_1}\, dt.\end{aligned}$$

Using (2.25) and Lemma 2.7 we obtain

$$E_1 = -\beta \int_{-\infty}^{\infty} |w'_{1,t_1}|^2\, dt + \beta \int_{-\infty}^{\infty} w'_{1,t_1} w'_{2,t_2}\, dt = -\beta \int_{-\infty}^{\infty} |w'|^2\, dt + o(\beta).$$

Note that $\gamma - \gamma_0 = -\beta^2/4$ and using (2.25) we get

$$E_2 = \frac{\beta^2}{4} \int_{-\infty}^{\infty} w_{\epsilon,\mathbf{t}} w'_{1,t_1}\, dt = O(\beta^2).$$

To estimate E_3, following the argument in the proof of Lemma 2.8. We divide $(-\infty,\infty)$ into two intervals I_1, I_2 defined by

$$I_1=(-\infty,\frac{t_1+t_2}{2}),\quad I_2=[\frac{t_1+t_2}{2},\infty).$$

On I_1 the following equality holds:

$$\begin{aligned}&\left|w_{\epsilon,\mathbf{t}}\right|^{p-1}w_{\epsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\\&=\left[(w_{1,t_1}-w_{2,t_2})^p-w_{1,t_1}^p+pw_{1,t_1}^{p-1}w_{2,t_2}\right]-pw_{1,t_1}^{p-1}w_{2,t_2}\\&+\left[(w_{t_1}+\phi_{1,t_1})^p-w_{t_1}^p-pw_{t_1}^{p-1}\phi_{1,t_1}\right]+pw_{t_1}^{p-1}\phi_{1,t_1}.\end{aligned}$$

We use inequality (2.28) to get

$$\begin{aligned}\left|(w_{1,t_1}-w_{2,t_2})^p-w_{1,t_1}^p+pw_{1,t_1}^{p-1}w_{2,t_2}\right|&\le Cw_{t_1}^{p-\delta}w_{t_2}^{\delta},\\\left|(w_{t_1}+\phi_{1,t_1})^p-w_{t_1}^p-pw_{t_1}^{p-1}\phi_{1,t_1}\right|&\le Cw_{t_1}^{p-\delta}\phi_{1,t_1}^{\delta},\end{aligned}$$

for any $1<\delta<2$.

Then using Lemma 2.7 and integrating by parts, we get

$$\begin{aligned}&\int_{I_1}\left[\left|w_{\epsilon,\mathbf{t}}\right|^{p-1}w_{\epsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\right]w_{1,t_1}'\,dt\\&=\int_{-\infty}^{\infty}w_{t_1}^pw_{t_2}'\,dt-\int_{-\infty}^{\infty}w_{t_1}^p\phi_{1,t_1}'\,dt+o(e^{-|t_1-t_2|/2})+o(e^{t_1}).\end{aligned}$$

On the other hand, on I_2, using $w_{1,t_1}\le w_{2,t_2}$, (2.25) and inequality (2.28) we get

$$\left|\left[\left|w_{\epsilon,\mathbf{t}}\right|^{p-1}w_{\epsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\right]w_{1,t_1}'\right|\le Cw_{t_1}^{\delta}w_{t_2}^{p+1-\delta}+C\phi_{2,t_2}^{\delta}w_{t_2}^{p+1-\delta},$$

for any $1<\delta<2$.

Using Lemma 2.7 we get

$$\int_{I_2}\left[\left|w_{\epsilon,\mathbf{t}}\right|^{p-1}w_{\epsilon,\mathbf{t}}-w_{t_1}^p+w_{t_2}^p\right]w_{t_1}'\,dt=o(e^{-|t_1-t_2|/2})+o(e^{t_2}).$$

Thus

$$\begin{aligned}E_3&=\frac{1}{2}e^{-|t_1-t_2|/2}A_{\epsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt+\frac{1}{2}e^{t_1}A_{\epsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt\\&+\ o(e^{-|t_1-t_2|/2})+o(e^{t_2}),\end{aligned}$$

and then

$$\begin{aligned}\int_{-\infty}^{\infty}S_\epsilon[w_{\epsilon,\mathbf{t}}]w_{1,t_1}'\,dt&=-\beta\int_{-\infty}^{\infty}|w'|^2\,dt+\frac{1}{2}e^{-|t_1-t_2|/2}A_{\epsilon,3}\int_{-\infty}^{\infty}w^pe^{t/2}\,dt\\&+\ o(\beta)+o(e^{-|t_1-t_2|/2})+o(e^{t_2}).\end{aligned}\tag{2.40}$$

Similarly,

$$\int_{-\infty}^{\infty} S_\epsilon[w_{\epsilon,\mathbf{t}}]w'_{2,t_2}\,dt = \beta\int_{-\infty}^{\infty}|w'|^2\,dt + \frac{1}{2}\Big[e^{-|t_1-t_2|/2} - e^{t_2}\Big]A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta) + o(e^{-|t_1-t_2|/2}) + o(e^{t_2}). \tag{2.41}$$

Combining all the estimates above, one can see that β, e^{t_2} and $e^{-|t_1-t_2|/2}$ must have the same order.

Therefore,

$$\begin{cases} -\beta\int_{-\infty}^{\infty}|w'|^2\,dt + \frac{1}{2}e^{-|t_1-t_2|/2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt = o(\beta); \\ \beta\int_{-\infty}^{\infty}|w'|^2\,dt + \frac{1}{2}\Big[e^{-|t_1-t_2|/2} - e^{t_2}\Big]A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt = o(\beta). \end{cases}$$

Let

$$e^{t_2} = a\beta, \quad e^{-|t_1-t_2|/2} = b\beta,$$

then

$$a = \frac{4\int_{-\infty}^{\infty}|w'|^2\,dt}{A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt} + o(1) \to a_{0,3},$$

and

$$b = \frac{2\int_{-\infty}^{\infty}|w'|^2\,dt}{A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt} + o(1) \to b_{0,3},$$

where $a_{0,3}, b_{0,3}$ are positive constants.

Thus

$$\begin{cases} t_1 = \log a + 2\log b + 3\log\beta; \\ t_2 = \log a + \log\beta, \end{cases}$$

where

$$a \to a_{0,3}, \;\; b \to b_{0,3}.$$

□

We now introduce the following configuration space:

$$\Lambda \equiv \begin{cases} \Big\{\mathbf{t} = (t_1,t_2) | \frac{1}{2}a_{0,3}\beta < e^{t_2} < \frac{3}{2}a_{0,3}\beta, \frac{1}{2}b_{0,3}\beta < e^{(t_1-t_2)/2} < \frac{3}{2}b_{0,3}\beta\Big\} \text{ for } N=3; \\ \Big\{\mathbf{t} = (t_1,t_2) | \frac{1}{2}a_{0,4}\beta < -2t_2e^{2t_2} < \frac{3}{2}a_{0,4}\beta, \frac{1}{2}b_{0,4}\beta < e^{t_1-t_2} < \frac{3}{2}b_{0,4}\beta\Big\} \text{ for } N=4; \\ \Big\{\mathbf{t} = (t_1,t_2) | \frac{1}{2}a_{0,N}\beta < e^{2t_2} < \frac{3}{2}a_{0,N}\beta, \frac{1}{2}b_{0,N}\beta < e^{(N-2)(t_1-t_2)/2} < \frac{3}{2}b_{0,N}\beta\Big\} \text{ for } N\geq 5. \end{cases} \tag{2.42}$$

Then by Lemma 2.10, for ϵ sufficient small, $\mathbf{t} = (t_1,t_2) \in \Lambda$ if v_ϵ is a sign-changing solution to equation (2.2). In the next section, we will show an one-to-one correspandence between the sign-changing solution of (1.1) and the critical points of some functional in Λ.

3 The existence result: Liapunov-Schmidt reduction

In this section we outline the main steps of the so called Liapunov-Schmidt reduction method or localized energy method, which reduces the infinite problem to finding a critical point for a functional on a finite dimensional space. A very important observation is the reduction Lemma 3.6. To achieve this, we first study the solvability of a linear problem and then apply some standard fixed point theorem for contraction mapping to solve the nonlinear problem. Since the procedure has been by now standard (see for example [22] and the references therein), we will omit most of the details.

3.1 An auxiliary linear problem

In this subsection we study a linear theory which allows us to perform the finite-dimensional reduction procedure.

First observing that orthogonality to $\frac{\partial w_{\epsilon,\mathbf{t}}}{\partial t_j}$ in H, $j=1,2$, is equivalent to orthogonality to the following functions

$$Z_{\epsilon,t_j} := -(\partial_{t_j} w_{\epsilon,\mathbf{t}})'' + \beta(\partial_{t_j} w_{\epsilon,\mathbf{t}})' + (\gamma + e^{2t})\partial_{t_j} w_{\epsilon,\mathbf{t}}, \quad j=1,2, \tag{3.1}$$

in the weighted L^2-product $\langle\ ,\ \rangle_\epsilon$.

By (2.27) and elementary computations, we obtain for $j=1,2$,

$$\partial_{t_j} w_{\epsilon,\mathbf{t}} = (-1)^{j+1}\partial_{t_j} w_{j,t_j} = (-1)^{j+1}\big(\partial_{t_j} w_{t_j} + \partial_{t_j}\phi_{t,t_j}\big) + O(e^{2t_j}), \tag{3.2}$$

and

$$Z_{\epsilon,t_j} = (-1)^j\Big[p w_{t_j}^{p-1} w_{t_j}' - \beta(\partial_{t_j} w_{j,t_j})' - (\gamma-\gamma_0)\partial_{t_j} w_{j,t_j}\Big]. \tag{3.3}$$

In this section, we consider the following linear problem:

$$\begin{cases} L_{\epsilon,\mathbf{t}}[\phi] := \phi'' - \beta\phi' - (\gamma + e^{2t})\phi + p|w_{\epsilon,\mathbf{t}}|^{p-1}\phi = h + \sum\limits_{j=1}^{2} c_j Z_{\epsilon,t_j}; \\ \\ \langle \phi, Z_{\epsilon,t_j}\rangle_\epsilon = 0,\ \ j=1,2, \end{cases} \tag{3.4}$$

where $\mathbf{t}\in\Lambda$.

For the above linear problem, we have the following result:

Proposition 3.1. *Let ϕ satisfy (3.4) with $\|h\|_\infty < \infty$. Then for ϵ sufficiently small, we have*

$$\|\phi\|_\infty \le C\|h\|_\infty, \tag{3.5}$$

where C is a positive constant independent of ϵ and $\mathbf{t}\in\Lambda$.

Proof. The proof is now standard, see for example [22].

□

Using Fredholm's alternative we can show the following existence result:

Proposition 3.2. *There exists $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$ the following property holds true. Given $h \in L^\infty(\mathbb{R})$, there exists a unique pair (ϕ, c_1, c_2) such that*

$$\begin{cases} L_{\epsilon,\boldsymbol{t}}[\phi] = h + \sum_{j=1}^{2} c_j Z_{\epsilon,t_j}; \\ \langle \phi, Z_{\epsilon,t_j} \rangle_\epsilon = 0, \;\; j = 1,2. \end{cases} \tag{3.6}$$

Moreover, we have

$$\|\phi\|_\infty + \sum_{j=1}^{2} |c_j| \leq C\|h\|_\infty \tag{3.7}$$

for some positive constant C.

Proof. The result follows from Proposition 3.1 and the Fredholm's alternative theorem, see for example [22].

□

In the following , if ϕ is the unique solution given in Proposition 3.2, we set

$$\phi = A_\epsilon(h). \tag{3.8}$$

Note that (3.7) implies

$$\|A_\epsilon(h)\|_\infty \leq C\|h\|_\infty. \tag{3.9}$$

3.2 The nonlinear projected problem

This subsection is devoted to the solvability of the following non-linear projected problem:

$$\begin{cases} (w_{\epsilon,\mathbf{t}} + \phi)'' - \beta(w_{\epsilon,\mathbf{t}} + \phi)' - (\gamma + e^{2t})(w_{\epsilon,\mathbf{t}} + \phi) + |w_{\epsilon,\mathbf{t}} + \phi|^{p-1}(w_{\epsilon,\mathbf{t}} + \phi) = \sum_{j=1}^{2} c_j Z_{\epsilon,t_j}; \\ \\ \langle \phi, Z_{\epsilon,t_j} \rangle_\epsilon = 0, \;\; j = 1,2. \end{cases} \tag{3.10}$$

The first equation in (3.10) can be written as

$$\phi'' - \beta\phi' - (\gamma + e^{2t})\phi + p|w_{\epsilon,\mathbf{t}}|^{p-1}\phi = -S_\epsilon[w_{\epsilon,\mathbf{t}}] - N_\epsilon[\phi] + \sum_{j=1}^{2} c_j Z_{\epsilon,t_j},$$

where

$$N_\epsilon[\phi] = |w_{\epsilon,\mathbf{t}} + \phi|^{p-1}(w_{\epsilon,\mathbf{t}} + \phi) - |w_{\epsilon,\mathbf{t}}|^{p-1}w_{\epsilon,\mathbf{t}} - p|w_{\epsilon,\mathbf{t}}|^{p-1}\phi_{\epsilon,\mathbf{t}}. \tag{3.11}$$

First, we have the following estimates:

Lemma 3.3. *For $\boldsymbol{t} \in \Lambda$ and ϵ sufficiently small, we have for $\|\phi\|_\infty + \|\phi_1\|_\infty + \|\phi_2\|_\infty \le 1$,*

$$\|N_\epsilon[\phi]\|_\infty \le C\|\phi\|_\infty^{\min\{p,2\}}; \tag{3.12}$$

$$\|N_\epsilon[\phi_1] - N_\epsilon[\phi_2]\|_\infty \le C(\|\phi_1\|_\infty^{\min\{p-1,1\}} + \|\phi_2\|_\infty^{\min\{p-1,1\}})\|\phi_1 - \phi_2\|_\infty. \tag{3.13}$$

Proof. These inequalities follows from the mean-value theorem and inequality (2.28). □

By the standard fixed point theorem for contraction mapping and Implicit Function Theorem, Lemma 2.8 and 3.3, we have the following Proposition:

Proposition 3.4. *For $\boldsymbol{t} \in \Lambda$ and ϵ sufficiently small, there exists a unique $\phi = \phi_{\epsilon,\boldsymbol{t}}$ such that (3.10) holds. Moreover, $\boldsymbol{t} \to \phi_{\epsilon,\boldsymbol{t}}$ is of class C^1 as a map into H, and we have*

$$\|\phi_{\epsilon,\boldsymbol{t}}\|_\infty + \sum_{j=1}^{2}|c_j| \le \begin{cases} C\big[\beta + e^{\tau t_2} + e^{-\tau|t_1-t_2|/2}\big] & \text{for } N=3; \\ C\big[\beta + t_2^\tau e^{2\tau t_2} + e^{-\tau|t_1-t_2|}\big] & \text{for } N=4; \\ C\big[\beta + e^{2\tau t_2} + e^{-\tau(N-2)|t_1-t_2|/2}\big] & \text{for } N\ge 5, \end{cases} \tag{3.14}$$

where τ satisfies $\frac{1}{2} < \tau < \frac{\min\{p,2\}}{2}$.

3.3 Energy expansion for reduced energy functional

In this subsection we expand the quantity

$$K_\epsilon(\mathbf{t}) = E_\epsilon[w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}] : \ \Lambda \to \mathbb{R} \tag{3.15}$$

in terms of ϵ and $\mathbf{t}$, where $\phi_{\epsilon,\mathbf{t}}$ is obtained in Proposition 3.4.

Lemma 3.5. *For $\boldsymbol{t} \in \Lambda$ and ϵ sufficiently small, we have for $N=3$,*

$$\begin{aligned} K_\epsilon(\boldsymbol{t}) &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)(e^{-\beta t_1} + e^{-\beta t_2})\int_{-\infty}^{\infty} w^{p+1}\,dt + \frac{1}{2}e^{t_2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt \\ &\quad + e^{-|t_1-t_2|/2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}) \\ &= \widetilde{K}_\epsilon(\boldsymbol{t}) + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}). \end{aligned}$$

For $N=4$,

$$\begin{aligned} K_\epsilon(\boldsymbol{t}) &= \Big(\frac{1}{2} - \frac{1}{p+1}\Big)(e^{-\beta t_1} + e^{-\beta t_2})\int_{-\infty}^{\infty} w^{p+1}\,dt - \frac{1}{4}t_2 e^{2t_2}A_{\epsilon,4}\int_{-\infty}^{\infty} w^p e^{t}\,dt \\ &\quad + e^{-|t_1-t_2|}A_{\epsilon,4}\int_{-\infty}^{\infty} w^p e^{t}\,dt + o(\beta) + o(t_2e^{2t_2}) + o(e^{-|t_1-t_2|}) \\ &= \widetilde{K}_\epsilon(\boldsymbol{t}) + o(\beta) + o(t_2e^{2t_2}) + o(e^{-|t_1-t_2|}). \end{aligned}$$

For $N \geq 5$,

$$
\begin{aligned}
K_\epsilon(\boldsymbol{t}) &= \big(\frac{1}{2} - \frac{1}{p+1}\big)(e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1}\, dt + \frac{1}{2} e^{2t_2} \int_{-\infty}^{\infty} w^2 e^{2t}\, dt \\
&\quad + e^{-(N-2)|t_1-t_2|/2} A_{\epsilon,N} \int_{-\infty}^{\infty} w^p e^{(N-2)t/2}\, dt + o(\beta) + o(e^{2t_2}) + o(e^{-(N-2)|t_1-t_2|/2}) \\
&= \widetilde{K}_\epsilon(\boldsymbol{t}) + o(\beta) + o(e^{2t_2}) + o(e^{-(N-2)|t_1-t_2|/2}).
\end{aligned}
$$

Proof. Here again, we only give the proof for $N = 3$. By the definition of $K_\epsilon(\mathbf{t})$, we can re-write it as

$$
K_\epsilon(\mathbf{t}) = E_\epsilon[w_{\epsilon,\mathbf{t}}] + K_1 + K_2 - K_3, \tag{3.16}
$$

where

$$
K_1 = \int_{-\infty}^{\infty} \Big[w'_{\epsilon,\mathbf{t}} \phi'_{\epsilon,\mathbf{t}} + (\gamma + e^{2t}) w_{\epsilon,\mathbf{t}} \phi_{\epsilon,\mathbf{t}} \Big] e^{-\beta t}\, dt - \int_{-\infty}^{\infty} |w_{\epsilon,\mathbf{t}}|^{p-1} w_{\epsilon,\mathbf{t}} \phi_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt;
$$

$$
K_2 = \frac{1}{2} \int_{-\infty}^{\infty} \Big[|\phi'_{\epsilon,\mathbf{t}}|^2 + (\gamma + e^{2t}) |\phi_{\epsilon,\mathbf{t}}|^2 - p |w_{\epsilon,\mathbf{t}}|^{p-1} |\phi_{\epsilon,\mathbf{t}}|^2 \Big] e^{-\beta t}\, dt;
$$

$$
\begin{aligned}
K_3 = \frac{1}{p+1} \int_{-\infty}^{\infty} \Big[&|w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}|^{p+1} - |w_{\epsilon,\mathbf{t}}|^{p+1} - (p+1)|w_{\epsilon,\mathbf{t}}|^{p-1} w_{\epsilon,\mathbf{t}} \phi_{\epsilon,\mathbf{t}} \\
&- \frac{1}{2} p(p+1) |w_{\epsilon,\mathbf{t}}|^{p-1} |\phi_{\epsilon,\mathbf{t}}|^2 \Big] e^{-\beta t}\, dt.
\end{aligned}
$$

Integrating by parts and using Lemma 2.8, 2.9, we have

$$
|K_1| = \Big| - \int_{-\infty}^{\infty} S_\epsilon[w_{\epsilon,\mathbf{t}}] \phi_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt \Big| = o\Big(\big[\beta + e^{t_2} + e^{-|t_1-t_2|/2} \big] \Big). \tag{3.17}
$$

To estimate K_2, note that $\phi_{\epsilon,\mathbf{t}}$ satisfies

$$
\begin{aligned}
\phi''_{\epsilon,\mathbf{t}} - \beta \phi'_{\epsilon,\mathbf{t}} - (\gamma + e^{2t}) \phi_{\epsilon,\mathbf{t}} &= -|w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}|^{p-1} (w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}) \\
&\quad + |w_{\epsilon,\mathbf{t}}|^{p-1} w_{\epsilon,\mathbf{t}} - S_\epsilon[w_{\epsilon,\mathbf{t}}] + \sum_{j=1}^{2} c_j Z_{\epsilon,t_j}.
\end{aligned} \tag{3.18}
$$

Integrating by parts and using the orthogonality condition (3.10), we have

$$
\begin{aligned}
2K_2 = \int_{-\infty}^{\infty} \Big[&|w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}|^{p-1} (w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}) - |w_{\epsilon,\mathbf{t}}|^{p-1} w_{\epsilon,\mathbf{t}} \\
&- p |w_{\epsilon,\mathbf{t}}|^{p-1} \phi_{\epsilon,\mathbf{t}} + S_\epsilon[w_{\epsilon,\mathbf{t}}] \Big] \phi_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt.
\end{aligned}
$$

By the mean value theorem and inequality (2.28) we get

$$
\Big| |w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}|^{p-1} (w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}) - |w_{\epsilon,\mathbf{t}}|^{p-1} w_{\epsilon,\mathbf{t}} - p |w_{\epsilon,\mathbf{t}}|^{p-1} \phi_{\epsilon,\mathbf{t}} \Big| \leq C |\phi_{\epsilon,\mathbf{t}}|^{\min\{p,2\}}.
$$

So using Lemma 2.8 and 2.9 we deduce

$$K_2 = o\Big(\big[\beta + e^{t_2} + e^{-|t_1-t_2|/2}\big]\Big). \tag{3.19}$$

For K_3, using the mean value theorem and inequality (2.28),

$$\Big||w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}|^{p+1} - |w_{\epsilon,\mathbf{t}}|^{p+1} - (p+1)|w_{\epsilon,\mathbf{t}}|^{p-1}w_{\epsilon,\mathbf{t}}\phi_{\epsilon,\mathbf{t}} \\ - \frac{1}{2}p(p+1)|w_{\epsilon,\mathbf{t}}|^{p-1}|\phi_{\epsilon,\mathbf{t}}|^2\Big| \leq C|\phi_{\epsilon,\mathbf{t}}|^{\min\{p+1,3\}}.$$

So, again, using Lemma 2.8 and 2.9 it follows that

$$K_3 = o\Big(\big[\beta + e^{t_2} + e^{-|t_1-t_2|/2}\big]\Big). \tag{3.20}$$

Combing with (3.16), (3.17), (3.19), (3.20), and the estimates in Appendix B, we obtain the desired estimates.

□

We will end this section with a reduction lemma which is important for both the existence and uniqueness:

Lemma 3.6. *$v_{\epsilon,\boldsymbol{t}} = w_{\epsilon,\boldsymbol{t}} + \phi_{\epsilon,\boldsymbol{t}}$ is a critical point of E_ϵ if and only if $\boldsymbol{t}$ is a critical point of K_ϵ in Λ.*

Proof. The proof follows from the proofs in [16], [38]. For the sake of completeness, we include a proof here.

By Proposition 3.4, there exists an ϵ_0 such that, for $0 < \epsilon < \epsilon_0$, we have a C^1 map $\mathbf{t} \to \phi_{\epsilon,\mathbf{t}}$ from Λ into H such that

$$S_\epsilon[v_{\epsilon,\mathbf{t}}] = \sum_{j=1}^{2} c_j(\mathbf{t}) Z_{\epsilon,t_j}, \quad v_{\epsilon,\mathbf{t}} = w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}, \tag{3.21}$$

for some constants c_j, which also are of class C^1 in $\mathbf{t}$.

First integrating by parts we get

$$\begin{aligned} \partial_{t_j} K_\epsilon(\mathbf{t}) &= \int_{-\infty}^{\infty} \Big[v'_{\epsilon,\mathbf{t}}\big(\partial_{t_j} w_{\epsilon,\mathbf{t}} + \partial_{t_j}\phi_{\epsilon,\mathbf{t}}\big)' + (\gamma + e^{2t})v_{\epsilon,\mathbf{t}}\big(\partial_{t_j} w_{\epsilon,\mathbf{t}} + \partial_{t_j}\phi_{\epsilon,\mathbf{t}}\big)\Big]e^{-\beta t}\,dt \\ &\quad - \int_{-\infty}^{\infty} \big|v_{\epsilon,\mathbf{t}}\big|^{p-1} v_{\epsilon,\mathbf{t}}\big(\partial_{t_j} w_{\epsilon,\mathbf{t}} + \partial_{t_j}\phi_{\epsilon,\mathbf{t}}\big)e^{-\beta t}\,dt \\ &= -\int_{-\infty}^{\infty} S_\epsilon[v_{\epsilon,\mathbf{t}}]\big(\partial_{t_j} w_{\epsilon,\mathbf{t}} + \partial_{t_j}\phi_{\epsilon,\mathbf{t}}\big)e^{-\beta t}\,dt. \end{aligned} \tag{3.22}$$

If $v_{\epsilon,\mathbf{t}} = w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}$ is a critical point of E_ϵ, then $S_\epsilon[v_{\epsilon,\mathbf{t}}] = 0$. By (3.22) we get

$$\partial_{t_j} K_\epsilon(\mathbf{t}) = -\int_{-\infty}^{\infty} S_\epsilon[v_{\epsilon,\mathbf{t}}]\big(\partial_{t_j} w_{\epsilon,\mathbf{t}} + \partial_{t_j}\phi_{\epsilon,\mathbf{t}}\big)e^{-\beta t}\,dt = 0,$$

which means that $\mathbf{t}$ is a critical point of K_ϵ.

On the other hand, let $\mathbf{t}_\epsilon \in \Lambda$ be a critical point of K_ϵ, that is $\partial_{t_j} K_\epsilon(\mathbf{t}_\epsilon) = 0$, $j = 1, 2$, by (3.22) we get

$$0 = \partial_{t_j} K_\epsilon(\mathbf{t}_\epsilon) = -\int_{-\infty}^{\infty} S_\epsilon[v_{\epsilon,\mathbf{t}_\epsilon}]\big(\partial_{t_j} w_{\epsilon,\mathbf{t}_\epsilon} + \partial_{t_j} \phi_{\epsilon,\mathbf{t}_\epsilon}\big) e^{-\beta t}\, dt$$

for $j = 1, 2$. Hence by (3.21) we have

$$\sum_{i=1}^{2} c_i(\mathbf{t}_\epsilon) \int_{-\infty}^{\infty} Z_{\epsilon,t_{\epsilon,i}}\big(\partial_{t_j} w_{\epsilon,\mathbf{t}_\epsilon} + \partial_{t_j} \phi_{\epsilon,\mathbf{t}_\epsilon}\big) e^{-\beta t}\, dt = 0.$$

By Proposition 3.4 and the fact $\langle \phi_{\epsilon,\mathbf{t}_\epsilon}, Z_{\epsilon,t_{\epsilon,i}} \rangle_\epsilon = 0$,

$$\langle Z_{\epsilon,t_{\epsilon,i}}, \partial_{t_j} \phi_{\epsilon,\mathbf{t}_\epsilon} \rangle_\epsilon = -\langle \phi_{\epsilon,\mathbf{t}_\epsilon}, \partial_{t_j} Z_{\epsilon,t_{\epsilon,i}} \rangle_\epsilon = o(1). \tag{3.23}$$

On the other hand,

$$\int_{-\infty}^{\infty} Z_{\epsilon,t_{\epsilon,i}} \partial_{t_j} w_{\epsilon,\mathbf{t}_\epsilon} e^{-\beta t}\, dt = \langle Z_{\epsilon,t_{\epsilon,i}}, \partial_{t_j} w_{\epsilon,\mathbf{t}_\epsilon} \rangle_\epsilon = \delta_{ij} p \int_{-\infty}^{\infty} w^{p-1} |w'|^2\, dt + o(1). \tag{3.24}$$

By (3.23) and (3.24), the matrix

$$\int_{-\infty}^{\infty} Z_{\epsilon,t_{\epsilon,i}}\big(\partial_{t_j} w_{\epsilon,\mathbf{t}_\epsilon} + \partial_{t_j} \phi_{\epsilon,\mathbf{t}_\epsilon}\big) e^{-\beta t}\, dt$$

is diagonally dominant and thus is non-singular, which implies $c_i(\mathbf{t}_\epsilon) = 0$ for $i = 1, 2$. Hence $v_{\epsilon,\mathbf{t}_\epsilon} = w_{\epsilon,\mathbf{t}_\epsilon} + \phi_{\epsilon,\mathbf{t}_\epsilon}$ is a critical point of E_ϵ. This finishes the proof.

$\square$

Remark. *Note that in the proof the theorem, we assume that the solution v_ϵ of equation (2.2) can be written as $v_\epsilon = w_{\epsilon,\boldsymbol{t}} + \phi_\epsilon$ with ϕ_ϵ satisfying*

$$\langle \phi_\epsilon, Z_{\epsilon,t_j} \rangle_\epsilon = 0, \ \ j = 1, 2. \tag{3.25}$$

In general, using (3.24) we can decompose

$$\phi_\epsilon = \overline{\phi_\epsilon} + \sum_{j=1}^{2} d_j \partial_{t_j} w_{\epsilon,\boldsymbol{t}},$$

where $\overline{\phi_\epsilon}$ satisfies (3.25) and $d_j = O(\|\phi_\epsilon\|_\infty)$. Thus we can write

$$v_\epsilon = w_{\epsilon,\boldsymbol{t}} + \sum_{j=1}^{2} d_j \partial_{t_j} w_{\epsilon,\boldsymbol{t}} + \overline{\phi_\epsilon}$$

and get the desired result using the same argument for $w_{\epsilon,\boldsymbol{t}} + \sum_{j=1}^{2} d_j \partial_{t_j} w_{\epsilon,\boldsymbol{t}}$.

4 The uniqueness result

By Lemma 3.6, the number of sign-changing once solutions of (2.2) equals to the number of critical points of $K_\epsilon(\mathbf{t})$. To count the number of critical points of $K_\epsilon(\mathbf{t})$, we need to compute $\partial K_\epsilon(\mathbf{t})$ and $\partial^2 K_\epsilon(\mathbf{t})$.

Recall that $K_\epsilon(\mathbf{t})$ and $\widetilde{K}_\epsilon(\mathbf{t})$ are defined in (3.15) and Lemma 3.5. The crucial estimate to prove uniqueness of v_ϵ and u_ϵ is the following Proposition:

Proposition 4.1. *$K_\epsilon(\boldsymbol{t})$ is of C^2 in Λ and for ϵ sufficiently small, we have*

(1) $K_\epsilon(\boldsymbol{t}) - \widetilde{K}_\epsilon(\boldsymbol{t}) = o(\beta)$;

(2) $\partial K_\epsilon(\boldsymbol{t}) - \partial\widetilde{K}_\epsilon(\boldsymbol{t}) = o(\beta)$ uniformly for $\boldsymbol{t} \in \Lambda$;

(3) if $\boldsymbol{t}_\epsilon \in \Lambda$ is a critical point of K_ϵ, then

$$\partial^2 K_\epsilon(\boldsymbol{t}_\epsilon) - \partial^2 \widetilde{K}_\epsilon(\boldsymbol{t}_\epsilon) = o(\beta). \tag{4.1}$$

The proof of Proposition 4.1 will be delayed until the end of this section. Let us now use it to prove the uniqueness of v_ϵ.

Proof of theorem 1.1 By lemma 3.6, we just need to prove that $K_\epsilon(\mathbf{t})$ has only one critical point in Λ. We prove it in the following steps as in [38].

Step 1. By (2) of proposition 4.1, both $K_\epsilon(\mathbf{t})$ and $\widetilde{K}_\epsilon(\mathbf{t})$ have no critical points on $\partial\Lambda$ and a continuous deformation argument shows that $\partial K_\epsilon(\mathbf{t})$ has the same degree as $\partial\widetilde{K}_\epsilon(\mathbf{t})$ on Λ. By the definition of $\widetilde{K}_\epsilon(\mathbf{t})$, we have $\deg(\widetilde{K}_\epsilon(\mathbf{t}), \Lambda, 0) = 1$ and thus $\deg(\partial K_\epsilon(\mathbf{t}), \Lambda, 0) = 1$.

Step 2. At each critical point $\mathbf{t}_\epsilon$ of $K_\epsilon(\mathbf{t})$, we have

$$\deg(\partial K_\epsilon(\mathbf{t}), \Lambda \cap B_{\delta_\epsilon}(\mathbf{t}_\epsilon), 0) = 1,$$

for δ_ϵ sufficiently small.

This follows from (3) of Proposition 4.1 and the fact that the eigenvalues of the matrix $\beta^{-1}(\partial_{t_i}\partial_{t_j}\widetilde{K}_\epsilon(\mathbf{t}_\epsilon))$ are positive and away from 0.

Step 3. From step 2, we deduce that $K_\epsilon(\mathbf{t})$ has only a finite number of critical points in Λ, say, k_ϵ. By the properties of degree, we have

$$\deg(\partial K_\epsilon(\mathbf{t}), \Lambda, 0) = k_\epsilon.$$

By step 1, $k_\epsilon = 1$ and then Theorem 1.1 is thus proved.

□

In the rest of this section, we shall prove Proposition 4.1.

Proof of Proposition 4.1 The proof of part (1) follows from Lemma 3.5. We now prove part (2) of Proposition 4.1 as follows:

$$\begin{aligned}
\partial_{t_j} K_\epsilon(\mathbf{t}) &= \int_{-\infty}^{\infty} \Big[v'_{\epsilon,\mathbf{t}}(\partial_{t_j} v_{\epsilon,\mathbf{t}})' + (\gamma + e^{2t}) v_{\epsilon,\mathbf{t}} \partial_{t_j} v_{\epsilon,\mathbf{t}}\Big] e^{-\beta t}\, dt - \int_{-\infty}^{\infty} |v_{\epsilon,\mathbf{t}}|^{p-1} v_{\epsilon,\mathbf{t}} \partial_{t_j} v_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt \\
&= -\int_{-\infty}^{\infty} S_\epsilon[v_{\epsilon,\mathbf{t}}] \partial_{t_j} v_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt \\
&= J_1 + J_2,
\end{aligned} \tag{4.2}$$

where

$$J_1 \equiv -\int_{-\infty}^{\infty} S_\epsilon[w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}] \partial_{t_j} w_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt,$$

and

$$J_2 \equiv -\int_{-\infty}^{\infty} S_\epsilon[w_{\epsilon,\mathbf{t}} + \phi_{\epsilon,\mathbf{t}}] \partial_{t_j} \phi_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt.$$

Using similar argument as in Lemma 2.10,for $N = 3$, we can obtain

$$J_1 = \begin{cases} -\beta \int_{-\infty}^{\infty} |w'|^2\, dt + \frac{1}{2} e^{(t_1 - t_2)/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), j = 1; \\ \\ -\beta \int_{-\infty}^{\infty} |w'|^2\, dt - \frac{1}{2} \Big[e^{(t_1 - t_2)/2} - e^{t_2}\Big] A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), j = 2. \end{cases} \tag{4.3}$$

By (3.10) and Proposition 3.4,

$$\begin{aligned}
J_2 &= -\sum_{i=1}^{2} c_i(\mathbf{t}) \int_{-\infty}^{\infty} Z_{\epsilon,t_i} \partial_{t_j} \phi_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt \\
&= \sum_{i=1}^{2} c_i(\mathbf{t}) \int_{-\infty}^{\infty} \phi_{\epsilon,\mathbf{t}} \partial_{t_j} Z_{\epsilon,t_i} e^{-\beta t}\, dt = o(\beta).
\end{aligned} \tag{4.4}$$

Combining the above two estimates (4.3) and (4.4), part (2) of Proposition 4.1 is thus proved.

In the rest we shall prove part (3) of Proposition 4.1. Using (4.2),

$$\begin{aligned}
\partial_{t_i} \partial_{t_j} K_\epsilon(\mathbf{t}) &= \partial_{t_i} \Big[-\int_{-\infty}^{\infty} S_\epsilon[v_{\epsilon,\mathbf{t}}] \partial_{t_j} v_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt\Big] \\
&= -\int_{-\infty}^{\infty} S_\epsilon[v_{\epsilon,\mathbf{t}}] \partial_{t_i} \partial_{t_j} v_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt - \int_{-\infty}^{\infty} \partial_{t_i} S_\epsilon[v_{\epsilon,\mathbf{t}}] \partial_{t_j} v_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt.
\end{aligned}$$

By (3.21) we get

$$\partial_{t_i} S_\epsilon[v_{\epsilon,\mathbf{t}}] = \sum_{k=1}^{2} c_k(\mathbf{t}) \partial_{t_i} Z_{\epsilon,t_k} + \sum_{k=1}^{2} \partial_{t_i} c_k(\mathbf{t}) Z_{\epsilon,t_k}$$

Let $\mathbf{t}_\epsilon$ be a critical point of $K_\epsilon(\mathbf{t})$ in Λ, then

$$S_\epsilon[v_{\epsilon,\mathbf{t}_\epsilon}] = 0 \quad \text{and } c_k(\mathbf{t}_\epsilon) = 0,$$

which implies

$$\partial_{t_i} S_\epsilon[v_{\epsilon,\mathbf{t}}]\Big|_{\mathbf{t}=\mathbf{t}_\epsilon} = \sum_{k=1}^{2} \partial_{t_i} c_k(\mathbf{t}_\epsilon) Z_{\epsilon,t_{\epsilon,k}}.$$

Note that

$$\partial_{t_i} S_\epsilon[v_{\epsilon,\mathbf{t}}] = L_\epsilon[\partial_{t_i} v_{\epsilon,\mathbf{t}}] + p\Big[|v_{\epsilon,\mathbf{t}}|^{p-1} - |w_{\epsilon,\mathbf{t}}|^{p-1}\Big]\partial_{t_i} v_{\epsilon,\mathbf{t}} =: \overline{L}_\epsilon[\partial_{t_i} v_{\epsilon,\mathbf{t}}]. \tag{4.5}$$

As in Lemma 3.2, multiplying (4.5) by $\partial_{t_j} w_{j,t_j}$ and integrating by parts, we get $\partial_{t_i} c_k(\mathbf{t}_\epsilon) = O(\beta)$. Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \partial_{t_i} S_\epsilon[v_{\epsilon,\mathbf{t}}] \partial_{t_j} \phi_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt\Big|_{\mathbf{t}=\mathbf{t}_\epsilon} &= \sum_{k=1}^{2} \partial_{t_i} c_k(\mathbf{t}_\epsilon) \int_{-\infty}^{\infty} Z_{\epsilon,t_{\epsilon,k}} (\partial_{t_j} \phi_{\epsilon,\mathbf{t}_\epsilon}) e^{-\beta t}\, dt \\
&= -\sum_{k=1}^{2} \partial_{t_i} c_k(\mathbf{t}_\epsilon) \int_{-\infty}^{\infty} (\partial_{t_j} Z_{\epsilon,t_{\epsilon,k}}) \phi_{\epsilon,\mathbf{t}_\epsilon} e^{-\beta t}\, dt = o(\beta),
\end{aligned}$$

and then

$$\begin{aligned}
\partial_{t_i}\partial_{t_j} K_\epsilon(\mathbf{t}_\epsilon) &= -\int_{-\infty}^{\infty} \partial_{t_i} S_\epsilon[v_{\epsilon,\mathbf{t}}] \partial_{t_j} v_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt\Big|_{\mathbf{t}=\mathbf{t}_\epsilon} \\
&= -\int_{-\infty}^{\infty} \overline{L}_\epsilon[\partial_{t_i} w_{\epsilon,\mathbf{t}} + \partial_{t_i} \phi_{\epsilon,\mathbf{t}}] \partial_{t_j} w_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt\Big|_{\mathbf{t}=\mathbf{t}_\epsilon} + o(\beta).
\end{aligned}$$

Note that

$$\int_{-\infty}^{\infty} \overline{L}_\epsilon[\partial_{t_i} \phi_{\epsilon,\mathbf{t}}] \partial_{t_j} w_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt = \int_{-\infty}^{\infty} \partial_{t_i} \phi_{\epsilon,\mathbf{t}} \overline{L}_\epsilon[\partial_{t_j} w_{\epsilon,\mathbf{t}}] e^{-\beta t}\, dt = o(\beta),$$

since

$$\overline{L}_\epsilon[\partial_{t_j} w_{\epsilon,\mathbf{t}}] = -Z_{\epsilon,t_j} + p|v_{\epsilon,\mathbf{t}}|^{p-1} \partial_{t_j} w_{\epsilon,\mathbf{t}} = O(\beta^\tau).$$

Therefore,

$$\partial_{t_i}\partial_{t_j}K_\epsilon(\mathbf{t}_\epsilon) = -\int\limits_{-\infty}^{\infty} \overline{L}_\epsilon[\partial_{t_i}w_{\epsilon,\mathbf{t}}]\partial_{t_j}w_{\epsilon,\mathbf{t}}e^{-\beta t}\,dt\Big|_{\mathbf{t}=\mathbf{t}_\epsilon} + o(\beta).$$

Using the following important estimate:

$$\int\limits_{-\infty}^{\infty} \overline{L}_\epsilon[\partial_{t_i}w_{\epsilon,\mathbf{t}}]\partial_{t_j}w_{\epsilon,\mathbf{t}}e^{-\beta t}\,dt = \begin{cases} -\frac{1}{4}e^{(t_1-t_2)/2}A_{\epsilon,3}\int\limits_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta), & \text{for } i=j=1; \\ \\ \frac{1}{4}e^{(t_1-t_2)/2}A_{\epsilon,3}\int\limits_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta), & \text{for } i\neq j; \\ \\ -\Big[\frac{1}{4}e^{(t_1-t_2)/2}+\frac{1}{2}e^{t_2}\Big]A_{\epsilon,3}\int\limits_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta), & \text{for } i=j=2, \end{cases} \tag{4.6}$$

which will be proved in Appendix C, we get the desired result.

□

5 The non-degeneracy result and eigenvalue estimates

In this section we shall study the eigenvalue estimates for

$$L_\epsilon(\phi) := \Delta\phi - \phi + p|u_\epsilon|^{p-1}\phi \tag{5.1}$$

and prove Theorem 1.2.

Proof of Theorem 1.2. Let $\lambda_k, e_k(\theta)$ with $\theta\in S^{N-1}$ be the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on S^{N-1}. Then

$$\lambda_0 = 0 < \lambda_1 = \cdots = \lambda_N = N-1 < \lambda_{N+1} \leq \cdots,$$

and e_k are normalized so that they form a complete orthonormal basis of $L^2(S^{N-1})$. In fact the set of eigenvalues is given by $\{j(N-2+j) \mid j\geq 0\}$.

Suppose ϕ satisfies

$$L_\epsilon(\phi) = 0 \text{ in } R^N,\ \phi(x)\to 0 \text{ as } |x|\to\infty.$$

Put

$$\phi_k(r) = \int_{S^{N-1}} \phi(r,\theta)e_k(\theta)d\theta,$$

then $\phi_k(r)\to 0$ as $r\to\infty$, and it satisfies

$$\phi_k'' + \frac{N-1}{r}\phi_k' - \phi_k + p|u_\epsilon|^{p-1}\phi_k + \frac{(-\lambda_k)}{r^2}\phi_k = 0 \quad \text{in } (0,\infty) \quad \text{and } \lim_{r\to\infty}\phi_k(r) = 0, \tag{5.2}$$

for $k = 0,1,\cdots$. We claim that $\phi_k = 0$ for $k\geq N+1$.

To this end, let us consider the eigenvalues of the problem

$$\phi_k'' + \frac{N-1}{r}\phi_k' - \phi_k + p|u_\epsilon|^{p-1}\phi_k + \frac{\nu}{r^2}\phi_k = 0 \quad \text{in } (0,\infty) \quad \text{and } \lim_{r\to\infty}\phi_k(r) = 0. \tag{5.3}$$

The l-th eigenvalue of (5.3) can be characterized variationally as

$$\nu_l(p) = \max_{\dim(V)<l}\inf_{\phi\in V^\perp} \frac{\int_0^\infty \big[|\phi'|^2 + |\phi|^2\big] r^{N-1}\,dr - p\int_0^\infty |u_\epsilon|^{p-1}|\phi|^2 r^{N-1}\,dr}{\int_0^\infty |\phi|^2 r^{N-3}\,dr}, \tag{5.4}$$

where V runs through subspaces of $H^1_r(\mathbb{R}^N)$ and $V^\perp$ is the set of $\phi \in H^1_{0,r}(\mathbb{R}^N)$ satisfying $\int_0^\infty \phi u r^{N-3} = 0$ for all $u \in V$, and $H^1_r(\mathbb{R}^N)$ be the space of radial functions in $H^1(\mathbb{R}^N)$. Thanks to Hardy's inequality:

$$\frac{(N-2)^2}{4}\int_0^\infty |\phi|^2 r^{N-3}\,dr \le \int_0^\infty |\phi'|^2 r^{N-1}\,dr,$$

the eigenvalues $\nu_1(p) \le \nu_2(p) \le \cdots$ are well defined. Using Hardy's embedding and a simple compactness argument involving the fast decay of $|u_\epsilon|^{p-1}$, there is an extremal for $\nu_l(p)$ which represents a solution to problem (5.3) for $\nu = \nu_l(p)$.

To prove Theorem 1.2 we need to know whether and when $\nu_l(p)$ equals $-\lambda_k$. To show this more information about solutions is required. So we consider the corresponding problems for v_ϵ using the Emden-Fowler transformation. Then the eigenvalue problem (5.3) becomes

$$\tilde{L}_\epsilon[\psi] := \psi'' - \beta\psi' - (\gamma + e^{2t})\psi + p|v_\epsilon|^{p-1}\psi = -\nu\psi \text{ in } (-\infty,\infty) \quad \text{and } \lim_{|t|\to\infty}\psi(t) = 0. \tag{5.5}$$

For the proof of Theorem 1.2, let us consider first the radial mode $k = 0$, namely $\lambda_k = 0$. The following result, which contains elements of independent interest, gives the small eigenvalue estimates of L_ϵ and shows that $\psi_k = 0$ for the mode $k = 0$.

Proposition 5.1. *For ϵ small enough, the eigenvalue problem*

$$L_\epsilon\phi_\epsilon = \mu_\epsilon\phi_\epsilon$$

has exactly two small eigenvalues μ_ϵ^j, $j = 1,2$, which satisfy

$$\frac{\mu_\epsilon^j}{\epsilon} \to -c_0\xi_j, \quad \text{up to a subsequence as } \epsilon \to 0, \text{ for } j = 1,2, \tag{5.6}$$

where ξ_j's are the eigenvalues of the Hessian matrix $\nabla^2\widetilde{K}_\epsilon$ and c_0 is a positive constant. Furthermore, the corresponding eigenfunctions ϕ_ϵ^j's satisfy

$$\phi_\epsilon^j = \sum_{i=1}^2 \big[a_{ij} + o(1)\big]\partial_{t_i} w_{\epsilon,\boldsymbol{t}} + O(\epsilon), \quad j = 1,2,$$

where $\boldsymbol{a}_j = (a_{1,j},\ldots,a_{2,j})^T$ is the eigenvector associated with ξ_j, namely,

$$\nabla^2\widetilde{K}_\epsilon\boldsymbol{a}_j = \xi_j\boldsymbol{a}_j.$$

Remark. *By (5.6) we know that $\mu_\epsilon \neq 0$ and then obtain the non-degeneracy of v_ϵ in the space of H^1-radial symmetric functions.*

Proof of proposition 5.1. To prove this Proposition, one may follow the arguments given in Section 5 of [38] or Section 2 of [19] and the following estimates

$$\int_{-\infty}^{\infty} \overline{L}_\epsilon[\partial_{t_i} w_{\epsilon,\mathbf{t}}]\partial_{t_j} w_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt = \begin{cases} -\frac{1}{4}e^{(t_1-t_2)/2}A_{\epsilon,3}\int\limits_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), & \text{for } i=j=1; \\ \\ \frac{1}{4}e^{(t_1-t_2)/2}A_{\epsilon,3}\int\limits_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), & \text{for } i \neq j; \\ \\ -\left[\frac{1}{4}e^{(t_1-t_2)/2} + \frac{1}{2}e^{t_2}\right]A_{\epsilon,3}\int\limits_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta), & \text{for } i=j=2, \end{cases} \tag{5.7}$$

given in Appendix C.

□

Let us consider now mode 1 for (5.2), namely $k = 1, \dots, N$, for which $\lambda_k = N-1$. In this case we have an explicit solution $u'_\epsilon(r)$. Now we show that $\phi_k = C_k u'_\epsilon$ for some constants C_k for $k = 1, \dots, N$. This is not trivial since $u'_\epsilon(r)$ changes sign once. Suppose that ϕ_k solve (5.2). We first multiply equation of ϕ_k by u'_ϵ and the equation of u'_ϵ by ϕ_k, and integrate over the ball B_r centered at the origin with radius r. Since they satisfy the same equation, we get

$$\phi'_k(r)u'_\epsilon(r) - \phi_k(r)u''_\epsilon(r) = 0,$$

from which we get $\phi_k = C_k u'_\epsilon$ for some constants C_k.

Finally let us consider modes 2 and higher. Assume now that $k \geq N+1$ for which $\lambda_k \geq 2N$. Since $u'_\epsilon(r)$ has exactly one zero in $(0,\infty)$ and $\lambda_k > \lambda_1$, by the standard Sturm-Liouville comparison theorem, ϕ_k does not change sign in $(0,\infty)$. On the other hand, by Sturm-Liouville theory, it is well known that the eigenfunctions corresponding to ν_l much change sign in $(0,\infty)$ at least $l-1$ times. Thus the only possibility for equation (5.2) to have a nontrivial solution for a given $k \geq N+1$ is that $\lambda_k = -\nu_1(p)$. In the next proposition we shall show that $-\nu_1(p) \to \lambda_1 = N-1$ as $p \to \frac{N+2}{N-2}$. Therefore we get $\lambda_k \neq -\nu_1(p)$ for $k \geq N+1$ when p is closed to $\frac{N+2}{N-2}$ and then complete the proof of Theorem 1.2.

□

Proposition 5.2. *As $p \uparrow \frac{N+2}{N-2}$, we have that $-\nu_l(p) \to \lambda_1 = N-1$ for $l \leq 2$.*

Proof of proposition 5.2. One may follow the arguments in Section 3 of [11]. Note that by the Emden-Fowler transformation, the eigenvalues have a variational characterization

$$\nu_l(p) = \max_{\dim(W)<l}\ \inf_{\psi \in W^\perp} \frac{\int_{-\infty}^{\infty}\left[|\psi'|^2 + (\gamma + e^{2t})|\psi|^2\right]e^{-\beta t}\, dt - p\int_{-\infty}^{\infty}|v_\epsilon|^{p-1}|\psi|^2 e^{-\beta t}\, dt}{\int_{-\infty}^{\infty}|\psi|^2 e^{-\beta t}\, dt}, \tag{5.8}$$

where W runs through the subspaces of H and $W^\perp$ is the set of $\psi \in W$ satisfying $\int_{-\infty}^{\infty} \psi v e^{-\beta t}\, dt = 0$ for all $v \in W$. Note that the term involving the weight is relatively compact and it follows from a previous argument that the eigenvalues exist.

Observe that the limiting eigenvalue problem

$$\psi'' - \frac{(N-2)^2}{4}\psi + \frac{N+2}{N-2} w_0^{\frac{4}{N-2}} \psi = \mu\psi, \quad \psi(\pm\infty) = 0,$$

admits eigenvalues

$$\mu_1 = N-1, \quad \mu_2 = 0, \quad \mu_3 < 0, \quad \cdots, \tag{5.9}$$

where the corresponding eigenfunction for the principal eigenvalue μ_1 is positive and denoted by Ψ_1. A simple computation shows that we can take $\Psi_1 = w_0^{\frac{N}{N-2}}$. Now we take $\psi_j = w_{j,\mathbf{t}_{\epsilon,j}}^{\frac{p+1}{2}}$, $j = 1, 2$. Let W be a given one-dimensional subspace. Then there exists c_1, c_2 (not all equal to 0) such that $\int_{-\infty}^{\infty} \big(\sum_{j=1}^{2} c_j\psi_j\big) v e^{-\beta t}\, dt = 0$ for all $v \in W$. We then compute that

$$\begin{aligned} &\int_{-\infty}^{\infty} \big[|\psi'|^2 + (\gamma + e^{2t})|\psi|^2\big] e^{-\beta t}\, dt - p\int_{-\infty}^{\infty} |v_\epsilon|^{p-1}|\psi|^2 e^{-\beta t}\, dt \\ &\qquad \le \sum_{j=1}^{2} c_j^2\big(-\mu_1 + o(1)\big)\int_{-\infty}^{\infty} |\psi|^2 e^{-\beta t}\, dt, \end{aligned}$$

and hence by variational characterization of ν_2 we deduce that

$$\nu_l(p) \le \nu_2(p) \le -(N-1) + o(1), \quad l = 1, 2. \tag{5.10}$$

On the other hand, according to (5.9), $v_l(p) \to \mu_k \ge -(N-1)$ for some k. Thus we have $\nu_l(p) \to -(N-1)$ as $p \to \frac{N+2}{N-2}$ for $l \le 2$.

□

6 Appendices

6.1 Appendix A

In this subsection we shall give the estimates of w_{j,t_j}, $j = 1, 2$. Recall that w_{j,t_j} is the unique solution to the following equation

$$v'' - (\gamma_0 + e^{2s})v + w_{t_j}^p = 0, \ v \in H \tag{6.1}$$

whose existence is given by the Riesz's representation Theorem. Here w is the unique positive even solution of

$$w'' - \gamma_0 w + w^p = 0. \tag{6.2}$$

In fact, the function $w(t)$ can be written explicitly and has the following form

$$w(t) = \gamma_0^{\frac{1}{p-1}} \big(\frac{p+1}{2}\big)^{\frac{1}{p-1}} \Big[\cosh\big(\frac{p-1}{2}\gamma_0^{1/2} t\big)\Big]^{-\frac{2}{p-1}} = A_{\epsilon,N}\Big[e^{\frac{p-1}{2}\gamma_0^{1/2} t} + e^{-\frac{p-1}{2}\gamma_0^{1/2} t}\Big]^{-\frac{2}{p-1}}.$$

Note that now w has the following expansion

$$\begin{cases} w(t) = A_{\epsilon,N} e^{-\sqrt{\gamma_0} t} + O(e^{-p\sqrt{\gamma_0} t}), \quad t \geq 0; \\ w'(t) = -\sqrt{\gamma_0} A_{\epsilon,N} e^{-\sqrt{\gamma_0} t} + O(e^{-p\sqrt{\gamma_0} t}), \quad t \geq 0, \end{cases}$$

where $A_{\epsilon,N} > 0$ is a constant depending on ϵ and N.

To get the estimates of w_{j,t_j}, we write $w_{j,t_j} = w_{t_j} + \phi$, then by (6.1) and (6.2), ϕ satisfies

$$\phi'' - (\gamma_0 + e^{2s})\phi - e^{2s} w_{t_j} = 0. \tag{6.3}$$

Note that as $s \to \infty$, $e^{2s} w_{t_j}(s) \to e^{\frac{N-2}{2} t_j} A_{\epsilon,N} e^{-\frac{N-6}{2} s}$. Hence when $N > 6$, $\phi \in H$ and $\phi = O(e^{2t_j})$. Therefore,

$$w_{j,t_j} = w_{t_j} + O(e^{2t_j}), \quad \text{when } N > 6. \tag{6.4}$$

Next we consider $N \leq 6$, let ϕ_N be the unique solution of

$$\phi'' - (\gamma_0 + e^{2s})\phi - e^{-\frac{N-6}{2} s} = 0, \quad |\phi(s)| \to 0, \text{ as } |s| \to \infty, \tag{6.5}$$

then

$$w_{j,t_j} = w_{t_j} + e^{\frac{N-2}{2} t_j} A_{\epsilon,N} \phi_N + O(e^{2t_j}) =: w_{t_j} + \phi_{j,t_j} + O(e^{2t_j}), \quad \text{when } N \leq 6. \tag{6.6}$$

The rest of this subsection will be devoted to the solvability of ϕ_N. A key observation is that

$$\phi_0 = -e^{-\frac{N-2}{2} s} \tag{6.7}$$

is a special solution of (6.5). Thus if we write

$$\phi_N = \phi_0 + \phi,$$

in order to find a solution of (6.5) which satisfies the decay condition at ∞, let

$$\phi(s) = e^{-\frac{N-2}{2} s} \widetilde{\phi}\big(\lambda_N e^{(N-2)s}\big), \quad \text{where } \lambda_N = (N-2)^{-(N-2)}. \tag{6.8}$$

Then $\widetilde{\phi}$ satisfies

$$\widetilde{\phi}''(s) = s^{-\frac{2N-6}{N-2}} \widetilde{\phi}(s), \quad \widetilde{\phi}(0) = 1, \quad \widetilde{\phi}(\infty) = 0 \tag{6.9}$$

and thus

$$\phi_N = -e^{-\frac{N-2}{2} s} \Big[1 - \widetilde{\phi}\big(\lambda_N e^{(N-2)s}\big)\Big].$$

In the case of $N = 3$, $\lambda_3 = 1$ and $\widetilde{\phi} = e^{-s}$. Then

$$\phi_3 = -e^{-s/2}\big(1 - e^{-e^s}\big).$$

In the case of $N = 4$, $\lambda_4 = 1/4$ and

$$\widetilde{\phi}(r) = 2\sqrt{r}K_1(2\sqrt{r}) =: \rho_0,$$

where $K_1(z)$ is the modified Bessel function of second kind and satisfies

$$z^2K_1''(z) + zK_1'(z) - (z^2+1)K_1(z) = 0,$$

see for example [25]. Then

$$\phi_4 = -e^{-s}\Big[1 - \rho_0(\frac{1}{4}e^{2s})\Big].$$

For $N = 5$,

$$\phi_5 = -e^{-3s/2}\Big[1 - (1+e^s)e^{-e^s}\Big].$$

In the case of $N = 6$,

$$\phi_6 = -e^{-2s}\big[1 - u_0(\frac{1}{16^2}e^{4s})\big],$$

where u_0 satisfies

$$u''(r) = \frac{u(r)}{r^{3/2}},\ u(0) = 1, u(\infty) = 0.$$

Actually, we have

$$u_0(r) = 8\sqrt{r}K_2(4r^{1/4}),$$

where $K_2(z)$ is the modified Bessel function of second kind and satisfies

$$z^2K_2''(z) + zK_2'(z) - (z^2+4)K_2(z) = 0.$$

6.2 Appendix B

In this appendix we expand the quality $E_\epsilon[w_{\epsilon,\mathbf{t}}]$ in terms of ϵ and $\mathbf{t}$.

Lemma 6.1. *For $\boldsymbol{t} \in \Lambda$ and ϵ sufficiently small, we have for $N = 3$,*

$$E_\epsilon[w_{\epsilon,\boldsymbol{t}}] = \big(\frac{1}{2} - \frac{1}{p+1}\big)(e^{-\beta t_1} + e^{-\beta t_2})\int_{-\infty}^{\infty} w^{p+1}\,dt + \frac{1}{2}e^{t_2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt$$
$$+ e^{-|t_1-t_2|/2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}).$$

For $N = 4$,

$$E_\epsilon[w_{\epsilon,\boldsymbol{t}}] = \big(\frac{1}{2} - \frac{1}{p+1}\big)(e^{-\beta t_1} + e^{-\beta t_2})\int_{-\infty}^{\infty} w^{p+1}\,dt - \frac{1}{4}t_2e^{2t_2}A_{\epsilon,4}\int_{-\infty}^{\infty} w^p e^{t}\,dt$$
$$+ e^{-|t_1-t_2|}A_{\epsilon,4}\int_{-\infty}^{\infty} w^p e^{t}\,dt + o(\beta) + o(t_2e^{2t_2}) + o(e^{-|t_1-t_2|}).$$

For $N \geq 5$,

$$E_\epsilon[w_{\epsilon,\boldsymbol{t}}] = \big(\frac{1}{2} - \frac{1}{p+1}\big)(e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1}\, dt + \frac{1}{2} e^{2t_2} \int_{-\infty}^{\infty} w^2 e^{2t}\, dt$$
$$+ e^{-(N-2)|t_1-t_2|/2} A_{\epsilon,N} \int_{-\infty}^{\infty} w^p e^{(N-2)t/2}\, dt + o(\beta) + o(e^{2t_2}) + o(e^{-(N-2)|t_1-t_2|/2}).$$

Proof. Since the proofs are similar for different cases, we give the details for $N = 3$ here. Integrating by parts we get

$$\begin{aligned} E_\epsilon[w_{\epsilon,\mathbf{t}}] &= \frac{1}{2} \int_{-\infty}^{\infty} \Big[- S_\epsilon[w_{\epsilon,\mathbf{t}}] + |w_{\epsilon,\mathbf{t}}|^{p-1} w_{\epsilon,\mathbf{t}} \Big] w_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt - \frac{1}{p+1} \int_{-\infty}^{\infty} |w_{\epsilon,\mathbf{t}}|^{p+1} e^{-\beta t}\, dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \Big[\beta w'_{\epsilon,\mathbf{t}} + (\gamma - \gamma_0) w_{\epsilon,\mathbf{t}} + w_{t_1}^p - w_{t_2}^p \Big] w_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt - \frac{1}{p+1} \int_{-\infty}^{\infty} |w_{\epsilon,\mathbf{t}}|^{p+1} e^{-\beta t}\, dt \\ &= E_1 + E_2 + E_3 - E_4 + E_5, \end{aligned}$$

where

$$E_1 = \frac{\beta}{2} \int_{-\infty}^{\infty} w'_{\epsilon,\mathbf{t}} w_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt = \frac{\beta^2}{4} \int_{-\infty}^{\infty} w^2_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt = O(\beta^2);$$

$$E_2 = \frac{(\gamma - \gamma_0)}{2} \int_{-\infty}^{\infty} w^2_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt = -\frac{\beta^2}{8} \int_{-\infty}^{\infty} w^2_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt = O(\beta^2);$$

$$E_3 = -\frac{1}{2} \int_{-\infty}^{\infty} w_{t_1}^p w_{2,t_2} e^{-\beta t}\, dt - \frac{1}{2} \int_{-\infty}^{\infty} w_{1,t_1} w_{t_2}^p e^{-\beta t}\, dt;$$

$$E_4 = \frac{1}{p+1} \int_{-\infty}^{\infty} \Big[|w_{1,t_1} - w_{2,t_2}|^{p+1} - w_{t_1}^p w_{1,t_1} - w_{t_2}^p w_{2,t_2} \Big] e^{-\beta t}\, dt;$$

$$E_5 = \big(\frac{1}{2} - \frac{1}{p+1}\big) \Big[\int_{-\infty}^{\infty} w_{t_1}^p w_{1,t_1} e^{-\beta t}\, dt + \int_{-\infty}^{\infty} w_{t_2}^p w_{2,t_2} e^{-\beta t}\, dt \Big].$$

First for E_3, by Lemma 2.6 we have

$$E_3 = -e^{-|t_1-t_2|/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2}\, dt + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}).$$

To estimate E_4, we divide $\mathbb{R}$ into two intervals I_1, I_2 defined by

$$I_1 = (-\infty, \frac{t_1+t_2}{2}), \quad I_2 = [\frac{t_1+t_2}{2}, \infty).$$

So on I_1 the following equality holds:

$$\begin{aligned}
&\frac{1}{p+1}\Big[|w_{1,t_1}-w_{2,t_2}|^{p+1}-w_{t_1}^p w_{1,t_1}-w_{t_2}^p w_{2,t_2}\Big]\\
&=\frac{1}{p+1}\big[(w_{1,t_1}-w_{2,t_2})^{p+1}-w_{1,t_1}^{p+1}+(p+1)w_{1,t_1}^p w_{2,t_2}\big]-w_{1,t_1}^p w_{2,t_2}\\
&+\frac{1}{p+1}\big[(w_{t_1}+\phi_{1,t_1})^p-w_{t_1}^p-pw_{t_1}^{p-1}\phi_{1,t_1}\big]w_{1,t_1}+\frac{p}{p+1}w_{t_1}^p\phi_{1,t_1}\\
&+\frac{p}{p+1}w_{t_1}^{p-1}\phi_{1,t_1}^2-\frac{1}{p+1}w_{t_2}^p w_{2,t_2}.
\end{aligned}$$

As in the proof of Lemma 2.10, by the mean value theorem and inequality (2.28) we have

$$\begin{aligned}
&\Big|\frac{1}{p+1}\Big[|w_{1,t_1}-w_{2,t_2}|^{p+1}-w_{t_1}^p w_{1,t_1}-w_{t_2}^p w_{2,t_2}\Big]\\
&+w_{1,t_1}^p w_{2,t_2}-\frac{p}{p+1}w_{t_1}^p\phi_{1,t_1}\Big|\le Cw_{t_1}^{p+1-\delta}w_{t_2}^{\delta},
\end{aligned}$$

for any $1<\delta<2$.

Using Lemma 2.7 and integrating by parts, we get

$$\begin{aligned}
&\frac{1}{p+1}\int_{I_1}\Big[|w_{t_1}-w_{t_2}|^{p+1}-w_{t_1}^p w_{1,t_1}-w_{t_2}^p w_{2,t_2}\Big]e^{-\beta t}\,dt\\
&=-\frac{p}{p+1}e^{t_1}A_{\epsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt-e^{-|t_1-t_2|/2}A_{\epsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(e^{-|t_1-t_2|/2}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\frac{1}{p+1}\int_{I_2}\Big[|w_{t_1}-w_{t_2}|^{p+1}-w_{t_1}^p w_{1,t_1}-w_{t_2}^p w_{2,t_2}\Big]e^{-\beta t}\,dt\\
&=-\frac{p}{p+1}e^{t_2}A_{\epsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt-e^{-|t_1-t_2|/2}A_{\epsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(e^{-|t_1-t_2|/2}).
\end{aligned}$$

Hence

$$E_4=-\frac{p}{p+1}e^{t_2}A_{\epsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt-2e^{-|t_1-t_2|/2}A_{\epsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(e^{-|t_1-t_2|/2}).$$

Regarding the term E_5, by the Lemma 2.6 we have

$$\begin{aligned}
E_5=&\,(\frac{1}{2}-\frac{1}{p+1})\Big[\int_{-\infty}^{\infty}w_{t_1}^{p+1}e^{-\beta t}\,dt+\int_{-\infty}^{\infty}w_{t_1}^p\phi_{1,t_1}e^{-\beta t}\,dt\\
&+\int_{-\infty}^{\infty}w_{t_2}^{p+1}e^{-\beta t}\,dt+\int_{-\infty}^{\infty}w_{t_2}^p\phi_{2,t_2}e^{-\beta t}\,dt\Big]\\
=&\,(\frac{1}{2}-\frac{1}{p+1})(e^{-\beta t_1}+e^{-\beta t_2})\int_{-\infty}^{\infty}w^{p+1}\,dt\\
&-(\frac{1}{2}-\frac{1}{p+1})e^{t_2}A_{\epsilon,3}\int_{-\infty}^{\infty}w^p e^{t/2}\,dt+o(\beta).
\end{aligned}$$

Combining the above estimates for E_1, E_2, E_3, E_4 and E_5, we obtain

$$E_\epsilon[w_{\epsilon,\mathbf{t}}] = \big(\frac{1}{2} - \frac{1}{p+1}\big)(e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1}\,dt + \frac{1}{2}e^{t_2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt$$
$$+ e^{-|t_1-t_2|/2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}).$$

□

6.3 Appendix C

In this section we give the technical proof of (4.6) for $N = 3$, that is,

$$\int_{-\infty}^{\infty} \overline{L}_\epsilon[\partial_{t_i} w_{\epsilon,\mathbf{t}}]\partial_{t_j} w_{\epsilon,\mathbf{t}} e^{-\beta t}\,dt = \begin{cases} -\frac{1}{4}e^{(t_1-t_2)/2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta), & \text{for } i=j=1; \\ \frac{1}{4}e^{(t_1-t_2)/2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta), & \text{for } i \neq j; \\ -\Big[\frac{1}{4}e^{(t_1-t_2)/2} + \frac{1}{2}e^{t_2}\Big]A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta), & \text{for } i=j=2. \end{cases} \tag{6.10}$$

Proof. Note that by (3.1) and (3.3), we obtain

$$\begin{aligned} \overline{L}_\epsilon[\partial_{t_j} w_{\epsilon,\mathbf{t}}] =& - Z_{\epsilon,t_j} + p|v_{\epsilon,\mathbf{t}}|^{p-1}\partial_{t_j} w_{\epsilon,\mathbf{t}} \\ =&(-1)^j\Big[- p w_{t_j}^{p-1} w'_{t_j} + \beta(\partial_{t_j} w_{j,t_j})' + (\gamma-\gamma_0)\partial_{t_j} w_{j,t_j} - p|v_{\epsilon,\mathbf{t}}|^{p-1}\partial_{t_j} w_{j,t_j}\Big], \end{aligned} \tag{6.11}$$

and by the definition of $w_{\epsilon,\mathbf{t}}$,

$$\partial_{t_j} w_{\epsilon,\mathbf{t}} = (-1)^{j+1}\partial_{t_j} w_{j,t_j} = (-1)^{j+1}\big(\partial_{t_j} w_{t_j} + \partial_{t_j}\phi_{j,t_j}\big) + O(e^{2t_j}). \tag{6.12}$$

In order to calculate the integration, we divide $(-\infty,\infty)$ into two intervals I_1, I_2 defined by

$$I_1 = (-\infty, \frac{t_1+t_2}{2}), \quad I_2 = [\frac{t_1+t_2}{2}, \infty).$$

First we computer the case of $i \neq j$. By (6.11) and (6.12) we get

$$\begin{aligned} \int_{-\infty}^{\infty} \overline{L}_\epsilon[\partial_{t_1} w_{\epsilon,\mathbf{t}}]\partial_{t_2} w_{\epsilon,\mathbf{t}} e^{-\beta t}\,dt &= \int_{-\infty}^{\infty} p w_{t_1}^{p-1} w'_{t_1} w'_{t_2} - \int_{I_1} p|v_{\epsilon,\mathbf{t}}|^{p-1} w'_{t_1} w'_{t_2} - \int_{I_2} p|v_{\epsilon,\mathbf{t}}|^{p-1} w'_{t_1} w'_{t_2} + o(\beta) \\ &= -\int_{I_2} p w_{t_2}^{p-1} w'_{t_1} w'_{t_2} + o(\beta) \\ &= \int_{-\infty}^{\infty} w_{t_2}^p w''_{t_1} + o(\beta) \\ &= \frac{1}{4}e^{(t_1-t_2)/2}A_{\epsilon,3}\int_{-\infty}^{\infty} w^p e^{t/2}\,dt + o(\beta). \end{aligned}$$

For the case of $i=j=1$, recall that $v_{\epsilon,\mathbf{t}} = w_{\epsilon,\mathbf{t}} + \phi$, where $\phi = \phi_{\epsilon,\mathbf{t}}$ is given by Proposition 3.4. Then on I_1:

$$\begin{aligned} & p|v_{\epsilon,\mathbf{t}}|^{p-1}(w_{t_1}')^2 - p|w_{t_1}|^{p-1}(w_{t_1}')^2 \\ = & -p(p-1)w_{t_1}^{p-2}(w_{t_1}')^2 w_{t_2} + p(p-1)w_{t_1}^{p-2}(w_{t_1}')^2\phi + o(\beta). \end{aligned}$$

So by (6.11) and (6.12) we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \overline{L}_\epsilon[\partial_{t_1} w_{\epsilon,\mathbf{t}}]\partial_{t_1} w_{\epsilon,\mathbf{t}} e^{-\beta t}\, dt \qquad (6.13) \\ = & -\int_{I_1} p w_{t_1}^{p-1}(w_{t_1}')^2 e^{-\beta t}\, dt + \int_{I_1} p|v_{\epsilon,\mathbf{t}}|^{p-1}(w_{t_1}')^2 e^{-\beta t}\, dt + o(\beta) \\ = & -\int_{I_1} p(p-1)w_{t_1}^{p-2}(w_{t_1}')^2 w_{t_2}\, dt + \int_{I_1} p(p-1)w_{t_1}^{p-2}(w_{t_1}')^2\phi + o(\beta) \\ = & \; T_1 + T_2 + o(\beta). \end{aligned}$$

Recall that w_{j,t_j} satisfies

$$w_{j,t_j}'' - (\gamma_0 + e^{2t})w_{j,t_j} + w_{t_j}^p = 0.$$

So $\partial_{t_j} w_{j,t_j}$ and $\partial_{t_j}^2 w_{j,t_j}$ satisfy

$$(\partial_{t_j} w_{j,t_j})'' - (\gamma_0 + e^{2t})(\partial_{t_j} w_{j,t_j}) + p w_{t_j}^{p-1}(\partial_{t_j} w_{t_j}) = 0,$$

and

$$(\partial_{t_j}^2 w_{j,t_j})'' - (\gamma_0 + e^{2t})(\partial_{t_j}^2 w_{j,t_j}) + p w_{t_j}^{p-1}(\partial_{t_j}^2 w_{t_j}) + p(p-1)w_{t_j}^{p-2}(\partial_{t_j} w_{t_j})^2 = 0,$$

which implies

$$p(p-1)w_{t_1}^{p-2}(w_{t_1}')^2 = -L_\epsilon[\partial_{t_1}^2 w_{1,t_1}] + o(\beta).$$

Hence

$$T_2 = -\int_{I_1} \phi L_\epsilon[\partial_{t_1}^2 w_{1,t_1}] + o(\beta).$$

By (2.31) and Proposition 3.4, on I_1 we have on I_1

$$L_\epsilon[\phi] = \beta w_{t_1}' + p w_{t_1}^{p-1} w_{t_2} + o(\beta).$$

Thus

$$\begin{aligned} T_2 & = -\textstyle\int_{\mathbb{R}} w_{t_1}''\left[\beta w_{t_1}' + p w_{t_1}^{p-1} w_{t_2}\right] dt + o(\beta) \qquad (6.14) \\ & = -\textstyle\int_{\mathbb{R}} w_{t_1}'' p w_{t_1}^{p-1} w_{t_2}\, dt + o(\beta). \end{aligned}$$

On the other hand,

$$
\begin{aligned}
T_1 &= -\int_{\mathbb{R}} p(p-1) w_{t_1}^{p-2} (w_{t_1}')^2 w_{t_2}\, dt + o(\beta) \\
&= \int_{\mathbb{R}} L_0[w_{t_1}''] w_{t_2}\, dt + o(\beta) \\
&= \int_{\mathbb{R}} w_{t_1}'' L_0[w_{t_2}]\, dt + o(\beta) \\
&= \int_{\mathbb{R}} w_{t_1}'' \left[w_{t_2}^p + p w_{t_1}^{p-1} w_{t_2} \right] dt + o(\beta) \\
&= -\int_{\mathbb{R}} w_{t_1}'' w_{t_2}^p\, dt + \int_{\mathbb{R}} w_{t_1}'' p w_{t_1}^{p-1} w_{t_2}\, dt + o(\beta), \qquad (6.15)
\end{aligned}
$$

where

$$
L_0[\phi] := \phi'' - \gamma_0 \phi + p w_{t_1}^{p-1} \phi.
$$

Combining (6.13), (6.14) and (6.15), we get the desired result for $i = j = 1$. The proof for $i = j = 2$ is similar, we omit the details here.

□

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