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ASYMPTOTIC BEHAVIOR OF ENERGY SOLUTIONS TO A TWO-DIMENSIONAL SEMILINEAR PROBLEM WITH MIXED BOUNDARY CONDITION

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1. INTRODUCTION

This work is concerned with the asymptotic behavior of the energy solutions of the mixed boundary value problem

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1, \end{cases} \tag{1.1}$$

where:

- Ω is a $C^{0,1}$ and bounded domain in R^2;
- $\partial\Omega$ consists of two pieces Γ_0 and Γ_1, where the one-dimensional Hausdorff measure of Γ_0 is greater than 0;
- Γ_0 is smooth and Γ_1 is piecewise smooth;
- Γ_0 and Γ_1 are relatively closed in $\partial\Omega$;
- ν is the unit outer normal of Ω;
- p is a large parameter.

In this work, we shall only consider the least energy solutions, although the method can be used to study other solutions with the same decay rate of energies. Let

$$\mathcal{A}_p = \{v \in W^{1,2}(\Omega) \colon v = 0 \text{ on } \Gamma_0,\ \|v\|_L^{p+1}(\Omega) = 1\}$$

be the admissible set. Define the energy

$$J_p(v) := \int_\Omega |\nabla v|^2 \, dx$$

on the admissible set $\mathcal{A}_p$. Standard argument shows that for any $p > 1$ J_p is bounded from below and the infimum is obtained by a function u_p' in $\mathcal{A}_p$. By the inhomogeneity of (1.1) we know that a positive multiple of u_p' solves (1.1). Throughout the rest of this paper we denote such least energy solutions by u_p.

Our goal here is to understand the asymptotic behavior of u_p as p, serving as a parameter, approaches ∞. It is known in [1] that for the pure Dirichlet problem, i.e. $\Gamma_1 = \emptyset$, the solutions

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u_p develop single or double bounded peaks in the interior of Ω as $p \to \infty$. In the current mixed problem, we shall see peaks on the Neumann boundary Γ_1 and show that u_p can develop no more than either one interior peak or two boundary peaks on Γ_1. We start to investigate c_p where

$$c_p := \inf\left\{\left[\int_\Omega |\nabla u|^2 \, dx\right]^{1/2} : u \in \mathcal{A}_p\right\}. \tag{1.2}$$

According to the construction of the least energy solution u_p,

$$c_p^2 = \frac{\int_\Omega |\nabla u_p|^2 \, dx}{[\int_\Omega u_p^{p+1} \, dx]^{2/(p+1)}}, \tag{1.3}$$

and c_p^{-1} is the optimal constant of the Sobolev embedding

$$V(\Gamma_1, \Omega) \hookrightarrow L^{p+1}(\Omega),$$

where $V(\Gamma_1, \Omega) = \{v \in W^{1,2}(\Omega) : v = 0 \text{ on } \Gamma_0\}$ is a Hilbert space equipped with the inner product

$$\langle u, v\rangle = \int_\Omega \langle \nabla u, \nabla v\rangle \, dx.$$

We shall see that c_p possesses nice decay property as $p \to \infty$. Next we extend some L^1 estimates of Brezis and Merle [2] for Δ with Dirichlet boundary condition in R^2 to mixed boundary condition. After these preparations we shall prove the following theorem.

THEOREM 1.1. There exist C_1, C_2, independent of p, such that

$$0 < C_1 < \|u_p\|_{L^\infty} < C_2 < \infty$$

for large p. Indeed

$$1 \le \liminf_{p\to\infty} \|u_p\|_{L^\infty(\Omega)} \le \limsup_{p\to\infty} \|u_p\|_{L^\infty(\Omega)} \le \exp \frac{1 + \alpha_0}{2},$$

where α_0, defined later in (4.4), is a constant dependent only on the pair (Γ_1, Ω).

To state our second result, we need a few definitions. Let

$$v_p = \frac{u_p}{\int_\Omega u_p^p}. \tag{1.4}$$

For a sequence $\{u_{p_n}\}$ of $\{u_p\}$ with $p_n \to \infty$ as $n \to \infty$, we define the blow-up set S to be the subset of $\bar{\Omega}$ such that $x \in S$ if there exist a subsequence, still denoted by $\{p_n\}$, and a sequence x_n in Ω with

$$v_{p_n}(x_n) \to \infty \quad \text{and} \quad x_n \to x. \tag{1.5}$$

Define

$$\begin{aligned} S_I &= S \cap \Omega, \\ S_C &= S \cap (\Gamma_0 \cap \Gamma_1), \\ S_D &= S \cap (\Gamma_0 \backslash (\Gamma_0 \cap \Gamma_1)), \\ S_N &= S \cap (\Gamma_1 \backslash (\Gamma_0 \cap \Gamma_1)). \end{aligned} \tag{1.6}$$

So every blow-up point must fall in one and only one of the above four classes. We shall see later that S contains the set of peaks of the sequence $\{u_{p_n}\}$. By a peak $P \in \bar{\Omega}$ we mean that $\{u_{p_n}\}$ does not vanish in the L^∞ norm in any small neighborhood of P. Theorem 1.1 in particular implies that the set of peaks of $\{u_p\}$ is not empty. In this paper we are mainly concerned with S_I and S_N. We will use $\#S_I$ ($\#S_N$) to denote the cardinality of S_I (S_I, respectively). Our second result is the following theorem.

THEOREM 1.2. For a domain Ω with the properties stated in the beginning of this paper, we have

$$S_D = \varnothing, \qquad \#(S_I \cup S_C \cup S_N) \geq 1; \tag{1}$$

$$\#S_I + \tfrac{1}{2}\#S_N \leq 1 \tag{2}$$

if Γ_1 is smooth;

$$S_I = \varnothing, \qquad \text{and} \qquad \#S_N = 1 \tag{3}$$

if Γ_1 has convex corners; furthermore in this case if x_0 is the point in S_N, x_0 must be a corner point with the least angle among all the corners on Γ_1.

Here by a convex corner, we mean a corner having angle less than π.

We shall also see that under the extra condition of Ω, Γ_0 and Γ_1, u_p can develop only one peak on the Neumann boundary Γ_1. We would like to point out that as in [3], most of our results can be extended to higher dimensions with Δ replaced by Δ_N, the N-Laplacian operator ($\Delta_N u = \operatorname{div}(|\nabla u|^{N-2} \nabla u)$), in (1.1) if Ω is a domain in R^N. However, we do not know anything about S_C if $\Gamma_0 \cap \Gamma_1$ is nonempty.

Our paper is organized as follows. In Section 2, we give some background materials for the mixed boundary value problem. Then in Section 3, we prove the decay rate of c_p. We prove theorem 1.1 in Section 4. In Section 5, we present some L^1 estimates. Section 6 is devoted to the proof of theorem 1.2. Finally we consider some special domains and some examples in Section 7.

2. PRELIMINARIES

Let Ω be a domain in R^2 with conditions stated in the beginning of this article. Let Γ_0 and Γ_1 be two parts of the boundary of Ω with Γ_0 having positive one dimensional Hausdorff measure. We recall that the isoperimetric constant of Ω relative to Γ_1, $Q(\Gamma_1, \Omega)$, is defined to be

$$Q(\Gamma_1, \Omega) = \sup \frac{|E|^{1/2}}{P_\Omega(E)}, \tag{2.1}$$

where the supremum is taken over all measurable sets of Ω such that $\partial E \cap \Gamma_0$ has one dimensional Hausdorff measure 0, and $P_\Omega(E)$ denotes the De Giorgi perimeter of E relative to Ω, i.e.

$$P_\Omega(E) = \sup\left\{\left|\int_E \operatorname{div} \psi \, dx\right| : \psi \in [C_0^\infty(\Omega)]^2, \ |\psi| \leq 1\right\}. \tag{2.2}$$

Some properties of $P_\Omega(E)$ are stated in [4, 5]. We also refer to [6] and [7] for more information about the De Giorgi perimeter and isoperimetric inequalities. In particular we notice that

$$Q(\Gamma_1, \Omega) \geq (2\pi^{1/2})^{-1},$$

where the second is the *absolute* isoperimetric constant; and if $H^1(\Gamma_1) > 0$,

$$Q(\Gamma_1, \Omega) \geq (2\pi/2)^{-1/2}.$$

From here we deduce that if $H^1(\Gamma_1) > 0$ and $Q(\Gamma_1, \Omega) < \infty$, there exists $\alpha \in [0, \pi]$ such that $Q(\Gamma_1, \Omega) = (\sqrt{2\alpha})^{-1}$ where α is the angle of the unitary sector

$$\Sigma(\alpha, 1) = \{x = (r, \theta) \in R^2 : 0 \leq r \leq 1,\ \theta \in [0, \alpha]\}.$$

We denote by $\mathcal{E}_\alpha$ the class of all pairs (Γ_1, Ω) of the type considered above such that

$$Q(\Gamma_1, \Omega) = (\sqrt{2\alpha})^{-1}. \tag{2.3}$$

By virtue of an isoperimetric inequality described in [5], any pair of a convex sector and its noncircular boundary $(\Gamma_1, \Sigma(\alpha, 1))$ belongs to $\mathcal{E}_\alpha$ once we denote by Γ_0 the circular part of $\Sigma(\alpha, 1)$. Therefore,

$$Q(\Gamma_1, \Sigma(\alpha, 1)) = (\sqrt{2\alpha})^{-1}$$

if $\Sigma(\alpha, 1)$ is a convex sector. By the way, if $(\Gamma_1, \Omega) \in \mathcal{E}_\alpha$ and β is the smallest angle among all convex corners on Γ_1,

$$\beta \geq \alpha. \tag{2.4}$$

Recall $V(\Gamma_1, \Omega)$ the Hilbert space defined in Section 1. Assuming $(\Gamma_1, \Omega) \in \mathcal{E}_0$ for some $\alpha \in [0, \pi]$, we have the following two dimensional Moser type embedding while the proof of this result in any dimension can be found in [5]. See also [8].

Proposition 2.1. There exists a universal constant C such that

$$\int_\Omega \exp\left[\frac{(2\alpha)|u|^2}{\|\nabla u\|^2_{L^2(\Omega)}}\right] \leq C|\Omega|$$

for any $u \in V(\Gamma_1, \Omega)$ with $(\Gamma_1, \Omega) \in \mathcal{E}_\alpha$.

We also need some results concerning the relative isoperimetric constants near the boundary Γ_1. Let us fix our notation first. For each smooth point $x \in \Gamma_1$, we can associate a smooth flattening map Φ_x in a neighborhood of x that maps the neighborhood of x to a neighborhood of $(0, 0)$ in

$$\{y \in R^2 : y = (y_1, y_2),\ y_2 > 0\}$$

and maps Γ near x to

$$\{y \in R^2 : y = (y_1, y_2),\ y_2 = 0\}$$

near $(0, 0)$. For a corner point x on Γ_1 we associate a similar map Φ_x in a neighborhood of x that maps the neighborhood of x to a neighborhood of $(0, 0)$ in

$$\{y \in R^2 : y = (\rho\cos\theta, \rho\sin\theta),\ 0 \leq \theta \leq \beta\},$$

where β is the angle of the corner at x and that maps the boundary near x to the boundary near $(0, 0)$. We further require that $D\Phi_x = I$ at x, and Φ_x varies smoothly with respect to x. From now on throughout the rest of this paper, for any x on Γ_1, by a ball $B_r(x_0)$, we mean $\Phi_x^{-1}(B_r(0, 0))$. Clearly it is well-defined if r is small. We can now state the following result concerning the asymptotic behavior of the relative isoperimetric constants and the quantities α defined in (2.3) of $(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0))$.

PROPOSITION 2.2. (1) Let $x_0 \in \Gamma_2$ such that Γ_2 is smooth near x_0. Then as $r \to 0$,

$$Q(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \frac{1}{\sqrt{2\pi}},$$

i.e.

$$\alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \pi,$$

where $\alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0))$ is the angle of the unit sector whose relative isoperimetric constant is the same as the one of $(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0))$.

(2) Let $x_0 \in \Gamma_2$ such that x_0 is the vertex of a convex corner with angle β_0 in Γ_2. Then as $r \to 0$,

$$Q(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \frac{1}{\sqrt{2\beta_0}},$$

i.e.

$$\alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \beta_0.$$

To prove this proposition, one just invokes the variable change formula in standard integration theory to compare the relative isoperimetric constants above with the relative isoperimetric constants of the sectors computed in [5]. We leave the details of this argument to the reader.

3. SOME ESTIMATES FOR c_p

Recall c_p defined in (1.2). We have the following refined Sobolev embedding.

LEMMA 3.1. For every $t \geq 2$ there is D_t such that

$$\|u\|_{L^t} \leq D_t t^{1/2} \|\nabla u\|_{L^2}$$

for all $u \in V(\Gamma_1, \Omega)$ with $(\Gamma_1, \Omega) \in \mathcal{E}_\alpha$; furthermore,

$$\lim_{t \to \infty} D_t = (4\alpha e)^{-1/2}.$$

Proof. Let $u \in V(\Gamma_1, \Omega)$. We know

$$\frac{1}{\Gamma(s+1)} x^s \leq e^x$$

for all $x \geq 0$, $s \geq 0$ where Γ is the Γ function. Using proposition 2.1, we have

$$\int_\Omega \exp\left[2\alpha\left(\frac{u}{\|\nabla u\|_{L^2}}\right)^2\right] dx \leq C|\Omega|,$$

where C does not depend on anything and $|\Omega|$ is the Lebesgue measure of Ω. Therefore,

$$\begin{aligned}\frac{1}{\Gamma(t/2+1)}\int_\Omega u^t\,dx &= \frac{1}{\Gamma(t/2+1)}\int_\Omega \left[2\alpha\left(\frac{u}{\|\nabla u\|_{L^2}}\right)^2\right]^{t/2} dx(2\alpha)^{-t/2}\|\nabla u\|_{L^2}^t \\ &\leq \int_\Omega \exp\left[2\alpha\left(\frac{u}{\|\nabla u\|_{L^2}}\right)^2\right] dx\,(2\alpha)^{-t/2}\|\nabla u\|_{L^2}^t \\ &\leq C|\Omega|(2\alpha)^{-t/2}\|\nabla u\|_{L^2}^t.\end{aligned}$$

Hence

$$\left(\int_\Omega u^t\,dx\right)^{1/t} \leq \left(\Gamma\left(\frac{t}{2}+1\right)\right)^{1/t} C^{1/t}(2\alpha)^{-1/2}|\Omega|^{1/t}\|\nabla u\|_{L^2(\Omega)}.$$

Notice that, according to Stirling's formula,

$$\left(\Gamma\left(\frac{t}{2}+1\right)\right)^{1/t} \sim \left(\left(\frac{t/2}{e}\right)^{t/2}\sqrt{tee^{\theta_t}}\right)^{1/t} \sim \left(\frac{1}{2e}\right)^{1/2} t^{1/2},$$

where $0 < \theta_t < \frac{1}{12}$. Choosing D_t to be

$$\left(\Gamma\left(\frac{t}{2}+1\right)\right)^{1/t} C^{1/t}(2\alpha)^{-1/2}|\Omega|^{1/t}t^{-1/2}$$

we get the desired result. ■

An immediate consequence is the following corollary.

Corollary 3.2.

$$\liminf_{p\to\infty} p^{1/2}c_p \geq (4\alpha e)^{1/2}.$$

Next we prove an upper bound for $p^{1/2}c_p$.

Lemma 3.3. For domains Ω with smooth Γ_1

$$\limsup_{p\to\infty} p^{1/2}c_p \leq (4\pi e)^{1/2};$$

if the domain Ω has convex corners on Γ_1,

$$\limsup_{p\to\infty} p^{1/2}c_p \leq (4\beta e)^{1/2},$$

where β is the smallest angle among all convex corners on Γ_1.

Proof. Let us first assume that Ω contains $\{(x_1, x_2): x_2 > 0, x_1^2 + x_2^2 \le L\}$ with $\{(x_1, x_2): x_2 = 0, x_1^2 + x_2^2 \le L\}$ being part of the Neumann boundary. We construct a Moser type test function near $(0, 0)$. Letting

$$m_l(x) = \frac{1}{\sqrt{\pi}} \begin{cases} (\log L - \log l)^{1/2}, & 0 \le |x| \le l \\ \dfrac{\log l - \log|x|}{[\log L - \log l]^{1/2}}, & l \le |x| \le L \\ 0, & |x| \ge L, \end{cases} \tag{3.1}$$

we have $m_l \in V(\Gamma_1, \Omega)$, $\|\nabla m_l\|_{L^2(\Omega)} = 1$ and

$$\int_\Omega m_l^{p+1}(x)\,\mathrm{d}x = \left[\frac{1}{\sqrt{\pi}}\left(\log\frac{L}{l}\right)^{1/2}\right]^{p+1} |B_l| + \left[\frac{1}{\sqrt{2\pi}}\left(\log\frac{L}{l}\right)^{-1/2}\right]^{p+1} \int_{l<|x|<L} \left(\log\frac{L}{|x|}\right)^{p+1} \mathrm{d}x$$
$$:= I_1 + I_2,$$

where

$$I_1 = \left[\frac{1}{\sqrt{\pi}}\left(\log\frac{L}{l}\right)^{1/2}\right]^{p+1} \pi l^2$$

$$I_2 = \left[\frac{1}{\sqrt{\pi}}\left(\log\frac{L}{l}\right)^{-1/2}\right]^{p+1} \int_{l<|x|<L} \left(\log\frac{L}{|x|}\right)^{p+1} \mathrm{d}x.$$

Choosing $l = Le^{-(p+1)/4}$, we have

$$\|m_l\|_{L^{p+1}} \ge I_1^{1/(p+1)} \ge \left[\frac{1}{4\pi e}\right]^{1/2} (p+1)^{1/2} (\pi L^2)^{1/(p+1)}.$$

Hence

$$c_p \le [4\pi e]^{1/2} (p+1)^{-1/2} (\pi L^2)^{-1/(p+1)},$$

i.e.

$$\limsup_{p\to\infty} p^{1/2} c_p \le (4\pi e)^{1/2}.$$

For a domain Ω with smooth Γ_1, we can first flatten the boundary and construct the same test function with small L. Sending L to 0, we still get the desired result.

If the domain Ω has a corner on Γ_1, we can first transform it into a sector by a smooth map. Then we construct a similar test function on that sector. Finally, we let L tend to 0. ■

COROLLARY 3.4. (1) For domains Ω with smooth Γ_1,

$$\limsup_{p\to\infty} p \int_\Omega u_p^{p+1} \le (4\pi e) \quad \text{and} \quad \limsup_{p\to\infty} p \int_\Omega |\nabla u_p|^2 \le 4\pi e.$$

(2) For domains Ω having convex corners on Γ_1,

$$\limsup_{p\to\infty} p \int_\Omega u_p^{p+1} \le 4\beta e \quad \text{and} \quad \limsup_{p\to\infty} p \int_\Omega |\nabla u_p|^2 \le 4\beta e,$$

where β is the smallest angle among all convex corners on Γ_1.

Proof. From (1.3), we know that

$$c_p = \frac{\|\nabla u_p\|_{L^2(\Omega)}}{\|u_p\|_{L^{p+1}(\Omega)}}.$$

If we multiply (1.1) by u_p and integrate by parts, we have

$$\int_\Omega |\nabla u_p|^2 = \int_\Omega u_p^{p+1}.$$

Therefore,

$$\int_\Omega u_p^{p+1} = c_p^{(2(p+1))/(p-1)} \quad \text{and} \quad \int_\Omega |\nabla u_p|^2 = c_p^{(2(p+1))/(p-1)}.$$

The results follow immediately from lemma 3.3. ■

As another consequence of lemma 3.3, we prove a crucial estimate for the quantity

$$L_0 = \limsup_{p\to\infty} \frac{p\int_\Omega u_p^p}{e}. \tag{3.2}$$

The proof follows easily from lemma 3.3 and Holder's inequality.

COROLLARY 3.5. (1) For the domains Ω with smooth Γ_1,

$$L_0 \le 4\pi;$$

(2) for domains Ω having convex corners on Γ_1,

$$L_0 \le 4\beta,$$

where β is the smallest angle among all convex corners on Γ_1.

4. PROOF OF THEOREM 1.1

A uniform lower bound indeed exists for *any* positive solutions to (1.1). Let λ_1 be the first eigenvalue of $-\Delta$ with the same boundary condition as the one in (1.1) and φ be a corresponding positive eigenfunction. Then for any solution u

$$\int_\Omega [u\,\Delta\varphi - \varphi\,\Delta u] = \int_{\partial\Omega} \left[u\frac{\partial\varphi}{\partial\nu} - \varphi\frac{\partial u}{\partial\nu}\right] = 0. \tag{4.1}$$

Therefore,

$$\int_\Omega (u^p - \lambda_1 u)\varphi = 0.$$

Hence

$$\|u\|_{L^\infty(\Omega)} \ge \lambda_1^{1/(p-1)} \to 1 \tag{4.2}$$

as $p \to \infty$ which yields a uniform lower bound in p for $\|u\|_{L^\infty(\Omega)}$ when $p > 1 + \varepsilon$, $\varepsilon > 0$.

To get an upper bound for $\{u_p\}$, we use an iteration argument. Define

$$\gamma_0 = \beta/a, \tag{4.3}$$

where β is the smallest angle among all convex corners on Γ_1 and (Γ_1, Ω) is in class $\mathcal{E}_\alpha$. Then $\gamma_0 \geq 1$ by (2.4). Let α_0 be such that

$$\exp \alpha_0 = \gamma_0(1 + \alpha_0). \tag{4.4}$$

Fix t and ε that will be chosen later. Letting $\nu = (1 + t)(p + 1)$, from lemma 3.1, we have

$$\left[\int_\Omega u_p^\nu\right]^{1/\nu} \leq (4\alpha e)^{-1/2} E_{(1+t)(p+1)} \nu^{1/2} \|\nabla u_p\|_{L^2(\Omega)},$$

where

$$\lim_{p\to\infty} E_{(1+t)(p+1)} = 1.$$

However, from corollary 3.4 we know that

$$\limsup_{p\to\infty} p \int_\Omega |\nabla u_p|^2 \leq 4\beta e.$$

Hence, there exists P_0 such that for all $p > P_0$,

$$\int_\Omega u_p^\nu \leq [\gamma_0(1 + t + \varepsilon)]^{\nu/2}. \tag{4.5}$$

Multiplying both sides of (1.1) by u_p^{2s-1}, we get, after integrating by parts,

$$\frac{2s-1}{s^2} \int_\Omega |\nabla u_p^s|^2 = \int_\Omega u_p^{p-1+2s}. \tag{4.6}$$

Using lemma 3.1 again, we have

$$\left[\int_\Omega u_p^{\nu s}\right]^{1/\nu} \leq D_{\nu s} \nu^{1/2} \|\nabla u_p^s\|_{L^2(\Omega)};$$

$$\left[\int_\Omega u_p^{\nu s}\right]^{2/\nu} \leq C_0 \nu \frac{s^2}{2s-1} \int_\Omega u_p^{p-1+2s} \leq C_1 \nu s \int_\Omega u_p^{p-1+2s},$$

where $D_{\nu s}$ is defined in lemma 3.1 and C_0 and C_1 are constants independent of $p > P_0$. Hence, we have

$$\left[\int_\Omega u_p^{\nu s}\right]^{2/\nu} \leq C_1 \nu s \int_\Omega u_p^{p-1+2s}. \tag{4.7}$$

We now define two sequences $\{s_j\}$ and $\{M_j\}$ by

$$\begin{cases} p - 1 + 2s_0 = \nu \\ p - 1 + 2s_{j+1} = \nu s_j \\ M_0 = [\gamma_0(1 + t + \varepsilon)]^{\nu/2} \\ M_{j+1} = [C_1 \nu s_j M_j]^{\nu/2}, \end{cases} \tag{4.8}$$

where C_1 is the constant in (4.7). From (4.5) and (4.7), we have, by induction, that

$$\int_\Omega u_p^{\nu s_{j-1}} \le M_j. \tag{4.9}$$

Next we claim that

$$M_j \le \exp[m(\gamma_0, t, p, \varepsilon)\nu s_{j-1}], \tag{4.10}$$

where $m(\gamma_0, t, p, \varepsilon)$ is a constant depending on γ_0, t, p, ε and

$$\lim_{p\to\infty} m(\gamma_0, t, p, \varepsilon) = \frac{1+t}{2t}\log[\gamma_0(1+t+\varepsilon)].$$

In fact, we can write down $\{s_j\}$ explicitly

$$s_j = \frac{1}{\nu-2}\left\{\left(\frac{\nu}{2}\right)^{j+1}(\nu-1-p-1)+p-1\right\}. \tag{4.11}$$

Put

$$\sigma_j = \frac{\nu}{2}\log(C_1\nu s_j), \qquad \mu_j = \log M_j.$$

Hence,

$$\mu_{j+1} = \frac{\nu\mu_j}{2} + \sigma_j.$$

Therefore, it is easy to see that

$$\sigma_j = \frac{\nu}{2}\left(\log\left[\frac{C_1\nu}{\nu-2}\right] + \log\left[\left(\frac{\nu}{2}\right)^{j+1}(\nu-p-1)+p-1\right]\right) \le [\nu\log\sqrt{2C_1}\,\nu](j+1).$$

Now we define $\{\tau_j\}$ by

$$\tau_0 = \mu_0 \qquad \tau_{j+1} = \tfrac{1}{2}\nu\tau_j + (\nu\log\sqrt{2C_1}\,\nu)(j+1). \tag{4.12}$$

Clearly, $\mu_j \le \tau_j$. Moreover, we have

$$\begin{aligned}
\tau_j &= \left(\frac{\nu}{2}\right)^j\left[\mu_0 + 2\nu\log(\sqrt{2C_1}\,\nu)\frac{\nu}{(\nu-2)^2}\right] - \frac{2}{\nu-2}\left(\nu\log(\sqrt{2C_1}\,\nu)\left(j+\frac{\nu}{\nu-2}\right)\right)\\
&\le \frac{\mu_0 + 2\nu\log(\sqrt{2C_1}\,\nu)\nu/(\nu-2)^2}{(\nu-2)^{-1}(\nu-p-1)}s_{j-1}\\
&\le \frac{\mu_0 + 2\nu\log(\sqrt{2C_1}\,\nu)\nu/(\nu-1)^2}{\nu-p-1}\,\frac{\nu-2}{\nu}\,\nu s_{j-1}\\
&:= m(\gamma_0, t, p, \varepsilon)\nu s_{j-1},
\end{aligned}$$

where

$$\lim_{p\to\infty} m(\gamma_0, t, p, \varepsilon) = \frac{1+t}{2t}\log[\gamma_0(1+t+\varepsilon)].$$

Therefore, we get

$$\|u_p\|_{L^{\nu s^{j-1}}(\Omega)} \leq \exp[m(\gamma_0, t, p, \varepsilon)].$$

Sending $j \to \infty$, we see

$$\|u_p\|_{L^\infty(\Omega)} \leq \exp[m(\gamma_0, t, p, \varepsilon)].$$

Sending $p \to \infty$, we have

$$\limsup_{p \to \infty} \|u_p\|_{L^\infty} \leq [\gamma_0(1 + t + \varepsilon)]^{(1+t)/2t}.$$

Sending $\varepsilon \to 0$, we deduce

$$\limsup_{p \to \infty} \|u_p\|_{L^\infty} \leq [\gamma_0(1 + t)]^{(1+t)/2t}.$$

If we let $f(t) = [\gamma_0(1 + t)]^{(1+t)/2t}$, the standard calculus argument shows that $\log f(t)$ achieves its minimum at α_0, where

$$\alpha_0 = \log[\gamma_0(1 + \alpha_0)]$$

defined in (4.4). So we obtain

$$\limsup_{p \to \infty} \|u_p\|_{L^\infty} \leq \exp \frac{1 + \alpha_0}{2}. \quad \blacksquare$$

We include a consequence of theorem 1.1 here which will be used later.

Corollary 4.1. There exist C_1 and C_2 such that

$$\frac{C_1}{p} \leq \int_\Omega u_p^p \leq \frac{C_2}{p}.$$

Proof. The first inequality follows from theorem 1.1 and the first limit of corollary 2.3; the second inequality follows from the first limit of corollary 2.3 through an interpolation argument. $\blacksquare$

5. SOME A PRIORI ESTIMATES

In this section we collect some less well-known estimates for Δ on two dimensional domains.

We first state a boundary estimate lemma. The proof of the lemma is standard. One combines the moving plane method in [9] with a Kelvin transform. We refer to [9, 10] for details. This lemma actually excludes the possibility that u_p develop a peak on Γ_0. See remark 6.5.

Lemma 5.1. Let u be a positive solution of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \subset R^2 \\ u|_{\Gamma_0} = 0, \end{cases}$$

where Γ_0 is a smooth piece of $\partial\Omega$ and f is a smooth function. Then for every $\Gamma \subset\subset \operatorname{int}(\Gamma_0)$ with respect to the relative topology of $\partial\Omega$ there exist a neighborhood ω of Γ and a constant C both depending on the geometry of Ω and Γ only such that

$$\|u\|_{L^\infty(\omega)} \le C\|u\|_{L^1(\Omega)}.$$

Next we state an L^1 estimate of Brezis and Merle, theorem 1 [2].

LEMMA 5.2. Let u be a solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a smooth bounded domain in R^2. We have for $0 < \varepsilon < 4\pi$

$$\int_\Omega \exp\left[\frac{(4\pi - \varepsilon)|u(x)|}{\|f\|_{L^1}}\right] dx \le \frac{4\pi \operatorname{Area}(\Omega)}{\varepsilon}.$$

Remark 5.3. In their paper, Brezis and Merle used $(\operatorname{Diameter}(\Omega))^2$ instead of $\operatorname{Area}(\Omega)$ in lemma 5.2. It turns out from the following symmetrization approach that $\operatorname{Area}(\Omega)$ is more appropriate.

We need a similar L^1 estimate as above to take care of the mixed boundary condition.

LEMMA 5.4. Let u be a solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u|_{\Gamma_0} = 0 \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma_1} = 0, \end{cases}$$

where the boundary condition is the same as the one in (1.1) and $(\Gamma_1, \Omega) \in \mathcal{E}_\alpha$. Then for every $0 < \varepsilon < 2\alpha$,

$$\int_\Omega \exp\left[\frac{(2\alpha - \varepsilon)|u(x)|}{\|f\|_{L^1}}\right] dx \le \frac{2\alpha \operatorname{Area}(\Omega)}{\varepsilon}.$$

Proof. Owing to the maximum principle, we may assume $f \ge 0$. Otherwise, we just replace f by $|f|$. We use the symmetrization approach here. Let $\Sigma(\alpha, R)$ be the sector having the same areas as Ω and the same relative isoperimetric constant as Ω. Define as in [4] the α-symmetrization to be the transformation that associates $u(x)$ with

$$u_\alpha := u_*\left(\frac{\alpha}{2}|x|^2\right)$$

for $x \in \Sigma(\alpha, R)$, where u_* is the standard decreasing rearrangement. Namely

$$u_* := \inf\{t \ge 0 : \mu(s) < t\}$$

and

$$\mu(t) = \text{meas}\{x \in \Omega \colon |u(x)| > t\}.$$

u_α has similar properties to those of the standard Schwartz symmetrization. In particular

$$\int_\Omega F(u(x))\,\mathrm{d}x = \int_{\Sigma(\alpha,R)} F(u_\alpha(x))\,\mathrm{d}x \tag{5.1}$$

for real Borel function F. Moreover, let u be a solution to the equation in lemma 5.4, and v be the solution of

$$\begin{cases} -\Delta v = f_\alpha & \text{in } \Sigma(\alpha, R) \\ v|_{\tilde{\Gamma}_0} = 0 & \\ \left.\dfrac{\partial v}{\partial \nu}\right|_{\tilde{\Gamma}_0} = 0, & \end{cases}$$

where

$$\tilde{\Gamma}_0 = \{x \in \partial\Sigma(\alpha, R) \colon |x| = R\},$$
$$\tilde{\Gamma}_1 = \{x \in \partial\Sigma(\alpha, R) \colon |x| \le R\}$$

and f_α is the α-symmetrization of f. Standard argument shows that v is radially symmetric. From [4], we assert that

$$u_\alpha(x) \le v(x), \tag{5.2}$$

where u_α is the α-symmetrization of the solution u in lemma 5.4. However, since it is radially symmetric, v satisfies

$$\begin{cases} v''(t) + \dfrac{1}{t}v'(t) + f_\alpha(t) = 0 \\ v'(0) = 0 \\ v(R) = 0. \end{cases}$$

Therefore, solving the O.D.E., we have

$$v(r) \le \log\left(\frac{R}{r}\right)\int_0^R s f_\alpha(s)\,\mathrm{d}s$$

$$\int_{\Sigma(\alpha,R)} \exp\left[\frac{(2\alpha - \varepsilon)v}{\|f_\alpha\|_{L^1(\Omega)}}\right] \le \frac{2\alpha\,\text{Area}(\Sigma(\alpha, R))}{\varepsilon} = \frac{2\alpha\,\text{Area}(\Omega)}{\varepsilon}.$$

Combining this with (5.1) and (5.2), we have the desired result. ■

6. PROOF OF THEOREM 1.2

Lemma 5.4 implies that $\{v_p\}$ is uniformly bounded in $L^1(\Omega)$. Therefore, lemma 5.1 implies that $\{v_p\}$ is uniformly bounded in $L^\infty(\omega)$ where ω is a neighborhood of any compact subset of $\text{int}(\Gamma_0)$. Since

$$\max_{x \in \bar{\Omega}} v_n(x) \ge \frac{C}{\nu_{p_n}} \to \infty,$$

from theorem 1.1 and corollary 4.1, we deduce $S \neq \emptyset$. However, since $S_D = \emptyset$, we conclude that $\#(S_I \cup S_C \cup S_N) \geq 1$. This proves part 1. To prove the rest of the theorem, define

$$L_0 = \overline{\lim_{p \to \infty}} \frac{p v_p}{e}, \tag{6.1}$$

where

$$v_p = \int_\Omega u_p^p. \tag{6.2}$$

We denote any sequence u_{p_n} of u_p with $p_n \to \infty$ by u_n. Let

$$v_n := v_{p_n} := \frac{u_n}{v_{p_n}}; \tag{6.3}$$

$$f_n := f_{p_n} := \frac{u_n^{p_n}}{\int_\Omega u_n^{p_n}} = v_{p_n}^{p_n - 1} v_n. \tag{6.4}$$

Since

$$\int_{\Omega \cup \Gamma_1} f_n = 1,$$

we can subtract a subsequence of f_n, still denoted by f_n, so that there is a positive bounded measure μ in $M(\Omega \cup \Gamma_1)$, the set of all real bounded Borel measures on $\Omega \cup \Gamma_1$, such that

$$\int_{\Omega \cup \Gamma_1} f_n \varphi \to \int_{\Omega \cup \Gamma_1} \varphi \, \mathrm{d}\mu \tag{6.5}$$

for all

$$\varphi \in C_0^\infty(\Omega \cup \Gamma_1).$$

Recall S_I and S_N defined in (1.6). For any $\delta > 0$ we call $x_0 \in \Omega \cup (\Gamma_1 \backslash (\Gamma_1 \cap \Gamma_0))$ a δ-regular point if:

- $x_0 \in \Omega$ and there is $\varphi \in C_0(\Omega)$, $0 \leq \varphi \leq 1$, $\varphi = 1$ in a neighborhood of x_0, such that

$$\int_{\Omega \cup \Gamma_1} \varphi \, \mathrm{d}\mu \leq \frac{4\pi}{L_0 + 2\delta}, \tag{6.6}$$

where L_0 is the quantity defined in (3.2); or

- $x_0 \in \Gamma_1 \backslash (\Gamma_1 \cap \Gamma_0)$ and there is $\varphi \in C_0(\Omega \cup \Gamma_1)$, $0 \leq \varphi \leq 1$, $\varphi = 1$ in a neighborhood of x_0, such that

$$\int_{\Omega \cup \Gamma_1} \varphi \, \mathrm{d}\mu \leq \frac{2\alpha(x_0)}{L_0 + 2\delta}, \tag{6.7}$$

where $\alpha(x_0) := \lim_{r \to 0} \alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0))$ considered in proposition 2.2.

We let $\alpha(x_0) = 2\pi$ if $x_0 \in \Omega$. We say that $x_0 \in \Omega \cup \Gamma_1 \backslash (\Gamma_0 \cap \Gamma_1)$ is δ-irregular if x_0 is not δ-regular.

LEMMA 6.1. If x_0 is a δ-regular point for $\delta > 0$, then $\{v_n\}$ is uniformly bounded in $L^\infty(B_{R_0}(x_0) \cup \bar{\Omega})$ for some $R_0 > 0$.

Proof. We first consider the case where $x_0 \in \Gamma_1 \backslash (\Gamma_0 \cap \Gamma_1)$. Let x_0 be a δ-regular point on $\Gamma_1 \backslash (\Gamma_0 \cap \Gamma_1)$. Then there exists R_0 such that

$$\int_{B_{R_0}(x_0) \cup \bar{\Omega}} f_n \leq \frac{2\alpha(x_0)}{L_0 + \delta}$$

for n large enough.

Split v_n into two parts, $v_n = v_{1n} + v_{2n}$ where v_{1n} solves

$$\begin{cases} \Delta v_{1n} + f_n = 0 & \text{in } B_{R_0}(x_0) \cap \Omega \\ v_{1n} = 0 & \text{on } \partial B_{R_0}(x_0) \cap \Omega \\ \dfrac{\partial v_{1n}}{\partial \nu} = 0 & \text{on } B_{R_0}(x_0) \cap \Gamma_1 \end{cases} \tag{6.8}$$

and v_{2n} solves

$$\begin{cases} \Delta v_{2n} = 0 & \text{in } B_{R_0}(x_0) \cap \Omega \\ v_{2_n} = v_n & \text{on } \partial B_{R_0}(x_0) \cap \Omega \\ \dfrac{\partial v_{2n}}{\partial \nu} = 0 & \text{on } B_{R_0}(x_0) \cap \Gamma_1 . \end{cases} \tag{6.9}$$

Then $v_{1n} \leq v_n$ and $v_{2n} \leq v_n$ by the maximum principle. Now from the standard elliptic boundary estimate for harmonic functions with Neumann data, we have

$$\|v_{2n}\|_{L^\infty(B_{R_0/2}(x_0) \cap \bar{\Omega})} \leq C \|v_{2n}\|_{L^1(B_{R_0} \cap \bar{\Omega})} \leq C',$$

where C' is a constant independent of n and the last inequality follows from lemma 5.4. So we only need to estimate v_{1n}.

We first claim that when n is large enough

$$f_n(x) \leq \exp(L_0 + \delta/2) v_n(x) \tag{6.10}$$

for all $x \in \Omega$.

Now observe that

$$\log x \leq \frac{x}{e} \tag{6.11}$$

for $x > 0$. We have

$$p_n \log \frac{u_n}{v_n^{1/p_n}} \leq \frac{p_n}{e} \frac{u_n}{v_n^{1/p_n}} \leq \frac{L_0 + \delta/3}{v_n} \frac{u_n}{v_n^{1/p_n}} \leq \frac{t' - \delta/6}{v_n^{1/p_n}} \frac{u_n}{v_n} \leq t' \frac{u_n}{v_n}$$

for n large enough because

$$\lim_{n \to \infty} v_n^{1/p_n} = 1,$$

which follows from corollary 4.1. Hence,

$$f_n \le \exp[(L_0 + \delta/2)v_n].$$

Next we claim that $\{f_n\}$ is uniformly bounded in $L^{1+\delta_0}(B_{R_1/2})$ for δ_0 sufficiently small. Since $\{v_{2n}\}$ is uniformly bounded in $B_{R_{1/2}}(x_0)$, we see from the previous claim that

$$\begin{aligned}\int_{B_{R_1}/2} f_n^{1+\delta_0} &\le \int_{B_{R_1}/2} \exp[(1 + \delta_0)(L_0 + 0.5\delta)v_n]\\ &\le C\int_{B_{R_1}/2} \exp[(1 + \delta_0)(L_0 + 0.5\delta)v_{1n}]\\ &\le C\int_{B_{R_1}/2} \exp\frac{4\pi(1 + \delta_0)(L_0 + 0.5\delta)/(L_0 + \delta)\, v_{1n}}{\int_{B_{R_1/2}(x_0)} f_n} \le C'\end{aligned}$$

with the aid of lemma 5.4 if we choose δ_0 sufficiently small. So we have proved lemma 6.1.

Now take $B_{R_1/4}(x_0)$. We conclude from the weak Hanack inequality [11, theorem 8.17]

$$\|v_n\|_{L^\infty(B_{R_1/4}(x_0))} \le C[\|v_n\|_{L^2(B_{R_1/2}(x_0))} + \|f_n\|_{L^{1+\delta_0}(B_{R_1/2}(x_0))}] \le C.$$

Here the boundedness of $\{v_n\}$ in $L^2(B_{R_1/2}(x_0))$ follows from lemma 5.4.

The case where $x_0 \in \Omega$ is similar. We just use lemma 5.2 in place of lemma 5.4. ∎

LEMMA 6.2. For any $\delta > 0$, $x_0 \in S_I \cup S_N$ if and only if x_0 is δ-irregular.

Proof. Let x_0 be a δ-irregular point. Then by lemma 6.1, $\{v_n\}$ is bounded in $L^\infty(B_{R_1} \cap \Omega)$ for some R_1. Hence, $x_0 \notin S_I \cup S_N$. Conversely, suppose x_0 is a δ-irregular point. Then we have for every $R > 0$

$$\lim_{n\to\infty} \|v_n\|_{L^\infty(B_R(x_0)\cap\Omega)} = \infty.$$

Otherwise, there would be some $R_0 > 0$ and a subsequence, still denoted by $\{v_n\}$, such that

$$\|v_{1n}\|_{L^\infty(B_R(x_0)\cap\bar\Omega)} \le C$$

for some C independent of n. Then

$$f_n = v_n^{p_n-1}v_n^{p_n} \le \left(\frac{M}{p_n}\right)^{p_n-1} C^{p_n} \to 0$$

uniformly as $n \to \infty$ on $B_{R_0}(x_0) \cap \bar\Omega$. Here M is a uniform upper bound of u_p obtained in theorem 1.1. Then

$$\int_{B_{R_0}(x_0)\cap\bar\Omega} f_n \le \varepsilon_0 \le \frac{2\alpha(x_0)}{L_0 + 2\delta}$$

which implies that x_0 is a δ-regular point. A contradiction. ∎

Back to the measure μ defined earlier in this section. Clearly, we have

$$1 \geq \mu(\Gamma_1 \cup \Omega) \geq \sum_{x_0 \in S_I \cup S_N} \frac{2\alpha(x_0)}{L_0 + 2\delta}$$

which in turn, if we let $\delta \to 0$, implies the following proposition.

PROPOSITION 6.3.

$$\sum_{x_0 \in S_I \cup S_N} \alpha(x_0) \leq \tfrac{1}{2} L_0 .$$

From this proposition, with the aid of proposition 2.2 and corollary 3.5, we obtain part 2 and part 3 of theorem 1.2.

Remark 6.4. We see that every peak P in Ω is a blow-up point of $v_p = u_p/v_p$ because by corollary 4.1 $v_p \to 0$ as $p \to \infty$.

7. FURTHER RESULTS AND EXAMPLES

In this section we shall focus on some special domains Ω where the corresponding quantities L_0 are indeed smaller than what we get in corollary 3.5. In these special cases, we can actually prove that the solutions of (1.1) possess single-peaks on the Neumann boundary of Ω. Let us first formulate a general result.

THEOREM 7.1. Let (Γ_1, Ω) be a pair such that α_0, defined in (4.4), with respect to this pair is strictly less than 1, i.e. $\gamma_0 < e/2$. Then for every sequence $\{u_{p_n}\}$ of solutions on Ω with the Neumann boundary Γ_1, there is a subsequence, again denoted by $\{u_{p_n}\}$, such that the interior blow-up set S_I is empty and the Γ_1-boundary blow-up set S_N contains at most one point.

Proof. If we check the proof of lemma 6.1 carefully, we can see that we can use a refined inequality

$$\frac{\log x}{x} \leq \frac{\log y}{y}$$

if $x \leq y \leq e$ instead of (6.11). Notice that since we assume $\alpha_0 < 1$,

$$\limsup_{n \to \infty} \frac{u_n}{v_n^{1/p_n}} \leq \exp \frac{1 + \alpha_0}{2} < e.$$

Let

$$L_0' = \frac{\limsup_{n \to \infty} (1 + \alpha_0) p \int_\Omega u_p^p}{2 \exp[(1 + \alpha_0)/2]} .$$

We still have, as proposition 6.3, with the aid of corollary 3.5,

$$\sum_{x_0 \in S_I \cup S_N} \alpha(x_0) \leq \tfrac{1}{2} L_0' < 2\beta. \tag{7.1}$$

If $S_I \neq \emptyset$, then, with the aid of proposition 2.2, $\alpha(x_0) = 2\pi$ for some $x_0 \in S_I$. If $\# S_N \geq 2$, then, with the aid of proposition 2.2 again, $\alpha(x_1) + \alpha(x_2) \geq 2\beta$ for two different x_1 and x_2 in S_N. In any case, we reach a contradiction to (7.1). ■

Example 7.2. Let

$$\Omega = \{x \in R^2 : r < |x| < R\}, \qquad \Gamma_1 = \{x \in R^2 : |x| = r\} \qquad \text{and} \qquad \Gamma_0 = \{x \in R^2 : |x| = R\}.$$

In this case the constant α with respect to (Γ_1, Ω) is equal to π (see [4, example 3.3]) and the constant β is clearly π. Hence, $\gamma_0 = 1 < e/2$ and the condition of theorem 7.1 is satisfied. Indeed, since the two boundaries has no intersection, passing to a subsequence if necessary, $S_N = \{x_0\}$.

Example 7.3. Let $\Omega = \Sigma(\alpha, R)$, $0 \leq \alpha \leq \pi$, and Γ_1 be the union of two sides of the sector.

In this case $\beta = \alpha$ (see [5]). hence, $\gamma_0 = 1 \leq e/2$ and the condition of theorem 7.1 is again satisfied.

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