Non-compactness of the Prescribed $Q-$curvature Problem in Large Dimensions

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Abstract

Let (M, g) be a compact Riemannian manifold of dimension $N \geq 5$ and Q_g be its Q curvature. The prescribed Q curvature problem is concerned with finding metric of constant Q curvature in the conformal class of g. This amounts to finding a positive solution to

$$P_g(u) = cu^{\frac{N+4}{N-4}}, \quad u > 0 \qquad \text{on } M$$

where P_g is the Paneitz operator. We show that for dimensions $N \geq 25$, the set of all positive solutions to the prescribed Q curvature problem is *non-compact*.

1 Introduction

Let (M, g) be a Riemannian manifold of dimension N. A basic question in conformal geometry is the following: can one change the original metric g conformally into a new metric g' with prescribed properties? This means that one searches for some positive function ψ (conformal factor) such that $g' = \psi g$ and the new metric g' has prescribed properties.

A best known example is the so-called Yamabe problem. For $N \geq 3$, let $L_g := -\frac{4(N-1)}{N-2}\Delta_g + S_g$ be the conformal Laplacian, where Δ_g is the Laplace-Beltrami operator and S_g is the scalar curvature. If one sets the conformal factor $\psi = u^{\frac{4}{N-2}}$ $(u > 0)$, then it is well known that L_g has the following conformal covariance property:

$$L_g(u\varphi) = u^{\frac{N+2}{N-2}} L_{g'}(\varphi) \qquad \forall\, \varphi \in C^\infty(M).$$

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If one prescribes the scalar curvature $S_{g'}$ for the metric g' then u has to satisfy the second-order equation

$$L_g(u) = u^{\frac{N+2}{N-2}} L_{g'}(1) = S_{g'} u^{\frac{N+2}{N-2}}. \tag{1}$$

In the case when $S_{g'}$ is a constant, this is the Yamabe problem. In the case when $S_{g'}$ is a prescribed function, it is called the Nirenberg problem.

Precisely, the Yamabe equation is

$$\frac{4(N-1)}{N-2}\Delta_g u - S_g u + c u^{\frac{N+2}{N-2}} = 0. \tag{2}$$

The question that whether the set of all solutions to the Yamabe problem (2) is compact in the $C^\infty -$ topology has been widely studied. It has been conjectured that this should be true unless (M, g) is conformally equivalent to the round sphere (see [27, 28, 29]). The case of the round sphere $\mathbb{S}^N$ is special in that (2) is invariant under the action of the conformal group on $\mathbb{S}^N$, which is non-compact. The Compactness Conjecture has been verified in low dimensions and locally conformally flat by R. Schoen [28, 29]. He also proposed a strategy to proving the conjecture in the non-locally conformally flat case. Developing further this strategy, the conjecture is proved in low dimensions: $N = 3$ by Li-Zhu [22], $N = 4, 5$ by Druet [10], $N = 6, 7$ by Li-Zhang [20] and Marques [23], $N = 10, 11$ by Li-Zhang [21] under the Positive Mass Theorem assumption. Recently this conjecture is shown to be true by Khuri-Marques-Schoen [16] for dimensions $N \leq 24$ under the Positive Mass Theorem condition. On the other hand, the Compactness Conjecture is not true for $N \geq 25$ in the recent papers by Brendle [3] ($N \geq 52$) and Brendle-Marques [4] ($25 \leq N \leq 51$). More precisely, given any integer $N \geq 25$, there exists a smooth Riemannian metric g on $\mathbb{S}^N$ such that set of constant scalar curvature metrics in the conformal class of g is non-compact. Moreover, the blowing-up sequences obtained in [3, 4] form exactly one bubble. The construction relies on a gluing procedure based on some local metric. The non-compactness of Yamabe problem in the $C^k -$ topology is studied by Ambrosetti-Malchiodi [1] and Berti-Malchiodi [2]. A complete description of blow-up behavior of Yamabe type problems can be found in the book Druet-Hebey-Robert [11].

Besides the conformal Laplacian L_g, there are many other operators which enjoy a conformal covariance property. A particularly interesting one is the fourth order operator P_g which was discovered by Paneitz in 1983 ([24]), which can be written for $N \geq 5$ as follows:

$$P_g = \Delta_g^2 - \mathrm{div}_g(a_N S_g g + b_N \mathcal{R}_g)d + \frac{N-4}{2} Q_g, \tag{3}$$

where div_g is the divergence of a vector-field, d is the differential operator,

$$a_N = \frac{(N-2)^2+4}{2(N-1)(N-2)}, \qquad b_N = -\frac{4}{N-2},$$

and

$$Q_g = -\frac{1}{2(N-1)}\Delta_g S_g + \frac{N^3-4N^2+16N-16}{8(N-1)^2(N-2)^2}S_g^2 - \frac{2}{(N-2)^2}|\mathcal{R}_g|^2. \quad (4)$$

Here $\mathcal{R}_g$ the Ricci tensor, $|\mathcal{R}_g|^2 = \sum_{i,j}\mathcal{R}^{ij}\mathcal{R}_{ij}$ where $\mathcal{R}^{ij} = \sum_{s,t} g^{is}\mathcal{R}_{st}g^{tj}$. Q_g is the so called $Q-$curvature. In the case $N > 4$, the conformal factor is usually chosen in the form $\psi = u^{\frac{4}{N-4}}$ $(u > 0)$ and the conformal covariance property of the Paneitz operator reads as follows:

$$P_g(u\varphi) = u^{\frac{N+4}{N-4}}P_{g'}(\varphi) \qquad \forall\, \varphi \in C^\infty(M).$$

If one prescribes the $Q-$curvature for the metric g' by a function $Q_{g'}$ this leads to the equation

$$P_g(u) = u^{\frac{N+4}{N-4}}P_{g'}(1) = \frac{N-4}{2}Q_{g'}u^{\frac{N+4}{N-4}},$$

which is a fourth-order analogue of (1). We refer to the survey articles of Chang [5] and Chang-Yang [8] and the lecture notes [6, 7] for more background information on the Paneitz operator. Recently, there are more and more interests in using higher order partial differential equations in the study of conformal geometry. See [9, 13, 30] and references therein.

There is an analogue problem to the Yamabe problem, that is, to find metrics of constant $Q-$curvature in the conformal class of g. The problem can be transformed to solving the following *$Q-$curvature equation*, for $N \geq 5$,

$$P_g u = \frac{N-4}{2}u^{\frac{N+4}{N-4}}, \quad u > 0 \qquad \text{on } M. \quad (5)$$

Clearly, every solution of (5) is a critical point of the functional

$$\mathcal{E}_g(u) = \frac{\int_M(\Delta_g u)^2 + \sum_{i,j}(a_N S_g g^{ij} + b_N\mathcal{R}^{ij})\partial_i u\partial_j u + \frac{N-4}{2}Q_g u^2 \mathrm{d}vol_g}{\left(\int_M u^{\frac{2N}{N-4}}\mathrm{d}vol_g\right)^{\frac{N-4}{N}}}. \quad (6)$$

Consider

$$P(g) = \inf\Big\{\mathcal{E}_g(u) : \ u \in H^2(M),\ u > 0\Big\}.$$

We refer to $P(g)$ as the Paneitz energy. Clearly it is a conformal invariant.

A similar question is whether or not the set of all positive solutions to the $Q-$curvature equation is compact. As far as we know, the only results in this direction are given by Hebey-Robert [15] and Qing-Raske [25]. Both papers give positive answers provided M is locally conformally flat and satisfies some additional assumptions. Compactness is also studied for non-geometric potentials of Paneitz operator in Hebey-Robert-Wen [14].

In this paper, we prove the non-compactness of the set of solutions to the $Q-$ curvature problem in large dimensions. We construct a blowing-up sequence consisting of exactly one bubble. More precisely we prove the following theorem.

Theorem 1.1. *Assume $N \geq 25$. Then there exists a C^∞ Riemannian metric g on $\mathbb{S}^N$ and a sequence of positive function u_n ($n \in \mathbb{N}$) with the following properties:*

i) g is not conformally flat,

ii) u_n is a positive solution of the $Q-$ curvature equation (5) for all n,

iii) $\mathcal{E}_g(u_n) < P(\mathbb{S}^N)$ for all n and $\mathcal{E}_g(u_n) \to P(\mathbb{S}^N)$ as $n \to \infty$,

iv) $\sup_{\mathbb{S}^N} u_n \to \infty$ as $n \to \infty$.

Here $P(\mathbb{S}^N)$ denotes the Paneitz energy of the round metric on $\mathbb{S}^N$.

For convenience (and by stereo-graphic projection), we will work on $\mathbb{R}^N$ instead of $\mathbb{S}^N$. Let g be a smooth metric on $\mathbb{R}^N$ which agrees with the Euclidean metric outside a ball of radius 1. We will assume throughout the paper that $\det g(x) = 1$ for all $x \in \mathbb{R}^N$, so that the volume form associated with g agrees with the Euclidean volume form. Precisely, we will consider

$$P_g u = \frac{N-4}{2} u^{\frac{N+4}{N-4}}, \quad u > 0 \qquad \text{in } \mathbb{R}^N. \tag{7}$$

Our goal is to construct solutions to the $Q-$ curvature equation (7) on $(\mathbb{R}^N, g)$. Though we shall follow the main ideas in [3] and [4], there are some major difficulties in fourth order equations. The main difficulty is that we need to ensure that u is strictly positive on $\mathbb{R}^N$. In the Yamabe problem case, one constructs solutions of the form

$$u = u_0 + \phi$$

where u_0 is the standard bubble and ϕ is the error. As long as $\|\phi\|_{H^1}$ is small, it can be shown easily that $u > 0$. In the prescribed Q curvature problem, even if we can show that $\|\phi\|_{H^2(\mathbb{R}^N)}$ is small, it is not guaranteed that u is positive. To overcome this difficulty, we need to use a weighted L^∞ norm

$$\|\phi\|_* = \|u_0^{-1}\phi\|_{L^\infty(\mathbb{R}^N)}$$

and we have to show that $\|\phi\|_*$ is small which then implies that u is positive. We use a technical framework which is more closely related to the finite dimensional Liapunov-Schmidt reduction procedure, as in [12], [26], and [31]. Another difficulty is the choice of the auxiliary function $f(s)$. In the second order Yamabe problem case, a linear function is chosen to obtain $N \geq 52$ ([3]) and a cubic polynomial is chosen to obtain $N \geq 25$ ([4]). While in the fourth order case, the best choice for f seems to be fourth order polynomial. (Linear function only gives $N \geq 52$.) A surprising fact is that in both Yamabe and $Q-$curvature problems, the two dimensions 52 and 25 are the same.

The organization of the paper is as follows: In Section 2, we introduce the special metric g in this paper. In Section 3, the first approximation of the solutions is given. In Section 4, we calculate the corresponding Δ_g, S_g, R_g, Q_g under this special metric g, and then acquire the estimate of the energy functional. In Section 5, the invertibility of the linearized operator is settled. In Section 6, we solve the nonlinear problem. In Section 7, a variational reduction procedure is processed. In Section 8, we show that the energy can be approximated by an reduced energy functional. In Section 9, we compute the reduced energy functional in terms of an auxiliary function f. In Section 10, we choose a linear auxilliary function to show that the reduced energy functional has a strict local minimum when $N \geq 52$. In Section 11, the dimension is reduced to $N \geq 25$ by choosing a fourth order polynomial. In Section 12, we prove the main theorem by a gluing method. Finally in the appendix, some inequalities used in the paper is proven.

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2 The Special Metric g

In this section we introduce the metric which will be used in this paper.

In what follows, we fix a multi-linear form W: $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$. We assume that $W_{ikj\ell}$ satisfy all the algebraic properties of the Weyl tensor. Moreover, we assume that some components of W are non-zero, so that

$$\sum_{i,j,k,\ell=1}^{n} (W_{ikj\ell} + W_{i\ell jk})^2 > 0.$$

For simplicity, we put

$$H_{ij}(y) = \sum_{p,q=1}^{n} W_{ipjq} y_p y_q$$

and

$$\overline{H}_{ij}(y) = f(|y|^2) H_{ij}(y),$$

where $f(|y|^2)$ will be chosen later. (Specifically, we shall choose $f(s) = \tau - 12000s + 2411s^2 - 135s^3 + s^4$. The number τ depends only on the dimension N and will be chosen later.) It is easy to see that $H_{ij}(y)$ is trace-free, $\sum_{i=1}^{n} y_i H_{ij}(y) = 0$ and $\sum_{i=1}^{n} \partial_i H_{ij}(y) = 0$ for all $y \in \mathbb{R}^N$.

We consider a Riemannian metric of the form $g(x) = e^{h(x)}$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^N$ satisfying $h(0) = 0$, $h(x) = 0$ for $|x| \geq 1$,

$$|h(x)| + |\partial h(x)| + |\partial^2 h(x)| + |\partial^3 h(x)| + |\partial^4 h(x)| \leq \alpha$$

for all $x \in \mathbb{R}^N$, where $\alpha > 0$ is a fixed small number, and

$$h_{ij}(x) = \mu \varepsilon^8 f(\varepsilon^{-2}|x|^2) H_{ij}(x)$$

for $|x| \leq \rho$. We assume that the parameter ε, μ and ρ are chosen such that $\mu \leq 1$ and $\varepsilon \leq \rho \leq 1$. Note that $\sum_{i=1}^{N} x_i h_{ij}(x) = 0$ and $\sum_{i=1}^{N} \partial_i h_{ij}(x) = 0$ for $|x| \leq \rho$.

For later purpose, we need to understand the Green's function of Δ_g^2. Denote $\widetilde{G}(x,y)$ be the Green's function of Δ_g in $\mathbb{R}^N$. Then the Green's function of Δ_g^2 is

$$G(x,y) = \int_{\mathbb{R}^N} \widetilde{G}(x,z)\widetilde{G}(y,z) \mathrm{d}z.$$

Since $|\widetilde{G}(x,y)| \leq C[d(x,y)]^{2-N}$, $|\nabla \widetilde{G}(x,y)| \leq C[d(x,y)]^{1-N}$ and $|\nabla^2 \widetilde{G}(x,y)| \leq C[d(x,y)]^{-N}$, we know that

$$\begin{aligned} |G(x,y)| &\leq C[d(x,y)]^{4-N}, \\ |\nabla G(x,y)| &\leq C[d(x,y)]^{3-N}, \\ |\nabla^2 G(x,y)| &\leq C[d(x,y)]^{2-N}. \end{aligned}$$

Here $d(x,y)$ is the distance between x and y under the metric g. It is easy to see that

$$d(x,y) = [1 + O(\alpha)]|x - y|$$

since $g = e^h$ and α is sufficiently small.

Notations In what follows, we use C to denote the variable constant which is independent of α and ε. $|O(A)| \leq CA$ and $o(A)/A \to 0$ as $\varepsilon \to 0$.

3 First Approximation of the Solutions

In this section we will provide an ansatz for solutions of Problem (7). Denote

$$u_0(x) = \gamma_N \left(\frac{\lambda' \varepsilon}{\lambda'^2 \varepsilon^2 + |x - \xi|^2} \right)^{\frac{N-4}{2}},$$

where $\gamma_N = \left[N(N-4)^2(N-2)(N+2)/2 \right]^{-\frac{N-4}{8}}$, $\lambda' > 0$ and $\varepsilon > 0$. It is well known [17] that u_0 is the only positive solution of

$$\Delta^2 u_0 = \frac{N-4}{2} u_0^{\frac{N+4}{N-4}} \qquad \text{in } \mathbb{R}^N.$$

Observe that $u(x)$ satisfies (7) if and only if $v(y) = \varepsilon^{\frac{N-4}{2}} u(\varepsilon y)$ satisfies

$$P_{\tilde{g}} v = \frac{N-4}{2} v^{\frac{N+4}{N-4}} \tag{8}$$

where $\tilde{g}(y) = g(\varepsilon y)$. Denote

$$\tilde{u}_0(y) = \varepsilon^{\frac{N-4}{2}} u_0(\varepsilon y) = \gamma_N \left(\frac{\lambda'}{\lambda'^2 + |y - \xi'|^2} \right)^{\frac{N-4}{2}},$$

where $\xi' = \xi/\varepsilon$, $\lambda' = \lambda/\varepsilon$. The configuration set for (ξ', λ') is

$$\Lambda = \left\{ (\xi', \lambda') \in \mathbb{R}^N \times \mathbb{R} : \ |\xi'| \leq 1, \ \frac{1}{2} < \lambda' < \frac{3}{2} \right\}.$$

We will look for a solution to (7) with the form $\tilde{u}_0(y) + \phi(y)$. It is easy to check that ϕ must be a solution of

$$P_{\tilde{g}} \phi - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \phi = -R + N(\phi) \qquad \text{in } \mathbb{R}^N, \tag{9}$$

where

$$R(y) = P_{\tilde{g}} \tilde{u}_0 - \frac{N-4}{2} \tilde{u}_0^{\frac{N+4}{N-4}}, \tag{10}$$

$$N(\phi) = \frac{N-4}{2} (\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} - \frac{N-4}{2} \tilde{u}_0^{\frac{N+4}{N-4}} - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \phi. \tag{11}$$

A main step in solving (9) for small ϕ is that of a solvability theory for the linearized operator $P_{\tilde{g}} - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}}$.

4 Preliminary Estimates

In this section we will mainly estimate the energy functional. From now on, denote $\tilde{h}(y) = h(\varepsilon y)$ and $\widetilde{\mathcal{R}}$, $S_{\tilde{g}}$, $Q_{\tilde{g}}$ are the corresponding Ricci tensor, scaler curvature, $Q-$ curvature under the metric $\tilde{g}$.

Lemma 4.1. *For $|y| \le \rho/\varepsilon$, it holds*

$$\begin{aligned}
\widetilde{\mathcal{R}}_{ij} = & -\sum_m \frac{1}{2}\partial_{mm}\tilde{h}_{ij} + \sum_{m,s}\frac{1}{2}[\tilde{h}_{ms}(\partial_{ms}\tilde{h}_{ij}) - (\partial_s\tilde{h}_{mj})(\partial_m\tilde{h}_{si})] \\
& + \sum_{m,s}\frac{1}{4}\Big[(\partial_i\tilde{h}_{ms})(\partial_m\tilde{h}_{sj}) + (\partial_j\tilde{h}_{ms})(\partial_m\tilde{h}_{si}) - (\partial_j\tilde{h}_{ms})(\partial_i\tilde{h}_{sm}) \\
& \qquad - \tilde{h}_{ms}(\partial_{mi}\tilde{h}_{sj}) - \tilde{h}_{ms}(\partial_{mj}\tilde{h}_{si}) - (\partial_{mm}\tilde{h}_{js})\tilde{h}_{si} - \tilde{h}_{js}(\partial_{mm}\tilde{h}_{si})\Big] \\
& + O(|\tilde{h}|^2|\partial^2\tilde{h}| + |\tilde{h}||\partial\tilde{h}|^2).
\end{aligned}$$

For $|y| \ge \rho/\varepsilon$, we have

$$\widetilde{\mathcal{R}}_{ij} = O(\alpha\varepsilon^2).$$

Proof. Recall that $\widetilde{\Gamma}^i_{k\ell} = \sum_m \frac{1}{2}\tilde{g}^{im}(\partial_\ell\tilde{g}_{mk} + \partial_k\tilde{g}_{m\ell} - \partial_m\tilde{g}_{k\ell})$. Since $\tilde{h}$ is trace-free, we have $\det\tilde{g} = 1$ for all $y \in \mathbb{R}^N$. This implies $\sum_i \widetilde{\Gamma}^i_{ik} = \sum_{i,\ell}\frac{1}{2}\tilde{g}^{i\ell}\partial_k\tilde{g}_{i\ell} = \frac{1}{2}\partial_k\log|\tilde{g}| = 0$. Therefore, we obtain

$$\begin{aligned}
\widetilde{\mathcal{R}}_{ij} &= \sum_m \partial_m\widetilde{\Gamma}^m_{ji} - \sum_m \partial_j\widetilde{\Gamma}^m_{mi} + \sum_{\ell,m}\widetilde{\Gamma}^m_{m\ell}\widetilde{\Gamma}^\ell_{ji} - \sum_{\ell,m}\widetilde{\Gamma}^m_{j\ell}\widetilde{\Gamma}^\ell_{mi} \\
&= \sum_m \partial_m\widetilde{\Gamma}^m_{ji} - \sum_{\ell,m}\widetilde{\Gamma}^m_{j\ell}\widetilde{\Gamma}^\ell_{mi}. \qquad (12)
\end{aligned}$$

Direct calculation shows

$$\begin{aligned}
\sum_m \partial_m\widetilde{\Gamma}^m_{ji} = & \sum_m \frac{1}{2}\Big[\partial_{mi}\tilde{h}_{mj} + \partial_{mj}\tilde{h}_{mi} - \partial_{mm}\tilde{h}_{ij}\Big] \\
& + \sum_{m,s}\frac{1}{2}\Big[(\partial_m\tilde{h}_{ms})(\partial_s\tilde{h}_{ij}) + \tilde{h}_{ms}(\partial_{ms}\tilde{h}_{ij})\Big] \\
& + \sum_{m,s}\frac{1}{4}\Big[(\partial_i\tilde{h}_{ms})(\partial_m\tilde{h}_{sj}) - (\partial_m\tilde{h}_{ms})(\partial_i\tilde{h}_{sj}) + (\partial_j\tilde{h}_{ms})(\partial_m\tilde{h}_{si}) \\
& \qquad - (\partial_m\tilde{h}_{ms})(\partial_j\tilde{h}_{si}) - 2(\partial_m\tilde{h}_{js})(\partial_m\tilde{h}_{si}) + (\partial_{mi}\tilde{h}_{ms})\tilde{h}_{sj} \\
& \qquad - (\partial_{mi}\tilde{h}_{sj})\tilde{h}_{ms} + (\partial_{mj}\tilde{h}_{ms})\tilde{h}_{si} - (\partial_{mj}\tilde{h}_{si})\tilde{h}_{ms} - (\partial_{mm}\tilde{h}_{js})\tilde{h}_{si}
\end{aligned}$$

$$-(\partial_{mm}\tilde{h}_{si})\tilde{h}_{js}\Big] + O(|\tilde{h}|^2|\partial^2\tilde{h}| + |\tilde{h}||\partial\tilde{h}|^2),$$

and

$$\begin{aligned}\sum_{\ell,m}\widetilde{\Gamma}^m_{j\ell}\widetilde{\Gamma}^\ell_{mi} = & \sum_{\ell,m}\frac{1}{2}\Big[(\partial_\ell\tilde{h}_{mj})(\partial_m\tilde{h}_{\ell i}) - (\partial_\ell\tilde{h}_{mj})(\partial_\ell\tilde{h}_{mi})\Big] + \frac{1}{4}(\partial_j\tilde{h}_{m\ell})(\partial_i\tilde{h}_{\ell m}) \\ & + O(|\tilde{h}||\partial\tilde{h}|^2).\end{aligned}$$

Since $\sum_m \partial_m\tilde{h}_{mk} = 0$ for $|y| \le \rho/\varepsilon$, the lemma follows from (12). □

Thus we have the following calculations for the Ricci tensor $\widetilde{\mathcal{R}}^{ij} = \sum_{s,t}\tilde{g}^{is}\widetilde{\mathcal{R}}_{st}\tilde{g}^{tj}$.

Corollary 4.2. *For $|y| \le \rho/\varepsilon$, we have*

$$\begin{aligned}\widetilde{\mathcal{R}}^{ij} = & -\sum_m \frac{1}{2}\partial_{mm}\tilde{h}_{ij} + \sum_{m,s}\frac{1}{2}[\tilde{h}_{ms}(\partial_{ms}\tilde{h}_{ij}) - (\partial_s\tilde{h}_{mj})(\partial_m\tilde{h}_{si})] \\ & + \sum_{m,s}\frac{1}{4}\Big[(\partial_i\tilde{h}_{ms})(\partial_m\tilde{h}_{sj}) + (\partial_j\tilde{h}_{ms})(\partial_m\tilde{h}_{si}) - (\partial_j\tilde{h}_{ms})(\partial_i\tilde{h}_{sm}) \\ & \quad - \tilde{h}_{ms}(\partial_{mi}\tilde{h}_{sj}) - \tilde{h}_{ms}(\partial_{mj}\tilde{h}_{si}) + (\partial_{mm}\tilde{h}_{js})\tilde{h}_{si} + h_{js}(\partial_{mm}\tilde{h}_{si})\Big] \\ & + O(|\tilde{h}|^2|\partial^2\tilde{h}| + |\tilde{h}||\partial\tilde{h}|^2).\end{aligned}$$

For $|y| \ge \rho/\varepsilon$, it holds

$$\widetilde{\mathcal{R}}^{ij} = O(\alpha\varepsilon^2).$$

Lemma 4.3. *For $|y| \le \rho/\varepsilon$, there holds*

$$S_{\tilde{g}} = -\frac{1}{4}\sum_{k,\ell,m}(\partial_\ell\tilde{h}_{mk})^2 + O(|\tilde{h}|^2|\partial^2\tilde{h}| + |\tilde{h}||\partial\tilde{h}|^2).$$

For $|y| \ge \rho/\varepsilon$, we have

$$S_{\tilde{g}} = O(\alpha\varepsilon^2).$$

Proof. The detailed proof can be found in [3, Prop. 26], noting that $\sum_m \partial_m\tilde{h}_{mk} = 0$ in $|y| \le \rho/\varepsilon$. □

The above lemma and a direct computation show the following conclusion.

Corollary 4.4. *For $|y| \leq \rho/\varepsilon$, we have*

$$\begin{aligned}\Delta_{\tilde g} S_{\tilde g} = & -\frac{1}{2}\sum_{i,k,\ell,m}(\partial_{i\ell}\tilde h_{mk})^2 - \frac{1}{2}\sum_{i,k,\ell,m}(\partial_\ell \tilde h_{mk})(\partial_{ii\ell}\tilde h_{mk}) \\ & + O(|\partial\tilde h|^2|\partial^2\tilde h| + |\tilde h||\partial^2\tilde h|^2 + |\tilde h||\partial\tilde h||\partial^3\tilde h| + |\tilde h|^2|\partial^4\tilde h|).\end{aligned}$$

For $|y| \geq \rho/\varepsilon$, it holds

$$\Delta_{\tilde g} S_{\tilde g} = O(\alpha\varepsilon^4).$$

Now it is ready to estimate $Q_{\tilde g}$.

Lemma 4.5. *For $|y| \leq \rho/\varepsilon$, we have*

$$\begin{aligned}Q_{\tilde g} = & \frac{1}{4(N-1)}\sum_{i,k,\ell,m}\Big[(\partial_{i\ell}\tilde h_{mk})^2 + (\partial_\ell\tilde h_{mk})(\partial_{ii\ell}\tilde h_{mk})\Big] \\ & - \frac{1}{2(N-2)^2}\sum_{i,j,m,s}(\partial_{mm}\tilde h_{ij})(\partial_{ss}\tilde h_{ij}) \\ & + O(|\partial\tilde h|^2|\partial^2\tilde h| + |\tilde h||\partial^2\tilde h|^2 + |\tilde h||\partial\tilde h||\partial^3\tilde h| + |\tilde h|^2|\partial^4\tilde h|).\end{aligned}$$

For $|y| \geq \rho/\varepsilon$, there holds

$$Q_{\tilde g} = O(\alpha\varepsilon^4).$$

Proof. This is a direct result of (4), Lemma 4.1, Corollary 4.2 and 4.4. □

Our next goal is to estimate $R(y)$ defined in (10).

Lemma 4.6. *For $|y| \leq \frac{\rho}{\varepsilon}$,*

$$\begin{aligned}\Delta^2_{\tilde g}\tilde u_0 - \Delta^2\tilde u_0 = & -(\partial_{ss}\tilde h_{ij})(\partial_{ij}\tilde u_0) - 2(\partial_s\tilde h_{ij})(\partial_{sij}\tilde u_0) - 2\tilde h_{ij}(\partial_{ssij}\tilde u_0) \\ & + O(|\tilde h||\partial^2\tilde h|)|\partial^2\tilde u_0| + O(|\tilde h||\partial\tilde h|)|\partial^3\tilde u_0| + O(|\tilde h|^2)|\partial^4\tilde u_0|.\end{aligned}$$

For $|y| \geq \frac{\rho}{\varepsilon}$,

$$\Delta^2_{\tilde g}\tilde u_0 - \Delta^2\tilde u_0 = O\left(\frac{\alpha\varepsilon}{(1+|y-\xi'|)^{N-1}}\right).$$

Proof. The computations follow easily from the definition of $\Delta_{\tilde g}$ and the properties of h. □

By Lemma 4.3, Corollary 4.2 and the properties of h, it is also not difficult to verify the following two lemmas.

Lemma 4.7. *For* $|y| \le \frac{\rho}{\varepsilon}$,

$$\sum_{i,j} \partial_j (S_{\tilde g} \tilde g^{ij} \partial_i \tilde u_0) = O\left(\frac{\mu^2 \varepsilon^{20}}{(1+|y-\xi'|)^{N-20}} \right).$$

For $|y| \ge \frac{\rho}{\varepsilon}$,

$$\sum_{i,j} \partial_j (S_{\tilde g} \tilde g^{ij} \partial_i \tilde u_0) = O\left(\frac{\alpha \varepsilon^{2}}{(1+|y-\xi'|)^{N-2}} \right).$$

Lemma 4.8. *For* $|y| \le \frac{\rho}{\varepsilon}$,

$$\begin{aligned}\sum_{i,j} \partial_j (\widetilde{\mathcal{R}}^{ij} \partial_i \tilde u_0) = & -\frac{1}{2}(\partial_{jmm} \tilde h_{ij})(\partial_i \tilde u_0) - \frac{1}{2}(\partial_{mm} \tilde h_{ij})(\partial_{ij} \tilde u_0) \\ & + O\left(\frac{\mu^2 \varepsilon^{20}}{(1+|y-\xi'|)^{N-20}} \right).\end{aligned}$$

For $|y| \ge \frac{\rho}{\varepsilon}$,

$$\sum_{i,j} \partial_j (\widetilde{\mathcal{R}}^{ij} \partial_i \tilde u_0) = O\left(\frac{\alpha \varepsilon^{2}}{(1+|y-\xi'|)^{N-2}} \right).$$

Combining the above results, we have the following estimate for $R(y)$.

Proposition 4.9. *It holds*

$$R(y) \le \begin{cases} C \dfrac{\mu \varepsilon^{10}}{(1+|y-\xi'|)^{N-10}} & \text{for } |y| \le \dfrac{\rho}{\varepsilon}, \\ C \dfrac{\alpha \varepsilon}{(1+|y-\xi'|)^{N-1}} & \text{for } \dfrac{\rho}{\varepsilon} \le |y| \le \dfrac{1}{\varepsilon}, \\ 0 & \text{for } |y| \ge \dfrac{1}{\varepsilon}. \end{cases}$$

Let us consider the energy functional $E_{\tilde g}(v)$ associated to Problem (8), namely

$$\begin{aligned} E_{\tilde g}(v) = & \frac{1}{2} \int_{\mathbb{R}^N} (\Delta_{\tilde g} v)^2 + \sum_{i,j} (a_N S_{\tilde g} \tilde g^{ij} + b_N \widetilde{\mathcal{R}}^{ij}_{\tilde g}) \partial_i v \partial_j v + \frac{N-4}{2} Q_{\tilde g} v^2 \mathrm{d}y \\ & - \frac{(N-4)^2}{4N} \int_{\mathbb{R}^N} v^{\frac{2N}{N-4}} \mathrm{d}y. \end{aligned}$$

In what follows, we will calculate the energy $E_{\tilde g}(\tilde u_0)$, which is a important step for the existence of the solutions of our equation. First we have

Lemma 4.10. *For $|y| \le \rho/\varepsilon$,*

$$
\begin{aligned}
& (\Delta_{\tilde g}\tilde u_0)^2 - (\Delta \tilde u_0)^2 \\
= \; & \sum_{i,j,k,\ell} \tilde h_{i\ell}(\partial_i \tilde h_{j\ell})(\partial_{kk}\tilde u_0)(\partial_j \tilde u_0) - \sum_{i,j,k} 2\tilde h_{ij}(\partial_{kk}\tilde u_0)(\partial_{ij}\tilde u_0) \\
& + \sum_{i,j,k,\ell} \tilde h_{i\ell}\tilde h_{j\ell}(\partial_{kk}\tilde u_0)(\partial_{ij}\tilde u_0) + \left(\sum_{i,j} \tilde h_{ij}(\partial_{ij}\tilde u_0)\right)^2 \\
& + O(|\tilde h||\partial \tilde h|^2)|\partial \tilde u_0|^2 + O(|\tilde h|^2|\partial \tilde h|)|\partial \tilde u_0||\partial^2 \tilde u_0| + O(|\tilde h|^3)|\partial^2 \tilde u_0|^2. \qquad (13)
\end{aligned}
$$

While for $|y| \ge \rho/\varepsilon$, we have

$$(\Delta_{\tilde g}\tilde u_0)^2 - (\Delta \tilde u_0)^2 = O(\alpha(1 + |y - \xi'|)^{4-2N}). \qquad (14)$$

Proof. It is easy to check that

$$
\begin{aligned}
(\Delta_{\tilde g}\tilde u_0)^2 - (\Delta \tilde u_0)^2 = \; & \left(\sum_{i,j} (\partial_i \tilde g^{ij})(\partial_j \tilde u_0)\right)^2 + \sum_{i,j,k,\ell} 2(\partial_k \tilde g^{k\ell})\tilde g^{ij}(\partial_\ell \tilde u_0)(\partial_{ij}\tilde u_0) \\
& + \sum_{i,j,k,\ell} (\tilde g^{ij} - \delta_{ij})(\tilde g^{k\ell} + \delta_{k\ell})(\partial_{ij}\tilde u_0)(\partial_{k\ell}\tilde u_0).
\end{aligned}
$$

By direct calculation, we have

$$
\begin{aligned}
& \left(\sum_{i,j} \partial_i \tilde g^{ij} \partial_j \tilde u_0\right)^2 \\
= \; & \sum_{i,j,k,\ell} (\partial_k \tilde h_{ik})(\partial_\ell \tilde h_{j\ell})\partial_i \tilde u_0 \partial_j \tilde u_0 + O(|\tilde h||\partial \tilde h|^2)|\partial \tilde u_0|^2,
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{i,j,k,\ell} 2\tilde g^{ij}(\partial_k \tilde g^{k\ell})\partial_\ell \tilde u_0 \partial_{ij}\tilde u_0 \\
= \; & -\sum_{i,k,\ell} 2(\partial_k \tilde h_{k\ell})\partial_\ell \tilde u_0 \partial_{ii}\tilde u_0 + \sum_{i,k,\ell,m} \left[(\partial_k \tilde h_{km})\tilde h_{k\ell} + (\partial_k \tilde h_{m\ell})\tilde h_{km}\right]\partial_\ell \tilde u_0 \partial_{ii}\tilde u_0 \\
& + \sum_{i,j,k,\ell} 2\tilde h_{ij}(\partial_k \tilde h_{k\ell})\partial_\ell \tilde u_0 \partial_{ij}\tilde u + O(|\tilde h|^2|\partial \tilde h|)|\partial \tilde u_0||\partial^2 \tilde u_0|
\end{aligned}
$$

and

$$\sum_{i,j,k,\ell} (\tilde g^{ij} - \delta_{ij})(\tilde g^{k\ell} + \delta_{k\ell})(\partial_{ij}\tilde u_0)(\partial_{k\ell}\tilde u_0)$$

$$
\begin{aligned}
= & -\sum_{i,j,k} 2\tilde{h}_{ij}\partial_{ij}\tilde{u}_0\partial_{kk}\tilde{u}_0 + \sum_{i,j,k,m} \tilde{h}_{im}\tilde{h}_{mj}\partial_{ij}\tilde{u}_0\partial_{kk}\tilde{u}_0 \\
& + \left(\sum_{i,j} \tilde{h}_{ij}\partial_{ij}\tilde{u}_0\right)^2 + O(|\tilde{h}|^3)|\partial^2\tilde{u}_0|^2.
\end{aligned}
$$

Therefore, for $|y| \le \rho/\varepsilon$, $\sum_i \partial_i \tilde{h}_{ij} = 0$ yields (13). Since $\tilde{h} = 0$ for $|y| \ge 1/\varepsilon$, (14) can be easily gotten. This concludes the proof. □

Lemma 4.11. *It holds*

$$
\begin{aligned}
& \int_{\mathbb{R}^N} (\Delta_{\tilde{g}}\tilde{u}_0)^2 - (\Delta\tilde{u}_0)^2 \\
= & -\int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,k,\ell} \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{ikk}\tilde{u}_0)(\partial_j\tilde{u}_0) + \int_{B_{\frac{\rho}{\varepsilon}}} \left(\sum_{i,j} \tilde{h}_{ij}(\partial_{ij}\tilde{u}_0)\right)^2 \\
& + O(\mu^3\varepsilon^{\frac{20N}{N-1}}) + O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).
\end{aligned}
$$

Proof. Since for $|y| \le \rho/\varepsilon$,

$$
\begin{aligned}
& |\tilde{h}||\partial\tilde{h}|^2|\partial\tilde{u}_0|^2 + |\tilde{h}|^2|\partial\tilde{h}||\partial\tilde{u}_0||\partial^2\tilde{u}_0| + |\tilde{h}|^3|\partial^2\tilde{u}_0|^2 \\
\le & C\mu^3\varepsilon^{30}(1+|y-\xi'|)^{34-2N} \\
\le & C\mu^3\varepsilon^{\frac{20N}{N-1}}(1+|y-\xi'|)^{\frac{20N}{N-1}+4-2N},
\end{aligned}
$$

from Lemma 4.10 we have

$$
\begin{aligned}
& \int_{\mathbb{R}^N} (\Delta_{\tilde{g}}\tilde{u}_0)^2 - (\Delta\tilde{u}_0)^2 \\
= & \int_{B_{\frac{\rho}{\varepsilon}}} \sum_{i,j,k,\ell} \tilde{h}_{i\ell}(\partial_i\tilde{h}_{j\ell})(\partial_{kk}\tilde{u}_0)(\partial_j\tilde{u}_0) - \sum_{i,j,k} 2\tilde{h}_{ij}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) \\
& + \sum_{i,j,k,\ell} \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) + \left(\sum_{i,j} \tilde{h}_{ij}(\partial_{ij}\tilde{u}_0)\right)^2 \mathrm{d}y \\
& + O(\mu^3\varepsilon^{\frac{20N}{N-1}}) + O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right). \qquad (15)
\end{aligned}
$$

On the other hand, integrating by parts and using $\sum_i \partial_i \tilde{h}_{ij} = 0$ for $|y| \le \rho/\varepsilon$ and $\tilde{h}(y) = 0$ for $|y| \ge 1/\varepsilon$, we know

$$
\sum_{i,j,k,\ell} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{i\ell}(\partial_i\tilde{h}_{j\ell})(\partial_{kk}\tilde{u}_0)(\partial_j\tilde{u}_0) + \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0)
$$

$$
\begin{aligned}
&= \sum_{i,j,k,\ell} \int_{\mathbb{R}^N} \partial_i(\tilde{h}_{i\ell}\tilde{h}_{j\ell})(\partial_{kk}\tilde{u}_0)(\partial_j\tilde{u}_0) + \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) + O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right) \\
&= -\sum_{i,j,k,\ell} \int_{\mathbb{R}^N} \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{ikk}\tilde{u}_0)(\partial_j\tilde{u}_0) + O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right) \\
&= -\sum_{i,j,k,\ell} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{ikk}\tilde{u}_0)(\partial_j\tilde{u}_0) + O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right). \qquad (16)
\end{aligned}
$$

Next, direct computation shows

$$
\begin{aligned}
\tilde{u}_0(\partial_{ijkk}\tilde{u}_0) = &\frac{N}{(N-3)\,(N^2-4N+8)}(\partial_{ijkk}\tilde{u}_0^2) + \frac{N^2+4N}{N^2-4N+8}(\partial_{ij}\tilde{u}_0)(\partial_{kk}\tilde{u}_0) \\
&+ \frac{4(N-4)^2(N-2)N}{N^2-4N+8}\tilde{u}_0^2\frac{|y-\xi'|^2\delta_{ij}}{(\varepsilon^2+|y-\xi'|^2)^3} \\
&- \frac{4(N-4)^2\,(N^2-2)}{N^2-4N+8}\tilde{u}_0^2\frac{\delta_{ij}}{(\varepsilon^2+|y-\xi'|^2)^2}.
\end{aligned}
$$

Recalling $\tilde{h}$ is divergence-free, we get

$$
\begin{aligned}
&\sum_{i,j,k} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{ij}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) \\
&= \sum_{i,j,k} \int_{\mathbb{R}^N} \partial_{ij}(\tilde{h}_{ij}\tilde{u}_0)(\partial_{kk}\tilde{u}_0) + O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right) \\
&= \sum_{i,j,k} \int_{\mathbb{R}^N} \tilde{h}_{ij}\tilde{u}_0(\partial_{ijkk}\tilde{u}_0) + O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right).
\end{aligned}
$$

Since $\sum_{i=1}^N \partial_i\tilde{h}_{ij} = 0$ for $|y| \le \frac{\rho}{\varepsilon}$, it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \tilde{h}_{ij}(\partial_{ijkk}\tilde{u}_0^2) = \int_{\mathbb{R}^N} (\partial_{ijkk}\tilde{h}_{ij})\tilde{u}_0^2 \\
&= \int_{\mathbb{R}^N\setminus B_{\frac{\rho}{\varepsilon}}} (\partial_{ijkk}\tilde{h}_{ij})\tilde{u}_0^2 = O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right).
\end{aligned}
$$

Thus

$$
\sum_{i,j,k} \int_{B_{\frac{\rho}{\varepsilon}}} \tilde{h}_{ij}(\partial_{kk}\tilde{u}_0)(\partial_{ij}\tilde{u}_0) = O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right). \qquad (17)
$$

The proof of the lemma is completed by (15), (16) and (17). □

Lemma 4.12. *For $|y| \le \rho/\varepsilon$,*

$$\sum_{i,j} S_{\tilde g}\tilde g^{ij}\partial_i\tilde u_0\partial_j\tilde u_0 = -\frac{1}{4}\sum_{i,k,\ell,m}(\partial_\ell\tilde h_{mk})^2(\partial_i\tilde u_0)^2 + O(|\tilde h||\partial\tilde h|^2)|\partial\tilde u_0|^2.$$

For $|y| \ge \rho/\varepsilon$,

$$\sum_{i,j} S_{\tilde g}\tilde g^{ij}\partial_i\tilde u_0\partial_j\tilde u_0 = O(\alpha\varepsilon^2(1+|y-\xi'|)^{6-2N}).$$

Proof. Recalling that $g = e^h$, this lemma is an easy consequence of Lemma 4.3. □

Lemma 4.13. *We have*

$$\begin{aligned}
&\int_{\mathbb{R}^N}\sum_{i,j} a_N S_{\tilde g}\tilde g^{ij}\partial_i\tilde u_0\partial_j\tilde u_0\\
= &-\frac{a_N}{4}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,k,\ell,m}(\partial_\ell\tilde h_{mk})^2(\partial_i\tilde u_0)^2 + O(\mu^3\varepsilon^{\frac{20N}{N-1}}) + O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right).
\end{aligned}$$

Proof. This follows from Lemma 4.12 by direct calculation. □

Lemma 4.14. *It holds*

$$\begin{aligned}
&\int_{\mathbb{R}^N}\sum_{i,j} b_N\widetilde{\mathcal{R}}^{ij}\partial_i\tilde u_0\partial_j\tilde u_0\\
= &-\frac{b_N}{4}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}(\partial_j\tilde h_{ms})(\partial_i\tilde h_{sm})(\partial_i\tilde u_0)(\partial_j\tilde u_0)\\
&-\frac{b_N}{2}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}\Big[\tilde h_{ms}(\partial_s\tilde h_{ij}) - \tilde h_{si}(\partial_s\tilde h_{mj}) + \tilde h_{sj}(\partial_i\tilde h_{ms})\\
&\qquad\qquad - \tilde h_{ms}(\partial_i\tilde h_{sj})\Big]\partial_m(\partial_i\tilde u_0\partial_j\tilde u_0)\\
&+\frac{b_N}{2}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}\tilde h_{is}(\partial_{mm}\tilde h_{js})(\partial_i\tilde u_0)(\partial_j\tilde u_0)\\
&+O(\mu^3\varepsilon^{\frac{20N}{N-1}}) + O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right).
\end{aligned}$$

Proof. From Corollary 4.2, we have

$$\int_{\mathbb{R}^N}\sum_{i,j} b_N\widetilde{\mathcal{R}}^{ij}\partial_i\tilde u_0\partial_j\tilde u_0$$

$$
\begin{aligned}
= & -\frac{b_N}{2}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m}(\partial_{mm}\tilde{h}_{ij})\partial_i\tilde{u}_0\partial_j\tilde{u}_0 \\
& -\frac{b_N}{4}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}(\partial_j\tilde{h}_{ms})(\partial_i\tilde{h}_{sm})(\partial_i u_0)(\partial_j u_0) \\
& +\frac{b_N}{2}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}\Big[\tilde{h}_{ms}(\partial_{ms}\tilde{h}_{ij})-(\partial_s\tilde{h}_{mj})(\partial_m\tilde{h}_{si})+(\partial_i\tilde{h}_{ms})(\partial_m\tilde{h}_{sj}) \\
& \qquad\qquad\qquad -\tilde{h}_{ms}(\partial_{mi}\tilde{h}_{sj})\Big](\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0) \\
& +\frac{b_N}{2}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}\tilde{h}_{is}(\partial_{mm}\tilde{h}_{js})(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0) \\
& +O(\mu^3\varepsilon^{\frac{20N}{N-1}})+O\left(\alpha\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right).
\end{aligned}
$$

Since

$$
\partial_i\tilde{u}_0\partial_j\tilde{u}_0-\frac{(N-4)}{4(N-3)}\partial_{ij}\tilde{u}_0^2=\frac{(N-4)^2}{2(N-3)}\tilde{u}_0^2\frac{\delta_{ij}}{\lambda'^2+|y-\xi'|^2},
$$

it is easy to check, noting that $\sum_{i=1}^N\partial_i\tilde{h}_{ij}=0$,

$$
\begin{aligned}
\left|\sum_{i,j,m}\int_{B_{\frac{\varrho}{\varepsilon}}}(\partial_{mm}\tilde{h}_{ij})\partial_i\tilde{u}_0\partial_j\tilde{u}_0\right| &\le C\int_{\mathbb{R}^n\setminus B_{\frac{\varrho}{\varepsilon}}}|\partial^4\tilde{h}|\tilde{u}_0^2+O\left(\alpha\rho^2\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right) \\
&=O\left(\alpha\rho^2\Big(\frac{\varepsilon}{\rho}\Big)^{N-4}\right).
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
& \sum_{i,j,m,s}\int_{B_{\frac{\varrho}{\varepsilon}}}\tilde{h}_{ms}(\partial_{ms}\tilde{h}_{ij})(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0) \\
= & \sum_{i,j,m,s}\int_{\mathbb{R}^n}\partial_m[\tilde{h}_{ms}(\partial_s\tilde{h}_{ij})]\partial_i\tilde{u}_0\partial_j\tilde{u}_0+O(\alpha^2\rho^2(\frac{\varepsilon}{\rho})^{N-4}) \\
= & -\sum_{i,j,m,s}\int_{B_{\frac{\varrho}{\varepsilon}}}\tilde{h}_{ms}(\partial_s\tilde{h}_{ij})\partial_m(\partial_i\tilde{u}_0\partial_j\tilde{u}_0)+O(\alpha^2\rho(\frac{\varepsilon}{\rho})^{N-4}),
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{i,j,m,s}\int_{B_{\frac{\varrho}{\varepsilon}}}(\partial_s\tilde{h}_{jm})(\partial_m\tilde{h}_{is})(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0) \\
= & \sum_{i,j,m,s}\int_{\mathbb{R}^n}\partial_m[\tilde{h}_{is}(\partial_s\tilde{h}_{jm})]\partial_i\tilde{u}_0\partial_j\tilde{u}_0+O(\alpha^2\rho^2(\frac{\varepsilon}{\rho})^{N-4})
\end{aligned}
$$

$$= -\sum_{i,j,m,s}\int_{B_{\frac{\varrho}{\varepsilon}}}\tilde{h}_{is}(\partial_s\tilde{h}_{jm})\partial_m(\partial_i\tilde{u}_0\partial_j\tilde{u}_0)+O(\alpha\rho(\frac{\varepsilon}{\rho})^{N-4}),$$

$$\sum_{i,j,m,s}\int_{B_{\frac{\varrho}{\varepsilon}}}(\partial_i\tilde{h}_{ms})(\partial_m\tilde{h}_{sj})(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0)$$
$$= -\sum_{i,j,m,s}\int_{B_{\frac{\varrho}{\varepsilon}}}\tilde{h}_{sj}(\partial_i\tilde{h}_{ms})(\partial_m(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0))+O(\alpha\rho(\frac{\varepsilon}{\rho})^{N-4})$$

and

$$\sum_{i,j,m,s}\int_{B_{\frac{\varrho}{\varepsilon}}}\tilde{h}_{ms}(\partial_{mi}\tilde{h}_{sj})(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0)$$
$$= -\sum_{i,j,m,s}\int_{B_{\frac{\varrho}{\varepsilon}}}\tilde{h}_{ms}(\partial_i\tilde{h}_{js})(\partial_m(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0))+O(\alpha\rho(\frac{\varepsilon}{\rho})^{N-4}).$$

The proof is complete. □

Lemma 4.15.

$$\frac{N-4}{2}\int_{\mathbb{R}^N}Q_{\tilde{g}}\tilde{u}_0^2$$
$$=\frac{N-4}{8(N-1)}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,k,\ell,m}\Big[(\partial_{i\ell}\tilde{h}_{mk})^2+(\partial_\ell\tilde{h}_{mk})(\partial_{ii\ell}\tilde{h}_{mk})\Big]\tilde{u}_0^2$$
$$-\frac{N-4}{4(N-2)^2}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}(\partial_{mm}\tilde{h}_{ij})(\partial_{ss}\tilde{h}_{ij})u_0^2$$
$$+O\left(\mu^3\varepsilon^{\frac{20N}{N-1}}\right)+O\left(\alpha\rho^4\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).$$

Proof. This is an easy consequence of Lemma 4.5. □

Now we have the following estimate of $E_{\tilde{g}}(\tilde{u}_0)$.

Proposition 4.16.

$$2E_{\tilde{g}}(\tilde{u}_0)=2E-\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,k,\ell}\tilde{h}_{i\ell}\tilde{h}_{j\ell}(\partial_{ikk}\tilde{u}_0)(\partial_j\tilde{u}_0)+\int_{B_{\frac{\varrho}{\varepsilon}}}\Big(\sum_{i,j}\tilde{h}_{ij}(\partial_{ij}\tilde{u}_0)\Big)^2$$
$$-\frac{a_N}{4}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,k,\ell,m}(\partial_\ell\tilde{h}_{mk})^2(\partial_i\tilde{u}_0)^2$$

$$
\begin{aligned}
&-\frac{b_N}{4}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}(\partial_j\tilde{h}_{ms})(\partial_i\tilde{h}_{sm})(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0)\\
&-\frac{b_N}{2}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}\Big[\tilde{h}_{ms}(\partial_s\tilde{h}_{ij})-\tilde{h}_{si}(\partial_s\tilde{h}_{mj})+\tilde{h}_{sj}(\partial_i\tilde{h}_{ms})\\
&\qquad\qquad\qquad\qquad\qquad-\tilde{h}_{ms}(\partial_i\tilde{h}_{sj})\Big]\partial_m(\partial_i\tilde{u}_0\partial_j\tilde{u}_0)\\
&+\frac{b_N}{2}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}\tilde{h}_{is}(\partial_{mm}\tilde{h}_{js})(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0)\\
&+\frac{N-4}{8(N-1)}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,k,\ell,m}\Big[(\partial_{i\ell}\tilde{h}_{mk})^2+(\partial_\ell\tilde{h}_{mk})(\partial_{ii\ell}\tilde{h}_{mk})\Big]\tilde{u}_0^2\\
&-\frac{N-4}{4(N-2)^2}\int_{B_{\frac{\varrho}{\varepsilon}}}\sum_{i,j,m,s}(\partial_{mm}\tilde{h}_{ij})(\partial_{ss}\tilde{h}_{ij})\tilde{u}_0^2\\
&+O\left(\mu^3\varepsilon^{\frac{20N}{N-1}}\right)+O\left(\alpha\big(\frac{\varepsilon}{\rho}\big)^{N-4}\right),
\end{aligned}
$$

where E is the constant such that

$$
E=\frac{N-4}{N}\int_{\mathbb{R}^N}\left(\frac{1}{1+|y|^2}\right)^N \mathrm{d}y.
$$

Remark: *Note that*

$$
E=P(\mathbb{S}^N). \tag{18}
$$

5 Linearized Operator

In this section we develop the invertibility theory for the linearized operator $P_{\tilde{g}}-\frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}$ in suitable weighted L^∞ spaces.

We define two norms

$$
\|\phi\|_*=\sup_{y\in\mathbb{R}^N}\sum_{i=0}^{2}\left[\frac{1}{\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14+i}}+\alpha(\frac{\varepsilon}{\rho})^{N-4+i}}+\frac{(1+|y-\xi'|)^{N-4+i}}{\alpha}\right]|\partial^i\phi(y)|,
$$

$$
\begin{aligned}
\|\zeta\|_{**}=\sup_{y\in\mathbb{R}^N}\Bigg[&\frac{\chi_{\{|y-\xi'|\le\frac{\varrho}{\varepsilon}\}}(1+|y-\xi'|)^{N-10}}{\mu\varepsilon^{10}}+\frac{\chi_{\{\frac{\varrho}{\varepsilon}\le|y-\xi'|\le\frac{1}{\varepsilon}\}}(1+|y-\xi'|)^{N-1}}{\alpha\varepsilon}\\
&+\frac{\chi_{\{|y-\xi'|\ge\frac{1}{\varepsilon}\}}(1+|y-\xi'|)^{N+\sigma}}{\alpha}\Bigg]|\zeta(y)|,
\end{aligned}
$$

where χ_S is the characteristic function on the set S, and $0<\sigma<1$ is a small constant.

Denote

$$Z_0 = \frac{\partial \tilde{u}_0}{\partial \lambda'}, \qquad Z_j = \frac{\partial \tilde{u}_0}{\partial \xi'_j} \quad j = 1, \cdots, N.$$

First, we consider the following problem. Given $\zeta \in C^\alpha(\mathbb{R}^N)$, find a function ϕ such that for certain constants c_i, $i = 0, 1, \cdots, N$,

$$\begin{cases} P_{\tilde{g}}\phi - \frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}\phi = \zeta + \sum_{i=0}^{N} c_i \chi Z_i & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \phi \chi Z_i = 0, \end{cases} \tag{19}$$

where $\chi(y) = \chi(|y-\xi'|)$ is a cut-off function satisfying $\chi(y) = 1$ for $|y-\xi'| \le r_0$, $\chi(y) = 0$ for $|y - \xi'| \ge r_0 + 1$. Here $r_0 > 0$ is large but fixed.

Proposition 5.1. *Assume $N \ge 18$, $(\xi', \lambda') \in \Lambda$ and α is small and fixed. Then for small ε, there is a unique solution ϕ to (19). Moreover*

$$\|\phi\|_* \le C\|\zeta\|_{**}$$

where C is independent of α and ε.

To prove the above proposition, we need the following priori estimate.

Lemma 5.2. *Under the assumptions of Proposition 5.1, for any solution ϕ to (19), there exists a constant C such that*

$$\|\phi\|_* \le C\|\zeta\|_{**}.$$

Proof. We use the contradiction argument as in [12] and [26]. Assume there are sequences $\varepsilon_n \to 0$ and the corresponding ζ_n, ϕ_n such that $\|\zeta_n\|_{**} \to 0$ but $\|\phi_n\|_* = 1$. For abbreviation, we omit the subscript n in the following proof. Testing the equation against $\bar{\chi}Z_j$ and integrating by parts four times, where $\bar{\chi}(y)$ is a smooth cut-off function satisfying $\bar{\chi}(y) = 1$ for $|y - \xi'| \le \frac{\rho}{4\varepsilon}$, $\bar{\chi}(y) = 0$ for $|y - \xi'| \ge \frac{\rho}{2\varepsilon}$ and $|\nabla^i \bar{\chi}| \le C(\frac{\varepsilon}{\rho})^i$, $1 \le i \le 4$. we get

$$\sum_i c_i \int_{\mathbb{R}^N} \chi Z_i Z_j = \int_{\mathbb{R}^N} \left(P_{\tilde{g}_n}(\bar{\chi}Z_j) - \frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}\bar{\chi}Z_j \right)\phi - \int_{\mathbb{R}^N} \zeta \bar{\chi} Z_j.$$

This defines a linear system in the c_i which is “almost diagonal” as ε approaches zero, since we have

$$\int_{\mathbb{R}^N} \chi Z_0 Z_j = \delta_{0j} \int \chi \left(\frac{\partial \tilde{u}_0}{\partial \lambda'} \right)^2,$$

$$\int_{\mathbb{R}^N} \chi Z_i Z_j = \delta_{ij} \int \chi \left(\frac{\partial \tilde{u}_0}{\partial y_j} \right)^2 \qquad \forall\ i = 1, \cdots, N.$$

On the other hand, using $\Delta^2 Z_j - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} Z_j = 0$ and the estimates of $S_{\tilde{g}}$, $\widetilde{\mathcal{R}}^{st}$, $Q_{\tilde{g}}$ in the previous section, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(P_{\tilde{g}}(\bar{\chi} Z_j) - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \bar{\chi} Z_j \right) \phi \\
= &\int_{\mathbb{R}^N} \left\{ \Delta_{\tilde{g}}^2 (\bar{\chi} Z_j) - \bar{\chi} \Delta^2 (Z_j) - \sum_{s,t} \partial_s \left[\left(a_N S_{\tilde{g}} \tilde{g}^{st} + b_N \widetilde{\mathcal{R}}^{st} \right) \partial_t (\bar{\chi} Z_j) \right] \right. \\
&\qquad\qquad \left. + \frac{N-4}{2} Q_{\tilde{g}} \bar{\chi} Z_j \right\} \phi \\
= &\ o(\mu \varepsilon^{10}) \|\phi\|_*.
\end{aligned}$$

It is also easy to get

$$\int_{\mathbb{R}^N} \zeta \bar{\chi} Z_j \leq C \mu \varepsilon^{10} \|\zeta\|_{**}.$$

Thus we conclude

$$|c_i| \leq o(\mu \varepsilon^{10}) \|\phi\|_* + C \mu \varepsilon^{10} \|\zeta\|_{**} \qquad \forall\ i = 0, \cdots, N,$$

so $c_i = o(\mu \varepsilon^{10})$.

Next we claim that, for any fixed $R > 0$,

$$\|\phi\|_{L^\infty(B_R(\xi'))} = o(\mu \varepsilon^{10}).$$

Indeed, by elliptic regularity we can get a $\hat{\phi}$ such that $\frac{\phi}{\mu \varepsilon^{10}} \to \hat{\phi}$ in $C^4_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\Delta^2 \hat{\phi} - \frac{N+4}{2} \tilde{u}_0^{\frac{8}{N-4}} \hat{\phi} = 0 \qquad \text{in } \mathbb{R}^N.$$

This implies $\hat{\phi}$ is a linear combination of the functions Z_j, $j = 0, 1, \cdots, N$, see [19]. On the other hand, the assumed orthogonality conditions on ϕ yields $\int_{\mathbb{R}^N} \hat{\phi} \chi Z_j = 0$ for all j. Hence $\hat{\phi} \equiv 0$, which concludes the claim.

Now rewrite the equation in the following form

$$\phi(y) = \int_{\mathbb{R}^N} G(y,z) \sum_{i,j} \partial_j \left[(a_N S_{\tilde{g}} \tilde{g}^{ij} + b_N \widetilde{\mathcal{R}}^{ij}) \partial_i \phi \right] (z) \mathrm{d}z$$

$$- \int_{\mathbb{R}^N} \frac{N-4}{2} G(y,z) Q_{\tilde{g}}(z)\phi(z)\mathrm{d}z + \int_{\mathbb{R}^N} G(y,z)\frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}\phi(z)\mathrm{d}z$$
$$+ \int_{\mathbb{R}^N} G(y,z)\zeta(z)\mathrm{d}z + \sum_i c_i \int_{\mathbb{R}^N} G(y,z)\chi Z_i(z)\mathrm{d}z. \qquad (20)$$

We make now the following observations:

Owing to $S_{\tilde{g}}(z) = O(\alpha\varepsilon^2)$, $\partial S_{\tilde{g}} = O(\alpha\varepsilon^3)$ and $\widetilde{\mathcal{R}}^{ij}(z) = O(\alpha\varepsilon^2)$, $\partial\widetilde{\mathcal{R}}^{ij}(z) = O(\alpha\varepsilon^3)$, we have

$$\left| \int_{\mathbb{R}^N} G(y,z) \sum_{i,j} \partial_j \left[(a_N S_{\tilde{g}} \tilde{g}^{ij} + b_N \widetilde{\mathcal{R}}^{ij})\partial_i\phi \right](z)\mathrm{d}z \right|$$
$$= O(\alpha\varepsilon^3)\int_{B_{1/\varepsilon}} G(y,z)|\partial\phi(z)|\mathrm{d}z + O(\alpha\varepsilon^2)\int_{B_{1/\varepsilon}} G(y,z)|\partial^2\phi(z)|\mathrm{d}z$$
$$:= I + II. \qquad (21)$$

For $|y-\xi'| \le \frac{\rho}{2\varepsilon}$,

$$|I| \le C\alpha\varepsilon^3\|\phi\|_* \Bigg\{ \int_{|z|\le\frac{\rho}{\varepsilon}} \frac{1}{|y-z|^{N-4}} \left[\frac{\mu\varepsilon^{10}}{(1+|z-\xi'|)^{N-13}} + \alpha(\frac{\varepsilon}{\rho})^{N-3} \right] \mathrm{d}z$$
$$+ \int_{\frac{\rho}{\varepsilon}\le|z|\le\frac{1}{\varepsilon}} \frac{1}{|y-z|^{N-4}} \frac{\alpha}{(1+|z-\xi'|)^{N-3}}\mathrm{d}z \Bigg\}$$
$$\le C\alpha\|\phi\|_* \left[\frac{\mu\varepsilon^{10}\varepsilon^3(1+|y-\xi'|)^3}{(1+|y-\xi'|)^{N-14}} + \alpha\rho^3(\frac{\varepsilon}{\rho})^{N-4} \right]$$
$$\le C\alpha\rho^3\|\phi\|_* \left[\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}} + \alpha(\frac{\varepsilon}{\rho})^{N-4} \right], \qquad (22)$$

where we use, for any $0 < s, k < N$ such that $s + k < N$,

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-s}} \frac{1}{(1+|z-\xi'|)^{N-k}}\mathrm{d}z \le C(1+|y-\xi'|)^{k+s-N}. \qquad (23)$$

The proof of (23) is standard and is given in the appendix. Similarly, for $|y-\xi'| \le \frac{\rho}{2\varepsilon}$,

$$|II| \le C\alpha\rho^2\|\phi\|_* \left[\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}} + \alpha(\frac{\varepsilon}{\rho})^{N-4} \right]. \qquad (24)$$

On the other hand, for $|y-\xi'| \ge \frac{\rho}{2\varepsilon}$,

$$|I| \le C\alpha\varepsilon^3\|\phi\|_* \Bigg\{ \int_{|z|\le\frac{\rho}{\varepsilon}} \frac{1}{|y-z|^{N-4}} \left[\frac{\mu\varepsilon^{10}}{(1+|z-\xi'|)^{N-13}} + \alpha(\frac{\varepsilon}{\rho})^{N-3} \right] \mathrm{d}z$$

$$+\int_{\frac{\rho}{\varepsilon}\le|z|\le\frac{1}{\varepsilon}}\frac{1}{|y-z|^{N-4}}\frac{\alpha}{(1+|z-\xi'|)^{N-3}}\mathrm{d}z\Bigg\}$$

$$\le C\rho^3\|\phi\|_*\frac{\alpha}{(1+|y-\xi'|)^{N-4}}, \tag{25}$$

where we use, for any $0<s,k<N$ such that $s+k<N$,

$$\int_{B_r}\frac{1}{|y-z|^{N-s}}\frac{1}{(1+|z-\xi'|)^{N-k}}\mathrm{d}z\le Cr^k(1+|y-\xi'|)^{s-N}, \tag{26}$$

which is a direct result of (23) and the proof is also given in the appendix. A similar proof also gives

$$|II|\le C\rho^2\|\phi\|_*\frac{\alpha}{(1+|y-\xi'|)^{N-4}}\qquad \text{for } |y-\xi'|\ge\frac{\rho}{2\varepsilon}. \tag{27}$$

Since $Q_{\tilde g}(z)=O(\alpha\varepsilon^4)$, we similarly have

$$\left|\int_{\mathbb{R}^N}\frac{N-4}{2}G(y,z)Q_{\tilde g}(z)\phi(z)\mathrm{d}z\right|$$
$$\le C\alpha\rho^4\|\phi\|_*\left[\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}}+\alpha(\frac{\varepsilon}{\rho})^{N-4}\right]\qquad \text{for } |y-\xi'|\le\frac{\rho}{2\varepsilon}, \tag{28}$$

and for $|y-\xi'|\ge\frac{\rho}{2\varepsilon}$,

$$\left|\int_{\mathbb{R}^N}\frac{N-4}{2}G(y,z)Q_{\tilde g}(z)\phi(z)\mathrm{d}z\right|\le C\rho^4\|\phi\|_*\frac{\alpha}{(1+|y-\xi'|)^{N-4}}. \tag{29}$$

Therefore, combining (21), (22), (24), (25), (27)-(29), we finally have

$$\Bigg|\int_{\mathbb{R}^N}G(y,z)\sum_{i,j}\partial_i\left[(a_NS_{\tilde g}\tilde g^{ij}+b_N\widetilde{\mathcal{R}}^{ij})\partial_j\phi\right](z)\mathrm{d}z$$
$$-\int_{\mathbb{R}^N}\frac{N-4}{2}G(y,z)Q_{\tilde g}(z)\phi(z)\mathrm{d}z\Bigg|$$

$$\le\begin{cases}C\alpha\rho^2\|\phi\|_*\left[\dfrac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}}+\alpha(\dfrac{\varepsilon}{\rho})^{N-4}\right] & \text{for } |y-\xi'|\le\dfrac{\rho}{2\varepsilon},\\ C\rho^2\|\phi\|_*\dfrac{\alpha}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\ge\dfrac{\rho}{2\varepsilon}.\end{cases} \tag{30}$$

Next note that

$$\left|\int_{\mathbb{R}^N}G(y,z)\frac{N+4}{2}\tilde u_0(z)^{\frac{8}{N-4}}\phi(z)\mathrm{d}z\right|$$

$$\leq C\left\{\int_{B_R(\xi')}+\int_{R<|z-\xi'|<\frac{\rho}{\varepsilon}}+\int_{|z-\xi'|>\frac{\rho}{\varepsilon}}\right\}\frac{1}{|y-z|^{N-4}}\frac{\phi(z)}{(1+|z-\xi'|)^8}.$$

Since $\|\phi\|_{L^\infty(B_R(\xi'))}=o(\mu\varepsilon^{10})$, it is easy to check that

$$\int_{B_R(\xi')}\frac{1}{|y-z|^{N-4}}\frac{\phi(z)}{(1+|z-\xi'|)^8}\mathrm{d}z$$
$$=\begin{cases} o(1)\left[\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}}+\alpha(\frac{\varepsilon}{\rho})^{N-4}\right] & \text{for } |y-\xi'|\leq\frac{\rho}{2\varepsilon},\\ o(\mu\varepsilon^{10})\frac{\alpha}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\geq\frac{\rho}{2\varepsilon}.\end{cases}$$

$$\int_{R<|z-\xi'|<\frac{\rho}{\varepsilon}}\frac{1}{|y-z|^{N-4}}\frac{\phi(z)}{(1+|z-\xi'|)^8}\mathrm{d}z$$
$$\leq\begin{cases} \frac{C}{R^3}\|\phi\|_*\left[\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}}+\alpha(\frac{\varepsilon}{\rho})^{N-4}\right] & \text{for } |y-\xi'|\leq\frac{\rho}{2\varepsilon},\\ C\alpha(\frac{\varepsilon}{\rho})^4\|\phi\|_*\frac{\alpha}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\geq\frac{\rho}{2\varepsilon}.\end{cases}$$

$$\int_{|z-\xi'|\geq\frac{\rho}{\varepsilon}}\frac{1}{|y-z|^{N-4}}\frac{\phi(z)}{(1+|z-\xi'|)^8}\mathrm{d}z$$
$$\leq\begin{cases} C(\frac{\varepsilon}{\rho})^4\|\phi\|_*\left[\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}}+\alpha(\frac{\varepsilon}{\rho})^{N-4}\right] & \text{for } |y-\xi'|\leq\frac{\rho}{2\varepsilon},\\ C(\frac{\varepsilon}{\rho})^3\|\phi\|_*\frac{\alpha}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\geq\frac{\rho}{2\varepsilon}.\end{cases}$$

Thus

$$\left|\int_{\mathbb{R}^N}G(y,z)\frac{N+4}{2}\tilde{u}_0(z)^{\frac{8}{N-4}}\phi(z)\mathrm{d}z\right|$$
$$\leq\begin{cases} \left(\frac{C}{R^3}\|\phi\|_*+o(1)\right)\left[\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}}+\alpha(\frac{\varepsilon}{\rho})^{N-4}\right] & \text{for } |y-\xi'|\leq\frac{\rho}{2\varepsilon},\\ C(\frac{\varepsilon}{\rho})^3\|\phi\|_*\frac{\alpha}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\geq\frac{\rho}{2\varepsilon}.\end{cases} \tag{31}$$

Similarly,

$$\int_{\mathbb{R}^N}G(y,z)\zeta(z)\mathrm{d}z=\left\{\int_{|z-\xi'|<\frac{\rho}{\varepsilon}}+\int_{\frac{\rho}{\varepsilon}<|z-\xi'|<\frac{1}{\varepsilon}}+\int_{|z-\xi'|>\frac{1}{\varepsilon}}\right\}G(y,z)\zeta(z).$$

Using (23) and (26), we have

$$\left|\int_{|z-\xi'|<\frac{\rho}{\varepsilon}} G(y,z)\zeta(z)\right| \le \begin{cases} C\|\zeta\|_{**}\dfrac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}} & \text{for } |y-\xi'|\le \dfrac{\rho}{2\varepsilon},\\ C\|\zeta\|_{**}\dfrac{\mu\rho^{10}}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\ge \dfrac{\rho}{2\varepsilon},\end{cases}$$

$$\left|\int_{\frac{\rho}{\varepsilon}<|z-\xi'|<\frac{1}{\varepsilon}} G(y,z)\zeta(z)\right| \le \begin{cases} C\|\zeta\|_{**}\alpha(\dfrac{\varepsilon}{\rho})^{N-4} & \text{for } |y-\xi'|\le \dfrac{\rho}{2\varepsilon},\\ C\|\zeta\|_{**}\dfrac{\alpha}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\ge \dfrac{\rho}{2\varepsilon}\end{cases}$$

and

$$\left|\int_{|z-\xi'|\ge\frac{1}{\varepsilon}} G(y,z)\zeta(z)\right| \le \begin{cases} C\|\zeta\|_{**}\alpha(\dfrac{\varepsilon}{\rho})^{N-4} & \text{for } |y-\xi'|\le \dfrac{\rho}{2\varepsilon},\\ C\|\zeta\|_{**}\dfrac{\alpha}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\ge \dfrac{\rho}{2\varepsilon}.\end{cases}$$

So

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} G(y,z)\zeta(z)\mathrm{d}z\right| \\ \le &\begin{cases} C\|\zeta\|_{**}\left[\dfrac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}}+\alpha(\dfrac{\varepsilon}{\rho})^{N-4}\right] & \text{for } |y-\xi'|\le \dfrac{\rho}{2\varepsilon},\\ C\|\zeta\|_{**}\dfrac{\alpha}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\ge \dfrac{\rho}{2\varepsilon}.\end{cases} \end{aligned} \tag{32}$$

Since we have know $c_i = o(\mu\varepsilon^{10})$, it holds

$$\begin{aligned} &\sum_i c_i\int_{\mathbb{R}^N} G(y,z)\chi(z)Z_i(z)\mathrm{d}z \\ &\le C|c_i|\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-4}}\chi(z)\frac{1}{(1+|z-\xi'|)^{N-4}}\mathrm{d}z \\ &= \begin{cases} o(1)\left[\dfrac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}}+\alpha(\dfrac{\varepsilon}{\rho})^{N-4}\right] & \text{for } |y-\xi'|\le \dfrac{\rho}{2\varepsilon},\\ o(\mu\varepsilon^{10})\dfrac{\alpha}{(1+|y-\xi'|)^{N-4}} & \text{for } |y-\xi'|\ge \dfrac{\rho}{2\varepsilon}.\end{cases} \end{aligned} \tag{33}$$

Now we obtain that, by combining (30)-(33) and choosing R large enough,

$$\left[\frac{1}{\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-14}}+\alpha(\frac{\varepsilon}{\rho})^{N-4}}+\frac{(1+|y-\xi'|)^{N-4}}{\alpha}\right]|\phi| \le C\|\zeta\|_{**}+o(1).$$

Taking the derivative in (20), we have

$$\begin{aligned}\partial_{y_i}\phi(y) = & \int_{\mathbb{R}^N} \partial_{y_i}G(y,z)\sum_{s,t}\partial_s\left[(a_N S_{\tilde{g}}\tilde{g}^{st} + b_N\widetilde{\mathcal{R}}^{st})\partial_t\phi\right](z)\mathrm{d}z \\ & - \int_{\mathbb{R}^N} \frac{N-4}{2}\partial_{y_i}G(y,z)Q_{\tilde{g}}(z)\phi(z)\mathrm{d}z \\ & + \int_{\mathbb{R}^N} \partial_{y_i}G(y,z)\frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}\phi(z)\mathrm{d}z \\ & + \int_{\mathbb{R}^N} \partial_{y_i}G(y,z)\zeta(z)\mathrm{d}z \\ & + \sum_i c_i\int_{\mathbb{R}^N} \partial_{y_i}G(y,z)Z_i(z)\mathrm{d}z.\end{aligned}$$

Since $|\partial_{y_i}G(y,z)| \leq C[1+O(\alpha)]|x-y|^{3-N}$, similarly we can prove that

$$\left[\frac{1}{\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-13}} + \alpha(\frac{\varepsilon}{\rho})^{N-3}} + \frac{(1+|y-\xi'|)^{N-3}}{\alpha}\right]|\partial\phi| \leq C\|\zeta\|_{**} + o(1).$$

It is also similar to get that

$$\left[\frac{1}{\frac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-12}} + \alpha(\frac{\varepsilon}{\rho})^{N-2}} + \frac{(1+|y-\xi'|)^{N-2}}{\alpha}\right]|\partial^2\phi| \leq C\|\zeta\|_{**} + o(1).$$

So we finally have

$$\|\phi\|_* \leq C\|\zeta\|_{**} + o(1),$$

which is a contradiction. □

Proof of Proposition 5.1. Consider the space

$$\mathcal{H} = \left\{\phi \in H^2(\mathbb{R}^N):\ \int_{\mathbb{R}^N}\chi Z_j\phi = 0 \quad \forall\ j = 0,1,\cdots,N\right\}$$

endowed with the inner product $(\phi,\psi) = \int_{\mathbb{R}^N}\Delta_g\phi\Delta_g\psi$. Problem (19) expressed in weak form is equivalent to that of finding a $\phi \in \mathcal{H}$ such that, for any $\psi \in \mathcal{H}$,

$$\begin{aligned}(\phi,\psi) = & -\int_{\mathbb{R}^N}\sum_{i,j}(a_N S_{\tilde{g}}\tilde{g}^{ij} + b_N\widetilde{\mathcal{R}}^{ij})\partial_i\phi\partial_j\psi + \frac{N-4}{2}Q_{\tilde{g}}\phi\psi \\ & + \int_{\mathbb{R}^N}\frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}\phi\psi + \int_{\mathbb{R}^N}\zeta\psi.\end{aligned}$$

With the aid of Riesz's representation theorem, this equation can be rewritten in $\mathcal{H}$ in the operator form

$$\phi = K(\phi) + \tilde{\zeta}$$

with certain $\tilde{\zeta} \in \mathcal{H}$ which depends linearly on ζ and K is a compact operator in $\mathcal{H}$. Fredholm's alternative guarantees unique solvability of this problem for any $\tilde{\zeta}$ provided that the homogeneous equation $\phi = K(\phi)$ has only the zero solution in $\mathcal{H}$, which is equivalent to (19) with $\zeta = 0$. Thus existence of a unique solution follows from Lemma 5.2. This finishes the proof. □

Remark 5.3. *The result of Proposition 5.1 implies that the unique solution $\phi = T(\zeta)$ of (19) defines a continuous linear map from the weighted L^∞ space L^∞_{**}, equipped with norm $\|\cdot\|_{**}$, into the weighted L^∞ space L^∞_*, equipped with $\|\cdot\|_*$.*

It is important for later purposes to understand the differentiability of the operator T with respect to the variables ξ' and λ'.

Proposition 5.4. *Assume $(\xi', \lambda') \in \Lambda$. We have*

$$\|\nabla_{(\xi',\lambda')} T(\zeta)\|_* \leq C\|\zeta\|_{**}.$$

Proof. Denote formally $Z = \partial_{\xi'}\phi$. We seek for an expression for Z. Then Z satisfies the following equation:

$$P_g Z - \frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}} Z = \frac{N+4}{2}\partial_{\xi'}(\tilde{u}_0^{\frac{8}{N-4}})\phi + \sum_{i=0}^{N} d_i \chi Z_i + \sum_{i=0}^{N} c_i \partial_{\xi'}(\chi Z_i) \qquad \text{in } \mathbb{R}^N,$$

where $d_i = \partial_{\xi'} c_i$. Besides, from differentiating the orthogonality condition $\int_{\mathbb{R}^N} \phi\chi Z_j = 0$, we further get

$$\int_{\mathbb{R}^N} \phi \; \partial_{\xi'}(\chi Z_j) + \int_{\mathbb{R}^N} Z\chi Z_j = 0.$$

Choose b_ℓ such that

$$\sum_\ell b_\ell \int_{\mathbb{R}^N} \chi Z_\ell Z_j = \int_{\mathbb{R}^N} \phi \; \partial_{\xi'}(\chi Z_j).$$

Since this system is diagonal dominant with uniformly bounded coefficients, we see that it is uniquely solvable and that

$$|b_\ell| \le C\mu\varepsilon^{10}\|\phi\|_*$$

uniformly in $(\xi', \lambda') \in \Lambda$.

Let us now set

$$\eta = Z + \sum_{i=0}^{N} b_i \chi Z_i. \tag{34}$$

Then η satisfies

$$\begin{cases} P_g\eta - \dfrac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}\eta = \tilde{\zeta} + \displaystyle\sum_{i=0}^{N} d_i\chi Z_i & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} \eta\chi Z_j = 0 & \forall\ j. \end{cases}$$

where

$$\tilde{\zeta} = \sum_i b_i \left(P_g(\chi Z_i) - \frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}(\chi Z_i) \right) + \frac{N+4}{2}\partial_{\xi'}(\tilde{u}_0^{\frac{8}{N-4}})\phi$$
$$+ \sum_{i=0}^{N} c_i \partial_{\xi'}(\chi Z_i).$$

Applying Lemma 5.2,

$$\|\eta\|_* \le C\|\tilde{\zeta}\|_{**}.$$

It can be directly checked that

$$|b_i| \left\| P_g(\chi Z_i) - \frac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}(\chi Z_i) \right\|_{**} \le C\|\phi\|_*,$$

$$\left\| \frac{N+4}{2}\partial_{\xi'}(\tilde{u}_0^{\frac{8}{N-4}})\phi \right\|_{**} \le C\|\phi\|_*,$$

and

$$\|c_i\partial_{\xi'}(\chi Z_i)\|_{**} \le C\|\zeta\|_{**}.$$

Thus $\|\tilde{\zeta}\|_{**} \le C\|\zeta\|_{**}$, and then

$$\|\eta\|_* \le C\|\zeta\|_{**}.$$

Obviously $\|b_i Z_i\|_* \le C\|\phi\|_* \le C\|\zeta\|_{**}$. Therefore, we get by (34) that

$$\|Z\|_* \le C\|\zeta\|_{**}.$$

The corresponding result for differentiation with respect to λ' follows similarly. This concludes the proof. $\square$

6 Nonlinear Problem

We recall that our aim is to solve Problem (10). Rather than doing so directly, we shall solve first the intermediate problem

$$\begin{cases} P_{\tilde{g}}\phi - \dfrac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}\phi = -R + N(\phi) + \displaystyle\sum_{i=0}^{N} c_i \chi Z_i & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} \phi \chi Z_j = 0 \qquad \forall\, j = 0, 1, \cdots, N. \end{cases} \tag{35}$$

Proposition 6.1. *There exists a unique solution to (35) such that*

$$\|\phi\|_* \leq \beta$$

where $\beta > 0$ is a large number independent of α and ε.

Proof. In terms of the operator T defined in Remark 5.3, Problem (35) becomes

$$\phi = T[-R + N(\phi)] := A(\phi).$$

For a given large number $\beta > 0$, let us set

$$\mathcal{S} = \left\{ \phi \in \mathcal{H} \cap L_*^\infty(\mathbb{R}^N) : \ \|\phi\|_* \leq \beta \right\}.$$

From Proposition 5.1, we get

$$\|A(\phi)\|_* \leq C(\|R\|_{**} + \|N(\phi)\|_{**}).$$

According to Lemma 4.9, direct computation shows

$$\|R\|_{**} \leq C. \tag{36}$$

Here C is independent of α and ε. By the mean value theorem, we also easily have

$$\|N(\phi)\|_{**} \leq C\varepsilon^{4-\sigma}\|\phi\|_*^2. \tag{37}$$

Thus $A(\phi) \in \mathcal{S}$.

Furthermore, it is easy to check that for any $\phi_1, \phi_2 \in \mathcal{S}$,

$$\|N(\phi_1) - N(\phi_2)\|_{**} \leq C\varepsilon^{4-\sigma}\|\phi_1 - \phi_2\|_*.$$

So

$$\|A(\phi_1) - A(\phi_2)\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_{**} \leq C\varepsilon^{4-\sigma}\|\phi_1 - \phi_2\|_*,$$

which implies that A is a contraction mapping with the norm $\|\cdot\|_*$ inside $\mathcal{S}$. Therefore the contraction mapping theorem yields the proposition. □

Our purpose in the remains of this section is to analyze the differentiability properties of the function ϕ defined in Proposition 6.1

Proposition 6.2. *The function $(\xi', \lambda') \mapsto \phi(\xi', \lambda')$ provided by Proposition 6.1 is of class C^1 for the norm $\|\cdot\|_*$. Moreover*

$$\|\nabla_{(\xi',\lambda')}\phi\|_* \leq C.$$

Proof. First, we come to the differentiability of $\phi_{(\xi',\lambda')}$. Consider the following map H: $\Lambda \times \mathcal{H} \cap L^\infty_*(\mathbb{R}^N) \times \mathbb{R}^{N+1} \longrightarrow L^\infty_{**}(\mathbb{R}^N) \times \mathbb{R}^{N+1}$ of class C^1:

$$H((\xi', \lambda'), \phi, \boldsymbol{c}) = \begin{pmatrix} P_{\tilde{g}}(\tilde{u}_0 + \phi) - \frac{N-4}{2}(\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} - \sum_{i=0}^{N} c_i \chi Z_i \\ \int_{\mathbb{R}^N} \chi Z_0 \phi \\ \vdots \\ \int_{\mathbb{R}^N} \chi Z_N \phi \end{pmatrix}.$$

Problem (35) is then equivalent to $H((\xi', \lambda'), \phi, \boldsymbol{c}) = 0$. We know that given $(\xi', \lambda') \in \Lambda$, there is a unique solution $\phi_{(\xi',\lambda')}$. We will prove that the linear operator

$$\left.\frac{\partial H((\xi', \lambda'), \phi, \boldsymbol{c})}{\partial(\phi, c)}\right|_{((\xi',\lambda'),\phi_{(\xi',\lambda')},\boldsymbol{c}_{(\xi',\lambda')})} : \mathcal{H} \cap L^\infty_*(\mathbb{R}^N) \times \mathbb{R}^{N+1} \longrightarrow L^\infty_{**}(\mathbb{R}^N) \times \mathbb{R}^{N+1}$$

is invertible for small ε and α. Then the C^1 regularity $(\xi', \lambda') \mapsto \phi(\xi', \lambda')$ follows from the implicit function theorem. Indeed, we have

$$\left.\frac{\partial H((\xi', \lambda'), \phi, \boldsymbol{c})}{\partial(\phi, \boldsymbol{c})}\right|_{((\xi',\lambda'),\phi_{(\xi',\lambda')},\boldsymbol{c}_{(\xi',\lambda')})} [\varphi, \boldsymbol{d}] = \begin{pmatrix} P_{\tilde{g}}\varphi - \frac{N+4}{2}(\tilde{u}_0 + \phi_{(\xi',\lambda')})^{\frac{8}{N-4}}\varphi - \sum_{i=0}^{N} d_i \chi Z_i \\ \int_{\mathbb{R}^N} \chi Z_0 \varphi \\ \vdots \\ \int_{\mathbb{R}^N} \chi_N Z_N \varphi \end{pmatrix}.$$

Since $\|\phi_{(\xi',\lambda')}\|_* \leq C$, the same proof as that of Proposition 5.1 shows that this operator is invertible for ε and α small.

Since now $\phi = T[-R + N(\phi)]$, we have

$$\partial_{\xi'}\phi = \partial_{\xi'}T[-R + N(\phi)] + T[-\partial_{\xi'}R + \partial_{\xi'}N(\phi)].$$

This implies, by Proposition 5.1 and 5.4,

$$\|\partial_{\xi'}\phi\|_* \le C(\|R\|_{**} + \|N(\phi)\|_{**} + \|\partial_{\xi'}R\|_{**} + \|\partial_{\xi'}N(\phi)\|_{**}).$$

Direct calculation shows

$$\begin{aligned}\partial_{\xi'}N(\phi) = \frac{N+4}{2}&(\partial_{\xi'}\tilde{u}_0 + \partial_{\xi'}\phi)\Big[(\tilde{u}_0+\phi)^{\frac{8}{N-4}} - \tilde{u}_0^{\frac{8}{N-4}}\Big] \\ &- \frac{4(N+4)}{N-4}\tilde{u}_0^{-\frac{N-12}{N-4}}(\partial_{\xi'}\tilde{u}_0)\phi.\end{aligned}$$

Hence we can obtain

$$\|\partial_{\xi'}N(\phi)\|_{**} \le C\varepsilon^{4-\sigma}\|\phi\|_*(1 + \|\partial_{\xi'}\phi\|_*) \le C\varepsilon^{4-\sigma}(1 + \|\partial_{\xi'}\phi\|_*). \tag{38}$$

It is easy to check

$$\|\partial_{\xi'}R\|_{**} \le C. \tag{39}$$

Since we already have (36) and (37), we can conclude

$$\|\partial_{\xi'}\phi\|_* \le C.$$

The corresponding result for differentiation with respect to λ' can be gotten similarly. The proof is concluded. □

7 Variational reduction

As we have said, after Problem (35) has been solved, we find a solution to Problem (7) if (ξ', λ') is such that

$$c_i(\xi', \lambda') = 0 \qquad \text{for all } i. \tag{40}$$

This problem is indeed variational: it is equivalent to finding critical points of a function of (ξ', λ'). To see this, we define

$$\mathcal{F}_{\tilde{g}}(\xi', \lambda') = E_{\tilde{g}}[\tilde{u}_0(\xi', \lambda') + \phi(\xi', \lambda')]$$

where ϕ is the solution given by Proposition 6.1.

Lemma 7.1. *(ξ', λ') satisfies System (40) if and only if (ξ', λ') is a critical point of $\mathcal{F}_{\tilde{g}}$.*

Proof. Since

$$P_{\tilde{g}}(\tilde{u}_0 + \phi) - \frac{N-4}{2}(\tilde{u}_0 + \phi)^{\frac{N+4}{N-4}} = \sum_{i=0}^{N} c_i \chi Z_i,$$

we have

$$\partial_{(\xi',\lambda')}\mathcal{F}_{\tilde{g}}(\xi', \lambda') = \sum_{i=0}^{N} c_i \int_{\mathbb{R}^N} \chi Z_i[\partial_{(\xi',\lambda')}\tilde{u}_0 + \partial_{(\xi',\lambda')}\phi],$$

from which the necessity follows. In what follows we assume $\partial_{(\xi',\lambda')}\mathcal{F}_{\tilde{g}}(\xi', \lambda') = 0$. Then

$$\sum_{i=0}^{N} c_i \int_{\mathbb{R}^N} \chi Z_i \Big[\partial_{(\xi',\lambda')}\tilde{u}_0 + \partial_{(\xi',\lambda')}\phi\Big] = 0.$$

Using Proposition 6.2, we can directly check that

$$\partial_{\xi'_i}\tilde{u}_0 + \partial_{\xi'_i}\phi = Z_i + o(1),$$
$$\partial_{\lambda'}\tilde{u}_0 + \partial_{\lambda'}\phi = Z_0 + o(1).$$

Thus the above system for c_i is diagonal dominant, which gives $c_i = 0$ for all $i = 0, \ldots, N$. This concludes the proof. □

8 Energy Expansion

In this section we obtain an expansion of $\mathcal{F}_{\tilde{g}}$.

We first need to acquire a more refined estimate for ϕ. Let $w(y)$ satisfies

$$\begin{cases} P_{\tilde{g}} w - \dfrac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}} w = R_1 + \displaystyle\sum_i c_i \chi Z_i & \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} \chi Z_i w = 0, \end{cases} \tag{41}$$

where

$$\begin{aligned} R_1(y) = & -\mu\varepsilon^{10}\hat{\chi}(y)\Big[2\overline{H}_{ij}(\partial_{ijss}\tilde{u}_0) + 2(\partial_s\overline{H}_{ij})(\partial_{ijs}\tilde{u}_0) + (\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) \\ & - \frac{b_N}{2}(\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{jss}\overline{H}_{ij})(\partial_i\tilde{u}_0)\Big]. \end{aligned}$$

Here $\hat{\chi}$ is the cut-off function such that $\hat{\chi}(y) = 1$ for $|y| \le \frac{\rho}{\varepsilon}$ and $\hat{\chi}(y) = 0$ for $|y| \ge \frac{2\rho}{\varepsilon}$. We define

$$\|w\|'_* = \sup_{\mathbb{R}^N} \sum_{i=0}^{2} \frac{(1+|y-\xi'|)^{N-14+i}}{\mu\varepsilon^{10}} |\partial^i w(y)|,$$
$$\|R_1\|'_{**} = \sup_{\mathbb{R}^N} \frac{(1+|y-\xi'|)^{N-10}}{\mu\varepsilon^{10}} |R_1(y)|.$$

A similar proof as that of Proposition 5.1 shows that there exists a unique solution w to (41) such that

$$\|w\|'_* \le C\|R_1\|'_{**} \le C. \tag{42}$$

We introduce again that

$$\|\varphi_1\|''_* = \sup_{\mathbb{R}^N} \sum_{i=0}^{2} \left\{ \frac{1}{\frac{\mu^2\varepsilon^{20}}{1+(|y-\xi'|)^{N-24+i}} + \alpha(\frac{\varepsilon}{\rho})^{N-4+i}} + \frac{(1+|y-\xi'|)^{N-4+i}}{\alpha} \right\} |\partial^i \varphi_1(y)|,$$
$$\|\varphi_2\|''_{**} = \sup_{\mathbb{R}^N} \left\{ \frac{\chi_{\{|y|\le\frac{\rho}{\varepsilon}\}}(1+|y-\xi'|)^{N-20}}{\mu^2\varepsilon^{20}} + \frac{\chi_{\{\frac{\rho}{\varepsilon}\le|y|\le\frac{1}{\varepsilon}\}}(1+|y-\xi'|)^{N-1}}{\alpha\varepsilon} \right.$$
$$\left. + \frac{\chi_{\{|y|\ge\frac{1}{\varepsilon}\}}(1+|y-\xi'|)^{N+\sigma}}{\alpha} \right\} |\varphi_2(y)|.$$

Lemma 8.1. *The function $\phi - w$ satisfies the estimate*

$$\|\phi - w\|''_* \le C.$$

Proof. Obviously,

$$\begin{cases} P_{\tilde{g}}(\phi - w) - \dfrac{N+4}{2}\tilde{u}_0^{\frac{8}{N-4}}(\phi - w) = -R_2 + N(\phi) + \displaystyle\sum_{i=0}^{N} c_i\chi Z_i \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} \chi Z_i(\phi - w) = 0, \end{cases} \tag{43}$$

where

$$R_2(y) = \Delta_{\tilde{g}}^2\tilde{u}_0 - \Delta^2\tilde{u}_0 - \mu\varepsilon^{10}\hat{\chi}\sum_{i,j,s}\Big[2\overline{H}_{ij}(\partial_{ijss}\tilde{u}_0) + 2(\partial_s\overline{H}_{ij})(\partial_{ijs}\tilde{u}_0)$$
$$+ (\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{jss}\overline{H}_{ij})(\partial_i\tilde{u}_0)\Big].$$

It is easy to check

$$|R_2(y)| \le \begin{cases} C\dfrac{\mu^2\varepsilon^{20}}{(1+|y-\xi'|)^{N-20}} & \text{for } |y| \le \dfrac{\rho}{\varepsilon}, \\ C\dfrac{\alpha\varepsilon}{(1+|y-\xi'|)^{N-1}} & \text{for } \dfrac{\rho}{\varepsilon} \le |y| \le \dfrac{1}{\varepsilon}, \\ 0 & \text{for } |y| \ge \dfrac{1}{\varepsilon}. \end{cases}$$

On the other hand, since $|N(\phi)| \le C(1+|y-\xi'|)^{N-12}|\phi|^2$ and $\|\phi\|_* \le C$, we have

$$|N(\phi)| \le \begin{cases} C\dfrac{\mu^2\varepsilon^{20}}{(1+|y-\xi'|)^{N-16}} + \alpha(\dfrac{\varepsilon}{\rho})^{N+4} & \text{for } |y| \le \dfrac{\rho}{\varepsilon}, \\ C\dfrac{\alpha}{(1+|y-\xi'|)^{N+4}} & \text{for } |y| \ge \dfrac{\rho}{\varepsilon}. \end{cases}$$

Similar to the proof of Lemma 5.2, we obtain

$$\|\phi - w\|''_* \le C\|R_2\|''_{**} + C\|N(\phi)\|''_{**} \le C. \qquad \square$$

Proposition 8.2. *The following expansion holds*

$$\begin{aligned} 2\mathcal{F}_{\tilde g}(\xi',\lambda') = 2E &- \int_{B_{\frac{\varrho}{\varepsilon}}} \sum_{i,j,k,\ell} \tilde h_{i\ell}\tilde h_{j\ell}(\partial_{ikk}\tilde u_0)(\partial_j \tilde u_0) + \int_{B_{\frac{\varrho}{\varepsilon}}} \Big(\sum_{i,j} \tilde h_{ij}(\partial_{ij}\tilde u_0)\Big)^2 \\ &- \frac{a_N}{4}\int_{B_{\frac{\varrho}{\varepsilon}}} \sum_{i,k,\ell,m} (\partial_\ell \tilde h_{mk})^2(\partial_i \tilde u_0)^2 \\ &- \frac{b_N}{4}\int_{B_{\frac{\varrho}{\varepsilon}}} \sum_{i,j,m,s} (\partial_j \tilde h_{ms})(\partial_i \tilde h_{sm})(\partial_i \tilde u_0)(\partial_j \tilde u_0) \\ &- \frac{b_N}{2}\int_{B_{\frac{\varrho}{\varepsilon}}} \sum_{i,j,m,s} \Big[\tilde h_{ms}(\partial_s \tilde h_{ij}) - \tilde h_{si}(\partial_s \tilde h_{mj}) + \tilde h_{sj}(\partial_i \tilde h_{ms}) \\ &\qquad\qquad - \tilde h_{ms}(\partial_i \tilde h_{sj})\Big]\partial_m(\partial_i \tilde u_0 \partial_j \tilde u_0) \\ &+ \frac{b_N}{2}\int_{B_{\frac{\varrho}{\varepsilon}}} \sum_{i,j,m,s} \tilde h_{is}(\partial_{mm}\tilde h_{js})(\partial_i \tilde u_0)(\partial_j \tilde u_0) \\ &+ \frac{N-4}{8(N-1)}\int_{B_{\frac{\varrho}{\varepsilon}}} \sum_{i,k,\ell,m} \Big[(\partial_{i\ell} h_{mk})^2 + (\partial_\ell h_{mk})(\partial_{ii\ell} h_{mk})\Big]\tilde u_0^2 \\ &- \frac{N-4}{4(N-2)^2}\int_{B_{\frac{\varrho}{\varepsilon}}} \sum_{i,j,m,s} (\partial_{mm}\tilde h_{ij})(\partial_{ss}\tilde h_{ij}) u_0^2 \end{aligned}$$

$$
\begin{aligned}
&+\int_{B_{\frac{\rho}{\varepsilon}}}\sum_{i,j,s}\Big[2\tilde{h}_{ij}(\partial_{ijss}\tilde{u}_0)+2(\partial_s\tilde{h}_{ij})(\partial_{ijs}\tilde{u}_0)+(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0)\\
&\qquad-\frac{b_N}{2}(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0)-\frac{b_N}{2}(\partial_{jss}\tilde{h}_{ij})(\partial_i\tilde{u}_0)\Big]w\\
&+O\left(\mu^3\varepsilon^{\frac{20N}{N-1}}\right)+O\left(\mu^{\frac{2N}{N-4}}\varepsilon^{\frac{20N}{N-4}}\right)+O\left(\alpha\left(\frac{\varepsilon}{\rho}\right)^{N-4}\right).
\end{aligned}
$$

Proof. Since ϕ is a solution to (35),

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Delta_{\tilde{g}}(\tilde{u}_0+\phi)\Delta_{\tilde{g}}\phi+\sum_{i,j}(a_N S_{\tilde{g}}g^{ij}+b_N\widetilde{\mathcal{R}}^{ij})\partial_i(\tilde{u}_0+\phi)\partial_j\phi\\
&\qquad+\frac{N-4}{2}Q_{\tilde{g}}(\tilde{u}_0+\phi)\phi\\
&=\frac{N-4}{2}\int_{\mathbb{R}^N}(\tilde{u}_0+\phi)^{\frac{N+4}{N-4}}\phi.
\end{aligned}
\tag{44}
$$

On the other hand

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\Delta_{\tilde{g}}\tilde{u}_0\Delta_{\tilde{g}}\phi+\sum_{i,j}(a_N S_{\tilde{g}}g^{ij}+b_N\widetilde{\mathcal{R}}^{ij})\partial_i\tilde{u}_0\partial_j\phi+\frac{N-4}{2}Q_{\tilde{g}}\tilde{u}_0\phi\\
=&\int_{\mathbb{R}^N}\Big\{P_{\tilde{g}}\tilde{u}_0-\Delta^2\tilde{u}_0-\sum_{i,j,s}\Big[2h_{ij}(\partial_{ijss}\tilde{u}_0)+2(\partial_s\tilde{h}_{ij})(\partial_{ijs}\tilde{u}_0)+(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0)\\
&\qquad-\frac{b_N}{2}(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0)-\frac{b_N}{2}(\partial_{jss}\tilde{h}_{ij})(\partial_i\tilde{u}_0)\Big]\Big\}\phi\\
&+\int_{\mathbb{R}^N}\sum_{i,j,s}\Big[2\tilde{h}_{ij}(\partial_{ijss}\tilde{u}_0)+2(\partial_s\tilde{h}_{ij})(\partial_{ijs}\tilde{u}_0)+(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0)\\
&\qquad-\frac{b_N}{2}(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0)-\frac{b_N}{2}(\partial_{jss}\tilde{h}_{ij})(\partial_i\tilde{u}_0)\Big](\phi-w)\\
&+\int_{\mathbb{R}^N}\sum_{i,j,s}\Big[2\tilde{h}_{ij}(\partial_{ijss}\tilde{u}_0)+2(\partial_s\tilde{h}_{ij})(\partial_{ijs}\tilde{u}_0)+(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0)\\
&\qquad-\frac{b_N}{2}(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0)-\frac{b_N}{2}(\partial_{jss}\tilde{h}_{ij})(\partial_i\tilde{u}_0)\Big]w\\
&+\frac{N-4}{2}\int_{\mathbb{R}^N}\tilde{u}_0^{\frac{N+4}{N-4}}\phi\\
:=&\,I_1+I_2+I_3+\frac{N-4}{2}\int_{\mathbb{R}^N}\tilde{u}_0^{\frac{N+4}{N-4}}\phi.
\end{aligned}
\tag{45}
$$

Since

$$\Big| P_{\tilde g}\tilde u_0 - \Delta^2\tilde u_0 - \sum_{i,j,s}\Big[2\tilde h_{ij}(\partial_{ijss}\tilde u_0) + 2(\partial_s\tilde h_{ij})(\partial_{ijs}\tilde u_0) + (\partial_{ss}\tilde h_{ij})(\partial_{ij}\tilde u_0)$$
$$- \frac{b_N}{2}(\partial_{ss}\tilde h_{ij})(\partial_{ij}\tilde u_0) - \frac{b_N}{2}(\partial_{jss}\tilde h_{ij})(\partial_i\tilde u_0)\Big]\Big|$$
$$\le \begin{cases} C\dfrac{\mu^2\varepsilon^{20}}{(1+|y-\xi'|)^{N-20}} & \text{for } |y|\le\dfrac{\rho}{\varepsilon},\\ C\dfrac{\alpha\varepsilon}{(1+|y-\xi'|)^{N-1}} & \text{for } |y|\ge\dfrac{\rho}{\varepsilon},\end{cases}$$

we have

$$|I_1| \le C\mu^3\varepsilon^{21}\rho^9|\log\varepsilon| + C\alpha^2\rho(\frac{\varepsilon}{\rho})^{N-4}. \tag{46}$$

Since

$$\Big|\sum_{i,j,s}\Big[2\tilde h_{ij}(\partial_{ijss}\tilde u_0) + 2(\partial_s\tilde h_{ij})(\partial_{ijs}\tilde u_0) + (\partial_{ss}\tilde h_{ij})(\partial_{ij}\tilde u_0)$$
$$- \frac{b_N}{2}(\partial_{ss}\tilde h_{ij})(\partial_{ij}\tilde u_0) - \frac{b_N}{2}(\partial_{jss}\tilde h_{ij})(\partial_i\tilde u_0)\Big]\Big|$$
$$\le \begin{cases} C\dfrac{\mu\varepsilon^{10}}{(1+|y-\xi'|)^{N-10}} & \text{for } |y|\le\dfrac{\rho}{\varepsilon},\\ C\dfrac{\alpha\varepsilon}{(1+|y-\xi'|)^{N-1}} & \text{for } |y|\ge\dfrac{\rho}{\varepsilon}\end{cases}$$

and $\|\phi - w\|''_* \le C$, we obtain that

$$|I_2| \le C\mu^3\varepsilon^{21}\rho^9|\log\varepsilon| + C\alpha^2\rho(\frac{\varepsilon}{\rho})^{N-4}. \tag{47}$$

Since $\|w\|'_* \le C$ and $h = 0$ for $|y| \ge \frac{1}{\varepsilon}$.

$$I_3 = \int_{|y|\le\frac{\rho}{\varepsilon}}\sum_{i,j,s}\Big[2\tilde h_{ij}(\partial_{ijss}\tilde u_0) + 2(\partial_s\tilde h_{ij})(\partial_{ijs}\tilde u_0) + (\partial_{ss}\tilde h_{ij})(\partial_{ij}\tilde u_0)$$
$$- \frac{b_N}{2}(\partial_{ss}\tilde h_{ij})(\partial_{ij}\tilde u_0) - \frac{b_N}{2}(\partial_{jss}\tilde h_{ij})(\partial_i\tilde u_0)\Big]w$$
$$+ O\left(\alpha\mu\rho^{10}(\frac{\varepsilon}{\rho})^{N-4}\right). \tag{48}$$

Thus, combining (44)-(48), we have

$$2\mathcal{F}_{\tilde g}(\xi',\lambda') - 2E_{\tilde g}(\tilde u_0)$$

$$
\begin{aligned}
&= \int_{|y|\le\frac{\rho}{\varepsilon}} \sum_{i,j,s} \Big[2\tilde{h}_{ij}(\partial_{ijss}\tilde{u}_0) + 2(\partial_s\tilde{h}_{ij})(\partial_{ijs}\tilde{u}_0) + (\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0) \\
&\qquad - \frac{b_N}{2}(\partial_{ss}\tilde{h}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{jss}\tilde{h}_{ij})(\partial_i\tilde{u}_0)\Big]w \\
&\quad + \frac{N-4}{2}\int_{\mathbb{R}^N}\left\{\frac{4}{N}\left[(\tilde{u}_0+\phi)^{\frac{2N}{N-4}} - \tilde{u}_0^{\frac{2N}{N-4}}\right]\right. \\
&\qquad \left. - \left[(\tilde{u}_0+\phi)^{\frac{N+4}{N-4}}\tilde{u}_0 - \tilde{u}_0^{\frac{N+4}{N-4}}(\tilde{u}_0+\phi)\right]\right\} \\
&\quad + O\left(\mu^3\varepsilon^{21}\rho^9|\log\varepsilon|\right) + O\left(\alpha^2\rho(\frac{\varepsilon}{\rho})^{N-4}\right). \qquad (49)
\end{aligned}
$$

Using the fact that $\|\phi\|_* \le C$, we have the pointwise estimate

$$
\left|\frac{4}{N}\left[(\tilde{u}_0+\phi)^{\frac{2N}{N-4}} - \tilde{u}_0^{\frac{2N}{N-4}}\right] - \left[(\tilde{u}_0+\phi)^{\frac{N+4}{N-4}}\tilde{u}_0 - \tilde{u}_0^{\frac{N+4}{N-4}}(\tilde{u}_0+\phi)\right]\right| \le C|\phi|^{\frac{2N}{N-4}},
$$

which implies

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left\{\frac{4}{N}\left[(\tilde{u}_0+\phi)^{\frac{2N}{N-4}} - \tilde{u}_0^{\frac{2N}{N-4}}\right] - \left[(\tilde{u}_0+\phi)^{\frac{N+4}{N-4}}\tilde{u}_0 - \tilde{u}_0^{\frac{N+4}{N-4}}(\tilde{u}_0+\phi)\right]\right\} \\
&= O(\mu^{\frac{2N}{N-4}}\varepsilon^{\frac{20N}{N-4}}) + O\left(\alpha(\frac{\varepsilon}{\rho})^N\right).
\end{aligned}
$$

Proposition 4.16 then gives the result. □

9 Finding a Critical Point for Reduced Energy Functional

We define a function F: $\mathbb{R}^N \times (0,\infty) \to \mathbb{R}$ as follows: given an pair $(\xi',\lambda') \in \mathbb{R}^N \times (0,\infty)$,

$$
\begin{aligned}
F(\xi',\lambda') = &-\int_{\mathbb{R}^N}\sum_{i,j,k,\ell}\overline{H}_{i\ell}\overline{H}_{j\ell}(\partial_{ikk}\tilde{u}_0)(\partial_j\tilde{u}_0) + \int_{\mathbb{R}^N}\left(\sum_{i,j}\overline{H}_{ij}(\partial_{ij}\tilde{u}_0)\right)^2 \\
&- \frac{a_N}{4}\int_{\mathbb{R}^N}\sum_{i,k,\ell,m}(\partial_\ell\overline{H}_{mk})^2(\partial_i\tilde{u}_0)^2 \\
&- \frac{b_N}{4}\int_{\mathbb{R}^N}\sum_{i,j,m,s}(\partial_j\overline{H}_{ms})(\partial_i\overline{H}_{sm})(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{b_N}{2} \int_{\mathbb{R}^N} \sum_{i,j,m,s} \Big[\overline{H}_{ms}(\partial_s \overline{H}_{ij}) - \overline{H}_{si}(\partial_s \overline{H}_{mj}) + \overline{H}_{sj}(\partial_i \overline{H}_{ms}) \\
&\qquad\qquad - \overline{H}_{ms}(\partial_i \overline{H}_{sj}) \Big] \partial_m(\partial_i \tilde{u}_0 \partial_j \tilde{u}_0) \\
&+ \frac{b_N}{2} \int_{\mathbb{R}^N} \sum_{i,j,m,s} \overline{H}_{is}(\partial_{mm} \overline{H}_{js})(\partial_i \tilde{u}_0)(\partial_j \tilde{u}_0) \\
&+ \frac{N-4}{8(N-1)} \int_{\mathbb{R}^N} \sum_{i,k,\ell,m} \Big[(\partial_{i\ell} \overline{H}_{mk})^2 + (\partial_\ell \overline{H}_{mk})(\partial_{ii\ell} \overline{H}_{mk}) \Big] \tilde{u}_0^2 \\
&- \frac{N-4}{4(N-2)^2} \int_{\mathbb{R}^N} \sum_{i,j,m,s} (\partial_{mm} \overline{H}_{ij})(\partial_{ss} \overline{H}_{ij}) u_0^2 \\
&+ \int_{\mathbb{R}^N} \sum_{i,j,s} \Big[2\overline{H}_{ij}(\partial_{ijss} \tilde{u}_0) + 2(\partial_s \overline{H}_{ij})(\partial_{ijs} \tilde{u}_0) + (\partial_{ss} \overline{H}_{ij})(\partial_{ij} \tilde{u}_0) \\
&\qquad - \frac{b_N}{2}(\partial_{ss} \overline{H}_{ij})(\partial_{ij} \tilde{u}_0) - \frac{b_N}{2}(\partial_{jss} \overline{H}_{ij})(\partial_i \tilde{u}_0) \Big] \bar{w}
\end{aligned}
$$

where $\bar{w}(y) = w(y)/\mu\varepsilon^{10}$ and $w(y)$ is defined in (41).

Proposition 4.16 shows that the reduced energy functional $\mathcal{F}_{\tilde{g}}$ is close to F.

Our goal in this section is to show that the function $F(\xi', \lambda')$ has a strict (nondegenerate) local minimum point. First we have the following symmetry result.

Proposition 9.1. *The function $F(\xi', \lambda')$ satisfies $F(\xi', \lambda') = F(-\xi', \lambda')$ for all $(\xi', \lambda') \in \mathbb{R}^N \times (0, \infty)$. Consequently, we have $\frac{\partial}{\partial \xi'} F(0, \lambda') = 0$ and $\frac{\partial^2}{\partial \xi' \partial \lambda'} F(0, \lambda') = 0$ for all $\lambda' > 0$.*

Proof. This follows immediately from the relation $\overline{H}_{ik}(-y) = \overline{H}_{ik}(y)$. □

To find a local minimum for F, we list some useful identities, which are all direct consequences of definitions.

Lemma 9.2.

$$
\begin{aligned}
\partial_i \tilde{u}_0 &= -(N-4)\tilde{u}_0(y) \frac{y_i - \xi'_i}{\lambda'^2 + |y - \xi'|^2}, \\
\partial_{ij} \tilde{u}_0 &= (N-2)(N-4)\tilde{u}_0(y) \frac{(y_i - \xi'_i)(y_j - \xi'_j)}{(\lambda'^2 + |y - \xi'|^2)^2} - (N-4)\tilde{u}_0(y) \frac{\delta_{ij}}{\lambda'^2 + |y - \xi'|^2}, \\
\partial_{ijs} \tilde{u}_0 &= -N(N-2)(N-4)\tilde{u}_0(y) \frac{(y_i - \xi'_i)(y_j - \xi'_j)(y_s - \xi'_s)}{(\lambda'^2 + |y - \xi'|^2)^3} \\
&\quad + (N-2)(N-4)\tilde{u}_0(y) \frac{\delta_{js}(y_i - \xi'_i) + \delta_{is}(y_j - \xi'_j) + \delta_{ij}(y_s - \xi'_s)}{(\lambda'^2 + |y - \xi'|^2)^2}.
\end{aligned}
$$

Lemma 9.3.

$$\begin{aligned}
\partial_\ell \overline{H}_{mk} &= 2f'(|y|^2)H_{mk}y_\ell + f(|y|^2)(\partial_\ell H_{mk}),\\
\partial_{i\ell} \overline{H}_{mk} &= 2\delta_{i\ell} f'(|y|^2)H_{mk} + 2f'(|y|^2)(\partial_i H_{mk})y_\ell + 2f'(|y|^2)(\partial_\ell H_{mk})y_i\\
&\qquad + f(|y|^2)(\partial_{i\ell}H_{mk}) + 4f''(|y|^2)H_{mk}y_iy_\ell.
\end{aligned}$$

Lemma 9.4.

$$\sum_i y_i\partial_i H_{ms} = 2H_{ms}, \qquad \sum_j y_j \partial_s H_{tj} = -H_{st},$$

$$\sum_j y_j\partial_{ms} H_{ij} = -\partial_s H_{im} - \partial_m H_{is}, \qquad \sum_\ell y_\ell(\partial_{i\ell}H_{mk}) = \partial_i H_{mk}.$$

Lemma 9.5.

$$\begin{aligned}
\int_{\partial B_1} \sum_{i,j} H_{ij}^2 &= \frac{|S^{N-1}|}{2N(N+2)} \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2,\\
\int_{\partial B_1} \sum_{i,j,k} (\partial_k H_{ij})^2 &= \frac{|S^{N-1}|}{N} \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2,\\
\int_{\partial B_1} \sum_{i,j,k,\ell} (\partial_{k\ell} H_{ij})^2 &= |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2.
\end{aligned}$$

Lemma 9.6.

$$\begin{aligned}
\int_{\partial B_1} \sum_{i,j} H_{ij}^2 y_p y_q &= \frac{2|S^{N-1}|}{N(N+2)(N+4)} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk})\\
&\qquad + \frac{|S^{N-1}|}{2N(N+2)(N+4)} (W_{ikj\ell} + W_{i\ell jk})^2 \delta_{pq},\\
\int_{\partial B_1} \sum_t H_{pt}H_{qt} &= \frac{|S^{N-1}|}{2N(N+2)} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}),\\
\int_{\partial B_1} \sum_{i,j,k} (\partial_k H_{ij})^2 y_p y_q &= \frac{2|S^{N-1}|}{N(N+2)} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk})\\
&\qquad + \frac{|S^{N-1}|}{N(N+2)} (W_{ikj\ell} + W_{i\ell jk})^2 \delta_{pq},\\
\int_{\partial B_1} \sum_{i,j} (\partial_p H_{ij})(\partial_q H_{ij}) &= \frac{|S^{N-1}|}{N} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}),\\
\int_{\partial B_1} \sum_{i,j} H_{ij}(\partial_q H_{ij}) y_p &= \frac{|S^{N-1}|}{N(N+2)} \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}),
\end{aligned}$$

$$\int_{\partial B_1} \sum_{i,j,k,\ell} (\partial_{k\ell} H_{ij})^2 y_p y_q = \frac{|S^{N-1}|}{N} \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \delta_{pq}.$$

We mention that the proof of some identities in Lemma 9.5 and 9.6 can be found in [3, 4]. The others may be proved by the same method.

Lemma 9.7. *It holds*

$$\int_{\mathbb{R}^N} \left\{ \sum_{i,j,s,t} \overline{H}_{tj} \overline{H}_{it} (\partial_{iss} \tilde{u}_0)(\partial_j \tilde{u}_0) \right\} \Bigg|_{\xi'=0} = 0,$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,s,t} \overline{H}_{tj} \overline{H}_{it} (\partial_{iss} \tilde{u}_0)(\partial_j \tilde{u}_0) \right\} \Bigg|_{\xi=0} \\ &= |S^{N-1}| \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda'^{N-4} \\ &\quad \cdot \frac{(N-2)(N-4)^2}{N(N+2)} \int_0^\infty f(r^2)^2 \left[\frac{N r^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{(N+2) r^{N+3}}{(\lambda'^2 + r^2)^{N-1}} \right] \mathrm{d}r. \end{aligned}$$

Proof. Since

$$\begin{aligned} &\sum_{i,j,s,t} \overline{H}_{tj} \overline{H}_{it} (\partial_{iss} \tilde{u}_0)(\partial_j \tilde{u}_0) \\ &= f(|y|^2)^2 H_{tj} H_{it} (N-2)(N-4)^2 \tilde{u}_0^2 \left[\frac{N|y-\xi'|^2}{(\lambda'^2 + |y-\xi'|^2)^4} - \frac{(n+2)}{(\lambda'^2 + |y-\xi'|^2)^3} \right] \\ &\qquad \cdot (y_i - \xi'_i)(y_j - \xi'_j) \\ &= N(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4} |y-\xi'|^2}{(\lambda'^2 + |y-\xi'|^2)^N} \sum_{i,j,t} H_{it} H_{jt} \xi'_i \xi'_j \\ &\quad - (N-2)(N+2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y'-\xi'|^2)^{N-1}} \sum_{i,j,t} H_{it} H_{jt} \xi'_i \xi'_j \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,s,t} \overline{H}_{tj} \overline{H}_{it} (\partial_{iss} \tilde{u}_0)(\partial_j \tilde{u}_0) \right\} \Bigg|_{\xi'=0} \\ &= 2N(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4} |y|^2}{(\lambda'^2 + |y|^2)^N} \sum_t H_{pt} H_{qt} \end{aligned}$$

$$- 2(N-2)(N+2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-1}} \sum_t H_{pt} H_{qt},$$

the lemma follows directly by Lemma 9.5 and 9.6. □

Lemma 9.8. *We have*

$$\int_{\mathbb{R}^N} \left\{ \sum_{i,j,s,t} \overline{H}_{ij} \overline{H}_{st} \partial_{ij} \tilde{u}_0 \partial_{st} \tilde{u}_0 \right\} \Bigg|_{\xi'=0} = 0,$$

$$\int_{\mathbb{R}^n} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,s,t} \overline{H}_{ij} \overline{H}_{st} \partial_{ij} \tilde{u}_0 \partial_{st} \tilde{u}_0 \right\} \Bigg|_{\xi'=0} = 0.$$

Proof. The lemma easily follows from

$$\sum_{i,j,s,t} \overline{H}_{ij} \overline{H}_{st} \partial_{ij} \tilde{u}_0 \partial_{st} \tilde{u}_0$$
$$= (N-2)^2 (N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^N} \sum_{i,j,s,t} H_{ij} H_{st} \xi'_i \xi'_j \xi'_s \xi'_t.$$ □

Lemma 9.9. *There hold*

$$\int_{\mathbb{R}^N} \left\{ \sum_{i,\ell,k,m} (\partial_\ell \overline{H}_{mk})^2 (\partial_i \tilde{u}_0)^2 \right\} \Bigg|_{\xi=0}$$
$$= |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4}$$
$$\cdot \left\{ \frac{2(N-4)^2}{N(N+2)} \int_0^\infty \left[r^2 f'(r^2)^2 + 2 f(r^2) f'(r^2) \right] \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-2}} \mathrm{d}r \right.$$
$$\left. + \frac{(N-4)^2}{N} \int_0^\infty f(r^2)^2 \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-2}} \mathrm{d}r \right\},$$

and

$$\int_{\mathbb{R}^N} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,\ell,k,m} (\partial_\ell \overline{H}_{mk})^2 (\partial_i \tilde{u}_0)^2 \right\} \Bigg|_{\xi'=0}$$
$$= |S^{N-1}| \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda'^{N-4}$$
$$\cdot \left\{ \frac{32(N-2)(N-4)^2}{N(N+2)(N+4)} \int_0^\infty \left[r^2 f'(r^2)^2 + 2 f(r^2) f'(r^2) \right] \right.$$

$$
\cdot \left[\frac{(N-1)r^{N+7}}{(\lambda'^2+r^2)^N} - \frac{2r^{N+5}}{(\lambda'^2+r^2)^{N-1}} \right] \mathrm{d}r
$$

$$
+ \frac{8(N-2)(N-4)^2}{N(N+2)} \int_0^\infty f(r^2)^2 \left[\frac{(N-1)r^{N+5}}{(\lambda'^2+r^2)^N} - \frac{2r^{N+3}}{(\lambda'^2+r^2)^{N-1}} \right] \mathrm{d}r \Bigg\}
$$

$$
+ |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4} \delta_{pq}
$$

$$
\cdot \Bigg\{ \frac{4(N-4)^2}{N(N+2)(N+4)} \int_0^\infty \left[r^2 f'(r^2)^2 + 2f(r^2)f'(r^2) \right]
$$

$$
\cdot \left[\frac{2(N-1)(N-2)r^{N+7}}{(\lambda'^2+r^2)^n} - \frac{(N-2)(N+8)r^{N+5}}{(\lambda'^2+r^2)^{N-1}} + \frac{(N+4)r^{N+3}}{(\lambda'^2+r^2)^{N-2}} \right] \mathrm{d}r
$$

$$
+ \frac{2(N-4)^2}{N(N+2)} \int_0^\infty f(r^2)^2
$$

$$
\cdot \left[\frac{2(N-1)(N-2)r^{N+5}}{(\lambda'^2+r^2)^N} - \frac{(N-2)(N+6)r^{N+3}}{(\lambda'^2+r^2)^{N-1}} + \frac{(N+2)r^{N+1}}{(\lambda'^2+r^2)^{N-2}} \right] \mathrm{d}r \Bigg\}.
$$

Proof. Direct computation shows

$$
\sum_{i,\ell,k,m} (\partial_\ell \overline{H}_{mk})^2 (\partial_i \tilde{u}_0)^2
$$

$$
= 4(N-4)^2 \Big[|y|^2 f'(|y|^2)^2 + 2f(|y|^2)f'(|y|^2) \Big] \lambda'^{N-4} \frac{|y-\xi'|^2}{(\lambda'^2+|y-\xi'|^2)^{N-2}} \sum_{m,k} H_{mk}^2
$$

$$
+ (N-4)^2 f(|y|^2)^2 \lambda'^{N-4} \frac{|y-\xi'|^2}{(\lambda'^2+|y-\xi'|^2)^{N-2}} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2,
$$

$$
\frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,\ell,k,m} (\partial_\ell \overline{H}_{mk})^2 (\partial_i \tilde{u}_0)^2 \right\} \Bigg|_{\xi'=0}
$$

$$
= 16(N-1)(N-2)(N-4)^2 \Big[|y|^2 f'(|y|^2)^2 + 2f(|y|^2)f'(|y|^2) \Big] \frac{|y|^2 \lambda'^{N-4}}{(\lambda'^2+|y|^2)^N} \sum_{k,m} H_{km}^2 y_p y_q
$$

$$
- 32(N-2)(N-4)^2 \Big[|y|^2 f'(|y|^2)^2 + 2f(|y|^2)f'(|y|^2) \Big] \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-1}} \sum_{k,m} H_{km}^2 y_p y_q
$$

$$
- 8(N-2)(N-4)^2 \Big[|y|^2 f'(|y|^2)^2 + 2f(|y|^2)f'(|y|^2) \Big] \frac{|y|^2 \lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-1}} \sum_{k,m} H_{km}^2 \delta_{pq}
$$

$$
+ 8(N-4)^2 \Big[|y|^2 f'(|y|^2)^2 + 2f(|y|^2)f'(|y|^2) \Big] \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-2}} \sum_{k,m} H_{km}^2 \delta_{pq}
$$

$$+ 4(N-1)(N-2)(N-4)^2 f(|y|^2)^2 \frac{|y|^2 \lambda'^{N-4}}{(\lambda'^2+|y|^2)^N} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2 y_p y_q$$

$$- 8(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-1}} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2 y_p y_q$$

$$- 2(N-2)(N-4)^2 f(|y|^2)^2 \frac{|y|^2 \lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-1}} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2 \delta_{pq}$$

$$+ 2(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-2}} \sum_{k,\ell,m} (\partial_\ell H_{mk})^2 \delta_{pq}.$$

Using Lemma 9.5 and 9.6, we finish the proof. □

Lemma 9.10. *We have*

$$\int_{\mathbb{R}^N} \left\{ \sum_{i,j,m,s} (\partial_i \overline{H}_{ms})(\partial_j \overline{H}_{ms}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\} \Bigg|_{\xi'=0}$$
$$= |S^{N-1}| (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4}$$
$$\cdot \frac{2(N-4)^2}{N(N+2)} \int_0^\infty \left[r^2 f'(r^2) + f(r^2) \right]^2 \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-2}} \mathrm{d}r,$$

$$\int_{\mathbb{R}^n} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,m,s} (\partial_i \overline{H}_{ms})(\partial_j \overline{H}_{ms}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\} \Bigg|_{\xi'=0}$$
$$= |S^{N-1}| \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda'^{N-4}$$
$$\cdot \left\{ \frac{32(N-1)(N-2)(N-4)^2}{N(N+2)(N+4)} \int_0^\infty \left[r^2 f'(r^2) + f(r^2) \right]^2 \frac{r^{N+5}}{(\lambda'^2+r^2)^N} \mathrm{d}r \right.$$
$$- \frac{64(N-2)(N-4)^2}{N(N+2)(N+4)} \int_0^\infty \left[r^2 f'(r^2)^2 + f(r^2) f'(r^2) \right] \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-1}} \mathrm{d}r$$
$$- \frac{16(N-2)(N-4)^2}{N(N+2)} \int_0^\infty \left[r^2 f(r^2) f'(r^2) + f(r^2)^2 \right] \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-1}} \mathrm{d}r$$
$$+ \frac{16(N-4)^2}{N(N+2)(N+4)} \int_0^\infty f'(r^2)^2 \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-2}} \mathrm{d}r$$
$$+ \frac{2(N-4)^2}{N} \int_0^\infty f(r^2)^2 \frac{r^{N+1}}{(\lambda'^2+r^2)^{N-2}} \mathrm{d}r$$
$$\left. + \frac{8(N-4)^2}{N(N+2)} \int_0^\infty f(r^2) f'(r^2) \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-2}} \mathrm{d}r \right\}$$

$$
\begin{aligned}
&+ |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4} \delta_{pq} \\
&\cdot \left\{ \frac{4(N-2)(N-4)^2}{N(N+2)} \int_0^\infty \left[r^2 f'(r^2) + f(r^2) \right]^2 \right. \\
&\qquad\qquad \cdot \left[\frac{2(N-1)}{N+4} \frac{r^{N+5}}{(\lambda'^2 + r^2)^N} - \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-1}} \right] \mathrm{d}r \\
&\quad - \frac{16(N-2)(N-4)^2}{N(N+2)(N+4)} \int_0^\infty \left[r^2 f'(r^2)^2 + f(r^2) f'(r^2) \right] \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-1}} \mathrm{d}r \\
&\quad \left. + \frac{4(N-4)^2}{N(N+2)(N+4)} \int_0^\infty f'(r^2)^2 \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-2}} \mathrm{d}r \right\}.
\end{aligned}
$$

Proof. Direct calculation shows

$$
\begin{aligned}
&\sum_{i,j,m,s} (\partial_i \overline{H}_{ms})(\partial_j \overline{H}_{ms}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \\
= &\sum_{i,j,m,s} [2f'(|y|^2) y_i H_{ms} + f(|y|^2)\partial_i H_{ms}][2f'(|y|^2) y_j H_{ms} + f(|y|^2) \partial_j H_{ms}] \\
&\qquad \cdot (N-4)^2 \tilde{u}_0^2(y) \frac{(y_i - \xi'_i)(y_j - \xi'_j)}{(\lambda'^2 + |y-\xi'|^2)^2} \\
= &\, 4(N-4)^2 \Big[|y|^2 f'(|y|^2) + f(|y|^2) \Big]^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^{N-2}} \sum_{m,s} H_{ms}^2 \\
&- 8(N-4)^2 \Big[|y|^2 f'(|y|^2)^2 + f(|y|^2) f'(|y|^2) \Big] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^{N-2}} \sum_{j,m,s} H_{ms}^2 y_j \xi'_j \\
&- 4(N-4)^2 \Big[|y|^2 f(|y|^2) f'(|y|^2) + f(|y|^2)^2 \Big] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^{N-2}} \sum_{j,m,s} H_{ms} (\partial_j H_{ms}) \xi'_j \\
&+ 4(N-4)^2 f'(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^{N-2}} \sum_{i,j,m,s} H_{ms}^2 y_i y_j \xi'_i \xi'_j \\
&+ (N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^{N-2}} \sum_{i,j,m,s} (\partial_i H_{ms})(\partial_j H_{ms}) \xi'_i \xi'_j \\
&+ 4(N-4)^2 f(|y|^2) f'(|y|^2) \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^{N-2}} \sum_{i,j,m,s} H_{ms} (\partial_j H_{ms}) y_i \xi'_i \xi'_j,
\end{aligned}
$$

$$
\frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,m,s} (\partial_i \overline{H}_{ms})(\partial_j \overline{H}_{ms}) \partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\} \Bigg|_{\xi=0}
$$

$$
\begin{aligned}
&= 16(N-1)(N-2)(N-4)^2\Big[|y|^2 f'(|y|^2) + f(|y|^2)\Big]^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^N} \sum_{m,s} H_{ms}^2 y_p y_q \\
&\quad - 8(N-2)(N-4)^2\Big[|y|^2 f'(|y|^2) + f(|y|^2)\Big]^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-1}} \sum_{m,s} H_{ms}^2 \delta_{pq} \\
&\quad - 32(N-2)(N-4)^2\Big[|y|^2 f'(|y|^2)^2 + f(|y|^2) f'(|y|^2)\Big] \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-1}} \sum_{m,s} H_{ms}^2 y_p y_q \\
&\quad - 8(N-2)(N-4)^2\Big[|y|^2 f(|y|^2) f'(|y|^2) + f(|y|^2)^2\Big] \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-1}} \\
&\qquad \cdot \sum_{m,s} \Big[H_{ms}(\partial_q H_{ms}) y_p + H_{ms}(\partial_p H_{ms}) y_q\Big] \\
&\quad + 8(N-4)^2 f'(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-2}} \sum_{m,s} H_{ms}^2 y_p y_q \\
&\quad + 2(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-2}} \sum_{i,j,m,s} (\partial_p H_{ms})(\partial_q H_{ms}) \\
&\quad + 4(N-4)^2 f(|y|^2) f'(|y|^2) \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-2}} \sum_{m,s} \left[H_{ms}(\partial_q H_{ms}) y_p + H_{ms}(\partial_p H_{ms}) y_q\right].
\end{aligned}
$$

Lemma 9.5 and 9.6 then yield the result. □

Lemma 9.11. *It holds*

$$
\begin{aligned}
\sum_{i,j,m,s} \Big[&\overline{H}_{ms}(\partial_s \overline{H}_{ij}) - \overline{H}_{si}(\partial_s \overline{H}_{mj}) + \overline{H}_{sj}(\partial_i \overline{H}_{ms}) \\
&- \overline{H}_{ms}(\partial_i \overline{H}_{sj})\Big] \partial_m(\partial_i \tilde{u}_0 \partial_j \tilde{u}_0) = 0.
\end{aligned}
$$

Proof. Direct computation shows

$$
\begin{aligned}
&\sum_{i,j,m,s} \overline{H}_{ms}(\partial_s \overline{H}_{ij}) \partial_m[\partial_i \tilde{u}_0 \partial_j \tilde{u}_0] \\
&= 2 \sum_{i,j,m,s} \overline{H}_{ms}(\partial_s \overline{H}_{ij}) \partial_{im} \tilde{u}_0 \partial_j \tilde{u}_0 \\
&= -2(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-2}} \sum_{i,j} H_{ij}^2 \\
&\quad - 2(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-2}} \sum_{i,j,m} H_{im}(\partial_m H_{ij}) \xi'_j
\end{aligned}
$$

$$+ 4(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-1}} \sum_{i,j,m} H_{im}H_{jm}\xi'_i\xi'_j$$
$$+ 2(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-1}} \sum_{i,j,m,s} H_{ms}(\partial_s H_{ij})\xi'_i\xi'_j\xi'_m$$

and

$$\begin{aligned}
&\sum_{i,j,m,s} \overline{H}_{is}(\partial_s \overline{H}_{jm})\partial_m[\partial_i\tilde{u}_0\partial_j\tilde{u}_0] \\
= &\sum_{i,j,m,s} \overline{H}_{is}(\partial_s \overline{H}_{jm})[\partial_{im}\tilde{u}_0\partial_j\tilde{u}_0 + \partial_i\tilde{u}_0\partial_{jm}\tilde{u}_0] \\
= &-(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-2}} \sum_{i,j} H_{ij}^2 \\
&-(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-2}} \sum_{i,j,m} H_{im}(\partial_m H_{ij})\xi'_j \\
&+ 4(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-1}} \sum_{i,j,m} H_{im}H_{jm}\xi'_i\xi'_j \\
&+ 2(N-2)(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-1}} \sum_{i,j,m,s} H_{is}(\partial_s H_{jm})\xi'_i\xi'_j\xi'_m.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{i,j,m,s} \left[\overline{H}_{ms}(\partial_s \overline{H}_{ij}) - \overline{H}_{is}(\partial_s \overline{H}_{jm})\right] \partial_m[\partial_i\tilde{u}_0\partial_j\tilde{u}_0] \\
= &-(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-2}} \sum_{i,j} H_{ij}^2 \\
&-(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-2}} \sum_{i,j,m} H_{im}(\partial_m H_{ij})\xi'_j.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{i,j,m,s} \left[\overline{H}_{sj}(\partial_i \overline{H}_{ms}) - \overline{H}_{ms}(\partial_i \overline{H}_{js})\right] (\partial_m(\partial_i\tilde{u}_0)(\partial_j\tilde{u}_0)) \\
= &\sum_{i,j,m,s} f(|y|^2)^2[H_{js}(\partial_i H_{ms}) - H_{ms}(\partial_i H_{js})](\partial_{im}\tilde{u}_0\partial_j\tilde{u}_0 + \partial_i\tilde{u}_0\partial_{jm}\tilde{u}_0) \\
= &(N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y-\xi'|^2)^{N-2}} \sum_{i,j} H_{ij}^2
\end{aligned}$$

$$+ (N-4)^2 f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^{N-2}} \sum_{i,j,s} H_{is}(\partial_i H_{js})\xi'_j.$$

The lemma follows immediately. □

Lemma 9.12. *We have*

$$\int_{\mathbb{R}^N} \left\{ \sum_{i,j,m,s} \overline{H}_{si}(\partial_{mm}\overline{H}_{js})\partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\}\Bigg|_{\xi'=0} = 0,$$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,m,s} \overline{H}_{si}(\partial_{mm}\overline{H}_{js})\partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\}\Bigg|_{\xi'=0} \\
&= |S^{N-1}|(W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk})\lambda'^{N-4} \\
&\quad \cdot \frac{2(N-4)^2}{N(N+2)} \int_0^\infty \Big[(N+4)f(r^2)f'(r^2) + 2r^2 f(r^2)f''(r^2)\Big] \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-2}} \mathrm{d}r.
\end{aligned}$$

Proof. It is directly checked that

$$\begin{aligned}
&\sum_{i,j,m,s} \overline{H}_{si}(\partial_{mm}\overline{H}_{js})\partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \\
&= (N-4)^2 \Big[(2N+8)f(|y|^2)f'(|y|^2) + 4|y|^2 f(|y|^2)f''(|y|^2)\Big] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^{N-2}} \\
&\qquad \cdot \sum_{i,j,s} H_{is}H_{js}\xi'_i \xi'_j,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,j,m,s} \overline{H}_{si}(\partial_{mm}\overline{H}_{js})\partial_i \tilde{u}_0 \partial_j \tilde{u}_0 \right\}\Bigg|_{\xi=0} \\
&= 4(N-4)^2 \Big[(N+4)f(|y|^2)f'(|y|^2) + 2|y|^2 f(|y|^2)f''(|y|^2)\Big] \frac{\lambda'^{N-4}}{(\lambda'^2 + |y-\xi'|^2)^{n-2}} \\
&\qquad \cdot \sum_{s} H_{ps}H_{qs}.
\end{aligned}$$

we conclude the proof by using Lemma 9.5 and 9.6. □

Lemma 9.13. *The following hold*

$$\int_{\mathbb{R}^N} \left\{ \sum_{i,k,\ell,m} \left[(\partial_{i\ell}\overline{H}_{mk})^2 + (\partial_\ell \overline{H}_{mk})(\partial_{ii\ell}\overline{H}_{mk})\right] \tilde{u}_0^2 \right\}\Bigg|_{\xi'=0}$$

$$
\begin{aligned}
&= |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4} \\
&\quad \cdot \Bigg\{ \frac{2}{N(N+2)} \int_0^\infty \Big[3(N+8) f'(r^2)^2 + 2(N+8) f(r^2) f''(r^2) \\
&\qquad\qquad + 2(N+18) r^2 f'(r^2) f''(r^2) + 4r^4 f''(r^2)^2 \\
&\qquad\qquad + 4r^2 f(r^2) f'''(r^2) + 4r^4 f'(r^2) f'''(r^2) \Big] \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-4}} \mathrm{d}r \\
&\quad + \frac{2}{N} \int_0^\infty \Big[4r^2 f'(r^2)^2 + (N+8) f(r^2) f'(r^2) + 2r^2 f(r^2) f''(r^2) \Big] \frac{r^{N+1}}{(\lambda'^2 + r^2)^{N-4}} \mathrm{d}r \\
&\quad + \int_0^\infty f(r^2)^2 \frac{r^{N-1}}{(\lambda'^2 + r^2)^{N-4}} \mathrm{d}r \Bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \frac{\partial^2}{\partial \xi'_p \partial \xi'_q} \left\{ \sum_{i,k,\ell,m} \left[(\partial_{i\ell} \overline{H}_{mk})^2 + (\partial_\ell \overline{H}_{mk})(\partial_{ii\ell} \overline{H}_{mk}) \right] \tilde{u}_0^2 \right\} \Bigg|_{\xi=0} \\
&= |S^{N-1}| (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda'^{N-4} \\
&\quad \Bigg\{ \frac{8(N-3)(N-4)}{N(N+2)(N+4)} \\
&\quad \cdot \int_0^\infty \Big[12(N+8) f'(r^2)^2 + 16r^4 f''(r^2)^2 + 8(N+18) r^2 f'(r^2) f''(r^2) \\
&\qquad\quad + 8(N+8) f(r^2) f''(r^2) + 16r^2 f(r^2) f'''(r^2) + 16r^4 f'(r^2) f'''(r^2) \Big] \\
&\qquad \cdot \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} \\
&\quad + \frac{8(N-3)(N-4)}{N(N+2)} \\
&\quad \cdot \int_0^\infty \Big[8r^2 f'(r^2)^2 + 2(N+8) f(r^2) f'(r^2) + 4r^2 f(r^2) f''(r^2) \Big] \frac{r^{N+3}}{(\lambda'^2 + |x|^2)^{N-2}} \Bigg\} \\
&\quad + |S^{N-1}| (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4} \delta_{pq} \\
&\quad \Bigg\{ \frac{N-4}{N(N+2)} \\
&\quad \cdot \int_0^\infty \Big[12(N+8) f'(r^2)^2 + 16r^4 f''(r^2)^2 + 8(N+18) r^2 f'(r^2) f''(r^2) \\
&\qquad + 8(N+8) f(r^2) f''(r^2) + 16r^2 f(r^2) f'''(r^2) + 16r^4 f'(r^2) f'''(r^2) \Big]
\end{aligned}
$$

$$
\cdot \left[\frac{2(N-3)}{N+4}\frac{r^{N+5}}{(\lambda'^2+r^2)^{N-2}} - \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-3}}\right]
$$

$$
+ \frac{2(N-4)}{N}\int_0^\infty \Big[8r^2 f'(r^2)^2 + 2(N+8)f(r^2)f'(r^2) + 4r^2 f(r^2)f''(r^2)\Big]
$$

$$
\cdot \left[\frac{2(N-3)}{N+2}\frac{r^{N+3}}{(\lambda'^2+r^2)^{N-2}} - \frac{r^{N+1}}{(\lambda'^2+r^2)^{N-3}}\right]
$$

$$
+ 2(N-4)\int_0^\infty f(r^2)^2 \left[\frac{2(N-3)}{N}\frac{r^{N+1}}{(\lambda'^2+r^2)^{N-2}} - \frac{r^{N-1}}{(\lambda'^2+r^2)^{N-3}}\right]\Bigg\}.
$$

Proof. We have

$$
\begin{aligned}
&\sum_{i,\ell}(\partial_{i\ell}\overline{H}_{mk})^2 \\
= &\sum_{i,\ell}\Big[2\delta_{i\ell}f'(|y|^2)H_{mk} + 2f'(|y|^2)y_\ell(\partial_i H_{mk}) + 2f'(|y|^2)y_i(\partial_\ell H_{mk}) \\
&\qquad + f(|y|^2)\partial_{i\ell}H_{mk} + 4f''(|y|^2)y_\ell y_i H_{mk}\Big]^2 \\
= &\, 4Nf'(|y|^2)^2H_{mk}^2 + 8|y|^2f'(|y|^2)^2\sum_i(\partial_i H_{mk})^2 + f(|y|^2)^2\sum_{i,\ell}(\partial_{i\ell}H_{mk})^2 \\
&+ 16f'(|y|^2)^2\sum_i y_i H_{mk}(\partial_i H_{mk}) + 4f'(|y|^2)f(|y|^2)\sum_i H_{mk}(\partial_{ii}H_{mk}) \\
&+ 8f'(|y|^2)^2\sum_{i,\ell}y_i y_\ell(\partial_i H_{mk})(\partial_\ell H_{mk}) + 8f'(|y|^2)f(|y|^2)\sum_{i,\ell}y_\ell(\partial_i H_{mk})(\partial_{i\ell}H_{mk}) \\
&+ 16f''(|y|^2)^2|y|^2H_{mk}^2 + 80f'(|y|^2)f''(|y|^2)|y|^2H_{mk}^2 \\
&+ 8f(|y|^2)f''(|y|^2)\sum_{i,\ell}y_i y_\ell H_{mk}(\partial_{i\ell}H_{mk}) \\
= &\Big[(4N+64)f'(|y|^2)^2 + 16|y|^4f''(|y|^2)^2 + 80|y|^2f'(|y|^2)f''(|y|^2) + 16f(|y|^2)f''(|y|^2)\Big]H_{mk}^2 \\
&+ 8\Big[|y|^2f'(|y|^2)^2 + f(|y|^2)f'(|y|^2)\Big]\sum_i(\partial_i H_{mk})^2 + f(|y|^2)^2\sum_{i,\ell}(\partial_{i\ell}H_{mk})^2,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i,\ell}(\partial_\ell\overline{H}_{mk})(\partial_{ii\ell}\overline{H}_{mk}) \\
= &\sum_\ell\Bigg\{\Big[(2N+8)f'(|y|^2) + 4|y|^2f''(|y|^2)\Big](\partial_\ell H_{mk}) \\
&\qquad + \Big[(4N+24)f''(|y|^2) + 8|y|^2f'''(|y|^2)\Big]y_\ell H_{mk}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \cdot \Big[2f'(|y|^2)y_\ell H_{mk} + f(|y|^2)\partial_\ell H_{mk}\Big] \\
&= \Big[8(N+4)f'(|y|^2)^2 + 8(N+8)|y|^2 f'(|y|^2)f''(|y|^2) + 8(N+6)f(|y|^2)f''(|y|^2) \\
&\quad + 16|y|^2 f(|y|^2)f'''(|y|^2) + 16|y|^4 f'(|y|^2)f'''(|y|^2)\Big] H_{mk}^2 \\
&\quad + \Big[2(N+4)f(|y|^2)f'(|y|^2) + 4|y|^2 f(|y|^2)f''(|y|^2)\Big] \sum_\ell (\partial_\ell H_{mk})^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial^2}{\partial\xi'_p\partial\xi'_q}\left\{\sum_{i,k,\ell,m}\left[(\partial_{i\ell}\overline{H}_{mk})^2 + (\partial_\ell\overline{H}_{mk})(\partial_{ii\ell}\overline{H}_{mk})\right]\tilde{u}_0^2\right\}\Bigg|_{\xi'=0} \\
&= 4(N-3)(N-4)\Big[12(N+8)f'(|y|^2)^2 + 16|y|^4 f''(|y|^2)^2 + 8(N+18)|y|^2 f'(|y|^2)f''(|y|^2) \\
&\quad + 8(N+8)f(|y|^2)f''(|y|^2) + 16|y|^2 f(|y|^2)f'''(|y|^2) + 16|y|^4 f'(|y|^2)f'''(|y|^2)\Big] \\
&\quad \cdot \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-2}}\sum_{k,m} H_{km}^2 y_p y_q \\
&\quad - 2(N-4)\Big[12(N+8)f'(|y|^2)^2 + 16|y|^4 f''(|y|^2)^2 + 8(N+18)|y|^2 f'(|y|^2)f''(|y|^2) \\
&\quad + 8(N+8)f(|y|^2)f''(|y|^2) + 16|y|^2 f(|y|^2)f'''(|y|^2) + 16|y|^4 f'(|y|^2)f'''(|y|^2)\Big] \\
&\quad \cdot \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-3}}\sum_{k,m} H_{km}^2 \delta_{pq} \\
&\quad + 4(N-3)(N-4)\Big[8|y|^2 f'(|y|^2)^2 + 2(N+8)f(|y|^2)f'(|y|^2) + 4|y|^2 f(|y|^2)f''(|y|^2)\Big] \\
&\quad \cdot \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-2}}\sum_{k,\ell,m} (\partial_\ell H_{km})^2 y_p y_q \\
&\quad - 2(N-4)\Big[8|y|^2 f'(|y|^2)^2 + 2(N+8)f(|y|^2)f'(|y|^2) + 4|y|^2 f(|y|^2)f''(|y|^2)\Big] \\
&\quad \cdot \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-3}}\sum_{k,\ell,m} (\partial_\ell H_{km})^2 \delta_{pq} \\
&\quad + 4(N-3)(N-4)f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-2}}\sum_{i,k,\ell,m} (\partial_{i\ell} H_{km})^2 y_p y_q \\
&\quad - 2(N-4)f(|y|^2)^2 \frac{\lambda'^{N-4}}{(\lambda'^2+|y|^2)^{N-3}}\sum_{i,k,\ell,m} (\partial_{i\ell} H_{km})^2 \delta_{pq}.
\end{aligned}
$$

Lemma 9.5 and 9.6 then give the result. □

Lemma 9.14. *We have*

$$\int_{\mathbb{R}^N} \left\{ \sum_{i,j,m,s} (\partial_{mm}\overline{H}_{ij})(\partial_{ss}\overline{H}_{ij})\tilde{u}_0^2 \right\}\Bigg|_{\xi'=0}$$

$$= |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4}$$

$$\cdot \frac{1}{2N(N+2)} \int_0^\infty \Big[2(N+4)f'(r^2) + 4r^2 f''(r^2)\Big]^2 \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-4}} \mathrm{d}r$$

and

$$\int_{\mathbb{R}^N} \frac{\partial^2}{\partial\xi_p \partial\xi_q} \left\{ \sum_{i,j,m,s} (\partial_{mm}\overline{H}_{ij})(\partial_{ss}\overline{H}_{ij})\tilde{u}_0^2 \right\}\Bigg|_{\xi'=0}$$

$$= |S^{N-1}| \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) \lambda'^{N-4}$$

$$\cdot \frac{8(N-3)(N-4)}{N(N+2)(N+4)} \int_0^\infty \big[2(N+4)f'(r^2) + 4r^2 f''(r^2)\big]^2 \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} \mathrm{d}r$$

$$+ |S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \lambda'^{N-4} \delta_{pq}$$

$$\cdot \frac{N-4}{N(N+2)} \int_0^\infty \big[2(N+4)f'(r^2) + 4r^2 f''(r^2)\big]^2$$

$$\cdot \left[\frac{2(N-3)}{N+4} \frac{r^{N+5}}{(\lambda'^2 + r^2)^{N-2}} - \frac{r^{N+3}}{(\lambda'^2 + r^2)^{N-3}} \right] \mathrm{d}r.$$

Proof. Since

$$\sum_m \partial_{mm}\overline{H}_{ij} = \sum_m \Bigg(4f'(|y|^2) y_m \partial_m H_{ij} + 2f'(|y|^2) H_{ij} + 4f''(|y|^2) y_m^2 H_{ij}$$

$$+ f(|y|^2)\partial_{mm} H_{ij} \Bigg)$$

$$= \Big[2(N+4)f'(|y|^2) + 4|y|^2 f''(|y|^2)\Big] H_{ij},$$

and

$$\frac{\partial^2}{\partial\xi'_p \partial\xi'_q} \left\{ \sum_{i,j,m,s} (\partial_{mm}\overline{H}_{ij})(\partial_{ss}\overline{H}_{ij}) u_{(\xi',\varepsilon)}^2 \right\}\Bigg|_{\xi'=0}$$

$$= 4(N-3)(N-4)\Big[2(N+4)f'(|y|^2) + 4|y|^2 f''(|y|^2)\Big]^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-2}} \sum_{i,j} H_{ij}^2 y_p y_q$$

$$- 2(N-4)\Big[2(N+4)f'(|y|^2) + 4|y|^2 f''(|y|^2)\Big]^2 \frac{\lambda'^{N-4}}{(\lambda'^2 + |y|^2)^{N-3}} \sum_{i,j} H_{ij}^2 \delta_{pq},$$

we obtain the result by Lemma 9.5 and 9.6. □

Lemma 9.15. *There hold*

$$\sum_{i,j,s} (\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0)w\Big|_{\xi'=0} = 0, \qquad \frac{\partial^2}{\partial\xi'_p\partial\xi'_q} \sum_{i,j,s} (\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0)w\Big|_{\xi'=0} = 0,$$

$$\sum_{i,j,s} (\partial_{s}\overline{H}_{ij})(\partial_{sij}\tilde{u}_0)w\Big|_{\xi'=0} = 0, \qquad \frac{\partial^2}{\partial\xi'_p\partial\xi'_q} \sum_{i,j,s} (\partial_{s}\overline{H}_{ij})(\partial_{sij}\tilde{u}_0)w\Big|_{\xi'=0} = 0,$$

$$\sum_{i,j,s} \overline{H}_{ij}(\partial_{ssij}\tilde{u}_0)w\Big|_{\xi'=0} = 0, \qquad \frac{\partial^2}{\partial\xi'_p\partial\xi'_q} \sum_{i,j,s} \overline{H}_{ij}(\partial_{ssij}\tilde{u}_0)w\Big|_{\xi'=0} = 0,$$

$$\sum_{i,j,m} (\partial_{jmm}\overline{H}_{ij})(\partial_{i}u)w\Big|_{\xi'=0} = 0, \qquad \frac{\partial^2}{\partial\xi'_p\partial\xi'_q} \sum_{i,j,m} (\partial_{jmm}\overline{H}_{ij})(\partial_{i}u)w\Big|_{\xi'=0} = 0.$$

Proof. Direct computation shows

$$\begin{aligned}
&\sum_{i,j,s} (\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) \\
&= \Big[2Nf'(|y|^2) + 8f'(|y|^2) + 4|y|^2 f''(|y|^2)\Big] \sum_{i,j} H_{ij}(\partial_{ij}\tilde{u}_0) \\
&= (N-2)(N-4)\Big[(2N+8)f'(|y|^2) + 4|y|^2 f''(|y|^2)\Big] \\
&\qquad \cdot \frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2 + |y-\xi'|^2)^{\frac{N}{2}}} \sum_{i,j} H_{ij}\xi'_i\xi'_j,
\end{aligned}$$

$$\begin{aligned}
&\sum_{i,j,s} (\partial_{s}\overline{H}_{ij})(\partial_{sij}\tilde{u}_0) \\
= &\sum_{i,j,s} \Big[2f'(|y|^2)y_s H_{ij} + f(|y|^2)(\partial_s H_{ij})\Big](N-2)(N-4)\tilde{u}_0 \\
&\cdot \left[\frac{\delta_{is}(y_j-\xi'_j) + \delta_{js}(y_i-\xi'_i) + \delta_{ij}(y_s-\xi'_s)}{(\lambda'^2 + |y-\xi'|^2)^2} - N\frac{(y_s-\xi'_s)(y_i-\xi'_i)(y_j-\xi'_j)}{(\lambda'^2 + |y-\xi'|^2)^3}\right] \\
= &- 2N(N-2)(N-4)|y|^2 f'(|y|^2) \frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2 + |y-\xi'|^2)^{\frac{N+2}{2}}} \sum_{i,j} H_{ij}\xi'_i\xi'_j
\end{aligned}$$

$$+ 2N(N-2)(N-4)f'(|y|^2)\frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2+|y-\xi'|^2)^{\frac{N+2}{2}}}\sum_{i,j,s} H_{ij}y_s\xi'_s\xi'_i\xi'_j$$

$$+ N(N-2)(N-4)f(|y|^2)\frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2+|y-\xi'|^2)^{\frac{N+2}{2}}}\sum_{i,j,s}(\partial_s H_{ij})\xi'_s\xi'_i\xi'_j,$$

$$\sum_{i,j,s}\overline{H}_{ij}(\partial_{ssij}\tilde{u}_0)$$

$$= N(N-2)(N+2)(N-4)f(|y|^2)\frac{|y-\xi'|^2\lambda'^{\frac{N-4}{2}}}{(\lambda'^2+|y-\xi'|^2)^{\frac{N+4}{2}}}\sum_{i,j} H_{ij}\xi'_i\xi'_j$$

$$- N(N-2)(N-4)(N+4)f(|y|^2)\frac{\lambda'^{\frac{N-4}{2}}}{(\lambda'^2+|y-\xi'|^2)^{\frac{N+2}{2}}}\sum_{i,j} H_{ij}\xi'_i\xi'_j,$$

$$\sum_{i,j,m}(\partial_{jmm}\overline{H}_{ij})(\partial_i u) = 0.$$

Recall the equation for w and note that if $\xi' = 0$, $R_1(y) = 0$, so $w|_{\xi'=0} \equiv 0$. The Lemma is easily concluded. □

Combining the above identities, we have the following proposition.

Proposition 9.16. *It holds that*

$$\begin{aligned}
&F(0,\lambda')\\
=& \frac{N-4}{2}|S^{N-1}|\sum_{i,j,k,\ell}(W_{ikj\ell}+W_{i\ell jk})^2\lambda'^{N-4}\\
&\cdot\Bigg\{-\frac{a_N(N-4)}{N(N+2)}\int_0^\infty\left[r^2 f'(r^2)^2+2f(r^2)f'(r^2)\right]\frac{r^{N+5}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r\\
&\quad-\frac{a_N(N-4)}{2N}\int_0^\infty f(r^2)^2\frac{r^{N+3}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r\\
&\quad-\frac{b_N(N-4)}{N(N+2)}\int_0^\infty\left[r^2 f'(r^2)+f(r^2)\right]^2\frac{r^{N+3}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r\\
&\quad+\frac{1}{2N(N-1)(N+2)}\int_0^\infty\Big[3(N+8)f'(r^2)^2+2(N+8)f(r^2)f''(r^2)\\
&\qquad\qquad+2(N+18)r^2 f'(r^2)f''(r^2)+4r^4 f''(r^2)^2\\
&\qquad\qquad+4r^2 f(r^2)f'''(r^2)+4r^4 f'(r^2)f'''(r^2)\Big]\frac{r^{N+3}}{(\lambda'^2+r^2)^{N-4}}\mathrm{d}r
\end{aligned}$$

$$+\frac{1}{2N(N-1)}\int_0^\infty \Big[4r^2f'(r^2)^2+(N+8)f(r^2)f'(r^2)+2r^2f(r^2)f''(r^2)\Big] \cdot\frac{r^{N+1}}{(\lambda'^2+r^2)^{N-4}}\mathrm{d}r$$

$$+\frac{1}{4(N-1)}\int_0^\infty f(r^2)^2\frac{r^{N-1}}{(\lambda'^2+r^2)^{N-4}}\mathrm{d}r$$

$$-\frac{1}{N(N-2)^2(N+2)}\int_0^\infty \Big[(N+4)f'(r^2)+2r^2f''(r^2)\Big]^2\frac{r^{N+3}}{(\lambda'^2+r^2)^{N-4}}\mathrm{d}r\Bigg\}.$$

Next we compute the Hessian of F at $(0,\lambda')$. Because of Lemma 9.15, it is obvious that

$$\frac{\partial^2}{\partial\xi_p'\partial\xi_q'}\Bigg\{\sum_{i,j,s}\Big[2\overline{H}_{ij}(\partial_{ijss}\tilde{u}_0)+2(\partial_s\overline{H}_{ij})(\partial_{ijs}\tilde{u}_0)+(\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0) - \frac{b_N}{2}(\partial_{ss}\overline{H}_{ij})(\partial_{ij}\tilde{u}_0)-\frac{b_N}{2}(\partial_{jss}\overline{H}_{ij})(\partial_i\tilde{u}_0)\Big]\bar{w}\Bigg\}\Bigg|_{\xi'=0}$$

$$=0.$$

Proposition 9.17. *The second order partial derivatives of $F(\xi',\lambda')$ at $(0,\lambda')$ are given by*

$$\frac{\partial^2}{\partial\xi_p'\partial\xi_q'}F(0,\lambda')$$

$$=\frac{(N-4)^2}{N(N+2)}|S^{N-1}|\sum_{i,j,k}(W_{ikjp}+W_{ipjk})(W_{ikjq}+W_{iqjk})\lambda'^{N-4}$$

$$\cdot\Bigg\{-(N-2)\int_0^\infty f(r^2)^2\left[\frac{Nr^{N+5}}{(\lambda'^2+r^2)^N}-\frac{(N+2)r^{N+3}}{(\lambda'^2+r^2)^{N-1}}\right]\mathrm{d}r$$

$$-\frac{8a_N(N-2)}{N+4}\int_0^\infty\left[r^2f'(r^2)^2+2f(r^2)f'(r^2)\right] \cdot\left[\frac{(N-1)r^{N+7}}{(\lambda'^2+r^2)^N}-\frac{2r^{N+5}}{(\lambda'^2+r^2)^{N-1}}\right]\mathrm{d}r$$

$$-2a_N(N-2)\int_0^\infty f(r^2)^2\left[\frac{(N-1)r^{N+5}}{(\lambda'^2+r^2)^N}-\frac{2r^{N+3}}{(\lambda'^2+r^2)^{N-1}}\right]\mathrm{d}r$$

$$-\frac{8b_N(N-1)(N-2)}{N+4}\int_0^\infty\left[r^2f'(r^2)+f(r^2)\right]^2\frac{r^{N+5}}{(\lambda'^2+r^2)^N}\mathrm{d}r$$

$$+\frac{16b_N(N-2)}{N+4}\int_0^\infty\left[r^2f'(r^2)^2+f(r^2)f'(r^2)\right]\frac{r^{N+5}}{(\lambda'^2+r^2)^{N-1}}\mathrm{d}r$$

$$+ 4b_N(N-2)\int_0^\infty \left[r^2 f(r^2)f'(r^2) + f(r^2)^2\right] \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-1}}\mathrm{d}r$$
$$- \frac{4b_N}{N+4}\int_0^\infty f'(r^2)^2 \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r$$
$$- \frac{b_N(N+2)}{2}\int_0^\infty f(r^2)^2 \frac{r^{N+1}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r$$
$$- 2b_N\int_0^\infty f(r^2)f'(r^2) \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r$$
$$+ b_N\int_0^\infty \Big[(N+4)f(r^2)f'(r^2) + 2r^2 f(r^2)f''(r^2)\Big] \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r$$
$$+ \frac{4(N-3)}{(N-1)(N+4)}$$
$$\cdot \int_0^\infty \Big[3(N+8)f'(r^2)^2 + 4r^4 f''(r^2)^2 + 2(N+18)r^2 f'(r^2)f''(r^2)$$
$$+ 2(N+8)f(r^2)f''(r^2) + 4r^2 f(r^2)f'''(r^2) + 4r^4 f'(r^2)f'''(r^2)\Big]$$
$$\cdot \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r$$
$$+ \frac{2(N-3)}{N-1}\int_0^\infty \Big[4r^2 f'(r^2)^2 + (N+8)f(r^2)f'(r^2) + 2r^2 f(r^2)f''(r^2)\Big]$$
$$\cdot \frac{r^{N+3}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r$$
$$- \frac{8(N-3)}{(N-2)^2(N+4)}\int_0^\infty \left[(N+4)f'(r^2) + 2r^2 f''(r^2)\right]^2 \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-2}}\mathrm{d}r \Bigg\}$$
$$+ \frac{(N-4)^2}{N(N+2)}|S^{N-1}| \sum_{i,j,k,\ell}(W_{ikj\ell}+W_{i\ell jk})^2 \lambda'^{N-4}\delta_{pq}$$
$$\cdot \Bigg\{ - \frac{a_N}{N+4}\int_0^\infty \left[r^2 f'(r^2)^2 + 2f(r^2)f'(r^2)\right]$$
$$\cdot \left[\frac{2(N-1)(N-2)r^{N+7}}{(\lambda'^2+r^2)^{N}} - \frac{(N-2)(N+8)r^{N+5}}{(\lambda'^2+r^2)^{N-1}} + \frac{(N+4)r^{N+3}}{(\lambda'^2+r^2)^{N-2}}\right]\mathrm{d}r$$
$$- \frac{a_N}{2}\int_0^\infty f(r^2)^2$$
$$\cdot \left[\frac{2(N-1)(N-2)r^{N+5}}{(\lambda'^2+r^2)^{N}} - \frac{(N-2)(N+6)r^{N+3}}{(\lambda'^2+r^2)^{N-1}} + \frac{(N+2)r^{N+1}}{(\lambda'^2+r^2)^{N-2}}\right]\mathrm{d}r$$

$$
\begin{aligned}
&- \frac{b_N(N-2)}{N+4} \int_0^\infty \left[r^2 f'(r^2) + f(r^2)\right]^2 \\
&\qquad\qquad \cdot \left[\frac{2(N-1)r^{N+5}}{(\lambda'^2+r^2)^N} - \frac{(N+4)r^{N+3}}{(\lambda'^2+r^2)^{N-1}}\right] \mathrm{d}r \\
&+ \frac{4b_N(N-2)}{N+4} \int_0^\infty \left[r^2 f'(r^2)^2 + f(r^2)f'(r^2)\right] \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-1}} \mathrm{d}r \\
&- \frac{b_N}{N+4} \int_0^\infty f'(r^2)^2 \frac{r^{N+5}}{(\lambda'^2+r^2)^{N-2}} \mathrm{d}r \\
&+ \frac{1}{2(N-1)(N+4)} \\
&\cdot \int_0^\infty \Big[3(N+8)f'(r^2)^2 + 4r^4 f''(r^2)^2 + 2(N+18)r^2 f'(r^2)f''(r^2) \\
&\qquad + 2(N+8)f(r^2)f''(r^2) + 4r^2 f(r^2)f'''(r^2) + 4r^4 f'(r^2)f'''(r^2)\Big] \\
&\qquad\qquad \cdot \left[\frac{2(N-3)r^{N+5}}{(\lambda'^2+r^2)^{N-2}} - \frac{(N+4)r^{N+3}}{(\lambda'^2+r^2)^{N-3}}\right] \mathrm{d}r \\
&+ \frac{1}{2(N-1)} \int_0^\infty \left[4r^2 f'(r^2)^2 + (N+8)f(r^2)f'(r^2) + 2r^2 f(r^2)f''(r^2)\right] \\
&\qquad\qquad \cdot \left[\frac{2(N-3)r^{N+3}}{(\lambda'^2+r^2)^{N-2}} - \frac{(N+2)r^{N+1}}{(\lambda'^2+r^2)^{N-3}}\right] \mathrm{d}r \\
&+ \frac{N+2}{4(N-1)} \int_0^\infty f(r^2)^2 \left[\frac{2(N-3)r^{N+1}}{(\lambda'^2+r^2)^{N-2}} - \frac{N r^{N-1}}{(\lambda'^2+r^2)^{N-3}}\right] \mathrm{d}r \\
&- \frac{1}{(N-2)^2(N+4)} \int_0^\infty \left[(N+4)f'(r^2) + 2r^2 f''(r^2)\right]^2 \\
&\qquad\qquad \cdot \left[\frac{2(N-3)r^{N+5}}{(\lambda'^2+r^2)^{N-2}} - \frac{(N+4)r^{N+3}}{(\lambda'^2+r^2)^{N-3}}\right] \mathrm{d}r \Bigg\}.
\end{aligned}
$$

In summary, we have reduced the derivative of F and the Hessian of F to integrals in terms of an auxilliary function f. In the next two sections, we choose the auxilliary function f so that F has a strict local minimum at $(0,1)$. More precisely, we have to choose a function (which is a polynomial) so that the following conditions are satisfied

(F_1) $\frac{\partial F}{\partial \lambda'}(0,1) = 0$;

(F_2) the matrix $(\frac{\partial^2}{\partial \xi'_p \partial \xi'_q} F(0,1))$ is positive definite;

(F_3) $\frac{\partial^2}{\partial \lambda'^2} F(0,1) > 0$.

We remark that by Proposition 9.1, it holds that $\frac{\partial F}{\partial \xi'_p}(0,1) = 0$, $\frac{\partial^2 F}{\partial \lambda' \xi'_p}(0,1) = 0$. Conditions (F_1)-(F_2) ensure that F has a nondegenerate local minimum at $(0,1)$.

Our first choice is a linear function.

10 Linear function and the case of $N \geq 52$

In this section, we show that when $N \geq 52$ the choice of suitable linear fuction satisfies (F_1)-(F_2). (Surprisingly, this dimension 52 also agrees with the second order Yamabe problem by Brendle [3] in which he also chose a linear function.) The computations are unfortunately complicated even in this case. Many of the computations below are carried out by Mathematica. Since all these computations only involve finding the roots of certain polynomials, the computing errors can be controlled.

Let the auxiliary function be

$$f(s) = \tau + s.$$

Using the software *Mathematica*, we get the following two propositions.

Proposition 10.1. *Assume $N > 12$, we have*

$$\begin{aligned} &F(0,\lambda') \\ &= \frac{(N-4)^2}{4(N^2-4)(N-8)}|S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \frac{\Gamma\left[\frac{N}{2}-3\right]\Gamma\left[\frac{N}{2}+3\right]}{\Gamma[N+1]} I(\lambda'), \end{aligned}$$

where Γ denotes the usual $\Gamma-$function and

$$\begin{aligned} &I(\lambda') \\ &= -\frac{\lambda'^8}{(N-12)(N-10)}\left(N^5 - 4N^4 - 80N^3 + 208N^2 - 32N - 192\right) \\ &\qquad - \frac{2(N-2)\lambda'^6\tau}{N-10}\left(N^3 - 8N^2 + 16\right) \\ &\qquad\qquad - \frac{(N-2)^2\lambda'^4\tau^2}{N+4}\left(N^2 - 4N - 4\right). \end{aligned}$$

Proposition 10.2. *Assume $N > 12$, we have*

$$\begin{aligned} &\frac{\partial^2}{\partial\xi'_p\partial\xi'_q} F(0,\lambda') \\ &= \frac{(N-4)^2}{N(N-8)(N+4)(N^2-4)}|S^{N-1}| \frac{\Gamma\left(\frac{N}{2}-3\right)\Gamma\left(\frac{N}{2}+3\right)}{\Gamma(N)} \end{aligned}$$

$$\cdot \left\{ \sum_{i,j,k} (W_{ikjp} + W_{ipjk})(W_{ikjq} + W_{iqjk}) J_1(\lambda') \right.$$
$$\left. + \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \delta_{pq} J_2(\lambda') \right\},$$

where

$$\begin{aligned} & J_1(\lambda') \\ = & -\frac{6\lambda'^6}{(N-10)} \left(N^5 - 4N^4 - 44N^3 + 112N^2 - 32N - 96 \right) \\ & - 4(N-2)^2 \lambda'^4 \left(N^2 - 4N - 4 \right) \tau \end{aligned}$$

and

$$\begin{aligned} & J_2(\lambda') \\ = & -\frac{\lambda'^6 (N-2) \left(N^4 - N^3 - 56N^2 + 40N + 88 \right)}{N-10} \\ & - \lambda'^4 (N-2)^2 \left(N^2 - 4N - 4 \right) \tau. \end{aligned}$$

Remark 10.3. *In the above two propositions, the assumption $N > 12$ guarantees the integrations in two propositions are finite.*

Lemma 10.4. *Assume that $N \geq 52$. Then there exists a $\tau_N \in \mathbb{R}$ such that $I'(1) = 0$, $I''(1) > 0$, $I(1) < 0$, $J_1(1) > 0$ and $J_2(1) > 0$.*

Proof. $I'(1) = 0$ is equivalent to

$$\begin{aligned} & \gamma_N + \beta_N \tau + \alpha_N \tau^2 := \\ & -\frac{8}{(N-12)(N-10)} \left(N^5 - 4N^4 - 80N^3 + 208N^2 - 32N - 192 \right) \\ & \quad - \frac{12(N-2)}{N-10} \left(N^3 - 8N^2 + 16 \right) \tau \\ & \quad\quad - \frac{4(N-2)^2}{N+4} \left(N^2 - 4N - 4 \right) \tau^2 = 0. \end{aligned}$$

The assumption $N \geq 52$ guarantees that there exists a solution $\tau_N \in \mathbb{R}$. Indeed, the discriminant is (after simplified) that

$$\begin{aligned} & \Delta_N := \\ & \frac{16(N-2)^2}{(N-10)^2(N-12)(N+4)} \Big(N^8 - 72N^7 + 1200N^6 - 7200N^5 + 15616N^4 \end{aligned}$$

$$- 31744N^3 + 34048N^2 + 47104N - 49152\Big).$$

Using Mathematica, we may solve the algebraic equation

$$\mathcal{A}(N) := N^8 - 72N^7 + 1200N^6 - 7200N^5 + 15616N^4 - 31744N^3 \\ + 34048N^2 + 47104N - 49152 = 0 \quad (50)$$

and find that the largest real solution is about $N \approx 51.1957$. Thus a real τ_N satisfying $I'(1) = 0$ must exist for $N \geq 52$. We choose

$$\tau_N = \frac{-\beta_N + \sqrt{\Delta_N}}{2\alpha_N} < -\frac{99}{50}, \quad (51)$$

since

$$\begin{aligned} &\gamma_N + \beta_N(-\frac{99}{50}) + \alpha_N(-\frac{99}{50})^2 \\ = &\frac{49N^6 + 26730N^5 - 1865112N^4 + 18883712N^3 - 36441904N^2 - 9453152N + 45467520}{625(N-12)(N-10)(N+4)} \\ > &\ 0 \end{aligned}$$

for $N \geq 52$. In fact, we can solve the algebraic equation

$$49N^6 + 26730N^5 - 1865112N^4 + 18883712N^3 \\ - 36441904N^2 - 9453152N + 45467520 = 0$$

and get the largest $N \approx 51.7253$, while $\alpha_N < 0$ for $N \geq 52$.

For this τ_N and $N \geq 52$, we have

$$\begin{aligned} &I''(1)|_{\tau_N} = I''(1) - 3I'(1)|_{\tau_N} \\ = &-\frac{24(N-2)\left(N^3 - 8N^2 + 16\right)\tau_N}{N-10} \\ &\quad - \frac{32\left(N^5 - 4N^4 - 80N^3 + 208N^2 - 32N - 192\right)}{(N-12)(N-10)} \\ > &\ \frac{24(N-2)\left(N^3 - 8N^2 + 16\right)}{N-10}\frac{99}{50} \\ &\quad - \frac{32\left(N^5 - 4N^4 - 80N^3 + 208N^2 - 32N - 192\right)}{(N-12)(N-10)} \\ := &\ \mathcal{A}_{1N} > 0, \end{aligned}$$

since the largest N satisfying $\mathcal{A}_{1N} = 0$ is $N \approx 47.248$.

Similarly, we can check that, for $N \geq 52$,

$$\begin{aligned} J_1(1)|_{\tau_N} > & -\frac{6}{(N-10)}\left(N^5-4N^4-44N^3+112N^2-32N-96\right) \\ & -4(N-2)^2\left(N^2-4N-4\right)\left(-\frac{99}{50}\right) \\ > & \ 0. \end{aligned}$$

Also we have that

$$\begin{aligned} J_2(1)|_{\tau_N} > & -\frac{(N-2)\left(N^4-N^3-56N^2+40N+88\right)}{N-10} \\ & -(N-2)^2\left(N^2-4N-4\right)\left(-\frac{99}{50}\right) \\ > & \ 0. \end{aligned}$$

Finally, we compute the discriminant of $I(1)$ and get

$$-\frac{24(N-2)^2\left(N^7-22N^6+156N^5-400N^4+672N^3-448N^2-1024N+768\right)}{(N-12)(N-10)^2(N+4)},$$

which is checked always negative for $N \geq 52$. So $I(1) < 0$.

The proof is complete. □

Proposition 10.5. *For τ_N chosen in Lemma 10.4, the function $F(\xi', \lambda')$ has a strict local minimum at $(0, 1)$.*

Proof. Since $I'(1) = 0$, we have $\frac{\partial}{\partial \lambda'} F(0, 1) = 0$. In addition Proposition 9.1 shows $\frac{\partial}{\partial \xi'} F(0, 1) = 0$. Therefore, $(0, 1)$ is a critical point of $F(\xi', \lambda')$.

Since $J_1(1) > 0$ and $J_2(1) > 0$, it follows from Lemma 10.4 that the matrix $(\frac{\partial^2}{\partial \xi'_p \partial \xi'_q} F(0, 1))$ is positive definite. Lemma 10.4 again shows that $I''(1) > 0$, which implies $\frac{\partial^2}{\partial \lambda'^2} F(0, 1) > 0$. Consequently, $(0, 1)$ is a strict local minimum point. □

11 Fourth polynomials and the case of $25 \leq N \leq 52$

Our ultimate goal is to reduce the dimension assumption $N \geq 52$ to $N \geq 25$. Unlike [4], where a cubic polynomial is chosen, we have to select a 4th order polynomial

$$f(s) = \tau - 12000s + 2411s^2 - 135s^3 + s^4. \tag{52}$$

Remark 11.1. *The coefficients in $f(s)$ are not unique. And they are chosen in order to verify the conditions (F_1)-(F_3). We have also tried cubic and fifth polynomials. They give larger bounds on N.*

Using the software *Mathematica*, we get

Proposition 11.2. *Assume $N \geq 25$,*

$$\begin{aligned} &F(0,\lambda') \\ &= \frac{N-4}{16(N^2-4)}|S^{N-1}| \sum_{i,j,k,\ell} (W_{ikj\ell} + W_{i\ell jk})^2 \frac{\Gamma\left[\frac{N}{2}-9\right]\Gamma\left[\frac{N}{2}+7\right]}{\Gamma[N+1]} I(\lambda') \end{aligned}$$

where Γ denotes the usual $\Gamma -$ function and

$$\begin{aligned} &I(\lambda') \\ = &- \frac{(N-4)(N+14)\lambda'^{20}}{(N-24)(N-22)(N-20)}\Big(N^6 + 42N^5 - 768N^4 - 17248N^3 + 38768N^2 \\ &\qquad - 2336N - 38400\Big) \\ &+ \frac{270(N-4)\lambda'^{18}}{(N-22)(N-20)}\Big(N^6 + 32N^5 - 612N^4 - 10768N^3 + 24672N^2 - 640N \\ &\qquad - 25600\Big) \\ &- \frac{(N-4)\lambda'^{16}}{(N-20)(N+12)}\Big(23047N^6 + 543484N^5 - 10985408N^4 - 146678256N^3 \\ &\qquad + 351063488N^2 - 16180224N - 363260160\Big) \\ &+ \frac{30(N-4)\lambda'^{14}}{(N+10)(N+12)}\Big(22499N^6 + 356784N^5 - 7984044N^4 - 76228592N^3 \\ &\qquad + 193344928N^2 - 4902016N - 209193984\Big) \\ &- \frac{(N-18)(N-4)\lambda'^{12}}{(N+8)(N+10)(N+12)}\Big(2\tau N^6 + 9052921N^6 + 4\tau N^5 + 84049210N^5 \\ &\qquad - 384\tau N^4 - 2265707776N^4 - 1856\tau N^3 - 14204127072N^3 + 4704\tau N^2 \\ &\qquad + 40189627120N^2 + 7360\tau N - 1125216800N - 12800\tau - 45033619968\Big) \\ &+ \frac{30(N-18)(N-16)(N-4)\lambda'^{10}}{(N+8)(N+10)(N+12)}\Big(9\tau N^5 + 1928800N^5 - 54\tau N^4 \\ &\qquad - 3857600N^4 - 864\tau N^3 - 293177600N^3 + 1584\tau N^2 + 648076800N^2 \\ &\qquad + 2880\tau N + 61721600N - 4608\tau - 740659200\Big) \end{aligned}$$

$$- \frac{2(N-18)(N-16)(N-14)(N-4)\lambda'^8}{(N+6)(N+8)(N+10)(N+12)}\Big(2411\tau N^5 + 72000000N^5$$
$$- 14466\tau N^4 - 288000000N^4 - 135016\tau N^3 - 5760000000N^3$$
$$+ 308608\tau N^2 + 14976000000N^2 + 347184\tau N - 2304000000N$$
$$- 694368\tau - 13824000000\Big)$$
$$+ \frac{2400(N-18)(N-16)(N-14)(N-12)(N-4)(N-2)\tau\lambda'^6}{(N+6)(N+8)(N+10)(N+12)}\Big(N^3 - 8N^2$$
$$+ 16\Big)$$
$$- \frac{(N-18)(N-16)(N-14)(N-12)(N-10)(N-4)(N-2)^2\tau^2\lambda'^4}{(N+4)(N+6)(N+8)(N+10)(N+12)}\Big(N^2$$
$$- 4N - 4\Big)\Bigg\}.$$

Also by *Mathematica*, the following holds.

Proposition 11.3. *Assume $N \geq 25$,*

$$\frac{\partial^2}{\partial\xi'_p\partial\xi'_q}F(0,\lambda')$$
$$= \frac{2(N-4)^2}{(N+2)(N+4)}|S^{N-1}|\frac{\Gamma\left(\frac{N}{2}-7\right)\Gamma\left(\frac{N}{2}+5\right)}{\Gamma(N+1)}$$
$$\cdot\Bigg\{\sum_{i,j,k}(W_{ikjp}+W_{ipjk})(W_{ikjq}+W_{iqjk})J_1(\lambda')$$
$$+ \sum_{i,j,k,\ell}(W_{ikj\ell}+W_{i\ell jk})^2\delta_{pq}J_2(\lambda')\Bigg\}$$

where

$$J_1(\lambda')$$
$$= -\frac{6\lambda'^{18}(N+10)(N+12)(N+14)(N+16)}{(N-22)(N-20)(N-18)(N-16)(N-2)}\big(N^5 - 4N^4 - 380N^3$$
$$+ 784N^2 - 32N - 768\big)$$
$$+ \frac{135\lambda'^{16}(N+10)(N+12)(N+14)}{2(N-20)(N-18)(N-16)(N-2)}(19N^5 - 76N^4 - 5776N^3 + 11872N^2$$
$$- 16N - 12160)$$
$$- \frac{\lambda'^{14}(N+10)(N+12)}{4(N-18)(N-16)(N-2)}(340883N^5 - 1363532N^4 - 80448388N^3$$

$$
\begin{aligned}
& + 167226320N^2 - 2575840N - 170104608) \\
& + \frac{135e^{12}(N+10)}{2(N-16)(N-2)}(27321N^5 - 109284N^4 - 4808496N^3 + 10054304N^2 \\
& + 153424N - 10644864) \\
& - \frac{2\lambda'^{10}}{(N-2)}(\tau N^5 + 8647921N^5 - 4\tau N^4 - 34591684N^4 - 124\tau N^3 \\
& - 1072342204N^3 + 208\tau N^2 + 2287434512N^2 + 416\tau N + 60226528N \\
& - 640\tau - 2486027776) \\
& + \frac{15\lambda'^8(N-14)}{2(N-2)(N+8)}(27\tau N^5 + 9644000N^5 - 108\tau N^4 - 38576000N^4 \\
& - 2160\tau N^3 - 771520000N^3 + 4320\tau N^2 + 1728204800N^2 \\
& + 5616\tau N - 30860800N - 10368\tau - 1851648000) \\
& - \frac{\lambda'^6(N-14)(N-12)}{(N-2)(N+6)(N+8)}(2411\tau N^5 + 108000000N^5 - 9644\tau N^4 \\
& - 432000000N^4 - 106084\tau N^3 - 4752000000N^3 + 270032\tau N^2 \\
& + 12096000000N^2 + 154304\tau N - 3456000000N - 462912\tau - 10368000000) \\
& + \frac{6000\lambda'^4(N-14)(N-12)(N-10)(N-2)\tau}{(N+6)(N+8)}\left(N^2 - 4N - 4\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& J_2(\lambda') \\
= & - \frac{\lambda'^{18}(N+10)(N+12)(N+14)}{2(N-22)(N-20)(N-18)(N-16)(N-2)}(N^6 + 40N^5 - 612N^4 \\
& - 14416N^3 + 33088N^2 - 1344N - 33536) \\
& + \frac{135\lambda'^{16}(N+10)(N+12)}{8(N-20)(N-18)(N-16)(N-2)}(7N^6 + 214N^5 - 3384N^4 - 61888N^3 \\
& + 145840N^2 - 864N - 154880) \\
& - \frac{\lambda'^{14}(N+10)}{8(N-18)(N-16)(N-2)}(69141N^6 + 1566133N^5 - 25714074N^4 \\
& - 353077328N^3 + 877267784N^2 - 17765904N - 935422272) \\
& + \frac{15\lambda'^{12}}{8(N-16)(N-2)}(112495N^6 + 1742122N^5 - 30701832N^4 \\
& - 297250720N^3 + 795284368N^2 + 714144N - 887282432) \\
& - \frac{\lambda'^{10}}{4(N-2)(N+8)}(2\tau N^6 + 9052921N^6 + 8\tau N^5 + 85669210N^5
\end{aligned}
$$

$$
\begin{aligned}
&- 312\tau N^4 - 1709000832N^4 - 1568\tau N^3 - 10658431936N^3 + 4160\tau N^2\\
&+ 32630916432N^2 + 5376\tau N - 7615136N - 10240\tau - 37788230528)\\
&+ \frac{15\lambda'^8(N-14)}{8(N-2)(N+8)}(27\tau N^5 + 5786400N^5 - 108\tau N^4 - 7715200N^4 - 2160\tau N^3\\
&- 648076800N^3 + 4320\tau N^2 + 1543040000N^2 + 5616\tau N\\
&+ 216025600N - 10368\tau - 1851648000)\\
&- \frac{\lambda'^6(N-14)(N-12)}{4(N+6)(N+8)}(2411\tau N^4 + 72000000N^4 - 4822\tau N^3\\
&- 72000000N^3 - 115728\tau N^2 - 4032000000N^2 + 38576\tau N\\
&+ 2880000000N + 231456\tau + 6336000000)\\
&+ \frac{1500\lambda'^4(N-14)(N-12)(N-10)(N-2)\tau}{(N+6)(N+8)}\left(N^2 - 4N - 4\right)\\
&\quad \left(N^5 + 7N^4 - 48N^3 + 16N^2 + 384N - 864\right).
\end{aligned}
$$

Remark 11.4. *Similarly, the assumption $N \geq 25$ is necessary in Proposition 11.2 and Proposition 11.3, otherwise the integrations diverge.*

Lemma 11.5. *Assume that $N \geq 25$. Then there exists a $\tau_N \in \mathbb{R}$ such that $I'(1) = 0$, $I''(1) > 0$, $I(1) < 0$, $J_1(1) > 0$ and $J_2(1) > 0$.*

Proof. By the software *Mathematica*, $I'(1) = 0$ is equivalent to the following quadratic polynomial

$$
\begin{aligned}
&\mathcal{B}_{1N}\tau^2 + \mathcal{B}_{2N}\tau + \mathcal{B}_{3N}\\
=&\ \big(N^{12} - 144N^{11} + 9112N^{10} - 332800N^9 + 7744896N^8 - 119529600N^7\\
&+ 1232785280N^6 - 8335672320N^5 + 35049651968N^4 - 81419810816N^3\\
&+ 72565198848N^2 + 41687285760N - 81749606400\big)\tau^2\\
&+ \big(- 27025N^{12} + 3697664N^{11} - 218286488N^{10} + 7232839376N^9\\
&- 145917536656N^8 + 1798916204864N^7 - 12415144280448N^6\\
&+ 29852476048896N^5 + 160699860860416N^4 - 1111179746377728N^3\\
&+ 1284360017780736N^2 + 1838260974256128N - 2852825357352960\big)\tau\\
&+ 168227346N^{12} - 21204491480N^{11} + 1108805978944N^{10}\\
&- 30273127750912N^9 + 427225222251424N^8 - 1914627449132672N^7\\
&- 2581051648570035 2N^6 + 354281472809361408N^5 - 800700785348505600N^4\\
&- 5829654487294640128N^3 + 15038003695249195008N^2\\
&- 13384541579520 24576N - 14134493112544788480 = 0.
\end{aligned}
$$

After simplified, the discriminant is

$$(N-24)(N-22)(N-20)(N-18)(N-2)^2(N+4)\big(57441241N^{17}$$
$$-13316757740N^{16}+1421208951488N^{15}$$
$$-92295209252880N^{14}+4060887517487792N^{13}$$
$$-127528377218205952N^{12}+2932837691854966528N^{11}$$
$$-49861018904126426112N^{10}+624738394629537111040N^{9}$$
$$-5683787728574744100864N^{8}+36513302074683044208640N^{7}$$
$$-158739757047539234324480N^{6}+443941679903779546513408N^{5}$$
$$-788934839032708877123584N^{4}+947159822427449128648704N^{3}$$
$$-70877049300753252876288N^{2}-1727615795557515443306496N$$
$$+1156307714218965199749120\big)=0.$$

By Mathematica, we may check that the biggest zero of the last term is $N \approx 24.9422$. Since $N \geq 25$, the discriminant is positive and there exists a real $\tau_N > 17000$ such that $I'(1) = 0$. In fact, the largest N satisfying $17000\mathcal{B}_{1N} + 17000\mathcal{B}_{2N} + \mathcal{B}_{3N} = 0$ is about $N \approx 24.9982$ and $17000\mathcal{B}_{1N} + 17000\mathcal{B}_{2N} + \mathcal{B}_{3N} \to -\infty$ as $N \to +\infty$. Thus

$$17000\mathcal{B}_{1N} + 17000\mathcal{B}_{2N} + \mathcal{B}_{3N} < 0 \qquad \text{when } N \geq 25. \tag{53}$$

On the other hand, the largest one to $\mathcal{B}_{1N} = 0$ is $N = 24$. So

$$\mathcal{B}_{1N} > 0 \qquad \text{for } N \geq 25. \tag{54}$$

Therefore a $\tau_N > 17000$ does exist from (53) and (54).

As for $I''(1)$, we have

$$I''(1)\Big|_{\tau_N} = I''(1) - 3I'(1)\Big|_{\tau_N}$$
$$= \frac{(N-4)}{2(N-24)(N-22)(N-20)(N+6)(N+8)(N+10)(N+12)}$$
$$\Big[\big(18713N^{11} - 2782116N^{10} + 180731080N^{9} - 6710115152N^{8}$$
$$+ 156124054384N^{7} - 2341982210432N^{6} + 22336952299904N^{5}$$
$$- 126901060502528N^{4} + 358317913417728N^{3} - 193547489329152N^{2}$$
$$- 744698653507584N + 844209499668480\big)\tau_N$$
$$- 239387740N^{11} + 33343448176N^{10} - 1965147182144N^{9}$$
$$+ 62606999818560N^{8} - 1119986901305472N^{7} + 9710408237796864N^{6}$$
$$+ 2497983280251904N^{5} - 775332925643090944N^{4} + 5322471076556407808N^{3}$$

$$- 845433085940285030 4N^2 - 1265620067537485824N + 8423950663121633280\Big]$$

$$:= \frac{(N-4)\left[\mathcal{B}_{4N}\tau_N + \mathcal{B}_{5N}\right]}{2(N-24)(N-22)(N-20)(N+6)(N+8)(N+10)(N+12)}.$$

On account that $\mathcal{B}_{4N} > 0$ and $17000\mathcal{B}_{4N} + \mathcal{B}_{5N} > 0$ for $N \geq 25$, we know that $I''(1)\Big|_{\tau_N} > 0$.

Come to $J_1(1)$. Direct computation shows that

$$\frac{4(N-22)(N-20)(N-18)(N-16)(N-2)(N+4)(N+6)(N+8)}{N} J_1(1)|_{\tau_N}$$
$$= (15158N^{11} - 1928488N^{10} + 106956432N^9 - 3385235008N^8$$
$$+ 67144207872N^7 - 861032971776N^6 + 7082375502592N^5$$
$$- 35544691776512N^4 + 95816075389440N^3 - 91338406885376N^2$$
$$- 68140868911104N + 124511268372480)\tau_N$$
$$- 204822475N^{11} + 24606405950N^{10} - 1253228747592N^9$$
$$+ 34711856215872N^8 - 549030905462832N^7 + 4489294448546432N^6$$
$$- 7165699950704832N^5 - 169724096918379392N^4 + 1156114211550506752N^3$$
$$- 1776295972609390592N^2 - 817567887178997776N + 1606525147689615360$$
$$:= \mathcal{B}_{6N}\tau_N + \mathcal{B}_{7N}.$$

In respect that $\mathcal{B}_{6N} > 0$ and $17000\mathcal{B}_{6N} + \mathcal{B}_{7N} > 0$ for $N \geq 25$, we obtain that $J_1(1)|_{\tau_N} > 0$.

As for $J_2(1)$, we can check by Mathematica that

$$\frac{8(N-22)(N-20)(N-18)(N-16)(N-2)(N+6)(N+8)}{N} J_2(1)|_{\tau_N}$$
$$= (7579N^{11} - 964244N^{10} + 53478216N^9 - 1692617504N^8 + 33572103936N^7$$
$$- 430516485888N^6 + 3541187751296N^5 - 17772345888256N^4$$
$$+ 47908037694720N^3 - 45669203442688N^2 - 34070434455552N$$
$$+ 62255634186240)\tau_N$$
$$- 73690617N^{11} + 8757708965N^{10} - 439144014720N^9$$
$$+ 11871532517728N^8 - 179607080990760N^7 + 1303031622196560N^6$$
$$+ 730732776846400N^5 - 81834334806699648N^4 + 487269552686137472N^3$$
$$- 729001679303608320N^2 - 294593226517125120N + 897917009560289280$$
$$:= \mathcal{B}_{8N}\tau_N + \mathcal{B}_{9N}.$$

Since $\mathcal{B}_{8N} > 0$ and $17000\mathcal{B}_{8N} + \mathcal{B}_{9N} > 0$ when $N \geq 25$, it holds that $J_2(1)|_{\tau_N} > 0$.

Finally, direct calculation gives that

$$
\begin{aligned}
&-\frac{(N-24)(N-22)(N-20)(N+4)(N+6)(N+8)(N+10)(N+12)}{N-4} I(1) \\
= &\,(N^{12} - 144N^{11} + 9112N^{10} - 332800N^9 + 7744896N^8 - 119529600N^7 \\
&+ 1232785280N^6 - 8335672320N^5 + 35049651968N^4 - 81419810816N^3 \\
&+ 72565198848N^2 + 41687285760N - 81749606400)\tau^2 \\
&+ (-19446N^{12} + 2634424N^{11} - 153791984N^{10} + 5031783008N^9 \\
&- 100052361056N^8 + 1212544446208N^7 - 8182468308224N^6 \\
&+ 18630407205888N^5 + 108938164472832N^4 - 719625359429632N^3 \\
&+ 817980109406208N^2 + 1176909695483904N - 1816259033825280)\tau \\
&+ 94536729N^{12} - 11656568130N^{11} + 593919303092N^{10} - 15702815300040N^9 \\
&+ 211574113757984N^8 - 815960309177792N^7 - 1386645015197248 0N^6 \\
&+ 171265848827756032N^5 - 341975328615126528N^4 \\
&- 279942886292011 2128N^3 + 7040619868808921088N^2 \\
&- 6076546479211806 72N - 6549150991381954560 \\
:= &\,\mathcal{B}_{10}\tau^2 + \mathcal{B}_{11}\tau + \mathcal{B}_{12}.
\end{aligned}
$$

Because $\mathcal{B}_{11}^2 - 4\mathcal{B}_{10}\mathcal{B}_{12} < 0$ and $\mathcal{B}_{10} > 0$ for $N \geq 25$, $I(1)$ then must be negative.

The proof is complete. □

Similar to the proof of Proposition 10.5, we obtain the following

Proposition 11.6. *Let $N \geq 25$. For τ_N chosen in Lemma 11.5, the function $F(\xi', \lambda')$ has a strict local minimum at $(0, 1)$.*

12 Proof of the main theorem

In this section we prove the main result of this paper by a gluing method.

Proposition 12.1. *Assume $N \geq 25$. Moreover, let g be a smooth metric on $\mathbb{R}^N$ of the form $g(x) = e^{h(x)}$, where $h(x)$ is a trace-free symmetric two-tensor on $\mathbb{R}^N$ such that*

$$|h(x)| + |\partial h(x)| + |\partial^2 h(x)| + |\partial^3 h(x)| + |\partial^4 h(x)| \leq \alpha$$

for all $x \in \mathbb{R}^N$, $h(0) = 0$, $h(x) = 0$ for $|x| \geq 1$, and

$$h_{ik}(x) = \mu \varepsilon^8 f(\varepsilon^{-2}|x|^2) H_{ik}(x)$$

for $|x| \le \rho$. If α and $\rho^{4-n}\mu^{-2}\varepsilon^{N-24}$ are sufficiently small, then there exists a positive solution $u(x)$ to

$$P_g u = \frac{N-4}{2} u^{\frac{N+4}{N-4}} \qquad \text{in } \mathbb{R}^N,$$
$$\int_{\mathbb{R}^N} u^{\frac{2N}{N-4}} < \int_{\mathbb{R}^N} \left(\frac{1}{1+|y|^2}\right)^N$$
$$\sup_{|x|\le\varepsilon} u \ge C\varepsilon^{\frac{4-N}{2}}.$$

Proof. By Proposition 11.6, the function $F(\xi', \lambda')$ has a strict local minimum at $(0,1)$ and $F(0,1) < 0$. Hence, we can find an open set $\mathcal{M} \subset \Lambda$ such that $(0,1) \in \mathcal{M}$ and

$$F(0,1) < \inf_{\partial\mathcal{M}} F(\xi', \lambda') < 0.$$

Using Lemma 8.2, we obtain

$$2\mathcal{F}_{\tilde{g}}(\xi', \lambda') = 2E + \mu^2\varepsilon^{20}F(\xi', \lambda') + O\left(\mu^3\varepsilon^{\frac{20N}{N-1}} + \mu^{\frac{20N}{N-4}}\varepsilon^{\frac{20N}{N-4}}\right) + O(\alpha(\frac{\varepsilon}{\rho})^{N-4}).$$

Hence, if $\rho^{4-n}\mu^{-2}\varepsilon^{N-24}$ is sufficiently small, we have

$$\mathcal{F}_{\tilde{g}}(0,1) < \inf_{\partial\mathcal{M}} \mathcal{F}_{\tilde{g}}(\xi', \lambda') < E.$$

Consequently, there exists a point $(\bar{\xi}', \bar{\lambda}') \in \mathcal{M}$ such that

$$\mathcal{F}_{\tilde{g}}(\bar{\xi}', \bar{\lambda}') < \inf_{\partial\mathcal{M}} \mathcal{F}_{\tilde{g}}(\xi', \lambda') < E.$$

Then $\tilde{u}_0(y) + \phi(y)$ is a solution of (8) with $\|\phi\|_* \le C$.

Note that since $\|\phi\|_* \le C$ we have

$$|\phi(y)| \le C\frac{\alpha}{(| + |y-\xi'|)^{N-4}} \le C\alpha u_0 \tag{55}$$

which shows that $u_0 + \phi > 0$ provided α is small. Thus $u_0(x) + \varepsilon^{-\frac{N-4}{2}}\phi(x/\varepsilon)$ is the positive solution we need. □

Proposition 12.2. *Let $N \ge 25$. Then there exists a smooth metric g on $\mathbb{R}^N$ with the following properties:*

1) $g_{ij}(x) = \delta_{ij}$ for $|x| \ge \frac{1}{2}$,

2) g is not conformally flat,

3) There exists a sequence of positive function u_n $(n \in \mathbb{R}^N)$ such that

$$P_g u_n = \frac{N-4}{2} u_n^{\frac{N+4}{N-4}},$$
$$\int_{\mathbb{R}^N} u_n^{\frac{2N}{N-4}} < \int_{\mathbb{R}^N} \left(\frac{1}{1+|y|^2}\right)^N,$$
$$\sup_{|x|\le 1} u_n \to \infty.$$

Proof. Choose a smooth cut-off function η such that $\eta(r) = 1$ for $r \le 1$ and $\eta(r) = 0$ for $r \ge 2$. We define a trace-free symmetric two-tensor on $\mathbb{R}^N$ by

$$h_{ij}(x) = \sum_{n=N_0}^{\infty} \eta(4n^2|x-x_n|) 2^{-\frac{25}{3}n} f(2^{2n}|x-x_n|) H_{ij}(x-x_n),$$

where $x_n = (\frac{1}{n}, 0, \ldots, 0)$. Clearly $h(x)$ is C^∞.

Moreover, if N_0 is sufficiently large, then we have $h(x) = 0$ for $|x| \ge \frac{1}{2}$ and $|h|+|\partial h|+|\partial^2 h|+|\partial^3 h|+|\partial^4 h| \le \alpha$. Provided that $n \ge N_0$ and $|x-x_n| \le \frac{1}{4n^2}$, we have

$$h_{ij}(x) = 2^{-\frac{25}{3}n} f(2^{2n}|x-x_n|) H_{ij}(x-x_n).$$

Hence we can apply Proposition 12.1 with $\mu = 2^{-n/3}$, $\varepsilon = 2^{-n}$, $\rho = \frac{1}{4n^2}$. From this the assertion follows. □

13 Appendix

In this section we will give the proof of (23) and (26). The proof of (23) can be found in [18] and we repeat it here for the sake of convience.

Lemma 13.1. *Assume that $0 < s < N$ and $t > s$. Then*

$$\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-s}} \frac{1}{(1+|y|)^t} \mathrm{d}y \le \begin{cases} C(1+|x|)^{s-t} & \text{if } t < N, \\ C(1+|x|)^{s-N}\Big[1+\log(1+|x|)\Big] & \text{if } t = N, \\ C(1+|x|)^{s-N} & \text{if } t > N. \end{cases}$$

Proof. First, observe that the above integral is well defined since $t > s$. So we only need to consider the case that $|x|$ is large. Next we decompose it as follows:

$$\left(\int_{|y-x|\le\frac{|x|}{2}} + \int_{\frac{|x|}{2}\le|y-x|\le 2|x|} + \int_{|y-x|\ge 2|x|}\right) \frac{1}{|x-y|^{N-s}} \frac{1}{(1+|y|)^t} \mathrm{d}y$$

$$:= I_1 + I_2 + I_3.$$

I_1 may be estimated as follows. Since $|y-x| \le |x|/2$ implies $|y| \ge |x|/2$,

$$\begin{aligned} I_1 &\le \int_{|y-x|\le \frac{|x|}{2}} \frac{1}{|x-y|^{N-s}} \frac{1}{(1+|x|/2)^t} \mathrm{d}y \\ &\le \frac{1}{(1+|x|/2)^t} \int_0^{|x|/2} \frac{1}{r^{N-s}} r^{N-1} \mathrm{d}r \\ &\le C|x|^{s-t}. \end{aligned}$$

I_3 may be estimated similarly. Because $|y-x| \ge 2|x|$, $|y-x| \le |y| + |x| \le |y| + |y-x|/2$. Thus $|y-x| \le 2|y|$ and

$$\begin{aligned} I_3 &\le \int_{|y-x|\ge 2|x|} \frac{1}{|x-y|^{N-s}} \frac{1}{(1+|x-y|/2)^t} \mathrm{d}y \\ &\le \int_{2|x|}^{\infty} \frac{1}{r^{N-s}} \frac{1}{(1+r/2)^t} r^{N-1} \mathrm{d}r \\ &\le C|x|^{s-t}. \end{aligned}$$

Finally, we observe that

$$\begin{aligned} I_2 &\le \frac{C}{|x|^{N-s}} \int_{\frac{|x|}{2}\le|y-x|\le 2|x|} \frac{1}{(1+|y|)^t} \mathrm{d}y \\ &\le \frac{C}{|x|^{N-s}} \left(\int_{|y|\le 1} + \int_{1\le|y|\le 3|x|} \right) \frac{1}{(1+|y|)^t} \mathrm{d}y \\ &\le C \frac{C}{|x|^{N-s}} \left(C + C \int_1^{3|x|} r^{N-t-1} \mathrm{d}r \right) \\ &\le \begin{cases} C|x|^{s-t} & \text{if } t < N, \\ C|x|^{s-N} \log|x| & \text{if } t = N, \\ C|x|^{s-N} & \text{if } t > N. \end{cases} \end{aligned}$$

Now it is easily seen that the lemma holds. □

Next we come to the proof of (26).

Proof of (26). Now $t = N-k$. Let $L > 0$ is a large number. If $|y| \le Lr$, we have

$$\int_{B_r} \frac{1}{|y-z|^{N-s}} \frac{1}{(1+|z|)^{N-k}} \mathrm{d}z$$

$$\begin{aligned} &\le \int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-s}} \frac{1}{(1+|z|)^{N-k}} \mathrm{d}z \\ &\le C(1+|y|)^{s+k-N} \le Cr^k(1+|y|)^{s-N}. \end{aligned}$$

For $|y| \ge Lr$, obvious $\frac{(1+|y|)^{N-s}}{|y-z|^{N-s}} \le C$ since $|z| \le r$. Thus, recalling that $k > 0$, we get

$$\begin{aligned} &\int_{B_r} \frac{1}{|y-z|^{N-s}} \frac{1}{(1+|z|)^{N-k}} \mathrm{d}z \\ &\le C(1+|y|)^{s-N} \int_{B_r} \frac{1}{(1+|z|)^{N-k}} \mathrm{d}z \\ &\le Cr^k(1+|y|)^{s-N}. \end{aligned}$$

This concludes the proof. □

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