QUALITATIVE PROPERTIES OF ENTIRE RADIAL SOLUTIONS FOR A BIHARMONIC EQUATION WITH SUPCRITICAL NONLINEARITY

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Abstract. We study some qualitative properties of entire positive radial solutions of the supercritical semilinear biharmonic equation:

$$(*) \qquad \Delta^2 u = u^p \text{ in } \mathbb{R}^n, \ \ n \geq 5, \ \ p > \frac{n+4}{n-4}.$$

It is known from [2] that there is a critical value $p_c > (n+4)/(n-4)$ of (*) for $n \geq 13$ and (*) has a singular solution $u_s(r) = K_0^{1/(p-1)} r^{-4/(p-1)}$. We show that for $5 \leq n \leq 12$ or $n \geq 13$ and $p < p_c$, any regular positive radial entire solution u of (*) intersects with $u_s(r)$ infinitely many times. On the other hand, if $n \geq 13$ and $p \geq p_c$ then $u(r) < u_s(r)$ for all $r > 0$. Moreover, the solutions are strictly ordered with respect to the initial value $a = u(0)$.

1. Introduction

We consider some qualitative properties of entire positive radial solutions to the following supercritical biharmonic equation

$$(1.1) \qquad \Delta^2 u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n$$

where $n \geq 5$ and $p > \frac{n+4}{n-4}$.

The corresponding supercritical second order elliptic equation

$$(1.2) \qquad -\Delta u = u^p, \quad u > 0 \text{ in } \mathbb{R}^n$$

with $p > \frac{n+2}{n-2}$ and $n \geq 3$ was intensively studied. In particular, we mention the following theorem on the classification of positive radial entire solutions of (1.2).

Theorem 1.1. *(Wang [4], Gui-Ni-Wang [3].) Let $n \geq 3$ and assume that $p > \frac{n+2}{n-2}$. Then for any $a > 0$ the equation (1.2) admits a unique radial solution $u = u(r)$ such that $u(0) = a$ and $u(r) \to 0$ as $r \to +\infty$. The solution u satisfies $u'(r) < 0$ for all $r > 0$ and*

$$(1.3) \qquad \lim_{r \to +\infty} r^{2/(p-1)} u(r) = L := \left[\frac{2}{p-1}\left(n - 2 - \frac{2}{p-1}\right)\right]^{1/(p-1)}.$$

1991 *Mathematics Subject Classification.* Primary 35B45; Secondary 53C21 35J40.

Key words and phrases. Entire radial solutions, biharmonic equations, singular solution .

Moreover, if $n \leq 10$ or if $n \geq 11$ and

$$p < p^c := \frac{n^2 - 8n + 4 + 8\sqrt{n-1}}{(n-2)(n-10)} \tag{1.4}$$

then $u(r) - Lr^{-2/(p-1)}$ changes sign infinitely many times. If $n \geq 11$ and $p \geq p^c$ then $u(r) < Lr^{-2/(p-1)}$ for all $r > 0$ and the solutions are strictly ordered with respect to the initial value $a = u(0)$.

The main purpose of this paper is to establish a similar theorem for entire solutions of (1.1).

Let us recall some known results on (1.1). In a recent paper [2], Gazzola and Grunau studied the existence and uniqueness of entire radial solutions to (1.1). They found the corresponding critical exponent p_c for (1.1). To state their results, we first define p_c to be the unique value of $p > \frac{n+4}{n-4}$ such that

$$-(n-4)(n^3 - 4n^2 - 128n + 256)(p-1)^4 + 128(3n-8)(n-6)(p-1)^3$$

$$+256(n^2 - 18n + 52)(p-1)^2 - 2048(n-6)(p-1) + 4096 = 0. \tag{1.5}$$

It has been shown in [2] that such a p_c exists (and is unique) only when $n \geq 13$. Let

$$u_s(r) = K_0^{1/(p-1)} r^{-4/(p-1)} \tag{1.6}$$

where $K_0 = \frac{8}{(p-1)^4}[(n-2)(n-4)(p-1)^3 + 2(n^2 - 10n + 20)(p-1)^2 - 16(n-4)(p-1) + 32]$. It is easy to see that u_s is a singular solution to (1.1) in $\mathbb{R}^n \setminus \{0\}$.

The main results in [2] are the following theorem.

Theorem 1.2. *([2]) Let $n \geq 5$ and assume that $p > \frac{n+4}{n-4}$. Then for any $a > 0$ the equation (1.1) admits a unique radial positive solution $u = u(r)$ such that $u(0) = a$ and $u(r) \to 0$ as $r \to +\infty$. Moreover, u satisfies $u'(r) < 0$, $\Delta u(r) < 0$, $(\Delta u)'(r) > 0$ for all $r > 0$, and*

$$u(r) < \left(\frac{p+1}{2}\right)^{1/(p-1)} u_s(r) \text{ for all } r \geq 0, \tag{1.7}$$

$$\lim_{r \to +\infty} \frac{u(r)}{u_s(r)} = 1 \tag{1.8}$$

Furthermore, for all $n \geq 13$, $p > p_c$, $u(r) - u_s(r)$ does not change sign infinitely many times.

In this paper we completely characterize the asymptotic behavior of the radial entire solutions of (1.1). We have the following theorem:

Theorem 1.3. *Let $n \geq 5$ and assume that $p > \frac{n+4}{n-4}$. Then for any $a > 0$ the equation (1.1) admits a unique radial solution $u = u(r)$ such that $u(0) = a$ and $u(r) \to 0$ as $r \to +\infty$. The solution u satisfies $u'(r) < 0$ for all $r > 0$ and*

$$\lim_{r \to +\infty} r^{4/(p-1)} u(r) = K_0^{1/(p-1)}. \tag{1.9}$$

Moreover, if $n \leq 12$ or if $n \geq 13$ and $p < p_c$, where p_c is given by (1.5), then $u - K_0^{1/(p-1)} r^{-4/(p-1)}$ changes sign infinitely many times. If $n \geq 13$ and $p \geq p_c$ then $u(r) < K_0^{1/(p-1)} r^{-4/(p-1)}$ for all $r > 0$ and the solutions are strictly ordered with respect to the initial value $a = u(0)$. Namely, if $u_1(r)$ and $u_2(r)$ are two radial solutions of (1.1) with $u_1(0) < u_2(0)$, then $u_1(r) < u_2(r)$ for $r > 0$.

Since the existence and uniqueness of entire radial solution to (1.1) are already given by Theorem 1.2, we shall assume that u_a is the unique entire radial solution of (1.1) with $u_a(0) = a$. If there is no confusion, we drop the index a.

In the rest of the paper, we proceed to prove Theorems 1.2 and 1.3. In Section 2, we collect some important preliminaries. In Section 3, we prove Theorem 1.2 and in Section 4, we prove Theorem 1.3.

After the completion of this paper, we came across the paper Ferrero-Grunau-Karageordis [1] in which they proved the first part of Theorem 1.3, i.e., when $p < p_c$. Their method, based on dynamical system, is quite different from ours. Our method generalizes the Sturm-Liouville comparison theorems to fourth order equations. In fact, our method in this paper in the case of $p \geq p_c$ also gives a new and more direct proof even in the second order case (Theorem 1.1).

Acknowledgments: The research of the first author is supported by a grant of NSFC (10571022). The research of the second author is partially supported by an Earmarked Grant from RGC of Hong Kong. The second author thanks Professor Nassif Ghoussoub for many useful discussions.

2. Emden-Fowler Transformations, Eigenvalues and Some Preliminaries

As in [2], we use the Emden-Fowler transformation:

$$u(r) = r^{-\frac{4}{p-1}} v(t), \quad t = \log r \ (r > 0). \tag{2.1}$$

Therefore, after the change of (2.1), the equation in (1.1) may be rewritten as

$$v^{(4)}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = v^p(t), \quad t \in \mathbb{R} \tag{2.2}$$

where the coefficients are given in [pp. 911, [2]]. The characteristic polynomial (linearized at $K_0^{1/(p-1)}$) is

$$\nu \mapsto \nu^4 + K_3\nu^3 + K_2\nu^2 + K_1\nu + (1-p)K_0$$

and the eigenvalues are given by

$$\nu_1 = \frac{N_1 + \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)}, \quad \nu_2 = \frac{N_1 - \sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)},$$

$$\nu_3 = \frac{N_1 + \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)}, \quad \nu_4 = \frac{N_1 - \sqrt{N_2 - 4\sqrt{N_3}}}{2(p-1)}$$

where

$$N_1 := -(n-4)(p-1) + 8, \quad N_2 := (n^2 - 4n + 8)(p-1)^2,$$

$$N_3 := (9n-34)(n-2)(p-1)^4 + 8(3n-8)(n-6)(p-1)^3 + (16n^2 - 288n + 832)(p-1)^2 - 128(n-6)(p-1) + 256.$$

Let us also define

$$\tilde{\nu}_j = \nu_j - \frac{4}{p-1}, j = 1,2,3,4. \tag{2.3}$$

A direct computation shows that $r^{\tilde{\nu}_j}$ are the four fundamental solutions to

$$\Delta^2 \psi = p u_s^{p-1} \psi \tag{2.4}$$

Using Proposition 2 of [2] and direct verifications, we have the following proposition:

Proposition 2.1. *(i) For any $n \geq 5$ and $p > \frac{n+4}{n-4}$, we have*

$$\tilde{\nu}_2 < 2 - n < 0 < \tilde{\nu}_1 \tag{2.5}$$

(ii) For any $5 \leq n \leq 12$ or $n \geq 13$, $p < p_c$, we have $\tilde{\nu}_3, \tilde{\nu}_4 \notin \mathbb{R}$ and $Re(\tilde{\nu}_3) = Re(\tilde{\nu}_4) = \frac{4-n}{2} < 0$

(iii) For any $n \geq 13, p = p_c$, we have $\tilde{\nu}_3 = \tilde{\nu}_4 = \frac{4-n}{2}$

(iv) For any $n \geq 13, p > p_c$, then

$$4 - n < \tilde{\nu}_4 < \frac{4-n}{2} < \tilde{\nu}_3 < 0, \tilde{\nu}_3 + \tilde{\nu}_4 = 4 - n. \tag{2.6}$$

Let us also recall the following theorem

Theorem 2.2. *([2]) The following limits hold:*

$$\lim_{t \to +\infty} v(t) = K_0^{1/(p-1)}, \quad \lim_{t \to +\infty} v^{(k)}(t) = 0 \tag{2.7}$$

for any $k \geq 1$.

3. The case of $p < p_c$

In this section, we prove that for $p < p_c$, $u(r) - u_s(r)$ must have infinitely many intersections (and hence prove Theorem 1.2). This amounts to the study of the following linearized equation

$$\Delta^2 \phi = p u^{p-1} \phi, \quad \phi(r) \to 0 \text{ as } r \to +\infty. \tag{3.1}$$

First we have

Lemma 3.1. *(1) If $\phi(0) = 0$, then $\phi \equiv 0$.*
(2) If $\phi(0) = 1$. Then $\Delta\phi(0) < 0$.

Proof: (1) Suppose $\phi(0) = 0$ and $\Delta\phi(0) \neq 0$. We may assume that $\Delta\phi(0) > 0$. Since $\phi(0) = \phi'(0) = 0$, we may assume that $\phi(r) > 0$ for $r \in (0, R)$ and $\phi(R) = 0$. (R can be $+\infty$.) Then in $(0, R)$, $(\Delta\phi)' > 0$ and hence $\Delta\phi(r) > 0$ for $r \in (0, R)$. This implies that $\phi'(r) > 0$ and $\phi(r) > 0$ for $r \in (0, R]$. This contradicts with $\phi(R) = 0$.

(2) follows from the same arguments. □

As a consequence of (1) of Lemma 3.1, we have

Lemma 3.2. *The solution to (3.1) is given by*

$$\phi(r) = c(\frac{4}{p-1} u(r) + r u'(r)) \tag{3.2}$$

for some $c \neq 0$.

The following theorem gives the asymptotic behavior of u, which is of independent interest.

Theorem 3.3. *Let u be the unique solution of (1.1). Then we have for r large*

$$u(r) = (-K_0)^{-1/3} r^{-4/(p-1)} + M_1 r^{\alpha} \cos(\beta \ln r) + M_2 r^{\alpha} \sin(\beta \ln r) + O(r^{\alpha - \delta}) \tag{3.3}$$

where $\delta = \frac{\sqrt{N_2 + 4\sqrt{N_3}}}{2(p-1)} > 0$, $\nu_3 = \alpha + \frac{4}{p-1} + i\beta$ and $M_1^2 + M_2^2 \neq 0$. (Note that it is known from Proposition 2 of [2] *that $\alpha + \frac{4}{p-1} < 0$.)*

Proof: Using the Emden-Fowler transformation:

$$u(r) = r^{-\frac{4}{p-1}} v(t), \quad t = \log r \ (r > 0), \tag{3.4}$$

and letting $v(t) = (K_0)^{1/(p-1)} - h(t)$, we see that $h(t)$ satisfies

$$h^{(4)}(t) + K_3 h'''(t) + K_2 h''(t) + K_1 h'(t) + (1-p) K_0 h(t) + O(h^2) = 0, \quad t > 1. \tag{3.5}$$

Therefore in the leading order, we can write

$$h(t) = M_1 e^{(\alpha + \frac{4}{p-1})t} \cos \beta t + M_2 e^{(\alpha + \frac{4}{p-1})t} \sin \beta t + M_3 e^{\nu_2 t} + o(e^{\nu_2 t}) \tag{3.6}$$

(note that we have from Theorem 2.2 that $\lim_{t\to+\infty} h(t) = 0$). This then implies that as $r \to +\infty$,

$$\varphi(r) = M_1 r^{\alpha}\cos(\beta\ln r) + M_2 r^{\alpha}\sin(\beta\ln r) + M_3 r^{\tilde{\nu}_2} + o(r^{\tilde{\nu}_2}) \tag{3.7}$$

where $\varphi(r) = r^{-\frac{4}{p-1}}h(t) := u_s(r) - u(r)$.

We now show that $M_1^2 + M_2^2 \neq 0$.

Suppose now that $M_1 = M_2 = 0$. Then we have

$$\varphi \sim r^{2-n-\kappa} \quad \text{as } r \to +\infty \tag{3.8}$$

where $\kappa = -\tilde{\nu}_2 - (n-2) > 0$ by Proposition 2.1. Furthermore, $\varphi(r)$ has no zeroes for r large. We show that this is impossible. In fact, it is easy to see that φ must change sign in $(0, +\infty)$. Otherwise, we assume $\varphi(r) > 0$ for $r \geq 0$ (note that $u(r) < u_s(r)$ for r small). Then using the behavior of φ near ∞ and integrating the equation $\Delta^2\varphi = u_s^p(r) - u^p(r)$ over $\mathbb{R}^n$, we see that

$$\int_0^{\infty} (u_s^p(r) - u^p(r))r^{n-1}dr = 0$$

which contradicts with $\varphi = u_s - u > 0$. Here we need to use the fact that $p > (n+4)/(n-4)$.

Suppose $\varphi(r)$ has exactly k zeroes in $(0, +\infty)$ (recalling that φ has no zeroes when r is large) and $\varphi(r) \sim r^{2-n-\kappa}$ as $r \to +\infty$, we easily see that $r^{n-1}\varphi'(r)$ has at least k zeroes. On the other hand, since the function $\eta(r) := r^{n-1}\varphi'(r)$ satisfies $\eta(0) = 0$ (note $p > (n+2)/(n-2)$) and $\eta(r) \to 0$ as $r \to +\infty$, we see that $\eta'(r)$ has at least $k+1$ zeroes. Thus $\Delta\varphi(r) = \frac{1}{r^{n-1}}\eta'(r)$ has at least $k+1$ zeroes. Similar idea implies that $r^{n-1}(\Delta\varphi)'(r)$ has at least k zeroes and $(r^{n-1}(\Delta\varphi)'(r))'$ has at least $k+1$ zeroes. Therefore, $\Delta^2\varphi = \frac{1}{r^{n-1}}(r^{n-1}(\Delta\varphi)'(r))'$ has at least $k+1$ zeroes. This contradicts our assumption that φ has k zeroes, since $\Delta^2\varphi = p\xi^{p-1}\varphi$, where $\xi(r) \in (\min\{u(r), u_s(r)\}, \max\{u(r), u_s(r)\}) > 0$ for all $r > 0$. This proves our claim and completes the proof of Theorem 3.3. □

4. The case of $p \geq p_c$

In this section, we consider the case $p \geq p_c$. We prove the following two theorems:

Theorem 4.1. *For $p \geq p_c$, we have $u(r) < u_s(r)$ for $r > 0$.*

Theorem 4.2. *For $p \geq p_c$, the solution to (3.1) remains of constant sign, that is,*

$$\frac{4}{p-1}u + ru'(r) < 0. \tag{4.1}$$

The proofs of both theorems depend on the use of comparison principle for fourth order equations.

We prove Theorem 4.2 first.

Proof of Theorem 4.2: Assume that Theorem 4.1 holds, i.e., $u(r) < u_s(r)$. Let $\phi(r)$ be a solution of (3.1). By Lemma 3.1, we may assume that $\phi(0) = 1$, $\Delta\phi(0) < 0$. Let $\psi(r) = r^{\tilde{\nu}_4}$. Then it is easy to see that

$$\Delta^2\psi = pu_s^{p-1}\psi \tag{4.2}$$

By Proposition 2.1, we have $\tilde{\nu}_4 > 4 - n$. This implies that $\int_{B_r(0)} r^{-4}|\phi|\psi < +\infty$. So we can multiply (3.1) by ψ and (4.2) by ϕ and integrate over $B_r(0)$ to obtain

$$\begin{aligned} 0 &= \int_{B_r(0)} p(u_s^{p-1} - u^{p-1})\phi\psi + \int_{\partial B_r(0)} [(\Delta\phi)'\psi - \Delta\phi\psi'] + \int_{\partial B_r(0)} [\Delta\psi\phi' - (\Delta\psi)'\phi] \\ &= I_1(r) + I_2(r) + I_3(r) \end{aligned} \tag{4.3}$$

where $I_i(r)$ are defined at the last equality.

Let us assume that there exist $r_1, r_2 \in (0, +\infty]$ such that

$$\phi(r) > 0, r \in (0, r_1), \phi(r_1) = 0, \quad \Delta\phi(r) < 0,\ r \in (0, r_2),\ \Delta\phi(r_2) = 0 \tag{4.4}$$

We divide our proof into three cases:

Case 1: $r_1 = r_2$.

In this case, we take $r = r_1 = r_2$. Then we have $I_1(r) > 0, I_2(r) \geq 0, I_3(r) \geq 0$. The identity (4.3) gives a contradiction.

Case 2: $r_2 < r_1$.

In this case, we take $r = r_2$. Then it is easy to see that $I_1(r_2) \geq 0$, $I_2(r_2) = \int_{\partial B_{r_2}(0)} [(\Delta\phi)'\psi - \Delta\phi\psi'] \geq 0$. It remains to estimate $I_3(r_2)$.

To this end, we first show that $\Delta\phi > 0$ for $r \in (r_2, r_1)$. In fact, since $\Delta^2\phi = pu^{p-1}\phi > 0$ in $(0, r_1)$, we see that $\Delta\phi$ must be positive for $r > r_2$ and near r_2. Suppose that there exists $r_3 \leq r_1$ such that $\Delta\phi(r_3) = 0$. Then we have $\Delta\phi > 0, \Delta(\Delta\phi) > 0$ in (r_2, r_3). This is impossible (since $\Delta\phi$ must attain its maximum in (r_2, r_3) where $\Delta(\Delta\phi) \leq 0$).

Now we consider the function $\Phi(r) = r^{n-1}(\Delta\psi\phi' - (\Delta\psi)'\phi)$. Its derivative is given by

$$\begin{aligned} \Phi'(r) &= (r^{n-1}\phi'(r))'\Delta\psi(r) - (r^{n-1}(\Delta\psi)'(r))'\phi(r) \\ &= r^{1-n}[\Delta\phi(r)\Delta\psi(r) - \phi(r)\Delta^2\psi(r)] < 0 \quad \text{for } r \in (r_2, r_1). \end{aligned}$$

(Here we have used the fact that $\Delta\psi < 0$.) So $\Phi(r_2) > \Phi(r_1) = r_1^{n-1}\Delta\psi(r_1)\phi'(r_1) \geq 0$. As a consequence, we have proved that $I_3(r_2) = r_2^{1-n}\int_{\partial B_{r_2}(0)} \Phi(r_2) \geq 0$. So again, we have $I_1(r_2) > 0$, $I_2(r_2) \geq 0$, $I_3(r_2) \geq 0$ and this gives a contradiction to the identity (4.3).

Case 3: $r_1 < r_2$.

The proof is similar to Case 2. In this case, we take $r = r_1$. Then it is easy to see that $I_1(r_1) \geq 0$, $I_3(r_1) = \int_{\partial B_{r_1}(0)} [\Delta \psi \phi'] \geq 0$. It remains to estimate $I_2(r_1)$.

As before, we first show that $\phi(r) < 0$ for $r \in (r_1, r_2)$. In fact, since $\Delta\phi < 0$ in $(0, r_2)$, we see that ϕ must be negative for $r > r_1$ and near r_1. Suppose that there exists $r_3 \leq r_2$ such that $\phi(r_3) = 0$. Then we have $\Delta\phi < 0$, $\phi < 0$ in (r_1, r_3). This is impossible (since ϕ must attain its minimum in (r_3, r_2) where $\Delta\phi \geq 0$).

Now we consider the function $\Psi(r) = r^{n-1}((\Delta\phi)'\psi - \Delta\phi\psi')$. Its derivative is given by

$$\begin{aligned}\Psi'(r) &= (r^{n-1}(\Delta\phi)'(r))'\psi(r) - (r^{n-1}\psi'(r))'\Delta\phi(r)\\ &= r^{1-n}[\Delta^2\phi(r)\psi(r) - \Delta\phi(r)\Delta\psi(r)] < 0 \text{ for } r \in (r_1, r_2).\end{aligned}$$

So $\Psi(r_1) > \Psi(r_2) = r_2^{n-1}(\Delta\phi)'(r_2)\psi(r_2) \geq 0$. As a consequence, we have proved that $I_2(r_1) = r_1^{1-n} \int_{\partial B_{r_1}(0)} \Psi(r_1) \geq 0$. So again, we have $I_1(r_1) > 0$, $I_2(r_1) \geq 0$, $I_3(r_1) \geq 0$ and a contradiction to the identity (4.3). These contradictions imply that ϕ remains constant sign and this completes the proof. □

Proof of Theorem 4.1: The proof is similar to that of Theorem 4.2. Let $\phi_0 = u_s(r) - u(r)$. Then it is easy to see that ϕ_0 satisfies

$$\Delta^2\phi_0 = u_s^p - (u_s - \phi_0)^p \leq p u_s^{p-1}\phi_0, r > 0 \tag{4.5}$$

Now let $\psi_0(r) = r^{\tilde{\nu}_3}$. Then by Proposition 2.1, $\tilde{\nu}_3 \geq \frac{4-n}{2}$. Thus $\int_{B_R(0)} r^{-4}|\phi_0|\psi_0 \leq \int_{B_R(0)} r^{-4} r^{-4/(p-1)} r^{(4-n)/2} < +\infty$ since $p > \frac{n+4}{n-4}$. Thus the integral $u_s^{p-1}\phi_0\psi_0$ is integrable. Similar to (4.3), we have the following identity

$$\int_{\partial B_r(0)} [(\Delta\phi_0)'\psi_0 - \Delta\phi_0\psi_0'] + \int_{\partial B_r(0)} [\Delta\psi_0\phi_0' - (\Delta\psi_0)'\phi_0] \leq 0 \tag{4.6}$$

Now note that $\phi_0 > 0$, $\Delta\phi_0 < 0$ for r small. So we may assume (4.4). The case $r_1 = r_2$ is easy to be excluded. We just need to consider the case $r_2 < r_1$. To this end, we first show that $\Delta\phi_0 > 0$ for $r \in (r_2, r_1)$. In fact, since $\Delta^2\phi_0 = u_s^p - (u_s - \phi_0)^p > 0$ in $(0, r_1)$, we see that $\Delta\phi_0$ must be positive for $r > r_2$ and near r_2. Suppose that there exists $r_3 \leq r_1$ such that $\Delta\phi_0(r_3) = 0$. Then we have $\Delta\phi_0 > 0, \Delta(\Delta\phi_0) > 0$ in (r_2, r_3). This is impossible (since $\Delta\phi_0$ must attain its maximum in (r_2, r_3) where $\Delta(\Delta\phi_0) \leq 0$). The rest of the proof is exactly the same as before. We omit the details. □

Theorem 4.2 yields very important estimates on the asymptotic behavior of u.

Corollary 4.3. *(1). Assume that $p \geq p_c$. Then the set of of solutions $\{u_a(r)\}$ to (1.1) is well ordered. That is if $a > b$ then $u_a(r) > u_b(r)$ for all $r > 0$.*

(2). If $p > p_c$, then we have the following asymptotic expansion for u:

$$u(r) = K_0^{1/(p-1)} r^{-4/(p-1)} + M_1 r^{\tilde{\nu}_3} + O(r^{\max(2\tilde{\nu}_3, \tilde{\nu}_4)}) \tag{4.7}$$

where $M_1 \neq 0$. If $p = p_c$, then we have the following asymptotic expansion for u:

$$u(r) = K_0^{1/(p-1)} r^{-4/(p-1)} + (M_1 + M_2 \log r) r^{\frac{4-n}{2}} + O(r^{4-n}) \tag{4.8}$$

Proof of Corollary 4.3: For (1), we note that $\phi = \frac{\partial u_a}{\partial a}$ satisfies (3.1) with $\phi(0) = 1 > 0$. By Theorem 4.2, $\phi > 0$. Thus $u_a(r) > u_b(r)$ for $a > b$.

For (2), if $p > p_c$, we have

$$u(r) = K_0^{1/(p-1)} r^{-4/(p-1)} + M_1 r^{\tilde{\nu}_3} + M_2 r^{\tilde{\nu}_4} + O(r^{\max(2\tilde{\nu}_3, \tilde{\nu}_4)}). \tag{4.9}$$

If $M_1 = 0$, then

$$u(r) = K_0^{1/(p-1)} r^{-4/(p-1)} + O(r^{\tilde{\nu}_4}) \tag{4.10}$$

which implies that $\phi = O(r^{\tilde{\nu}_4})$. Now as in the proof of Theorem 4.2, we have

$$\int_0^\infty p(u_s^{p-1} - u^{p-1}) \phi r^{\tilde{\nu}_4} r^{n-1} dr = 0 \tag{4.11}$$

where the integral is finite because $2\tilde{\nu}_4 < 4 - n$. This is impossible since $\phi > 0$. So $M_1 \neq 0$.

When $p = p_c$, (4.8) follows from the fact that $\tilde{\nu}_3 = \tilde{\nu}_4 = \frac{4-n}{2}$. □

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