A NONLOAL EIGENVALUE PROBLEM AND THE STABILITY OF SPIKES FOR REACTION-DIFFUSION SYSTEMS WITH FRACTIONAL REACTION RATES

JUNCHENG WEI AND MATTHIAS WINTER

Abstract. We consider a nonlocal eigenvalue problem which arises in the study of stability of spike solutions for reaction-diffusion systems with fractional reaction rates such as the Sel'kov model, the Gray-Scott system, the hypercycle Eigen and Schuster, angiogenesis, and the generalized Gierer-Meinhardt system. We give some sufficient and explicit conditions for stability by studying the corresponding nonlocal eigenvalue problem in a new range of parameters.

1. Motivation: The Sel'kov Model

We consider a nonlocal eigenvalue problem which arises in the study of spike solutions for reaction-diffusion systems in many areas of applied science.

We begin by considering the so-called Sel'kov model [Sel'kov, 1968] which was derived by Sel'kov to describe the enzyme reaction of glycolysis. Starting from a simple kinetic scheme with substrate inhibition and product activation, Sel'kov derived this reaction-diffusion system according to the law of mass action and the law of mass conservation. The model and some modifications of it have also been used in the study of morphogenesis and population dynamics (see [Hunding & Sorensen, 1988] or [Murray, 1989], respectively). In its simplified and

1991 *Mathematics Subject Classification.* Primary 35B40, 35B45; Secondary 35J40.

Key words and phrases. Nonlocal Eigenvalue Problem, Stability, Spike Solution, Reaction-Diffusion Systems.

nondimensionalized form, the system in 1-D becomes

$$\begin{cases} u_t = D_u u'' + 1 - uv^p \text{ in } (-1,1), \\ v_t = D_v v'' - v + uv^p \text{ in } (-1,1), \\ u^{'}(\pm 1, t) = v^{'}(\pm, 1, t) = 0, \end{cases} \tag{1.1}$$

where $D_u, D_v > 0$ are the diffusion coefficients of u and v, respectively, and $p > 1$. We are particularly interested in the case $D_u >> 1$, $D_v = \epsilon^2 << 1$, when spike layer solutions of (1.1) exist.

We remark that the properties of stationary solutions of (1.1) in general domains have been studied in a number of papers. See [Davidson & Rynne, 2000], [Eilbeck & Furter, 1995], and the references therein.

A possible way to study (1.1) is to consider its *shadow system* first: Assume that $D_u \to +\infty$. Thus $u(x,t) \to \xi(t)$. Now integrating the first Eq. over $(-1,1)$, we (formally) obtain the so-called shadow system (similar ideas were also used in other reaction-diffusion systems, see for example [Ni, 1998], [Nishiura, 1982], [Wei 1999a], [Wei, 1999b]):

$$\begin{cases} \xi_t = 1 - \frac{1}{2}\xi \int_{-1}^{1} v^p(x)dx \text{ in } (-1,1), \\ v_t = D_v v'' - v + \xi v^p \text{ in } (-1,1), \\ v^{'}(\pm 1, t) = 0. \end{cases} \tag{1.2}$$

As $D_v = \epsilon^2 \to 0$, we (asymptotically) have a symmetric stationary solution of (1.2) which takes the following form

$$(v_\epsilon, \xi_\epsilon) \sim \left(\xi_\epsilon^{-\frac{1}{p-1}} w\left(\frac{x}{\epsilon}\right), \left(\frac{\epsilon \int_R w^p}{2}\right)^{p-1} \right), \tag{1.3}$$

where w is the unique solution of

$$w^{''} - w + w^p = 0,\ w^{'}(0) = 0,\ w(y) > 0,\ w(y) \to 0 \text{ as } |y| \to +\infty. \tag{1.4}$$

Note that w is a homoclinic orbit connecting zero with itself. The existence and uniqueness of the solution w of (1.4) follows from standard dynamical systems theory: Multiplication of (1.4) by w' and integration with respect to y shows that

$$\frac{1}{2}(w')^2 - \frac{1}{2}w^2 + \frac{1}{p+1}w^{p+1} = C. \tag{1.5}$$

For $|y| \to \infty$, we have $w(y) \to 0$ and $w^{'}(y) \to 0$; this implies C. For $y = 0$, we have $w^{'}(0) = 0$ and therefore $w(0) = \left(\frac{p+1}{2}\right)^{1/(p-1)}$. In fact, it is easy to see that $w(y)$ can be written explicitly as

$$w(y) = \left(\frac{p+1}{2}\right)^{1/(p-1)} \left(\cosh\left(\frac{p-1}{2}y\right)\right)^{-2/(p-1)}. \tag{1.6}$$

Such a solution has a spike at the center of the domain. We call it a spike-layer solution.

If we linearize the shadow system (1.2) with respect to the solutions given by (1.3), then we obtain the following linear system for $(\phi_\epsilon, \eta_\epsilon) \in H^1(-1,1) \times L^\infty(-1,1)$

$$-\frac{1}{2}\eta_\epsilon \int_{-1}^{1} v_\epsilon^p(x)\,dx - \frac{p}{2}\xi_\epsilon \int_{-1}^{1} v_\epsilon^{p-1}(x)\phi_\epsilon(x)\,dx = \lambda_\epsilon \eta_\epsilon, \tag{1.7}$$

$$\epsilon^2 \phi_\epsilon'' - \phi_\epsilon + \xi_\epsilon p v_\epsilon^{p-1}\phi_\epsilon + \eta_\epsilon v_\epsilon^p = \lambda_\epsilon \phi_\epsilon. \tag{1.8}$$

Substituting (1.7) into (1.8), we have the following nonlocal eigenvalue problem

$$\epsilon^2 \phi_\epsilon'' - \phi_\epsilon + \xi_\epsilon p v_\epsilon^{p-1}\phi_\epsilon - \frac{\frac{p}{2}\xi_\epsilon \int_{-1}^{1} v_\epsilon^{p-1}(x)\phi_\epsilon(x)\,dx}{\lambda_\epsilon + \frac{1}{2}\int_{-1}^{1} v_\epsilon^p(x)\,dx} v_\epsilon^p = \lambda_\epsilon \phi_\epsilon. \tag{1.9}$$

Now we scale the space variable as follows

$$x = \epsilon y$$

and assume that (a suitable extension to R of) $\phi_\epsilon(\epsilon y) \to \phi(y)$ in $H^2(R)$ and $\lambda_\epsilon \to \lambda$ as $\epsilon \to 0$. Then noting that

$$\xi_\epsilon v_\epsilon^{p-1}(x) \sim w^{p-1}\left(\frac{x}{\epsilon}\right), \quad \int_{-1}^{1} v_\epsilon^p(x)\,dx = \frac{2}{\xi_\epsilon} \sim \epsilon^{1-p} 2^p \left(\int_R w^p(y)\,dy\right)^{1-p}$$

in the limit $\epsilon \to 0$ we obtain the following nonlocal eigenvalue problem

$$\begin{cases} \phi'' - \phi + pw^{p-1}\phi - \gamma(p-1)\frac{\int_R w^{p-1}\phi}{\int_R w^p} w^p = \lambda\phi, \\ \phi \in H^2(R), \quad \lambda \in \mathcal{C}, \end{cases} \tag{1.10}$$

where $\gamma > 1$, $p > 1$ and $\gamma = \frac{p}{p-1}$. The limit $\epsilon \to 0$ in fact requires some detailed justification which is given in [Ni *et. al.*, 2002] and [Wei, 1999b] for the case of the Gierer-Meinhardt system.

We remark that since w is an even function, we may assume that ϕ is also an even function. From now on we shall work only with even functions unless otherwise stated.

2. Main Result

The following is the main result in this paper:

Theorem 1. *For $p > 1$ and $1 < \gamma \le \frac{p}{p-1}$, problem (1.10) is stable.*

Let us discuss what is new in the above theorem.

Problem (1.10) is a special case of the following general nonlocal eigenvalue problem in $R^N, N \ge 1$

$$\Delta\phi - \phi + pw^{p-1}\phi - \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r}w^p = \lambda\phi, \quad \phi \in H^2(R^N), \tag{2.11}$$

where

$$\gamma > 1,\, p > 1,\, r > 1,\, \lambda \in \mathcal{C}.$$

The case $r = 2$ or $r = p+1$ is studied in [Ni *et. al.*, 2002], [Wei, 1999a], [Wei, 1999b]. The Hopf bifurcation is studied in [Dancer, 2002]. In general, it is quite difficult to study the general r case. The reason is that problem (2.11) is not self-adjoint for $r \neq p + 1$ and therefore it may have complex eigenvalues.

In [Wei,1999a] and [Wei, 1999b], it is proved that for $r = 2$ and $1 < p < 1 + \frac{4}{N}$, problem (2.11) is stable for any $\gamma > 1$. Here we say (2.11) is stable if there exists a positive constant $c_1 > 0$ such that Re $(\lambda) < -c_1$ for any eigenvalue λ. It is unstable if there exists an eigenvalue λ with Re $(\lambda) > 0$. This implies that problem (1.10) is stable when $p = 2$ and $\gamma > 1$.

In [Wei & Zhang, 2001], some stability results of problem (2.11) are obtained in the case of $2 < r < p+1$ and $1 < p < 1+\frac{2r}{N}$. We will recall and make use of the results of [Wei & Zhang, 2001] in Section 4.

Theorem 1 is the first result on (1.10) which covers the full range $1 < p < +\infty$ and $1 < \gamma \le \frac{p}{p-1}$. This is important to all applications since the physically relevant parameters are within.

We remark that the lower bound $\gamma = 1$ in Theorem 1 is sharp. Like in [Wei, 2000a] it can be shown here that the solution is linearly unstable for $\gamma < 1$. Note also that for $\gamma = 1$ the function w is an eigenfunction to the eigenvalue zero. On the other hand, the upper bound $\gamma \le \frac{p}{p-1}$ can be relaxed to $\gamma \le \gamma_0(p)$ for some $\gamma_0(p) > \frac{p}{p-1}$. See remark after the proof of Theorem 1 in Section 4.

We conjecture that for $p > 5$ there is a $\gamma_h(p)$ such that for $\gamma < \gamma_h(p)$, problem (1.10) is stable and for $\gamma > \gamma_h(p)$, problem (1.10) becomes unstable. At such a point $\gamma_h(p)$, problem (1.10) has a Hopf bifurcation. Our numerical computations strongly suggest that this conjecture hold true.

From Theorem 1, we immediately deduce that the spike solution given by (1.3) is metastable to the shadow system (1.2) for all $p > 1$. For the definition of metastability and the proof of this statement, we refer to [Wei, 1999b].

We will sketch the proof of Theorem 1 in Sec. 4 and will provide the proofs of some technical lemmas in three appendices.

We remark that there is another approach in studying (1.10) in one dimension, that is by using hypergeometric functions to (formally) reduce (1.10) to algebraic Eqs. and then solving these with Mathematica or Maple, [Doelman *et. al.*, 1998], [Doelman *et. al.*, 2000a], [Doelman *et. al.*, 2000b],

Our results here are derived by rigorous proofs which combine functional analysis and hypergeometric functions.

3. Applications

In this Sec. we describe four other applications to which (1.10) is the main result to understand the stability of spike solutions.

We consider the following autocatalytic reaction

$$A + pB \to (p+1)B, \quad p \geq 1$$

with reaction rate

$$\alpha[A][B]^p, \quad \alpha > 0,$$

where $[A]$ denotes the concentration of A and the same of B; A is the substrate and B is the product as well as its own catalyst. We assume further a positive influx of the substrate A into the system and a decay of the product B. Then using the mass action law and the mass conservation law we obtain the generalized Gray-Scott system which in its nondimensionalized form can be stated as follows:

$$\begin{cases} a_t = D_A a'' + \mu_a(1-a) - ab^p \text{ in } (-1,1), \\ b_t = D_B b'' - \mu_b b + ab^p \text{ in } (-1,1), \\ a^{'}(\pm 1, t) = b^{'}(\pm, 1, t) = 0. \end{cases} \tag{3.12}$$

We note that for $p = 2$, this becomes the classical Gray-Scott model [Gray & Scott, 1983]. Similar to the previous arguments (for details, see [Wei 1999b]), we arrive at problem (1.10) with $\gamma \leq \frac{p}{p-1}$.

Our second application is the following hypercycle reaction-diffusion system with nonlinear rate:

$$\begin{cases} \frac{\partial X_i}{\partial t} = D_X \frac{\partial^2 X_i}{\partial x^2} - g_X X_i + M \sum_{j=1}^N k_{ij} X_i X_j^n, \quad i = 1, 2, \ldots, N, \quad x \in R, \\ \frac{\partial M}{\partial t} = D_M \frac{\partial^2 M}{\partial x^2} + k_M - g_M M - LM \sum_{i,j=1}^N k_{ij} X_i X_j^n, \quad x \in R, \end{cases} \tag{3.13}$$

where X_i denotes the concentration of the polymers, and M is the concentration of activated monomers. N is the number of different polymer species. The replication of each polymer X_i is catalysed by each X_j at a constant rate k_{ij}. Linear (non-catalytic) growth terms are neglected. The activated monomers are produced at a constant rate, k_M; g_X and g_M are decay rate constants. L is the number of monomers in each polymer, and D_X and D_M are constant diffusion coefficients. The exponent n is a positive number. We assume that the coefficients k_{ij} are represented by a hypercyclical $N \times N$ matrix,

$$(k_{ij}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & k_0 \\ k_0 & 0 & \cdots & 0 & 0 \\ 0 & k_0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & k_0 & 0 \end{pmatrix}_{N \times N}, \quad k_0 > 0.$$

The system (3.13) is very similar to the hypercycle of Eigen and Schuster modeling pre-biotic evolution [Eigen & Schuster, 1977], [Eigen & Schuster, 1978a], [Eigen & Schuster, 1978b]. The main difference is in (3.13) the equations for X_i are coupled in a nonlocal way by the equation for M, whereas Eigen and Schuster makee the simpler assumption that the sum of X_i is constant. The effect of this coupling is very similar in both cases: The concentrations X_i are bounded.

When $n \neq 1$, we call (3.13) a hypercycle system with nonlinear rate. The reason is the following: For each X_i, the kinetic reaction rate is given by

$$\Gamma_i = -g_X + M \sum_{j=1}^{N} k_{ij} X_j^n. \tag{3.14}$$

When $n = 1$, we have a linear growth rate for Γ_i and the system is called classical hypercycle system. If $n < 1$, the growth rate is sublinear, while if $n > 1$, the growth rate is superlinear.

In our paper [Wei & Winter, 2001b], (1.10) with $n = p - 1$ was derived from a system of nonlocal eigenvalue problems, which arose from (3.13) by taking the limit $\epsilon \to 0$.

The third application comes from the study of the onset of capillary formation initiating angenesis [Levine & Sleeman, 1997] and [Levine *et. al.*, 2001]. The following mathematical model was proposed in [Levine & Sleeman, 1997] and [Levine *et. al.*, 2001]:

$$\begin{cases} P_t = D_1(P(\log \frac{P}{W^a})_x)_x \quad \text{in } (-1,1), \\ W_t = D_2 W_{xx} - W + \frac{PW}{1+\gamma W} \quad \text{in } (-1,1), \end{cases} \tag{3.15}$$

where $a > 0, \gamma > 0$ are constant coefficients, and D_1 and D_2 are diffusion coefficients. (D_2 is set to be zero in [Levine & Sleeman, 1997] since D_2 is assumed to be small.) Here $P(x,t)$ denotes the particle density of a particular species and $W(x,t)$ is the concentration of the "active agent". The study of the stability of a spatially non-uniform steady-state of (3.15) can be reduced to (1.10). Here we take $p = a > 1$. See the recent work [Sleeman *et. al.*, 2002].

As a fourth application, we remark that problem (1.10) also arises in the stability analysis for the generalized Gierer-Meinhardt system modeling morphogenesis in living organisms.

$$\begin{cases} A_t = \epsilon^2 \Delta A - A + \frac{A^p}{H^q}, \quad A > 0 \quad \text{in } \Omega, \\ \tau H_t = D\Delta H - H + \frac{A^r}{H^s}, \quad H > 0 \quad \text{in } \Omega, \\ \frac{\partial A}{\partial \nu} = \frac{\partial H}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

where the exponents (p, q, r, s) satisfy the following conditions

$$p > 1, \quad q > 0, \quad r > 0, \quad s \geq 0, \quad \frac{qr}{(p-1)(s+1)} > 1.$$

For details see [Gierer & Meinhardt, 1972], [Iron et. al., 2001], [Ni, 1998], [Ni *et. al.*, 2002], [Wei, 1999a], [Wei, 2000b], [Wei & Winter, 1999], [Wei & Winter, 2001a], [Wei & Winter, 2002], and the references therein.

These four applications of the nonlocal eigenvalue problem (1.10) show that it is relevant for stability in various branches of applied science. Therefore it is important to understand the stability behaviour of (1.10).

4. Sketch of the Proof of Theorem 1

In this section we sketch the proof of our main result. To this end, we need to define some functions.

Let

$$D(r) := \frac{(p-1)\int_{R^N}\left((L_0^{-1}w^{r-1})w^{r-1}\right)\int_{R^N} w^2}{(\int_{R^N} w^r)^2} \tag{4.1}$$

where $L_0 = \Delta - 1 + pw^{p-1}$. (Note that L_0^{-1} exists in

$$H^2_{radial}(R^N) := \{u \in H^2(R^N) | u(x) = u(|x|)\})$$

and

$$F(r) = 1 - \frac{p-1}{2r}N. \tag{4.2}$$

We will make use of the following theorem due to [Wei & Zhang, 2001]:

Theorem A. *[Wei & Zhang, 2001] Suppose that there exists an interval $(r_1, r_2) \subset (1, +\infty)$ such that either $2 \in (r_1, r_2)$ or $p+1 \in (r_1, r_2)$, and for any $r \in (r_1, r_2)$, we have*

(i) $\gamma^2 D(r) - F(p+1) - 2\gamma(\gamma-1)F(r) + (\gamma-1)^2 F(2) < 0$,
(ii) $F(p+1) + \gamma F(r) - (\gamma-1)F(2) > 0$,
(iii) $\gamma^2 D(r) > (\gamma-2)^2 F(p+1) - \frac{(\gamma F(r) - (\gamma-2)F(p+1))^2}{F(p+1)-F(2)}$.

Then for any $r \in (r_1, r_2)$ and any nonzero eigenvalue λ of problem (2.11), we have $Re(\lambda) < -c_1 < 0$ for some $c_1 > 0$.

For the convenience of the reader, a proof of Theorem A is included in Appendix A.

Let return to our primal problem (1.10). We would like to apply Theorem A to problem (1.10). To this end, we need to know the sign of $D(r)$. In [Wei & Zhang, 2001], it is proved that if $2 < r < p+1$

and $1 < p < \frac{2r}{N}$, then $D(r) > 0$. A key observation of this paper is the following lemma which gives the explicit formula for $D(r)$ when $r = p$.

Lemma 2. *For $p > 1$ and $N = 1$, we have*

$$D(p) = \frac{(p-1)(p+1)^2}{p^2} B(\frac{2}{p-1}, \frac{1}{2})(B(\frac{1}{p-1}, \frac{1}{2}))^{-2}, \tag{4.3}$$

where $B(\alpha, \beta)$ is the usual Beta function, i.e. $B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$.

Lemma 2 will be proved in Appendix B.

With the help of Lemma 2, we also prove that

Lemma 3. *For all $1 < p < +\infty$ and all $\gamma > 1$, condition (i) of Theorem A is satisfied for $r = p$. Namely we have*

$$\gamma^2 D(p) - F(p+1) - 2\gamma(\gamma-1)F(p) + (\gamma-1)^2 F(2) < 0. \tag{4.4}$$

A proof of Lemma 3 will be given in Appendix C.

Let us now use Lemma 2 and 3 to finish the proof of Theorem 1.

Proof of Theorem 1.

Using Lemma (2) and (3), we verify the conditions of Theorem A.

By Lemma (3), condition (i) of Theorem A holds true for all $p > 1$ and $\gamma > 1$.

It is easy to see that condition (ii) of Theorem A is true for $r = p$ and any $\gamma > 1$ since $F(r) > 0$.

To finish the proof of Theorem 1, it remains to check condition (iii) of Theorem A. To this end, set

$$\rho(t) = t^2 F(p+1) - \frac{(tF(p+1) - F(p))^2}{F(p+1) - F(2)}, \; t = \frac{\gamma - 2}{\gamma} \in (-1, 1). \tag{4.5}$$

Note that

$$\rho^{'}(t) = \frac{2F(p+1)}{F(p+1) - F(2)}(F(p) - tF(2)).$$

We first consider the case $1 < p < 2$. To check condition (iii), we note that for $1 < \gamma \le \frac{p}{p-1}$ and $p < 2$, we have that $F(2) > 0$ and

$t = \frac{\gamma-2}{\gamma} \leq \frac{2-p}{p}$. There are two roots for $\rho(t) = 0$:

$$t_1 = \frac{2(p+1)}{p(5-p)}(1 - \frac{p-1}{\sqrt{2(p+3)}}), t_2 = \frac{2(p+1)}{p(5-p)}(1 + \frac{p-1}{\sqrt{2(p+3)}}). \tag{4.6}$$

We have $\rho(t) < 0$ for all $t < t_1$. But it is easy to check that for $1 < p < 2$

$$\frac{2-p}{p} < t_1.$$

So for all $1 < \gamma \leq \frac{p}{p-1}$, we have $\rho(t) \leq 0 < D(p)$. Therefore condition (iii) of Theorem A is satisfied.

Finally we consider the case $p > 2$.

For $2 < p \leq 5$ and $\gamma \leq \frac{p}{p-1}$, $t = \frac{\gamma-2}{\gamma} \leq \frac{2-p}{p} \leq 0$. Since $F(2) \geq 0$, $\rho(t) \leq \rho(0) \leq 0$ for $t \leq 0$. Condition (iii) is satisfied.

For $p > 5$, we have $F(2) < 0$. Let t_1, t_2 be the two roots for $\rho(t) = 0$. If $|t_2| \geq 1$ (i.e. $p \leq p_1 \sim 11.38$), then we have $\rho(t) \leq 0$ for $-1 < t \leq 0$. If $|t_2| < 1$, then the maximum of $\rho(t)$ in $[-1, 0]$ is $\rho(-1)$. Therefore we only need to show that

$$D(p) > \rho(-1) = F(p+1) - \frac{(F(p+1) + F(p))^2}{F(p+1) - F(2)}. \tag{4.7}$$

We write (4.7) as follows

$$\frac{(p+1)^2}{2\pi p^2} \frac{(B(\frac{1}{p-1} + \frac{1}{2}, \frac{1}{2}))^2}{B(\frac{2}{p-1} + \frac{1}{2}, \frac{1}{2})} > \frac{p+3}{2(p+1)} - \frac{(2p^2+5p+1)^2}{p^2(p+1)(p-1)^2}, \ p > p_1. \tag{4.8}$$

To show (4.8), we apply the following inequality due to Luke (page 18, [Luke, 1975]):

$$\frac{z+1}{2z^2+z+1} < \frac{2^{-2z}\pi}{B(z+\frac{1}{2}, \frac{1}{2})}$$

$$< \frac{(1-z)(2z^2+5z+1)}{(z+1)^2(2z+1)} + \frac{8z^2(z+2)^2}{(z+1)^2(2z+1)(2z^2+5z+5)}, \tag{4.9}$$

for $0 \leq z \leq 1$. From (4.9), it is easy to deduce that

$$2^{-2z}\pi < B(z+\frac{1}{2}, \frac{1}{2}) < 2^{-2z}\pi \frac{2z^2+z+1}{z+1}, \quad 0 \leq z \leq 1. \tag{4.10}$$

Applying (4.10), we obtain that

$$\frac{(p+1)^2}{2\pi p^2}\frac{(B(\frac{1}{p-1}+\frac{1}{2},\frac{1}{2}))^2}{B(\frac{2}{p-1}+\frac{1}{2},\frac{1}{2})} > \frac{(p+1)^2}{2p^2}\frac{p^2-1}{p^2+7}.$$

On the other hand, for $p > p_1$,

$$\frac{p+3}{2(p+1)} - \frac{(2p^2+5p+1)^2}{p^2(p+1)(p-1)^2} < \frac{p+3}{2(p+1)} - \frac{4}{p}.$$

It remains to show that

$$\frac{(p+1)^2}{2p^2}\frac{p^2-1}{p^2+7} > \frac{p+3}{2(p+1)} - \frac{4}{p} \quad \text{for } p > p_1,$$

which is an easy exercise.

In conclusion, we have shown that for $p > 1$ and $1 < \gamma \leq \frac{p}{p-1}$, all the conditions in Theorem A are satisfied. Theorem 1 follows Theorem A. □

Remark: In general, the condition that $\gamma \leq \frac{p}{p-1}$ may be relaxed to the following: $\gamma \leq \gamma_0(p)$ where γ_0 satisfies

$$\gamma_0^2 D(p) = (\gamma_0 - 2)^2 F(p+1) - \frac{(\gamma_0 F(p) - (\gamma_0 - 2)F(p+1))^2}{F(p+1) - F(2)}, \gamma_0 > 1.$$

Since $D(p)$ is explicitly known, $\gamma_0(p)$ can be computed explicitly. We remark that $\gamma_0(p) > 2$. Hence for all $p > 1$ and $1 < \gamma \leq \max(2, \frac{p}{p-1})$, problem (1.10) is stable.

5. Appendix A: Proof of Theorem A

In this appendix, we include a proof of Theorem A. The main idea is that we introduce a quadratic form which is positive definite at $r = p+1$ and $r = 2$. Then we use a continuation argument for r.

We first introduce a quadratic form.

To this end, let us suppose that (λ, ϕ) is a solution of (2.11) with $\lambda \neq 0$. Set $\lambda = \lambda_R + i\lambda_I$ and $\phi = \phi_R + i\phi_I$. Then we obtain two Eqs.

$$L_0\phi_R - (p-1)\gamma\frac{\int_{R^N} w^{r-1}\phi_R}{\int_{R^N} w^r} w^p = \lambda_R\phi_R - \lambda_I\phi_I, \tag{5.1}$$

$$L_0\phi_I - (p-1)\gamma\frac{\int_{R^N} w^{r-1}\phi_I}{\int_{R^N} w^r}w^p = \lambda_R\phi_I + \lambda_I\phi_R, \tag{5.2}$$

where $L_0 = \Delta - 1 + pw^{p-1}$.

Multiplying (5.1) by ϕ_R and (5.2) by ϕ_I and adding them up, we obtain

$$-\lambda_R\int_{R^N}(\phi_R^2+\phi_I^2) \tag{5.3}$$

$$= \int_{R^N}(|\nabla\phi_R|^2+\phi_R^2) - p\int_{R^N} w^{p-1}\phi_R^2 + \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi_R\int_{R^N} w^p\phi_R}{\int_{R^N} w^r}$$

$$+\int_{R^N}(|\nabla\phi_I|^2+\phi_I^2) - p\int_{R^N} w^{p-1}\phi_I^2 + \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi_I\int_{R^N} w^p\phi_I}{\int_{R^N} w^r}.$$

Multiplying (5.1) by w and (5.2) by w we obtain

$$(p-1)\int_{R^N} w^p\phi_R - \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi_R}{\int_{R^N} w^r}\int_{R^N} w^{p+1} = \lambda_R\int_{R^N} w\phi_R - \lambda_I\int_{R^N} w\phi_I, \tag{5.4}$$

$$(p-1)\int_{R^N} w^p\phi_I - \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi_I}{\int_{R^N} w^r}\int_{R^N} w^{p+1} = \lambda_R\int_{R^N} w\phi_I + \lambda_I\int_{R^N} w\phi_R. \tag{5.5}$$

For $t > -1$, let us set

$$I^t(\varphi) = \mathcal{L}_0(\varphi,\varphi) + \frac{(p-1)\gamma}{\int_{R^N} w^r}\int_{R^N} w^{r-1}\varphi\int_{R^N} w^p\varphi - t\frac{(p-1)}{\int_{R^N} w^2}\int_{R^N} w^p\varphi\int_{R^N} w\varphi$$

$$+t\frac{\gamma(p-1)}{\int_{R^N} w^r\int_{R^N} w^2}\int_{R^N} w^{p+1}\int_{R^N} w^{r-1}\varphi\int_{R^N} w\varphi, \tag{5.6}$$

where $\mathcal{L}_0$ is defined by

$$\mathcal{L}_0(u,v) = \int_{R^N}(\nabla u\nabla v + uv - pw^{p-1}uv). \tag{5.7}$$

From (5.3), (5.4) and (5.5) we obtain that

$$I^t(\phi_R) + I^t(\phi_I) = -\lambda_R\left[\int_{R^N}(|\phi_R|^2+|\phi_I|^2) + t\frac{(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2}{\int_{R^N} w^2}\right]. \tag{5.8}$$

To prove Theorem A, our main idea is to find a continuous function $t = t(r) > -1$ such that $I^{t(r)}$ is positive definite. This is achieved by the following lemma.

Lemma 4. *Suppose that for all $r \in (r_1, r_2)$ assumptions (i), (ii) and (iii) of Theorem A hold. Moreover either $2 \in (r_1, r_2)$ or $p+1 \in (r_1, r_2)$. Then there exists a continuous function $t = t(r) > -1, r \in (r_1, r_2)$ such that $I^{t(r)}(\varphi) > 0$ for any $\varphi \in H^1_{radial}(R^N), \varphi \not\equiv 0$.*

Proof: We first note that

$$I^t(\varphi) = ((-L^t)\varphi, \varphi)$$

where

$$L^t\varphi := L_0\varphi - \frac{(p-1)\gamma}{2\int_{R^N} w^r}\Big(w^p \int w^{r-1}\varphi + w^{r-1}\int_{R^N} w^p\varphi\Big)$$
$$+t\frac{p-1}{2\int w^2}\Big(w^p\int_{R^N} w\varphi + w\int_{R^N} w^p\varphi\Big)$$
$$-t\frac{(p-1)\gamma}{2\int_{R^N} w^r \int_{R^N} w^2}\int_{R^N} w^{p+1}\Big(w^{r-1}\int_{R^N} w\varphi + w\int_{R^N} w^{r-1}\varphi\Big).$$

Since L^t is self-adjoint, it is easy to see that I^t is positive definite if and only if L^t has only negative eigenvalues.

We now study the following zero eigenvalue problem for L^t on $L^2_{radial}(R^N)$:

$$L^t\varphi = 0, \varphi \in L^2_{radial}(R^N), \varphi \not\equiv 0. \tag{5.9}$$

It is easy to see that $\varphi \in H^2_{radial}(R^N)$. Since L_0 is invertible in $H^1_{radial}(R^N)$, we invert (5.9) and obtain that

$$\varphi = \left(\frac{\gamma\int_{R^N} w^{r-1}\varphi}{2\int_{R^N} w^r} - \frac{t\int_{R^N} w\varphi}{2\int_{R^N} w^2}\right)w \tag{5.10}$$
$$+\left(\frac{(p-1)\gamma\int_{R^N} w^p\varphi}{2\int_{R^N} w^r} + \frac{t\gamma(p-1)\int_{R^N} w^{p+1}\int_{R^N} w\varphi}{2\int_{R^N} w^r\int_{R^N} w^2}\right)L_0^{-1}w^{r-1}$$
$$+\left(\frac{t\gamma\int_{R^N} w^{p+1}\int_{R^N} w^{r-1}\varphi}{2\int_{R^N} w^r\int_{R^N} w^2} - \frac{t\int_{R^N} w^p\varphi}{2\int_{R^N} w^2}\right)(w + \frac{p-1}{2}x\cdot\nabla w).$$

Set $A = \int_{R^N} w\varphi,\ B = \int_{R^N} w^p\varphi,\ C = \int_{R^N} w^{r-1}\varphi$. Then we have

$$A = \frac{\gamma\int_{R^N} w^2}{2\int_{R^N} w^r}C - \frac{t}{2}A + \Big(\frac{\gamma}{2}B + \frac{\gamma t}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}A\Big)F(r)$$
$$+\Big(\frac{\gamma t\int_{R^N} w^{p+1}}{2\int_{R^N} w^r}C - \frac{t}{2}B\Big)F(2), \tag{5.11}$$

$$B = \frac{\gamma}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^r}C - \frac{t}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}A + \Big(\frac{\gamma}{2}B + \frac{\gamma t}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}A\Big)$$

$$+\Big(\frac{\gamma t\int_{R^N} w^{p+1}}{2\int_{R^N} w^r}C-\frac{t}{2}B\Big), \tag{5.12}$$

$$C=\frac{\gamma}{2}C-\frac{t}{2}\frac{\int_{R^N} w^r}{\int_{R^N} w^2}A+\Big(\frac{\gamma}{2}B+\frac{\gamma t}{2}\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^r}A\Big)\frac{(p-1)\int_{R^N}(L_0^{-1}w^{r-1})w^{r-1}}{\int_{R^N} w^r}$$

$$+\Big(\frac{rt\int_{R^N} w^{p+1}}{2\int_{R^N} w^2}C-\frac{t}{2}\frac{\int_{R^N} w^r}{\int_{R^N} w^2}B\Big)F(r). \tag{5.13}$$

Recall that $D(r):=\dfrac{(p-1)\int_{R^N}(L_0^{-1}w^{r-1})w^{r-1}\int_{R^N} w^2}{(\int_{R^N} w^r)^2}$.

Since $A^2+B^2+C^2\neq 0$ (otherwise, by (5.10), $\varphi\equiv 0$), we have by (5.11), (5.12) and (5.13) that

$$\begin{vmatrix}\gamma tF(r)-F(p+1)(t+2) & \gamma F(r)-tF(2) & \gamma F(p+1)+\gamma F(2)t\\ (\gamma-1)t & \gamma-2-t & \gamma+\gamma t\\ \big(\gamma D(r)-F(p+1)\big)t & \gamma D(r)-F(r)t & (\gamma-2)F(p+1)+\gamma F(r)t\end{vmatrix}=0.$$

That is

$$I_1(t):=\begin{vmatrix}\big(\gamma F(r)-F(p+1)\big)t-2F(p+1) & \gamma F(r)-tF(2) & \gamma F(p+1)+\gamma^2F(r)\\ (\gamma-1)t & \gamma-2-t & \gamma^2-\gamma\\ \big(\gamma D(r)-F(p+1)\big)t & \gamma D(r)-F(r)t & \gamma^2D+(\gamma-2)F(p+1)\end{vmatrix}=0.$$

It is easy to check that

$$I_1(0)=2F(p+1)\Big(\gamma^2D(r)-(\gamma-2)^2F(p+1)\Big),$$

$$I'_1(0)=4F(p+1)\Big(\gamma^2D(r)+(\gamma-2)F(p+1)-\gamma(\gamma-1)F(r)\Big),$$

$$I''_1(0)=4F(p+1)\Big(\gamma^2D(r)-F(p+1)-2\gamma(\gamma-1)F(r)+(\gamma-1)^2F(2)\Big).$$

Thus we obtain that L^t has a zero eigenvalue if and only if

$$I_1(t)=\frac{1}{2}I''_1(0)t^2+I_1'(0)t+I_1(0)=0.$$

Note that $I_1(-1)=2(\gamma-1)^2F(p+1)(F(2)-F(p+1))$. Assumption (i) implies that $I_1(t)$ is concave while assumption (ii) implies that the maximum point

$$t_{max}:=-\frac{I_1'(0)}{I_1''(0)}$$

is greater than -1. Finally simple computations show that

$$I_1(t_{max})=I_1(0)-\frac{(I_1'(0))^2}{2I_1''(0)}$$

$$= \frac{(4(\gamma-1)F(p+1))^2}{2I_1''(0)}$$

$$\times\left[(\gamma^2 D(r)-(\gamma-2)^2F(p+1))(F(2)-F(p+1))-(\gamma F(r)-(\gamma-2)F(p+1))^2\right] > 0$$

by assumption (iii).

Let (r_1, r_2) be defined in Theorem A. Without loss of generality, we may assume that $2 \in (r_1, r_2)$. Let us now choose

$$t(r) := t_{max} = -\frac{I_1'(0)}{I''_1(0)}. \tag{5.14}$$

Then $t(r) > -1$ and $I_1(t(r)) > 0$.

We first prove Lemma 4 for $r = 2$. We need to show that

$$I^{t(2)}(\varphi) > 0, \forall\ \varphi \in H^1_{radial}(R^N), \varphi \not\equiv 0. \tag{5.15}$$

To this end, we use a continuation argument. By Lemma 5.1 of [Wei, 1999b], if $F(2) > 0$, then $I^{\gamma-2}$ is positive definite which implies that $L^{\gamma-2}$ has no nonnegative eigenvalues. Moreover, when $r = 2$,

$$I_1(\gamma-2) = 8F(p+1)(\gamma-1)^2F(2) > 0 \tag{5.16}$$

and

$$t(2) = \frac{\gamma F(2) + (\gamma-2)F(p+1)}{F(p+1) - F(2)} > \gamma - 2. \tag{5.17}$$

Since $I_1(t)$ is concave, we have that $I_1(t) > 0$ for $t \in [\gamma-2, t(2)]$.

Let us now vary t. We claim that

$$I^t(\varphi) > 0, \forall t \in [\gamma-2, t_{max}], \text{ and } \ \varphi \in H^1_{radial}(R^N), \varphi \not\equiv 0. \tag{5.18}$$

In fact, suppose not. Then at some point $t = t_0 \in (\gamma-2, t_{max}]$, we must have that L^{t_0} has a zero eigenvalue, which implies that $I_1(t_0) = 0$. This is impossible.

So (5.15) is proved. Next we vary r. Assume that $r = r_0 > 1$ is the first value for which $I^{t(r)}(\phi) = 0$ and that r_0 satisfies assumptions (i)-(iii). Then at $r = r_0$, $L^{t(r_0)}$ must have a zero eigenvalue which implies that $I_1(t(r_0)) = 0$. This is in contradiction to the fact that $I_1(t(r_0)) > 0$. Thus we deduce that $I^{t(r)}(\varphi) > 0$ for any $\varphi \in H^1_r(R^N)$ and r satisfying the assumptions (i)-(iii).

Similarly we can prove the case when $p + 1 \in (r_1, r_2)$.

Lemma 4 is thus proved.

□

Finally, Theorem A follows directly from Lemma 4 and (5.8).

□

6. Appendix B: Computation of $D(p)$ and Proof of Lemma 2

In this appendix, we prove Lemma 2.

We use hypergeometric functions to compute $D(p)$. We just need to compute $\int_R w^{p-1} L_0^{-1} w^{p-1}$. Let $\phi_0 := L_0^{-1} w^{p-1}$ and $\alpha = \frac{p-2}{p-1}$. We first assume that

$$\alpha + \frac{1}{2} > 0. \tag{6.19}$$

Set

$$\phi_0 = wG.$$

Then G satisfies

$$G'' + 2\frac{w'}{w} G' + (p-1) w^{p-1} G = w^{p-2}.$$

Next we perform the following change of variables

$$z = \frac{2}{p+1} w^{p-1}. \tag{6.20}$$

(The transformation (6.20) has been used in [Doelman *et. al.*, 1999] [Doelman *et. al.*, 2001a] [Doelman *et. al.*, 2001b] .)

Note that due to the remarks after (1.4), z is a homeomorphism from $[0, +\infty]$ to $[0, 1]$.

By some lengthy computations, we obtain the following Eq. for $G(z)$:

$$z(1-z)G'' + (c - (a+b+1)z)G' - abG = \frac{1}{(p-1)^2} (\frac{p+1}{2})^{\alpha} z^{\alpha - 1} \tag{6.21}$$

where

$$a = \frac{p+1}{p-1}, b = -\frac{1}{2}, c = \frac{p+1}{p-1}. \tag{6.22}$$

To solve (6.21), we take a power series

$$G(z) = z^s \sum_{k=0}^{+\infty} c_k z^k$$

and substitute it into (6.21). We obtain that

$$\sum_{k=0}^{+\infty} c_k z^{s+k-1}(s+k)(s+k-1+c)$$

$$-\sum_{k=1}^{+\infty} c_k z^{s+k}(s+k+a)(s+k+b) = \frac{1}{(p-1)^2}(\frac{p+1}{2})^{\alpha} z^{\alpha-1}.$$

So

$$s-1=\alpha-1, \quad c_0 s(s-1+c) = \frac{1}{(p-1)^2}(\frac{p+1}{2})^{\alpha},$$

$$c_k(s+k)(s+k-1+c) = c_{k-1}(s+k-1+a)(s+k-1+b), \quad k=1,2,....$$

Hence we have

$$s=\alpha, \quad c_0 = \frac{1}{p(p-2)}(\frac{p+1}{2})^{\alpha}$$

and

$$c_k = \frac{\alpha+k-\frac{3}{2}}{\alpha+k} c_{k-1}, \quad k=1,2,....$$

Therefore we obtain

$$G(z) = \frac{1}{p(p-2)}(\frac{p+1}{2})^{\alpha} z^{\alpha}\left(1+\frac{\alpha-\frac{1}{2}}{\alpha+1}z+\frac{(\alpha-\frac{1}{2})(\alpha+\frac{1}{2})}{(\alpha+1)(\alpha+2)}z^2+...\right).$$

In terms of the so-called hypergeometric function (see [Erdelyi et. al., 1953]), we can write $G(z)$ as

$$G(z) = \frac{1}{p(p-2)}(\frac{p+1}{2})^{\alpha} z^{\alpha} F(1,\alpha-\frac{1}{2};\alpha+1;z). \tag{6.23}$$

So

$$\int_R w^{p-1}\phi_0 dy = 2\int_0^{+\infty} w^{p-1}\phi_0 dy$$

$$= \frac{p+1}{p-1}\int_0^1 w(1-z)^{-\frac{1}{2}} G(z)dz \text{ (by (6.20), (1.5))}$$

$$= \frac{(p+1)^2}{2p(p-1)(p-2)}\int_0^1 z(1-z)^{-\frac{1}{2}} F(1,\alpha-\frac{1}{2};\alpha+1;z)dz$$

$$= \frac{(p+1)^2}{2p(p-1)(p-2)}\left(B(2,\frac{1}{2})+\frac{\alpha-\frac{1}{2}}{\alpha+1}\int_0^1 z^2(1-z)^{-\frac{1}{2}} F(1,\alpha+\frac{1}{2};\alpha+2;z)dz\right)$$

$$= \frac{(p+1)^2}{2p(p-1)(p-2)}\left(B(2,\frac{1}{2})+\frac{\alpha-\frac{1}{2}}{\alpha+1}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+\frac{1}{2})}I\right) \tag{6.24}$$

where

$$I=\sum_{n=0}^{+\infty}\int_0^1 z^2(1-z)^{-\frac{1}{2}}\frac{\Gamma(\alpha+\frac{1}{2}+n)}{\Gamma(\alpha+2+n)}z^n dz. \qquad (6.25)$$

We can rewrite I as

$$I=\frac{1}{\Gamma(\frac{3}{2})}\sum_{n=0}^{+\infty}\int_0^1 z^2(1-z)^{-\frac{1}{2}}z^n\int_0^1 t^{\alpha+\frac{1}{2}+n-1}(1-t)^{\frac{1}{2}}dt$$

$$=\frac{1}{\Gamma(\frac{3}{2})}\int_0^1 t^{\alpha-\frac{1}{2}}(1-t)^{\frac{1}{2}}dt\int_0^1 z^2(1-z)^{-\frac{1}{2}}(1-tz)^{-1}dz$$

$$=\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}\int_0^1 t^{\alpha-\frac{1}{2}}(1-t)^{\frac{1}{2}}F(1,3;\frac{7}{2};t)dt.$$

Let us now compute $F(1,3;\frac{7}{2};t)$. First it is easy to see that

$$F(1,1;\frac{3}{2};(\sin z)^2)=\frac{z}{\sin z\cos z}. \qquad (6.26)$$

(See Page 101 of [Erdelyi *et. al.*, 1953].) On the other hand

$$F(1,1;\frac{3}{2};t)=1+\frac{2}{3}t+\frac{8}{15}t^2+\frac{8}{15}t^2(F(1,3;\frac{7}{2};t)-1). \qquad (6.27)$$

Thus

$$F(1,3;\frac{7}{2};t)=1+\frac{15}{8}t^{-2}\left(F(1,1;\frac{3}{2};t)-1-\frac{2}{3}t-\frac{8}{15}t^2\right).$$

Substituting that into I, we obtain that

$$I=\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}(\int_0^1 t^{\alpha-\frac{1}{2}}(1-t)^{\frac{1}{2}}$$

$$+\frac{15}{8}\int_0^1 t^{\alpha-\frac{1}{2}-2}(1-t)^{\frac{1}{2}}(F(1,1;\frac{3}{2};t)-1-\frac{2}{3}t-\frac{8}{15}t^2))dt$$

$$=\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}B(\alpha+\frac{1}{2},\frac{3}{2})$$

$$+\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}\Bigg(\frac{15}{4}\int_0^1(\sin z)^{2\alpha-5}(\cos z)$$

$$\times[z-\sin z\cos z-\frac{2}{3}(\sin z)^3\cos z-\frac{8}{15}(\sin z)^5\cos z]dz\Bigg)$$

$$=\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}B(\alpha+\frac{1}{2},\frac{3}{2})$$

$$+\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}\left(\frac{15}{4}(-\frac{4}{3(\alpha-2)}B(\alpha+\frac{1}{2},\frac{1}{2})-\frac{4}{15}B(\alpha+\frac{1}{2},\frac{3}{2})\right)$$

$$= -\frac{16}{3(\alpha-2)}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}.$$

So we have

$$\int_R w^{p-1}L_0^{-1}w^{p-1}$$

$$= \frac{(p+1)^2}{2p(p-1)(p-2)}\left(\frac{4}{3}+\frac{\alpha-\frac{1}{2}}{\alpha+1}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+\frac{1}{2})}(-\frac{16}{3(\alpha-2)}\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)}\right))$$

$$= \frac{2(p+1)^2}{p^2(p-1)}. \tag{6.28}$$

Next we note that

$$\int_R w^r = \frac{2}{p-1}(\frac{p+1}{2})^{\frac{r}{p-1}}\frac{\Gamma(\frac{r}{p-1})\Gamma(\frac{1}{2})}{\Gamma(\frac{r}{p-1}+\frac{1}{2})}.$$

Hence

$$\int_R w^p = \int_R w = \frac{2}{p-1}(\frac{p+1}{2})^{\frac{1}{p-1}}\frac{\Gamma(\frac{1}{p-1})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{p-1}+\frac{1}{2})} \tag{6.29}$$

and

$$\int_R w^2 = \frac{2}{p-1}(\frac{p+1}{2})^{\frac{2}{p-1}}\frac{\Gamma(\frac{2}{p-1})\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{p-1}+\frac{1}{2})}. \tag{6.30}$$

Thus

$$D(p) = (p-1)\int_R w^{p-1}L_0^{-1}w^{p-1}\frac{\int_R w^2}{(\int_R w^p)^2}$$

$$= \frac{(p-1)(p+1)^2}{p^2}B(\frac{2}{p-1},\frac{1}{2})(B(\frac{1}{p-1},\frac{1}{2}))^{-2},$$

which proves Lemma 2.

If $\alpha < -\frac{1}{2}$, we need to choose a positive integer k such that $\alpha + k > -\frac{1}{2}$. Then we expand as in (6.24) until the $k-$th term and then compute the sum. The proof is similar. We omit the details.

7. Appendix C: Proof of Lemma 3

In this appendix, we prove Lemma 3.

By Lemma 9 of [Wei & Zhang, 2001], condition (i) is true for $p \geq 2$ and $\gamma > 1$. So we only need to consider the case when $1 < p < 2$.

Let $s = \frac{1}{\gamma} \in (0, 1]$ and

$$\beta(s) = s^2F(p+1) + 2(1-s)F(p) - (1-s)^2F(2)$$

$$= s^2\Big(F(p+1) - F(2)\Big) + 2s(F(2) - F(p)) + 2F(2) - F(2).$$

Since

$$\beta'(0) = 2\Big(F(2) - F(p)\Big) \geq 0, \beta'(1) = 2(F(p+1) - F(p)) \geq 0,$$

$\beta(s)$ has a minimum at 0, i.e.

$$\beta(s) \geq \beta(0) = 2F(p) - F(2). \tag{7.31}$$

To prove (4.4), it is enough to show that

$$D(p) < 2F(p) - F(2). \tag{7.32}$$

We follow the idea in [Wei & Zhang, 2001]. Set

$$\mathcal{L}(u, v) = \int_R (u^{'} v^{'} + uv - pw^{p-1}uv). \tag{7.33}$$

We first claim that for $p > 1$, we have

$$\inf\left\{\mathcal{L}(\varphi,\varphi)\middle|\varphi \in H^1_{even}(R), \int_R \varphi^2 = 1 \text{ and } \int_R w^{p-1}\varphi = 0\right\} > 0. \tag{7.34}$$

Here $H^1_{even}(R)$ consists of functions in $H^1(R)$ which are even.

In fact this is true for $p \geq 2$ by Lemma 8 of [Wei & Zhang, 2001]. Suppose that there exists $p \in (1, 2)$ such that

$$\inf\left\{\mathcal{L}(\varphi,\varphi)\middle|\varphi \in H^1_{even}(R)\ , \int_R \varphi^2 = 1 \text{ and } \int_R w^{p-1}\varphi = 0\right\} = 0. \tag{7.35}$$

Then the function φ for which this infimum is attained satisfies

$$\varphi^{''} - \varphi + pw^{p-1}\varphi = c_1 w^{p-1} + c_2\varphi, \int_R w^{p-1}\varphi = 0, \int_R \varphi^2 = 1,$$

for some constants c_1, c_2.

Multiplying the last Eq. by φ and integrating, by (7.35) we have $c_2 = 0$. So $\varphi = c_1 L_0^{-1} w^{p-1}$. We note that $c_1 \neq 0$, otherwise $\varphi = cw^{'}(y)$ which is impossible. Thus $\int_R w^{p-1} L_0^{-1} w^{p-1} = 0$, which contradicts the fact that $D(p) > 0$.

Next we consider the following variational problem:

$$\inf\left\{\mathcal{L}(\varphi,\varphi)\middle|\varphi \in H^1_{even}(R), \int_R w^{p-1}\varphi = 1\right\}. \tag{7.36}$$

We claim that the infimum is attained by some function φ_0. In fact, if we put $\varphi = \frac{1}{\int_R w^p} w + \psi$, then $\int_R w^{p-1}\psi = 0$. Therefore by (7.34) there exists a $c_0 > 0$ such that

$$\int_R [|\psi^{'}|^2 + \psi^2 - pw^{p-1}\psi^2] \geq c_0 \int_R \psi^2.$$

Then by the direct method in the calculus of variations, we can easily show that there exists a φ_0 for which the infimum in (7.36) is attained and which satisfies

$$\varphi_0^{''} - \varphi_0 + pw^{p-1}\varphi_0 = \lambda w^{p-1}$$

where $\lambda = -\mathcal{L}(\varphi_0, \varphi_0)$. By uniqueness, $\varphi_0 = \lambda(L_0^{-1} w^{p-1})$ and thus

$$D(p) = \frac{(p-1)\int_R (L_0^{-1} w^{p-1} w^{p-1}) \int_R w^2}{(\int_R w^p)^2}$$

$$= \frac{(p-1)\int_R w^2}{(\int_R w^p)^2} \frac{1}{\lambda}. \tag{7.37}$$

We now choose some special test functions to compute a lower bound for λ. In fact, we take

$$\varphi = c\Big(\lambda_1 w + \lambda_2 (w + \frac{p-1}{2} y w^{'}(y))\Big),$$

where

$$c(\lambda_1 + \lambda_2 F(p)) \int_R w^p = 1, \tag{7.38}$$

and λ_1 and λ_2 are to be chosen later. It follows that

$$\int_R w^{p-1}\varphi = c\Big(\lambda_1 + \lambda_2 F(p)\Big) \int_R w^p = 1.$$

Let us compute

$$\mathcal{L}(\varphi,\varphi) = c^2\Big[\lambda_1^2\mathcal{L}(w,w) + \lambda_2^2\mathcal{L}(w+\frac{p-1}{2}yw^{'}(y),\ w+\frac{p-1}{2}yw^{'}(y))$$
$$+ 2\lambda_1\lambda_2\mathcal{L}(w, w+\frac{p-1}{2}yw^{'}(y))\Big]$$
$$= c^2\Big[\lambda_1^2(1-p)\int_R w^{p+1} + \lambda_2^2(1-p)F(2)\int_R w^2$$
$$+ 2\lambda_1\lambda_2(1-p)\int_R w^p(w+\frac{p-1}{2}yw^{'}(y))\Big]$$
$$= c^2\Big[\lambda_1^2(1-p)\frac{1}{F(p+1)} + \lambda_2^2(1-p)F(2) + 2\lambda_1\lambda_2(1-p)\Big]\int_R w^2$$
$$= \frac{\lambda_1^2\frac{1}{F(p+1)} + \lambda_2^2F(2) + 2\lambda_1\lambda_2}{(\lambda_1+\lambda_2F(p))^2}\,\frac{(1-p)\int_R w^2}{(\int_R w^p)^2}$$
$$= \frac{\lambda_1^2 + \lambda_2^2F(2)F(p+1) + 2\lambda_1\lambda_2F(p+1)}{(\lambda_1+\lambda_2F(p))^2}\,\frac{(1-p)\int_R w^2}{(\int_R w^p)^2F(p+1)}.$$

Set $\frac{\lambda_1}{\lambda_2} = \eta$ and

$$h(\eta) := \frac{(\eta+F(p))^2}{\eta^2 + 2\eta F(p+1) + F(2)F(p+1)}\cdot F(p+1). \tag{7.39}$$

Then we obtain that

$$\mathcal{L}(\varphi,\varphi) = \frac{1}{h(\eta)}\frac{(1-p)\int_R w^2}{(\int_R w^p)^2}. \tag{7.40}$$

We now choose an η such that

$$h(\eta) = 2F(p) - F(2).$$

A simple computation shows that we may choose

$$\eta = \eta_0 := \frac{\sqrt{F(p+1)(2F(p)-F(2))} - F(p+1)}{2F(p+1) - 2F(p) + F(2)}(F(2)-F(p)).$$

(Note that $2F(p) - F(2) > 0$ for $1 < p < 2$.) Then

$$h(\eta_0) = 2F(p) - F(2).$$

By (7.37) and the definition of λ, we have

$$D(p) = \frac{(p-1)\int_R w^2}{(\int_R w^p)^2\lambda} < h(\eta_0), \tag{7.41}$$

which proves Lemma 3.

□

Acknowledgments: The research of the first author is supported by an Earmarked Research Grant from RGC of Hong Kong. The second author thanks the Department of Mathematics at The Chinese University of Hong Kong for their kind hospitality.

References

Dancer, E. N. [2002] "On stability and Hopf bifurcations for chemotaxis systems", to appear in "Proceedings of an IMS Workshop on Reaction-Diffusion Systems".

Doelman, A., Gardner, R. A. & Kaper, T. J. [1998] "Stability of singular patterns in the 1-D Gray-Scott model: A matched asymptotic approach", *Phys. D* **122**, 1-36.

Doelman, A., Gardner, R. A. & Kaper, T. J. [2001a] "A stability index analysis of 1-D patterns of the Gray-Scott model", *Methods Appl. Anal.* **7**, 105-149.

Doelman, A., Gardner, R. A. & Kaper, T. J. [2001b] "Large stable pulse solutions in reaction-diffusion equations", *Indiana Univ. Math. J.* **50**, 443-507.

Davidson, F. A. & Rynne, B. P. [2000] "*A priori* bounds and global existence of solutions of the steady-state Sel'kov model", *Proc. Roy. Soc. Edinb. A* **130**, 507-516.

Eigen, M. and Schuster, P. [1977] "The hypercycle. A principle of natural self organisation. Part A. Emergence of the hypercycle", *Naturwissenschaften* **64**, 541-565.

Eigen, M. and Schuster, P. [1978a] "The hypercycle. A principle of natural self organisation. Part B. The abstract hypercycle", *Naturwissenschaften* **65**, 7-41.

Eigen, M. and Schuster, P. [1978b] "The hypercycle. A principle of natural self organisation. Part C. The realistic hypercycle", *Naturwissenschaften* **65**, 341-369.

Eilbeck, J. C. & Furter, J. E. [1995] "Analysis of bifurcations in reaction-diffusion systems with no fixed boundary conditions-the Sel'kov model", *Proc. Roy. Soc. Edinb. A* **125**, 413-438.

Erdelyi, A., Magnus, W., Oberhettinger, F., & Tricomi, F. G. [1953] *Higher Transcendental Functions, Vol. 1*, McGraw-Hill Book, New York.

Gierer, A. & Meinhardt, H. [1972] "A theory of biological pattern formation", *Kybernetik (Berlin)* **12**, 30-39.

Gray, P. & Scott, S. K. [1983] "Autocatalytic reactions in the isothermal, continuous stirred tank reactor: isolas and other forms of multistability", *Chem. Eng. Sci.* **38**, 29-43.

Hunding, A. & Sorensen, P. G. [1988] "Size adaptation of Turing patterns", *J. Math. Biol.* **26**, 27-39.

Iron, D., Ward, M. J. & Wei, J. [2001] The stability of spike solutions to the one-dimensional Gierer-Meinhardt model, *Phys. D* **150**, 25-62.

Levine, H. A. & Sleeman, B. D. [1977] "A system of reaction diffusion equations arising in the theory of reinforced random walks", *SIAM J. Appl. Math.* **57**, 683-730.

Levine, H. A., Sleeman, B. .D. & Nilsen-Hamilton, M. [2001] "Mathematical modeling of the onset of capillary formation initiating angiogenesis", *J. Math. Biol.* **42**, 195-238.

Luke, Y. [1975] *Mathematical Functions and Their Approximations*, Academic Press.

Murray, J. D. [1989] *Mathematical Biology*, Springer.

Ni, W.-M. [1998] "Diffusion, cross-diffusion, and their spike-layer steady states", *Notices of Amer. Math. Soc.* **45**, 9-18.

Ni, W.-M., Takagi, I. & Yanagida [2002], E. *Tohoku Math. J.*, to appear.

Nishiura, Y. [1982] "Global structure of bifurcation solutions of some reaction-diffusion systems", *SIAM J. Math. Anal.* **13**, 555-593.

Sel'kov, E. E. [1968] "Self-osillations in glycolysis", *Eur. J. Biochem.* **4**, 79-86.

Sleeman, B. D., Ward, M. J. & Wei, J. [2002] "Stability and dynamics of spiky pattern in angiogenesis", preprint.

Wei, J. [1999a] "On single interior spike solutions of Gierer-Meinhardt system: uniqueness, spectrum estimates and stability analysis", *Euro. J. Appl. Math.* **10**, 353-378.

Wei, J. [1999b] "Existence, stability and metastability of point condensation patterns generated by Gray-Scott system", *Nonlinearity* **12**, 593-616.

Wei, J. [2000a] "On a nonlocal eigenvalue problem and its applications to point-condensations in reaction-diffusion systems", *Int. J. Bifur. and Chaos* **10**, 1485-1496.

Wei, J. [2000b] "Point condensations generated by Gierer-Meinhardt system: a brief survey", book chapter in *New Trends in Nonlinear Partial Differential Equations 2000* (Y. Morita, H. Ninomiya, E. Yanagida, and S. Yotsutani editors), 46-59.

Wei, J. & Winter, M. [1999] "On the two-dimensional Gierer-Meinhardt system with strong coupling", *SIAM J. Math. Anal.* **30**, 1241-1263.

Wei, J. & Winter, M. [2001a] "Spikes for the two-dimensional Gierer-Meinhardt system: the weak coupling case", *J. Nonlinear Sci.* **11**, 415-458.

Wei, J. & Winter, M. [2001b] "On a hypercycle system wih nonlinear rate", *Meth. Appl. Anal.*, **7**, 257-278.

Wei, J. & Winter, M. [2002] "Spikes for the two-dimensional Gierer-Meinhardt system: the strong coupling case", *J. Differential Equations*, to appear.

Wei, J. & Zhang, L. [2001] "On a nonlocal eigenvalue problem", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **30**, 41-61.

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: wei@math.cuhk.edu.hk

Mathematisches Institut, Universität Stuttgart, D-70511 Stuttgart, Germany

E-mail address: winter@mathematik.uni-stuttgart.de