INFINITELY MANY POSITIVE SOLUTIONS FOR NONLINEAR EQUATIONS WITH NON-SYMMETRIC POTENTIALS

WEIWEI AO AND JUNCHENG WEI

ABSTRACT. We consider the following nonlinear Schrödinger equation

$$\begin{cases} \Delta u-(1+\delta V)u+f(u)=0 \quad \text{in } \mathbb{R}^N, \\ u>0 \text{ in } \mathbb{R}^N, u\in H^1(\mathbb{R}^N) \end{cases}$$

where V is a continuous potential and $f(u)$ is a nonlinearity satisfying some decay condition and some non-degeneracy condition, respectively. Using localized energy method, we prove that there exists a δ_0 such that for $0<\delta<\delta_0$, the above problem has infinitely many positive solutions. This generalizes and gives a new proof of the results by Cerami-Passaseo-Solimini [13]. The new techniques allow us to establish the existence of infinitely many positive bound states for elliptic systems.

1. INTRODUCTION

In this paper, we consider nonlinear Schrödinger equations and systems with non-symmetric potentials. We are interested in the multiplicity of positive solutions.

1.1. Nonlinear Schrödinger equation with non-symmetric potential.

We first consider the following equation:

$$\begin{cases} \Delta u-(1+\delta V(x))u+f(u)=0 \quad \text{in } \mathbb{R}^N \\ u>0 \text{ in } \mathbb{R}^N, \quad u\in H^1(\mathbb{R}^N) \end{cases} \tag{1.1}$$

where $N\geq 2$, δ is a positive constant and the potential V is a continuous function satisfying suitable decay assumption, but without any symmetry. We are interested in the existence of *infinitely many* positive solutions of equation (1.1).

Equation (1.1) arises in the study of solitary waves in nonlinear equations of the Klein-Gordon or Schrödinger type and has been under extensive studies in recent years.

Consider the following problem first:

$$\Delta u-V(x)u+u^p=0, \quad u>0 \quad \text{in } \mathbb{R}^N, \quad u\in H^1(\mathbb{R}^N), \tag{1.2}$$

where p is subcritical, i.e., $1 < p < (\frac{N+2}{N-2})_+ (= \frac{N+2}{N-2}$ if $N \geq 3; = +\infty$ if $N = 2)$. There have been many results in the literature on conditions imposed on $V(x)$ to ensure the existence of a positive (mountain-pass or least energy) solution. For example, if $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies

$$0 < \inf_{x \in \mathbb{R}^N} V(x) \leq V(x) \leq \lim_{|x| \to \infty} \inf V(x), \text{ and } V(x) \not\equiv \lim_{|x| \to \infty} \inf V(x) \tag{1.3}$$

Rabinowitz ([Theorem 4.27, [42]]) proved the existence of a least energy solution. Other conditions to ensure the existence of a least energy positive solution can also be found in [9], [24], [44] and the references therein. But if (1.3) does not hold, (1.2) may not have a least energy solution. So, one needs to find solutions with higher energy levels. For results in this direction, the readers can refer to [7, 8].

On the other hand, if we consider the following semi-classical problem:

$$\varepsilon^2 \Delta u - V(y) u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N, \quad \lim_{|y| \to +\infty} u(y) = 0, \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter and p is subcritical, then the number of the critical points of $V(y)$ (see for example [1, 38],[20]–[23],[26, 38, 49]), the type of the critical points of $V(y)$ (see for example [10, 29, 37]), and the topology of the level set of $V(y)$ (see for example [2, 3, 11, 27]), can affect the number of the solutions. The construction of single and multiple spikes in the degenerate setting is done by Byeon-Tanaka [10, 11]. In particular, we mention the following multiplicity result due to Kang-Wei [29] (see [10] for general $f(u)$): If $V(x)$ has a local maximum point, then for any fixed integer K, there exists $\varepsilon_K > 0$ such that for $\varepsilon < \varepsilon_K$ there are solutions with K spikes. Therefore, for the singularly perturbed problems (1.4), the parameter ε will tend to zero as the number of the solutions tends to infinity. Thus all these results do not give multiplicity results for (1.2).

About the existence of infinitely many positive solutions, Coti-Zelati and Rabinowitz [15, 16] first proved the existence of arbitrary many number of bumps (hence infinitely many solutions) for (1.2) when V is a periodic function in $\mathbb{R}^N$, (see Sere [43] for related work on Hamiltonian systems). As far as we know, without periodicity nor smallness of the parameters, the first result on the existence of infinitely many positive solutions was due to Wei-Yan [50]. (Another variational proof was given in [19].) They proved the existence of infinitely many non-radial positive bump solutions for (1.2) under the following assumption at infinity:

$$V(x) = V(|x|) = V_\infty + \frac{a}{|x|^m} + O(\frac{1}{|x|^{m+\sigma}}), \quad m > 1, V_\infty, a, \sigma > 0.$$

In a recent remarkable paper [13], Cerami-Passasseo-Solimini developed a localized Nehari's manifold argument and localized variational method to prove the existence of infinitely many positive solutions of the following equation

$$\begin{cases} \Delta u - (1+\delta V)u + u^p = 0 \quad \text{in } \mathbb{R}^N \\ u > 0 \text{ in } \mathbb{R}^N, \ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.5}$$

where the potential V satisfies suitable decay assumption (see below (H1)-(H3)) and p is subcritical. (The existence of a positive solution with infinitely many bumps was also proved in [14].)

The aim of the first part of this paper is two folds. Firstly, we want to generalize the results of [13] to more general nonlinearities, i.e, we consider the more general equation (1.1). Secondly, we will give another proof of the results of [13], using Lyapunov-Schmidt reduction.

In this paper, we assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the following two conditions:

- (f_1) $f : \mathbb{R} \to \mathbb{R}$ is of class $C^{1+\sigma}$ for some $0 < \sigma \le 1$ and $f(u) = 0$ for $u \le 0$, $f'(0) = 0$.
- (f_2) The equation

$$\begin{cases} \Delta w - w + f(w) = 0, \ w > 0 \text{ in } \ \mathbb{R}^N \\ w(0) = \max_{y \in \mathbb{R}^N} w(y), \ w \to 0 \text{ as } |y| \to \infty \end{cases} \tag{1.6}$$

has a non-degenerate solution w, i.e.,

$$\text{Ker}(\Delta - 1 + f'(w)) \cap L^\infty(\mathbb{R}^N) = \text{Span}\{\frac{\partial w}{\partial y_1}, \cdots, \frac{\partial w}{\partial y_N}\}. \tag{1.7}$$

From the well-known results of [28], we know that w is radial symmetric with exponential decay. Moreover, we have the following asymptotic behavior of w

$$w(r) = A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})), \ w'(r) = -A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})), \tag{1.8}$$

for r large, where $A_N > 0$ a generic constant.

We note that the function

$$f(w) = w^p - aw^q, \quad \text{for} \ \ w \ge 0 \tag{1.9}$$

with a constant $a \ge 0$ satisfies the above assumptions $(f_1) - (f_2)$ if $1 < q < p < (\frac{N+2}{N-2})_+$. (See Appendix C of [35].) We should point out that f need not to be superlinear, only existence and non-degeneracy are needed. For example, the following function

$$f(w) = \frac{w^2}{1+w^2}, \quad \text{for} \quad w \geq 0 \tag{1.10}$$

was considered in [48], and it has been proved that $f(w)$ satisfies the above assumptions $(f_1) - (f_2)$. See Lemma 2.2 in [48]. Of course $f(w)$ is not superlinear. Nondegeneracy is a generic condition. We should remark that there do exist nonlinearities with degenerate ground states; the first example seems to be given by Dancer [18]. See also Polacik [40].

Under the nondegeneracy condition (f_2), the spectrum of the linearized operator

$$\Delta\phi - \phi + f'(w)\phi = \lambda\phi, \ \phi \in H^1(\mathbb{R}^N) \tag{1.11}$$

admits the following decompositions

$$\lambda_1 > \lambda_2 > ...\lambda_m > \lambda_{m+1} = 0 > \lambda_{m+2} \tag{1.12}$$

where each of the eigenfunction corresponding to the positive eigenvalue λ_j decays exponentially. These eigenfunctions will play important role in our secondary Lyapunov-Schmidt reduction (see Section 2.2 below).

The energy functional associated with (1.1) is

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (1+\delta V)u^2 dx - \int_{\mathbb{R}^N} F(u)dx, \tag{1.13}$$

where $F(u) = \int_0^u f(s)ds$.

Let us now introduce the assumptions on $V(x)$ (similar to [13])

$$\begin{cases} (H1) \ \ V(x) \to 0 \quad \text{as } |x| \to \infty, \\ (H2) \ \ \exists \ 0 < \bar{\eta} < 1, \lim_{|x|\to\infty} V(x)\ e^{\bar{\eta}|x|} = +\infty, \\ (H3) \ \ V \ \text{ is continuous in } \mathbb{R}^N. \end{cases} \tag{1.14}$$

We now state the main theorem in this paper:

Theorem 1.1. *Let f satisfy assumptions $(f_1)-(f_2)$ and the potential V satisfy assumptions $(H1)-(H3)$. Then there exists a positive constant δ_0, such that for $0 < \delta < \delta_0$, problem (1.1) has infinitely many positive solutions.*

In the following we sketch the main steps in the proof of Theorem 1.1.

1.2. **Sketch of the proof of Theorem 1.1.** We introduce some notations first.

Let w be the nondegenerate solution of (1.6) and $k \geq 1$ be an integer. Let $\rho > 0$ be a real number such that $w(x) \leq e^{-|x|}$ for $|x| > \rho$. Now we define the configuration space,

$$\Lambda_1 = \mathbb{R}^N, \quad \Lambda_k := \{(Q_1, \cdots, Q_k) \in \mathbb{R}^N | \min_{i \neq j} |Q_i - Q_j| \geq \rho\}, \forall k > 1. \tag{1.15}$$

Fixing $\mathbf{Q}_k = (Q_1, \cdots, Q_k) \in \Lambda_k$, we define the sum of k spikes as

$$w_{Q_1, \cdots, Q_k} = \sum_{i=1}^{k} w_{Q_i} \quad \text{where } w_{Q_i} = w(x - Q_i). \tag{1.16}$$

Define the operator

$$S(u) = \Delta u - (1 + \delta V)u + f(u). \tag{1.17}$$

We also define the following functions as the approximate kernels:

$$Z_{ij} = \frac{\partial w_{Q_i}}{\partial x_j} \chi_i(x), \text{ for } i = 1, \cdots, k, \ j = 1, \cdots, N, \tag{1.18}$$

where $\chi_i(x) = \chi(\frac{2|x-Q_i|}{(\rho-1)})$ and $\chi(t)$ is a cut-off function such that $\chi(t) = 1$ for $|t| \leq 1$ and $\chi(t) = 0$ for $|t| \geq \frac{\rho^2}{\rho^2-1}$. Note that the support of Z_{ij} is contained in $B_{\frac{\rho^2}{2(\rho+1)}}(Q_i)$.

Using $w_{Q_1, \cdots, Q_k}$ as the approximate solution and performing the Lyapunov-Schmidt reduction, we can show that there exists a constant ρ_0, such that for $\rho \geq \rho_0$, and $\delta < c_\rho$ (for some constant c_ρ depending on ρ but independent of k and $\mathbf{Q}_k$), we can find a $\phi_{\mathbf{Q}_k}$ such that

$$S(w_{Q_1, \cdots, Q_k} + \phi_{\mathbf{Q}_k}) = \sum_{i=1, \cdots, k, j=1, \cdots, N} c_{ij} Z_{ij}, \tag{1.19}$$

and we can show that $\phi_{\mathbf{Q}_k}$ is C^1 in $\mathbf{Q}_k$. This is done is Section 2.1.

Next, for any k, we define a new function

$$\mathcal{M}(\mathbf{Q}_k) = J(w_{Q_1, \cdots, Q_k} + \phi_{\mathbf{Q}_k}), \tag{1.20}$$

and maximize $\mathcal{M}(\mathbf{Q}_k)$ over $\bar{\Lambda}_k$. At the maximum point of $\mathcal{M}(\mathbf{Q}_k)$, we prove that $c_{ij} = 0$ for all i, j. Thus we prove that the corresponding $w_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}$ is a solution to (1.1). The preceding discussions imply that there exists $\rho_0 > 0$ large such that for $\rho \geq \rho_0$ and $\delta \leq c_\rho$, and for any k, there exists a spike solution to (1.1) with k spikes in Λ_k. Since k is arbitrary, it follows that there exists infinitely many spike solutions for $\delta < c_{\rho_0}$ independent of k.

There are two main difficulties in the maximization process. First, we need to show that the maximum points in $\bar{\Lambda}_k$ will not go to infinity. This is guaranteed by the slow decay assumption on the potential V. Second, we have to detect the difference in the energy when the spikes move to the boundary of the configuration space. In the second step, we use the induction method and detect the difference in energy between the k-spikes energy and the $k+1$-spikes energy. A crucial estimate is Lemma 2.2, in which we prove that the accumulated error can be controlled from step k to step $k+1$. To prove this, we perform a secondary Lyapunov-Schmidt reduction. This is done is Section 2.2 and 2.3. Finally in Section 2.4, we give the proof of Theorem 1.1.

Unlike the variational method and Nehari's manifold arguments in [13], our main idea is to use the Lyapunov-Schmidt reduction method. The only assumption we need is the nondegeneracy of the bump. We have no requirements on the structure of the nonlinearity. Note that the nondegeneracy is also needed in arguments of [13]. Our approach is different. It handles more general nonlinearities and can be readily applied to other similar problems such as elliptic systems and magnetic Ginzburg-Landau equations ([39]).

In the following we present the applications of our techniques to elliptic systems in which the bump can have higher Morse index.

1.3. **Nonlinear Schrödinger system with non-symmetric potentials.** As we mentioned above, our approach can be applied to other problems such as elliptic systems. So in this section, we apply our method to the elliptic systems. We consider the following nonlinear Schrödinger system in $\mathbb{R}^N$ $(N \leq 3)$

$$\begin{cases} -\Delta u + (1+\delta a(x))u = \mu_1 u^3 + \beta v^2 u \\ -\Delta v + (1+\delta b(x))v = \mu_2 v^3 + \beta u^2 v \end{cases} \tag{1.21}$$

where μ_1, μ_2 and δ are positive constants, $\beta \in \mathbb{R}$ and the potentials $a(x), b(x)$ are continuous functions satisfying suitable decay assumption, but without any symmetry property.

This type of system arises when one considers the standing wave solutions of the time dependent $M-$coupled Schrodinger systems of the form with $M = 2$

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_j = \Delta\Phi_j - V_j(x)\Phi_j + \mu_j|\Phi_j|^2\Phi_j + \Phi_j\sum_{l=1,k\neq j}^{M}\beta_{jk}|\Phi_l|^2, \text{ in } \mathbb{R}^N \\ \Phi_j = \Phi_j(x,t) \in C, \ t>0, \ j=1,\cdots,M, \end{cases} \tag{1.22}$$

where μ_j and $\beta_{jl} = \beta_{lj}$ are constants. The system (1.22) arises in applications of many physical problems, especially in the study of incoherent solitons in nonlinear optics. Physically, the solution Φ_j denotes the $j-$th component of the beam in Kerr-like photorefractive media. The positive constant μ_j is for self-focusing in the $j-$th component of the beam. The coupling constant β is the interaction between the first and the second component of the beam. As $\beta > 0$, the interaction is attractive, while the interaction is repulsive if $\beta < 0$.

Mathematical work on systems of nonlinear Schrödinger equations have been studied extensively in recent years, see for example [6, 17, 32, 36, 45, 46, 47] and references therein. Phase separation has been proved in several cases with constant potentials such as in the work [6, 17, 25, 36, 46, 47] as the coupling constant β tends to negative infinity. In symmetric case $(a = b = 0, \mu_1 = \mu_2)$, [47] gives infinitely many non-radial positive solutions for $\beta \leq -1$ which are potentially segregated type. In a recent paper of Peng and Wang [41], the authors considered the multiplicity of solutions. They proved the existence of infinitely many solutions of synchronized type to (1.21) for radially symmetric potentials $a(|x|), b(|x|)$ satisfying some algebraic decay assumption. Their proof is in the spirit of the work [50].

The second result of this paper concerns the existence of infinitely many synchronized solutions for potentials without any symmetry assumption.

We assume that $a(x), b(x)$ satisfy the following conditions:

$$\begin{cases} (H_1') \ \ a(x),\ b(x) \text{ are continuous functions in } \mathbb{R}^N, \\ (H_2') \ \ a(x),\ b(x) \to 0 \quad \text{as } |x| \to \infty, \quad a(x), b(x) \geq 0 \ \ as \ \ |x| \to \infty, \\ (H_3') \ \ \exists\, 0 < \bar{\eta} < 1, \ \ \lim_{|x|\to\infty} (\alpha^2 a(x) + \gamma^2 b(x)) e^{\bar{\eta}|x|} = +\infty, \end{cases} \tag{1.23}$$

where $\alpha = \sqrt{\frac{\mu_2 - \beta}{\mu_1\mu_2 - \beta^2}}$ and $\gamma = \sqrt{\frac{\mu_1 - \beta}{\mu_1\mu_2 - \beta^2}}$.

The energy functional associated with problem (1.21) is

$$\begin{aligned} J_1(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (1+\delta a)u^2 + |\nabla v|^2 + (1+\delta b)v^2 dx \quad (1.24) \\ &- \frac{1}{4}\int_{\mathbb{R}^N} \mu_1 u^4 + \mu_2 v^4 dx - \frac{\beta}{2}\int_{\mathbb{R}^N} u^2 v^2 dx, \quad u, v \in H^1(\mathbb{R}^N). \end{aligned}$$

The second result of this paper is as follows:

Theorem 1.2. *Let potentials a, b satisfy assumptions $(H_1') - (H_3')$. Then there exists $\beta^* > 0$, and $\delta_0 > 0$, such that for $\beta \in (-\beta^*, 0) \cup$*

$(0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$, *and* $0 < \delta < \delta_0$, *system (1.21) has infinitely many positive synchronized solutions.*

For the Schrödinger system, we will use the special solution $(U, V) = (\alpha, \gamma)w$ of the system (1.21) (when $\delta = 0$) as the bump profile. The condition on β guarantees the existence and non-degeneracy of the bump solution.

The main technical difference between the scalar problem (1.1) and the system (1.21) is that the system has higher Morse index for the bump profile. Since we have the non-degeneracy of the bump, we can still perform the secondary Lyapunov-Schmidt reduction.

The rest of the paper is organized as follows: Theorem 1.1 and 1.2 are proven in Sections 2 and 3, respectively.

Throughout this paper, unless otherwise stated, the letters c, C will always denote various generic constants that are independent of $k \geq 1$ and $\delta < 1$.

Acknowledgment. Juncheng Wei was supported by a GRF grant from RGC of Hong Kong. We thank the anonymous referee for a careful and thorough reading of the manuscript.

2. Infinitely many solutions and the proof of Theorem 1.1

2.1. **Lyapunov-Schmidt reduction.** In this section, we use the standard Lyapunov-Schmidt reduction procedure to solve problem (1.1). Since this has become a rather routine procedure, we omit most of the proofs. (The only part we need to pay attention to is the independence of all the coefficients on the number of spikes k.) We refer to [4], [34] and [31] for technical details.

For a fixed $\eta \in (0, 1)$ and $\mathbf{Q}_k = (Q_1, \cdots, Q_k) \in \Lambda_k$, we define a barrier function

$$W(\cdot) := \sum_{i=1}^{k} e^{-\eta|\cdot - Q_i|}. \tag{2.1}$$

Consider the norm

$$\|h\|_* = \sup_{x \in \mathbb{R}^N} |W(x)^{-1} h(x)| \tag{2.2}$$

which was introduced and used in [34].

We first estimate the error of the approximate solution in the above norm.

Lemma 2.1. *For any $0 < \eta < 1$, there exists $\rho_0 > 0$ such that for $\rho > \rho_0$ and for any $\mathbf{Q}_k \in \Lambda_k$ and $\delta < e^{-2\rho}$, the following estimate holds:*

$$\|S(w_{\mathbf{Q}_k})\|_* \leq ce^{-\xi\rho}, \tag{2.3}$$

for some constants $\xi > 0$ and $c > 0$, both of which are independent of ρ, k and $\mathbf{Q}_k$.

Proof. Observe that

$$S(w_{\mathbf{Q}_k}) = f(w_{\mathbf{Q}_k}) - \sum_{i=1}^{k} f(w_{Q_i}) - \delta V w_{\mathbf{Q}_k}. \tag{2.4}$$

Firstly, fix $j \in \{1, \dots, k\}$ and consider the region $|x - Q_j| \leq \frac{\rho}{2}$. In this region, using the exponential decay of w, the assumption (f_1) on f, i.e. f is $C^{1+\sigma}$ and $f(0) = f'(0) = 0$, and the definition of the configuration space (similar to the estimate in (2.87) in subsection 2.3), we have

$$\begin{aligned} |f(w_{\mathbf{Q}_k}) - \sum_{i=1}^{k} f(w_{Q_i})| &\leq C\left[f'(w_{Q_j})\sum_{Q_i\neq Q_j} w(x-Q_i) + \sum_{Q_i\neq Q_j} f(w_{Q_i})\right] \\ &\leq C[f'(w_{Q_j})e^{-\frac{1}{2}\rho} + e^{-\frac{(1+\sigma)\rho}{2}}] \\ &\leq Ce^{-\min\{\frac{1}{6},\frac{\sigma}{3}\}\rho}e^{-\eta|x-Q_j|} \\ &\leq Ce^{-\xi_1\rho}e^{-\eta|x-Q_j|} \end{aligned} \tag{2.5}$$

for a proper choice of $\xi_1 > 0$ depending on σ and independent of ρ large, k and $\mathbf{Q}_k$.

Consider now the region $|x - Q_j| > \frac{\rho}{2}$, for all j. In this region, we have

$$\begin{aligned} |f(w_{\mathbf{Q}_k}) - \sum_{i=1}^{k} f(w_{Q_i})| &\leq C\left[\sum_j f(w_{Q_j})\right] \leq C\left[\sum_j e^{-(1+\sigma)|x-Q_j|}\right] \\ &\leq \sum_j e^{-\eta|x-Q_j|}e^{-\frac{1+\sigma-\eta}{2}\rho} \\ &\leq ce^{-\xi_2\rho}\sum_j e^{-\eta|x-Q_j|} \end{aligned} \tag{2.6}$$

for some $\xi_2 > 0$. Finally, it is easy to see that under the assumption on δ that $\delta < e^{-2\rho}$ and $V \to 0$ as $|x| \to \infty$, there holds

$$|\delta V w_{\mathbf{Q}_k}| \leq ce^{-\rho}\sum_j e^{-\eta|x-Q_j|} \leq ce^{-\xi_3\rho}\sum_j e^{-\eta|x-Q_j|}, \tag{2.7}$$

for some $\xi_3 > 0$.

Let $\xi = \min(\xi_1, \xi_2, \xi_3)$. From the above estimates (2.5), (2.6) and (2.7), we obtain

$$\|S(w_{\mathbf{Q}_k})\|_* \le ce^{-\xi\rho}. \tag{2.8}$$

□

From now on, ξ will denote a positive constant depending on σ and η but independent of $\rho, k, \mathbf{Q}_k$ and may vary from line to line.

The following proposition is standard. We refer to [34], [31] and further improvements in [4]. Note that this new norm $\|\cdot\|_*$ has been used in [34] in a different setting.

Proposition 2.1. *Let $0 < \eta < 1$ be fixed. There exist positive numbers ρ_0, C and $\xi > 0$, (independent of ρ, k and $\mathbf{Q}_k \in \Lambda_k$), such that for all $\rho \ge \rho_0$, and for any $\mathbf{Q}_k \in \Lambda_k$, $\delta < e^{-2\rho}$, there is a unique solution $(\phi_{\mathbf{Q}_k}, \{c_{ij}\})$ to the following problem:*

$$\begin{cases} \Delta(w_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}) - (1+\delta V)(w_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}) + f(w_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}) = \sum_{i=1,\cdots,k, j=1,\cdots,N} c_{ij} Z_{ij}, \\ \int_{\mathbb{R}^N} \phi_{\mathbf{Q}_k} Z_{ij} dx = 0 \text{ for } i = 1, \cdots, k, j = 1, \cdots, N. \end{cases} \tag{2.9}$$

Furthermore $\phi_{\mathbf{Q}_k}$ is C^1 in $\mathbf{Q}_k$ and we have

$$\|\phi_{\mathbf{Q}_k}\|_* \le C\|S(w_{\mathbf{Q}_k})\|_* \le Ce^{-\xi\rho}, \quad |c_{ij}| \le Ce^{-\xi\rho}. \tag{2.10}$$

Proof. The proof follows exactly the same one as in Sections 3 and 4 of [34] in which they constructed multi-bump sign-changing finite energy solutions to the autonomous problem

$$\Delta u - u + f(u) = 0 \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N). \tag{2.11}$$

The same norm $\|\cdot\|_*$ was introduced and used in [34].

For completeness, we give a sketch of the proof here. We notice that the only difference between the equation (2.11) (as studied in [34]) and the equation (1.1) is the extra term $-\delta V u$.

We shall follow the main lines of the proofs given in Section 3 and Section 4, pages 1936-1946 of [34] and identify the differences here.

First, as in [34], the solvability of (2.9) is equivalent to the following nonlinear problem:

$$\begin{cases} \mathcal{L}(\phi, c) + E + N(\phi) = 0, \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} \phi Z_{ij} dx = 0 \text{ for } i = 1, \cdots, k, j = 1, \cdots, N, \end{cases} \tag{2.12}$$

where

$$\mathcal{L}(\phi, c) = L(\phi) - M(c),$$

$$L(\phi) = \Delta\phi - \phi + f^{'}(w_{\mathbf{Q}_k})\phi, \ M(c) = \sum_{ij} c_{ij} Z_{ij}$$

$$E = S(w_{\mathbf{Q}_k})$$

$$N(\phi) = (f(w_{\mathbf{Q}_k} + \phi) - f(w_{\mathbf{Q}_k}) - f'(w_{\mathbf{Q}_k})\phi) - \delta V \phi,$$

where $S(w_{\mathbf{Q}_k})$ is the error term given in Lemma 2.1.

The basic idea of solving (2.12) is to use the contraction mapping theorem. To this end, we first obtain *a priori estimates* for the linear problem. Then we write (2.12) as nonlinear operator and solve it using contraction mapping theorem. We summarize in the following three steps.

Step 1: We first obtain *a priori estimates* for the linear operator $\mathcal{L}$ in the norm $\| \cdot \|_*$. We claim that there exists $\rho_0 > 0, C > 0$ (all depend on the choice of $\eta > 0$) such that for $\rho > \rho_0$ and for ϕ satisfying the orthogonal conditions there holds

$$\|\phi\|_* \leq C \|\mathcal{L}(\phi, c)\|_* \tag{2.13}$$

The proof is by contradiction and the use of barrier function $W(\cdot)$, and it follows exactly the same proof as in Proposition 3.1, page 1937-1938 of [34]. Note that here the extra term $-\delta V u$ does not appear.

As a consequence of (2.13) and Lax-Milgram theorem, we prove the **Step 2:** existence of solution for the linear problem. There exists $\rho_0 > 0, C > 0$ such that for all $\rho > \rho_0$ and for all $g \in L^\infty(\mathbb{R}^N)$ with $\|g\|_* < +\infty$, there exists a unique pair (ϕ, c) such that

$$\mathcal{L}(\phi, c) + g = 0, \quad \text{in } \mathbb{R}^N, \ \int_{\mathbb{R}^N} \phi Z_{ij} dx = 0 \text{ for } i = 1, \cdots, k, j = 1, \cdots, N. \tag{2.14}$$

Moreover

$$\sum_{i,j} |c_{ij}| + \|\phi\|_* \leq C \|g\|_*. \tag{2.15}$$

This follows exactly the same argument in the proof of Proposition 3.2, page 1939-1940 in [34].

We denote the solution of the linear problem (2.14) by $(\phi, c) := \mathcal{L}^{-1}(g)$. Then the solvability of the full nonlinear equation (2.12) is reduced to finding a fixed point of

$$(\phi, c) = \mathcal{L}^{-1} \circ [S(w_{\mathbf{Q}_k}) + N(\phi) - \delta V \phi] \tag{2.16}$$

where $\mathcal{L}^{-1}$ denote the inverse operator of (2.14).

Stpe 3: We use the contraction mapping theorem to prove the existence of solutions to (2.16).

Recall that we have already obtained the error estimate $S(w_{\mathbf{Q}_k})$ in Lemma 2.1. Now define

$$\mathcal{B} = \{(\phi, c) \in L^{\infty}(\mathbb{R}^N) \times (\mathbb{R}^{kN}), \sum_{i,j} |c_{ij}| + \|\phi\|_* < Ce^{-\xi\rho}\} \tag{2.17}$$

where C is a large number.

Since

$$\|\mathcal{L}^{-1}(-\delta V\phi)\|_* \leq C\delta\|\phi\|_*, \tag{2.18}$$

as in [34] (pages 1940-1941), we can check that the mapping in (2.16) is a contraction mapping from $\mathcal{B}$ to itself. Thus by contraction mapping theorem we obtain a solution of (2.16) in $\mathcal{B}$ and the estimate in (2.10) follows from the above estimates. The continuity and differentiability parts of the proposition follow from last part of Proposition 4.1 (pages 1940-1941) of [34].

□

2.2. **A secondary Lyapunov-Schmidt reduction.** In this section, we present a key estimate on the difference between the solutions in the $k-$th step and the $(k+1)-$th step. This secondary Lyapunov-Schmidt reduction has been used in the paper [4].

For $(Q_1, \cdots, Q_k) \in \Lambda_k$, we denote $u_{Q_1,\cdots,Q_k}$ as $w_{Q_1,\ldots,Q_k} + \phi_{Q_1,\ldots,Q_k}$, where $\phi_{Q_1,\cdots,Q_k}$ is the unique solution given by Proposition 2.1. The main estimate below states that the difference between $u_{Q_1,\cdots,Q_{k+1}}$ and $u_{Q_1,\cdots,Q_k} + w_{Q_{k+1}}$ is small globally in $H^1(\mathbb{R}^N)$ norm.

To this end, we now write

$$\begin{aligned} u_{Q_1,\cdots,Q_{k+1}} &= u_{Q_1,\cdots,Q_k} + w_{Q_{k+1}} + \varphi_{k+1} \\ &= \bar{W} + \varphi_{k+1}, \end{aligned} \tag{2.19}$$

where

$$\bar{W} = u_{Q_1,\cdots,Q_k} + w_{Q_{k+1}}.$$

As one can see from the definition of the $\|\cdot\|_*$ norm in (2.2) that it depends on the spikes $\mathbf{Q}$, we now denote the norm by $\|\cdot\|_{*,\mathbf{Q}_k}$ to indicate the dependence. By the definition of φ_{k+1}, we have

$$\|\varphi_{k+1}\|_{*,\mathbf{Q}_{k+1}} = \|\phi_{Q_1,\cdots,Q_{k+1}} - \phi_{Q_1,\cdots,Q_k}\|_{*,\mathbf{Q}_{k+1}}. \tag{2.20}$$

By Proposition 2.1, we have

$$\|\phi_{Q_1,\cdots,Q_{k+1}}\|_{*,\mathbf{Q}_{k+1}} \leq Ce^{-\xi\rho}, \tag{2.21}$$

and

$$\|\phi_{Q_1,\cdots,Q_k}\|_{*,\mathbf{Q}_{k+1}} \leq \|\phi_{Q_1,\cdots,Q_k}\|_{*,\mathbf{Q}_k} \leq Ce^{-\xi\rho}. \tag{2.22}$$

So by the above three estimates (2.20), (2.21) and (2.22), one obtains

$$\|\varphi_{k+1}\|_* \leq Ce^{-\xi\rho} \tag{2.23}$$

for some $\xi > 0$ independent of ρ, k and $\mathbf{Q}_{k+1}$.

However, estimate (2.23) is not sufficient. For example, if we use (2.23) to estimate the L^2 norm of φ_{k+1}, we get

$$\int_{\mathbb{R}^N} |\varphi_{k+1}|^2 \leq Ce^{-2\xi\rho} \int_{\mathbb{R}^N} W^2 \leq Cke^{-2\xi\rho}. \tag{2.24}$$

This estimate depends linearly on k. The following key estimate shows that not only we have global H^1 estimate for φ_{k+1} but also we have localized H^1 estimate. (In the following we will always assume that $\eta > \frac{1}{2}$.)

Lemma 2.2. *Let ρ, $\delta, \eta, \mathbf{Q}_k$ be as in Proposition 2.1. Then it holds*

$$\begin{aligned} \int_{\mathbb{R}^N} (|\nabla \varphi_{k+1}|^2 + \varphi_{k+1}^2) \ &\leq \ Ce^{-\xi\rho} \sum_{i=1}^{k} w(|Q_{k+1} - Q_i|) \\ &+ \ C\delta^2 \Big(\int_{\mathbb{R}^N} V^2 w_{Q_{k+1}}^2 dx + \big(\int_{\mathbb{R}^N} |V| w_{Q_{k+1}} dx\big)^2\Big), \end{aligned} \tag{2.25}$$

for some constants $C > 0, \xi > 0$ independent of ρ, k and $\mathbf{Q}_{k+1} \in \Lambda_{k+1}$ (the constants may depend on the choice of η).

Before we proceed with the proof, we recall the following formula:

Lemma 2.3. *([Proposition 1.2, [7]]) Let $f \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $g \in C(\mathbb{R}^N)$ be radially symmetric and satisfy for some $\alpha \geq 0$, $\beta \geq 0$, $\gamma_0 \in \mathbb{R}$,*

$$f(x) exp(\alpha|x|)|x|^\beta \to \gamma_0 \ \text{as} \ |x| \to \infty,$$
$$\int_{\mathbb{R}^N} |g(x)| exp(\alpha|x|)(1 + |x|^\beta) dx < \infty.$$

Then

$$exp(\alpha|y|)|y|^\beta \int_{\mathbb{R}^N} g(x+y) f(x) dx \to \gamma_0 \int_{\mathbb{R}^N} g(x) exp(-\alpha x_1) dx \ \text{as} \ |y| \to \infty.$$

Using the above lemma, we have the following:

Lemma 2.4. *For $|Q_i - Q_j| \geq \rho$ large, it holds that*

$$\int_{\mathbb{R}^N} f(w(x - Q_i)) w(x - Q_j) dx = (\gamma_1 + o(1)) w(|Q_i - Q_j|) \tag{2.26}$$

as $\rho \to \infty$ *and*

$$\gamma_1 = \int_{\mathbb{R}^N} f(w) e^{-y_1} dy > 0. \tag{2.27}$$

Proof. This can be deduced from the above lemma using the fact that f satisfies (f_1). Since ρ is large enough, by (1.8), we have

$$\begin{aligned} w(x+Q_i-Q_j) &= (A_N+o(1))(\frac{1}{|x+Q_i-Q_j|})^{\frac{N-1}{2}} e^{-|x+Q_i-Q_j|} \\ &= w(Q_i-Q_j) e^{-\langle x, \frac{Q_i-Q_j}{|Q_i-Q_j|}\rangle + o(|x|)} \end{aligned}$$

Thus by Lebesgue's dominated convergence theorem

$$\begin{aligned} &\int_{\mathbb{R}^N} f(w(x-Q_i)) w(x-Q_j) dx \\ &= \int_{\mathbb{R}^N} f(w(x)) w(x+Q_i-Q_j) dx \\ &= (1+o(1)) w(Q_i-Q_j) \int_{\mathbb{R}^N} f(w(x)) e^{-\langle x, \frac{Q_i-Q_j}{|Q_i-Q_j|}\rangle} dx \\ &= (\gamma_1+o(1)) w(Q_i-Q_j). \end{aligned}$$

□

Proof of Lemma 2.2. To prove (2.25), we need to perform a further decomposition. The basic idea is the following: around each spike, we project φ_{k+1} into the orthogonal space of the unstable eigenfunctions and kernels. In this way, we obtain a linear operator which is positively definite. Thus we need to estimate three components of φ_{k+1}: the coefficients of projections to the unstable eigenfunctions and kernels, and the orthogonal part. In the following, we carry out this procedure in details.

By the non-degeneracy assumption (f_2), the following eigenvalue problem

$$\Delta\phi - \phi + f'(w)\phi = \lambda\phi, \ \phi \in H^1(\mathbb{R}^N) \tag{2.28}$$

admits the following set of eigenvalues

$$\lambda_1 > \cdots > \lambda_{m+1} = 0 > \lambda_{m+2} \cdots . \tag{2.29}$$

We denote the eigenfunctions corresponding to the positive eigenvalues λ_j as $\phi_0^j(x), j = 1, ..., m$. By the non-degeneracy assumption (f_2) we infer that there exists a positive generic constant c_0 such that

$$\int_{\mathbb{R}^N} [(|\nabla\phi|^2 + \phi^2) - f'(w)\phi^2] \geq c_0 \|\phi\|^2_{H^1(\mathbb{R}^N)} \tag{2.30}$$

for all H^1 functions satisfying $\int_{\mathbb{R}^N} \phi\phi_0^j = \int_{\mathbb{R}^N} \phi \frac{\partial w}{\partial x_i} = 0, i = 1, ..., N, j = 1, ..., m$. We fix ϕ_0^j such that $\max_{x\in\mathbb{R}^N} \phi_0^j = 1$. Denote by $\phi_{ij} = \chi_i \phi_0^j(x - Q_i)$, where χ_i is the cut-off function introduced in Section 1.2.

By the equations satisfied by φ_{k+1}, we deduce that

$$\bar{L}\varphi_{k+1} = -\bar{S} + \sum_{i=1,\cdots,k+1, j=1,\cdots,N} c_{ij} Z_{ij} \tag{2.31}$$

for some constants $\{c_{ij}\}$, where

$$\bar{L} = \Delta - (1 + \delta V) + f'(\tilde{W}),$$

$$f'(\tilde{W}) = \begin{cases} \frac{f(\bar{W}+\varphi_{k+1})-f(\bar{W})}{\varphi_{k+1}}, \text{ if } \varphi_{k+1} \neq 0 \\ f'(\bar{W}), \text{ if } \varphi_{k+1} = 0, \end{cases} \tag{2.32}$$

and

$$\bar{S} = f(u_{Q_1,\cdots,Q_k} + w_{Q_{k+1}}) - f(u_{Q_1,\cdots,Q_k}) - f(w_{Q_{k+1}}) - \delta V w_{Q_{k+1}}. \tag{2.33}$$

Here we may write $\tilde{W}$ as $\tilde{W} = \bar{W} + \tau\varphi_{k+1}$ where $\tau \in [0, 1]$.

We proceed with the proof in multiple steps. The L^2-norm of $\bar{S}$ is estimated first:

By the estimates in the proof of Lemma 2.1 and Proposition 2.1, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |f(u_{Q_1,\cdots,Q_k} + w_{Q_{k+1}}) - f(u_{Q_1,\cdots,Q_k}) - f(w_{Q_{k+1}})|^2 dx \\ &\leq C \int_{\mathbb{R}^N} |f'(u_{Q_1,\cdots,Q_k})|^2 w_{Q_{k+1}}^2 + |f'(w_{Q_{k+1}})|^2 u_{Q_1,\cdots,Q_k}^2 dx \\ &\leq C \sum_{i=1}^k w(|Q_{k+1} - Q_i|)^{1+\sigma} + e^{-\xi\rho} w(|Q_{k+1} - Q_i|)^{1+\eta\sigma} + e^{-\xi\rho} w(|Q_{k+1} - Q_i|)^{\sigma+\eta} \\ &\leq C e^{-\xi\rho} \sum_{i=1}^k w(|Q_{k+1} - Q_i|), \end{aligned}$$

for some $\xi > 0$ if we choose $\eta + \sigma > 1$, and the last term in (2.33) can be estimated as

$$\int_{\mathbb{R}^N} (\delta V w_{Q_{k+1}})^2 dx \leq C\delta^2 \int_{\mathbb{R}^N} V^2 w_{Q_{k+1}}^2 dx.$$

The above two estimates give

$$\|\bar{S}\|_{L^2(\mathbb{R}^N)}^2 \leq C(e^{-\xi\rho} \sum_{i=1}^k w(|Q_{k+1} - Q_i|) + \delta^2 \int_{\mathbb{R}^N} V^2 w_{Q_{k+1}}^2 dx). \tag{2.34}$$

By virtue of (2.23), we have

$$\tilde{W} = \sum_{i=1}^{k+1} w(x - Q_i) + O(e^{-\xi\rho}). \tag{2.35}$$

Decompose φ_{k+1} as

$$\varphi_{k+1} = \psi + \sum_{i=1,\cdots,k+1, l=1,\cdots,m} \ell_{il}\phi_{il} + \sum_{i=1,\cdots,k+1, j=1,\cdots,N} d_{ij}Z_{ij} \tag{2.36}$$

for some ℓ_{il}, d_{ij} such that

$$\int_{\mathbb{R}^N} \psi\phi_{il} = \int_{\mathbb{R}^N} \psi Z_{ij} = 0,\ i = 1, ..., k+1,\ j = 1, ..., N,\ l = 1, \cdots, m. \tag{2.37}$$

In the following, we carry out estimates for each of the three terms at the right hand side of (2.36).

Estimates of the coefficients d_{ij}: Since

$$\varphi_{k+1} = \phi_{Q_1,\cdots,Q_{k+1}} - \phi_{Q_1,\cdots,Q_k}, \tag{2.38}$$

we have for $i = 1, \cdots, k$,

$$\begin{aligned} d_{ij} &= \int_{\mathbb{R}^N} \varphi_{k+1} Z_{ij} \\ &= \int_{\mathbb{R}^N} (\phi_{Q_1,\cdots,Q_{k+1}} - \phi_{Q_1,\cdots,Q_k}) Z_{ij} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} d_{k+1,j} &= \int_{\mathbb{R}^N} \varphi_{k+1} Z_{k+1,j} \\ &= \int_{\mathbb{R}^N} (\phi_{Q_1,\cdots,Q_{k+1}} - \phi_{Q_1,\cdots,Q_k}) Z_{k+1,j} \\ &= -\int_{\mathbb{R}^N} \phi_{Q_1,\cdots,Q_k} Z_{k+1,j}, \end{aligned}$$

where we have used the orthogonality conditions satisfied by $\phi_{Q_1,\cdots,Q_k}$ and $\phi_{Q_1,\cdots,Q_{k+1}}$. So by Proposition 2.1, we obtain

$$\begin{cases} |d_{ij}| = 0 \text{ for } i = 1, \cdots, k, \\ \\ |d_{k+1,j}| \le ce^{-\xi\rho} \sum_{i=1}^{k} e^{-\eta|Q_i - Q_{k+1}|}. \end{cases} \tag{2.39}$$

By (2.36), we can rewrite (2.31) as

$$\bar{L}\psi + \sum_{i=1,\cdots,k+1,l=1,\cdots,m} \ell_{il}\bar{L}\phi_{il} + \sum_{i=1,\cdots,k+1,j=1,\cdots,N} d_{ij}\bar{L}Z_{ij} = -\bar{S} + \sum_{i=1,\cdots,k+1,j=1,\cdots,N} c_{ij}Z_{ij}. \tag{2.40}$$

Estimates of the coefficients ℓ_{il}: we use the equation (2.40) to estimate ℓ_{il}. First, multiplying (2.40) by ϕ_{il} and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned} \ell_{il}\int_{\mathbb{R}^N}\bar{L}(\phi_{il})\phi_{il} &= -\sum_{j=1}^{N} d_{ij}\int_{\mathbb{R}^N}\bar{L}(Z_{ij})\phi_{il} - \int_{\mathbb{R}^N}\bar{S}\phi_{il} \\ &- \sum_{j\neq l}\ell_{ij}\int_{\mathbb{R}^N}\bar{L}(\phi_{ij})\phi_{il} - \int_{\mathbb{R}^N}\bar{L}(\psi)\phi_{il} \end{aligned} \tag{2.41}$$

where we have used the fact that $Z_{ij}(x+Q_i)$ is odd in the j-th variable, and

$$\begin{cases} |\int_{\mathbb{R}^N}\bar{S}\phi_{il}| \le ce^{-\xi\rho}e^{-\eta|Q_i-Q_{k+1}|} + \delta|\int_{\mathbb{R}^N}Vw_{Q_{k+1}}\phi_{il}dx| \text{ for } i=1,\cdots,k \\ |\int_{\mathbb{R}^N}\bar{S}\phi_{k+1,l}| \le ce^{-\xi\rho}\sum_{i=1}^{k}e^{-\eta|Q_i-Q_{k+1}|} + \delta|\int_{\mathbb{R}^N}Vw_{Q_{k+1}}\phi_{k+1,l}dx|. \end{cases} \tag{2.42}$$

By the equation satisfied by ϕ_0^l, and the definition of ϕ_{il}, we have

$$\bar{L}(\phi_{ij}) = \lambda_j\phi_{ij} + O(e^{-\xi\rho}),$$

thus one has

$$\int_{\mathbb{R}^N}\bar{L}(\phi_{ij})\phi_{il} = -\delta_{jl}\lambda_j\int_{\mathbb{R}^N}\phi_0^l\phi_0^j + O(e^{-\xi\rho}). \tag{2.43}$$

Recall the definition of ψ, we have

$$\begin{aligned} \int_{\mathbb{R}^N}\bar{L}(\psi)\phi_{il} &= -\int_{\mathbb{R}^N}\psi\bar{L}(\phi_{il}) \\ &= -\lambda_l\int_{\mathbb{R}^N}\phi_{il}\psi + O(e^{-\xi\rho})\|\psi\|_{H^1(B_{\frac{\rho}{2}}(Q_i))} \\ &= O(e^{-\xi\rho})\|\psi\|_{H^1(B_{\frac{\rho}{2}}(Q_i))}. \end{aligned} \tag{2.44}$$

Thus, one can get that

$$|\ell_{il}| \le C(|\int_{\mathbb{R}^N}\bar{S}\phi_{il}| + O(e^{-\xi\rho})\sum_{j\neq l}|\ell_{ij}| + O(e^{-\xi\rho})\|\psi\|_{H^1(B_{\frac{\rho}{2}}(Q_i))}) \tag{2.45}$$

for $i=1,\cdots,k$ and

$$
\begin{aligned}
|\ell_{k+1,l}| &\leq C(|\int_{\mathbb{R}^N}\bar{S}\phi_{k+1,l}|+O(e^{-\xi\rho})\sum_{j\neq l}|\ell_{k+1,j}| \qquad (2.46)\\
&+ O(e^{-\xi\rho})\|\psi\|_{H^1(B_{\frac{\rho}{2}}(Q_{k+1}))}+\sum_{j=1}^N|d_{k+1,j}|.
\end{aligned}
$$

Combining (2.39) and (2.41)-(2.46), we have

$$
\begin{cases}
|\ell_{il}|\leq Ce^{-\xi\rho}e^{-\eta|Q_i-Q_{k+1}|}\\
+\delta|\int_{\mathbb{R}^N}Vw_{Q_{k+1}}\phi_{il}dx|+e^{-\xi\rho}\|\psi\|_{H^1(B_{\frac{\rho}{2}}(Q_i))},\ i=1,...,k,\\
\\
|\ell_{k+1,l}|\leq Ce^{-\xi\rho}\sum_{i=1}^k e^{-\eta|Q_i-Q_{k+1}|}\\
+\delta|\int_{\mathbb{R}^N}Vw_{Q_{k+1}}\phi_{k+1,l}dx|+e^{-\xi\rho}\|\psi\|_{H^1(B_{\frac{\rho}{2}}(Q_{k+1}))}.
\end{cases} \qquad (2.47)
$$

Finally, we consider the estimate for ψ.

Estimate of ψ: Multiplying (2.40) by ψ and integrating over $\mathbb{R}^N$, we find

$$
\begin{aligned}
\int_{\mathbb{R}^N}\bar{L}(\psi)\psi &= -\int_{\mathbb{R}^N}\bar{S}\psi-\sum_{i=1,\cdots,k+1,j=1,\cdots,N}d_{ij}\int_{\mathbb{R}^N}\bar{L}(Z_{ij})\psi\\
&- \sum_{i=1,\cdots,k+1,l=1,\cdots,m}\ell_{il}\int_{\mathbb{R}^N}\bar{L}(\phi_{il})\psi. \qquad (2.48)
\end{aligned}
$$

We claim that

$$
\int_{\mathbb{R}^N}[-\bar{L}(\psi)\psi]\geq c_0\|\psi\|^2_{H^1(\mathbb{R}^N)} \qquad (2.49)
$$

for some constant $c_0>0$ (independent of k and $\mathbf{Q}_{k+1}$).

Indeed, since the approximate solution is exponentially decaying away from the points Q_i, we have

$$
-\int_{\mathbb{R}^N\setminus\cup_i B_{\frac{\rho}{2}}(Q_i)}\bar{L}(\psi)\psi\geq\frac{1}{2}\int_{\mathbb{R}^N\setminus\cup_i B_{\frac{\rho}{2}}(Q_i)}|\nabla\psi|^2+|\psi|^2. \qquad (2.50)
$$

Now we only need to prove the above estimate in the domain $\cup_i B_{\frac{\rho}{2}}(Q_i)$. We prove it by contradiction. Otherwise, there exists a sequence $\rho_n\to+\infty$, and $Q_i^{(n)}$ such that

$$
\int_{B_{\frac{\rho_n}{2}}(Q_i^{(n)})}|\nabla\psi_n|^2+|\psi_n|^2=1,\ \int_{B_{\frac{\rho_n}{2}}(Q_i^{(n)})}\bar{L}(\psi_n)\psi_n\to 0,\ \text{as } n\to\infty.
$$

By Fatou's Lemma, we can extract from the sequence $\psi_n(\cdot - Q_i^{(n)})$ a subsequence which will converge weakly in $H^1(\mathbb{R}^N)$ to ψ_∞, such that

$$\int_{\mathbb{R}^N} |\nabla \psi_\infty|^2 + |\psi_\infty|^2 \leq \int_{\mathbb{R}^N} f'(w)\psi_\infty^2, \tag{2.51}$$

and

$$\int_{\mathbb{R}^N} \psi_\infty \phi_0^l = \int_{\mathbb{R}^N} \psi_\infty \frac{\partial w}{\partial x_i} = 0, \text{ for } i = 1, \cdots, N, l = 1, \cdots, m. \tag{2.52}$$

From (2.51), (2.52) and (2.30), we deduce that $\psi_\infty = 0$.

Therefore

$$\psi_n \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}^N), \tag{2.53}$$

and so

$$\int_{B_{\frac{\rho_n}{2}}(Q_i^{(n)})} f'(\tilde{W})\psi_n^2 \to 0 \text{ as } n \to \infty.$$

Hence

$$\|\psi_n\|_{H^1(B_{\frac{\rho_n}{2}})} \to 0 \text{ as } n \to \infty,$$

and this contradicts the assumption that $\|\psi_n\|_{H^1} = 1$.

Therefore, (2.49) is true and using this and (2.48), we obtain

$$\begin{aligned} \|\psi\|_{H^1(\mathbb{R}^N)}^2 &\leq c(\sum_{ij} |d_{ij}| |\int_{\mathbb{R}^N} \bar{L}(Z_{ij})\psi| + \sum_{il} |\ell_{il}| |\int_{\mathbb{R}^N} \bar{L}(\phi_{il})\psi| + |\int_{\mathbb{R}^N} \bar{S}\psi|) \\ &\leq c(\sum_{ij} |d_{ij}| \|\psi\|_{H^1(B_{\frac{\rho}{2}}(Q_i))} + \sum_{il} |\ell_{il}| \|\psi\|_{H^1(B_{\frac{\rho}{2}}(Q_i))} + \|\bar{S}\|_{L^2(\mathbb{R}^N)} \|\psi\|_{H^1(\mathbb{R}^N)}). \end{aligned} \tag{2.54}$$

The above estimate and estimate (2.47) imply that

$$\begin{aligned} \|\psi\|_{H^1(\mathbb{R}^N)} \quad &\leq \quad c(\sum_{ij} |d_{ij}| + e^{-\xi\rho} \sum_{i=1}^{k} e^{-\eta|Q_{k+1} - Q_i|} \\ &+ \quad \delta \int_{\mathbb{R}^N} |V| w_{Q_{k+1}} dx + \|\bar{S}\|_{L^2(\mathbb{R}^N)}). \end{aligned} \tag{2.55}$$

From (2.39) (2.34) and (2.55) (and recalling that $\eta > \frac{1}{2}$), we have

$$\begin{aligned}\|\varphi_{k+1}\|_{H^1(\mathbb{R}^N)} &\le C(e^{-\xi\rho}\sum_{i=1}^k e^{-\eta|Q_{k+1}-Q_i|} + \delta\int_{\mathbb{R}^N}|V|w_{Q_{k+1}}dx + \|\bar{S}\|_{L^2}) \\ &\le C(e^{-\xi\rho}\sum_{i=1}^k e^{-\eta|Q_{k+1}-Q_i|} + e^{-\xi\rho}(\sum_{i=1}^k w(|Q_{k+1}-Q_i|))^{\frac{1}{2}} \\ &+ \delta\int_{\mathbb{R}^N}|V|w_{Q_{k+1}}dx + \delta(\int_{\mathbb{R}^N}V^2w_{Q_{k+1}}^2dx)^{\frac{1}{2}}). \qquad (2.56)\end{aligned}$$

Since we choose $\eta > \frac{1}{2}$, we have

$$(\sum_{i=1}^k e^{-\eta|Q_i-Q_{k+1}|})^2 \le c\sum_{i=1}^k w(|Q_i-Q_{k+1}|) \qquad (2.57)$$

by the definition of the configuration space.

By (2.56) and (2.57), we thus obtain that

$$\begin{aligned}\|\varphi_{k+1}\|_{H^1(\mathbb{R}^N)} &\le C(e^{-\xi\rho}(\sum_{i=1}^k w(|Q_{k+1}-Q_i|))^{\frac{1}{2}} + \delta\int_{\mathbb{R}^N}|V|w_{Q_{k+1}}dx \\ &+ \delta(\int_{\mathbb{R}^N}V^2w_{Q_{k+1}}^2dx)^{\frac{1}{2}}). \qquad (2.58)\end{aligned}$$

The estimate (2.25) then follows, and we are done with the proof of Lemma 2.2.

Moreover, from the estimate (2.47) and (2.39), and taking into consideration that χ_i is supported in $B_{\frac{\rho}{2}}(Q_i)$, we can get a more accurate estimate on φ_{k+1} using Holder inequality:

$$\begin{aligned}\|\varphi_{k+1}\|_{H^1(\mathbb{R}^N)} &\le C(e^{-\xi\rho}(\sum_{i=1}^k w(|Q_{k+1}-Q_i|))^{\frac{1}{2}} + \delta\sum_{i=1,\cdots,k+1}(\int_{B_{\frac{\rho}{2}}(Q_i)}V^2w_{Q_{k+1}}^2dx)^{\frac{1}{2}} \\ &+ \delta(\int_{\mathbb{R}^N}V^2w_{Q_{k+1}}^2dx)^{\frac{1}{2}}). \qquad (2.59)\end{aligned}$$

□

2.3. **The reduced problem: a maximization procedure.** In this section, we study a maximization problem. Fix $\mathbf{Q}_k \in \Lambda_k$, we define the new functional $\mathcal{M} : \Lambda_k \to \mathbb{R}$

$$\mathcal{M}(\mathbf{Q}_k) = J(u_{\mathbf{Q}_k}) = J[w_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}]. \qquad (2.60)$$

Define

$$\mathcal{C}_k = \sup_{\mathbf{Q}_k \in \Lambda_k} \{\mathcal{M}(\mathbf{Q}_k)\}. \tag{2.61}$$

Note that $\mathcal{M}(\mathbf{Q}_k)$ is continuous in $\mathbf{Q}_k$. We will show below that the maximization problem has a solution. Let $\mathcal{M}(\bar{\mathbf{Q}}_k)$ be the maximum where $\bar{\mathbf{Q}}_k = (\bar{Q}_1, \cdots, \bar{Q}_k) \in \bar{\Lambda}_k$, that is

$$\mathcal{M}(\bar{Q}_1, \cdots, \bar{Q}_k) = \max_{\mathbf{Q}_k \in \Lambda_k} \mathcal{M}(\mathbf{Q}_k), \tag{2.62}$$

and we denote the solution by $u_{\bar{Q}_1, \cdots, \bar{Q}_k}$.

We first prove that the maximum can be attained at a finite point for each $\mathcal{C}_k$. This prevents noncompactness to infinity.

Lemma 2.5. *Let V satisfy $(H1)-(H3)$ and let ρ, δ be as in Proposition 2.1. Then, for all k:*

- *There exists $\mathbf{Q}_k = (Q_1, Q_2, \cdots, Q_k) \in \Lambda_k$ such that*

$$\mathcal{C}_k = \mathcal{M}(\mathbf{Q}_k); \tag{2.63}$$

- *There holds*

$$\mathcal{C}_{k+1} > \mathcal{C}_k + I(w), \tag{2.64}$$

where $I(w)$ is the energy of the solution w of (1.6):

$$I(w) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) - \int_{\mathbb{R}^N} F(w) dx. \tag{2.65}$$

Proof. In this part, we follow the proofs in [13] but we use the estimates we derived in Section 2.2. We prove Lemma 2.5 by induction.

Base step: We claim that $\mathcal{C}_1 > I(w)$ and $\mathcal{C}_1$ can be attained at a finite point. Indeed, first using standard Lyapunov-Schmidt reduction, similar to the derivation of Lemma 3.3 in [29], we have

$$\|\phi_Q\|_{H^1} \le c \|\delta V w_Q\|_{L^2}. \tag{2.66}$$

for some $c > 0$ independent of Q. Assuming that $|Q| \to \infty$, then we have

$$
\begin{aligned}
J(u_Q) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_Q|^2 + u_Q^2 - \int_{\mathbb{R}^N} F(u_Q)dx + \frac{1}{2}\int_{\mathbb{R}^N} \delta V u_Q^2 dx \\
&= I(w) + \int_{\mathbb{R}^N} \nabla w_Q \nabla \phi_Q + w_Q\phi_Q - f(w_Q)\phi_Q + \frac{1}{2}\int_{\mathbb{R}^N} \delta V w_Q^2 dx \\
&- \int_{\mathbb{R}^N} F(u_Q) - F(w_Q) - f(w_Q)\phi_Q + \|\phi_Q\|_{H^1}^2 \\
&= I(w) + \frac{1}{2}\int_{\mathbb{R}^N} \delta V w_Q^2 dx + O(\|\phi_Q\|_{H^1}^2) \\
&\geq I(w) + \frac{1}{2}\int_{\mathbb{R}^N} \delta V w_Q^2 dx - C\int_{\mathbb{R}^N} \delta^2 V^2 w_Q^2 dx \\
&\geq I(w) + \frac{1}{4}\int_{\mathbb{R}^N} \delta V w_Q^2 dx \\
&(V \text{ positive at } \infty) \\
&\geq I(w) + \frac{1}{4}[\int_{B_{\frac{\rho}{2}}(Q)} \delta V w_Q^2 dx - \sup_{B_{\frac{|Q|}{4}}(0)} |w_Q|^{\frac{3}{2}} \int_{suppV^-} \delta |V| w_Q^{\frac{1}{2}} dx] \\
&\geq I(w) + \frac{1}{4}\int_{B_{\frac{\rho}{2}}(Q)} \delta V w_Q^2 dx - O(\delta e^{-\frac{9}{8}|Q|})
\end{aligned}
$$

where for the last line, we use the exponential decay of w_Q.

By the slow decay assumption on the potential V, we obtain

$$
\frac{1}{4}\int_{B_{\frac{\rho}{2}}(Q)} \delta V w_Q^2 dx - O(\delta e^{-\frac{9}{8}|Q|}) > \delta[e^{-\rho}e^{-\overline{\eta}|Q|} - O(e^{-\frac{9}{8}|Q|})] > 0,
$$

for $|Q|$ large enough.

Therefore, one can get that $\mathcal{C}_1 \geq J(u_Q) > I(w)$, and we have proven the first part of our claim.

Let us prove now that $\mathcal{C}_1$ can be attained at a finite point. If not, then there exists a sequence $\{Q_i\} \to \infty$ such that $\lim_{i\to\infty} \mathcal{M}(Q_i) = \mathcal{C}_1$.

Observe that

$$
\begin{aligned}
&J(u_{Q_i})\\
&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla(w_{Q_i}+\phi_{Q_i})|^2+|w_{Q_i}+\phi_{Q_i}|^2dx-\int_{\mathbb{R}^N}F(w_{Q_i}+\phi_{Q_i})dx\\
&+\frac{1}{2}\int_{\mathbb{R}^N}\delta V(w_{Q_i}+\phi_{Q_i})^2dx\\
&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla w_{Q_i}|^2+|w_{Q_i}|^2dx-\int_{\mathbb{R}^N}F(w_{Q_i})dx\\
&+\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\phi_{Q_i}|^2+|\phi_{Q_i}|^2dx+\int_{\mathbb{R}^N}\nabla w_{Q_i}\nabla\phi_{Q_i}+w_{Q_i}\phi_{Q_i}-f(w_{Q_i})\phi_{Q_i}dx\\
&-\int_{\mathbb{R}^N}F(w_{Q_i}+\phi_{Q_i})-F(w_{Q_i})-f(w_{Q_i})\phi_{Q_i}dx+\frac{1}{2}\int_{\mathbb{R}^N}\delta V(w_{Q_i}+\phi_{Q_i})^2dx\\
&\le I(w)+c\|S(w_{Q_i})\|^2_{L^2(\mathbb{R}^N)}+\frac{1}{2}\int_{\mathbb{R}^N}\delta V(w_{Q_i}+\phi_{Q_i})^2dx\\
&\le I(w)+O(\int_{\mathbb{R}^N}\delta^2V^2w_{Q_i}^2dx)+\frac{1}{2}\int_{\mathbb{R}^N}\delta V(w_{Q_i}+\phi_{Q_i})^2dx.
\end{aligned}
$$

Since $V(x)\to 0$ as $|x|\to\infty$, we have

$$O(\int_{\mathbb{R}^N}\delta^2V^2w_{Q_i}^2dx)+\frac{1}{2}\int_{\mathbb{R}^N}\delta V(w_{Q_i}+\phi_{Q_i})^2dx\to 0 \text{ as } i\to\infty,$$

and therefore,

$$\mathcal{C}_1=\lim_{i\to\infty}J(u_{Q_i})\le I(w).$$

This is a contradiction. Thus $\mathcal{C}_1$ can be attained at a finite point.

Induction step: Assume that for $k\ge 1$, there exists $\mathbf{Q}_k=(\bar{Q}_1,\cdots,\bar{Q}_k)\in\Lambda_k$ such that $\mathcal{C}_k=\mathcal{M}(\mathbf{Q}_k)$, and we denote the solution by $u_{\bar{Q}_1,\cdots,\bar{Q}_k}$.

Next, we prove that there exists $(Q_1,\cdots,Q_{k+1})\in\Lambda_{k+1}$ such that $\mathcal{C}_{k+1}$ can be attained.

Let $((Q_1^{(n)},\cdots,Q_{k+1}^{(n)}))_n$ be a sequence such that

$$\mathcal{C}_{k+1}=\lim_{n\to\infty}\mathcal{M}(Q_1^{(n)},\cdots,Q_{k+1}^{(n)}). \tag{2.67}$$

We claim that $(Q_1^{(n)},\cdots,Q_{k+1}^{(n)})$ is bounded. We prove it by contradiction. Without loss of generality, we assume that $|Q_{k+1}^{(n)}|\to\infty$ as $n\to\infty$. In the following we omit the index n for simplicity.

First, we observe that

$$
\begin{aligned}
&J(u_{Q_1,\cdots,Q_{k+1}}) \qquad (2.68)\\
&= J(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}}+\varphi_{k+1})\\
&= J(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}})+\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\varphi_{k+1}|^2+|\varphi_{k+1}|^2+\delta V\varphi_{k+1}^2dx\\
&+\int_{\mathbb{R}^N}\nabla(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}})\nabla\varphi_{k+1}+(1+\delta V)(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}})\varphi_{k+1}\\
&-f(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}})\varphi_{k+1}dx\\
&-\int_{\mathbb{R}^N}F(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}}+\varphi_{k+1})-F(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}})\\
&-f(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}})\varphi_{k+1}dx\\
&= J(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}})+R_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&J(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}}) \qquad (2.69)\\
&= J(u_{Q_1,\cdots,Q_k})+I(w_{Q_{k+1}})+\frac{1}{2}\int_{\mathbb{R}^N}\delta V w_{Q_{k+1}}^2dx\\
&+\int_{\mathbb{R}^N}\nabla u_{Q_1,\cdots,Q_k}\nabla w_{Q_{k+1}}+(1+\delta V)u_{Q_1,\cdots,Q_k}w_{Q_{k+1}}dx\\
&-\int_{\mathbb{R}^N}F(u_{Q_1,\cdots,Q_k}+w_{Q_{k+1}})-F(u_{Q_1,\cdots,Q_k})-F(w_{Q_{k+1}})dx\\
&\leq \mathcal{C}_k+I(w)+\frac{1}{2}\int_{\mathbb{R}^N}\delta V w_{Q_{k+1}}^2dx\\
&-\int_{\mathbb{R}^N}\sum_{i=1,\cdots,k,j=1,\cdots,N}c_{ij}Z_{ij}w_{Q_{k+1}}dx-\int_{\mathbb{R}^N}f(w_{Q_{k+1}})(w_{Q_1,\cdots,Q_k}+\phi_{\mathbf{Q}_k})\\
&+O(e^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|))\\
&= \mathcal{C}_k+I(w)+\frac{1}{2}\int_{\mathbb{R}^N}\delta V w_{Q_{k+1}}^2dx+R_2+O(e^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|)),
\end{aligned}
$$

where R_1 and R_2 are defined by the above computations.

Note that by (2.25) and (2.34), we have

$$\begin{aligned} R_1 &= O(\|\varphi_{k+1}\|^2_{H^1(\mathbb{R}^N)} + \|\bar{S}(u_{Q_1,\cdots,Q_k} + w_{Q_{k+1}})\|_{H^1(\mathbb{R}^N)}\|\varphi_{k+1}\|_{H^1(\mathbb{R}^N)}) \\ &+ \sum_{i=1,\cdots,k,j=1,\cdots,N} \int_{\mathbb{R}^N} c_{ij} Z_{ij}\varphi_{k+1} dx \qquad (2.70) \\ &= O(e^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|) + \delta^2(\int_{\mathbb{R}^N} V^2 w^2_{Q_{k+1}}dx + (\int_{\mathbb{R}^N} V w_{Q_{k+1}}dx)^2), \end{aligned}$$

where we have used the orthogonality condition satisfied by φ_{k+1}.

Next, we estimate R_2. By estimate (2.10), and that the definition of Z_{ij}, we have

$$|\sum_{i=1,\cdots,k,j=1,\cdots,N} c_{ij}\int_{\mathbb{R}^N} Z_{ij} w_{Q_{k+1}} dx| \le c e^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|). \qquad (2.71)$$

By the equation satisfied by ϕ_k, we have

$$\Delta\phi_{\mathbf{Q}_k} - \phi_{\mathbf{Q}_k} + f'(w_{\mathbf{Q}_k})\phi_{\mathbf{Q}_k} = -S(w_{\mathbf{Q}_k}) - N(\phi_{\mathbf{Q}_k}) + \delta V\phi_{\mathbf{Q}_k} + \sum_{i=1,\cdots,k,j=1,\cdots,N} c_{ij}Z_{ij},$$

where

$$N(\phi_{\mathbf{Q}_k}) = f(w_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}) - f(w_{\mathbf{Q}_k}) - f'(w_{\mathbf{Q}_k})\phi_{\mathbf{Q}_k}. \qquad (2.72)$$

We derive that

$$\begin{aligned} &\int_{\mathbb{R}^N} f(w_{Q_{k+1}})\phi_{\mathbf{Q}_k} dx = -\int_{\mathbb{R}^N} (\Delta - 1) w_{Q_{k+1}}\phi_{\mathbf{Q}_k} dx \\ &= -\int_{\mathbb{R}^N} (\Delta-1)\phi_{\mathbf{Q}_k} w_{Q_{k+1}} dx \\ &= \int_{\mathbb{R}^N} (S(w_{\mathbf{Q}_k}) + N(\phi_{\mathbf{Q}_k}) - \delta V\phi_{\mathbf{Q}_k} \\ &- \sum_{i=1,\cdots,k,j=1,\cdots,N} c_{ij}Z_{ij} + f'(w_{\mathbf{Q}_k})\phi_{\mathbf{Q}_k}) w_{Q_{k+1}} dx. \end{aligned}$$

We can further choose η such that $(1+\sigma)\eta > 1$ and recall that $\eta + \sigma > 1$, one can get from the definition of $N(\phi_{\mathbf{Q}_k})$ and f that

$$|N(\phi_{\mathbf{Q}_k})| \le C\sum_{i=1}^k e^{-\xi\rho} e^{-(1+\sigma)\eta|x-Q_i|},$$

and

$$|f'(w_{\mathbf{Q}_k})\phi_{\mathbf{Q}_k}| \le C\sum_{i=1}^k e^{-\xi\rho} e^{-(\sigma+\eta)|x-Q_i|}.$$

Using the above two estimates and Lemma 2.3, we can obtain

$$\Big|\int_{\mathbb{R}^N}(N(\phi_{\mathbf{Q}_k})+f'(w_{\mathbf{Q}_k})\phi_{\mathbf{Q}_k})w_{Q_{k+1}}dx\Big|\leq Ce^{-\xi\rho}\sum_{i=1}^{k}w(|Q_{k+1}-Q_i|)).$$

Using the estimate on (2.10) on c_{ij}, and using Lemma 2.3, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}\sum c_{ij}Z_{ij}w_{Q_{k+1}}dx\Big| &\leq c\sum e^{-\frac{\xi}{2}\rho}\int_{\mathbb{R}^N}e^{-\frac{\xi}{2}\rho}Z_{ij}w_{Q_{k+1}}\\ \leq ce^{-\frac{\xi}{2}\rho}\sum_{i=1}^{k}w(|Q_{k+1}-Q_i|),\end{aligned}$$

and using the definition of S and the estimate on $\phi_{\mathbf{Q}_k}$ and using Lemma 2.3,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}(S(w_{\mathbf{Q}_k})-\delta V\phi_{\mathbf{Q}_k})w_{Q_{k+1}}dx\Big|\\ &\leq c(\delta\int_{\mathbb{R}^N}Vw_{\mathbf{Q}_k}w_{Q_{k+1}}dx+\delta e^{-\xi\rho}\int_{\mathbb{R}^N}\sum_{i=1}^{k}e^{-\eta|x-Q_i|}Vw_{Q_{k+1}}dx\\ &+|\int_{\mathbb{R}^N}(f(w_{\mathbf{Q}_k})-\sum_{i=1}^{k}f(w_{Q_i}))w_{Q_{k+1}}|).\end{aligned}$$

Since w_{Q_i} is exponentially decaying, and f is $C^{1+\sigma}$, using Lemma 2.3, we have

$$\begin{aligned}&|\int_{\mathbb{R}^N}(f(w_{\mathbf{Q}_k})-\sum_{i=1}^{k}f(w_{Q_i}))w_{Q_{k+1}}|\\ &\leq\sum_{i=1}^{k}[\int_{B_{\frac{\rho}{4}}(Q_i)}f'(w_{Q_i})\sum_{j\neq i}w_{Q_j}w_{Q_{k+1}}+\int_{\mathbb{R}^N\setminus\cup_iB_{\frac{\rho}{4}}(Q_i)}f(w_{Q_i})w_{Q_{k+1}}]\\ &\leq\sum_{i=1}^{k}[e^{-\frac{\rho}{4}}\int_{B_{\frac{\rho}{4}}(Q_i)}f(w_{Q_i})w_{Q_{k+1}}+e^{-\frac{\sigma}{2}\frac{\rho}{4}}e^{-(1+\frac{\sigma}{2})|x-Q_i|}w_{Q_{k+1}}]\\ &\leq e^{-\xi\rho}\sum_{i=1}^{k}w(|Q_i-Q_{k+1}|),\end{aligned}$$

one can obtain that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}(S(w_{\mathbf{Q}_k})-\delta V\phi_{\mathbf{Q}_k})w_{Q_{k+1}}dx\Big|\\&\le c(\delta\int_{\mathbb{R}^N}Vw_{\mathbf{Q}_k}w_{Q_{k+1}}dx+\delta e^{-\xi\rho}\int_{\mathbb{R}^N}\sum_{i=1}^k e^{-\eta|x-Q_i|}Vw_{Q_{k+1}}dx\\&+e^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|)).\end{aligned}$$

By the above three estimates, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}f(w_{Q_{k+1}})\phi_{\mathbf{Q}_k}dx\Big| &\le c(\delta e^{-\xi\rho}\int_{\mathbb{R}^N}\sum_{i=1}^k e^{-\eta|x-Q_i|}Vw_{Q_{k+1}}dx+\delta\int_{\mathbb{R}^N}Vw_{\mathbf{Q}_k}w_{Q_{k+1}}dx\\&+ ce^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|)).\end{aligned}\tag{2.73}$$

Therefore,

$$\begin{aligned}&R_2\le -\frac{1}{4}\gamma_1\sum_{i=1}^k w(|Q_{k+1}-Q_i|)+O(e^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|))\\&+\delta e^{-\xi\rho}\int_{\mathbb{R}^N}\sum_{i=1}^k e^{-\eta|x-Q_i|}Vw_{Q_{k+1}}dx+\delta\int_{\mathbb{R}^N}Vw_{\mathbf{Q}_k}w_{Q_{k+1}}dx.\end{aligned}\tag{2.74}$$

Thus combining (2.68), (2.69), (2.70) and (2.74), we obtain

$$\begin{aligned}&J(u_{Q_1,\cdots,Q_{k+1}})\\&\le \mathcal{C}_k+I(w)+\frac{1}{2}\int_{\mathbb{R}^N}\delta Vw^2_{Q_{k+1}}dx-\frac{1}{4}\gamma_1\sum_{i=1}^k w(|Q_{k+1}-Q_i|)\\&+O(e^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|)+\delta e^{-\xi\rho}\int_{\mathbb{R}^N}\sum_{i=1}^k e^{-\eta|x-Q_i|}Vw_{Q_{k+1}}dx\\&+\delta\int_{\mathbb{R}^N}Vw_{\mathbf{Q}_k}w_{Q_{k+1}}dx+\delta^2\int_{\mathbb{R}^N}V^2w^2_{Q_{k+1}}+\delta^2(\int_{\mathbb{R}^N}Vw_{Q_{k+1}})^2).\end{aligned}\tag{2.75}$$

By the assumption that $|Q^{(n)}_{k+1}|\to\infty$, we have

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N}\delta Vw^2_{Q^{(n)}_{k+1}}dx+\delta e^{-\xi\rho}\int_{\mathbb{R}^N}\sum_{i=1}^k e^{-\eta|x-Q_i|}Vw_{Q^{(n)}_{k+1}}dx+\delta\int_{\mathbb{R}^N}Vw_{\mathbf{Q}_k}w_{Q^{(n)}_{k+1}}dx\\&+\delta^2\int_{\mathbb{R}^N}V^2w^2_{Q^{(n)}_{k+1}}+\delta^2(\int_{\mathbb{R}^N}Vw_{Q^{(n)}_{k+1}})^2\to 0 \text{ as } n\to\infty,\end{aligned}\tag{2.76}$$

and

$$-\frac{1}{4}\gamma_1\sum_{i=1}^{k} w(|Q_{k+1}-Q_i|)+O(e^{-\xi\rho}\sum_{i=1}^{k} w(|Q_{k+1}-Q_i|))<0. \quad (2.77)$$

Combining (2.67), (2.75), (2.76)and (2.77), we have

$$\mathcal{C}_{k+1}\leq \mathcal{C}_k+I(w). \quad (2.78)$$

On the other hand, by the assumption that $\mathcal{C}_k$ can be attained, without loss of generality, we assume that $\mathcal{C}_k$ is attained at $(\bar{Q}_1,\cdots,\bar{Q}_k)$, then one can choose another point Q_{k+1} which is far away from the k points $\bar{Q}_1,\cdots,\bar{Q}_k$.

Next let's consider the solution concentrated at the points $(\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1})$, and we denote the solution by $u_{\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}}$. Then similar to the above argument, using the estimate (2.59) of φ_{k+1} instead of (2.25), we have the following estimate:

$$\begin{aligned} J(u_{\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}}) &= J(u_{\bar{Q}_1,\cdots,\bar{Q}_k})+I(w)+\Big(\frac{1}{2}\int_{\mathbb{R}^N}\delta V w_{Q_{k+1}}^2dx \quad (2.79)\\ &+ O(\sum_{i=1,\cdots,k+1}(\int_{B_{\frac{\rho}{2}}(Q_i)}\delta^2V^2w_{Q_{k+1}}^2dx)^{\frac{1}{2}})^2+O(\int_{\mathbb{R}^N}\delta^2V^2w_{Q_{k+1}}^2dx)\\ &- O(\sum_{i=1}^{k} w(|Q_{k+1}-\bar{Q}_i|))\\ &+ O(\delta e^{-\xi\rho}\int_{\mathbb{R}^N}\sum_{i=1}^{k}e^{-\eta|x-\bar{Q}_i|}Vw_{Q_{k+1}}dx+\delta\int_{\mathbb{R}^N}Vw_{\bar{\mathbf{Q}}_k}w_{Q_{k+1}}dx)\Big)\\ &= J(u_{\bar{Q}_1,\cdots,\bar{Q}_k})+I(w)+R_3, \end{aligned}$$

where R_3 is defined by the above terms in the big bracket.

By the asymptotic behavior of V at infinity, i.e. $\lim_{|x|\to\infty}V(x)e^{\bar{\eta}|x|}=+\infty$ as $|x|\to\infty$, for some $\bar{\eta}<1$, we further choose $\eta>\bar{\eta}$. Then we can choose Q_{k+1} such that

$$|Q_{k+1}|>>\frac{\max_{i=1}^{k}|\bar{Q}_i|+\ln\delta}{\eta-\bar{\eta}}. \quad (2.80)$$

Thus we obtain

$$R_3\geq C\delta e^{-\bar{\eta}|Q_{k+1}|}-O(\sum_{i=1,\cdots,k}e^{-\eta|\bar{Q}_i-Q_{k+1}|})>0. \quad (2.81)$$

So

$$\mathcal{C}_{k+1}\geq J(u_{\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}})>\mathcal{C}_k+I(w). \quad (2.82)$$

Combining (2.82) and (2.78), one gets that

$$\mathcal{C}_k + I(w) < \mathcal{C}_{k+1} \leq \mathcal{C}_k + I(w). \tag{2.83}$$

Obviously this is a contradiction. So we get that $\mathcal{C}_{k+1}$ can be attained at a finite point in Λ_{k+1}.

Moreover, from the proof above, we derived the following relation between $\mathcal{C}_{k+1}$ and $\mathcal{C}_k$:

$$\mathcal{C}_{k+1} > \mathcal{C}_k + I(w). \tag{2.84}$$

□

Next we have the following Proposition:

Proposition 2.2. *The maximization problem*

$$\max_{\mathbf{Q}\in\bar{\Lambda}_k} \mathcal{M}(\mathbf{Q}) \tag{2.85}$$

has a solution $\mathbf{Q} \in \Lambda_k^\circ$, *i.e., the interior of* Λ_k.

Proof. We prove it by contradiction again. If $\mathbf{Q} = (\bar{Q}_1, \cdots, \bar{Q}_k) \in \partial\Lambda_k$, then there exists (i, j) such that $|\bar{Q}_i - \bar{Q}_j| = \rho$. Without loss of generality, we assume $(i, j) = (i, k)$. Then following the estimates in (2.68), (2.69), (2.71) and (2.74), we have

$$\begin{aligned}
\mathcal{C}_k \quad &= J(u_{\bar{Q}_1,\cdots,\bar{Q}_k}) \\
&\leq \mathcal{C}_{k-1} + I(w) + \frac{1}{2}\int_{\mathbb{R}^N} \delta V w_{\bar{Q}_k}^2 dx \\
&-\frac{1}{4}\gamma_1 \sum_{i=1}^{k-1} w(|\bar{Q}_k - \bar{Q}_i|) + O(e^{-\xi\rho}\sum_{i=1}^{k-1} w(|\bar{Q}_k - \bar{Q}_i|)) + O(\delta) \\
&\leq \mathcal{C}_{k-1} + I(w) \\
&+O(\delta) - \frac{1}{4}\gamma_1 \sum_{i=1}^{k-1} w(|\bar{Q}_k - \bar{Q}_i|) + O(e^{-\xi\rho}\sum_{i=1}^{k-1} w(|\bar{Q}_k - \bar{Q}_i|)).
\end{aligned} \tag{2.86}$$

By the definition of the configuration set, we observe that given a ball of size ρ, there are at most $c_N := 6^N$ number of non-overlapping balls of size ρ surrounding this ball. Since $|\bar{Q}_i - \bar{Q}_k| = \rho$, we have

$$\sum_{i=1}^{k-1} w(|\bar{Q}_k - \bar{Q}_i|) = w(|\bar{Q}_i - \bar{Q}_k|) + \sum_{j\neq i} w(|\bar{Q}_j - \bar{Q}_k|)$$

and

$$\sum_{j\neq i} w(|\bar{Q}_j - \bar{Q}_k|) \leq e^{-\rho} + c_N e^{-\rho-\frac{\rho}{2}} + \cdots + c_N^j e^{-\rho-\frac{j\rho}{2}} + \cdots$$
$$\leq Ce^{-\rho}\sum_{j=0}^{\infty} e^{j\ln c_N - \frac{j\rho}{2}}$$
$$\leq Ce^{-\rho}, \tag{2.87}$$

if $c_N < e^{\frac{\rho}{2}}$, which is true for ρ large enough.

Therefore,

$$\mathcal{C}_k \leq \mathcal{C}_{k-1} + I(w) + c\delta - \frac{1}{4}\gamma_1 w(\rho) + O(e^{-(1+\xi)\rho}) \tag{2.88}$$
$$< \mathcal{C}_{k-1} + I(w),$$

which is a contradiction to (2.64) of Lemma 2.5. Thus we prove the Proposition. □

2.4. **Proof of Theorem 1.1.** In this section, we apply the results in Section 2.1, 2.2 and Section 2.3 to prove Theorem 1.1.

Proof of Theorem 1.1: By Proposition 2.1 in Section 2.2, there exists ρ_0 such that for $\rho > \rho_0$, we have C^1 map which, to any $\mathbf{Q}^\circ \in \Lambda_k$, associates $\phi_{\mathbf{Q}^\circ}$ such that

$$S(w_{\mathbf{Q}^\circ} + \phi_{\mathbf{Q}^\circ}) = \sum_{i=1,\cdots,k, j=1,\cdots,N} c_{ij} Z_{ij}, \quad \int_{\mathbb{R}^N} \phi_{\mathbf{Q}^\circ} Z_{ij} dx = 0, \tag{2.89}$$

for some constants $\{c_{ij}\} \in \mathbb{R}^{kN}$.

From Proposition 2.2 in Section 2.3, there is a $\mathbf{Q}^\circ \in \Lambda_k^\circ$ that achieves the maximum for the maximization problem in Proposition 2.2. Let $u_{\mathbf{Q}^\circ} = w_{\mathbf{Q}^\circ} + \phi_{\mathbf{Q}^\circ}$. Then we have

$$D_{Q_{ij}}|_{Q_i=Q_i^\circ}\mathcal{M}(\mathbf{Q}^\circ) = 0, \quad i = 1, \cdots, k, \quad j = 1, \cdots, N. \tag{2.90}$$

Hence we have

$$\int_{\mathbb{R}^N} \nabla u_{\mathbf{Q}} \nabla \frac{\partial(w_{\mathbf{Q}} + \phi_{\mathbf{Q}})}{\partial Q_{ij}}|_{Q_i=Q_i^\circ} + (1+\delta V) u_{\mathbf{Q}} \frac{\partial(w_{\mathbf{Q}} + \phi_{\mathbf{Q}})}{\partial Q_{ij}}|_{Q_i=Q_i^\circ}$$
$$-f(u_{\mathbf{Q}})\frac{\partial(w_{\mathbf{Q}} + \phi_{\mathbf{Q}})}{\partial Q_{ij}}|_{Q_i=Q_i^\circ} = 0,$$

which gives

$$\sum_{i=1,\cdots,k,\ j=1,\cdots,N} c_{ij} \int_{\mathbb{R}^N} Z_{ij} \frac{\partial(w_{\mathbf{Q}} + \phi_{\mathbf{Q}})}{\partial Q_{sl}}|_{Q_s=Q_s^\circ} = 0, \tag{2.91}$$

for $s = 1, \cdots, k, l = 1, \cdots, N$. We claim that (2.91) is a diagonally dominant system. In fact, since $\int_{\mathbb{R}^N} \phi_{\mathbf{Q}} Z_{sl} dx = 0$, we have that

$$\int_{\mathbb{R}^N} Z_{sl} \frac{\partial \phi_{\mathbf{Q}}}{\partial Q_{ij}} |_{Q_i = Q_i^\circ} = -\int_{\mathbb{R}^N} \phi_{\mathbf{Q}} \frac{\partial Z_{sl}}{\partial Q_{ij}} = 0, \text{ if } s \neq i,$$

the last equality is because $\frac{\partial Z_{sl}}{\partial Q_{ij}} = 0$ if $s \neq i$.

If $s = i$, we have

$$|\int_{\mathbb{R}^N} Z_{il} \frac{\partial \phi_{\mathbf{Q}}}{\partial Q_{ij}} |_{Q_i = Q_i^\circ}| = | - \int_{\mathbb{R}^N} \phi_{\mathbf{Q}} \frac{\partial Z_{il}}{\partial Q_{ij}}| \\ \leq C \|\phi_{\mathbf{Q}}\|_* = O(e^{-\xi \rho}).$$

For $s \neq i$, we have

$$\int_{\mathbb{R}^N} Z_{sl} \frac{\partial w_{\mathbf{Q}}}{\partial Q_{ij}} = O(e^{-\frac{|Q_i - Q_s|}{2}}).$$

For $s = i$, recalling the definition of Z_{ij}, we have

$$\begin{aligned} \int_{\mathbb{R}^N} Z_{sl} \frac{\partial w_{\mathbf{Q}}}{\partial Q_{sj}} &= -\int_{\mathbb{R}^N} \chi_s \frac{\partial w_{Q_s}}{\partial x_l} \frac{\partial w_{Q_s}}{\partial x_j} \\ &= -\delta_{lj} \int_{\mathbb{R}^N} (\frac{\partial w}{\partial x_j})^2 + O(e^{-\xi \rho}). \end{aligned} \tag{2.92}$$

Thus one can get that for each (s, l), the off-diagonal term gives

$$\begin{aligned} &\sum_{s \neq i} \int_{\mathbb{R}^N} Z_{sl} \frac{\partial (w_{\mathbf{Q}} + \phi_{\mathbf{Q}})}{\partial Q_{ij}} |_{Q_i = Q_i^\circ} + \sum_{s = i, l \neq j} \int_{\mathbb{R}^N} Z_{sl} \frac{\partial (w_{\mathbf{Q}} + \phi_{\mathbf{Q}})}{\partial Q_{sj}} |_{Q_i = Q_i^\circ} \\ &= O(e^{-\frac{\rho}{2}}) + O(e^{-\xi \rho}) \qquad (2.93) \\ &= O(e^{-\xi \rho}). \end{aligned}$$

So from (2.92) and (2.93), we can see that equation (2.91) becomes a system of homogeneous equations for c_{sl}, and the matrix of the system is nonsingular. Therefore $c_{sl} = 0$ for $s = 1, \cdots, k, l = 1, \cdots, N$. Hence $u_{\mathbf{Q}^\circ} = w_{\mathbf{Q}^\circ} + \phi_{\mathbf{Q}^\circ}$ is a solution of (1.1).

Similar to the argument in Section 6 of [31], one can show that $u_{\mathbf{Q}^\circ} > 0$ and it has exactly k local maximum points for ρ large enough.

3. Synchronized vector solutions and the proof of Theorem 1.2

In this section, we consider the elliptic system (1.21) and prove Theorem 1.2. Since we use the same method to deal with the system as in Section 2, we may use the same notation as in Section 2.

3.1. **Notations and Lyapunov-Schmidt reduction.** Let $N \leq 3$ and w be the unique solution of

$$\begin{cases} \Delta w - w + w^3 = 0, \\ w(0) = \max_{y\in\mathbb{R}^N} w(y), \ w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{3.1}$$

It is known that the following asymptotic behavior holds:

$$w(r) = A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})), \ w'(r) = -A_N r^{-\frac{N-1}{2}} e^{-r}(1 + O(\frac{1}{r})), \tag{3.2}$$

for r large, where $A_N > 0$ is a constant.

Note that the limit system as $\delta \to 0$ for (1.21) is

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta v^2 u, \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v. \end{cases} \tag{3.3}$$

It is easy to see that the following pair

$$(U, V) = (\alpha w, \gamma w) \tag{3.4}$$

solves (3.3) provided that $\beta > \max\{\mu_1, \mu_2\}$ or $-\sqrt{\mu_1\mu_2} < \beta < \min\{\mu_1, \mu_2\}$, where

$$\alpha = \sqrt{\frac{\mu_2 - \beta}{\mu_1\mu_2 - \beta^2}} > 0, \ \gamma = \sqrt{\frac{\mu_1 - \beta}{\mu_1\mu_2 - \beta^2}} > 0. \tag{3.5}$$

(It has been proved in [51] that for $\beta > \max\{\mu_1, \mu_2\}$, all positive solutions to (3.3) are given by (3.4).)

We will use (U, V) as the building blocks for the solution of (1.21).

Let $\rho > 0$ and the configuration space Λ_k be defined as in Section 1. For $\mathbf{Q}_k = (Q_1, \cdots, Q_k) \in \Lambda_k$, we define

$$(U_{Q_i}, V_{Q_i}) = (U(x - Q_i), V(x - Q_i)), \tag{3.6}$$

and the approximate solution to be $(U_{\mathbf{Q}_k}, V_{\mathbf{Q}_k})$ where

$$U_{\mathbf{Q}_k} = \sum_{i=1}^{k} U_{Q_i}, \ V_{\mathbf{Q}_k} = \sum_{i=1}^{k} V_{Q_i}. \tag{3.7}$$

Denote by

$$S\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \Delta u - (1 + \delta a(x))u + \mu_1 u^3 + \beta v^2 u, \\ \Delta v - (1 + \delta b(x))v + \mu_2 v^3 + \beta u^2 v \end{pmatrix}. \tag{3.8}$$

For $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, we denote by

$$\langle f, g\rangle = \int_{\mathbb{R}^N} f_1 g_1 + f_2 g_2 dx. \tag{3.9}$$

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1. Fixing $\mathbf{Q}_k = (Q_1, \cdots, Q_k) \in \Lambda_k$, our main idea is to use $(U_{\mathbf{Q}_k}, V_{\mathbf{Q}_k})$ as the approximate solution. First using the Lyapunov-Schmidt reduction, we can show that there exists a constant ρ_0, such that for $\rho \geq \rho_0$, and $\delta < c_\rho$, (for some constant c_ρ depending on ρ but independent of k and $\mathbf{Q}_k$,) we can find a $(\phi_{\mathbf{Q}_k}, \psi_{\mathbf{Q}_k})$ such that

$$S(\begin{pmatrix} U_{\mathbf{Q}_k} \\ V_{\mathbf{Q}_k} \end{pmatrix} + \begin{pmatrix} \phi_{\mathbf{Q}_k} \\ \psi_{\mathbf{Q}_k} \end{pmatrix}) = \sum_{i=1,\cdots,k, j=1,\cdots,N} c'_{ij} \bar{Z}_{ij}, \tag{3.10}$$

for some c'_{ij}, where $\bar{Z}_{ij}$ is defined as

$$\bar{Z}_{ij} = \begin{pmatrix} \bar{Z}_{ij,1} \\ \bar{Z}_{ij,2} \end{pmatrix} = \begin{pmatrix} \frac{\partial U_{Q_i}}{\partial x_j} \chi_i(x) \\ \frac{\partial V_{Q_i}}{\partial x_j} \chi_i(x) \end{pmatrix}, \text{ for } i = 1, \cdots, k, \ j = 1, \cdots, N, \tag{3.11}$$

where χ_i is defined in Section 1.2. We can show that $(\phi_{\mathbf{Q}_k}, \psi_{\mathbf{Q}_k})$ is C^1 in $\mathbf{Q}_k$. Then, for any k, we define a new function

$$\mathcal{M}_1(\mathbf{Q}_k) = J_1(\begin{pmatrix} U_{\mathbf{Q}_k} \\ V_{\mathbf{Q}_k} \end{pmatrix} + \begin{pmatrix} \phi_{\mathbf{Q}_k} \\ \psi_{\mathbf{Q}_k} \end{pmatrix}), \tag{3.12}$$

and maximize $\mathcal{M}_1(\mathbf{Q}_k)$ over $\bar{\Lambda}_k$. Here the energy functional J_1 is defined in (1.24).

For large ρ, and fixed point $\mathbf{Q}_k \in \Lambda_k$, we first show solvability in $\{\begin{pmatrix} \phi_{\mathbf{Q}_k} \\ \psi_{\mathbf{Q}_k} \end{pmatrix}, \{c_{ij}\}\}$ of the non linear projected problem

$$\begin{cases} \Delta(U_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}) - (1 + \delta a(x))(U_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}) + \mu_1 (U_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k})^3 \\ +\beta(V_{\mathbf{Q}_k} + \psi_{\mathbf{Q}_k})^2 (U_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k}) = \sum_{i=1,\cdots,k,j=1,\cdots,N} c_{ij} \bar{Z}_{ij,1}, \\ \Delta(V_{\mathbf{Q}_k} + \psi_{\mathbf{Q}_k}) - (1 + \delta b(x))(V_{\mathbf{Q}_k} + \psi_{\mathbf{Q}_k}) + \mu_2 (V_{\mathbf{Q}_k} + \psi_{\mathbf{Q}_k})^3 \\ +\beta(U_{\mathbf{Q}_k} + \phi_{\mathbf{Q}_k})^2 (V_{\mathbf{Q}_k} + \psi_{\mathbf{Q}_k}) = \sum_{i=1,\cdots,k,j=1,\cdots,N} c_{ij} \bar{Z}_{ij,2}, \\ \langle \begin{pmatrix} \phi_{\mathbf{Q}_k} \\ \psi_{\mathbf{Q}_k} \end{pmatrix}, \begin{pmatrix} \bar{Z}_{ij,1} \\ \bar{Z}_{ij,2} \end{pmatrix} \rangle = 0 \ \text{ for } i = 1, \cdots, k, j = 1, \cdots, N. \end{cases} \tag{3.13}$$

For a fixed $0 < \eta < 1$ and $\mathbf{Q}_k = (Q_1, \cdots, Q_k) \in \Lambda_k$, let W be the function defined in (2.1) (with $f(u) = u^3$) and the norm to be

$$\|h\|_{**} = \sup_{x \in \mathbb{R}^N} |W(x)^{-1} h_1(x)| + \sup_{x \in \mathbb{R}^N} |W(x)^{-1} h_2(x)|. \tag{3.14}$$

First we need the following nondegeneracy result:

Lemma 3.1. *There exists $\beta^* > 0$, such that for $\beta \in (-\beta^*, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$, (U, V) is non-degenerate for the system (3.3) in $H^1(\mathbb{R}^N)$ in the sense that the kernel is given by*

$$\text{Span}\{(\frac{\partial U}{\partial x_j}, \frac{\partial V}{\partial x_j})|j = 1, \cdots, N\}. \tag{3.15}$$

Proof. For the proof, see the proof of Proposition 2.3 in [41] or [6]. □

From now on, we will always assume that

$$\beta \in (-\beta^*, 0) \cup (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty). \tag{3.16}$$

Similar to Section 2, the following proposition is standard.

Proposition 3.1. *Given $0 < \eta < 1$. There exist positive numbers ρ_0, C and $\xi > 0$ (independent of k and $\mathbf{Q}_k$) such that for all $\rho \geq \rho_0$, and for any $\mathbf{Q}_k \in \Lambda_k$, $\delta < e^{-2\rho}$, there is a unique solution $(\begin{pmatrix} \phi_{\mathbf{Q}_k} \\ \psi_{\mathbf{Q}_k} \end{pmatrix}, \{c_{ij}\})$ to problem (3.13). Furthermore $(\phi_{\mathbf{Q}_k}, \psi_{\mathbf{Q}_k})$ is C^1 in $\mathbf{Q}_k$ and we have*

$$\|(\phi_{\mathbf{Q}_k}, \psi_{\mathbf{Q}_k})\|_{**} \leq C\|S \begin{pmatrix} U_{\mathbf{Q}_k} \\ V_{\mathbf{Q}_k} \end{pmatrix} \|_{**} \leq Ce^{-\xi\rho}, \tag{3.17}$$

$$|c_{ij}| \leq Ce^{-\xi\rho}. \tag{3.18}$$

3.2. **A Secondary Lyapunov-Schmidt Reduction.** Similar to the estimate in Section 2.2, we have the key estimate on the difference in energy between the $k-$spike and the $(k + 1)-$spike. From now on, we choose $\eta > \frac{1}{2}$.

For $(Q_1, \cdots, Q_k) \in \Lambda_k$, we denote $\begin{pmatrix} u_{Q_1,\cdots,Q_k} \\ v_{Q_1,\cdots,Q_k} \end{pmatrix}$ as $\begin{pmatrix} U_{Q_1,\ldots,Q_k} + \phi_{Q_1,\ldots,Q_k} \\ V_{Q_1,\cdots,Q_k} + \psi_{Q_1,\cdots,Q_k} \end{pmatrix}$, where $\begin{pmatrix} \phi_{Q_1,\cdots,Q_k} \\ \psi_{Q_1,\cdots,Q_k} \end{pmatrix}$ is the unique solution given by Proposition 3.1.

We now write

$$\begin{aligned} \begin{pmatrix} u_{Q_1,\cdots,Q_{k+1}} \\ v_{Q_1,\cdots,Q_{k+1}} \end{pmatrix} &= \begin{pmatrix} u_{Q_1,\cdots,Q_k} \\ v_{Q_1,\cdots,Q_k} \end{pmatrix} + \begin{pmatrix} U_{Q_{k+1}} \\ V_{Q_{k+1}} \end{pmatrix} + \varphi_{k+1} \\ &= \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} + \begin{pmatrix} \varphi_{k+1,1} \\ \varphi_{k+1,2} \end{pmatrix} \end{aligned} \tag{3.19}$$

where

$$\begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} = \begin{pmatrix} u_{Q_1,\cdots,Q_k} \\ v_{Q_1,\cdots,Q_k} \end{pmatrix} + \begin{pmatrix} U_{Q_{k+1}} \\ V_{Q_{k+1}} \end{pmatrix}.$$

We have the following estimate for φ_{k+1} whose proof is exactly the same as Lemma 2.2. We omit the details.

Lemma 3.2. *Let ρ, δ be as in Proposition 3.1. Then it holds*

$$\begin{aligned}&\int_{\mathbb{R}^N}(|\nabla\varphi_{k+1,1}|^2+\varphi_{k+1,1}^2)+(|\nabla\varphi_{k+1,2}|^2+\varphi_{k+1,2}^2)dx \qquad (3.20)\\ &\leq C(e^{-\xi\rho}\sum_{i=1}^{k}w(|Q_{k+1}-Q_i|)\\ &+\delta^2(\int_{\mathbb{R}^N}a^2U_{Q_{k+1}}^2+b^2V_{Q_{k+1}}^2dx+(\int_{\mathbb{R}^N}|a|U_{Q_{k+1}}+|b|V_{Q_{k+1}}dx)^2),\end{aligned}$$

for some constant $C>0,\xi>0$ independent of ρ,k and $\mathbf{Q}\in\Lambda_{k+1}$ (the constants may depend on the choice of η).

Similar to (2.59) in the proof of Lemma 2.2, we have a more accurate estimate of φ_{k+1} as follows:

$$\begin{aligned}\|\varphi_{k+1}\|_{H^1(\mathbb{R}^N)} &\leq C(e^{-\xi\rho}(\sum_{i=1}^{k}w(|Q_{k+1}-Q_i|))^{\frac{1}{2}}\\ &+\delta\sum_{i=1,\cdots,k+1}(\int_{B_{\frac{\rho}{2}}(Q_i)}a^2U_{Q_{k+1}}^2+b^2V_{Q_{k+1}}^2dx)^{\frac{1}{2}}\\ &+\delta(\int_{\mathbb{R}^n}a^2U_{Q_{k+1}}^2+b^2V_{Q_{k+1}}^2dx)^{\frac{1}{2}}). \qquad (3.21)\end{aligned}$$

3.3. **The reduced problem: a maximization procedure.** In this section, we study a maximization problem. Fix $\mathbf{Q}_k\in\Lambda_k$, we define the new functional

$$\mathcal{M}_1(\mathbf{Q}_k)=J_1(u_{\mathbf{Q}_k},v_{\mathbf{Q}_k}):\Lambda_k\to\mathbb{R}. \qquad (3.22)$$

Define

$$\mathcal{C}_k=\max_{\mathbf{Q}\in\Lambda_k}\{\mathcal{M}_1(\mathbf{Q}_k)\}. \qquad (3.23)$$

We will show below that the maximization problem has a solution.

We first prove that the maximum can be attained at a finite point for each $\mathcal{C}_k$.

Lemma 3.3. *Let $a(x),b(x)$ satisfy assumptions $(H_1')-(H_3')$ and ρ,δ be as in Proposition 3.1. Then, for all k:*

- *There exists $\mathbf{Q}_k=(Q_1,Q_2,\cdots,Q_k)\in\Lambda_k$ such that*

$$\mathcal{C}_k=\mathcal{M}_1(\mathbf{Q}_k); \qquad (3.24)$$

- *There holds*

$$\mathcal{C}_{k+1} > \mathcal{C}_k + I_1(U,V), \tag{3.25}$$

where $I_1(U,V)$ is the energy of (U,V):

$$\begin{aligned} I_1(U,V) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla U|^2 + U^2 + |\nabla V|^2 + V^2 dx \\ &- \frac{1}{4}\int_{\mathbb{R}^N} \mu_1 U^4 + \mu_2 V^4 dx - \frac{\beta}{2}\int_{\mathbb{R}^N} U^2V^2 dx \end{aligned} \tag{3.26}$$

Proof. The proof is similar to the proof of Lemma 2.5. We divide the proof into two steps.

Base step: We claim $\mathcal{C}_1 > I_1(U,V)$ and $\mathcal{C}_1$ can be attained at a finite point. First using standard Lyapunov-Schmidt reduction, similar to the derivation of (2.66), we have

$$\|(\phi_Q, \psi_Q\|_{H^1} \le C\|\delta(aU_Q, bV_Q)\|_{L^2}. \tag{3.27}$$

Assume that $|Q| \to \infty$, then we have

$$\begin{aligned} &J_1(u_Q, v_Q) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla(U_Q+\phi_Q)|^2 + (1+\delta a)(U_Q+\phi_Q)^2 dx \\ &+\frac{1}{2}\int_{\mathbb{R}^N} |\nabla(V_Q+\psi_Q)|^2 + (1+\delta b)(V_Q+\psi_Q)^2 dx \\ &-\frac{1}{4}\int_{\mathbb{R}^N} \mu_1(U_Q+\phi_Q)^4 + \mu_2(V_Q+\psi_Q)^4 dx - \frac{\beta}{2}\int_{\mathbb{R}^N}(U_Q+\phi_Q)^2(V_Q+\psi_Q)^2 dx \\ &= I_1(U_Q,V_Q) + \frac{\delta}{2}\int_{\mathbb{R}^N} aU_Q^2 + bV_Q^2 dx + \delta^2\|(aU_Q, bV_Q)\|^2_{L^2(\mathbb{R}^N)} \\ &\ge I_1(U_Q,V_Q) \\ &+\frac{1}{4}[\int_{B_{\frac{\rho}{2}}(Q)} \delta(aU_Q^2 + bV_Q^2)dx - \sup_{B_{\frac{|Q|}{4}}(0)} (U_Q^{\frac{3}{2}} + V_Q^{\frac{3}{2}})\int_{supp(\alpha^2 a+\gamma^2 b)^-} \delta(|a|U_Q^{\frac{1}{2}} + |b|V_Q^{\frac{1}{2}})dx] \\ &\ge I_1(U,V) + \frac{1}{4}\int_{B_{\frac{\rho}{2}}(Q)} \delta(aU_Q^2 + bV_Q^2)dx - O(\delta e^{-\frac{9}{8}|Q|}). \end{aligned}$$

By the slow decay assumption on the potentials a, b, we obtain

$$\frac{1}{4}\int_{B_{\frac{\rho}{2}}(Q)} \delta(aU_Q^2 + bV_Q^2)dx - O(\delta e^{-\frac{9}{8}|Q|}) > 0, \text{ for } |Q| \text{ large },$$

and therefore,

$$\mathcal{C}_1 \ge J_1(u_Q, v_Q) > I_1(U,V),$$

and we have proven the first part of our claim.

Let us prove now that $\mathcal{C}_1$ can be attained at a finite point. If not, then there exists a sequence $\{Q_i\} \to \infty$ such that $\lim_{i\to\infty} \mathcal{M}_1(Q_i) = \mathcal{C}_1$, then by the same argument as above, we obtain

$$\begin{aligned} J_1(u_{Q_i}, v_{Q_i}) &\le I_1(U,V) + \frac{1}{2}\delta \int_{\mathbb{R}^N} aU_{Q_i}^2 + bV_{Q_i}^2 dx \\ &+ O(\delta^2 \int_{\mathbb{R}^N} a^2U_{Q_i}^2 + b^2V_{Q_i}^2 dx). \end{aligned}$$

As $|Q_i| \to \infty$, by the decay assumption on a, b, we have

$$\frac{\delta}{2}\int_{\mathbb{R}^N} aU_{Q_i}^2 + bV_{Q_i}^2 dx + O(\delta^2 \int_{\mathbb{R}^N} a^2U_{Q_i}^2 + b^2V_{Q_i}^2 dx) \to 0 \text{ as } i \to \infty.$$

Thus,

$$\mathcal{C}_1 = \lim_{i\to\infty} J_1(u_{Q_i}, v_{Q_i}) \le I_1(U,V) \text{ as } i \to \infty, \tag{3.28}$$

which is a contradiction. So $\mathcal{C}_1$ can be attained at a finite point.

Induction step: Assume that there exists $\mathbf{Q}_k = (\bar{Q}_1, \cdots, \bar{Q}_k) \in \Lambda_k$ such that $\mathcal{C}_k = \mathcal{M}_1(\mathbf{Q}_k)$, and we denote the solution by $(u_{\bar{Q}_1,\cdots,\bar{Q}_k}, v_{\bar{Q}_1,\cdots,\bar{Q}_k})$.

Next, we prove that there exists $(Q_1, \cdots, Q_{k+1}) \in \Lambda_{k+1}$ such that $\mathcal{C}_{k+1}$ can be attained.

Let $((Q_1^{(n)}, \cdots, Q_{k+1}^{(n)}))_n$ be a sequence such that

$$\mathcal{C}_{k+1} = \lim_{n\to\infty} \mathcal{M}_1(Q_1^{(n)}, \cdots, Q_{k+1}^{(n)}). \tag{3.29}$$

We claim that $(Q_1^{(n)}, \cdots, Q_{k+1}^{(n)})$ is bounded. We prove it by contradiction. Without loss of generality, we assume that $|Q_{k+1}^{(n)}| \to \infty$ as $n \to \infty$. In the following we omit the index n for simplicity

First, we observe that

$$\begin{aligned} &J_1(u_{Q_1,\cdots,Q_{k+1}}, v_{Q_1,\cdots,Q_{k+1}}) \\ &= J_1(\begin{pmatrix} u_{Q_1,\cdots,Q_k} \\ v_{Q_1,\cdots,Q_k} \end{pmatrix} + \begin{pmatrix} U_{Q_{k+1}} \\ V_{Q_{k+1}} \end{pmatrix}) \\ &+\|\bar{S}(u_{Q_1,\cdots,Q_k} + U_{Q_{k+1}}, v_{Q_1,\cdots,Q_k} + V_{Q_{k+1}})\|_{L^2}\|\varphi_{k+1}\|_{H^1} \\ &+ \sum_{i=1,\cdots,k, j=1,\cdots,N} c_{ij}\langle \bar{Z}_{ij}, \varphi_{k+1}\rangle + O(\|\varphi_{k+1}\|_{H^1}^2) \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} \bar{S} &= \begin{pmatrix} \mu_1[(\bar{U}^3 - u_{Q_1,\cdots,Q_k}^3 - U_{Q_{k+1}}^3)] + \beta[\bar{V}^2\bar{U} - v_{Q_1,\cdots,Q_k}^2 u_{Q_1,\cdots,Q_k} - V_{Q_{k+1}}^2 U_{Q_{k+1}}] \\ \mu_2[(\bar{V}^3 - v_{Q_1,\cdots,Q_k}^3 - V_{Q_{k+1}}^3)] + \beta[\bar{U}^2\bar{V} - u_{Q_1,\cdots,Q_k}^2 v_{Q_1,\cdots,Q_k} - U_{Q_{k+1}}^2 V_{Q_{k+1}}] \end{pmatrix} \\ &- \delta \begin{pmatrix} aU_{Q_{k+1}} \\ bV_{Q_{k+1}} \end{pmatrix}. \end{aligned}$$

Similar to the estimate (2.34), one has the following L^2 estimate for $\bar{S}$:

$$\|\bar{S}\|^2_{L^2(\mathbb{R}^N)} \leq C(e^{-\xi\rho}\sum_{i=1}^{k} w(|Q_{k+1}-Q_i|) + \delta^2\int_{\mathbb{R}^N} a^2U^2_{Q_{k+1}} + b^2V^2_{Q_{k+1}}dx). \tag{3.31}$$

Thus we have

$$\begin{aligned}
&J_1(u_{Q_1,\cdots,Q_{k+1}}, v_{Q_1,\cdots,Q_{k+1}}) \\
&= J_1(\begin{pmatrix} u_{Q_1,\cdots,Q_k} \\ v_{Q_1,\cdots,Q_k}\end{pmatrix} + \begin{pmatrix} U_{Q_{k+1}} \\ V_{Q_{k+1}}\end{pmatrix}) \\
&+O(e^{-\xi\rho}\sum_{i=1}^{k} w(|Q_{k+1}-Q_i|) + \delta^2\int_{\mathbb{R}^N} a^2U^2_{Q_{k+1}} + b^2V^2_{Q_{k+1}}dx \\
&+\delta^2(\int_{\mathbb{R}^N} |a|U_{Q_{k+1}} + |b|V_{Q_{k+1}}dx)^2,
\end{aligned} \tag{3.32}$$

where we use the condition that $\langle \bar{Z}_{ij}, \varphi_{k+1}\rangle = 0$ for $i = 1, \cdots, k$.

Next, we have the following:

$$\begin{aligned}
&J_1(\begin{pmatrix} u_{Q_1,\cdots,Q_k} \\ v_{Q_1,\cdots,Q_k}\end{pmatrix} + \begin{pmatrix} U_{Q_{k+1}} \\ V_{Q_{k+1}}\end{pmatrix}) \\
&\leq \mathcal{C}_k + I_1(U_{Q_{k+1}}, V_{Q_{k+1}}) + \frac{\delta}{2}\int_{\mathbb{R}^N} aU^2_{Q_{k_1}} + bV^2_{Q_{k+1}}dx \\
&+ \sum_{i=1,\cdots,k,j=1,\cdots,N} c_{ij}\langle \bar{Z}_{ij}, \begin{pmatrix} U_{Q_{k+1}} \\ V_{Q_{k+1}}\end{pmatrix}\rangle \\
&-\int_{\mathbb{R}^N} \mu_1 U^3_{Q_{k+1}} u_{Q_1,\cdots,Q_k} + \mu_2 V^3_{Q_{k+1}} v_{Q_1,\cdots,Q_k}dx \\
&-\beta\int_{\mathbb{R}^N} U^2_{Q_{k+1}}V_{Q_{k+1}}v_{Q_1,\cdots,Q_k} + V^2_{Q_{k+1}}U_{Q_{k+1}}u_{Q_1,\cdots,Q_k}dx \\
&+O(e^{-\xi\rho}\sum_{i=1}^{k} w(|Q_{k+1}-Q_i|)).
\end{aligned} \tag{3.33}$$

Similar to the estimate (2.75), we have

$$\begin{aligned}
&J_1(u_{Q_1,\cdots,Q_{k+1}}, v_{Q_1,\cdots,Q_{k+1}}) \qquad (3.34)\\
&\le \mathcal{C}_k + I_1(U,V) + \frac{\delta}{2}\int_{\mathbb{R}^n} aU^2_{Q_{k+1}} + bV^2_{Q_{k+1}}dx\\
&-\frac{1}{4}A\gamma_1\sum_{i=1}^k w(|Q_{k+1}-Q_i|) + O(e^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|))\\
&+O(\delta e^{-\xi\rho}\int_{\mathbb{R}^N}\sum_{i=1}^k e^{-\eta|x-Q_i|}(aU_{Q_{k+1}}+bV_{Q_{k+1}})dx + \delta\int_{\mathbb{R}^N} aU_{\mathbf{Q}_k}U_{Q_{k+1}} + bV_{\mathbf{Q}_k}V_{Q_{k+1}}dx)\\
&+\delta^2\int_{\mathbb{R}^N} a^2U^2_{Q_{k+1}} + b^2V^2_{Q_{k+1}} + \delta^2(\int_{\mathbb{R}^N}|a|U_{Q_{k+1}} + |b|V_{Q_{k+1}})^2dx,
\end{aligned}$$

where γ_1 is defined in (2.27) with $f(t) = t^3$ and $A = \mu_1\alpha^4 + \mu_2\gamma^4 + 2\beta\alpha^2\gamma^2 > 0$.

By the assumption that $|Q^{(n)}_{k+1}| \to \infty$, we obtain that the last two lines in (3.34) tends to 0 as $n \to \infty$ and

$$-\frac{1}{4}A\gamma_1\sum_{i=1}^k w(|Q_{k+1}-Q_i|) + O(e^{-\xi\rho}\sum_{i=1}^k w(|Q_{k+1}-Q_i|)) < 0. \quad (3.35)$$

Combining (3.29), (3.34) and the above estimates, we obtain

$$\mathcal{C}_{k+1} \le \mathcal{C}_k + I_1(U,V). \quad (3.36)$$

On the other hand, by the assumption in the base step, $\mathcal{C}_k$ can be attained at $(\bar{Q}_1,\cdots,\bar{Q}_k)$, there exists another point Q_{k+1} which is far away from the k points which will be determined later.

Next let us consider the solution concentrated at the points $(\bar{Q}_1,\cdots,\bar{Q}_k, Q_{k+1})$. We denote the solution by $(u_{\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}}, v_{\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}})$. By similar argument as the above, using the estimate (3.21) instead of (3.20), we have the following estimate:

$$
\begin{aligned}
&J_1(u_{\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}}, v_{\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}}) \\
&= J_1(u_{\bar{Q}_1,\cdots,\bar{Q}_k}, v_{\bar{Q}_1,\cdots,\bar{Q}_k}) + I_1(U,V) \\
&+\frac{\delta}{2}\int_{\mathbb{R}^N} aU_{Q_{k+1}}^2 + bV_{Q_{k+1}}^2 dx + O(\sum_{i=1}^k w(|Q_{k+1}-\bar{Q}_i|)) \\
&+O(\delta e^{-\xi\rho}\int_{\mathbb{R}^N}\sum_{i=1}^k e^{-\eta|x-Q_i|}(aU_{Q_{k+1}}+bV_{Q_{k+1}})dx + \delta\int_{\mathbb{R}^N} aU_{\bar{\mathbf{Q}}_k}U_{Q_{k+1}} + bV_{\bar{\mathbf{Q}}_k}V_{Q_{k+1}}dx \\
&+\delta^2(\int_{\mathbb{R}^N} a^2U_{Q_{k+1}}^2 + b^2V_{Q_{k+1}}^2 dx + (\sum_{i=1,\cdots,k+1}(\int_{B_{\frac{\rho}{2}}(Q_i)} a^2U_{Q_{k+1}}^2 + b^2V_{Q_{k+1}}^2 dx)^{\frac{1}{2}})^2)) \\
&= J_1(u_{\bar{Q}_1,\cdots,\bar{Q}_k}, v_{\bar{Q}_1,\cdots,\bar{Q}_k}) + I_1(U,V) + R_4,
\end{aligned}
\tag{3.37}
$$

where R_4 is given in the above expressions.

By the slow decay assumption of a, b at infinity, i.e. $\lim_{|x|\to\infty}(\alpha^2 a + \gamma^2 b)e^{\bar{\eta}|x|} = +\infty$ as $|x| \to \infty$, for some $\bar{\eta} < 1$, we can further choose $\eta > \bar{\eta}$, and choose Q_{k+1} such that

$$
|Q_{k+1}| >> \frac{\max_{i=1}^k |\bar{Q}_i| + \ln\delta}{\eta - \bar{\eta}}. \tag{3.38}
$$

This implies that

$$
R_4 \geq Ce^{-\bar{\eta}|Q_{k+1}|} - O(\sum_{i=1}^k e^{-\eta|Q_i - Q_{k+1}|}) > 0.
$$

Therefore, we obtain

$$
\mathcal{C}_{k+1} \geq J_1(u_{\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}}, v_{\bar{Q}_1,\cdots,\bar{Q}_k,Q_{k+1}}) > \mathcal{C}_k + I_1(U,V). \tag{3.39}
$$

Combining (3.39) and (3.36), one gets that

$$
\mathcal{C}_k + I_1(U,V) < \mathcal{C}_{k+1} \leq \mathcal{C}_k + I_1(U,V). \tag{3.40}
$$

This is in contradiction to (3.36). Thus we get that $\mathcal{C}_{k+1}$ can be attained at a finite point in Λ_{k+1}.

Moreover, from the proof above, we derive the following relation between $\mathcal{C}_{k+1}$ and $\mathcal{C}_k$:

$$
\mathcal{C}_{k+1} > \mathcal{C}_k + I_1(U,V). \tag{3.41}
$$

□

The next Proposition excludes boundary maximization.

Proposition 3.2. *The maximization problem*

$$\max_{\mathbf{Q}_k\in\bar{\Lambda}_k} \mathcal{M}_1(\mathbf{Q}_k) \tag{3.42}$$

has a solution $\mathbf{Q}_k \in \Lambda_k^\circ$*, i.e., the interior of* Λ_k*.*

Proof. We prove it by contradiction again. If $\mathbf{Q}_k = (\bar{Q}_1, \cdots, \bar{Q}_k) \in \partial\Lambda_k$, then there exists (i,j) such that $|\bar{Q}_i - \bar{Q}_j| = \rho$. Without loss of generality, we assume $(i,j) = (i,k)$. Then from (3.34), we have

$$\begin{aligned}\mathcal{C}_k \quad &= J_1(u_{\bar{Q}_1,\cdots,\bar{Q}_k}, v_{\bar{Q}_1,\cdots,\bar{Q}_k}) \\ &\le \mathcal{C}_{k-1} + I_1(U,V) + \frac{\delta}{2}\int_{\mathbb{R}^N} aU_{Q_k}^2 + bV_{Q_k}dx \\ &-\frac{1}{4}A\gamma_1\sum_{i=1}^{k-1} w(|Q_k - Q_i|) + O(e^{-\xi\rho}\sum_{i=1}^{k-1} w(|Q_k-Q_i|)) + O(\delta) \\ &\le \mathcal{C}_{k-1} + I_1(U,V) \\ &+C\delta - \frac{1}{4}A\gamma_1\sum_{i=1}^{k-1} w(|Q_k - Q_i|) + O(e^{-\xi\rho}\sum_{i=1}^{k-1} w(|Q_k-Q_i|)).\end{aligned} \tag{3.43}$$

Similar to Section 2.3, by the definition of the configuration set, we have

$$\begin{aligned}\mathcal{C}_k \quad &\le \mathcal{C}_{k-1} + I_1(U,V) + c\delta - \frac{1}{8}\gamma_1 w(\rho) + O(e^{-(1+\xi)\rho}) \\ &< \mathcal{C}_{k-1} + I_1(U,V),\end{aligned} \tag{3.44}$$

which is a contradiction to Lemma 3.3. So we get the proof. □

3.4. **Proof of Theorem 1.2.** In this section, we apply the results in Section 3.1, 3.2 and 3.3 to prove Theorem 1.2.

Proof of Theorem 1.2: By Proposition 3.1 in Section 3.1, there exists ρ_0 such that for $\rho > \rho_0$, we have C^1 map which, to any $\mathbf{Q}^\circ \in \Lambda_k$, associates $\phi_{\mathbf{Q}^\circ}$ such that

$$S(\begin{pmatrix} U_{\mathbf{Q}^\circ} + \phi_{\mathbf{Q}^\circ} \\ V_{\mathbf{Q}^\circ} + \psi_{\mathbf{Q}^\circ} \end{pmatrix}) = \sum_{i=1,\cdots,k,j=1,\cdots,N} c_{ij}\bar{Z}_{ij}, \quad \langle \begin{pmatrix} \phi_{\mathbf{Q}^\circ} \\ \psi_{\mathbf{Q}^\circ} \end{pmatrix}, \bar{Z}_{ij}\rangle = 0, \tag{3.45}$$

for some constants $\{c_{ij}\} \in \mathbb{R}^{kN}$.

From Proposition 3.2 in Section 3.2, there is a $\mathbf{Q}^\circ \in \Lambda_k^\circ$ that achieves the maximum for the maximization problem in Proposition 3.2. Let

$$\begin{pmatrix} u_{\mathbf{Q}^\circ} \\ v_{\mathbf{Q}^\circ} \end{pmatrix} = \begin{pmatrix} U_{\mathbf{Q}^\circ} \\ V_{\mathbf{Q}^\circ} \end{pmatrix} + \begin{pmatrix} \phi_{\mathbf{Q}^\circ} \\ \psi_{\mathbf{Q}^\circ} \end{pmatrix}.$$

Then we have

$$D_{Q_{ij}}|_{Q_i=Q_i^\circ}\mathcal{M}(\mathbf{Q}^\circ)=0,\quad i=1,\cdots,k,\quad j=1,\cdots,N. \tag{3.46}$$

Similar to the proof of Theorem 1.1,

$$\sum_{i=1,\cdots,k,\ j=1,\cdots,N} c_{ij}\int_{\mathbb{R}^N}\bar{Z}_{ij,1}\frac{\partial(U_{\mathbf{Q}}+\phi_{\mathbf{Q}})}{\partial Q_{sl}}|_{Q_s=Q_s^\circ}+\bar{Z}_{ij,2}\frac{\partial(V_{\mathbf{Q}}+\psi_{\mathbf{Q}})}{\partial Q_{sl}}|_{Q_s=Q_s^\circ}dx=0, \tag{3.47}$$

for $s=1,\cdots,k, l=1,\cdots,N$.

Similar to the derivation of (2.92) and (2.93), one can show that (3.47) is a diagonally dominant system for c_{sl}. Thus $c_{sl}=0$ for $s=1,\cdots,k, l=1,\cdots,N$. Hence $(u_{\mathbf{Q}^\circ},v_{\mathbf{Q}^\circ})$ is a solution of (1.21).

References

[1] A. Ambrosetti, M. Badiale and S. Cingolani, Semiclassical states of nonlinear Schrödinger equations. *Arch. Rational Mech. Anal.* 140 (1997), 285–300.

[2] A. Ambrosetti, A. Malchiodi and W.M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. I. *Comm. Math. Phys.* 235 (2003), 427–466.

[3] A. Ambrosetti, A. Malchiodi and W.M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. II. *Indiana Univ. Math. J.* 53 (2004), 297–329.

[4] W.W. Ao, J.C. Wei and J. Zeng, An optimal bound on the number of interior spike solutions for the Lin-Ni-Takagi problem, preprint 2012.

[5] P.W. Bates, E.N. Dancer and J.P. Shi, Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability, *Adv. Differential Equations* 4 (1999),1-69.

[6] T. Bartsch, N. Dancer and Z.Q. Wang ,A Liouville theorem, a priori bounds, and bifurcating branches of positive solutions for a nonlinear elliptic system, *Cal. Var. PDE.* 37, No. 3-4 (2010), 345-361.

[7] A. Bahri and Y.Y. Li, On a min-max procedure for the existence of a positive solution for certain scalar field equations in R^N. *Rev. Mat. Iberoamericana* 6 (1990), 1–15.

[8] A. Bahri and P.L. Lions, On the existence of a positive solution of semilinear elliptic equations in unbounded domains. *Ann. Inst. H. Poincare Anal. Non Linaire* 14 (1997), 365-413.

[9] T. Bartsch and Z.Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on R^N, *Comm. Part. Diff. Eqns.* 20 (1995), 1725-1741.

[10] J.Y. Byeon and K. Tanaka, Semiclassical standing waves with clustering peaks for nonlinear Schrödinger equations, To appear in *Memoir. Am. Math. Soc.*.

[11] J.Y. Byeon and K. Tanaka, Semi-classical standing waves for nonlinear Schrödinger equations at structurally stable critical points of the potential, To appear in *J. Eur. Math. Soc.*.

[12] G.Cerami, G. Devillanova and S. Solimini, Infinitely many bound states for some nonlinear scalar field equations. *Calc. Var. PDE* 23 (2005), 139–168.

[13] G. Cerami, D. Passaseo and S. Solimini, Infinitely many positive solutions to some scalar field equations with non symmetric coefficients, *Comm. Pure Appl. Math.* 66 (2013), 372-413.

[14] G. Cerami, D. Passaseo and S. Solimini, Nonlinear scalar equations: existence of a positive solution with infinitely many bumps, preprint 2012.

[15] V. Coti Zelati and P. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials. *J. Amer. Math. Soc.* 4 (1992), 693–727.

[16] V. Coti Zelati and P.H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^n$. *Comm. Pure Appl. Math.* 45 (1992), 1217 - 1269.

[17] M. Conti, S. Terracini and G. Verzini, Nehari's problem and competing species systems, *Ann. Inst. H. Poincar Anal. Non Linaire* 19 (2002), 871-888.

[18] E.N. Dancer, On the uniqueness of the positive solution of a singularly perturbed problem, *Rocky Mountain J. Math.* 25(1995), 957-975.

[19] G. Devillanova and S. Solimini, Min-max solutions to some scalar field equations, Adv. Nonlinear Stud. 12 (2012), no. 1, 173186.

[20] M. del Pino and P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains. *Calc. Var. PDE*, 4 (1996), 121–137.

[21] M. del Pino and P. Felmer, Multi-peak bound states of nonlinear Schröinger equations. *Ann. Inst. H. Poincaré, Analyse Nonlineaire* 15(1998), 127–149.

[22] M. del Pino and P. Felmer, Semi-classcal states for nonlinear Schrödinger equations. *J. Funct. Anal.* 149(1997), 245–265.

[23] M. del Pino and P. Felmer, Semi-classical states of nonlinear Schröinger equations: a variational reduction method. *Math. Ann.* 324 (2002), 1–32.

[24] W.Ding and W.M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation. *Arch. Rational Mech. Anal.* 91 (1986), 283–308.

[25] N. Dancer, J.C. Wei and T. Weth, A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system, *Ann. Inst. H. Poincar Anal. Non Linaire* 27 (2010), 953-969.

[26] A. Floer and A. Weinstein, Nonspreading Wave Packets for the Cubic Schrödinger Equation with a Bounded Potential. *J. Funct. Anal.* 69 (1986), 397–408.

[27] P. Felmer and S. Martinez, Thick clusters for the radially symmetric nonlinear Schrödinger equation, *Cal. Var. PDE* 31(2008), 231-261.

[28] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$. Mathematical analysis and applications, Part A, pp. 369–402, Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London, 1981.

[29] X.S. Kang and J.C. Wei, On interacting bumps of semi-classical states of nonlinear Schrödinger equations. *Adv. Diff. Eqn.* 5(2000), 899–928.

[30] Y.Y. Li, On a singularly perturbed elliptic equation. *Adv. Diff. Eqn.* 2(1997), 955–980.

[31] F.H. Lin, W.M. Ni and J.C. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem, *Comm. Pure Appl. Math.* 60 (2007), 252–281.

[32] T.C. Lin and J.C. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n$, $n \leq 3$, *Comm. Math Phys.* 255 (2005), 629-653.

[33] T.C. Lin and J.C. Wei, Solitary and self similar solutions of two component systems of nonlinear Schrödinger equations, *Phy. D: Nonlinear Phenomena*, 220 (2006), 99-115.

[34] M. Musso, F. Pacard and J.C. Wei, Finite energy sigh-changing solutions with dihedral symmetry for the stationary non linear schrodinger equation, *J. Eur. Math. Soc.*, 14(2012), no.6, 1923-1953.

[35] W. M. Ni and I. Takagi, Locating the peaks of least energy solutions to a semilinear Neumann problem, *Duke Math. J.* 70 (1993), no. 2, 247-281.

[36] B. Noris, H. Tavares, S. Terracini and G. Verzini, Uniform holder bounds for nonlinear Schrödinger systems with strong competition, *Comm. Pure Appl. Math*, 63 (2010), 267-302.

[37] E.S. Noussair and S.S. Yan, On positive multipeak solutions of a nonlinear elliptic problem, *J. London Math. Soc.* 62 (2000), 213-227.

[38] Y.J. Oh, On positive multi-lump bound states nonlinear Schrödinger equations under multiple well potential, *Comm. Math. Phys.* 131 (1990), 223–253.

[39] A. Pakylak, F. Ting and J.C. Wei, Multi-vortex solutions to Ginzburg-Landau equations with external potential, *Arch. Rat. Mech. Anal.* 204(2012), no.1, 314-354.

[40] P. Polacik, Morse indices and bifurcations of positive solutions of $\Delta u + f(u) = 0$ on $\mathbb{R}^N$, *Indiana Univ. Math. J.* 50 (2001), 1407-1432.

[41] S.J. Peng and Z.Q. Wang, Segregated and synchronized vector solutions for nonlinear Schrödinger systems, *Arch. Rat. Mech. Anal.*, 208(2013), no. 1, 305-339.

[42] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.* 43 (1992) 270–291.

[43] E. Séré, Looking for the Bernoulli shift, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 10 (1993), 561 - 590.

[44] B. Sirakov, Standing wave solutions of the nonlinear Schrödinger equation in R^N, *Ann. Mat. Pura Appl.* 181(2002), no.1, 73-83.

[45] S. Terracini and G. Verzini, Multipulse phase in $k-$mixtures of Bose-Einstein condenstates, *Arch. Rat. Mech. Anal.* 194, No. 3 (2009), 717-741.

[46] J.C. Wei and T. Weth, Nonradial symmetric bound states for system of two coupled Schrodinger equations, *Rend. Lincei Mat. Appl.*, 18 (2007), 279-293.

[47] J.C. Wei and T. Weth, Radial solutions and phase separation in a system of two coupled Schrödinger equations, *Arch. Rat. Mech. Anal.*, 190 (2008), 83-106.

[48] J.C. Wei and M. Winter, On Gierer-Meinhardt system with saturation,*Comm. Contemp. Math.*, 6(2004), no.2, 259-277.

[49] X.F. Wang, On concentration of positive bound states of nonlinear Schrödinger equations. *Commun. Math. Phys.* 153(1993), 229–243.

[50] J.C. Wei and S.S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger equations in $\mathbb{R}^n$, *Calc. Var. PDE*, 37 (2010), 423-439.

[51] J.C. Wei and W. Yao, Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations, *CPAA* 11(2012), no.3, 1003-1011.

Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong. *wwao@math.cuhk.edu.hk*

Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong,and Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2. *wei@math.cuhk.edu.hk*