

# STABILITY AND MULTIPLE SOLUTIONS TO EINSTEIN-SCALAR FIELD LICHNEROWICZ EQUATION ON MANIFOLDS

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**ABSTRACT.** In this paper, we study the stability and multiple solutions to Einstein-scalar field Lichnerowicz equation on compact Riemannian manifolds. In particular, in dimension no more than 5, we can find a different way (comparing with the previous result of Hebey-Pacard-Pollack) by showing that there are at least two positive solutions or a unique positive solution according to the coercivity property of a quadratic form defined by the minimal solution obtained by the monotone method. When the coercive condition fails, we prove a uniqueness result. A positive solution of the Lichnerowicz equation is also found in a complete non-compact Riemannian manifold.

**Résumé:** Dans ce papier, nous étudions la stabilité et existence de solutions multiples pour l'équation scalaire de Einstein-Lichnerowicz, sur une variété riemannienne compacte. En dimension inférieure ou égale à 5, nous exposons une méthode différente (en comparaison avec celle de Hebey-Pacard-Pollack) pour démontrer l'existence de deux solutions positives, ou d'une unique solution positive, selon la coercivité d'une forme quadratique, définie par la solution minimale provenant de l'itération monotone. Quand la condition de coercivité n'est pas satisfaite, nous prouvons un résultat d'unicité. L'existence d'une solution positive pour l'équation de Lichnerowicz est aussi montrée sur une variété riemannienne complète non compacte.

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## 1. INTRODUCTION

In the mathematical analysis of the Einstein field equations in general relativity, an important part is to find reasonable initial data sets for solving the nonlinear wave system. The initial data has to satisfy the the Einstein constraint conditions, which are the Gauss and Codazzi equations. Using the conformal method, one is lead to one of the simplest scalar equation, which is named as the Einstein-scalar field Lichnerowicz equation (in short, we just call it the Lichnerowicz equation). In this paper we mainly consider

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the following Einstein-scalar field Lichnerowicz type equation on a compact Riemannian manifold  $(M, g)$  of dimension  $n \geq 3$ :

$$(1) \quad -\Delta u + hu = Bu^p + Au^{-(p+2)},$$

where  $p = \frac{n+2}{n-2}$ ,  $\Delta$  is the Laplacian operator on  $M$  (which is the standard Laplacian operator when  $M = \mathbb{R}^n$ ),  $h$ ,  $A$ , and  $B$  are nontrivial smooth functions on  $M$  with  $h > 0$ ,  $B > 0$  and  $A \geq 0$ . The existence results for equation (1) can be studied by the monotone method and the mountain pass theorem. For these, we refer to the works of Choquet-Bruhat-Isenberg-Pollack and their friends [3]-[6] [10] and Hebey-Pacard-Pollack [9] (see also [11] and [12] for related results).

As the first step we start from the minimal positive solution to (1). Based on the minimal solution, which will be assumed to be strictly stable, we can get second solution by using the mountain pass theorem (and see Theorem 4). Our construction is different from the mountain pass solution obtained in [9] (and the construction in [2]). In the interesting work [7], the authors developed the stability result which may be used to obtain the second solution as in Theorem 4. The result of Druet-Hebey is more general in the sense that there is no assumption on  $h$ . If the minimal solution  $\underline{u}$  is not strictly stable, we have a uniqueness result, which is stated in Theorem 5. We shall also obtain a positive solution to (1) on a complete non-compact Riemannian manifold  $(M^n, g)$ ,  $n \geq 3$ . The result is stated in Theorem 7.

Using the monotone method, we can easily get the following result.

**Theorem 1.** *Suppose that there is a positive super-solution  $v$  to (1). Then for sufficiently small  $\epsilon > 0$ , there is a positive (stable) solution  $\underline{u}$  to (1) such that  $\epsilon \leq \underline{u}(x) \leq v(x)$  in  $M$ .*

The proof of this result is below. In fact, for small  $0 < \epsilon < \inf_M v(x)$ , we know that  $u_0 = \epsilon$  is a sub-solution to (1). Then using the monotone method [14], we get a positive solution to (1) such that  $\epsilon \leq \underline{u}(x) \leq v(x)$  in  $M$ . Here we prefer to give a variational characterization of the solution  $\underline{u}$ . Recall that the equation (1) is the Euler-Lagrange equation of the functional

$$J(u) = \int \frac{1}{2} [|\nabla u|^2 + h(x)u^2] + \frac{A}{p+1} u^{-(p+1)} - \frac{B}{p+1} u^{p+1}.$$

Let  $\Sigma = \{u \in H^1(M); \epsilon \leq u(x) \leq v(x)\}$ . Then  $J(u)$  is bounded from below on  $\Sigma$  and by the direct method, we can get a minimizer  $u_*$  of the functional  $J(u)$  on  $\Sigma$  and by the standard regularity theory of elliptic equation of second order, we know that  $u_*$  is a smooth positive solution. We may denote this solution as  $\underline{u}$ . By the standard calculation we then obtain the stability of  $\underline{u}$ .

In practise, we may find the following result more useful.

**Theorem 2.** *Assume that  $A, B, h$  are positive functions on the compact Riemannian manifold  $(M^n, g)$ ,  $n \geq 3$ . Assume that there are positive constants  $c_0, A_1$  and  $B_1$  such that  $h(x) \geq c_0$ ,  $A(x) \leq A_1$ ,  $b(x) \leq B_1$  on  $M$ . Assume*

further that there is a positive constant  $X$  such that

$$c_0X - B_1X^p - A_1X^{-(p+2)} = 0.$$

Then for any  $\epsilon > 0$  small, there is a positive solution  $\underline{u}$  to (1) such that  $\epsilon \leq \underline{u}(x) \leq X$ .

If we assume that there are positive constants  $c_0$ ,  $A_1$  and  $B_1$  such that  $h(x) \geq c_0$ ,  $A(x) \leq A_1$ ,  $b(x) \leq B_1$  on  $M$ , then the curves  $y = c_0X$  and  $y = A_1X^p + BX^{-(p+2)}$  intersect at two points  $X_1$  and  $X_2$  with  $X_1 < X_2$ . Then we know that for  $X = X_j, j = 1, 2$ ,

$$h(x)X - B(x)X^p - A(x)X^{-(p+2)} \geq c_0X - B_1X^p - A_1X^{-(p+2)} = 0.$$

Hence  $u = X_j$  is a super-solution to (1). We may assume that  $X_1 \leq 1$ .

Note that there is a positive constant  $A_0$  such that  $A(x) \geq A_0$ . Then for any small constant  $\epsilon > 0$ , the constant function  $u = \epsilon$  is a sub-solution to (1). In fact, we always have

$$h(x)\epsilon - B(x)\epsilon^p - A(x)\epsilon^{-(p+2)} \leq 0.$$

Therefore, by the monotone method, we get a positive solution  $\underline{u}$  to (1) such that  $\epsilon \leq \underline{u}(x) \leq X_1$ .

We remark that similar result is obtained in Theorem 4 in [3]. One may also see the works [4][5][6] and [10] for more results by the monotone method.

We note that in some cases, the solution  $\underline{u}$  is strictly stable. Recall that

$$(J''(\underline{u})v, v) = \int |\nabla v|^2 + h(x)v^2 + (p+2)A\underline{u}^{-(p+3)}v^2 - pB\underline{u}^{p-1}v^2.$$

Note that for  $X \leq 1$  in Theorem 2,

$$(p+2)A\underline{u}^{-(p+3)} - pB\underline{u}^{p-1} \geq (p+2)AX_1^{-(p+3)} - pBX_1^{p-1} \geq 0.$$

Then the strictly stability of  $\underline{u}$  follows.

The plan of this paper is below. In section 2, we introduce the coercive condition about the solution  $\underline{u}$  and we introduce the mountain pass geometry in section 3. The main result is proved in section 4. The uniqueness result is proved in section 5. In the last section we obtain the stable solution on complete non-compact Riemannian manifold.

## 2. A MOUNTAIN PASS SOLUTION: ASSUMPTION

To obtain a mountain pass solution to (1), we now introduce a bubble solution. Fix  $a \in M^n$  such that  $B(a) = \max B(x)$ . Let  $U_{a,\epsilon}$  be the standard bubble solution to the equation

$$[-\Delta + \frac{n(n-2)}{4}]U_{a,\epsilon} = \frac{n(n-2)}{4}U_{a,\epsilon}^p, \quad \text{in } S^n$$

with the base point  $a \in S^n$ . Again in the the stereographic coordinates at  $-a$  in  $R^n$  we have

$$U_{a,\epsilon} = C_n \left[ \frac{\epsilon}{\epsilon^2 + |x|^2} \right]^{(n-2)/2}$$

where  $C_n$  is the well-known dimensional constant (see [1] and [15]).

We remark that  $h = \frac{n-2}{4(n-1)}S_g - |\nabla\Phi|^2$  for some nontrivial function  $\Phi$ , where  $S_g$  is the scalar curvature of the metric  $g$ .

We shall look for a large solution of the form  $u = \underline{u} + v$  and follow the mini-max principle used in the paper of Brezis-Nirenberg [1] (1983).

Then the problem (1) is reduced to finding positive solution to

$$(2) \quad -\Delta v + (h - f'(x, 0))v = f(x, v) - f'(x, 0)v,$$

where  $v_+ = \max(v, 0)$  and

$$f(x, v) = f_+(x, v) + f_-(x, v),$$

with

$$f_+(x, v) = B[(\underline{u} + v_+)^p - \underline{u}^p]$$

and

$$f_-(x, v) = A[(\underline{u} + v_+)^{-(p+2)} - \underline{u}^{-(p+2)}].$$

Note that for  $v > 0$  large the leading of  $f(x, v)$  is  $Bv^p$  and for  $v > 0$  small the leading term in  $f(x, v)$  is  $f'(x, 0)v = [pB - (p+2)A]\underline{u}v$ . For this reason we write it as

$$f(x, v) = f'(x, 0)v + g_+(x, v) + g_-(x, v)$$

with

$$g_+(x, v) = f_+(x, v) - f'_+(x, 0)v = Bv^p + \dots$$

and

$$|g_-(x, v)| = |f_-(x, v) - f'_-(x, 0)v| \leq Cv^2,$$

where  $C$  is a uniform constant depending only on  $\underline{u}$ .

For (2) the corresponding functional is

$$I(u) = \int \frac{1}{2}[|\nabla u|^2 + (h(x) - f'(x, 0))u^2] - G(x, u)$$

with

$$G(x, u) = G_+(x, u) + G_-(x, u) = \frac{1}{p+1} \int B u_+^{p+1} + \dots,$$

where

$$G_+(x, u) = \int_0^u (g_+(x, v))dv, \quad \text{and} \quad G_-(x, u) = \int_0^u (g_-(x, v))dv.$$

To obtain further result, we need to assume that  $(\mathfrak{F})$

**the least eigenvalue of  $-\Delta + h - f'(x, 0)$  is positive.**

Recall here that

$$f'(x, 0) = [pB\underline{u}^{p-1} - (p+2)A\underline{u}^{(p+3)}].$$

The importance of the condition  $(\mathfrak{F})$  is that it gives us a property that for some uniform constant  $\lambda_0 > 0$ , for any  $u$  with the norm  $|u|$  small,

$$I(u) \geq \lambda_0|u|^2 + o(|u|^2).$$

Because of the leading term in  $I(\cdot)$  is  $\int B|u|^{p+1}$ , we can see that

$$I(tu) \rightarrow -\infty, \text{ as } t \rightarrow \infty$$

for any fixed  $u = e_1 \neq 0$  in  $H^1(M)$ . This is the mountain pass property which will play a key role in our argument. However, because of the negative power term in  $I(\cdot)$ , we should be very careful to choose a class of paths for mountain pass value.

We remark that this assumption is not very strong since the solution  $\underline{u}$  is *stable* and we always have the conclusion that

**the least eigenvalue of  $-\Delta + h - f'(x, 0)$  is non-negative.**

We remark that, generally speaking, we don't know the sign of the term  $f'(x, 0) + |\nabla\phi|^2$ .

### 3. MOUNTAIN PASS SOLUTION: INTRODUCTION

The following basic fact is well-known in Riemannian geometry.

**Lemma 3.** *In the normal coordinates  $(x_1, \dots, x_n)$  centered at  $p \in M$ , we have the following expansion of the volume element*

$$dv_g = (1 - \frac{1}{6}R_{ij}x_i x_j + o(|x|^3))dx$$

where  $R_{ij}$  is the Ricci tensor of the metric  $g$  at  $p$ .

We now consider the Lichnerowicz equation

$$(3) \quad \Delta_g u - hu + Bu^p + Au^{-p-2} = 0$$

on the compact Riemannian manifold  $(M^n, g)$ ,  $n \geq 3$ , where  $\Delta_g$  is the Laplacian operator of the metric  $g$  on  $M$ ,  $p = \frac{n+2}{n-2}$ ,  $h, A > 0$ , and  $B \geq 0$  are smooth functions on  $M$ . We shall write  $f(x, u) = Bu^p + Au^{-p-2}$ .

Suppose  $\underline{u}$  is the positive solution to (3) obtained by the monotone method. We are looking for the mountain pass solution to (3). Let  $u = \underline{u} + v$ . Then we consider the following equation

$$(4) \quad \Delta_g v - hv + f(x, \underline{u} + v) - f(x, \underline{u}) = 0, \quad v > 0, \quad \text{on } M.$$

Let

$$F_1(x, v) = \frac{B}{p+1}[(\underline{u} + v)^{p+1} - \underline{u}^{p+1} - (p+1)\underline{u}^p v],$$

$$F_2(x, v) = \frac{A}{p+1}[(\underline{u} + v)^{-p-1} - \underline{u}^{-p-1} + (p+1)\underline{u}^{-p-2} v]$$

and

$$F(x, v) = F_1(x, v) - F_2(x, v).$$

One can easily see that  $F_2(x, v)$  is non-positive and we may drop it in our consideration of the mini-max argument.

Then by an easy computation we know that the equation (4) is the Euler-Lagrange equation for the functional

$$(5) \quad I(v) = \frac{1}{2} \int (|\nabla u|^2 + hu^2) - \int F(x, v)$$

on  $H^1(M)$ .

Fix  $a \in M$  which is the maximum point of  $B(x)$  on  $M$  and choose the normal coordinates  $(x_1, \dots, x_n)$  at  $a$  in  $B_r(a)$ ,  $r < inj(a)$  the injectivity radius of  $g$  at  $a$ . Let  $\xi$  be the cut-off function on  $M$  such that  $\xi(x) = 1$  in the ball  $B_\delta(a)$  and  $\xi(x) = 0$  outside the ball  $B_{2\delta}(a)$ . For  $\epsilon > 0$  small, consider

$$v_{\epsilon,a}(x) = \xi(x)U_{\epsilon,a}(x),$$

where

$$U_{\epsilon,a}(x) = \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^{(n-2)/2}$$

satisfies

$$\Delta U_{\epsilon,a}(x) = \frac{n(n-2)}{4} U_{\epsilon,a}(x)^p, \quad \text{in } \mathbb{R}^n.$$

In short, we write  $U = U_{1,a}$ . We shall omit the lower order term caused by  $0(|x|^2)dx$  in the volume form  $dv_g = (1 + 0(|x|^2))dx$  in the small ball  $B_{2\delta}(a)$ .

Though the term  $F_2(x, v)$  in  $I$  may not be very important in mountain pass construction, it is useful when we use the implicit function theorem. We may bound it below (since it may be useful in finding more solutions).

**Claim :** For  $n = 3$ ,

$$\int_M F_2(x, v_{\epsilon,a}) = 0(\epsilon^{n/2}) + 0(\epsilon);$$

for  $n = 4$ ,

$$\int_M F_2(x, v_{\epsilon,a}) = 0(\epsilon^{n/2}) + 0(\epsilon^2 \log \epsilon);$$

and for  $n \geq 5$ ,

$$\int_M F_2(x, v_{\epsilon,a}) = 0(\epsilon^{n/2}) + 0(\epsilon^2).$$

In fact, for  $|x| > \epsilon^{1/2}$ , we have  $v_{\epsilon,a} < 1$  and

$$F_2(x, v_{\epsilon,a}) \leq C v_{\epsilon,a}^2.$$

In short we write  $v = v_{\epsilon,a}$ . Then we have

$$\int_{|x| > \epsilon^{1/2}} F_2(x, v) \leq C \int_{|x| > \epsilon^{1/2}} \left(\frac{\epsilon}{\epsilon^2 + |x|^2}\right)^{n-2} \leq c\epsilon^2 \int_{\epsilon^{-1/2}}^{\delta/\epsilon} (1+r^2)^{2-n} dr,$$

which is of order  $0(\epsilon)$  for  $n = 3$ ,  $0(\epsilon^2 \log \epsilon)$  for  $n = 4$ , and  $o(\epsilon^2)$  for  $n \geq 5$ .

Note that

$$\int_{|x| \leq \epsilon^{1/2}} [(\underline{u} + v)^{-p-1} - \underline{u}^{-p-1}] = 0(\epsilon^{n/2})$$

and

$$(p+1) \int_{|x| \leq \epsilon^{1/2}} \underline{u}^{-p-2} v = 0(\epsilon^{n/2}).$$

Then

$$\int_{|x| \leq \epsilon^{1/2}} F_2(x, v) = \int_{|x| \leq \epsilon^{1/2}} [(\underline{u}+v)^{-p-1} - \underline{u}^{-p-1}] + (p+1) \int_{|x| \leq \epsilon^{1/2}} \underline{u}^{-p-2} v = 0(\epsilon^{n/2}).$$

Combining all above together we have proved the Claim.

Compute

$$\frac{1}{2} \int_M (|\nabla v|^2 + hv^2) = \frac{1}{2} \int_{R^n} |\nabla U|^2 + C(\epsilon)$$

where  $C(\epsilon) = \omega h(a) (\int_0^1 |\xi|^2(r) dr) \epsilon$ , where  $\omega$  is the area of the 2-sphere  $S^2$  for  $n = 3$ ,  $C(\epsilon) = K_3 h(a) \epsilon^2 |\log \epsilon| + o(\epsilon^2)$  for  $n = 4$ , and  $C(\epsilon) = K_3 \epsilon^2 h(a) + o(\epsilon^2)$  with  $K_3 = \int_{R^n} U^2$  for  $n \geq 5$ . Following the work of Brezis-Nirenberg [1] we shall write

$$K_1 = \int_{R^n} |\nabla U|^2, \quad K_2 = \left( \int_{R^n} U^{p+1} \right)^{2/(p+1)}, \quad K'_2 = \int_{R^n} U^{p+1}.$$

Recall that the best Sobolev constant is  $S = K_1/K_2$ .

We now compute  $\int_M F_1(x, v) = \frac{1}{p+1} \int Bv^{p+1} + I_1 + I_2 + I_3$ , where

$$I_1 = \int \frac{B}{p+1} [(\underline{u}+v)^{p+1} - \underline{u}^{p+1} - v^{p+1} - (p+1)(\underline{u}^p v + \underline{u} v^p) - \frac{p(p+1)}{2} \underline{u}^{p-1} v^2],$$

$$I_2 = \int_M B \underline{u} v^p, \quad \text{and} \quad I_3 = \frac{p}{2} \int_M B \underline{u} v^2.$$

It is well-known that

$$\frac{1}{p+1} \int Bv^{p+1} = B(a)K'_2 + 0(\epsilon^2).$$

In the dimension three to five, this expansion is enough for our use. For higher dimensions, the term  $0(\epsilon^2)$  can be further expanded via the use of  $\Delta K(a)$  and the curvature of the metric  $g$ .

We now compute or estimate  $I_1, I_2$ , and  $I_3$  one by one.

It is clear that  $I_2 = \underline{u}(a)B(a)\epsilon^{(n-2)/2} \int_{R^n} U^p + o(\epsilon^{(n-2)/2})$ .

For the computation of  $I_3$ , we have

$I_3 = O(\epsilon)$  for  $n = 3$ ,

$I_3 = \frac{p}{2} B(a) \underline{u}(a) \epsilon^2 |\log \epsilon| + 0(\epsilon^2)$  for  $n = 4$ , and

$I_3 = \frac{p}{2} B(a) \underline{u}(a)^{p-1} \epsilon^2 + o(\epsilon^2)$  for  $n \geq 5$ .

For  $n \leq 4$ , then  $\frac{n-2}{2} < 2$ ,  $p+1 > 2$ , and we have

$$|I_1| \leq \int_M \underline{u}^{p-1-\delta} v^{2+\delta} \leq C \epsilon^{n - \frac{n-2}{2}(2+\delta)} = O(\epsilon^{2 - \frac{n-2}{2}\delta}).$$

For  $n = 5$ ,

$$I_1 = -\frac{p}{2} B(a) \underline{u}(a)^{p-1} \epsilon^2 + o(\epsilon^2).$$

Then we have for  $n = 3$ , the leading term for  $F_1$  is in  $I_2$ ,

$$\int_M F_1(x, v) = \underline{u}(a)B(a)\epsilon^{1/2} \int_{R^3} U^p + o(\epsilon^{1/2}) + 0(\epsilon);$$

for  $n = 4$ , the leading term is in  $I_2$  too,

$$\int_M F_1(x, v) = \underline{u}(a)B(a)\epsilon \int_{R^4} U^p + o(\epsilon).$$

Then for  $n = 3$ ,

$$I(v) = \frac{K_1}{2} + C(\epsilon) - \frac{B(a)}{p+1}K_2' - \underline{u}(a)B(a)\epsilon^{1/2} \int_{R^3} U^p + o(\epsilon^{1/2}) + 0(\epsilon)$$

and for  $n = 4$ ,

$$I(v) = \frac{K_1}{2} + C(\epsilon) - \frac{B(a)}{p+1}K_2' - \underline{u}(a)B(a)\epsilon \int_{R^4} U^p + o(\epsilon).$$

For  $n = 5$ , using proposition 1 in [13],

$$\int_M F_1(x, v) = \frac{B(a)}{p+1}K_2' + \underline{u}(a)B(a)\epsilon^{3/2} \int_{R^5} U^p + o(\epsilon^{3/2}).$$

In this case, we have

$$I(v) = \frac{K_1}{2} + C(\epsilon) - \frac{B(a)}{p+1}K_2' - \underline{u}(a)B(a) \int_{R^n} U^p \epsilon^{3/2} + O(\epsilon^{3/2}).$$

When  $n = 6$ , we have  $I_1 = 0$  and

$$\int_M F_1(x, v) = o(\epsilon^2).$$

Then

$$I(v) = \frac{K_1}{2} + C(\epsilon) - \frac{B(a)}{p+1}K_2' + \gamma_1 \epsilon^2 + o(\epsilon^2)$$

where

$$\gamma_1 = K_3 h(a)B(a) - 2\underline{u}B(a) - pB(a)\underline{u}(a)^{p-1}.$$

When  $n \geq 7$ , we have

$$I_1 = \frac{p}{2} \int_M B \underline{u}^{p-1} v^2 + \int \frac{B}{p+1} [(\underline{u}+v)^{p+1} - \underline{u}^{p+1} - v^{p+1} - (p+1)(\underline{u}^p v + \underline{u} v^p)].$$

Note that

$$-\frac{p}{2} \int_M B \underline{u}^{p-1} v^2 = -\frac{p}{2} B(a) \underline{u}^{p-1}(a) \epsilon^2 \int_{R^n} U^2$$

and

$$\int \frac{B}{p+1} [(\underline{u}+v)^{p+1} - \underline{u}^{p+1} - v^{p+1} - (p+1)(\underline{u}^p v + \underline{u} v^p)] = o(\epsilon^2).$$

Then we have

$$I_1 = pK_3 B(a) \underline{u}^{p-1}(a) \epsilon^2 + o(\epsilon^2)$$

and

$$I(v) = \frac{K_1}{2} + C(\epsilon) - \frac{B(a)}{p+1}K_2' - pK_3 B(a) \underline{u}^{p-1}(a) \epsilon^2 + o(\epsilon^2).$$



Let  $t_0 = \frac{K_1}{B(a)K_2}$  and  $t_1 = t_0 + 2\sqrt{\epsilon}$ . With this computation result we can the mini-max construction for (1) via considering the value

$$(6) \quad I(tv) = \frac{K_1 + C(\epsilon)}{2}t^2 - \frac{B(a)t^{p+1}}{p+1}K_2' - \underline{u}(a)B(a)t^2 \int_{R^n} U^p \epsilon^{(n-2)/2} + O(\epsilon^{(n-2)/2}),$$

for  $n \leq 5$ ,

$$(7) \quad I(tv) = \frac{K_1 + C(\epsilon)}{2}t^2 - \frac{B(a)t^{p+1}}{p+1}K_2' + \gamma_1 t^2 \epsilon^2 + o(\epsilon^2), \quad \text{for } n = 6,$$

$$(8) \quad I(tv) = \frac{K_1 + C(\epsilon)}{2}t^2 - \frac{B(a)t^{p+1}}{p+1}K_2' - pK_3B(a)\underline{u}^{p-1}(a)\epsilon^2 t^2 + o(\epsilon^2), \quad \text{for } n \geq 7$$

for  $t \in [0, t_1]$  and get the following result.

**Theorem 4.** *Assume that  $A, B, h$  are positive functions on the compact Riemannian manifold  $(M^n, g)$ ,  $n \geq 3$ . Assume that  $(\mathfrak{F})$  is true. Assume that  $3 \leq n < 6$ . Then we can always define a mountain pass of  $I(\cdot)$  and get a positive solution to (2) provided the condition  $(\mathfrak{F})$  is true.*

We remark that for  $n = 6$ , one may assume that

$$\frac{\gamma_1}{B(a)} = K_3 h(a) - 2\underline{u} - p\underline{u}(a)^{p-1} < 0$$

with a curvature assumption to get the same conclusion as above. For  $n > 6$ , one need to assume the flat-ness condition about  $B$  as the scalar curvature problem. We shall not present this kind of result in this paper.

The proof of theorem 4 will be given in next section.

#### 4. MOUNTAIN PASS SOLUTION: PROOF

We now use the mountain pass theorem (see also the argument of Theorem 2.1 (also lemma 2.1) in [1]) to prove Theorem 4.

In fact, the solution corresponds to the minimax value defined by

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} I(u),$$

where  $\Gamma$  stands for the set of continuous paths joining 0 and  $t_1 \phi_{a, \epsilon}$  in  $H^1$ .

The plan to prove Theorem 4 is to use the mountain pass geometry of  $I$  and show that

$$(9) \quad c < \frac{1}{n(\max B)^{(n-2)/n}} S^{n/2},$$

where  $S$  is the best Sobolev constant in  $R^n$ . From the classical theorem of Ambrosetti-Rabinowitz (1973) (see [1]), we know that there is a sequence  $\{u_j\} \subset H^1$  such that

$$I(u_j) \rightarrow c, \quad \text{and} \quad I'(u_j) \rightarrow 0.$$

It is a classical argument [1] that we know that  $\{u_j\}$  is a bounded sequence in  $H^1$ . Then we may assume that  $u_j$  converges weakly to a limit  $u$  in  $H^1$  and in  $L^{p+1}$ , and strongly in  $L^q$  for  $1 < q < p + 1$ . We remark that the negative power term in  $F$  or in  $f$  converge strongly in corresponding spaces ([9]).

Suppose that  $u \equiv 0$ . We may assume that

$$\int |\nabla u_j|^2 \rightarrow l$$

for some  $l \geq 0$ . Then we must have

$$\int B|u_{j+}|^{p+1} \rightarrow l$$

and  $\frac{1}{3}l = c$  (and this implies that  $l > 0$ ).

Using the Sobolev inequality

$$\int |\nabla u_j|^2 + \frac{n(n-2)}{4}u_j^2 \geq S|u_j|_{p+1}^2 \geq \frac{S}{(\max B)^{(p+1)/2}}|Bu_{j+}|_{p+1}^2,$$

we get that

$$l \geq \frac{S}{(\max B)^{(p+1)/2}}l^{2/(p+1)}.$$

Then we have

$$c \geq \frac{1}{n(\max B)^{(n-2)/n}}S^{n/2}.$$

Assume that the limit  $u \neq 0$  and  $u_j$  does not converge strongly in  $H^1$  to  $u$ . Then  $u$  is a solution to (2). We claim that  $I(u) = c$ . In fact, by Bresiz-Lieb lemma [1] we know that

$$c \leftarrow I(u_j) = I(u) + I(u_j - u) + o(1).$$

Note that

$$I(u_j - u) = \int \frac{1}{2}|\nabla(u_j - u)|^2 - \frac{1}{p+1} \int B|u_{j+} - u_+|^{p+1} + o(1).$$

Let  $w_j = u_j - u$ . Then  $w_j \rightarrow 0$  strongly in  $L^2$ . Using  $(I'(u_j), u_j) \rightarrow 0$  we get that

$$\int |\nabla w_j|^2 = \int B|w_{j+}|^{p+1} + o(1).$$

This gives us that

$$I(u) + \frac{1}{n} \int |\nabla w_j|^2 = c + o(1).$$

We may assume that

$$\int |\nabla w_j|^2 \rightarrow k > 0, \quad \text{and} \quad \int B|w_{j+}|^{p+1} \rightarrow k.$$

Using the Sobolev inequality we have that

$$k \geq \frac{S}{(\max B)^{(p+1)/2}}l^{2/(p+1)}.$$

A contradiction. Hence we have that  $u_j$  does not converge strongly in  $H^1$  to  $u$ , which is a positive solution in  $H^1$  and  $I(u) = c$ . By the standard regularity theory we know that  $u > 0$  is a smooth solution to (2).

In the remaining part of this section we show that (9) is always true.

We now try to bound of the quantity

$$\sup_{t \in [0, t_1]} I(tv)$$

by using the computation results in (6-8).

Let  $C = K_1 - \gamma_1(\epsilon)$  for some small  $\gamma_1(\epsilon) > 0$  and  $D = B(a)K'_2$ .

Recall that the maximum value of  $\frac{t^2}{2}C - \frac{t^{p+1}}{p+1}D$  is taken at  $t_0 = (\frac{C}{D})^{1/p}$  with the value

$$\frac{1}{n} \left[ \frac{C}{D^{2/(p+1)}} \right]^{n/2} < \frac{1}{n \max B^{(n-2)/n}} S^{n/2}.$$

Using the implicit function theorem we know that the maximal value of  $I(t\phi_{a,\epsilon})$  is taken at  $t_0 + o(\epsilon)$ . Hence we have

$$I(tv) < \frac{1}{n \max B^{(n-2)/n}} S^{n/2}.$$

Therefore, the condition (9) is satisfied and the proof of theorem 4 is complete.

## 5. UNIQUENESS WHEN THE CONDITION $(\mathfrak{F})$ FAILS

The main question now is to show the assumption  $(\mathfrak{F})$  is not true at  $\underline{u}$ . Hence, there is a positive solution  $\eta > 0$  such that

$$[-\Delta + h - f'(x, 0)]\eta = 0, \quad \text{in } M.$$

Furthermore, we have by using the monotone method and the bifurcation theory of Crandall-Rabinowitz [2] that  $\underline{u}$  is the minimal solution to (1). In this case we always have a family of minimal positive solutions  $(\lambda, u(\lambda)) \in (0, 1] \times C^2(M)$  to the perturbation problem

$$-\Delta u + hu = \lambda[Bu^p + Au^{-(p+2)}], \quad \text{in } M$$

with  $u(1) = \underline{u}$ .

Assume that there is another positive solution  $w$  to (1). Then we have  $w > \underline{u}$  in  $M$ . Let  $\phi = w - \underline{u}$ . Then  $\phi > 0$  satisfies (2). Using the convexity of  $f(x, v)$  we know that

$$(10) \quad -\Delta \phi + (h - f'(x, 0))\phi > 0, \quad \text{in } M.$$

Then for any  $c \in \mathbb{R}$ , we have

$$-\Delta(\phi - c\eta) + (h - f'(x, 0))(\phi - c\eta) > 0, \quad \text{on } M.$$

Choose  $c \in \mathbb{R}$  such that  $\phi - c\eta \geq 0$  has its minimum value 0 at  $x_0 \in M$ . Then this implies that at  $x_0$ ,

$$-\Delta(\phi - c\eta) \leq 0,$$

however, by (10),

$$0 < -\Delta(\phi - c\eta) + (h - f'(x, 0))(\phi - c\eta) = -\Delta(\phi - c\eta),$$

which is a contradiction.

In conclusion we have

**Theorem 5.** *Assume that  $A, B, h$  are positive functions on the compact Riemannian manifold  $(M^n, g)$ ,  $n \geq 3$ . Assume that the condition  $(\mathfrak{F})$  fails, i.e., there is a positive solution  $\eta > 0$  such that*

$$[-\Delta + h - f'(x, 0)]\eta = 0, \quad \text{in } M.$$

*Then the problem (1) has a unique positive solution  $\underline{u}$ .*

## 6. EXISTENCE RESULT FOR LICHNEROWICZ EQUATION ON COMPLETE NON-COMPACT RIEMANNIAN MANIFOLDS

We now make a remark about the solvability of (1) on a general non-parabolic complete Riemannian manifold  $(M, g)$ . We make the following two assumptions about  $(M, g)$ .

(1). We shall assume that the Riemannian manifold  $(M^n, g)$ ,  $n \geq 3$ , is not parabolic, that is, the positive Green function  $G(x, y)$  exists on  $M \times M$  to the operator  $-\Delta$ .

(2). For the complete Riemannian manifold  $(M, g)$ , assume that there is a positive constant  $Z \geq 1$  such that the function

$$(11) \quad h(x) - B(x)Z - A(x)Z^{1-n} \geq 0.$$

The important feature about the assumption (1) is the following result:

**Proposition 6.** *Assume (1) above is true and assume that  $0 \leq h \in L^1(M, g)$  and  $h \neq 0$  with*

$$\int_M G(x, y)h(y)dv_g < \infty.$$

*Then the equation*

$$-\Delta u + hu = 0, \quad \text{in } M$$

*has a bounded positive solution  $u$ .*

This result has been proved by A.Grigor'yan [8]. With the help of the result above, we have

**Theorem 7.** *Assume (1) and (2) above. Assume also that  $0 \leq h \in L^1(M, g)$  and  $h \neq 0$  such that*

$$\int_M G(x, y)h(y)dv_g < \infty.$$

*Then there is a positive solution  $u$  to the Lichnerowicz equation (1) with  $0 < u < Z$ .*

*Proof.* By our assumption, we can get a bounded positive solution  $u_*$  to the equation

$$-\Delta u + hu = 0, \text{ in } M.$$

We may normalize  $u_*$  such that  $0 < u_* \leq 1$ . Using the strong maximum principle, we know that  $u(x) < 1$  on  $M$ .

It is now clear that  $u_- = u_* < Z^{(n-2)/4} = u_+$  are a pair of sub and super solutions to (1). Hence we get by the monotone method that there is a positive solution  $u$  to (1) with  $u_0 \leq u \leq Z$ . This completes the proof of Theorem 7.  $\square$

#### REFERENCES

- [1] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*. Comm. Pure Appl. Math. 36 (1983), no. 4, 437-477.
- [2] M.G.Crandall, P.H.Rabinowitz, *Bifurcation from simple eigenvalues*, J.Functional Analysis, 8(1971)321-340.
- [3] Yvonne Choquet-Bruhat; Isenberg, James; Pollack, Daniel, *The constraint equations for the Einstein-scalar field system on compact manifolds*. Classical Quantum Gravity 24 (2007), no. 4, 809-828.
- [4] Yvonne Choquet-Bruhat; Isenberg, James; Pollack, Daniel, *Applications of theorems of Jean Leray to the Einstein-scalar field equations*. J. Fixed Point Theory Appl. 1 (2007), no. 1, 31-46.
- [5] Yvonne Choquet-Bruhat; Isenberg, James; Pollack, Daniel, *The Einstein-scalar field constraints on asymptotically Euclidean manifolds*. Chinese Ann. Math. Ser. B 27 (2006), no. 1, 31-52.
- [6] Yvonne Choquet-Bruhat, James Isenberg; James W.York, Jr. *Einstein constraints on asymptotically Euclidean manifolds*. Phys. Rev. D (3) 61 (2000), no. 8, 084034
- [7] Olivier Druet, Emmanuel Hebey, *Stability and instability for Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds*, Mathematische Zeitschrift, 2009, Volume 263, Number 1, Pages 33-67
- [8] A.A.Grigor'yan, *Bounded solutions of the Schrodinger equation on non-compact riemannian manifolds*, Journal of Soviet Math. 51(1)(1990)2340-2349.
- [9] Emmanuel Hebey, Frank Pacard, Daniel Pollack, *A variational analysis of Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds*. Comm. Math. Phys. 278 (2008), no. 1, 117-132
- [10] James Isenberg, David Maxwell, Daniel Pollack, *A gluing construction for non-vacuum solutions of the Einstein-constraint equations*. Adv. Theor. Math. Phys. 9 (2005), no. 1, 129-172.
- [11] Li Ma, *Liouville type theorem and uniform bound for the Lichnerowicz equation and the Ginzburg-Landau equation*. C. R. Math. Acad. Sci. Paris 348 (2010), no. 17-18, 993-996
- [12] Li Ma, Xingwang Xu, *Uniform bound and a non-existence result for Lichnerowicz equation in the whole n-space*, C. R. Mathematique Ser. I 347 (2009), pp. 805-808.
- [13] Li Ma, H. Wang, *A minimization problem arising from prescribing scalar curvature functions*, Math. Z., 222(1996)1-6.
- [14] W.M. Ni, *On the elliptic equation  $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$ , its generalizations, and applications in geometry*. Indiana Univ. Math. J. 31 (1982), no. 4, 493-529.
- [15] G.Tarantello, *On non-homogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. Henri Poincare, Analyse non lineaire, 9(3)(1992)281-304.

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