Multi-vortex solutions to Ginzburg-Landau equations with external potential

A. Pakylak *F. Ting †J. Wei ‡

October 12, 2011

Abstract

We consider the existence of multi-vortex solutions to the Ginzburg-Landau equations with external potential on $\mathbb{R}^2$. These equations model equilibrium states of superconductors and stationary states of $U(1)$ Higgs model of particle physics. In the former case, the external potential models impurities and defects. We show that if the external potential is small enough and the magnetic vortices are widely spaced, then one can pin one or an arbitrary number of vortices in the vicinity of a critical point of the potential. In addition, one can pin an arbitrary number of vortices near infinity if the potential is radially symmetric and of algebraic order near infinity.

Keywords: Ginzburg-Landau equations, magnetic vortices, external potential, superconductivity, multi-vortex solutions, existence, pinning, reduced energy.

1 Introduction

In this section, we the introduce the physical phenomena in superconductors we are considering: the pinning phenomena in superconductors. We also review some facts about Ginzburg-Landau equations with and without external potential on *all* of $\mathbb{R}^2$ and some previous results on pinning a *single vortex* from [43].

*Department of Physics, Lakehead University, Thunder Bay, ON, Canada, P7B 5E1. Email: apakylak@lakeheadu.ca

†Department of Mathematical Sciences, Lakehead University, Thunder Bay, ON, Canada, P7B 5E1. E-mail: fridolin.ting@lakeheadu.ca.

‡Department of Mathematics, Chinese University of Hong Kong, Shatin, N.T., Hong Kong. Email: wei@math.cuhk.edu.hk

1.1 Pinning phenomena in superconductors

In this section, we introduce the pinning phenomena in superconductors.

When a superconductor of Type II is placed in an external magnetic field, the field penetrates the superconductor in thin tubes of magnetic flux called *magnetic vortices*. To date, superconductors of Type II have been used to produce large and steady magnetic fields over 100,000 Gauss. It is well known that one major obstacle in producing larger magnetic fields is the dissipation of energy due to *creeping* or *flow* of magnetic vortices [47]. One way to resolve this is to *pin* vortices down.

The Ginzburg-Landau equations with external potential models a superconductor with inhomogeneities, impurities or point defects. It is known that magnetic vortices get pinned down to the sites of the impurities [47, 40]. In [43], it is shown, within the standard macroscopic theory of superconductivity, that this indeed happens. Namely, it is shown that a vortex solution exists which is localized near a critical point of the potential. In this paper, we prove that under suitable assumptions on the potential, then multi-vortex configurations will be localized near multiple critical points of the impurity potential. In addition, we show that one can pin an arbitrary number of vortices to one critical point and pin an arbitrary number of vortices near infinity.

1.2 Ginzburg-Landau equations with and without potential

In this section, we review some facts about Ginzburg-Landau equations with and without external potential on *all* of $\mathbb{R}^2$.

The standard macroscopic (or mean field) theory of superconductivity is due to Ginzburg and Landau. Stationary states of superconductors occupying (for simplicity) the plane $\mathbb{R}^2$, are described by pairs (ψ, A), where $\psi : \mathbb{R}^2 \to \mathbb{C}$ is the *order parameter* and $A : \mathbb{R}^2 \to \mathbb{R}^2$ is the *magnetic potential*. These states satisfy the system of equations

$$-\Delta_A \psi + \lambda(|\psi|^2 - 1)\psi = 0 \tag{1}$$

$$\nabla \times \nabla \times A + \mathrm{Im}(\bar{\psi} \nabla_A \psi) = 0 \tag{2}$$

called the *Ginzburg-Landau* (GL) equations. Here $\lambda > 0$ is a constant depending on the material in question: when $\lambda < 1/2$ or $> 1/2$, the material is of type I or II superconductor, respectively; $\nabla_A = \nabla - iA$ is the covariant gradient, and $\Delta_A = \nabla_A \cdot \nabla_A$. For a vector field A, $\nabla \times A$ is the scalar $\partial_1 A_2 - \partial_2 A_1$ and for scalar ξ, $\nabla \times \xi$ is the vector $(-\partial_2 \xi, \partial_1 \xi)$. Equation (2) is the static Maxwell equation for the magnetic field $B = curl\, A$ and supercurrent $Im(\bar{\psi} \nabla_A \psi)$. We consider here configurations satisfying the superconducting boundary condition

$$|\psi(x)| \to 1 \qquad as \qquad |x| \to \infty.$$

The Ginzburg-Landau equations on the plane model superconductors which are spatially homogeneous in one direction, when neglecting boundary effects. They also describe equilibrium states of the $U(1)$ Higgs model of particle physics [27].

Equations (1) and (2) are Euler-Lagrange equations for the Ginzburg-Landau energy functional

$$\mathcal{E}(\psi, A) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_A \psi|^2 + (\nabla \times A)^2 + \frac{\lambda}{2}(|\psi|^2 - 1)^2,$$

i.e., solutions of (1) and (2) are critical points of $\mathcal{E}$: $\mathcal{E}'(\psi, A) = 0$. Here $\mathcal{E}'(u)$ denotes the L^2 gradient of the functional $\mathcal{E}$ at a point $u := (\psi, A)$.

Define the vorticity or the winding number of the vector field $\psi : \mathbb{R}^2 \to \mathbb{C}$ at infinity as $deg\, \psi := deg\left(\frac{\psi}{|\psi|}|_{|x|=R}\right) = \frac{1}{2\pi}\int_{|x|=R} d(arg\psi)$ for R sufficiently large. Assuming a pair (ψ, A) has finite energy, then the degree of the vector field ψ is related to the flux of the magnetic field $B = curl\, A$ as follows:

$$\int_{\mathbb{R}^2} B = 2\pi(deg\, \psi).$$

The only non-trivial, finite energy, rigorously known solutions to equations (1)-(2) in $\mathbb{R}^2$ are the radially symmetric, equivariant solutions of the form $u = (\psi^{(n)}, A^{(n)})$, with

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \qquad and \qquad A^{(n)}(x) = a_n(r)\nabla(n\theta) \tag{3}$$

called $n - vortices$. Here (r, θ) are the polar coordinates of the vector $x \in \mathbb{R}^2$ and $n = deg\, \psi_n$ is an integer. Existence of n-vortex solutions of the form (3) was proved in [37, 5] using variational methods. The stability and instability properties of n-vortices were established in [23], [22]. More specifically, in [23], they showed that for $\lambda < 1/2$, any integer degree vortex is stable; for $\lambda > 1/2$, only $n = \pm 1$ vortices are stable. When $\lambda = 1/2$, all integer degree vortices are stable [9]. Recently [48], there have been new developments in finding non-radial solutions to (1) and (2).

One has the following information on the vortex profiles f_n and a_n (see [37, 5]): $0 < f_n < 1, 0 < a_n < 1$ on $(0, \infty)$; $f_n', a_n' > 0$; and $1 - f_n, 1 - a_n \to 0$ as $r \to \infty$ with exponential rates of decay. In fact,

$$\begin{aligned} f_n(r) &= 1 + O(e^{-m_\lambda r}) \quad \text{and} \\ a_n(r) &= 1 + O(e^{-r}) \quad \text{with} \\ m_\lambda &:= \min(\sqrt{2\lambda}, 2). \end{aligned}$$

At the origin, $f_n \sim cr^n, a_n \sim dr^2$ ($c > 0, d > 0$ are constants) as $r \to 0$.

In addition, we have the asymptotics of the field components as established in [37] as $r = |x| \to \infty$:

$$\begin{aligned} j^{(n)}(r) &= n\beta_n K_1(r)[1 + o(e^{-m_\lambda |x|})]J\hat{x} \\ B^{(n)}(r) &= n\beta_n K_1(r)[1 - \frac{1}{2r} + O(1/r^2)] \\ |f_n'(r)| &\leq ce^{-m_\lambda r}. \end{aligned} \tag{4}$$

Here $j^{(n)} = \mathrm{Im}(\overline{\psi^{(n)}}(\nabla_A\psi)^{(n)}$ is the n-vortex supercurrent, $\beta_n > 0$ is a constant, and $K_1(r)$ the order one Bessel's function of the second kind which behaves like $e^{-r}/\sqrt{r}$ as $r \to \infty$.

Equations (1) and (2), in addition to being rotationally invariant, have translational and gauge symmetries: solutions are mapped to solutions under the transformations

$$\psi(x) \mapsto \psi(x - z), \qquad A(x) \mapsto A(x - z)$$

for any $z \in \mathbb{R}^2$, and

$$\psi \mapsto e^{i\gamma}\psi, \qquad A \mapsto A + \nabla\gamma$$

for twice differentiable $\gamma : \mathbb{R}^2 \to \mathbb{R}$. Consequently, solutions (3) lead to the following families of solutions

$$\psi_{nz\gamma}(x) = e^{i\gamma(x)}\psi^{(n)}(x - z) \qquad A_{nz\gamma}(x) = A^{(n)}(x - z) + \nabla\gamma(x) \tag{5}$$

where n is an integer, $z \in \mathbb{R}^2$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}$.

A superconductor with impurities can be modelled (see for e.g., [1, 12, 13, 43]) by modified GL equations:

$$-\Delta_A\psi + \lambda(|\psi|^2 - 1)\psi + W(x)\psi = 0 \tag{6}$$

$$\nabla \times \nabla \times A + \mathrm{Im}(\bar{\psi}\nabla_A\psi) = 0 \tag{7}$$

where $W : \mathbb{R}^2 \to \mathbb{R}$ is a potential of impurities. These are Euler-Lagrange equations for the (Ginzburg-Landau) energy functional with external potential

$$\mathcal{E}_W(\psi, A) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla_A\psi|^2 + (\nabla \times A)^2 + \frac{\lambda}{2}(|\psi|^2 - 1)^2 + W(x)(|\psi|^2 - 1). \tag{8}$$

We note here that $\mathcal{E}_0(\psi, A)$ is the usual G-L energy functional with $W = 0$, i.e., the Euler-Lagrange equations for $\mathcal{E}_0(\psi, A)$ are (1) and (2).

This type of model has been analyzed frequently in the applied mathematics literature (see, eg., [1, 12, 13, 43]), having appeared earlier in the physics literature (references can be found in the above-mentioned papers). The model considered here differs slightly from those in [1, 3, 12, 13] since we are considering the whole space $\mathbb{R}^2$ and there is no external applied field. In addition, we would like to emphasize that in the present work, we do not consider a point-vortex limit (i.e. $\lambda \to \infty$) – indeed our results are valid for *any* $\lambda > 0$ and this is an important difference between our analysis and much of the previous work.

Existence and uniqueness of ± 1 fundamental vortex-type solutions of the Ginzburg-Landau system (6)-(7) with small external potential W was shown in [43]. These solutions, which are near the vortex solutions (5), are called *pinned fundamental vortices.*

Notation: For the rest of the paper, when we write L^2 and H^s, we mean scalar/vector L^2 spaces and scalar/vector Sobolev spaces or order s. We denote the real inner product on $L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$ to be

$$\left\langle \begin{pmatrix} \xi \\ \alpha \end{pmatrix}, \begin{pmatrix} \varrho \\ \beta \end{pmatrix} \right\rangle := \int_{\mathbb{R}^2} \{\mathrm{Re}(\bar{\xi}\varrho) + \alpha \cdot \beta\}.$$

We will denote L^p norms as $\|\cdot\|_p = \|\cdot\|_{L^p}$ and H^s norms as $\|\cdot\|_{H^s}$. When we write $b << c$ for real numbers b and c, we mean $b < ac$ for some small constant $0 < a < 1$. Finally, we will denote the letter c or C for generic constants that do not depend on any small parameters present.

1.3 Pinning a single vortex

In this section, we review the existence result of Theorem 2.1 in [43]: pinning of a single vortex to one critical point. We *modify and expand* the conditions for existence of a pinned fundamental vortex in [43] to suit our present purposes. In particular, we state the Theorem in terms of the potential W instead of the effective potential W_{eff}; we extend the existence result to higher degree vortices for $\lambda < 1/2$. The same analysis follows through as in [43].

Theorem 1.1. *Let parameters μ and ν be taken to be small and $c > 0$ is fixed. Assume the external potential $W(x)$ satisfies*

- *(A) $\|W\|_{L^2} \leq \mu$;*
- *(B) $\sup_{x\in\mathbb{R}^2} |\partial_x^\alpha W(x)| \leq \mu\nu^{|\alpha|+1}$ for $0 \leq |\alpha| \leq 3$;*
- *(C) $\nabla W(z_0) = 0$ for some $z_0 \in \mathbb{R}^2$ and*
- *(D) $W(z_0)$ is invertible and satisfies $\|W''(z_0)^{-1}\| \leq c(\mu\nu^3)^{-1}$.*

Then for $\mu > 0$ and $\nu > 0$ sufficiently small, $\nu \ll 1$ and $\mu \ll \nu^4$, equations (6)-(7) with external potential W satisfying conditions (A) through (D) has a solution of the form

$$u_{pv} := (\psi_{pv}, A_{pv}),$$

where

$$\begin{aligned} \psi_{pv}(x) &= e^{i\gamma(x)}\psi^{(n)}(x-z)+\xi(x) \\ A_{pv}(x) &= A^{(n)}(x-z)+\gamma(x)+\beta(x) \end{aligned} \tag{9}$$

with $z = z_0 + O\left(\max\left(\nu, \frac{\mu}{\nu^3}\right)\right)$, $\gamma(x) \in H^1(\mathbb{R}^2;\mathbb{R})$, *and* $\xi(x)$ *and* $\beta(x)$ *of* $O(\mu)$ *in* $H^2(\mathbb{R}^2;\mathbb{C})$ *and* $H^2(\mathbb{R}^2;\mathbb{R}^2)$, *respectively. Here,* n *is any integer if* $0 < \lambda < 1/2$, *and* $n = \pm 1$ *if* $\lambda > 1/2$.

Examples of potentials satisfying these conditions for some c, and for arbitrarily small μ and ν, are given by $W(x) := \tilde{\mu}\tilde{\nu}V(\tilde{\nu}(x-z_0))$ for $V \in L^2(\mathbb{R}^2) \cap C^3(\mathbb{R}^2)$ with a non-degenerate critical point at the origin, with $\tilde{\mu}$ and $\tilde{\nu}$ sufficiently small.

2 Problem and results

In this section, we state our main results on existence of *multi-vortex* solutions to the Ginzburg-Landau equations with external potential in the macroscopic model of superconductivity and the Abelian Higgs model of particle physics.

Consider test functions describing $m \geq 2$ number of vortices glued together with centers at $z_1, z_2, ..., z_m$, and degrees $n_1, n_2, ..., n_m$. More specifically, let $m \in \mathbb{Z}^+$ denote the number of vortices with topological degrees $(n_1, n_2, ..., n_m) \in \mathbb{Z}^m$, $n_j \in \mathbb{Z} \setminus \{0\}$; denote the location of the center of each of these m vortices by $\underline{z} = (z_1, z_2, ..., z_m) \in \mathbb{R}^{2m}$, and let $\chi : \mathbb{R}^2 \to \mathbb{R}$ denote a gauge transformation. We associate with each $\underline{z}$ and χ, a function

$$v_{\underline{z}\chi} := (\psi_{\underline{z}\chi}, A_{\underline{z}\chi}) \tag{10}$$

where

$$\psi_{\underline{z}\chi} = e^{i\chi(x)}\prod_{j=1}^{m}\psi^{(n_j)}(x-z_j) \tag{11}$$

and

$$A_{\underline{z}\chi} = \sum_{j=1}^{m} A^{(n_j)}(x-z_j) + \nabla\chi(x). \tag{12}$$

For a given $\underline{z} \in \mathbb{R}^{2m}$, we take our gauge transformations to be of the form

$$\chi(x) = \sum_{j=1}^{m} z_j \cdot A^{(n_j)}(x-z_j) + \tilde{\chi}(x) \tag{13}$$

with $\tilde{\chi} \in H^2(\mathbb{R}^2;\mathbb{R})$. Equivalently,

$$\chi \in H^2_{\underline{z}}(\mathbb{R}^2;\mathbb{R}) := \{\chi \in H^2(\mathbb{R}^2;\mathbb{R}) \mid \chi - \sum_{j=1}^{m} z_j \cdot A^{(n_j)}(x-z_j) \in H^2(\mathbb{R}^2;\mathbb{R})\}$$

to ensure that $v_{\underline{z}\chi} \in X^{(n)}$, where

$$X^{(n)} := \{(\psi, A) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2 \mid (\psi, A) - (\psi^{(n)}, A^{(n)}) \in H^1(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)\}$$

is the affine space of degree n configurations (see (A.15) and (A.16) in the Appendix). The pair $(\underline{z}, \chi)$ (or sometimes just $\underline{z}$) is called a *multi-vortex configuration.*

We make the following definitions:

For $m \geq 2$ arbitrary points $x_1, ..., x_m \in \mathbb{R}^2$, define the *configuration* of $\underline{x} = (x_1, ..., x_m) \in \mathbb{R}^{2m}$ and the *separation of configuration* $\underline{x} \in \mathbb{R}^{2m}$ as

$$R(\underline{x}) = \min_{i \neq j} |x_i - x_j|.$$

For a configuration $\underline{x} \in \mathbb{R}^{2m}$, define

$$R_\lambda(\underline{x}) = \begin{cases} \frac{e^{-R(\underline{x})}}{\sqrt{R(\underline{x})}}, & \lambda > 1/2, \\ e^{-m_\lambda R(\underline{x})}, & \lambda < 1/2. \end{cases} \tag{14}$$

For $\underline{z} = (z_1, ..., z_m) \in \mathbb{R}^{2m}$ denoting the centers of m vortices in a vortex configuration, $R(\underline{z}) = \min_{i \neq j} |z_i - z_j|$ is called the *inter-vortex separation.* Note that if $R(\underline{z})$ is taken to be large enough, then the multi-vortex configurations are approximate solutions to the Ginzburg-Landau equations (see Theorem 3.1(a) below).

For real positive numbers e, f, g, h and ϵ, μ small, define error functions

$$\Gamma_\lambda^{e,f,g,h}(\epsilon) = \begin{cases} \epsilon^e \log^f(1/\epsilon), & \lambda > 1/2, \\ \epsilon^g \log^h(1/\epsilon^{q_\lambda}), & \lambda < 1/2. \end{cases} \tag{15}$$

with $q_\lambda := \frac{2}{m_\lambda}$ and

$$\Gamma_{\lambda,\mu}^{e,f,g,h}(\epsilon) = O(\max(\Gamma_\lambda^{e,f,g,h}(\epsilon), \mu)). \tag{16}$$

Note that the definition of q_λ here is different from the one in [46] since $R_\lambda(\underline{z}) = e^{-m_\lambda R(\underline{z})} R(\underline{z})^{3/4}$ in [46] while $R_\lambda(\underline{z}) = e^{-m_\lambda R(\underline{z})}$ here for $\lambda < 1/2$.

Let

$$v_{z_j \chi} = \begin{pmatrix} e^{i\chi} \psi^{(n_j)}(x - z_j) \\ A^{(n_j)}(x - z_j) + \nabla \chi \end{pmatrix}, \tag{17}$$

and the *self-energy of the* j^{th} *vortex* $(\psi^{(n_j)}, A^{(n_j)})$ as

$$E^{(n_j)} := \mathcal{E}_0(v_{z_j \chi}).$$

Define *inter-vortex/interaction energy* as

$$V_{\text{int}}(\underline{z}) := \mathcal{E}_0(v_{\underline{z}\chi}) - \sum_{j=1}^{m} E^{(n_j)} \tag{18}$$

and the *effective external potential* as

$$\mathrm{W}_{\mathrm{ext}}(\underline{z}) := \frac{1}{2}\sum_{j=1}^{m}\int_{\mathbb{R}^2} W(x)(|\psi^{(n_j)}(x-z_j)|^2-1)dx. \tag{19}$$

Note that if our multi-vortex configuration $\underline{z}$ satisfies the *large separation condition*

$$R_\lambda(\underline{z}) < \epsilon$$

for ϵ sufficiently small (see (14)), then the strength of the inter-vortex interaction energy is

$$|V_{\mathrm{int}}(\underline{z})| = O(\Gamma_\lambda^{1,0,1,0}(\epsilon)) = O(\epsilon) \tag{20}$$

by Lemma 6.1 below.

We now consider the Ginzburg-Landau equations with external potential W satisfying the following conditions below for some $\mu, \nu > 0$ small parameters.

- (A) (**Strength of external potential** W)
  $W \in L^2(\mathbb{R}^2)$, with $\|W\|_{L^2} \leq \mu$.
- (B) (**Smallness of derivatives of** W)
  $\sup_{x\in\mathbb{R}^2}|\partial_x^\alpha W(x)| \leq \mu\nu^{|\alpha|+1}$ for $0 \leq |\alpha| \leq 3$.

We will see in the theorems below that we can pin one vortex to one critical point if the strength of the external potential $W(x)$ (roughly μ) is stronger than the strength of the inter-vortex energy given in (20). If the strength of the external potential and inter-vortex potential are "comparable", then one can pin several vortices and pin vortices near infinity.

We begin with the following crucial reduction result and existence of a reduced energy.

Theorem 2.1 (**Reduced Energy**). *Suppose $\lambda > 0$ and W satisfies condition (A). Then there exists a constant $\epsilon_0 > 0$ such that for ϵ and μ satisfying $\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon) < \epsilon_0$, there exists a constant $\delta_0 > 0$ and a C^1 function $\Phi_W : \mathbb{R}^{2m} \to \mathbb{R}$ such that there is a one-to-one correspondence between critical points of $\mathcal{E}_W$ in the tube $\{v \in X^{(n)} \mid ||v - v_{\underline{z}\chi}||_{H^2} \leq \delta_0$ for some $\underline{z}$ with $R_\lambda(\underline{z}) < \epsilon$ and $\chi \in H^2_{\underline{z}}(\mathbb{R}^2;\mathbb{R})\}$ and critical points of Φ_W in the open set $\{\underline{z} \in \mathbb{R}^{2m} \mid R_\lambda(\underline{z}) < \epsilon\}$.*

With the existence of a reduced energy, we are left to find critical points of a real valued finite dimensional function $\Phi_W(\underline{z})$ in the open set $\{\underline{z} \in \mathbb{R}^{2m} \mid R_\lambda(\underline{z}) < \epsilon\}$. The next four theorems give sufficient conditions for existence of critical points in $\{\underline{z} \in \mathbb{R}^{2m} \mid R_\lambda(\underline{z}) < \epsilon\}$.

Theorem 2.2 (**Pinning one vortex to one critical point**). *Suppose $\lambda > 1/2$ ($\lambda < 1/2$) and $n_j = +1$ or -1 (n_j is any positive or negative integer) for $j = 1, ..., m$. Suppose the external potential W satisfies conditions (A) and (B) above with extra conditions (C) and (D) below for some $\mu, \nu > 0$.*

- *(C) (Widely spaced critical points)*

  W has m critical points widely spaced apart, i.e., $\nabla W(b_j) = 0$ for some $b_j \in \mathbb{R}^2$, $j = 1, ..., m$ with critical point separation $R(\underline{b}) = min_{j \neq k}|b_j - b_k|$ *satisfying*
  $$R_\lambda(\underline{b}) < \epsilon. \tag{21}$$
  Here $\underline{b} := (b_1, ..., b_m)$ is the configuration of critical points.

- *(D) (Non-degeneracy of critical points)*

  The critical points $b_1, b_2, ..., b_m$ of W are non-degenerate in the following sense: For all $j = 1, ..., m$, $W''(b_j)$ is invertible with bound $||W''(b_j)^{-1}|| \leq c(\mu\nu^3)^{-1}$.

Then there exists a constant $\epsilon_0 > 0$ such that for ϵ, μ and ν satisfying $\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon) < \epsilon_0$,
$$\Gamma_{\lambda}^{1,\frac{1}{2},1,\frac{3}{2}}(\epsilon) << \mu\nu^4, \quad \mu << \nu^4 \quad \text{and} \;\; \nu << 1, \tag{22}$$
(6) and (7) has solutions of the form
$$\begin{aligned} \psi_{mvp}(x) &= e^{i\chi(x)} \prod_{j=1}^{m} \psi^{(n_j)}(x - a_j) + \xi(x) \\ A_{mvp}(x) &= \sum_{j=1}^{m} A^{(n_j)}(x - a_j) + \nabla\chi(x) + \beta(x) \end{aligned}$$
where $a_j = b_j + O(\nu) \in \mathbb{R}^2$, $j = 1, ..., m$; $\chi \in H^1(\mathbb{R}^2; \mathbb{R})$ is an arbitrary gauge function and $(\xi(x), \beta(x))$ are of $O(\mu)$ in $H^2(\mathbb{R}^2; \mathbb{C}\times\mathbb{R}^2)$. The pair (ψ_{mvp}, A_{mvp}) is called a multi-vortex pinned solution.

Theorem 2.3 (**Pinning several vortices to a maximum/minimum**). *Suppose $\lambda > 1/2$ ($\lambda < 1/2$) and all the vortices have degree $n_j = +1$ or all $n_j = -1$ (any degree vortex). Suppose the external potential W satisfies condition (A) and (B) with $\mu = O(\epsilon^r)$ for some r satisfying $1 < r < 2$ and $\nu << 1$. If $x = 0$ is a strict local maximum (minimum) of W, then there exists a constant $\epsilon_0 > 0$ such that for ϵ and μ satisfying $\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon) < \epsilon_0$ and for every integer $2 \leq m < \infty$, there exists a vortex configuration $\underline{\tilde{z}}_m$ containing m number of vortices and a solution of (6) and (7) of the form*
$$u_m(x) = v_{\underline{\tilde{z}}_m\chi} + \eta_{\underline{\tilde{z}}_m\chi\epsilon}$$
with $R_\lambda(\underline{\tilde{z}}_m) < \epsilon$, $\chi \in H^1(\mathbb{R}^2; \mathbb{R})$ is an arbitrary gauge function and $\eta_{\underline{\tilde{z}}_m\chi\epsilon}$ is of $O(\Gamma_{\lambda}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))$ in $H^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$.

Theorem 2.4 (Pinning vortices near infinity). *Suppose $\lambda > 1/2$ ($\lambda < 1/2$) and all the vortices have degree $n_j = +1$ or all $n_j = -1$ (any degree vortex). Suppose W satisfies condition (A) with $\mu = O(\epsilon)$. If W is radially symmetric of the form*

$$W(r) = \mu\left(\frac{1}{r^q} + O\left(\frac{1}{r^{q+\varsigma}}\right)\right) \qquad \text{as } r \to \infty \tag{23}$$

for some $q > 1$ and $\varsigma > 0$ (for $\lambda < 1/2$, take the negative of (23)), then there exists a constant $\epsilon_0 > 0$ such that for ϵ and μ satisfying $\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon) < \epsilon_0$, there exists an integer k_0 such that for all $k > k_0$, there exists a solution of (6) and (7) of the form

$$u_k(x) = v_{\tilde{\underline{z}}_k\chi} + \eta_{\tilde{\underline{z}}_k\chi\epsilon}$$

where $\tilde{\underline{z}}_k$ is a vortex configuration containing k vortices equally spaced out on a circle of radius $\tilde{r}_k \in ((\frac{q-\beta}{2\pi})\, k\ln k, (\frac{q+\beta}{2\pi})\, k\ln k)$ for $\beta = o(1)$ small (for $\lambda < 1/2$, the denominator 2π in the interval is replaced by $2\pi m_\lambda$), $\eta_{\tilde{\underline{z}}_k\chi\epsilon} = O(\Gamma_\lambda^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))$ and u_k has energy

$$\mathcal{E}_W(u_k) = k\mathcal{E}_W(v^{\pm 1}) + O\left(\frac{e^{-R(\tilde{\underline{z}}_k)}}{\sqrt{R(\tilde{\underline{z}}_k)}}\right)$$

where $v^{\pm 1}$ is the ± 1 degree vortices.

Remarks:

1. All the theorems above are valid for both type I ($\lambda < 1/2$) and type II ($\lambda > 1/2$) superconductors.

2. Theorems 2.1 and 2.2 are the multi-vortex analogs of Theorems 2.3 and 2.1 in [43], respectively.

3. By Theorem 2.2, if the strength of the external potentials W is larger than the inter-vortex energy, i.e., for $\epsilon << \mu$ (by (22)), and for widely spaced critical points of W, vortices get pinned near the critical points of W. Note that pinning still occurs for vortex-antivortex pairs. This confirms numerical evidence that if the pinning force is strong enough and the distance between the vortex-antivortex is large enough, then pinning occurs [29].

4. Theorems 2.3 and 2.4 says that we can pin an arbitrary number of vortices at an inhomogeneity and near infinity when the magnitude of the external potential is "comparable" to the inter-vortex energy, i.e., $\mu = O(\epsilon^r)$ for $1 < r < 2$ for Theorem 2.3 and $\mu = O(\epsilon)$ for Theorem 2.4.

5. For type II superconductors ($\lambda > 1/2$), the multi-vortex solutions found in Theorems 2.3 and 2.4 are all local energy minimizers (see proof of these theorems in Section 6.2). Therefore, there exists an infinite number of stable states for (6) and (7).

Previous results for magnetic vortices ($A \neq 0$) include Aftalion, Sandier and Serfaty [1] and Andre, Bauman and Phillips [3] who have shown existence of stable configurations/minima of the energy functional correspond to maxima of $W(x)$ in the $\lambda \to \infty$ regime, for applied external magnetic fields, and bounded domains. In their model, $W(x) = \frac{1}{\epsilon^2}(1 - a_\epsilon(x))$, and the vortex centers are pinned near minima of a_ϵ as $\epsilon \to 0$. These are all static (time-independent) results, which consider the singular limit $\lambda \to \infty$ and vortices as zeros of ψ. Our results describe the vortex structure of the solutions and hold for *all* $\lambda > 0$, non-degenerate critical points and hold on all of $\mathbb{R}^2$.

Numerical evidence that fundamental magnetic vortices of the same degree are attracted to maxima of $W(x)$ can be found in works by Chapman, et. al., [12] Du, et. al., [19] and Kim [29]. Ting and Gustafson [25] have shown dynamic stability/instability of single pinned fundamental vortices. Sigal and Strauss [42] have derived the effective dynamics of the magnetic vortex in a local potential. Ting [46] has studied the effective dynamics of multi-vortices in an external potential for the strength of external potential $\mu = \epsilon^p$ for $0 < p < 1$ and $p > 1$ (strong and weak external potentials).

Work has also been done on non-magnetic vortices ($A = 0$) with pinning (see [4, 6]). For example, in the model considered by [4], a weight function $p(x)$ is introduced into the energy $E_\lambda(u) = \frac{1}{2}\int_\Omega p|\nabla u|^2 + \lambda(1 - |u|^2)^2$ with Ω a bounded domain and $\lambda \to \infty$. They show that non-magnetic vortices are localized near minima of $p(x)$.

For review articles on Ginzburg-Landau theory, see [8, 10, 11, 27, 38, 39, 40]. We mention briefly the numerous work on dynamics of (unpinned) vortices in the $\lambda \to \infty$ limit. For *non-magnetic* vortices ($A = 0$), dynamics were rigorously derived in [14, 32, 7, 28]; for *magnetic vortices* by [44, 41]. Dynamics were derived non-rigorously by [36, 20] for widely separated vortices and for $\lambda >> 1/2$; and rigorously for $\lambda > 1/2$ by [24]. Our work uses many crucial results from [24]. Dynamics for $\lambda \approx 1/2$ were derived in the gradient case by [17], and in the Hamiltonian case by [45].

Much work has been done on the nonlinear Schrödinger equation in an external potential beginning with the seminal paper by Floer and Weinstein [21]. Also, existence of single soliton solutions to single "well" potential for the Schrödinger equation was considered under various assumptions on the external potential by Ambrosetti et. al., [2], Oh [33] and Del Pino and Felmer [15]. In addition, Oh [35], Del Pino and Felmer [16], and Gui [26] have shown multi-soliton solutions to multi-well potential and Oh [34] has considered stability of a single pinned soliton. Lin, Ni and Wei have found multi-spike solutions for a singularly perturbed Neumann problem and Wei and Winter [49] have shown multi-bump solutions to the Gierer-Meinhardt System. In the work by Kang and Wei [30], they showed that arbitrary number of bumps can be pinned at a local maximum point for the non-linear Schrödinger equation. In [50], Wei and Yan showed that infinitely many non-radial positive solutions can be pinned at infinity if the potential has algebraic order at infinity. Theorems 2.3 and 2.4 were motivated by the latter two works [30] and [50], respectively. In most of these papers, they use Lyapunov-Schmidt reduction to prove existence of these multi-bump solutions, a method which we'll employ here too.

3 Main Steps of Proof of Theorems 2.1 to 2.4

In this section, we outline the main steps of the proof of Theorems 2.1 to 2.4.

First, define our infinite dimensional manifold of *widely spaced* multi-vortex configurations

$$M_{mv,\epsilon} = \{v_{\underline{z}\chi} \mid (\underline{z}, \chi) \in \Sigma_\epsilon\}$$

parameterized by the set of all centers of vortices and gauge transformations

$$\Sigma_\epsilon = \{(\underline{z}, \chi) \mid R_\lambda(\underline{z}) < \epsilon \ \text{ and } \ \chi \in H^2_{\underline{z}}(\mathbb{R}^2; \mathbb{R})\}.$$

With abuse of notation, we will also denote Σ_ϵ as the open subset $\{\underline{z} \in \mathbb{R}^{2m} \mid R_\lambda(\underline{z}) < \epsilon\}$ of $\mathbb{R}^{2m}$ without the gauge part. The tangent space to point $v_{\underline{z}\chi} \in M_{mv,\epsilon}$ is

$$T_{v_{\underline{z}\chi}} M_{mv,\epsilon} = \text{span}\left\{ \langle \gamma, \partial_\chi \rangle v_{\underline{z}\chi}, \partial_{z_{jk}}^{A_k^{(j)}} v_{\underline{z}\chi} \ \middle| \ j = 1, ..., m; \ k = 1, 2; \ \gamma \in H^2(\mathbb{R}^2; \mathbb{R}) \right\}.$$

consisting of the "almost zero-modes" defined by (10) to (13) as follows: the gauge-tangent "almost zero-modes" are

$$G_\gamma^{\underline{z}\chi} := \partial_\chi v_{\underline{z}\chi}|_\gamma := \langle \gamma, \partial_\chi \rangle v_{\underline{z}\chi} = \begin{pmatrix} i\gamma\psi_{\underline{z}\chi} \\ \nabla\gamma \end{pmatrix} \tag{24}$$

for $\gamma : \mathbb{R}^2 \to \mathbb{R}$. Here, the notation $\partial_\chi v_{\underline{z}\chi}|_\gamma := \langle \gamma, \partial_\chi \rangle v_{\underline{z}\chi}$ denotes the Fréchet derivative of the map $\chi \to (e^{i\chi}\psi, A + \nabla\chi)$ evaluated at χ in the direction of γ. The (gauge-invariant) translational-tangent "almost zero-modes" are

$$\begin{aligned} T_{jk}^{\underline{z}\chi} &:= \partial_{z_{jk}}^{A_k^{(j)}} v_{\underline{z}\chi} = \left(\partial_{z_{jk}} + \langle A_k^{(n_j)}(\cdot - z_j), \partial_\chi \rangle\right) v_{\underline{z}\chi} \qquad (25) \\ &= \begin{pmatrix} e^{i\chi(x)} \prod_{l\neq j} \psi^{(n_l)}(x - z_l)[\partial_{x_{jk}} - i(A^{(n_j)}(x - z_j))_k]\psi^{(n_j)}(x - z_j) \\ B^{(n_j)}(x - z_j)e_k^\perp \end{pmatrix} \end{aligned}$$

where $A_k^{(j)} := [A^{(n_j)}(\cdot - z_j)]_k$, $B^{(n)} = \nabla \times A^{(n)}$ and $e_1^\perp = (0, 1)$ and $e_2^\perp = (-1, 0)$. Note that $T_{jk}^{\underline{z}\chi}$ are defined by covariant differentiation to ensure that $\partial_{z_{jk}}^{A_k^{(j)}} v_{\underline{z}\chi} \in H^1 \times L^2$, while $\partial_{z_{jk}} v_{\underline{z}\chi}$ is not. These tangent vectors are called almost zero modes since they "almost solve" $\mathcal{E}_0''(v_{\underline{z}\chi})\eta = 0$ (see Theorem 3.1(c) below).

Let $u = (\psi, A)$, and denote $F_W(u) = \mathcal{E}'_W(u)$, defined as a map from H^2 to L^2. Explicitly,

$$F_W(u) = \begin{pmatrix} -\Delta_A \psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi + W(x)\psi \\ -\nabla \times \nabla \times A - Im(\bar{\psi}\nabla_A \psi) \end{pmatrix} \tag{26}$$

Thus, equations (6) and (7) can be written as $F_W(u) = 0$.

Define orthogonal projections

$$\begin{aligned} \pi_{\underline{z}\chi} &:= L^2 - \text{projection onto span}\{T^{\underline{z}\chi}_{jk}, G^{\underline{z}\chi}_{\gamma} \mid j = 1, ..., m,\ k = 1, 2,\ \gamma \in H^2(\mathbb{R}^2; \mathbb{R})\} \\ \pi^{\perp}_{\underline{z}\chi} &:= 1 - \pi_{\underline{z}\chi}. \end{aligned} \tag{27}$$

The operator $\pi^{\perp}_{\underline{z}\chi}$ projects onto the L^2 orthogonal complement of $\mathrm{Ran}(\pi_{\underline{z}\chi})$, i.e., $\pi^{\perp}_{\underline{z}\chi} : L^2 \to [\mathrm{Ran}(\pi_{\underline{z}\chi})]^{\perp}$.

Denote $\pi^g_{\underline{z}\chi}$ and $\pi^t_{\underline{z}\chi}$ the L^2-orthogonal projections onto the gauge and translational "almost" tangent vectors, respectively (see (67) and (68) below for explicit expressions). By definition, we have

$$\pi_{\underline{z}\chi} = \pi^g_{\underline{z}\chi} + \pi^t_{\underline{z}\chi} \tag{28}$$

and from (24), (25) and (27), we see that

$$T_{v_{\underline{z}\chi}} M_{mv,\epsilon} = \mathrm{Ran}\pi_{\underline{z}\chi}.$$

The proof of existence of multi-vortex solutions to (6) and (7) relies on the following three steps:

1. ***Liapunov-Schmidt Reduction and Solution in the Orthogonal Direction.*** We use Liapunov-Schmidt reduction to break the problem up into its tangential and orthogonal components. We will show there exists a solution in the orthogonal direction using an implicit function type argument. More specifically, we will show that for all widely spaced multi-vortex configurations $(\underline{z}, \chi) \in \Sigma_\epsilon$ with $\epsilon > 0$ small enough and for W satisfying condition (A) for $\mu > 0$ small enough, there exists a unique $\eta_{\underline{z}\chi\epsilon} \in \mathrm{Ran}(\pi^{\perp}_{\underline{z}\chi})$ such that

$$\pi^{\perp}_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}) = 0. \tag{29}$$

2. ***Equivalent Reduced Problems and Solution in the Tangential Direction.*** We will solve the corresponding problem in the tangential direction

$$\pi_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}) = 0 \tag{30}$$

in two ways:

(a) By showing that there exists a one to one correspondence between critical points of the reduced energy functional $\Phi_W : \Sigma_\epsilon \subset \mathbb{R}^{2m} \to \mathbb{R}$ defined by

$$\Phi_W(\underline{z}) := \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon})$$

and critical points of the full energy functional $\mathcal{E}_W$. More precisely, we show that $\Phi_W(\underline{z})$ attains a critical point at multi-vortex configuration $\underline{z}_p \in \Sigma_\epsilon$ if and only if (30) has $\underline{z} = \underline{z}_p$ as a solution. This is precisely the statement of Theorem 2.1. Therefore, we are reduced to finding critical points of $\Phi_W(\underline{z})$ to solve for (30) for $\underline{z} \in \Sigma_\epsilon \subset \mathbb{R}^{2m}$.

(b) First, we note that solving (30) for $\underline{z}$ and χ is equivalent to solving the *two* equations

$$\pi^g_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}) = 0 \quad \text{and} \tag{31}$$

$$\pi^t_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}) = 0 \tag{32}$$

for χ and $\underline{z} \in \Sigma_\epsilon \subset \mathbb{R}^{2m}$, respectively. We will show that (31) is true for all gauge functions χ (see Proposition 5.3(i)), and hence we are reduced to solving (32) for $\underline{z}_p \in \Sigma_\epsilon \subset \mathbb{R}^{2m}$.

Both reductions in steps 2a and 2b exist due to the gauge invariance of the reduced energy functional $\Phi_W(\underline{z}, \chi) := \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon})$ (see Proposition 5.2).

3. ***Solution to the reduced problem.*** We use the reduced methods in steps 2a and 2b above to show that the given conditions on the potential W imply that **(a)** the reduced energy Φ_W attains a critical point at some multi-vortex configuration $\underline{z}_p \in \Sigma_\epsilon$ or **(b)** (32) has a solution at $\underline{z}_p \in \Sigma_\epsilon$ using an implicit function argument. The method in step 2a will be used to prove Theorem 2.3 and 2.4; the method in step 2b will be used to prove Theorem 2.2.

Steps 1 and 2 will imply Theorem 2.1 and steps 1 to 3 imply $F_W(v_{\underline{z}_p\chi}+\eta_{\underline{z}_p\chi\epsilon}) = 0$ and hence Theorems 2.2 to Theorem 2.4 follows. Step 1, 2 and 3 will be carried out in Sections 4, 5 and 6 respectively. Theorem 2.1 will be proven in Section 5, and Theorem 2.2 to Theorem 2.4 will be proven in Section 6. Technical lemmas and estimates will be proven in the Appendix.

For the rest of the section, we will state a theorem from [24] which is *crucial* in our analysis: Theorem 3.1. From Theorem 3.1, we'll be able to prove the main result in step 1 above. More precisely, using Theorem 3.1, we'll be able to obtain two corollaries from which we can construct a solution in the orthogonal direction (see Theorem 3.2).

Now we state Theorem 3.1 below which is proven for $\lambda > 1/2$ and $\lambda < 1/2$ in [24] and Appendix A in [46], respectively.

Theorem 3.1. *For $\epsilon > 0$ sufficiently small and for $(\underline{z}, \chi) \in \Sigma_\epsilon$,*

(a) *(Almost solution)*

$$\|\mathcal{E}_0'(v_{\underline{z}\chi})\|_{L^2} = O(\Gamma_\lambda^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)); \tag{33}$$

(b) *(Almost Orthogonality)*

$$\langle T_{jk}^{\underline{z}\chi}, T_{lm}^{\underline{z}\chi}\rangle = \gamma_{(n_j)}\delta_{jl}\delta_{km} + O(\Gamma_\lambda^{1,\frac{1}{2},1,\frac{3}{2}}(\epsilon)), \tag{34}$$

where

$$\gamma_{(n_j)} \quad = \quad \frac{1}{2}\|\nabla_{A^{(n_j)}}\psi^{(n_j)}\|_2^2 + \|curl A^{(n_j)}\|_2^2, \tag{35}$$

and

$$\langle T_{jk}^{\underline{z}\chi}, G_\gamma^{\underline{z}\chi}\rangle = O(\Gamma_\lambda^{1,\frac{1}{4},1,0}(\epsilon))\|\gamma\|_{L^2}. \tag{36}$$

(c) *(Almost zero modes) Write*

$$L_{\underline{z}\chi} := \mathcal{E}_0''(v_{\underline{z}\chi}).$$

Then

$$\|L_{\underline{z}\chi}T_{jk}^{\underline{z}\chi}\|_{L^2} = O(\Gamma_\lambda^{1,\frac{1}{2},1,0}(\epsilon)) \ \ and \tag{37}$$

$$\|L_{\underline{z}\chi}G_\gamma^{\underline{z}\chi}\|_{L^2} \leq c\Gamma_\lambda^{1,\frac{1}{4},1,0}(\epsilon)\|\gamma\|_{L^2}. \tag{38}$$

Therefore, from (27), (37) and (38), it follows that

$$L_{\underline{z}\chi}\pi_{\underline{z}\chi} = O(\Gamma_\lambda^{1,\frac{1}{2},1,0}(\epsilon)) \ \ in\ L^2. \tag{39}$$

(d) *(Coercivity of Hessian) There exists an $\tilde{\epsilon}_0 > 0$ such that for $0 < \epsilon < \tilde{\epsilon}_0$, $(\underline{z}, \chi) \in \Sigma_\epsilon$ and $\eta \in Ran(\pi_{\underline{z}\chi}^\perp)$,*

$$\langle \eta, L_{\underline{z}\chi}\eta\rangle \geq c_1\|\eta\|_{H^1}^2 \geq c_2\|\eta\|_2^2. \tag{40}$$

(e) *(Invertibility of Hessian) There exists an $\tilde{\epsilon}_0$ such that for all $0 < \epsilon < \tilde{\epsilon}_0$, $(\underline{z}, \chi) \in \Sigma_\epsilon$ and $\eta \in Ran(\pi_{\underline{z}\chi}^\perp)$, we have*

$$\|L_{\underline{z}\chi}\eta\|_{L^2} \geq \omega\|\eta\|_{H^2}$$

for some $\omega > 0$.

We draw attention to two essential results from [24]: Theorem 3.1 (a) and (e), i.e., any widely spaced multivortex configuration is almost a solution to the Ginzburg-Landau equations and the linearized operator $F_0'(v_{\underline{z}\chi})$ is invertible on the L^2 orthogonal complement of $T_{v_{\underline{z}\chi}}M_{mv,\epsilon}$. From the above two results, we have the following two corollaries.

Corollary 3.1.1. *Suppose $W(x)$ satisfies condition (A). Then for $\epsilon > 0$ sufficiently small and $(\underline{z}, \chi) \in \Sigma_\epsilon$,*

$$\mathcal{E}'_W(v_{\underline{z}\chi}) = F_W(v_{\underline{z}\chi}) = O(\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)) \quad in \;\; L^2. \tag{41}$$

Proof. We use $F_W(v_{\underline{z}\chi}) = F_0(v_{\underline{z}\chi}) + \begin{pmatrix} W\psi_{\underline{z}\chi} \\ 0 \end{pmatrix}$, $||F_0(v_{\underline{z}\chi})||_{L^2} \leq c\Gamma_\lambda^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)$ (by Theorem 3.1 (a)) and $\|W\psi_{\underline{z}\chi}\|_{L^2} = O(\mu)$ (by Condition (A) and since $|\psi_{\underline{z}\chi}| \leq 1$) to obtain $\|F_W(v_{\underline{z}\chi})\|_{L^2} \leq c_1\Gamma_\lambda^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon) + \mu$. Therefore, $\|F_W(v_{\underline{z}\chi})\|_{L^2} = O(\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))$. □

Corollary 3.1.2. *Suppose $W(x)$ satisfies condition (A). Define*

$$L_{\underline{z}\chi W} = \pi_{\underline{z}\chi}^{\perp} F'_W(v_{\underline{z}\chi}) \mid_{Ran(\pi_{\underline{z}\chi}^{\perp}) \cap H^2} \tag{42}$$

and let $\omega, \tilde{\epsilon}_0$ be constants in Theorem 3.1(e). For ϵ and μ satisfying $0 < \epsilon < \tilde{\epsilon}_0$ and $0 < \mu < \omega$, $(\underline{z}, \chi) \in \Sigma_\epsilon$, and $\eta \in Ran(\pi_{\underline{z}\chi}^{\perp}) \cap H^2$,

$$\|L_{\underline{z}\chi W}\eta\|_{L^2} \geq \beta\|\eta\|_{H^2}$$

where $\beta := \omega - \mu > 0$ is a positive constant.

Proof. Since $L_{\underline{z}\chi W} = L_{\underline{z}\chi} + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}$, then by Theorem 3.1(e) and $\epsilon < \tilde{\epsilon}_0$, $\|L_{\underline{z}\chi W}\eta\|_2 \geq \|L_{\underline{z}\chi}\eta\|_2 - \|W\eta\|_2 \geq (\omega - \|W\|_2)\|\eta\|_{H^2} \geq (\omega - \mu)\|\eta\|_{H^2}$, and our result follows by condition (A) on our potential, $\epsilon < C_1$ and with $\beta := \omega - \mu$. □

Next, we will state precisely what we will show in Step 1 above. Denote $B_X(z, r)$ as the open ball in a Banach space X of radius r centered at z.

Theorem 3.2. *Suppose $W(x)$ satisfies condition (A). Then there exist positive constants ϵ_0 and δ_0 such that for every ϵ and μ satisfying $0 < \Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon) < \epsilon_0$ and for every $(\underline{z}, \chi) \in \Sigma_\epsilon$, there exists a unique element $\eta_{\underline{z}\chi\epsilon}$ in $B_{H^2}(0, \delta_0) \cap Ran(\pi_{\underline{z}\chi}^{\perp})$ such that equation (29) is satisfied.*

In addition, we have the following:

a)$\|\eta_{\underline{z}\chi\epsilon}\|_{H^2} \leq D\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)$ with a positive constant $D = D(\kappa, \beta)$ where

$$\kappa := \sup_{\epsilon>0,\mu>0} \frac{1}{\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)} ||F_W(v_{\underline{z}\chi})||_{L^2} < \infty \tag{43}$$

and β is defined in Corollary 3.1.2.

b) $\eta_{\underline{z}\chi\epsilon}$ is C^1 in $\underline{z}$ and $||\partial_{z_{jk}}\eta_{\underline{z}\chi\epsilon}||_{L^2} \leq c\Gamma_{\lambda,\mu}^{1,\frac{1}{2},1,\frac{3}{4}}(\epsilon)$ for $j = 1, ..., m$ and $k = 1, 2$.

4 Solution in the Orthogonal direction

In this section, we prove Theorem 3.2 and complete step 1 in Section 3. More precisely, we prove that for $\mu, \epsilon > 0$ sufficiently small, W satisfying condition (A) and $(\underline{z}, \chi) \in \Sigma_\epsilon$, the equation $\pi^{\perp}_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta) = 0$ has a unique solution for $\eta \in [Ran(\pi^{\perp}_{\underline{z}\chi})]$.

To this end we use an implicit function type argument. We begin with the following definitions. Let $v = v_{\underline{z}\chi} + \eta$, where $v_{\underline{z}\chi} = \begin{pmatrix} \psi_{\underline{z}\chi} \\ A_{\underline{z}\chi} \end{pmatrix} \in M_{mv,\epsilon}$, and $\eta = \begin{pmatrix} \xi \\ B \end{pmatrix} \in H^2$ with $\eta \perp T_{v_{\underline{z}\chi}} M_{mv,\epsilon}$. Using Taylor expansion, we have

$$F_W(v_{\underline{z}\chi} + \eta) = F_W(v_{\underline{z}\chi}) + F'_W(v_{\underline{z}\chi})\eta + N_W(v_{\underline{z}\chi}, \eta), \tag{44}$$

where

$$F'_W(v_{\underline{z}\chi})\eta = \tag{45}$$
$$\begin{pmatrix} [-\Delta_{A_{\underline{z}\chi}} + \frac{\lambda}{2}(2|\psi_{\underline{z}\chi}|^2 - 1)]\xi \ + \ \frac{\lambda}{2}\psi^2_{\underline{z}\chi}\bar{\xi} \ + \ i[2\nabla_{A_{\underline{z}\chi}}\psi_{\underline{z}\chi} + \psi_{\underline{z}\chi}\nabla] \cdot B \\ Im([\overline{\nabla_{A_{\underline{z}\chi}}\psi_{\underline{z}\chi}} - \overline{\psi_{\underline{z}\chi}}\nabla_{A_{\underline{z}\chi}}]\xi) \ + \ (-\Delta + \nabla\nabla + |\psi_{\underline{z}\chi}|^2) \cdot B \end{pmatrix}$$

and

$$N_W(v_{\underline{z}\chi}, \eta) = \begin{pmatrix} \lambda(2\psi\bar{\xi} + \bar{\psi}\xi + |\xi|^2)\xi \ + \ ||B||^2(\psi + \xi) \ + \ [i(\nabla \cdot B + B \cdot \nabla) + 2A \cdot B\eta\xi \\ -Im(\bar{\xi}\nabla_A\xi) \ + \ B(2Re(\bar{\psi}\xi) + |\xi|^2) \end{pmatrix}. \tag{46}$$

We need the following lemma:

Lemma 4.1. *There exist positive constants C_2, C_3, C_4 independent of $\underline{z}, \chi, \epsilon$ such that for all $\eta \in H^2$ with $\|\eta\|_{H^2} \leq C_2$,*

$$\|N_W(v_{\underline{z}\chi}, \eta)\|_{L^2} \leq C_3 \|\eta\|^2_{H^2}, \tag{47}$$

and

$$||\partial_\eta N_W(v_{\underline{z}\chi}, \eta)||_{H^2 \to L^2} \leq C_4 ||\eta||_{H^2}. \tag{48}$$

Proof. Lemma 4.1 follows directly from Sobolev embedding theorems and the mean value theorem (see [43]). □

Let ϵ and μ satisfy $0 < \epsilon < \tilde{\epsilon}_0$ and $0 < \mu < \omega$ so that $L_{\underline{z}\chi W}$ is invertible by Corollary 3.1.2. Using the Taylor expansion (44) and abbreviating $\pi^{\perp}_{\underline{z}\chi} F_W(v_{\underline{z}\chi})$ to $F^{\perp}_{\underline{z}\chi W}$ and $\pi^{\perp}_{\underline{z}\chi} N_W(v_{\underline{z}\chi}, \eta)$ to $N^{\perp}_{\underline{z}\chi W}(\eta)$, we rewrite equation (29) as a fixed point equation $\eta = S_{\underline{z}\chi W}(\eta)$ for the map $S_{\underline{z}\chi W}$ defined on H^2 by

$$S_{\underline{z}\chi W}(\eta) = -L^{-1}_{\underline{z}\chi W}\left[N^{\perp}_{\underline{z}\chi W}(\eta) + F^{\perp}_{\underline{z}\chi W}\right]. \tag{49}$$

Let β, C_2, C_3 and C_4 be the constants in Corollary 3.1.2 and Lemma 4.1. Set $\delta_0 = \min(C_2, \frac{\beta}{2C_3}, \frac{\beta}{2C_4})$ and $\epsilon_0 = \min(\tilde{\epsilon}_0, \omega, \frac{\delta_0}{2\kappa}\beta)$, where κ is defined in (43).

We will show that for ϵ satisfying $0 < \Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon) < \epsilon_0$, $S_{\underline{z}\chi W}$ maps the ball $B^{\perp}_{\delta_0} = B_{H^2}(0,\delta_0) \cap [\mathrm{Ran}(\pi^{\perp}_{\underline{z}\chi})]$ continuously into itself. Let $\eta \in B^{\perp}_{\delta_0}$. Then for ϵ and μ satisfying $\Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon) < \epsilon_0$ and $||\eta|| \leq \delta_0 \leq C_2$, we have by Corollary 3.1.2 and Lemma 4.1

$$
\begin{aligned}
\|S_{\underline{z}\chi W}(\eta)\|_{H^2} &\leq \frac{1}{\beta}\left\|N^{\perp}_{\underline{z}\chi W}(\eta) + F^{\perp}_{\underline{z}\chi W}\right\|_{L^2} \\
&\leq \frac{1}{\beta}\left(C_3\|\eta\|^2_{H^2} + \left\|F^{\perp}_{\underline{z}\chi W}\right\|_{L^2}\right) \\
&\leq \frac{1}{\beta}\left(C_3\delta_0^2 + \kappa\Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon)\right) \leq \delta_0,
\end{aligned}
$$

where in the second last inequality, we used (43), and in the last inequality, we used the definition of δ_0 and ϵ_0. Therefore $S_{\underline{z}\chi W}(\eta)$ is in $B^{\perp}_{\delta_0}$ too.

In addition, for η and η' in $B^{\perp}_{\delta_0}$, we have from (48) and the mean value theorem that

$$
\|N_W(v_{\underline{z}\chi},\eta) - N_W(v_{\underline{z}\chi},\eta')\|_{L^2} \leq C_4\delta_0\|\eta-\eta'\|_{H^2}. \tag{50}
$$

Hence, (50) and our choice of δ_0 imply

$$
\begin{aligned}
\|S_{\underline{z}\chi W}(\eta) - S_{\underline{z}\chi W}(\eta')\|_{H^2} &= \left\|L^{-1}_{\underline{z}\chi W}(N^{\perp}_{\underline{z}\chi W}(\eta) - N^{\perp}_{\underline{z}\chi W}(\eta'))\right\|_{L^2} \\
&\leq \frac{C_4\delta_0}{\beta}\|\eta-\eta'\|_{H^2} \leq \frac{1}{2}\|\eta-\eta'\|_{H^2}.
\end{aligned}
$$

Therefore, $S_{\underline{z}\chi W}$ is a contraction map and so $S_{\underline{z}\chi W}$ has a unique fixed point $\eta_{\underline{z}\chi\epsilon}$ in $B^{\perp}_{\delta_0}$. By the definition of the map $S_{\underline{z}\chi W}$, this fixed point solves (29) which proves the first part of Theorem 3.2.

For part a) of the second part of Theorem 3.2, we note that

$$
\|S_{\underline{z}\chi W}(0)\|_{H^2} = \left\|L^{-1}_{\underline{z}\chi W}F^{\perp}_{\underline{z}\chi W}\right\|_{H^2} \leq \beta^{-1}\|F_W(v_{\underline{z}\chi})\|_{L^2}.
$$

But for the fixed point $\eta_{\underline{z}\chi\epsilon}$, we have

$$
\eta_{\underline{z}\chi\epsilon} = S_{\underline{z}\chi W}(\eta_{\underline{z}\chi\epsilon}) = S_{\underline{z}\chi W}(0) + S_{\underline{z}\chi W}(\eta_{\underline{z}\chi\epsilon}) - S_{\underline{z}\chi W}(0).
$$

Consequently,

$$
\begin{aligned}
\|\eta_{\underline{z}\chi\epsilon}\|_{H^2} &\leq \|S_{\underline{z}\chi W}(0)\|_{H^2} + \|S_{\underline{z}\chi W}(\eta_{\underline{z}\chi\epsilon}) - S_{\underline{z}\chi W}(0)\|_{H^2} \\
&\leq \beta^{-1}\|F_W(v_{\underline{z}\chi})\|_{L^2} + \frac{1}{2}\|\eta_{\underline{z}\chi\epsilon}\|_{H^2}.
\end{aligned}
$$

Since $||F_W(v_{\underline{z}\chi})||_{L^2} \leq \kappa\Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon)$ by (43), the last inequality implies part a) with $D = 2\beta^{-1}\kappa$:

$$||\eta_{\underline{z}\chi\epsilon}||_{H^2} \leq 2\beta^{-1}\kappa\Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon). \tag{51}$$

To prove part b), we proceed in a standard way. Define $F^{\perp} : \mathbb{R}^{2m} \times (Ran(\pi_{\underline{z}\chi})^{\perp} \cap H^2) \to (Ran(\pi_{\underline{z}\chi})^{\perp} \cap L^2)$ by

$$F^{\perp}(\underline{z},\eta) = \pi^{\perp}_{\underline{z}\chi} F_W(v_{\underline{z}\chi} + \eta), \tag{52}$$

where we have suppressed the dependence of $F^{\perp}$ on W and χ for brevity. By part a), we have shown that for ϵ and μ satisfying $0 < \Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon) < \epsilon_0$ and $\chi \in H^2$, there exists $\eta = \eta(\underline{z}) \in Ran(\pi^{\perp}_{\underline{z}\chi}) \cap H^2$ such that

$$F^{\perp}(\underline{z},\eta(\underline{z})) = 0 \qquad \forall\, \underline{z} \in \mathbb{R}^{2m} \text{ with } \quad R_{\lambda}(\underline{z}) < \epsilon. \tag{53}$$

Fix j and k for some $j = 1, ..., m$, $k = 1, 2$. It is standard to show that $\eta(\underline{z})$ in (53) is C^1 in z_{jk} (see [18, 43]) and that $\partial_{z_{jk}}\eta(\underline{z})$ is given by the expression

$$\partial_{z_{jk}}\eta(\underline{z}) = -\partial_{\eta}F^{\perp}(\underline{z},\eta(\underline{z}))^{-1}\partial_{z_{jk}}F^{\perp}(\underline{z},\eta(\underline{z})). \tag{54}$$

Now we prove that $||\partial_{z_{jk}}\eta(\underline{z})||_{H^2} \leq c\Gamma^{1,\frac{1}{2},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon)$. Fix $\underline{z} \in \mathbb{R}^{2m}$ with $R_{\lambda}(\underline{z}) < \epsilon$ and write for convenience $\eta = \eta(\underline{z})$. We estimate the r.h.s of (54). By (53) and the implicit function theorem, we know that $\partial_{\eta}F^{\perp}(\underline{z},\eta)$ is invertible and hence $||\partial_{\eta}F^{\perp}(\underline{z},\eta)^{-1}|| \leq C$. Hence, it suffices to show that $||\partial_{z_{jk}}F^{\perp}(\underline{z},\eta)|| = O(\Gamma^{1,\frac{1}{2},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon))$ as $\epsilon \to 0$. Now by (52), we have

$$\partial_{z_{jk}}F^{\perp}(\underline{z},\eta) = (\partial_{z_{jk}}\pi^{\perp}_{\underline{z}\chi})F_W(v_{\underline{z}\chi} + \eta) + \pi^{\perp}_{\underline{z}\chi}\partial_{z_{jk}}F_W(v_{\underline{z}\chi} + \eta). \tag{55}$$

By explicit expressions for $\pi_{\underline{z}\chi} = \pi^{t}_{\underline{z}\chi} + \pi^{g}_{\underline{z}\chi}$ in (67) and (68) below, $||\pi^{\perp}_{\underline{z}\chi}|| \leq C$. Similarly one shows that $||\partial_{z_{jk}}\pi^{\perp}_{\underline{z}\chi}|| \leq C$. By (44), we have

$$||F_W(v_{\underline{z}\chi} + \eta)||_{L^2} \leq ||F_W(v_{\underline{z}\chi})||_{L^2} + C||\eta||_{H^2} + C||\eta||^2_{H^2} = O(\Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon)), \tag{56}$$

as $||\eta||_{H^2} = O(\Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon))$ by (51). In the above, we have used equation (41), for the first term, F_W is C^1 for the second term, and Lemma 4.1 for the last term. Now, recall that

$$F'_W(v_{\underline{z}\chi})\eta = F'_0(v_{\underline{z}\chi})\eta + \begin{pmatrix} W(x) & 0 \\ 0 & 0 \end{pmatrix}\eta. \tag{57}$$

By (57) and the fact that $F'_0(v_{\underline{z}\chi})\partial_{z_{jk}}v_{\underline{z}\chi} = L_{\underline{z}\chi}T^{\underline{z}\chi}_{jk} = O(\Gamma^{1,\frac{1}{2},1,0}_{\lambda}(\epsilon))$ (by Theorem 3.1 (c)), we have

$$\begin{aligned} ||\partial_{z_{jk}}F_W(v_{\underline{z}\chi} + \eta)||_{L^2} &= ||F'_W(v_{\underline{z}\chi} + \eta)\partial_{z_{jk}}v_{\underline{z}\chi}||_{L^2} \\ &\leq ||(F'_W(v_{\underline{z}\chi} + \eta) - F'_W(v_{\underline{z}\chi}))\partial_{z_{jk}}v_{\underline{z}\chi}||_{L^2} + ||F'_W(v_{\underline{z}\chi})\partial_{z_{jk}}v_{\underline{z}\chi}||_{L^2} \\ &\leq C \cdot max(||\partial_{z_{jk}}v_{\underline{z}\chi}||_{\infty}, ||\partial^2_{z_{jk}}v_{\underline{z}\chi}||_{\infty})||\eta||_{H^2} + \\ &\quad\; ||F'_0(v_{\underline{z}\chi})\partial_{z_{jk}}v_{\underline{z}\chi}||_{L^2} + ||W||_{L^2}||\partial_{z_{jk}}v_{\underline{z}\chi}||_{\infty} = O(\Gamma^{1,\frac{1}{2},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon)) \end{aligned} \tag{58}$$

since F_W is C^2, $\|\eta\|_{H^2} = O(\Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon))$, $W = O(\mu)$ in L^2 and $||\partial_{z_{jk}} v_{\underline{z}\chi}||_\infty$, $||\partial^2_{z_{jk}} v_{\underline{z}\chi}||_\infty < \infty$ (by the explicit form of $T^{\underline{z}\chi}_{jk} = \partial_{z_{jk}} v_{\underline{z}\chi}$ in (25)). Equations (55), (56) and (58) show that $||\partial_{z_{jk}} \eta_{\underline{z}\chi\epsilon}||_{L^2} \leq c\Gamma^{1,\frac{1}{2},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon)$ and we are done with the proof of the estimates in Theorem 3.2(b). ■

5 Equivalent reduced problems and solution in the tangential direction

In this section, we show that the problem in the tangential direction (30) can be solved by the reduction methods described in steps 2a and 2b in Section 3.

For the method in step 2a, we prove Theorem 2.1: a critical point of reduced energy is equivalent to a critical point of full energy. The proof of Theorem 2.1 is a straight forward modification of the proof of Theorem 2.3 in [43], Section 5. Therefore, we will just outline the proof here in section 5.1. Theorem 2.1 will be used to prove Theorem 2.3 and 2.4 in Section 6.2.

However, we will prove one fact in the proof of Theorem 2.1 (or equivalently, in the proof of Theorem 2.3 in [43] (Section 5, equation (5.2))) which is not as obvious: that the reduced energy $\Phi_W(\underline{z}) = \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon})$ is independent of the gauge function χ. This will be stated and proved in Proposition 5.2 in Section 5.2.

For the method in step 2b, we will show that the statement (31) is true for any $\chi \in H^2(\mathbb{R}^2, \mathbb{R})$ is equivalent to the independence of the reduced energy Φ_W on gauge χ. This will be stated and proved in Proposition 5.3 in Section 5.3. Proposition 5.3 will be used to prove Theorem 2.2 in Section 6.1.

5.1 Proof of Theorem 2.1: General Argument

In this section, we outline the proof of Theorem 2.1 and show that a critical point of reduced energy is equivalent to critical point of full energy. More precisely, we show that if (29) is true, then the reduced energy $\Phi_W(\underline{z}) := \mathcal{E}_W(v_{\underline{z}\chi\epsilon})$, where $v_{\underline{z}\chi\epsilon} := v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}$ has a critical point at $\underline{z}_p$ if and only if $\mathcal{E}'_W(v_{\underline{z}_p\chi\epsilon}) = 0$.

Equation (29) implies for any $(\underline{z}, \chi) \in \Sigma_\epsilon$,

$$\mathcal{E}'_W(v_{\underline{z}\chi\epsilon}) \in T_{v_{\underline{z}\chi}} M_{mv,\epsilon}. \tag{59}$$

By Proposition 5.2 below, the energy functional is independent of gauge and hence

$$0 = \partial_\chi \Phi_W(\underline{z}) =< \partial_\chi v_{\underline{z}\chi\epsilon}, \mathcal{E}'_W(v_{\underline{z}\chi\epsilon}) > . \tag{60}$$

We claim that, given (59) and (60),

$$\partial_{\underline{z}} \Phi_W(\underline{z})|_{\underline{z}=\underline{z}_p} = 0 \quad \Longleftrightarrow \quad \mathcal{E}'_W(v_{\underline{z}\chi\epsilon}|_{\underline{z}=\underline{z}_p}) = 0, \tag{61}$$

where we have used the direct sum notation

$$\partial_{\underline{z}} = \bigoplus_{j=1}^{m} \bigoplus_{k=1}^{2} \partial_{z_{jk}} = (\partial_{z_{11}}, \partial_{z_{12}}, ..., \partial_{z_{j1}}, \partial_{z_{j2}}, ..., \partial_{z_{m1}}, \partial_{z_{m2}}). \tag{62}$$

Note that statement (61) is equivalent to the statement in Theorem 2.1. The ($\Leftarrow$) part of statement (61) is trivial: if $\mathcal{E}'_W(v_{\underline{z}\chi\epsilon}|_{\underline{z}=\underline{z}_p}) = 0$, then

$$\partial_{\underline{z}}\Phi_W(\underline{z})|_{\underline{z}=\underline{z}_p} =< \partial_{\underline{z}} v_{\underline{z}\chi\epsilon}, \mathcal{E}'_W(v_{\underline{z}\chi\epsilon}) > |_{\underline{z}=\underline{z}_p} =< \partial_{\underline{z}} v_{\underline{z}\chi\epsilon}|_{\underline{z}=\underline{z}_p}, \mathcal{E}'_W(v_{\underline{z}\chi\epsilon})|_{\underline{z}=\underline{z}_p} >= 0.$$

Hence, it remains to prove the ($\Rightarrow$) part of statement (61). First, we observe that the relation

$$< \partial_{\underline{z}} v_{\underline{z}\chi\epsilon}, \mathcal{E}'_W(v_{\underline{z}\chi\epsilon}) > |_{\underline{z}=\underline{z}_p} = \partial_{\underline{z}}\Phi_W(\underline{z})|_{\underline{z}=\underline{z}_p} = 0 \tag{63}$$

together with (60) implies

$$\mathcal{E}'_W(v_{\underline{z}_p\chi\epsilon}) \perp T_{v_{\underline{z}_p\chi\epsilon}} M^W_{mv,\epsilon}, \tag{64}$$

where $M^W_{mv,\epsilon} := \{v_{\underline{z}\chi\epsilon} = v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon} | (\underline{z}, \chi) \in \Sigma_\epsilon\}$. Thus it remains to show that (59) and (64) imply $\mathcal{E}'_W(v_{\underline{z}\chi\epsilon})|_{\underline{z}=\underline{z}_p} = 0$. Denote $f_W = \mathcal{E}'_W(v_{\underline{z}_p\chi\epsilon})$ and

$$\begin{aligned} \pi &= L^2 - orthogonal\ projection\ onto\ T_{v_{\underline{z}_p\chi}} M_{mv,\epsilon}, \\ \pi_W &= L^2 - orthogonal\ projection\ onto\ T_{v_{\underline{z}_p\chi\epsilon}} M^W_{mv,\epsilon}. \end{aligned}$$

Then equations (59) and (64) can be written as

$$\pi f_W = f_W \ \ and \ \ \pi_W f_W = 0. \tag{65}$$

We want to show $f_W = 0$.

But by (65),

$$f_W = \pi f_W = (\pi - \pi_W) f_W. \tag{66}$$

Now by Proposition 5.1 below, we have

$$||f_W|| \leq ||\pi - \pi_W|| ||f_W|| \leq C\Gamma^{1,\frac{1}{2},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon)||f_W||.$$

This implies that $f_W = 0$ which completes the proof of the sufficient part of (61), modulo the proof of Proposition 5.1.

Proposition 5.1. *The operators* π *and* π_W *are bounded and* $||\pi - \pi_W|| = O(\Gamma^{1,\frac{1}{2},1,\frac{3}{4}}_{\lambda,\mu}(\epsilon))$.

The proposition follows from the explicit expressions for the projections $\pi = \pi^t_{\underline{z}\chi} + \pi^g_{\underline{z}\chi}$ and $\pi_W = (\pi_W)^t_{\underline{z}\chi} + (\pi_W)^g_{\underline{z}\chi}$ in Dirac notation (see [43]):

$$\pi^t_{\underline{z}\chi}\begin{pmatrix}\xi\\ B\end{pmatrix} = \sum_{jk}\sum_{rs}|T^{\underline{z}\chi}_{jk} > \left[(\beta^{\underline{z}\chi})^{-1}\right]_{(jk)(rs)}\left\langle T^{\underline{z}\chi}_{rs}|\begin{pmatrix}\xi\\ B\end{pmatrix}\right\rangle \tag{67}$$

$$(\pi_W)^t_{\underline{z}\chi}\begin{pmatrix}\xi\\ B\end{pmatrix} = \sum_{jk}\sum_{rs}|T^{\underline{z}\chi\epsilon}_{jk} > \left[(\beta^{\underline{z}\chi\epsilon})^{-1}\right]_{(jk)(rs)}\left\langle T^{\underline{z}\chi\epsilon}_{rs}|\begin{pmatrix}\xi\\ B\end{pmatrix}\right\rangle$$

$$\pi^g_{\underline{z}\chi}\begin{pmatrix}\xi\\ B\end{pmatrix} = \begin{pmatrix} i\psi^{\underline{z}\chi}J[Im(\bar{\psi}^{\underline{z}\chi}\xi) - \nabla\cdot B]\\ \nabla J[Im(\bar{\psi}^{\underline{z}\chi}\xi) - \nabla\cdot B]\end{pmatrix} \tag{68}$$

$$(\pi_W)^g_{\underline{z}\chi}\begin{pmatrix}\xi\\ B\end{pmatrix} = \begin{pmatrix} i\psi^{\underline{z}\chi\epsilon}J^\epsilon[Im(\bar{\psi}^{\underline{z}\chi\epsilon}\xi) - \nabla\cdot B]\\ \nabla J^\epsilon[Im(\bar{\psi}^{\underline{z}\chi\epsilon}\xi) - \nabla\cdot B]\end{pmatrix}.$$

Here, $[\beta^{\underline{z}\chi}]_{(jk)(rs)} = \langle T^{\underline{z}\chi}_{jk}|T^{\underline{z}\chi}_{rs}\rangle$ is invertible by (34) and $J = (-\Delta + |\psi^{\underline{z}\chi}|^2)^{-1}$. In addition, $T^{\underline{z}\chi\epsilon}_{jk} := \partial_{z_{jk}} v_{\underline{z}\chi\epsilon}$ and $[\beta^{\underline{z}\chi\epsilon}]_{(jk)(rs)} = \langle T^{\underline{z}\chi\epsilon}_{jk}|T^{\underline{z}\chi\epsilon}_{rs}\rangle$ is invertible by (34) and Theorem 3.2(b) and $J^\epsilon = (-\Delta + |\psi^{\underline{z}\chi\epsilon}|^2)^{-1}$.

One then proves the above Proposition 5.1 with analogs of Lemmas 5.4 and 5.5 in [43] for multi-vortices. ■

5.2 Independence of the reduced energy functional on gauge

In this section, we prove Proposition 5.2 below which states that the reduced energy functional Φ_W is independent of gauge χ. Proposition 5.2 was required in the general argument in proving Theorem 2.1 in the previous section 5.1 (see (60)). In addition, the validity of Proposition 5.2 below will establish the validity of Proposition 5.3(i) in Section 5.3 below.

We are ready to state our main result of this section.

Proposition 5.2. *For* $(\underline{z}, \chi) \in \Sigma_\epsilon$*, define the* reduced energy *by*

$$\Phi_W(\underline{z}, \chi) := \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}).$$

Then Φ_W *is independent of gauge* χ*.*

Before we prove Proposition 5.2, we need some definitions and a lemma. Define the symmetry action of g_χ on the vortex $v_0 = (\psi, A)$ as

$$g_\chi v_0 = \begin{pmatrix} e^{i\chi}\psi\\ A + \nabla\chi\end{pmatrix} =: v_\chi := \begin{pmatrix}\psi^\chi\\ A^\chi\end{pmatrix}, \tag{69}$$

where we have dropped (or equivalently fixed) the dependance on $\underline{z}$ everywhere for clarity.

Lemma 5.1. ***(a)*** *The explicit action of a gauge χ on fixed point vectors $\eta = \begin{pmatrix} \xi \\ B \end{pmatrix}$ in a tube of radius δ_0 (defined in Theorem 3.2) about the manifold $M_{mv,\epsilon}$ is*

$$g_\chi \eta = \begin{pmatrix} e^{i\chi}\xi \\ B \end{pmatrix}. \tag{70}$$

(b) *Set $\eta_{\underline{z}\chi\epsilon} = g_\chi \eta_{\underline{z}0\epsilon}$ for some $\eta_{\underline{z}0\epsilon} \in Ran(\pi^\perp_{\underline{z}0}) \cap B_{H^2}(\delta_0)$. Then*

$$\partial_\chi \eta_{\underline{z}\chi\epsilon}|_\gamma = \langle \gamma, \partial_\chi \rangle \eta_{\underline{z}\chi\epsilon} = \begin{pmatrix} i\gamma\xi_{\underline{z}\chi\epsilon} \\ 0 \end{pmatrix}. \tag{71}$$

Therefore, the action of gauge χ on vortices v and fixed point vectors η in a tube of radius δ_0 around the multi-vortex manifold $M_{mv,\epsilon}$ differ.

Proof. Part (b) immediately follows from part (a) and notation used in (24).

From Theorem 3.2, we know that for all $(\underline{z}, \chi) \in \Sigma_\epsilon$ and $\eta_{\underline{z}\chi\epsilon} \in Ran(\pi^\perp_{\underline{z}\chi}) \cap B_{H^2}(\delta_0)$, $\eta_{\underline{z}\chi\epsilon}$ is a fixed point of map $S_{\underline{z}\chi}$ defined in (49). Dropping the $\underline{z}$ and ϵ dependance everywhere (equivalently, fixing $\underline{z}$ and ϵ), we have

$$\eta_\chi = S_\chi(\eta_\chi) \quad \forall \eta_\chi \in Ran(\pi^\perp_{\underline{z}\chi}) \cap B_{H^2}(\delta_0). \tag{72}$$

For $\eta_0 = \begin{pmatrix} \xi \\ B \end{pmatrix} \in Ran(\pi^\perp_{\underline{z}0}) \cap B_{H^2}(\delta_0)$ a fixed point of the map S_0, set

$$\eta_\chi := g_\chi \eta_0 := \begin{pmatrix} e^{i\chi}\xi \\ B \end{pmatrix} \quad for\ \ any\ \chi \in H^2. \tag{73}$$

Then, given (69) and (73), one can show that

$$\begin{aligned} (i)\pi^\perp_\chi(g_\chi \eta_0) &= g_\chi \pi^\perp_0(\eta_0); \\ (ii)F_W(g_\chi v_0) &= g_\chi F_W(v_0); \\ (iii)N_W(v_\chi, \eta_\chi) &= g_\chi N_W(v_0, \eta_0); \\ (iv)L_\chi &= g_\chi L_0 g_\chi^{-1} \text{ for } L_\chi := F'_W(v_{\underline{z}\chi}) \implies (L^\perp_\chi)^{-1} = \pi^\perp_\chi L_\chi^{-1} \pi^\perp_\chi. \end{aligned}$$

Indeed, (i) comes from (28), the explicit expressions for π^t_χ and π^g_χ in (67) and (68), $|T^\chi_{jk}> = \begin{pmatrix} e^{i\chi}\prod_{l\neq j}\psi^{(l)}[\nabla_A\psi]^{(j)}_k \\ (\nabla \times A)e^\perp_j \end{pmatrix} = g_\chi |T^0_{jk}>$, $[\beta^\chi]_{(jk)(lm)} = \langle T^\chi_{jk}|T^\chi_{lm}\rangle = \langle T^0_{jk}|T^0_{lm}\rangle$ and $J = (-\Delta + |\psi^\chi|^2)^{-1}$. Now, (ii) to (iv) comes from the explicit expressions for F_W, L_χ and N_W from (26), (45) and (46), respectively. One can show using facts (i) to (iv) and (49) that for η_0 and η_χ given in (73),

$$S_\chi(\eta_\chi) = g_\chi S_0(\eta_0).$$

Therefore,

$$S_\chi(\eta_\chi) = g_\chi S_0(\eta_0) = g_\chi \eta_0 = \eta_\chi,$$

where in the 2nd and 3rd inequalities, we used that η_0 is a fixed point and (73), respectively. Since (72) is true for any χ, then the action of χ on the fixed point η_0 is given by (73). $\square$

From the above lemma, we are ready to prove the main proposition 5.2 above.

Proof of Proposition 5.2. By Lemma 5.1a, we have for $v_{\underline{z}\chi} = \begin{pmatrix} \psi_{\underline{z}\chi} \\ A_{\underline{z}\chi} \end{pmatrix} := \begin{pmatrix} e^{i\chi}\psi_{\underline{z}0} \\ A_{\underline{z}0} + \nabla\chi \end{pmatrix}$ and $\eta_{\underline{z}\chi\epsilon} = \begin{pmatrix} \xi_{\underline{z}\chi\epsilon} \\ B_{\underline{z}\chi\epsilon} \end{pmatrix} = \begin{pmatrix} e^{i\chi}\xi_{\underline{z}0\epsilon} \\ B_{\underline{z}0\epsilon} \end{pmatrix}$, we have

$$
\begin{aligned}
\Phi_W(\underline{z},\chi) &= \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}) \\
&= \mathcal{E}_W(g_{0\chi}v_{\underline{z}0} + g_{0\chi}\eta_{\underline{z}0\epsilon}) \\
&= \mathcal{E}_W\left(\begin{pmatrix} e^{i\chi}\psi_{\underline{z}0} \\ A_{\underline{z}0} + \nabla\chi \end{pmatrix} + \begin{pmatrix} e^{i\chi}\xi_{\underline{z}0\epsilon} \\ B_{\underline{z}0\epsilon} \end{pmatrix}\right) \\
&= \mathcal{E}_W\left(\begin{pmatrix} e^{i\chi}(\psi_{\underline{z}0} + \xi_{\underline{z}0\epsilon}) \\ (A_{\underline{z}0} + B_{\underline{z}0\epsilon}) + \nabla\chi \end{pmatrix}\right) \\
&= \mathcal{E}_W\left(\begin{pmatrix} \psi_{\underline{z}0} + \xi_{\underline{z}0\epsilon} \\ A_{\underline{z}0} + B_{\underline{z}0\epsilon} \end{pmatrix}\right) \\
&= \mathcal{E}_W(v_{\underline{z}0} + \eta_{\underline{z}0\epsilon}) \\
&= \Phi_W(\underline{z},0)
\end{aligned}
$$

where in the fifth equality, we used (70) and gauge invariance of the G-L+W energy functional (8), respectively. □

5.3 Step 2b: Independence of tangential problem on gauge

In this section, we prove Proposition 5.3 below which states that (31) is true for every χ is equivalent to the independence of the reduced energy Φ_W on gauge χ. By Proposition 5.2, the latter is true and hence the former is true too. Proposition 5.3(i) will be used to prove Theorem 2.2 in the next Section 6.1.

We are now ready to state our main proposition of this section.

Proposition 5.3. *For $\epsilon > 0$ sufficiently small and $(\underline{z},\chi) \in \Sigma_\epsilon$, the following statements are equivalent:*

(i) (31) is true for all $\chi \in H^2(\mathbb{R}^2;\mathbb{R})$.

(ii) The reduced energy $\Phi_W(\underline{z},\chi) = \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon})$ is independent of gauge χ.

Proof. Indeed, by chain rule, we have for any γ_0 (using the notation in (24)),

$$\begin{aligned}
\partial_\chi \Phi(\underline{z},\chi)|_{\gamma_0} &= \partial_\chi \mathcal{E}_W(v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon})|_{\gamma_0}\\
&= \langle \mathcal{E}'_W(v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon}), \partial_\chi(v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon})\rangle|_{\gamma_0}\\
&= \langle F_W(v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon}), \langle\gamma_0,\partial_\chi\rangle(v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon})\rangle\\
&= \left\langle F_W(v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon}), \begin{pmatrix} i\gamma_0\psi_{\underline{z}\chi}\\ \nabla\gamma_0\end{pmatrix} + \begin{pmatrix} i\gamma_0\xi_{\underline{z}\chi\epsilon}\\ 0\end{pmatrix}\right\rangle\\
&= \left\langle F_W(v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon}), \begin{pmatrix} i\gamma_0\psi_{\underline{z}\chi\epsilon}\\ \nabla\gamma_0\end{pmatrix}\right\rangle, \qquad (74)
\end{aligned}$$

where we have let

$$v_{\underline{z}\chi\epsilon} := v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon} =: (\psi_{\underline{z}\chi\epsilon}, A_{\underline{z}\chi\epsilon}) := \begin{pmatrix} \psi_{\underline{z}\chi}+\xi_{\underline{z}\chi\epsilon}\\ A_{\underline{z}\chi}+B_{\underline{z}\chi\epsilon}\end{pmatrix}. \qquad (75)$$

Observe that (31) is true for any χ if and only if the following statement is true:

$$F_W(v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon}) = \begin{pmatrix} i\gamma_0\psi_{\underline{z}\chi}\\ \nabla\gamma_0\end{pmatrix} \quad\Longrightarrow\quad \gamma_0\equiv 0. \qquad (76)$$

Now, taking the inner product of the first equation in the first part of (76) with $\begin{pmatrix} i\gamma_0\psi_{\underline{z}\chi\epsilon}\\ \nabla\gamma_0\end{pmatrix}$ and using (74) and (75), we obtain

$$\begin{aligned}
\partial_\chi \Phi(\underline{z},\chi)|_{\gamma_0} &= \left\langle \begin{pmatrix} i\gamma_0\psi_{\underline{z}\chi\epsilon}\\ \nabla\gamma_0\end{pmatrix}, \begin{pmatrix} i\gamma_0\psi_{\underline{z}\chi}\\ \nabla\gamma_0\end{pmatrix}\right\rangle \qquad (77)\\
&= \int \gamma_0[-\Delta+|\psi_{\underline{z}\chi}|^2+Re(\psi_{\underline{z}\chi}\overline{\xi}_{\underline{z}\chi\epsilon})]\gamma_0\,dx.
\end{aligned}$$

If (i) were true, the statement in (76) is true and hence $\gamma_0=0$. Therefore, (77) implies that $\partial_\chi\Phi(\underline{z},\chi)|_{\gamma_0}=0$ and hence (ii) follows.

If (ii) were true, then $\partial_\chi\Phi(\underline{z},\chi)|_{\gamma_0}=0$ in (77). Since $[-\Delta+|\psi_{\underline{z}\chi}|^2+Re(\psi_{\underline{z}\chi}\overline{\xi}_{\underline{z}\chi\epsilon})]$ is a positive operator for ϵ sufficiently small (by Theorem 3.2(a) and using Lemma 5.1 in [25]), then $\gamma_0\equiv 0$, which proves (76), and therefore (i) is true.

□

6 Solution to the reduced problem

In this section, we prove Theorem 2.2 in section 6.1 and Theorems 2.3 and 2.4 in section 6.2.

6.1 Proof of Theorem 2.2: Pinning one vortex to one critical point

In this section, we prove Theorem 2.2. Note that by Proposition 5.3(i) and Proposition 5.2, (31) is true for any $\chi\in H^2(\mathbb{R}^2;\mathbb{R})$ and hence we are reduced to finding a $\underline{z} \in \Sigma_\epsilon \subset \mathbb{R}^{2m}$ to solve (32) directly. To find $\underline{z} \in \Sigma_\epsilon \subset \mathbb{R}^{2m}$ to solve (32), we show by implicit function argument that for $\lambda > 1/2 (\lambda < 1/2)$ and external potential W satisfying conditions (A) to (D) (plus some technical assumptions), there exists a multi-vortex configuration $\underline{a}$ is close to $\underline{b}$ such that (32) is true. Therefore, $\pi_{\underline{a}\chi} F_W(v_{\underline{a}\chi} + \eta_{\underline{a}\chi\epsilon}) = 0$ and Theorem 3.2 will imply that $F_W(v_{\underline{a}\chi} + \eta_{\underline{a}\chi\epsilon}) = 0$ and hence there exists a solution to (6) and (7) of the form given in Theorem 2.2.

First, define functions $H, G : \Sigma_\epsilon \to \mathbb{R}^{2m}$ on the open set $\Sigma_\epsilon \subset \mathbb{R}^{2m}$ with component functions $H_j, G_j : \Sigma_\epsilon \to \mathbb{R}^2$ by

$$
\begin{aligned}
H_j(\underline{z}) &:= \langle F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}), \vec{T}_j^{\underline{z}\chi} \rangle \quad \text{for} \quad j = 1, \dots, m && (78)\\
G_j(\underline{z}) &:= \left\langle \begin{pmatrix} W\psi_{\underline{z}\chi} \\ 0 \end{pmatrix}, \vec{T}_j^{\underline{z}\chi} \right\rangle \quad \text{for} \quad j = 1, \dots, m && (79)
\end{aligned}
$$

where

$$
\vec{T}_j^{\underline{z}\chi} = (T_{j1}^{\underline{z}\chi}, T_{j2}^{\underline{z}\chi})^T.
$$

Note that from the expression for $\pi_{\underline{z}\chi}^t = \pi_\chi^t$ in (67),

$$
H(\underline{z}) = 0 \iff (32) \quad \text{is true.} \tag{80}
$$

Writing $\vec{T}^{\underline{z}\chi} = \left[\vec{T}_j^{\underline{z}\chi} \right]_{j=1}^m$ and Taylor expanding $F_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon})$ about $v_{\underline{z}\chi}$, we have

$$
\begin{aligned}
H(\underline{z}) &= G(\underline{z}) + \langle F_0(v_{\underline{z}\chi}), \vec{T}^{\underline{z}\chi} \rangle + \langle F_W'(v_{\underline{z}\chi}) \eta_{\underline{z}\chi\epsilon}, \vec{T}^{\underline{z}\chi} \rangle + \langle N_W(v_{\underline{z}\chi}, \eta_{\underline{z}\chi\epsilon}), \vec{T}^{\underline{z}\chi} \rangle \\
&= G(\underline{z}) + O(\Gamma_\lambda^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)) + O(\Gamma_{\lambda,\mu}^{1,\frac{1}{2},1,0}(\epsilon) \cdot \Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)) + O((\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))^2) \\
&= G(\underline{z}) + O(\Gamma_\lambda^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)) + O(\mu^2) && (81)
\end{aligned}
$$

where we used (22) in the last equality. Indeed, the second term in the first equation is of $O(\Gamma_\lambda^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))$ due to Theorem 3.1a and the last term is $O(\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))^2$ due to (47) and Theorem 3.2(a). Also, we can estimate the third term in the first equation by

$$
\begin{aligned}
\langle T_{jk}^{\underline{z}\chi}, F_W'(v_{\underline{z}\chi}) \eta_{\underline{z}\chi\epsilon} \rangle &= \langle F_W'(v_{\underline{z}\chi}) T_{jk}^{\underline{z}\chi}, \eta_{\underline{z}\chi\epsilon} \rangle && (82)\\
&\leq \| F_W'(v_{\underline{z}\chi}) T_{jk}^{\underline{z}\chi} \|_{L^2} \| \eta_{\underline{z}\chi\epsilon} \|_{L^2} = O(\Gamma_{\lambda,\mu}^{1,\frac{1}{2},1,0}(\epsilon) \cdot \Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))
\end{aligned}
$$

where we used self-adjointness of $F_W'(v_{\underline{z}\chi})$ in the first equality, Theorem 3.2(a) and

$$
\| F_W'(v_{\underline{z}\chi}) T_{jk}^{\underline{z}\chi} \|_{L^2} = O(\Gamma_{\lambda,\mu}^{1,\frac{1}{2},1,0}(\epsilon)) \tag{83}
$$

(by (37) and (57)) in the last estimate.

Using the definition of $T_{jk}^{\underline{z}\chi}$ in (25) and condition (B) on W, we compute

$$\begin{aligned}
G_j(\underline{z}) &= \int_{\mathbb{R}^2} W(x+z_j)[Re(\bar{\psi}\nabla_A\psi)^{(n_j)}](x)\,dx \\
&\quad + \int_{\mathbb{R}^2} W(x+z_j)\left(\prod_{l\neq j}|\psi^{(n_l)}(x-z_l)|^2-1\right)[Re(\bar{\psi}\nabla_A\psi)^{(n_j)}](x)\,dx \\
&= \int_{\mathbb{R}^2} W(x+z_j)[Re(\bar{\psi}\nabla_A\psi)^{(n_j)}](x)\,dx + O(\mu\nu e^{-m_\lambda R(\underline{z})}R(\underline{z})^{3/2})\,dx \\
&= \int_{\mathbb{R}^2} \nabla W(x+z_j)(1-|\psi^{(n_j)}(x)|^2)\,dx + O(\mu\nu e^{-m_\lambda R(\underline{z})}R(\underline{z})^{3/2}) \qquad (84)\\
&= \int_{\mathbb{R}^2} \left(\nabla W(z_j)+W''(z_j)x+O(\mu\nu^4x^2)\right)(1-|\psi^{(n_j)}(x)|^2)\,dx + O(\mu\nu\Gamma_\lambda^{1,0,1,\frac{3}{2}}(\epsilon))
\end{aligned}$$

where in the second equality, we used (4), (A.3), and Lemma A.1 in the Appendix (with $\alpha=\beta=m_\lambda, \gamma=\delta=0$). In the third and fourth equalities, we used integration by parts and expansion of $\nabla W(x+z_j)$ about z_j.

Now, by (84), condition (C) on W, vanishing of integrals of the form $\int_{\mathbb{R}^2} xf(r)dx$ and since $\underline{b}\in\Sigma_\epsilon = Dom(G) = Dom(F)$ by (21), we have

$$G_j(\underline{b}) = O(\mu\nu^4) \qquad (85)$$

for all $j=1,\dots,m$ by (22). Therefore, by (81), (85) and (22),

$$H(\underline{b}) = O(\max(\mu\nu^4, \Gamma_\lambda^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)+\mu^2)) = O(\mu\nu^4). \qquad (86)$$

Using (84) and condition (B) on W,

$$\nabla_{z_i}G_j(\underline{b}) = \delta_{ij}\int_{\mathbb{R}^2}\left(W''(b_j)+O(\mu\nu^5x^2)\right)(1-|\psi(x)|^2)\,dx + O(\mu\nu^2\Gamma_\lambda^{1,0,1,\frac{3}{2}}(\epsilon)) \qquad (87)$$

where $z_i=(z_{i1},z_{i2})$. By condition (D) on W, (87) and (22), the 2×2 matrix $\nabla_{z_i}G_j(\underline{b})$ is invertible with bound

$$\|(\nabla_{z_i}G_j(\underline{b}))^{-1}\| \le c(\mu\nu^3)^{-1}. \qquad (88)$$

Therefore, by (87) and (88), the $2m\times 2m$ matrix $G'(b)$ is invertible with bound

$$\|G'(\underline{b})^{-1}\| \le c(\mu\nu^3)^{-1}. \qquad (89)$$

Since $\eta_{\underline{z}\chi\epsilon}$ is C^1 in $\underline{z}$ by Theorem 3.2(b), then $H:\Sigma_\epsilon\to\mathbb{R}^{2m}$ defined in (78) is also C^1 in $\underline{z}$ and hence, by (81)

$$\begin{aligned}
H'(\underline{b}) &= G'(\underline{b}) + \partial_{\underline{z}}[H(\underline{z})-G(\underline{z})]|_{\underline{z}=\underline{b}} \\
&= G'(\underline{b}) + \partial_{\underline{z}}\langle F_0(v_{\underline{z}\chi})+F_W'(v_{\underline{z}\chi})\eta_{\underline{z}\chi\epsilon}+N_W(v_{\underline{z}\chi},\eta_{\underline{z}\chi\epsilon}),\vec{T}^{\underline{z}\chi}\rangle|_{\underline{z}=\underline{b}} \\
&= G'(\underline{b}) + O(\Gamma_\lambda^{1,\frac{1}{2},1,\frac{3}{4}}(\epsilon)) + O(\mu^2) \qquad (90)
\end{aligned}$$

where we have used the notation $\partial_{\underline{z}}$ in (62). Indeed (90) is true since $\|\partial_{\underline{z}}\vec{T}^{\underline{z}\chi}\|_{L^2} < \infty$ (by explicit form of $T_{jk}^{\underline{z}\chi}$ in (25)) and

$$
\begin{aligned}
\partial_{\underline{z}}\langle F_0(v_{\underline{z}\chi}), \vec{T}^{\underline{z}\chi}\rangle &= \langle F_0'(v_{\underline{z}\chi})\vec{T}^{\underline{z}\chi}, \vec{T}^{\underline{z}\chi}\rangle + \langle F_0(v_{\underline{z}\chi}), \partial_{\underline{z}}\vec{T}^{\underline{z}\chi}\rangle \\
&= O(\Gamma_{\lambda}^{1,\frac{1}{2},1,0}(\epsilon)) + O(\Gamma_{\lambda}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)) \\
&= O(\Gamma_{\lambda}^{1,\frac{1}{2},1,\frac{3}{4}}(\epsilon))
\end{aligned} \tag{91}
$$

by Theorem 3.1(a) and (c). Also,

$$
\begin{aligned}
\partial_{\underline{z}}\langle F_W'(v_{\underline{z}\chi})\eta_{\underline{z}\chi\epsilon}, \vec{T}^{\underline{z}\chi}\rangle &= \partial_{\underline{z}}\langle \eta_{\underline{z}\chi\epsilon}, F_W'(v_{\underline{z}\chi})\vec{T}^{\underline{z}\chi}\rangle \\
&= \langle \partial_{\underline{z}}\eta_{\underline{z}\chi\epsilon}, F_W'(v_{\underline{z}\chi})\vec{T}^{\underline{z}\chi}\rangle + \langle \eta_{\underline{z}\chi\epsilon}, \partial_{\underline{z}}(F_W'(v_{\underline{z}\chi})\vec{T}^{\underline{z}\chi})\rangle \\
&= O(\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)\cdot\Gamma_{\lambda,\mu}^{1,\frac{1}{2},1,0}(\epsilon)) \\
&\quad + O(\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)\cdot(\Gamma_{\lambda,\mu}^{1,\frac{1}{2},1,0}(\epsilon))) \\
&= O(\mu^2)
\end{aligned} \tag{92}
$$

where in the first equality, we use self-adjointness of $F_W'(v_{\underline{z}\chi})$; in the first term in the third equality, we used Theorem 3.2(b) and (83); in the second term in the third equality, we used Theorem 3.2(a) and Lemma A.3 in the Appendix; and in the fourth equality, we used (22). Finally,

$$
\partial_{\underline{z}}\langle N_W(v_{\underline{z}\chi}, \eta_{\underline{z}\chi\epsilon}), \vec{T}^{\underline{z}\chi}\rangle = O((\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))^2) = O(\mu^2) \tag{93}
$$

by (46) to (48), Theorem 3.2(a) and (b) and (22). Therefore, (90) is proven.

By (89), (90) and (22), $H'(\underline{b})$ is invertible with bound

$$
\|H'(\underline{b})^{-1}\| \leq c(\mu\nu^3)^{-1}. \tag{94}
$$

Therefore, by (86) and (94) and a standard implicit function argument (see e.g., [43]), there exists a solution to $H(\underline{z}) = 0$ at some $\underline{z} = \underline{a}$ where

$$
\underline{a} = \underline{b} + O(\nu). \tag{95}
$$

By (80) and Proposition 5.3(i), we have shown that (30) is true for $\underline{z} = \underline{a}$ and any $\chi \in H^2(\mathbb{R}^2;\mathbb{R})$. Combined with Theorem 3.2, we have shown that $F_W(v_{\underline{a}\chi} + \eta_{\underline{a}\chi\epsilon}) = 0$ with $\eta_{\underline{a}\chi\epsilon} = (\xi, \beta) = O(\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)) = O(\mu)$ (by (22)), and hence we have proven Theorem 2.2.

As a final note, by (95) and since $\underline{b} \in \Sigma_\epsilon$ and $R_\lambda(\underline{z})$ is a continuous function, then $\underline{a} \in \Sigma_\epsilon$ for ν sufficiently small.

6.2 Proof of Theorems 2.3 and 2.4

In this section, we prove theorems 2.3 and 2.4. To prove Theorems 2.3 and 2.4, we show that if the potential W satisfies the given assumptions of these theorems, then there exists a critical point $\underline{z} \in \Sigma_\epsilon$ for the reduced energy Φ_W.

Then using Theorem 2.1, these critical points $\underline{z} \in \Sigma_\epsilon$ of the reduced energy are also critical points of the full energy functional $\mathcal{E}_W$, and hence solutions to (6) and (7).

First, we begin with a very important lemma analyzing the reduced energy $\Phi_W(\underline{z})$, which will be proven in the Appendix.

Lemma 6.1. *For $\epsilon > 0$ sufficiently small and $(\underline{z}, \chi) \in \Sigma_\epsilon$,*

$$\Phi_W(\underline{z}) \;=\; C_{se}^m + V_{int}(\underline{z}) + W_{ext}(\underline{z}) + W_{ext,Rem}(\underline{z}) + R_W(\underline{z}) \tag{96}$$

where

$$C_{se}^m := \sum_{j=1}^m E^{(n_j)} \quad \text{with } \; E^{(n_j)} = \mathcal{E}_0(v_{z_j \chi}) \; \text{ is constant;}$$

$V_{int}(\underline{z})$ is defined in (18);

$$\begin{aligned}
W_{ext}(\underline{z}) &:= \sum_{j=1}^m W_{ext,j}(z_j) \quad \text{as defined in (19) with} \\
W_{ext,j}(z_j) &:= \frac{1}{2}\int_{\mathbb{R}^2} W(x)(|\psi^{(n_j)}(x-z_j)|^2 - 1)dx; && (97) \\
W_{ext,\,Rem}(\underline{z}) &:= \frac{1}{2}\sum_{j<l}^m \int_{\mathbb{R}^2} W(x)(f_j^2-1)(f_l^2-1)dx && (98) \\
&\quad + \frac{1}{2}\sum_{j<l<k}^m \int_{\mathbb{R}^2} W(x)(f_j^2-1)(f_l^2-1)(f_k^2-1)dx + \ldots \quad \text{and} \\
R_W(\underline{z}) &:= \mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}) - \mathcal{E}_W(v_{\underline{z}\chi}). && (99)
\end{aligned}$$

In addition, suppose W satisfies condition (A) and $(\underline{z}, \chi) \in \Sigma_\epsilon$. We have the following estimates on each term in (96).

(a) *For $\lambda > 1/2$,*

$$\begin{aligned}
V_{int}(\underline{z}) &= \sum_{l \neq k} n_l n_k \Psi^{\lambda>1/2}(|z_l - z_k|) \quad \text{where as } R(\underline{z}) \to \infty \\
\Psi^{\lambda>1/2}(|z_l - z_k|) &= c_{lk}^{\lambda>1/2} \frac{e^{-|z_l-z_k|}}{\sqrt{|z_l - z_k|}}\left(1 + O\left(\frac{1}{|z_j - z_k|}\right)\right) && (100) \\
&= O(\epsilon) \quad \text{and} \\
c_{lk}^{\lambda>1/2} &= \frac{1}{2}\beta_l \int_{\mathbb{R}^2} e^{x\cdot(z_l-z_k)/|z_l-z_k|}(-\Delta+1)B^{(n_k)}\,dx \\
&\quad \text{is a positive constant independent of } l \text{ and } k \text{ and} \\
&\quad \beta_l \text{ is a the constant in (4);} \\
W_{ext}(\underline{z}) &= O(\mu); && (101) \\
W_{ext,Rem}(\underline{z}) &= O\left(\mu\left(e^{-R(\underline{z})}/\sqrt{R(\underline{z})}\right)\right) = O(\mu\epsilon) && (102) \\
R_W(\underline{z}) &= O(\Gamma_{\lambda,\mu}^{2,\frac{1}{2},2,\frac{3}{2}}(\epsilon)^*) && (103)
\end{aligned}$$

where
$$\Gamma^{2,\frac{1}{2},2,\frac{3}{2}}_{\lambda,\mu}(\epsilon)^* = O((\max(\Gamma^{1,\frac{1}{4},1,\frac{3}{4}}_{\lambda}(\epsilon),\mu))^2)$$

(b) *For $\lambda < 1/2$, estimates on $W_{ext}(\underline{z})$ and $R_W(\underline{z})$ remain the same as for the $\lambda > 1/2$ case, i.e., (101) and (103) still hold true for $\lambda < 1/2$, respectively. However, for $V_{int}(\underline{z})$ and $W_{ext,Rem}(\underline{z})$, we have*

$$\begin{aligned}
V_{int}(\underline{z}) &= -\sum_{l\neq k}\Psi^{\lambda<1/2}(|z_l-z_k|) \quad \textit{where as } R(\underline{z})\to\infty \\
\Psi^{\lambda<1/2}(|z_l-z_k|) &= c^{\lambda<1/2}_{lk}\frac{e^{-m_\lambda|z_l-z_k|}}{\sqrt{m_\lambda|z_l-z_k|}}(1+O(e^{-m_\lambda|z_l-z_k|})) \\
&= O(\epsilon) \qquad (104)\\
\textit{and } c^{\lambda<1/2}_{lk} &= c_1\int_{\mathbb{R}^2} e^{m_\lambda x\cdot\frac{z_l-z_k}{|z_l-z_k|}}\left[\lambda(1-f_l^2)^2+|(\nabla_A\psi)_l|^2\right]dx \\
W_{ext,Rem}(\underline{z}) &= O(\mu e^{-m_\lambda R(\underline{z})}R(\underline{z})^{3/4}). \qquad (105)
\end{aligned}$$

We are now ready to prove Theorems 2.3 and 2.4.

6.3 Proof of Theorem 2.3: Pinning several vortices to a maximum/minimum

In this section, we prove Theorem 2.3.

Proof of Theorem 2.3. We prove this theorem for $\lambda > 1/2$ and mention the modifications in the proof for $\lambda < 1/2$ at the end.

Expanding $W_{\text{ext},j}(z_j)$ around the strict local maximum point 0 of W and using condition (B) on W, we obtain

$$\begin{aligned}
W_{\text{ext},j}(z_j) &= \frac{1}{2}\int_{\mathbb{R}^2} W(x+z_j)(|\psi^{(n_j)}(x)|^2-1)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left[W(0)+\underbrace{W'(0)}_{=0}(x+z_j)+W''(0)(x+z_j)^2\right](|\psi^{(n_j)}(x)|^2-1)dx \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^2} O(\mu\nu^4|x+z_j|^3)(|\psi^{(n_j)}(x)|^2-1)dx \\
&= c_j+\frac{1}{2}\int_{\mathbb{R}^2}\left[W''(0)z_j^2+O(\mu\nu^4|x+z_j|^3)\right](|\psi^{(n_j)}(x)|^2-1)dx
\end{aligned}$$

where $c_j = \int [W(0)+W''(0)x^2](|\psi^{(n_j)}(x)|^2-1)dx$ is a constant. For any fixed positive integer $m\geq 2$ and any configuration with m vortices $\underline{z}_m$,

$$W_{ext}(\underline{z}_m) = C^m_{ext} - \tilde{C}\sum_{j=1}^m\langle z_j, W''(0)z_j\rangle + O(\mu\nu^4) \qquad (106)$$

by (97) where $C^m_{ext} = \frac{1}{2}\sum_{j=1}^m c_j$ and $\tilde{C} = \frac{1}{2}\int_{\mathbb{R}^2}(1-|\psi^{(n_j)}(x)|^2)dx > 0$ are constants.

From Lemma 6.1(a), we see that for $C^m = C^m_{se} + C^m_{ext}$ constant,

$$\begin{aligned}
\Phi_W(\underline{z}_m) &= C^m - \tilde{C}\sum_{j=1}^m \langle z_j, W''(0)z_j\rangle + O(\mu\nu^4) \qquad (107)\\
&\quad + \sum_{l\neq k}^m n_k n_l \Psi^{\lambda>1/2}(|z_k - z_l|) + O(\epsilon\mu) + O(\Gamma^{2,\frac{1}{2},2,\frac{3}{2}}_{\lambda,\mu}(\epsilon)^*)\\
&= C^m - \tilde{C}\sum_{j=1}^m \langle z_j, W''(0)z_j\rangle + \sum_{l\neq k}^m \Psi^{\lambda>1/2}(|z_k - z_l|) + o(\epsilon^r)
\end{aligned}$$

since $\mu = O(\epsilon^r)$, $\nu << 1$ and $n_k n_l = 1$ by assumption of Theorem 2.3.

Choose positive real numbers

$$s, t > 0 \quad \text{with} \quad 1 < t < r < s < 2 \qquad (108)$$

and define the set containing m-vortices in the configuration by $\Sigma^s_{\epsilon,m}$ by

$$\Sigma^s_{\epsilon,m} := \{\underline{z}_m \,|\, R(\underline{z}_m) > \log(1/\epsilon) \ \text{ and } \ |z_j| < \epsilon^{-s/2} \ \ \forall j\}. \qquad (109)$$

Note that $\Sigma^s_{\epsilon,m} \subset \Sigma_\epsilon$ since if $\underline{z}_m \in \Sigma^s_{\epsilon,m}$, then $R(\underline{z}_m) > \log(1/\epsilon) > \log(1/\epsilon) - \frac{1}{2}\log R(\underline{z}_m)$ and therefore, $\frac{e^{-R(\underline{z}_m)}}{\sqrt{R(\underline{z}_m)}} < \epsilon$, i.e., $\underline{z}_m \in \Sigma_\epsilon$. The boundary of $\Sigma^s_{\epsilon,m}$ is

$$\partial\Sigma^s_{\epsilon,m} = \{\underline{z}_m \,|\, R(\underline{z}_m) = \log(1/\epsilon) \ \text{ or } \ |z_j| = \epsilon^{-s/2} \ \text{ for some } j\}. \qquad (110)$$

We will show that there exists a critical point of the reduced energy Φ_W in the interior of the set $\Sigma^s_{\epsilon,m}$.

Consider a test configuration $\underline{z}^t_m$ with $R(\underline{z}^t_m) = t\log(1/\epsilon)$. Certainly, $\underline{z}^t_m \in \Sigma^s_{\epsilon,m}$ since $t > 1$ by (108). An example of such a configuration would be the m equilateral polygon with sides of equal length $R(\underline{z}^t_m) = t\log(1/\epsilon)$ and center at the origin. For this configuration, the vortex centers are located at $z_j = \frac{R(\underline{z}^t_m)}{\sqrt{2\left(1-\cos\left(\frac{2\pi}{m}\right)\right)}}e_j$ where $e_j = (\cos(2\pi j/m), \sin(2\pi j/m))$ for $j = 1,\ldots,m$, and hence by (100) and (107),

$$\begin{aligned}
\Phi_W(\underline{z}^t_m) &= C^m - \tilde{C}\sum_{j=1}^m \frac{t^2\log^2(1/\epsilon)}{2\left(1-\cos\left(\frac{2\pi}{m}\right)\right)}\langle e_j, W''(0)e_j\rangle + c(t\log(1/\epsilon))^{-1/2}\epsilon^t + o(\epsilon^r)\\
&= O(\epsilon^t) \qquad (111)
\end{aligned}$$

by assumption that $\|W''(0)\| = O(\mu) = O(\epsilon^r)$ for $\lambda > 1/2$ and $1 < t < r$ by (108).

Now, consider configurations on the boundary of $\Sigma^s_{\epsilon,m}$. Suppose $\underline{z}^{bc1}_m \in \partial\Sigma^s_{\epsilon,m}$ is a configuration with $R(\underline{z}^{bc1}_m) = log(1/\epsilon)$. Using the fact that $x = 0$ is a strict maximum of W, we have

$$\begin{aligned}\Phi_W(\underline{z}_m^{bc1}) &\geq C^m + \tilde{C}\|W''(0)\| \sum_{j=1}^m |z_j|^2 + c(\log(1/\epsilon))^{-1/2}\epsilon + o(\epsilon^r) \\ &\geq C^m + m\tilde{C}\|W''(0)\| \left(\min_{j=1,\ldots,m} |z_j|^2\right) + c(\log(1/\epsilon))^{-1/2}\epsilon + o(\epsilon^r) \\ &= O(\epsilon) \qquad (112)\end{aligned}$$

by (100), (107), (108) and $\|W''(0)\| = O(\epsilon^r)$.

Suppose $\underline{z}_m^{bc2} \in \partial\Sigma_{\epsilon,m}^s$ is a configuration with $|z_i| = \epsilon^{-s/2}$ for some $i = 1, ..., m$. W.L.O.G., assume $R(\underline{z}_m^{bc2}) = \tilde{t}\log(1/\epsilon)$ with $1 < s < \tilde{t}$. Then by (107),

$$\begin{aligned}\Phi_W(\underline{z}_m^{bc2}) &\geq C + \tilde{C}\|W''(0)\|\frac{1}{\epsilon^s} + (m-1)\|W''(0)\| \left(\min_{j\neq i,\, j=1,\ldots,m} |z_j|^2\right) \\ &\quad + c(\tilde{t}\log(1/\epsilon))^{-1/2}\epsilon^{\tilde{t}} + o(\epsilon^r) = O(\epsilon^{r-s}) \qquad (113)\end{aligned}$$

where we used $r - s < 0$ by (108) and the fact that $\|W\| = O(\epsilon^r)$.

Comparing (111), (112) and (113), we see that

$$\Phi_W(\underline{z}_m^t) < \Phi_W(\underline{z}_m^{bc1,bc2})$$

by (108) and therefore, the minimum of Φ_W cannot be reached on the boundary of $\Sigma_{\epsilon,m}^s$. Hence, the minimum must be attained in the interior of $\Sigma_{\epsilon,m}^s$. Therefore, there exists a local minimum (and hence a critical point) $\underline{\tilde{z}}_m \in \Sigma_{\epsilon,m}^s \subset \Sigma_\epsilon$ of the reduced energy Φ_W. By Theorem 2.1, there exists a vortex configuration $\underline{\tilde{z}}_m$ containing m number of vortices and a solution of (6) and (7) of the form

$$u_m(x) = v_{\underline{\tilde{z}}_m\chi} + \eta_{\underline{\tilde{z}}_m\chi\epsilon}.$$

Note that by Theorem 3.2(a), $\eta_{\underline{\tilde{z}}_m\chi\epsilon} = O(\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)) = O(\Gamma_{\lambda}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))$ in H^2 since $\mu = O(\epsilon^r)$ for $1 < r < 2$. Hence we have proven Theorem 2.3 for $\lambda > 1/2$.

For $\lambda < 1/2$, the interaction energy is of the form $-\sum_{k\neq l} \frac{e^{-m_\lambda|z_j - z_k|}}{\sqrt{m_\lambda|z_j - z_k|}}$ by (104) in Lemma 6.1(b) and since $W''(0) > 0$, we are looking for a maximum of the reduced energy $\Phi_W(\underline{z})$. The only differences are that the configuration set is $\Sigma_{\epsilon,m}^s = \{\underline{z}_m \,|\, R(\underline{z}_m) > \frac{1}{m_\lambda} log(1/\epsilon) \text{ and } |z_j| < \epsilon^{-s/2}\, \forall j\}$ and that $s, t > 0$ in (108) satisfies $1 < m_\lambda t < r < s < 2$. Using the fact that for $\lambda < 1/2$, $\|W''(0)\| = O(\mu) = O(\epsilon^r)$ to control the errors in the remainders, one can show using (104) in Lemma 6.1 that $\Phi_W(\underline{z}_m^t) = -O(\epsilon^{m_\lambda t})$, $\Phi_W(\underline{z}_m^{bc1}) \leq -O(\epsilon)$ and $\Phi_W(\underline{z}_m^{bc2}) \leq -O(\epsilon^{r-s})$. Since $1 < m_\lambda t < r < s$, then $\Phi_W(\underline{z}_m^t) > \Phi_W(\underline{z}_m^{bc1,bc2})$ and there exists a maximum in the interior of $\Sigma_{\epsilon,m}^s$. $\square$

As a final note in this section, for $\lambda > 1/2$, $\underline{\tilde{z}}_m$ is local minimum of the reduced energy $\Phi_W(\underline{z})$. Therefore, $u_m(x)$ is a local minimizer of the full energy $\mathcal{E}_W$ by Theorem 2.1. Hence, $u_m(x)$ is a stable solution for every m and so there exists an infinite number of stable state solutions for (6) and (7) (see Remark 5 in Section 2).

6.4 Proof of Theorem 2.4: Pinning near infinity

In this section, we prove Theorem 2.4.

Proof of Theorem 2.4. First, we note that condition (A) on the potential W implies that $q > 1$ in the radial algebraic behavior of W at infinity. Again, we will prove it for $\lambda > 1/2$ only. The modification in the proof for $\lambda < 1/2$ will be outlined in the end.

Fix a positive integer $k > 0$ and let $\underline{z}_k^{r_k}$ be a vortex configuration with k vortices equally spaced on a circle of radius $r_k > 0$. More precisely, we place the centers of the vortices in configuration $\underline{z}_k^{r_k} := (z_1^{r_k}, ..., z_k^{r_k})$ at

$$z_j^{r_k} = r_k(\cos\theta_j, \sin\theta_j) \text{ where } \theta_j = \frac{2(j-1)\pi}{k}, \quad j = 1, ..., k$$
$$\text{with } r_k \in S_k := \left[\left(\frac{q-\beta}{2\pi}\right) k \ln k, \left(\frac{q+\beta}{2\pi}\right) k \ln k\right],$$

for some small constant $\beta > 0$. Note that by placing the k vortices equally spaced on a circle of radius r_k, we can identify the circular vortex configuration $\underline{z}_k^{r_k} \in \mathbb{R}^{2k}$ with the real variable r_k, i.e.,

$$k- \text{ vortex circular configuration } \underline{z}_k^{r_k} \in \mathbb{R}^{2k} \iff r_k \in S_k \subset \mathbb{R}.$$

This will simplify our reduced problem of finding a critical point of $\Phi_W(\underline{z})$ in $\mathbb{R}^{2k}$ to a critical point $r_k \in S_k \subset \mathbb{R}$.

By assumption on the radially symmetric form of W in Theorem 2.4 and since $|z_j^{r_k}| = r_k$ is large for every j, we can expand $W(x+z_j^{r_k}) = \mu|x+z_j^{r_k}|^{-q} + O(\mu|x+z_j^{r_k}|^{-(q+\varsigma)})$, $|x+z_j^{r_k}|^{-q} = |z_j^{r_k}|^{-q} + O(|z_j^{r_k}|^{-(q+2)})$ and write the function $W_{\text{ext},j}(z_j^{r_k})$ as a function of r_k as follows:

$$\begin{aligned}
W_{\text{ext},j}(z_j^{r_k}) &= \frac{1}{2}\int_{\mathbb{R}^2} W(x+z_j^{r_k})(|\psi^{(n_j)}(x)|^2 - 1)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{\mu}{|x+z_j^{r_k}|^q} + O\left(\frac{\mu}{|x+z_j^{r_k}|^{q+\varsigma}}\right)\right](|\psi^{(n_j)}(x)|^2 - 1)dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{\mu}{|z_j^{r_k}|^q} + O\left(\frac{\mu}{|z_j^{r_k}|^{q+\min(2,\varsigma)}}\right)\right](|\psi^{(n_j)}(x)|^2 - 1)dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^2} \left[\frac{\mu}{r_k^q} + O\left(\frac{\mu}{r_k^{q+\min(2,\varsigma)}}\right)\right](1 - |\psi^{(n_j)}(x)|^2)dx \\
&= -c_j\frac{\mu}{r_k^q} + O\left(\frac{\mu}{r_k^{q+\min(2,\varsigma)}}\right) = W_{\text{ext},j}(r_k).
\end{aligned} \tag{114}$$

Here $c_j = \int_{\mathbb{R}^2}(1 - |\psi^{(n_j)}(x)|^2)dx > 0$ is a constant by assumption that all the degrees are $n_j = 1$ or all $n_j = -1$ in Theorem 2.4. Therefore, $W_{\text{ext}}(\underline{z}_k^{r_k}) = W_{\text{ext}}(r_k)$ and

$$W_{\text{ext}}(r_k) = \sum_{j=1}^{k} W_{\text{ext},j}(r_k) = -k\left[c_j\frac{\mu}{r_k^q} + O\left(\frac{\mu}{r_k^{q+\min(2,\varsigma)}}\right)\right]. \tag{115}$$

Similarly, we can write $V_{\text{int}}(\underline{z}_k^{r_k}) = V_{\text{int}}(r_k)$ and since $R(\underline{z}_k^{r_k}) = \frac{2\pi r_k}{k}\left(1 - O(k^{-2})\right)$, then by (100), $n_l n_k = 1$ for all l, k, and nearest neighbor interactions, we have

$$\begin{aligned} V_{\text{int}}(\underline{z}_k^{r_k}) &= \sum_{j=1}^{k}\left(\Psi^{\lambda>1/2}\left(\frac{2\pi r_k}{k}\left(1 - O\left(k^{-2}\right)\right)\right) + O\left(e^{-2\pi r_k(1+\delta)/k}\right)\right) \\ &= k\left[\Psi^{\lambda>1/2}\left(\frac{2\pi r_k}{k}\left(1 - O\left(k^{-2}\right)\right)\right) + O\left(e^{-2\pi r_k(1+\delta)/k}\right)\right] \\ &= V_{\text{int}}(r_k) \end{aligned} \tag{116}$$

for some $\delta > 0$. Combining (96), Lemma 6.1(a), (115) and (116), we have for $\Phi_W(\underline{z}_k^{r_k}) = \Phi_W(r_k)$,

$$\begin{aligned} \Phi_W(r_k) &= k\left[C_{se}^k/k - \frac{c_j\mu}{r_k^q} + \Psi^{\lambda>1/2}\left(\frac{2\pi r_k}{k}\left(1 - O\left(k^{-2}\right)\right)\right)\right] \\ &\quad + k\left[O\left(e^{-2\pi r_k(1+\delta)/k}\right) + O\left(\frac{\mu}{r_k^{q+\min(2,\varsigma)}}\right)\right] + O\left(\mu\epsilon\right) + O(\Gamma_{\lambda,\mu}^{2,\frac{1}{2},2,\frac{3}{2}}(\epsilon)^*) \\ &= k\left[\tilde{C} - \frac{c_j\epsilon}{r_k^q} + \Psi^{\lambda>1/2}\left(\frac{2\pi r_k}{k}\left(1 - O\left(k^{-2}\right)\right)\right)\right] \\ &\quad + k\left[O\left(e^{-2\pi r_k(1+\delta)/k}\right) + O\left(\frac{\epsilon}{(k\ln k)^{q+\min(2,\varsigma)}}\right)\right] + o(\epsilon) \end{aligned}$$

by assumption that $W = O(\mu) = O(\epsilon)$ for $\lambda > 1/2$ and since $r \in [r_0 k\ln k, r_1 k\ln k]$, and where $\tilde{C} = C_{se}^k/k$. By the form of the interaction function $\Psi^{\lambda>1/2}(d)$ as $d \to \infty$ in (100), there exists an integer $k_0 > 0$ such that for all $k > k_0$, $\tilde{C} - \frac{c_j\epsilon}{r_k^q} + \Psi^{\lambda>1/2}\left(\frac{2\pi r_k}{k}\left(1 + O(k^{-2})\right)\right)$ has a minimum point at $\tilde{r}_k = \left(\frac{q+o(1)}{2\pi}\right) k\ln k$ which is in the interior of S_k. Therefore, we have proven that $\Phi_W(r_k)$ has a critical point at $\tilde{r}_k \in S_k$ which is equivalent to saying that $\Phi_W(\underline{z}_k^{r_k})$ has a critical point at the $k-$vortex circular configuration $\underline{\tilde{z}}_k \in \mathbb{R}^{2k}$. By Theorem 2.1, (6) and (7) has a solution of the form

$$u_k(x) = v_{\underline{\tilde{z}}_k\chi} + \eta_{\underline{\tilde{z}}_k\chi\epsilon}$$

where by Theorem 3.2(a), $\eta_{\underline{\tilde{z}}_k\chi\epsilon} = O(\Gamma_{\lambda,\mu}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon)) = O(\Gamma_{\lambda}^{1,\frac{1}{4},1,\frac{3}{4}}(\epsilon))$ in H^2 since $\mu = O(\epsilon)$. In addition, observing that $C_{se}^m + W_{\text{ext}}(\underline{z}) = \sum_{j=1}^m \mathcal{E}_W(v_{z_j\chi})$ by (96) and (97), then u_k has energy

$$\mathcal{E}_W(u_k) = k\mathcal{E}_W(v^{\pm 1}) + O\left(\frac{e^{-R(\underline{\tilde{z}}_k)}}{\sqrt{R(\underline{\tilde{z}}_k)}}\right)$$

where $v^{\pm 1}$ is the ± 1 degree vortices. Therefore, we have proven Theorem 2.4 for $\lambda > 1/2$.

For $\lambda < 1/2$, the proof is exactly the same except that the form of $\Phi_W(r_k)$ is $\tilde{C} + \frac{\tilde{c}_k\epsilon}{r_k^q} - \Psi^{\lambda<1/2}\left(\frac{2\pi r_k}{k}\right)$ and attains a *maximum* point at $\tilde{r}_k = \left(\frac{q+o(1)}{2\pi m_\lambda}\right) k\ln k$. Here, $\tilde{c}_k = \frac{1}{k}\sum_{j=1}^k c_j$ where the constant c_j is defined under (114). □

Again, as a final note, for $\lambda > 1/2$, $\underline{\tilde{z}}_k$ is a local minimizer of the reduced energy $\Phi_W(\underline{z})$, and hence $u_k(x)$ is a local minimizer of the full energy $\mathcal{E}_W$ by Theorem 2.1. Therefore, $u_k(x)$ is a stable solution for every k and so there exists an infinite number of stable state solutions for (6) and (7) (see Remark 5 in Section 2).

A Appendix

In this Appendix, we prove Lemma 6.1 and a technical estimate required in Section 6.1. We will prove Lemma 6.1 first and then leave the technical estimate to the end.

We will use the subindex j to denote vortex component functions which have degree n_j and is centered at z_j. For example, $1 - f_j^2 = 1 - |\psi^{(n_j)}(\cdot - z_j)|^2$ or $(\nabla_A \psi)_j = \nabla_{A^{(n_j)}(\cdot - z_j)} \psi^{(n_j)}(\cdot - z_j)$, etc... We will also denote $[u]_{\psi,A}$ as the complex and vector components of $u \in L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$.

To prove Lemma 6.1, we need the following two technical lemmas.

Lemma A.1. *For $0 < \alpha \le \beta$, $0 \le \delta, \gamma < 3/2$, we have*

$$\int_{\mathbb{R}^2} \frac{e^{-\alpha|x|} e^{-\beta|x-a|}}{|x|^\gamma |x-a|^\delta} \, dx \le c \frac{e^{-\alpha|a|}}{|a|^{\gamma+\delta-2}} \begin{cases} |a|^{-1/2}, & \alpha = \beta \\ |a|^{\delta-2}, & \alpha < \beta. \end{cases}$$

Lemma A.2. *Suppose $0 < m_1 < m_2$ and functions $b(x), e(x)$ satisfy $|b(x)| \le ce^{-m_2|x|}$ and $e(x) = c_1 \frac{e^{-m_1|x|}}{\sqrt{m_1|x|}}(1 + O(e^{-m_1|x|}))$ as as $|x| \to \infty$. Then as $|z| \to \infty$,*

$$I(z) := \int_{\mathbb{R}^2} b(x) e(x-z) \, dx = c_1 \frac{e^{-m_1|x|}}{\sqrt{m_1|x|}} \int_{\mathbb{R}^2} e^{m_1 x \cdot \frac{z}{|z|}} b(x) \, dx (1 + O(e^{-m_1|z|})).$$

Lemma A.1 was proven in [24] in Appendix A.3. Lemma A.2 is a straight forward modification of Lemma 13 in [24] with similar proof so we won't prove it here. Now we are ready to prove Lemma 6.1.

Proof of Lemma 6.1. We prove (96) first. Write

$$\mathcal{E}_W(v_{\underline{z}\chi} + \eta_{\underline{z}\chi\epsilon}) = \mathcal{E}_W(v_{\underline{z}\chi}) + R_W(\underline{z}), \tag{A.1}$$

where $R_W(\underline{z})$ is defined by this relation. But

$$\begin{aligned} \mathcal{E}_W(v_{\underline{z}\chi}) &= \mathcal{E}_0(v_{\underline{z}\chi}) + \frac{1}{2} \int_{\mathbb{R}^2} W(x)(|\psi_{\underline{z}\chi}(x)|^2 - 1) dx \\ &= C_{se}^m + V_{\text{int}}(\underline{z}) + \int_{\mathbb{R}^2} W(x)(|\psi_{\underline{z}\chi}(x)|^2 - 1) dx \end{aligned} \tag{A.2}$$

by definition of $V_{\text{int}}(\underline{z})$ in (18). From the fact that we can write

$$\begin{aligned} \prod_{j=1}^m f_j^2 - 1 &= \sum_{j=1}^m (f_j^2 - 1) + \sum_{j<l}^m (f_j^2 - 1)(f_l^2 - 1) \\ &\quad + \sum_{j<l<k}^m (f_j^2 - 1)(f_l^2 - 1)(f_k^2 - 1) + ..., \end{aligned} \tag{A.3}$$

we have

$$\frac{1}{2}\int_{\mathbb{R}^2} W(x)(|\psi_{\underline{z}\chi}(x)|^2-1)dx = W_{\text{ext}}(\underline{z}) + W_{\text{ext,Rem}}(\underline{z}). \tag{A.4}$$

Now we prove part (a) for $\lambda > 1/2$. Firstly, (101) and (102) follows from (97) and (98) and application of Lemma A.1, (4), and conditions (A) and (B) on potential W. For example, for (102), by (4), we have

$$\begin{aligned}
|W_{\text{ext,Rem}}(\underline{z})| &\leq \|W\|_{L^2}\sum_{l<j}\|e^{-m_\lambda(|x-z_j|+|x-z_l|)}\|_{L^2} \leq \mu e^{-m_\lambda R(\underline{z})}R(\underline{z})^{3/4} \\
&\ll \mu\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}} = O(\mu\epsilon) \qquad \text{(A.5)}
\end{aligned}$$

where in the second last inequality, we used condition (A) of Lemma 6.1 and applied Lemma A.1 with $\alpha=\beta=2m_\lambda>1$, $\delta=\gamma=0$.

Next, equation (100) follows directly from Lemma 7 in [24].

For (103), we have

$$\begin{aligned}
R_W(\underline{z}) &= \mathcal{E}_W(v_{\underline{z}\chi}+\eta_{\underline{z}\chi\epsilon}) - \mathcal{E}_W(v_{\underline{z}\chi}) = \underbrace{\langle\mathcal{E}'_W(v_{\underline{z}\chi}),\eta_{\underline{z}\chi\epsilon}\rangle}_{O(\Gamma^{1,\frac14,1,\frac34}_{\lambda,\mu}(\epsilon)\Gamma^{1,\frac14,1,\frac34}_{\lambda,\mu}(\epsilon))} + \underbrace{O(\|\eta_{\underline{z}\chi\epsilon}\|^2)}_{O(\Gamma^{1,\frac14,1,\frac34}_{\lambda,\mu}(\epsilon))^2} \\
&= O(\Gamma^{2,\frac12,2,\frac32}_{\lambda,\mu}(\epsilon))^*
\end{aligned}$$

by Corollary 3.1.1 and Theorem 3.1a).

Now we prove part (b) for $\lambda < 1/2$. We prove (104) first. A straight forward computation (see Lemma 7 in [24]) gives

$$\begin{aligned}
V_{\text{int}}(\underline{z}) &= \mathcal{E}_0(v_{\underline{z}\chi}) - \sum_{j=1}^m E^{(n_j)} = LO_{\lambda>1/2} + A + B + C + D \\
&= LO_{\lambda>1/2} + (A_1+A_2) + (B_1+B_2) + C + (D_1+D_2)
\end{aligned}$$

where by (3) and (4),

$$LO_{\lambda>1/2} = \frac{1}{2}\sum_{l\neq k}^m\int_{\mathbb{R}^2} j_l\cdot j_k + B_lB_k \leq c\sum_{l\neq k}^m\int_{\mathbb{R}^2}\frac{e^{-|x-z_l|}}{\sqrt{|x-z_l|}}\frac{e^{-|x-z_k|}}{\sqrt{|x-z_k|}};$$

$$\begin{aligned}
A &= \frac{1}{2}\sum_{j=1}^m\int_{\mathbb{R}^2}\left(\prod_{k\neq j} f_k^2 - 1\right)|(\nabla_A\psi)_j|^2 =: A_1 + A_2 \text{ where} \qquad \text{(A.6)}\\
A_1 &:= \frac{1}{2}\sum_{k\neq j}^m\int_{\mathbb{R}^2}(f_k^2-1)|(\nabla_A\psi)_j|^2 \leq c\sum_{j\neq k}^m\int_{\mathbb{R}^2} e^{-m_\lambda|x-z_k|}e^{-2m_\lambda|x-z_j|} \quad\text{and}\\
A_2 &:= \frac{1}{2}\sum_{i,k\neq j}^m\int_{\mathbb{R}^2}(f_k^2-1)(f_i^2-1)|(\nabla_A\psi)_j|^2 + \cdots \\
&\leq c\sum_{i,k\neq j}^m\int_{\mathbb{R}^2} e^{-m_\lambda|x-z_k|}e^{-m_\lambda|x-z_i|}e^{-2m_\lambda|x-z_j|};
\end{aligned}$$

$$
\begin{aligned}
B &= \frac{1}{2}\sum_{j\neq l}^{m}\int_{\mathbb{R}^2}\left(\prod_{k\neq j,l} f_k^2\right)[Re(\bar{\psi}\nabla_A\psi)]_j[Re(\bar{\psi}\nabla_A\psi)]_l =: B_1+B_2 \quad \text{where}\\
B_1 &:= \frac{1}{2}\sum_{j\neq l}^{m}\int_{\mathbb{R}^2}\left(\prod_{k\neq j,l} f_k^2-1\right)[Re(\bar{\psi}\nabla_A\psi)]_j[Re(\bar{\psi}\nabla_A\psi)]_l\\
&\leq c\sum_{j\neq l\neq k}^{m}\int_{\mathbb{R}^2} e^{-m_\lambda|x-z_k|}e^{-m_\lambda|x-z_j|}e^{-m_\lambda|x-z_l|} \quad \text{and} \qquad\qquad (A.7)\\
B_2 &:= \frac{1}{2}\sum_{j\neq l}^{m}\int_{\mathbb{R}^2}[Re(\bar{\psi}\nabla_A\psi)]_j[Re(\bar{\psi}\nabla_A\psi)]_l \leq c\sum_{l\neq j}^{m}\int_{\mathbb{R}^2} e^{-m_\lambda|x-z_l|}e^{-m_\lambda|x-z_j|};
\end{aligned}
$$

$$
C = \frac{1}{2}\sum_{j\neq l}^{m}\int_{\mathbb{R}^2}\left(\prod_{k\neq j} f_k^2-1\right) j_j\cdot j_l \leq c\sum_{j\neq l\neq k}^{m}\int_{\mathbb{R}^2} e^{-m_\lambda|x-z_k|}e^{-|x-z_j|}e^{-|x-z_l|}
$$

$$
\begin{aligned}
D &= \frac{\lambda}{4}\int_{\mathbb{R}^2}\left[\sum_{j=1}^{m}(f_j^2-1)+\sum_{l<j}^{m}(f_l^2-1)(f_j^2-1)+\cdots\right]^2 dx =: D_1+D_2 \quad \text{where}\\
D_1 &:= \frac{\lambda}{4}\sum_{l\neq j}^{m}\int_{\mathbb{R}^2}(f_l^2-1)(f_j^2-1)dx+\frac{\lambda}{2}\sum_{l\neq j}^{m}\int_{\mathbb{R}^2}(f_l^2-1)^2(f_j^2-1)dx \qquad\qquad (A.8)\\
&\leq c\int_{\mathbb{R}^2}\left[\sum_{l\neq j}^{m} e^{-m_\lambda|x-z_l|}e^{-m_\lambda|x-z_j|}+\sum_{l\neq j}^{m} e^{-2m_\lambda|x-z_l|}e^{-m_\lambda|x-z_j|}\right]dx \quad \text{and}\\
D_2 &= \frac{\lambda}{4}\int_{\mathbb{R}^2}\left[\sum_{j\neq l\neq k}^{m}(f_j^2-1)(f_l^2-1)(f_k^2-1)+\sum_{j\neq l}^{m}(f_j^2-1)^2(f_l^2-1)^2\cdots\right]dx\\
&\leq c\int_{\mathbb{R}^2}\left[\sum_{j\neq l\neq k}^{m} e^{-m_\lambda|x-z_j|}e^{-m_\lambda|x-z_l|}e^{-m_\lambda|x-z_k|}+\sum_{j\neq l}^{m} e^{-2m_\lambda|x-z_j|}e^{-2m_\lambda|x-z_l|}\cdots\right]dx.
\end{aligned}
$$

In (A.6), we used $|\nabla_A\psi|^2 = (f')^2+\frac{n^2f^2}{r^2}(1-a)^2$ and in terms $A = A_1+A_2, B_1, C$ and D, we used (A.3).

Since $m_\lambda < 1$ for $\lambda < 1/2$, then the leading order terms of V_{int} come from terms B_2 and D_1. But, we claim that $B_2 + D_1$ has the same order as A_1. Indeed, by (A.7),

$$
\begin{aligned}
B_2 &= \frac{1}{2}\sum_{l\neq j}^{m}\int_{\mathbb{R}^2}(f_l(\nabla_x f_l))\cdot(f_j(\nabla_x f_j))\,dx = \frac{1}{4}\sum_{l\neq j}^{m}\int_{\mathbb{R}^2}(f_l(\nabla_x f_l))\cdot\nabla_x(f_j^2-1)\,dx\\
&= -\frac{1}{4}\sum_{l\neq j}^{m}\int_{\mathbb{R}^2}\left[|\nabla_x f_l|^2+f_l\Delta_x f_l\right](f_j^2-1)\,dx\\
&= \frac{1}{4}\sum_{l\neq j}^{m}\int_{\mathbb{R}^2}\left[|(\nabla_A\psi)_l|^2+\lambda(f_l^2-1)^2+\lambda(f_l^2-1)\right](1-f_j^2)dx \qquad\qquad (A.9)
\end{aligned}
$$

where we used integration by parts in the second last equality and the equation

$$-\Delta_r f_n + n^2 \frac{(1-a_n)^2}{r^2} f_n + \lambda(f_n^2 - 1) f_n = 0$$

(by (1), (3) and $\Delta_x f = \Delta_r f$) in the last equality. Therefore by (A.8) and (A.9), we obtain

$$\begin{aligned} B_2 + D_1 &= \frac{1}{4} \sum_{l \neq j}^{m} \int_{\mathbb{R}^2} \left[|(\nabla_A \psi)_l|^2 - \lambda (f_l^2 - 1)^2 \right] (1 - f_j^2) \, dx \\ &\leq c \int_{\mathbb{R}^2} e^{-2m_\lambda |x - z_l|} e^{-m_\lambda |x - z_j|} \, dx. \end{aligned} \tag{A.10}$$

By (A.6) and (A.10), $B_2 + D_1$ has the same order as A_1, and therefore, the leading order term in V_{int} for $\lambda < 1/2$ comes from the addition of terms $A_1 + B_2 + D_1$. Computing

$$A_1 + B_2 + D_1 = -\frac{1}{4} \sum_{l \neq j}^{m} \int_{\mathbb{R}^2} \left[\lambda (1 - f_l^2)^2 + |(\nabla_A \psi)_l|^2 \right] (1 - f_j^2) \, dx < 0,$$

we see that to leading order, $V_{\text{int}}(\underline{z})$ is negative for $\lambda < 1/2$. Now, using the asymptotic expression

$$1 - f = \tilde{c}_1 \frac{e^{-m_\lambda r}}{\sqrt{m_\lambda r}} (1 + o(e^{-m_\lambda r})) \quad \text{as } r \to \infty$$

(see [37], Theorem II.5.3) for $\lambda < 1/2$, then $1 - f^2 = (2 - (1 - f))(1 - f)$ has the same asymptotic behavior as $r \to \infty$ with the constant $\tilde{c}_1$ replaced by $c_1 = 2\tilde{c}_1$. Therefore, by Lemma A.2,

$$\begin{aligned} V_{\text{int}}(\underline{z}) &= -\sum_{l > j} \Psi^{\lambda < 1/2}(|z_l - z_j|) \text{ where} \\ \Psi^{\lambda < 1/2}(|z_l - z_j|) &= c_{lj}^{\lambda < 1/2} \frac{e^{-m_\lambda |z_l - z_j|}}{\sqrt{m_\lambda |z_l - z_j|}} (1 + O(e^{-m_\lambda |z_l - z_j|})) \text{ with} \\ c_{lj}^{\lambda < 1/2} &= c_1 \int_{\mathbb{R}^2} e^{m_\lambda x \cdot \frac{z_l - z_j}{|z_l - z_j|}} \left[\lambda (1 - f_l^2)^2 + |(\nabla_A \psi)_l|^2 \right] dx > 0 \end{aligned} \tag{A.11}$$

and (104) follows.

For (105), we use the same estimate as in (A.5) for $\lambda > 1/2$ to obtain the result for $\lambda < 1/2$. $\square$

Now, we state a prove a technical estimate required in Section 6.1.

Lemma A.3. *Suppose $W(x)$ satisfies condition (A). Then*

$$\|\partial_{z_i} (F'_W(v_{\underline{z}\chi}) T_{jk}^{\underline{z}\chi})\|_{L^2} = O(\Gamma_{\lambda,\mu}^{1,\frac{1}{2},1,0}(\epsilon)). \tag{A.12}$$

Proof. Recall that for every $j = 1, ..., m$, we can decompose

$$\begin{aligned} F'_W(v_{\underline{z}\chi}) &= L_{\underline{z}\chi} + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \\ &= L_j + L^{1/2}_{(j)} + V_{(j)} + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{A.13}$$

with

$$L_j := \mathcal{E}''_{GL}(g_{\chi_{(j)}} u^{(n_j)}(x - z_j)),$$

$L^{1/2}_{(j)}$ is a first order differential operator at $g_{\chi_{(j)}} u^{(n_j)}(x - z_j)$ given by

$$L^{1/2}_{(j)}(g_{\chi_{(j)}} u^{(n_j)}(x - z_j)) \begin{pmatrix} \xi \\ B \end{pmatrix} = \begin{pmatrix} -2i \left[\Theta_{(j)}\right]_A \cdot \nabla_{[g_{\chi_{(j)}} u^{(n_j)}(x-z_j)]_A} \xi + \left[\Theta_{(j)}\right]_\psi \nabla \cdot B \\ \left[\Theta_{(j)}\right]_\psi \nabla_{[g_{\chi_{(j)}} u^{(n_j)}(x-z_j)]_A} \xi \end{pmatrix} \tag{A.14}$$

and

$$\Theta_j(x) := v_{\underline{z}\chi} - g_{\chi_{(j)}} u^{(n_j)}(x - z_j) \tag{A.15}$$

with $\chi_{(j)} := \chi + \sum_{k \neq j} \theta(\cdot - z_k)$ and $V_{(j)}$, $\Theta_{(j)}$ are multiplication operators satisfying

$$\|V_{(j)}\|_\infty, \|\Theta_{(j)}\|_\infty \leq \sum_{k \neq j} e^{-\min(1, m_\lambda)|x - z_k|} \tag{A.16}$$

(see [24] or [48]). Since we can write

$$\begin{aligned} T^{\underline{z}\chi}_{jk} &=: T^{z_j \chi_{(j)}}_{jk} + T^{Rem}_{jk} \\ &:= \begin{pmatrix} e^{i\chi_{(j)}} (\nabla_A \psi)^{(j)}_k \\ B^{(j)} e^\perp_k \end{pmatrix} + \begin{pmatrix} e^{i\chi_{(j)}} \left(\prod_{l \neq j} f^{(l)} - 1\right) (\nabla_A \psi)^{(j)}_k \\ 0 \end{pmatrix}, \end{aligned} \tag{A.17}$$

then

$$\begin{aligned} \|\partial_{z_i}(L_j T^{\underline{z}\chi}_{jk})\|_{L^2} &\leq c \|L_j T^{z_j \chi_{(j)}}_{jk} + L_j T^{Rem}_{jk}\|_{H^1} \\ &= c \|L_j T^{Rem}_{jk}\|_{H^1} \\ &\leq c e^{-\min(1, m_\lambda) R(\underline{z})} \end{aligned} \tag{A.18}$$

since $1 - f, f', f'', f''' \leq e^{-m_\lambda |x|}$ and $1 - a, a', a'' \leq e^{-|x|}$. Now due to gauge equivariance of $L^{1/2}_{(j)}$ in (A.14), we have

$$\|\partial_{z_i}(L^{1/2}_{(j)} T^{\underline{z}\chi}_{jk})\|_{L^2} \leq c e^{-\min(1, m_\lambda) R(\underline{z})} \tag{A.19}$$

by (A.15) to (A.17). Now

$$\|\partial_{z_i}(V_{(j)} T^{\underline{z}\chi}_{jk})\|_{L^2} \leq c e^{-\min(1, m_\lambda) R(\underline{z})} \tag{A.20}$$

by (A.16) and

$$\left\| \partial_{z_i} \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} T^{\underline{z}\chi}_{jk} \right\|_{L^2} = \left\| \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \partial_{z_i} T^{\underline{z}\chi}_{jk} \right\|_{L^2} \leq \|W\|_{L^2} \|\partial_{z_i} T^{\underline{z}\chi}_{jk}\|_\infty \leq c\mu \tag{A.21}$$

where we used condition (A) on W and $\|\partial_{z_i} T^{z\chi}_{jk}\|_\infty < \infty$ (by explicit expression of $T^{z\chi}_{jk}$ in (25)). Therefore, by (A.13) and (A.18) to (A.21), we have proven estimate (A.12).

□

Acknowledgements

F.T. would like to thank I.M. Sigal and S. Gustafson for fruitful collaboration and discussions on magnetic vortices. J.W. would like to thank F. Pacard and M. Kowalczyk for useful discussions. F.T.'s research on this paper was partly sponsored by NSERC and Lakehead University. J.W.'s research is partially supported by a GRF grant (Entire Solutions to Semilinear Elliptic PDEs) of RGC of Hong Kong. This paper was completed while F.T. was visiting J.W. at the Chinese University of Hong Kong. F.T. is grateful to J.W., Yao Wei, Ao Wei Wei for fruitful discussions and the Department of Mathematics at Chinese University of Hong Kong for their hospitality.

References

[1] A. Aftalion, E. Sandier, and S. Serfaty, *Pinning phenomena in Ginzburg-Landau model of superconductivity*, J. Math. Pures Appl., 80 (2001), pp. 339–372.

[2] A. Ambrosetti, M. Badiale, and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rational Mech. Anal., 140 (1997), pp. 285–300.

[3] N. Andre, P. Bauman, and D. Phillips, *Vortex pinning with bounded fields for the Ginzburg Landau equation*, Ann. Inst. H. Poincaré Anal. Non Linéare, 20 (2003), pp. 705–729.

[4] N. Andre and I. Shafrir, *Asymptotic behavior of minimizers for the GinzburgLandau functional with weight. I, II*, Arch. Rational Mech. Anal. 142, no. 1 (1998), pp. 45–73, 75–98.

[5] M. S. Berger and Y. Y. Chen, *Symmetric vortices for the nonlinear Ginzburg-Landau equations of superconductivity*, J. Funct. Anal., 82 (1989), pp. 259–295.

[6] A. Beaulieu and R. Hadiji, *Asymptotic behavior of minimizers of a GinzburgLandau equation with weight near their zeroes*, Asymptot. Anal. 22, no. 34 (2000), pp. 303–347.

[7] F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau Vortices*, Birkhauser, 1994.

[8] G. Blatter, M. V. Feigelman, et al., *Vortices in high temperature superconductors*, Rev. Mod. Phys., 66 (1994), p. 112.5.

[9] E. B. Bogomolńyi, *Stability of classical solutions*, Yad. Fiz., 24 (1976), pp. 861–870.

[10] S. J. Chapman, *Macroscopic models of superconductivity*, Proc. NATO ASI., 1999.

[11] ——, *Macroscopic models of superconductivity*, Proc. 4th ICIAM, 1999.

[12] S. J. Chapman, Q. Du, and M. D. Gunzburger, *A Ginzburg- Landau type model of superconducting/normal junctions including Josephson junctions*, Eur. J. Appl. Math., 6 (1995), pp. 97–114.

[13] S. J. Chapman and G. Richardson, *Vortex pinning by inhomogeneities in type-II superconductors*, Physica D, 108 (1997), pp. 397–407.

[14] J. Colliander and R. Jerrard, *Vortex dynamics for the Ginzburg-Landau Schrödinger equation*, Int. Math. Res. Not., 7 (1997), pp. 333–358.

[15] M. del Pino and P. Felmer, *Local mountain passes for semilinear elliptic problems on unbounded domains*, Calc. Var. PDEs, 4 (1996), pp. 121–137.

[16] M. del Pino and P. Felmer, *Multi-peak bound states for nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 15 (1998), pp. 127–149.

[17] S. Demoulini and D. Stuart, *Gradient flow of the superconducting Ginzburg-Landau functional of the plane*, Commun. Anal. Geom., 5 (1997), pp. 121–198.

[18] J. Diudonne, *Foundations of Modern Analysis*, Academic Press Inc., New York, USA, 1960.

[19] Q. Du, M. D. Gunzburger, and J. S. Peterson, *Computational simulations of type II superconductivity including pinning phenomena*, Phys. Rev. B, 51 (1995), pp. 16194–16203.

[20] W. E, *Dynamics of vortices in Ginzburg-Landau thories with applications to superconductivity*, Physica D, 77 (1994), pp. 384–404.

[21] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal., 69 (1986), pp. 397–408.

[22] S. Gustafson, *Dynamic stability of magnetic vortices*, Nonlinearity, 15 (2002), pp. 1717–1728.

[23] S. Gustafson and I. M. Sigal, *Stability of magnetic vortices*, Comm. Math. Phys., 212 (2000), pp. 257–275.

[24] ——, *Effective dynamics of magnetic vortices*, Adv. in Math., 199 (2006), pp. 448–498.

[25] S. Gustafson and F. S. T. Ting, *Dynamic stability and instability of pinned fundamental vortices*, 19 (2009), pp. 341–374. J. Nonlinear Science.

[26] C. Gui, *Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method*, Comm. in PDE, 21 (1997) pp. 787 – 820.

[27] J. Jaffe and C. Taubes, *Vortices and Monopoles*, Bikhäuser, Boston, USA, 1980.

[28] R. Jerrard and M. Soner, *Dynamics of Ginzburg-Landau vortices*, Arch. Rat. Mech. Anal., 142 (1998), pp. 99–125.

[29] S. Kim, PhD thesis, Texas A&M, Materials Science & Engineering, 2004.

[30] X. Kang and J. Wei, *On interacting bumps of semiclassical states of nonlinear Schrödinger equations*, Adv. Diff. Eqns., 5 (2000), pp. 899–928.

[31] F. H. Lin and W. M. Ni and J. Wei, *On the Number of Interior Peak Solutions for A Singularly Perturbed Neumann Problem*, Comm. Pure Appl. Math., 60, 2 (2007), pp. 252–281.

[32] F. H. Lin and J. Xin, *On the incompressible fluid limit and the vortex motion law of the nonlinear Schrödinger equation*, Comm. Math. Phys., 200 (1999), pp. 249–274.

[33] Y. G. Oh, *Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials of the class* $(V)_a$, Commun. Partial Diff. Eq., 13 (1988), pp. 1499–1519.

[34] ——, *Stability of semi-classical bound states of nonlinear Schrödinger equations with potentials*, Comm. Math. Phys., 121 (1989), pp. 11–33.

[35] ——, *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential*, Comm. Math. Phys., 131 (1990), pp. 223–253.

[36] L. Peres and J. Rubinstein, *Vortex dynamics in U(1) Ginzburg-Landau models*, Physica D, 64 (1993), pp. 299–309.

[37] B. Phlor, PhD thesis, Princeton University, 1978.

[38] T. Riviére, *Ginzburg-Landau vortices: the static model*, Séminaire Bourbaki, 1999/2000 (2002).

[39] ——, *Towards Jaffe and Taubes conjectures in the strongly repulsive limit*, Manuscripta Math, 108 (2002), pp. 217–273.

[40] J. Rubenstein, *Six lectures on superconductivity*, CRM Proc. and Lect. Notes, 13 (1998), pp. 163–184.

[41] E. Sandier and S. Serfaty, *Gamma-convergence of gradient flows with applications to Ginzburg-Landau*, Comm. Pure Appl. Math., 57.

[42] I. M. Sigal and Y. Strauss, *Effective dynamics of a magnetic vortex in a local potential*, Journal of Nonlinear Science, 16 (2006), pp. 123–157.

[43] I. M. Sigal and F. S. T. Ting, *Pinning of magnetic vortices by external potential*, Algebra i Analyz, 1 (2004), pp. 239–268, also reprinted in St. Petersburg Mathematical Journal, 16:1 (2005), pp. 211–236.

[44] D. Spirn, *Vortex dynamics of the full time-dependent Ginzburg-Landau equations*, Comm. Pure Appl. Math., 55 (2002), pp. 537–581.

[45] D. Stuart, *Dynamics of Abelian Higgs vortices in the near Bogomolny regime*, Comm. Math. Phys., 159 (1994), pp. 51–91.

[46] F. S. T. Ting, *Effective dynamics of multi-vortices in an external potential for Ginzburg-Landau gradient flow*, Nonlinearity, 23 (2010) pp. 179.

[47] M. Tinkham, *Introduction to Superconductivity*, McGraw-Hill, New York, USA, 1975.

[48] F. Ting and J. Wei, *Finite-energy degree-changing non-radial magnetic vortex solutions to the Ginzburg-Landau equations*, Submitted.

[49] J. Wei and M. Winter, *Existence, classification and stability analysis of multiple-peaked solutions for the Gierer-Meinhardt system in* $\mathbb{R}^1$, Methods Appl. Anal., 14 (2007), pp. 119–163.

[50] S. Yan and J. Wei, *Infinitely many positive solutions for the nonlinear Schrödinger equations in* $\mathbb{R}^N$, Cal.Var. PDE 37(2010), pp. 423–439.