INFINITELY MANY POSITIVE SOLUTIONS FOR AN NONLINEAR FIELD EQUATION WITH SUPER-CRITICAL GROWTH

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ABSTRACT. We consider the following nonlinear field equation with super critical growth:

$$(*)\begin{cases} -\Delta u + \lambda u = Q(y) u^{\frac{N+2}{N-2}},\ u > 0 & \text{in } \mathbb{R}^{N+m}, \\ u(y) \to 0 & \text{as } |y| \to +\infty, \end{cases}$$

where $m \geq 1$, $\lambda \geq 0$ and $Q(y)$ is a bounded positive function. We show that equation (*) has infinitely many positive solutions under certain symmetry conditions on $Q(y)$.

1. INTRODUCTION

In this paper, we consider the following nonlinear field equation with super critical growth:

$$\begin{cases} -\Delta u + \lambda u = Q(y) u^{\frac{N+2}{N-2}},\ u > 0 & \text{in } \mathbb{R}^{N+m}, \\ u(y) \to 0 & \text{as } |y| \to +\infty, \end{cases} \tag{1.1}$$

where $m \geq 1$, $\lambda \geq 0$ and $Q(y)$ is a bounded positive function. Note that $\frac{N+2}{N-2}$ is a super critical exponent in $\mathbb{R}^{N+m}$.

Equation (1.1) is a special case of the following problem:

$$\begin{cases} -\Delta u + \lambda u = Q(y) u^{p},\ u > 0 & \text{in } \mathbb{R}^{N}, \\ u(y) \to 0 & \text{as } |y| \to +\infty, \end{cases} \tag{1.2}$$

where $p > 1$. If $\lambda > 0$, (1.2) is the field equation. When $p < \frac{N+2}{N-2}$, existence of the ground state solution, or positive solutions with higher energy for the field equations were considered in [4, 5, 7, 13, 20, 26, 27]. If $\lambda = 0$ and $p = \frac{N+2}{N-2}$, (1.2) is the prescribed scalar curvature problem on $\mathbb{S}^N$, which was studied extensively in the last thirty years. See for example [2, 3, 6, 8, 9, 10, 11, 12, 21, 22, 23, 24, 25, 38, 40] and the references therein. For the super-critical case, no result is known for (1.2), except for the case $\lambda = 0$ and p is very close to $\frac{N+2}{N-2}$. See [39].

The aim of this paper is to prove that under some conditions on Q, (1.1) has infinitely many positive solutions. We will achieve this goal by constructing solutions concentrating along large number of m-dimensional manifolds. As far as we know, this seems to be the first result on the infinite multiplicity for nonlinear field equation with super crtical growth.

To simplify (1.1), we impose the following symmetry condition on $Q(y)$:

(Q): Suppose that $Q(y)=Q(r,d)$, where $y=(y^*,y^{**})$, $y^*\in\mathbb{R}^{N-1}$, $y^{**}\in\mathbb{R}^{m+1}$, $r=|y^*|$ and $d=|y^{**}|$.

In this paper, we assume that there is a pair (r_0,d_0), $r_0>0$, $d_0>0$, such that (r_0,d_0) is a non-degenerate critical point of the function $\frac{d^m}{Q^{\frac{N-2}{2}}(r,d)}$. That is, the pair (r_0,d_0) satisfies

$$Q_r(r_0,d_0)=0, \tag{1.3}$$

$$\frac{N-2}{2}Q_d(r_0,d_0)=\frac{mQ(r_0,d_0)}{d_0}, \tag{1.4}$$

and that the 2×2 matrix $(D^2\frac{d^m}{Q^{\frac{N-2}{2}}(r,d)})$ at (r_0,d_0) is invertible. Moreover, we assume

$$\begin{aligned}&\lambda-\frac{m}{d_0^2}\frac{3(N-2)}{4(N-1)}-\frac{(N-4)(N-2)}{8(N-1)}\frac{Q_{rr}(r_0,d_0)+Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}\\&+\frac{m^2}{d_0^2}\frac{5N-14}{4(N-1)(N-2)}>0.\end{aligned} \tag{1.5}$$

Our main result in this paper can be stated as follows:

Theorem 1.1. *Suppose that $N\ge 5$ and Q satisfies the symmetry condition (Q). Assume that the function $\frac{d^m}{Q^{\frac{N-2}{2}}(r,d)}$ has a non-degenerate critical point (r_0,d_0), satisfying $r_0>0$, $d_0>0$, and (1.5). Then problem (1.1) has infinitely many distinct positive solutions.*

In the end of the introduction, let us outline the proof of Theorem 1.1. We will construct solutions which concentrate at large number of spheres and we will use the number of the sphere as the parameter in the construction. This technique was first developed in [35] to study the prescribed scalar curvature problem and then was used to study other elliptic problems [33, 34, 36, 37].

Denote $2^*=\frac{2N}{N-2}$. It is well-known that the functions

$$\big(N(N-2)\big)^{\frac{N-2}{4}}\left(\frac{\mu}{1+\mu^2|y-x|^2}\right)^{\frac{N-2}{2}},\ \mu>0,\ x\in\mathbb{R}^N$$

are the only solutions to the following problem

$$-\Delta u=u^{2^*-1},\ u>0 \text{ in } \mathbb{R}^N.$$

For any positive integer k, let

$$\Gamma_j = \left(r_0 \cos \frac{2(j-1)\pi}{k}, r_0 \sin \frac{2(j-1)\pi}{k}, 0, \cdots, 0, y^{**}\right), \quad |y^{**}| = d_0.$$

Then Γ_j is an m-dimensional sphere in $\mathbb{R}^{N+m}$. Near Γ_j, we denote

$$w_j(y) = \big(N(N-2)\big)^{\frac{N-2}{4}} Q^{-\frac{N-2}{4}}(r_0, d_0) \left(\frac{\mu}{1 + \mu^2 |y^* - \tilde{x}_j|^2 + \mu^2 (|y^{**}| - d_0)^2} \right)^{\frac{N-2}{2}},$$

where

$$\tilde{x}_j = \left(r_0 \cos \frac{2(j-1)\pi}{k}, r_0 \sin \frac{2(j-1)\pi}{k}, 0, \cdots, 0\right) \in \mathbb{R}^{N-1}.$$

Then, if μ is large, w_j is a function concentrating at the sphere Γ_j. In this paper, we will prove that for $k > 0$ large, (1.1) has a solution u_i with

$$u_i \approx \sum_{j=1}^{k} w_j,$$

for some large μ.

If $m = 0$, it was shown in [35] that if the potential Q has a non-degenerate local minimum point $(r_0, 0)$ with $r_0 > 0$, one can construct positive solutions with large number of bubbles. These conditions correspond to (1.3), (1.4) and (1.5) in the case $m = 0$. But there are some striking differences between the case $m = 0$ and $m > 0$. When $m = 0$, the construction can be carried out starting from w_j as first approximation of the solution. In fact, in this case, the condition $DQ(r_0, 0) = 0$ gives that this approximation is good enough to construct bubbling solutions. On the other hand, when $m > 0$, we have $Q_d(r_0, d_0) \neq 0$. This is the reason why, in this case, if we just consider w_j as starting point for our construction, the error of the approximation is too big, being of order μ^{-1}. This implies that the contribution from the perturbation would be of order μ^{-2}. The calculations show that one needs to find the exact formula up to order μ^{-2} in order to be able to determine the concentration rate μ of the bubbling solutions. This fact forces us to find explicitly the second term in the expansion of the approximate solution in the case $m > 0$. Indeed, the second term in the expansion of the approximate solution is not negligible and it contributes to the constant in the left hand side of (1.5).

Using the symmetry of the function Q, we will look for solution of the form $u = u(y^*, |y^{**}|)$. Another major difference between the case $m = 0$ and $m > 0$ is that if $m > 0$, though the problem can be reduced to a problem in $\mathbb{R}^N$, the reduced equation has singularity at $|y^{**}|=0$. This will create some difficulty in carrying out the reduction argument. Note that in [37, 1], to avoid such difficulty, it requires that the domain does not intersect the hyper-plane $y^{**}=0$. Arguing in the same way as in this paper, we can get rid of this condition imposed on the domains in [37, 1].

The existence of positive solutions for elliptic problems involving super-critical nonlinearities is a very delicate problem. See for example [1, 14, 15, 16, 19, 31, 32, 37] for the results for problems in bounded domains. In [1, 16, 17, 37], existence results were obtained by constructing solutions concentrating at curves or higher dimensional manifolds at higher critical Sobolev exponents in bounded domains. The readers can find other results on solutions concentrating at curves or higher dimensional manifolds in [18, 28, 29, 30].

This paper is arranged as follows. In section 2, we will construct the second approximate of the solutions. In section 3, we will find the equations which determine the location and the concentration rate of the bubbles by neglecting the perturbation term. Section 4 is devoted to the study of a linear problem in a cylinder which plays a crucial role in carrying out the reduction argument in Section 5. Theorem 1.1 is proved in Section 6. We put all the other estimates needed in the proof of the main theorem to appendixes.

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2. The approximate solutions near a given curve

In this section, we construct approximate solutions for (1.1) concentrating at the m-dimensional sphere Γ_j, where

$$\Gamma_j=\big(r_0\cos\frac{2(j-1)\pi}{k},r_0\sin\frac{2(j-1)\pi}{k},0,\cdots,0,y^{**}\big),\quad |y^{**}|=d_0.$$

By the rotational invariance of the problem, we just need to construct an approximate solutions for (1.1) concentrating at Γ_1.

Using the symmetry condition on Q, we can look for solutions of the form $u(y^*,|y^{**}|)$ for (1.1). Then $u(y^*,t)$, $t=|y^{**}|$, satisfies

$$-\Delta u-\frac{m}{t}u_t+\lambda u=Q(|y^*|,t)u^{\frac{N+2}{N-2}}, \tag{2.1}$$

where Δ is the Laplace operator for $(y_1,\cdots,y_{N-1},t)$.

Remark 2.1. *Equation* (2.1) *can be written as the following divergent form*

$$-div(t^m Du) + \lambda t^m u = t^m Q(|y^*|, t) u^{\frac{N+2}{N-2}}. \tag{2.2}$$

For large positive integer k, we let $\varepsilon = k^{-\frac{N-2}{N-4}}$. We make the change of variable $\varepsilon^{\frac{N-2}{2}} u(\varepsilon y^*, \varepsilon t)$. Then (2.1) becomes

$$-\Delta u - \frac{m}{t} u_t + \varepsilon^2 \lambda u = Q(\varepsilon |y^*|, \varepsilon t) u^{\frac{N+2}{N-2}}. \tag{2.3}$$

To simplify the notations in the calculations, we use y_N to replace t in (2.3). So we have

$$-\Delta u - \frac{m}{y_N} u_{y_N} + \varepsilon^2 \lambda u = Q(\varepsilon y^*, \varepsilon y_N) u^{\frac{N+2}{N-2}}. \tag{2.4}$$

For any constant $\beta > 0$, let

$$\alpha = Q^{-\frac{N-2}{4}}(r_0, d_0) \beta^{\frac{N-2}{2}}. \tag{2.5}$$

We define $v(x)$ by the relation

$$u(y) = \alpha v(x), \tag{2.6}$$

where $y = (y_1, \cdots, y_N)$,

$$x_i = \beta y_i, \quad i = 2, \cdots, N-1,$$

$$x_1 = \beta(y_1 - \frac{r_0}{\varepsilon} - f_1), \quad x_N = \beta(y_N - \frac{d_0}{\varepsilon} - f_N),$$

and f_1 and f_N are small parameters.

We expand

$$\begin{aligned} Q(\varepsilon y) =& Q(r_0, d_0) + Q_d(r_0, d_0)\varepsilon \left(\frac{x_N}{\beta} + f_N \right) \\ &+ \frac{1}{2}\varepsilon^2 D^2 Q(r_0, d_0)(\frac{x}{\beta} + f, \frac{x}{\beta} + f) + O\big(\varepsilon^3 |y|^3\big), \end{aligned}$$

since $Q_r(r_0, d_0) = 0$. Here we let $f_i = 0$, $i = 2, \cdots, N-1$.

Let

$$\mathbb{S}(u) = \Delta u + \frac{m}{y_N} u_{y_N} - \varepsilon^2 \lambda u + Q(\varepsilon y) u^{2^*-1}. \tag{2.7}$$

Then $\mathbb{S}(u) = Q^{-\frac{N-2}{4}}(r_0, d_0)\beta^{\frac{N+2}{2}} S(v)$, where

$$S(v) \equiv B_1(v) + B_2(v) + \Delta v + v^{2^*-1}. \tag{2.8}$$

Here Δ denotes the Laplace operator for y, $B_1(v)$ is a linear differential operator defined by

$$B_1(v) = -\beta^{-2}\varepsilon^2\lambda v + \frac{m\varepsilon}{\beta d_0 + \varepsilon x_N + \varepsilon\beta f_N} v_{x_N}, \tag{2.9}$$

and

$$\begin{aligned} B_2(v) = &\left[\varepsilon \frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}\left(\frac{x_N}{\beta} + f_N\right) + \frac{\varepsilon^2}{2}\frac{D^2Q(r_0,d_0)}{Q(r_0,d_0)}\left(\frac{x}{\beta}+f, \frac{x}{\beta}+f\right)\right] v^{2^*-1} \\ &+ O\big(\varepsilon^3|x|^3|v|^{2^*-1}\big). \end{aligned} \tag{2.10}$$

The major term of an approximate solution concentrating along $\Gamma_{\varepsilon,1} := \frac{\Gamma_1}{\varepsilon}$ is given by

$$w(x) = \big(N(N-2)\big)^{\frac{N-2}{4}}\left(\frac{1}{1+|x|^2}\right)^{\frac{N-2}{2}}.$$

We have

$$\begin{aligned} S(w) =& B_2(w) + B_3(w) \\ =& \beta^{-1}\left[\frac{\varepsilon m}{d_0} - \frac{\varepsilon^2 m}{d_0^2}\left(\frac{x_N}{\beta} + f_N\right)\right] w_{x_N} - \beta^{-2}\varepsilon^2\lambda v \\ &+ \left[\varepsilon\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}\left(\frac{x_N}{\beta}+f_N\right) + \frac{\varepsilon^2}{2}\frac{D^2Q(r_0,d_0)}{Q(r_0,d_0)}\left(\frac{x}{\beta}+f, \frac{x}{\beta}+f\right)\right] w^{2^*-1} \\ &+ O\big(\varepsilon^3|x|^3|w|^{2^*-1} + \varepsilon^3|x|^2|w_{y_N}|\big). \end{aligned} \tag{2.11}$$

Write

$$S(w) = \varepsilon S_1 + \varepsilon S_2 + \varepsilon^2 S_3 + \varepsilon^2 S_4 + O\big(\varepsilon^3|x|^3|w|^{2^*-1} + \varepsilon^3|x|^2|w_{y_N}|\big),$$

where

$$S_1 = \beta^{-1}\big(\frac{m w_{x_N}}{d_0} + \frac{Q_d(r_0,d_0)}{Q(r_0,d_0)} x_N w^{2^*-1}\big), \quad S_2 = \frac{Q_d(r_0,d_0)}{Q(r_0,d_0)} f_N w^{2^*-1}, \tag{2.12}$$

$$S_3 = -\frac{m f_N w_{x_N}}{\beta d_0^2} + \frac{D^2Q(r_0,d_0)}{Q(r_0,d_0)}\left(\frac{x}{\beta}, f\right) w^{2^*-1} \tag{2.13}$$

and

$$
\begin{aligned}
S_4 = & -\frac{mx_N w_{x_N}}{\beta^2 d_0^2} - \beta^{-2}\lambda w \\
& + \frac{1}{2}\Big(\frac{D^2Q(r_0,d_0)}{\beta^2 Q(r_0,d_0)}(x,x) + \frac{D^2Q(r_0,0,\varepsilon z)}{Q(r_0,d_0)}(f,f)\Big) w^{2^*-1}.
\end{aligned} \tag{2.14}
$$

Note that S_1 and S_3 are odd, while S_2 and S_4 are even.

Next, we will make a correction for the first approximate w. We will choose ϕ, such that $S(w+\varepsilon\phi)$ is of order ε^2. For this aim, we calculate

$$
\begin{aligned}
S(w+\varepsilon\phi) = & B_2(w+\varepsilon\phi) + B_3(w+\varepsilon\phi) + \varepsilon L_0(\phi) + N_0(\varepsilon\phi) \\
= & S(w) + \varepsilon L_0(\phi) + N_0(\varepsilon\phi) + B_2(w+\varepsilon\phi) - B_2(w) + \varepsilon B_3(\phi),
\end{aligned}
$$

where

$$
L_0(\phi) = \Delta\phi + (2^*-1)w^{2^*-2}\phi \tag{2.15}
$$

and

$$
N_0(\phi) = (w+\phi)^{2^*-1} - w^{2^*-1} - (2^*-1)w^{2^*-2}\phi. \tag{2.16}
$$

Consider the following problem:

$$
-\Delta\phi - (2^*-1)w^{2^*-2}\phi = S_1 + S_2. \tag{2.17}
$$

We can write down the solutions for (2.17). First, we consider

$$
\begin{aligned}
& -\Delta\phi - (2^*-1)w^{2^*-2}\phi = S_1 = \beta^{-1}\big(\frac{mw_{x_N}}{d_0} + \frac{Q_d(r_0,d_0)}{Q(r_0,d_0)} x_N w^{2^*-1}\big) \\
= & \beta^{-1}\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)} x_N \big(\frac{N-2}{2}\frac{1}{r}w' + w^{2^*-1}\big),
\end{aligned} \tag{2.18}
$$

where $r=|x|$. For (2.18), we let $\phi = \beta^{-1}\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)} x_N \varphi(r)$. Then

$$
-\varphi'' - \frac{N+1}{r}\varphi' - (2^*-1)w^{2^*-2}\varphi = \frac{N-2}{2}\frac{1}{r}w' + w^{2^*-1}. \tag{2.19}
$$

It is easy to check that

$$
\varphi = -\frac{N-2}{4}w \tag{2.20}
$$

is a solution of (2.19). So,

$$
\phi = -\frac{N-2}{4}\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}\beta^{-1} x_N w \tag{2.21}
$$

is a solution of (2.18).

Now, we consider

$$-\Delta\phi-(2^*-1)w^{2^*-2}\phi=S_2=\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}f_N w^{2^*-1}. \tag{2.22}$$

So, it is easy to see

$$\phi=-\frac{N-2}{4}\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}f_N w \tag{2.23}$$

is a solution of (2.22).

Combining (2.21) and (2.23), we find that

$$\begin{aligned}\phi=&-\frac{N-2}{4}\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}\beta^{-1}x_N w-\frac{N-2}{4}\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}f_N w\\ =&-\frac{m}{2d_0}\beta^{-1}x_N w-\frac{m}{2d_0}f_N w\end{aligned} \tag{2.24}$$

is a solution of (2.17).

Let ϕ be defined in (2.24). Then

$$\begin{aligned}S(w+\varepsilon\phi)=&\varepsilon^2 S_3+\varepsilon^2 S_4+B_2(w+\varepsilon\phi)-B_2(w)+\varepsilon B_3(\phi)\\ &+N_0(\varepsilon\phi)+O\big(\varepsilon^3|x|^3|w|^{2^*-1}+\varepsilon^3|x|^2|w_{y_N}|\big).\end{aligned} \tag{2.25}$$

The approximate solution for $S(v)=0$, where $S(v)$ is defined in (2.8), near $\Gamma_{\varepsilon,1}$ to be

$$w+\varepsilon\phi. \tag{2.26}$$

To avoid the possible singularity of the term $\frac{m\varepsilon}{\beta d_0+\varepsilon x_N+\varepsilon\beta f_N}$, we further modify the approximate solution as follows. Let $\xi(s)$ be a function such that $\xi=1$ if $0\le|s|\le\frac{1}{2}\delta$, $\xi=0$ if $|s|\ge\delta$, and $0\le\xi\le1$. Define

$$\mathtt{w}(x)=\xi(\varepsilon x_N)(w+\varepsilon\phi)\ . \tag{2.27}$$

For any function ψ, let

$$\psi_{f,\beta}(y)=\alpha\psi\big(\beta(y_1-\frac{r_0}{\varepsilon}-f_1),\beta y_2,\cdots,\beta y_{N-1},\beta(y_N-\frac{d_0}{\varepsilon}-f_N)\big).$$

Define

$$W_k(y)=\sum_{j=1}^{k}\mathtt{w}_{f,\beta}((R_k^j)^{-1}y), \tag{2.28}$$

where R_k^j is the rotation of angle $\frac{2j\pi}{k}$ in the y_1y_2 plane, $j=1,\cdots,k$. We will use W_k as an approximate solution for (2.3).

Using the symmetry condition on the function Q, we introduce the following space:

$$H_s=\Big\{u:u=u(y^*,|y^{**}|),\ \ u(R_k\cdot)=u(\cdot), u \text{ is even in } y_h, h=2,\cdots,N-1$$
$$\int_{\mathbb{R}^{N+m}}\big(|Du|^2+\lambda u^2\big)<+\infty\Big\}.$$

It is easy to check that $W_k\in H_s$.

Theorem 1.1 is a direct consequence of the following result:

Theorem 2.2. *Under the same conditions as in Theorem 1.1, there exists a large constant $k_0>0$, such that for all $k\geq k_0$, (1.1) has a solution $u_k\in H_s$ satisfying*

$$u_k=W_k+o(1) \tag{2.29}$$

where $o(1)\to 0$ uniformly in $\mathbb{R}^{N+m}$ as $k\to\infty$.

3. Key Estimates

Let $\mathtt{w}$ be the approximate solution defined in (2.27). In this section, we will find the equations that determine the parameters in the approximate solutions.

Proposition 3.1. *We have the following estimates:*

$$\varepsilon^{-2}\int_{\mathbb{R}^N}S(\mathtt{w})\big(\frac{N-2}{2}w+xDw\big)=\frac{D}{\beta^2}+O\big(|f|^2+\varepsilon\big), \tag{3.1}$$

$$\varepsilon^{-2}\int_{\mathbb{R}^N}S(\mathtt{w})\frac{\partial w}{\partial x_1}=-\Big(\frac{Q_{rr}(r_0,d_0)}{Q(r_0,d_0)}f_1+\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}f_N\Big)\frac{1}{2^*\beta}\int_{\mathbb{R}^N}w^{2^*}+O(\varepsilon), \tag{3.2}$$

and

$$\begin{aligned}\varepsilon^{-2}\int_{\mathbb{R}^N}S(\mathtt{w})\frac{\partial w}{\partial x_N}=&-f_1\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\beta^{-1}\int_{\mathbb{R}^N}w^{2^*}\\&-f_N\big(\frac{m}{Nd_0^2}-\frac{2}{N(N-2)}\frac{m^2}{d_0^2}+\frac{Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\big)\beta^{-1}\int_{\mathbb{R}^N}w^{2^*}+O(\varepsilon)\end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
D =&\lambda B_2 - \frac{m}{d_0^2}\frac{3(N-2)}{4(N-1)}B_2 - \frac{(N-4)(N-2)}{8(N-1)}\frac{Q_{rr}(r_0,d_0)+Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}B_2 \\
&+ \frac{m^2}{d_0^2}\frac{5N-14}{4(N-1)(N-2)}B_2.
\end{aligned} \tag{3.4}
$$

Proof. Recall that

$$
\begin{aligned}
&S(w+\varepsilon\phi) \\
=&\varepsilon^2 S_3 + \varepsilon^2 S_4 + B_2(w+\varepsilon\phi) - B_2(w) + \varepsilon B_3(\phi) \\
&+ (w+\varepsilon\phi)^{2^*-1} - w^{2^*-1} - (2^*-1)w^{2^*-2}\varepsilon\phi \\
&+ O\big(\varepsilon^3|x|^3|w|^{2^*-1} + \varepsilon^3|x|^2|w_{y_N}|\big) \\
=&\varepsilon^2 S_3 + \varepsilon^2 S_4 + \varepsilon^2\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}\big(\frac{x_N}{\beta} + f_N\big)(2^*-1)w^{2^*-2}\phi \\
&+ \frac{\varepsilon^2 m}{\beta d_0}\phi_{x_N} + \varepsilon^2\frac{(2^*-1)(2^*-2)}{2}w^{2^*-3}\phi^2 \\
&+ O\big(\varepsilon^3|x|^3|w|^{2^*-1} + \varepsilon^3|x|^2|w_{y_N}|\big).
\end{aligned} \tag{3.5}
$$

Since $\frac{N-2}{2}w + xDw$ is even, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} S(\mathtt{w})\big(\frac{N-2}{2}w + xDw\big) \\
=&\int_{\mathbb{R}^N} S(w+\varepsilon\phi)\big(\frac{N-2}{2}w + xDw\big) + O(\varepsilon^3) \\
=&\varepsilon^2\int_{\mathbb{R}^N} S_4\big(\frac{N-2}{2}w + xDw\big) \\
&+ \varepsilon^2\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}\int_{\mathbb{R}^N}\big(\frac{x_N}{\beta} + f_N\big)(2^*-1)w^{2^*-2}\phi\big(\frac{N-2}{2}w + xDw\big) \\
&+ \varepsilon^2\frac{m}{\beta d_0}\int_{\mathbb{R}^N}\phi_{x_N}\big(\frac{N-2}{2}w + xDw\big) \\
&+ \varepsilon^2\int_{\mathbb{R}^N}\frac{(2^*-1)(2^*-2)}{2}w^{2^*-3}\phi^2\big(\frac{N-2}{2}w + xDw\big) + O(\varepsilon^3).
\end{aligned} \tag{3.6}
$$

From

$$
S_4 = -\frac{m x_N w_{x_N}}{\beta^2 d_0^2} + \frac{1}{2}\Big(\frac{D^2Q(r_0,d_0)}{Q(r_0,d_0)\beta^2}(x,x) + \frac{D^2Q(r_0,d_0)}{Q(r_0,d_0)}(f,f)\Big)w^{2^*-1} - \beta^{-2}\lambda w, \tag{3.7}
$$

we find

$$\int_{\mathbb{R}^N} S_4\big(\frac{N-2}{2}w + xDw\big) = \frac{\bar{D}}{\beta^2} + O\big(|f|^2\big), \tag{3.8}$$

where, in view of (A.1), (A.3) and (A.6),

$$\begin{aligned}\bar{D} =& -\lambda\int_{\mathbb{R}^N} w\big(\frac{N-2}{2}w + xDw\big) - \frac{m}{d_0^2}\int_{\mathbb{R}^N} x_N w_{x_N}\big(\frac{N-2}{2}w + xDw\big)\\ &+ \frac{1}{2}\int_{\mathbb{R}^N}\frac{D^2Q(r_0,d_0)}{Q(r_0,d_0)}(x,x)w^{2^*-1}\big(\frac{N-2}{2}w + xDw\big)\\ =&\lambda B_2 - \frac{m}{d_0^2}\frac{3(N-2)}{4(N-1)}B_2 - \frac{(N-4)(N-2)}{8(N-1)}\frac{\Delta Q(r_0,d_0)}{Q(r_0,d_0)}B_2,\end{aligned} \tag{3.9}$$

and $B_2 = \int_{\mathbb{R}^N} w^2$.

¿From (1.3) and (1.4), we find

$$\begin{aligned}\Delta Q(r_0,d_0) =& Q_{rr}(r_0,d_0) + \frac{N-2}{r_0}Q_r(r_0,d_0) + Q_{dd}(r_0,d_0) + \frac{m}{d_0}Q_d(r_0,d_0)\\ =& Q_{rr}(r_0,d_0) + Q_{dd}(r_0,d_0) + \frac{2}{N-2}\frac{m^2}{d_0^2}Q(r_0,d_0).\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{D} =& \lambda B_2 - \frac{m}{d_0^2}\frac{3(N-2)}{4(N-1)}B_2 - \frac{m^2}{d_0^2}\frac{N-4}{4(N-1)}B_2\\ &- \frac{(N-4)(N-2)}{8(N-1)}\frac{Q_{rr}(r_0,d_0) + Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}B_2.\end{aligned} \tag{3.10}$$

On the other hand,

$$\begin{aligned}&\frac{m}{\beta d_0}\int_{\mathbb{R}^N}\phi_{x_N}\big(\frac{N-2}{2}w + xDw\big)\\ =&\frac{m}{\beta d_0}\int_{\mathbb{R}^N}\big(-\frac{m}{2d_0}\beta^{-1}x_N w - \frac{m}{2d_0}f_N w\big)_{x_N}\big(\frac{N-2}{2}w + xDw\big)\\ =& -\beta^{-2}\frac{m^2}{2d_0^2}\int_{\mathbb{R}^N}\big(w + x_N w_{x_N}\big)\big(\frac{N-2}{2}w + xDw\big)\\ =&\beta^{-2}\frac{m^2}{d_0^2}\frac{N+2}{8(N-1)}B_2,\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}\int_{\mathbb{R}^N}\big(\frac{x_N}{\beta}+f_N\big)(2^*-1)w^{2^*-2}\phi\big(\frac{N-2}{2}w+xDw\big)\\
=&\frac{2(N+2)}{(N-2)^2}\frac{m}{d_0}\beta^{-1}\int_{\mathbb{R}^N}x_Nw^{2^*-2}\big(-\frac{1}{2}\frac{m}{d_0}\beta^{-1}x_Nw\big)\big(\frac{N-2}{2}w+xDw\big)+O(|f|^2)\\
=&\frac{m^2}{d_0^2}\beta^{-2}\frac{N+2}{(N-2)^2}\frac{(N-2)(N-4)}{4(N-1)}B_2+O(|f|^2),
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\frac{(2^*-1)(2^*-2)}{2}w^{2^*-3}\phi^2\big(\frac{N-2}{2}w+xDw\big)\\
=&\int_{\mathbb{R}^N}\frac{(2^*-1)(2^*-2)}{2}w^{2^*-3}\big(-\frac{1}{2}\frac{m}{d_0}\beta^{-1}x_Nw\big)^2\big(\frac{N-2}{2}w+xDw\big)+O(|f|^2)\\
=&-\frac{N+2}{2(N-2)^2}\frac{m^2}{d_0^2}\beta^{-2}\frac{(N-4)(N-2)}{4(N-1)}B_2+O(|f|^2).
\end{aligned}
\tag{3.13}
$$

So,

$$
\begin{aligned}
&\beta^{-1}\frac{m}{d_0}\int_{\mathbb{R}^N}\phi_{x_N}\big(\frac{N-2}{2}w+xDw\big)\\
&+\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}\int_{\mathbb{R}^N}\big(\frac{x_N}{\beta}+f_N\big)(2^*-1)w^{2^*-2}\phi\big(\frac{N-2}{2}w+xDw\big)\\
&+\int_{\mathbb{R}^N}\frac{(2^*-1)(2^*-2)}{2}w^{2^*-3}\phi^2\big(\frac{N-2}{2}w+xDw\big)\\
=&\frac{m^2}{d_0^2}\beta^{-2}\Big(\frac{(N+2)(N-4)}{8(N-1)(N-2)}+\frac{N+2}{8(N-1)}\Big)B_2+O(|f|^2)\\
=&\frac{m^2}{d_0^2}\beta^{-2}\frac{(N+2)(N-3)}{4(N-1)(N-2)}B_2+O(|f|^2).
\end{aligned}
\tag{3.14}
$$

Thus, using (3.6), (3.8), (3.10) and (3.14), we find

$$
\varepsilon^{-2}\int_{\mathbb{R}^N}S(w+\varepsilon\phi)\big(\frac{N-2}{2}w+xDw\big)=\frac{D}{\beta^2}+O(|f|^2),
\tag{3.15}
$$

where

$$
\begin{aligned}
D=&\lambda B_2-\frac{m}{d_0^2}\frac{3(N-2)}{4(N-1)}B_2-\frac{(N-4)(N-2)}{8(N-1)}\frac{Q_{rr}(r_0,d_0)+Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}B_2\\
&+\frac{m^2}{d_0^2}\frac{5N-14}{4(N-1)(N-2)}B_2.
\end{aligned}
\tag{3.16}
$$

So (3.1) follows from (3.15) and (3.6).

We now compute $\int_{\mathbb{R}^N} S(w+\varepsilon\phi)w_{x_i}$. Let us compute $\int_{\mathbb{R}^N} S_3 w_{x_i}$. Recall

$$S_3 = -\frac{mf_N w_{x_N}}{\beta d_0^2} + \frac{D^2Q(r_0,d_0)}{Q(r_0,d_0)}\left(\frac{x}{\beta}, f\right) w^{2^*-1}. \tag{3.17}$$

For $i=1$,

$$\begin{aligned}\int_{\mathbb{R}^N} S_3 w_{x_1} =&\Big(\frac{Q_{rr}(r_0,d_0)}{Q(r_0,d_0)}\frac{f_1}{\beta} + \frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}\frac{f_N}{\beta}\Big)\int_{\mathbb{R}^N} x_1 w_{x_1} w^{2^*-1}\\ =&-\Big(\frac{Q_{rr}(r_0,d_0)}{Q(r_0,d_0)}f_1 + \frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}f_N\Big)\frac{1}{2^*\beta}\int_{\mathbb{R}^N} w^{2^*}.\end{aligned} \tag{3.18}$$

For $i=N$,

$$\begin{aligned}\int_{\mathbb{R}^N} S_3 w_{x_N} =&-f_N\big(\frac{m}{N\beta d_0^2} + \frac{Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*\beta}\big)\int_{\mathbb{R}^N} w^{2^*}\\ &-f_1\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*\beta}\int_{\mathbb{R}^N} w^{2^*}.\end{aligned} \tag{3.19}$$

On the other hand,

$$\begin{aligned}&\beta^{-1}\int_{\mathbb{R}^N}\frac{m}{d_0}\phi_{x_N}w_{x_i} = \beta^{-1}\int_{\mathbb{R}^N}\frac{m}{d_0}\big(-\frac{1}{2}\frac{m}{d_0}\beta^{-1}x_N w - \frac{1}{2}\frac{m}{d_0}f_N w\big)_{x_N} w_{x_i}\\ =&-\beta^{-1}\frac{m^2}{2d_0^2}f_N\int_{\mathbb{R}^N} w_{x_N}w_{x_i} = -f_N\beta^{-1}\frac{m^2}{2d_0^2}\frac{\delta_{iN}}{N}\int_{\mathbb{R}^N} w^{2^*},\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}(2^*-1)\int_{\mathbb{R}^N}\big(\frac{x_N}{\beta}+f_N\big)w^{2^*-2}\phi w_{x_i}\\ =&\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}(2^*-1)\int_{\mathbb{R}^N}\big(-\frac{x_N}{\beta}\frac{1}{2}\frac{m}{d_0}f_N w - f_N\frac{1}{2}\frac{m}{d_0}\beta^{-1}x_N w\big)w^{2^*-2}w_{x_i}\\ =&f_N\frac{Q_d(r_0,d_0)}{Q(r_0,d_0)}(2^*-1)\frac{m}{d_0}\beta^{-1}\frac{\delta_{iN}}{2^*}\int_{\mathbb{R}^N} w^{2^*}\\ =&f_N\frac{m^2}{d_0^2}\beta^{-1}\frac{(N+2)\delta_{iN}}{N(N-2)}\int_{\mathbb{R}^N} w^{2^*},\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{(2^*-1)(2^*-2)}{2}w^{2^*-3}\phi^2 w_{x_i}\\ =&\frac{(2^*-1)(2^*-2)}{2}\int_{\mathbb{R}^N} w^{2^*-3}\frac{1}{2}\frac{m^2}{d_0^2}\beta^{-1}x_N w^2 f_N w_{x_i}\\ =&-f_N\frac{N+2}{2(N-2)N}\frac{m^2}{d_0^2}\beta^{-1}\delta_{iN}\int_{\mathbb{R}^N} w^{2^*}.\end{aligned} \tag{3.22}$$

So, we obtain

$$\varepsilon^{-2}\int_{\mathbb{R}^N} S(w+\varepsilon\phi)w_{x_1} = -\Big(\frac{Q_{rr}(r_0,d_0)}{Q(r_0,d_0)}f_1 + \frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}f_N\Big)\frac{1}{2^*\beta}\int_{\mathbb{R}^N} w^{2^*} + O(\varepsilon), \quad (3.23)$$

and

$$\begin{aligned}&\varepsilon^{-2}\int_{\mathbb{R}^N} S(w+\varepsilon\phi)w_{x_N}\\ =&-f_1\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\beta^{-1}\int_{\mathbb{R}^N} w^{2^*}\\ &-f_N\big(\frac{m}{Nd_0^2}-\frac{2}{N(N-2)}\frac{m^2}{d_0^2}+\frac{Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\big)\beta^{-1}\int_{\mathbb{R}^N} w^{2^*}+O(\varepsilon).\end{aligned} \quad (3.24)$$

□

Recall that

$$\mathbb{S}(u) = \Delta u + \frac{m}{y_N}u_{y_N} - \varepsilon^2\lambda u + Q(\varepsilon y)u^{2^*-1} = Q^{-\frac{N-2}{4}}(r_0,d_0)\beta^{\frac{N+2}{2}}S(v), \quad (3.25)$$

and for any function $\psi_{f,\beta}$, we use the notation

$$\psi_{f,\beta}(y) = \alpha\psi\big(\beta(y_1-\frac{r_0}{\varepsilon}-f_1),\beta y_2,\cdots,\beta y_{N-1},\beta(y_N-\frac{d_0}{\varepsilon}-f_N)\big),$$

and $\xi(s)$ is a function such that $\xi=1$ if $0\le|s|\le\frac{1}{2}\delta$, $\xi=0$ if $|s|\ge\delta$, and $0\le\xi\le1$.

To find the equations that determine the parameters f and β, we need the following result.

Proposition 3.2. *We have the following estimates:*

$$\begin{aligned}&\varepsilon^{-2}(Q(r_0,d_0))^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\mathbb{S}(W_m)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial\beta}\\ =&\frac{D}{\beta^2}-\frac{B}{\beta^{N-2}}+O\big(|f|^2+\varepsilon\big),\end{aligned} \quad (3.26)$$

$$\begin{aligned}&\varepsilon^{-2}(Q(r_0,d_0))^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\mathbb{S}(W_m)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial x_1}\\ =&-\Big(\frac{Q_{rr}(r_0,d_0)}{Q(r_0,d_0)}f_1+\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}f_N\Big)\frac{1}{2^*\beta}\int_{\mathbb{R}^N} w^{2^*}+O(\varepsilon),\end{aligned} \quad (3.27)$$

and

$$
\begin{aligned}
&\varepsilon^{-2}(Q(r_0,d_0))^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\mathbb{S}(W_m)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial x_N}\\
=&-f_1\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\beta^{-1}\int_{\mathbb{R}^N}w^{2^*}\\
&-f_N\Big(\frac{m}{Nd_0^2}-\frac{2}{N(N-2)}\frac{m^2}{d_0^2}+\frac{Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\Big)\beta^{-1}\int_{\mathbb{R}^N}w^{2^*}+O(\varepsilon),
\end{aligned}
\tag{3.28}
$$

where the constant D is the same as in Proposition 3.1, and B is a positive constant.

Proof. It follows from (2.8),

$$
\begin{aligned}
\mathbb{S}(W_k)=&\sum_{j=1}^{k}\mathbb{S}(\alpha\mathtt{w}_{f,\beta}((R_k^j)^{-1}y))\\
&+Q\Big(W_k^{2^*-1}-\sum_{j=1}^{k}(\alpha\mathtt{w}_{f,\beta}((R_k^j)^{-1}y))^{2^*-1}\Big).
\end{aligned}
\tag{3.29}
$$

To prove (3.26), we use (2.25) to find that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\sum_{j=2}^{k}S(\alpha\mathtt{w}_{f,\beta}((R_k^j)^{-1}y))\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial\beta}\\
=&\varepsilon^2O\Big(\sum_{j=2}^{k}\frac{\varepsilon^{N-2}}{|x_j-x_1|^{N-2}}\Big)=\varepsilon^2O\big(k^{N-2}\varepsilon^{N-2}\big),
\end{aligned}
\tag{3.30}
$$

On the other hand, it is standard to prove

$$
\begin{aligned}
&\int_{\mathbb{R}^N}Q\Big(W_k^{2^*-1}-\sum_{j=1}^{k}(\alpha\mathtt{w}_{f,\beta}((R_k^j)^{-1}y,z))^{2^*-1}\Big)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial\beta}\\
=&\frac{B}{(Q(r_0,d_)))^{\frac{N-2}{2}}\beta^{N-2}}k^{N-2}\varepsilon^{N-2},
\end{aligned}
\tag{3.31}
$$

where $B>0$ is a constant.

It is easy to see that (3.1), (3.30) and (3.31) imply (3.26).

To prove (3.27), we first note that

$$\begin{aligned}&\int_{\mathbb{R}^N}\sum_{j=2}^{k}\mathbb{S}(\alpha \mathtt{w}_{f,\beta}((R_k^j)^{-1}y))\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial x_i}\\ =&\varepsilon^2 O\Big(\sum_{j=2}^{k}\frac{\varepsilon^{N-2}}{|x_j-x_1|^{N-2}}\Big)=\varepsilon^2 O\big(k^{N-2}\varepsilon^{N-2}\big),\end{aligned}\tag{3.32}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}Q\Big(W_k^{2^*-1}-\sum_{j=1}^{k}(\alpha \mathtt{w}_{f,\beta}((R_k^j)^{-1}y,z))^{2^*-1}\Big)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial x_i}\\ =&O\big(k^{N-1}\varepsilon^{N-1}\big)\end{aligned}\tag{3.33}$$

which, together with (3.2), imply (3.27). We can prove (3.28) in a similar way.

□

4. A linear Problem

Let $\Omega=\{r_0-4\delta<|y^*|<r_0+4\delta\}\times\{d_0-4\delta<y_N<d_0+4\delta\}$, $y^*=(y_1,\cdots,y_{N-1})$. Define $\Omega_\varepsilon=\varepsilon^{-1}\Omega$. For $j=1,\cdots,k$, denote

$$x_j=\Big((\varepsilon^{-1}r_0-f_1)\cos\frac{2(j-1)\pi}{k},(\varepsilon^{-1}r_0-f_1)\sin\frac{2(j-1)\pi}{k},0,\cdots,0,\varepsilon^{-1}d_0-f_N\Big)\in\mathbb{R}^N.$$

Let

$$\begin{aligned}\|u\|_*=&\sup_{y\in\Omega_\varepsilon}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1}|u(y)|\\ &+\sup_{y\in\Omega_\varepsilon}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N}{2}+\tau}}\Big)^{-1}|Du(y)|,\end{aligned}\tag{4.1}$$

and

$$\|f\|_{**}=\sup_{y\in\Omega_\varepsilon}\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)^{-1}|f(y)|,\tag{4.2}$$

where $\tau=\frac{N-4}{N-2}$. For this choice of τ, we find that

$$\sum_{j=2}^{k}\frac{1}{|x_j-x_1|^\tau}\le Ck^\tau\varepsilon^\tau\sum_{j=2}^{k}\frac{1}{j^\tau}\le C\varepsilon^\tau k\le C'.$$

$$L_\varepsilon u=-\Delta u-\frac{m}{y_N}u_{y_N}+\lambda\varepsilon^2u. \tag{4.3}$$

Consider

$$\begin{cases}L_\varepsilon u-(2^*-1)Q(\varepsilon|y^*|,\varepsilon y_N)W_k^{2^*-2}u=h \ \text{ in } \Omega_\varepsilon,\\ u=0, \quad \text{on } \partial\Omega_\varepsilon,\\ u\in H_s,\end{cases} \tag{4.4}$$

Lemma 4.1. *Let u be the solution of (4.4). Then there is a constant $C>0$ and $\theta>0$, such that*

$$\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1}|u(y,z)|+\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N}{2}+\tau}}\Big)^{-1}|Du(y,z)|$$
$$\le C\Big(\|h_k\|_{**}+\|u\|_*\frac{\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}}\Big).$$

Proof. To use Lemma C.2, we first make the change of variable $\bar u(\cdot)=\varepsilon^{-\frac{N-2}{2}}u(\varepsilon^{-1}\cdot)$. Then, $\bar u$ satisfies

$$\begin{cases}\tilde L_\varepsilon\bar u-(2^*-1)Q(|y^*|,y_N)\bar W_k^{2^*-2}\tilde u=\varepsilon^{-\frac{N+2}{2}}h(\varepsilon^{-1}\cdot) \ \text{ in } \Omega,\\ u=0, \quad \text{on } \partial\Omega,\\ u\in H_s,\end{cases}$$

where $\bar W_k(\cdot)=\varepsilon^{-\frac{N-2}{2}}W_k(\varepsilon^{-1}\cdot)$

It is easy to check that

$$\tilde L_\varepsilon u=\frac{-div(y_N^mDu)+\lambda y_N^m u}{y_N^m},$$

and $-div(y_N^mDu)+\lambda y_N^m u$ is uniformly elliptic in Ω. It follows from Lemma C.2 that

$$
\begin{aligned}
|\bar u(\varepsilon x)| =&\Big|\int_\Omega G(\varepsilon x, y)\big)\big((2^*-1)Q(|y^*|, y_N)\bar W_k^{2^*-2}\bar u+\varepsilon^{-\frac{N+2}{2}}h(\varepsilon^{-1}y)\big)y_N^m\,dy\\
\le& C\int_\Omega \frac{1}{|\varepsilon x-y|^{N-2}}\big(\bar W_k^{2^*-2}|\bar u|+\varepsilon^{-\frac{N+2}{2}}|h(\varepsilon^{-1}y)|\big)\,dy\\
=&C\varepsilon^{-\frac{N-2}{2}}\int_{\Omega_\varepsilon}\frac{1}{|x-y|^{N-2}}\big(W_k^{2^*-2}|u|+|h(y)|\big)\,dy\\
\le& C\varepsilon^{-\frac{N-2}{2}}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-2}}\Big[\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)W_k^{2^*-2}\|u\|_*\\
&\qquad+\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)\|h\|_{**}\Big].
\end{aligned}
\tag{4.5}
$$

So, we obtain

$$
\begin{aligned}
|u(x)|\le& C\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-2}}\Big[\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)W_k^{2^*-2}\|u\|_*\\
&+\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)\|h\|_{**}\Big].
\end{aligned}
\tag{4.6}
$$

Using Lemma B.3, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-2}}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)W_m^{2^*-2}\\
\le& C\sum_{j=1}^k\frac{1}{(1+|x-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.
\end{aligned}
\tag{4.7}
$$

It follows from Lemma B.2 that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-2}}\Big(\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)\,dy\\
\le& C\sum_{j=1}^k\frac{1}{(1+|x-x_j|)^{\frac{N-2}{2}+\tau}},
\end{aligned}
\tag{4.8}
$$

So, from (4.7), (4.8) and (4.6), we find

$$
\begin{aligned}
|u(x)| \le & C\sum_{j=1}^{k}\frac{1}{(1+|x-x_j|)^{\frac{N-2}{2}+\tau+\theta}}\|u\|_* \\
& + C\sum_{j=1}^{k}\frac{1}{(1+|x-x_j|)^{\frac{N-2}{2}+\tau}}\|h\|_{**}
\end{aligned}
\tag{4.9}
$$

For the estimate of $|Du|$, we use

$$
\begin{aligned}
& |D\bar{u}(\varepsilon x)| \\
=& \Big|\int_\Omega DG(\varepsilon x,y)\big((2^*-1)Q(|y^*|,y_N)\bar{W}_k^{2^*-2}\bar{u}+\varepsilon^{-\frac{N+2}{2}}h(\varepsilon^{-1}y)\big)y_N^m\,dy\Big| \\
\le & C\varepsilon^{-\frac{N}{2}}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-1}}\Big[\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)W_k^{2^*-2}\|u\|_* \\
& + \Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)\|h\|_{**}\Big].
\end{aligned}
\tag{4.10}
$$

So, we obtain

$$
\begin{aligned}
& |Du(x)| \\
\le & C\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-1}}\Big[\Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)W_k^{2^*-2}\|u\|_* \\
& + \Big(\sum_{j=1}^{k}\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\Big)\|h\|_{**}\Big] \\
\le & C\sum_{j=1}^{k}\frac{1}{(1+|x-x_j|)^{\frac{N}{2}+\tau+\theta}}\|u\|_* + C\sum_{j=1}^{k}\frac{1}{(1+|x-x_j|)^{\frac{N}{2}+\tau}}\|h\|_{**},
\end{aligned}
\tag{4.11}
$$

and the result follows.

□

Recall that for any function ψ, we use the notation

$$
\psi_{f,\beta}(x)=\alpha\psi\big(\beta(x_1-\frac{r_0}{\varepsilon}-f_1),\beta x_2,\cdots,\beta x_{N-1},\beta(x_N-\frac{d_0}{\varepsilon}-f_N)\big),
$$

and $\xi(s)$ is a function such that $\xi=1$ if $0\le|s|\le\frac{1}{2}\delta$, $\xi=0$ if $|s|\ge\delta$, and $0\le\xi\le1$. In the following, we will use that notations:

$$Z_{1,1}=\partial_{x_1}(\xi(\varepsilon x_N)w)_{f,\beta},\ \ Z_{1,2}=\partial_{x_N}(\xi(\varepsilon x_N)w)_{f,\beta},\ \ Z_{1,3}=\partial_{\beta}(\xi(\varepsilon x_N)w)_{f,\beta},$$

and $Z_{i,t}(\cdot)=Z_{1,t}(R_i^{-1}\cdot)$, where R_i is the rotation of angle $\frac{2i\pi}{k}$ in the x_1x_2 plane.

Now we consider the *a priori estimates* for the following problem:

$$\begin{cases} L_\varepsilon\varphi-(2^*-1)Q(\varepsilon|y^*|,\varepsilon y_N)W_k^{2^*-2}\varphi=h+\sum_{t=1}^3 c_t\sum_{i=1}^k Z_{i,t}\ \text{ in } \Omega_\varepsilon,\\ \varphi=0\ \text{ on }\ \partial\Omega_\varepsilon,\\ \int_{\mathbb{R}^N}\varphi\sum_{i=1}^k Z_{i,j}=0,\ j=1,2,3,\\ \phi\in H_s. \end{cases} \tag{4.12}$$

Lemma 4.2. *Assume that φ_k solves (4.12). Then $\|\varphi_k\|_*\le C\|h\|_{**}$.*

Proof. We argue by contradiction. Suppose that there are $k\to+\infty$, and φ_k solving (4.12) with $\|h_k\|_{**}\to 0$, and $\|\varphi_k\|_*=1$. For simplicity, we drop the subscript k.

We estimate c_l, $l=1,2,3$. Multiplying (4.12) by $Z_{1,l}$ and integrating over $\mathbb{R}^N$, we see that c_t satisfies

$$\begin{aligned}&\sum_{t=1}^3\sum_{i=1}^k\int_{\mathbb{R}^N}Z_{i,t}Z_{1,l}c_t\\ =&\int_{\mathbb{R}^N}\big(L_\varepsilon\varphi-(2^*-1)W_k^{2^*-2}\varphi\big)Z_{1,l}-\int_{\mathbb{R}^N}hZ_{1,l}.\end{aligned} \tag{4.13}$$

It follows from Lemma B.1 that

$$\begin{aligned}&\big|\langle h,Z_{1,l}\rangle\big|\\ \le& C\|h\|_{**}\int_{\mathbb{R}^N}\frac{1}{(1+|y-x_1|)^{N-2}}\sum_{j=1}^k\frac{1}{(1+|y-x_j|)^{\frac{N+2}{2}+\tau}}\,dy\\ \le& C\|h\|_{**}.\end{aligned} \tag{4.14}$$

On the other hand, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\big(L_\varepsilon\varphi-(2^*-1)Q(\varepsilon|y^*|,\varepsilon y_N)W_k^{2^*-2}\varphi\big)Z_{1,l}\\ =&\int_{\mathbb{R}^N}\big(-\Delta Z_{1j}-(2^*-1)Q(\varepsilon|y^*|,\varepsilon y_N)W_k^{2^*-2}Z_{1j}\big)\varphi\\ &+\int_{\mathbb{R}^N}\big(-\frac{m}{x_N}\varphi_{x_N}+\varepsilon^2\lambda\varphi\big)Z_{1,l}=o(1)\|\varphi\|_*.\end{aligned} \tag{4.15}$$

Thus we obtain from (4.13), (4.14) and (4.15) that

$$c_l = o(\|\varphi\|_*) + O(\|h\|_{**}). \tag{4.16}$$

So, it follows from Lemma 4.1,

$$\begin{aligned}&\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1}|\varphi(y)| + \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N}{2}+\tau}}\Big)^{-1}|D\varphi(y)| \\ \le&\Big(\|h\|_{**} + \|\varphi\|_* \frac{\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}}{\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}}\Big).\end{aligned} \tag{4.17}$$

Next, we show that for any $R>0$,

$$|\varphi_k| + |D\varphi_k| \to 0, \quad \text{uniformly in } B_R(x_i). \tag{4.18}$$

Suppose that there is $(x^k) \in B_R(x_i)$, such that

$$|\varphi_k(x^k)| + |D\varphi_k(x^k)| \ge c' > 0. \tag{4.19}$$

Then, $\bar\varphi_k(\cdot) = \varphi_k(\cdot - x^k) \to \bar\varphi$ in $C^1_{loc}(\mathbb{R}^N)$ and $\bar\varphi$ satisfies

$$-\Delta\varphi - (2^*-1)w^{2^*-2}\varphi = 0.$$

Thus, $\bar\varphi$ is a linear combination of $\partial_{x_i} w$, $\frac{N-2}{2}w + xDw$. By the assumption, we can deduce

$$\int_{\mathbb{R}^N} \partial_{x_i} w\bar\varphi = 0, \quad \int_{\mathbb{R}^N} \big(\frac{N-2}{2}w + xDw\big)\bar\varphi = 0.$$

This implies $\bar\varphi = 0$. So we obtain a contradiction.

¿From (4.18) and (4.17), we find

$$\begin{aligned}\|\varphi_k\|_* =& \max_{x\in\mathbb{R}^N}\Big[\Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau}}\Big)^{-1}|\varphi_k(y)| \\ &+ \Big(\sum_{j=1}^k \frac{1}{(1+|y-x_j|)^{\frac{N}{2}+\tau}}\Big)^{-1}|D\varphi(y)|\Big] \\ \le& o(1) + C\|h_k\|_{**} + o_R(1)\|\varphi_k\|_*.\end{aligned}$$

This is a contradiction to $\|\varphi_k\|_* = 1$.

□

¿From Lemma 4.2, using the same argument as in the proof of Proposition 4.1 in [14], we can prove the following result :

Proposition 4.3. *There exists $k_0 > 0$ and a constant $C > 0$, independent of k, such that for all $k \geq k_0$, problem (4.12) has a unique solution φ_k satisfying*

$$\|\varphi_k\|_* \leq C\|h\|_{**}. \tag{4.20}$$

5. Finite-dimensional Reduction

Our objective is to construct a solution of the form

$$u = W_k + \varphi,$$

for

$$\begin{cases} -\Delta u + \varepsilon^2 \lambda u = Q(\varepsilon|y^*|, \varepsilon|y^{**}|) u^{\frac{N+2}{N-2}}, \ u > 0, & \text{in } \mathbb{R}^{N+m}, \\ u(y) \to 0 & \text{as } |y| \to +\infty, \end{cases} \tag{5.1}$$

where $\phi \in H_s$ is some small perturbation. Then ϕ satisfies

$$\mathcal{A}\varphi = E + N(\varphi), \tag{5.2}$$

where

$$\mathcal{A}\varphi = -\Delta\varphi + \varepsilon^2\lambda\varphi - (2^* - 1)Q(\varepsilon|y^*|, \varepsilon|y^{**}|)W_k^{2^*-2}\varphi, \tag{5.3}$$

$$E = Q(\varepsilon|y^*|, \varepsilon|y^{**}|)W_k^{2^*-1} + \Delta W_k - \varepsilon^2\lambda W_k^2, \tag{5.4}$$

and

$$N(\varphi) = Q(\varepsilon|y^*|, \varepsilon|y^{**}|)\Big(\big(W_k + \varphi\big)^{2^*-1} - W_k^{2^*-1} - (2^* - 1)W_k^{2^*-2}\varphi\Big). \tag{5.5}$$

The solutions we will construct are radially symmetric in y^{**}. So they satisfy (2.4). To overcome the difficulty caused by the singularity at $y_N = 0$ in (2.4), we need to separate this problem into two problems.

Let

$$D_{\varepsilon,\delta} = \bigcup_{j=1}^{k} \big\{x : \ d(y, \Gamma_{j,\varepsilon}) \leq \delta\varepsilon^{-1}\big\},$$

where

$$\Gamma_{j,\varepsilon}=\big(\varepsilon^{-1}r_0\cos\frac{2(j-1)\pi}{k},\varepsilon^{-1}r_0\sin\frac{2(j-1)\pi}{k},0,\cdots,0,y^{**}\big),\quad |y^{**}|=\varepsilon^{-1}d_0.$$

Similar to Section 4, we define the norms

$$\begin{aligned}\|u\|_{*,\Gamma}=&\sup_{y\in D_{\varepsilon,2\delta}}\Big(\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N-2}{2}+\tau}}\Big)^{-1}|u(y)|\\&+\sup_{y\in D_{\varepsilon,2\delta}}\Big(\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N}{2}+\tau}}\Big)^{-1}|Du(y)|\\&+\sup_{\mathbb{R}^{N+m}\setminus D_{\varepsilon,\delta}}|u(y)|,\end{aligned}\tag{5.6}$$

and

$$\begin{aligned}\|f\|_{**,\Gamma}=&\sup_{y\in D_{\varepsilon,2\delta}}\Big(\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{i,\varepsilon}))^{\frac{N+2}{2}+\tau}}\Big)^{-1}|f(y)|\\&+\sup_{\mathbb{R}^{N+m}\setminus D_{\varepsilon,\delta}}|f(y)|,\end{aligned}\tag{5.7}$$

where $\tau=\frac{N-4}{N-2}$.

In this section, we discuss the solvability of

$$\begin{cases}\mathcal{A}\varphi=E+N(\varphi)+\sum_{t=1}^{3}c_t\sum_{i=1}^{k}Z_{i,t}\ \text{ in }\mathbb{R}^{N+m},\\ \sum_{i=1}^{k}\int_{\mathbb{R}^{N+m}}\varphi Z_{i,j}=0,\ j=1,2,3,\\ \varphi\in H_s,\end{cases}\tag{5.8}$$

for some constants c_t.

The main result of this section is the following:

Proposition 5.1. *There is an integer $k_0>0$, such that for each $k\ge k_0$, (5.8) has a unique solution φ, satisfying*

$$\|\varphi\|_{*,\Gamma}\le C\varepsilon^{1+\sigma},$$

where $\sigma>0$ is a small constant.

Proof. Step 1. Decomposition of the problem.

To use the results obtained in section 4, we need to decompose (5.2) into two problems. Let $\eta_{1,\delta}(|y^*|)$ be a function satisfying $\eta_{1,\delta}=1$ if $0\le ||y^*|-\varepsilon^{-1}r_0|<\varepsilon^{-1}\delta$, $\eta_{1,\delta}=0$ if $||y^*|-\varepsilon^{-1}r_0|>2\varepsilon^{-1}\delta$, and $0\le\eta\le1$, where $y^*=(y_1,\cdots,y_{N-1})$. We take another function $\eta_{2,\delta}(|y^{**}|)$, satisfying $\eta_{2,\delta}=1$ if $0\le ||y^{**}|-\varepsilon^{-1}d_0|<\varepsilon^{-1}\delta$, $\eta_{2,\delta}=0$ if $||y^{**}|-\varepsilon^{-1}d_0|>2\varepsilon^{-1}\delta$, and $0\le\eta\le1$, where $y^{**}=(y_N,\cdots,y_{N+m})$. Define $\eta_\delta(y)=\eta(|y^*|,|y^{**}|)=\eta_{1,\delta}(y^*)\eta_{2,\delta}(y^{**})$. Then, $\varphi=\eta_{2\varepsilon^{-1}\delta}\omega+\psi$ satisfies (5.8), if ω and ψ satisfy

$$\begin{cases}\mathcal{A}\omega=\eta_{2\delta}\big(E+N(\eta_\delta\omega+\psi)-(2^*-1)Q(\varepsilon|y^*|,\varepsilon|y^{**}|)W_k^{2^*-2}\psi\big)+\sum_{t=1}^3 c_t\sum_{i=1}^k Z_{i,t},\\ \omega=0,\quad\text{on }\partial\Omega_\varepsilon,\end{cases}\tag{5.9}$$

where $\Omega_\varepsilon=\{\varepsilon^{-1}r_0-4\delta\varepsilon^{-1}<|y^*|<\varepsilon^{-1}r_0+4\delta\varepsilon^{-1}\}\times\{\varepsilon^{-1}d_0-4\delta\varepsilon^{-1}<|y^{**}|<\varepsilon^{-1}d_0+4\delta\varepsilon^{-1}\}$, and

$$\begin{aligned}&-\Delta\psi+\varepsilon^2\lambda\psi-(2^*-1)(1-\eta_{2\delta})Q(\varepsilon|y^*|,\varepsilon|y^{**}|)W_k^{2^*-2}\psi\\ =&(1-\eta_{2\delta}^2)\big(E+\sum_{t=1}^3 c_t\sum_{i=1}^k Z_{i,t}\big)\\ &+(1-\eta_{2\delta}^2)N(\eta_{2\delta}\omega+\psi)+2D\eta_{2\delta}D\omega+\omega\Delta\eta_{2\delta}\\ =&(1-\eta_{2\delta}^2)N(\eta_{2\delta}\omega+\psi)+2D\eta_{2\delta}D\omega+\omega\Delta\eta_{2\delta}.\end{aligned}\tag{5.10}$$

Step 2. Solving (5.10).

Given any ω satisfying $\|\omega\|_{*,\Gamma}\le C$ (see (5.6) for the definition of the norm $\|\cdot\|_{*,\Gamma}$), we find

$$|2D\eta_{2\delta}D\omega+\omega\Delta\eta_{2\delta}|\le C\|\omega\|_{*,\Gamma}\varepsilon^2k\varepsilon^{\frac{N-2}{2}+\tau}\le C\|\omega\|_{*,\Gamma}\varepsilon^{\frac{N+2}{2}}.\tag{5.11}$$

So,

$$\begin{aligned}&|(1-\eta_{2\delta}^2)N(\eta_{2\delta}\omega+\psi)+2D\eta_{2\delta}D\omega+\omega\Delta\eta_{2\delta}|\\ \le&C(1-\eta_{2\delta}^2)\big(|\eta_{2\delta}\omega|^{2^*-1}+|\psi|^{2^*-1}\big)+C\|\omega\|_{*,\Gamma}\varepsilon^{\frac{N+2}{2}}\\ \le&C\|\omega\|_{*,\Gamma}^{2^*-1}\varepsilon^{\frac{(2^*-1)(N-2)}{2}}+C|\psi|^{2^*-1}+C\|\omega\|_{*,\Gamma}\varepsilon^{\frac{N+2}{2}}\\ =&C\big(\|\omega\|_{*,\Gamma}^{2^*-1}+\|\omega\|_{*,\Gamma}\big)\varepsilon^{\frac{N+2}{2}}+C|\psi|^{2^*-1}.\end{aligned}\tag{5.12}$$

On the other hand, the operator $-\Delta\psi+\varepsilon^2\lambda\psi-(2^*-1)(1-\eta_{2\delta})Q(\varepsilon|y^*|,\varepsilon|y^{**}|)W_k^{2^*-2}\psi$ is invertible in $L^\infty(\mathbb{R}^{N+m})$. So using the contraction mapping theorem, we can prove that (5.10) has a solution $\psi=\psi(\omega)$, satisfying

$$\|\psi\|_{L^\infty(\mathbb{R}^{N+m})} \le C\big(\|\omega\|_{*,\Gamma}^{2^*-1} + \|\omega\|_{*,\Gamma}\big)\varepsilon^{\frac{N+2}{2}}. \tag{5.13}$$

Moreover, we can also prove

$$\|\psi(\omega_1) - \psi(\omega_2)\|_{L^\infty(\mathbb{R}^{N+m})} \le C\big(\|\omega_1\|_{*,\Gamma}^{2^*-2} + \|\omega_2\|_{*,\Gamma}^{2^*-2} + 1\big)\|\omega_1 - \omega_2\|_{*,\Gamma}\varepsilon^{\frac{N+2}{2}}. \tag{5.14}$$

Step 3. Solving (5.9).

We insert $\psi = \psi(\omega)$ into (5.9). Given ω, it follows from Proposition 4.3 that there exists $B(\omega)$, satisfying

$$\begin{cases} \mathcal{A}B(\omega) = h_\varepsilon + \sum_{t=1}^{3} c_t \sum_{i=1}^{k} Z_{i,t}, \\ B(\omega) = 0, \quad \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.15}$$

and

$$\|B(\omega)\|_{*,\Gamma} \le C\|h_\varepsilon\|_{**,\Gamma}, \tag{5.16}$$

where

$$h_\varepsilon = \eta_{2\delta}\big(E + N(\eta_\delta\omega + \psi) - (2^* - 2)Q(\varepsilon|y^*|, \varepsilon|y^{**}|)W_k^{2^*-2}\psi(\omega)\big).$$

It is easy to check that

$$\begin{aligned} &|\eta_{2\delta}W_k^{2^*-2}\psi| \\ \le& C\big(\|\omega\|_{*,\Gamma}^{2^*-1} + \|\omega\|_{*,\Gamma}\big)\varepsilon^{\frac{N+2}{2}} \sum_{j=1}^{k} \frac{\eta_{2\delta}}{(1 + d(y, \Gamma_{j,\varepsilon}))^4} \\ \le& C\big(\|\omega\|_{*,\Gamma}^{2^*-1} + C\|\omega\|_{*,\Gamma}\big)\varepsilon^{4-\tau} \sum_{j=1}^{k} \frac{1}{(1 + d(y, \Gamma_{j,\varepsilon}))^{\frac{N+2}{2}+\tau}}, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} &|\eta_{2\delta}N(\eta_{2\delta}\omega + \psi)| \le C\eta_{2\delta}\big(|\omega|^{2^*-1} + |\psi|^{2^*-1}\big) \\ \le& C\eta_{2\delta}|\omega|^{2^*-1} + C\eta_{2\delta}\big(\|\omega\|_{*,\Gamma}^{2^*-1} + \|\omega\|_{*,\Gamma}\big)^{2^*-1}\varepsilon^{\frac{(2^*-1)(N+2)}{2}} \\ \le& C\|\omega\|_{*,\Gamma}^{2^*-1} \sum_{j=1}^{k} \frac{1}{(1 + d(y, \Gamma_{j,\varepsilon}))^{\frac{N+2}{2}+\tau}}. \end{aligned} \tag{5.18}$$

So, we obtain

$$\|h_\varepsilon\|_{**,\Gamma} \le C\|\omega\|_{*,\Gamma}^{2^*-1} + C\|\omega\|_{*,\Gamma}\varepsilon^{4-\tau}. \tag{5.19}$$

Let

$$F = \big\{\omega : \ \|\omega\|_{*,\Gamma} \le \varepsilon, \ \omega \in H_s, \ \sum_{i=1}^{k}\int_{\mathbb{R}^{N+m}} \omega Z_{i,j} = 0, \ j = 1,2,3\big\}.$$

¿From (5.19) and (5.16), we find that B maps F to F.

On the other hand, $\tilde\omega =: B(\omega_1) - B(\omega_2)$ satisfying

$$\begin{aligned}\mathcal{A}\tilde\omega =& \eta_{2\delta}\big(N(\eta_\delta\omega_1 + \psi(\omega_1)) - N(\eta_\delta\omega_2 + \psi(\omega_2))\big)\\ &- (2^*-1)Q(\varepsilon|y^*|, \varepsilon|y^{**}|)W_k^{2^*-2}\big(\psi(\omega_1)) - \psi(\omega_2)\big) + \sum_{t=1}^{3} c_t \sum_{i=1}^{k} Z_{i,t},\end{aligned} \tag{5.20}$$

which, together with (5.14), gives

$$\|\tilde\omega\|_{*,\Gamma} \le C\big(\|\omega_1\|_{*,\Gamma}^{2^*-2} + \|\omega_2\|_{*,\Gamma}^{2^*-2}\big)\|\omega_1 - \omega_2\|_{*,\Gamma} + C\|\omega_1-\omega_2\|_{*,\Gamma}\varepsilon^{4-\tau}. \tag{5.21}$$

Thus, B is a contraction map.

Using the contraction mapping theorem, we find that (5.9) has a solution ω, satisfying $\omega \in F$, and

$$\|\omega\|_{*,\Gamma} \le C\|E\|_{**,\Gamma}.$$

So, the result follows from Lemma 5.3.

□

Lemma 5.2. *If $N \ge 5$, then*

$$\|N(\varphi)\|_{**,\Gamma} \le C\|\varphi\|_{*,\Gamma}^{\min(2^*-1,2)}.$$

Proof. If $N \ge 6$, we have

$$|N(\varphi)| \le C|\varphi|^{2^*-1}.$$

Using

$$\sum_{j=1}^{k} a_j b_j \le \big(\sum_{j=1}^{k} a_j^p\big)^{\frac1p}\big(\sum_{j=1}^{k} b_j^q\big)^{\frac1q}, \ \ \frac1p + \frac1q = 1, \ a_j, \, b_j \ge 0,$$

we obtain that for $y \in D_{\varepsilon,\delta}$,

$$
\begin{aligned}
|N(\varphi)| \le C\|\varphi\|_*^{2^*-1}\Big(\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\
\le C\|\varphi\|_*^{2^*-1}\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\tau}}\Big)^{\frac{4}{N-2}} \qquad (5.22)\\
\le C\|\varphi\|_*^{2^*-1}\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

Thus, the result follows for $N \ge 6$.

If $N = 5$, then

$$
|N(\varphi)| \le CW_k^{2^*-3}|\varphi|^2 + C|\varphi|^{2^*-1}.
$$

So, for $y \in D_{\varepsilon,\delta}$,

$$
\begin{aligned}
|N(\varphi)| \le& C\|\varphi\|_{*,\Gamma}^2 W_k^{2^*-3}\Big(\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N-2}{2}+\tau}}\Big)^2\\
&+ C\|\varphi\|_*^{2^*-1}\Big(\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\
=& C\big(\|\varphi\|_{*,\Gamma}^2 + \|\varphi\|_*^{2^*-1}\big)\Big(\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N-2}{2}+\tau}}\Big)^{2^*-1}\\
\le& C\|\varphi\|_*^2\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N+2}{2}+\tau}}.
\end{aligned}
$$

□

Next, we estimate E.

Lemma 5.3. *If $N \ge 5$, then*

$$
\|E\|_{**,\Gamma} \le C\varepsilon^{1+\sigma},
$$

where $\sigma > 0$ is a small constant.

Proof. Define

$$\Omega_j=\big\{y:\ y\in\Omega,\ \langle\frac{y'}{|y'|},\frac{x_j}{|x_j|}\rangle\ge\cos\frac{\pi}{k}\big\},\quad y'=(y_1,y_2,,0,\cdots,0).$$

Using the estimates in Section 3, we find for $y\in D_{\varepsilon,2\delta}$,

$$\begin{aligned}|E|\le&C\big|W_k^{2^*-1}-\sum_{j=1}^k\big(\alpha\mathtt{w}_{f,\beta}((R_m^j)^{-1}y)\big)^{2^*-1}\big|\\&+C\varepsilon^2\sum_{j=1}^k\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{N-2}}\\=:&J_0+J_1.\end{aligned}$$

¿From the symmetry, we can assume that $y\in\Omega_1$. Then,

$$|y-x_j|\ge|y-x_1|,\quad\forall\,y\in\Omega_1.$$

Firstly, we claim

$$\frac{1}{1+|y-x_j|}\le\frac{C}{|x_j-x_1|},\quad\forall\,y\in\Omega_1,\ j\ne1.\tag{5.23}$$

In fact, if $|y-x_1|\le\frac12|x_1-x_j|$, then $|y-x_j|\ge\frac12|x_1-x_j|$. If $|y-x_1|\ge\frac12|x_1-x_j|$, then $|y-x_j|\ge|y-x_1|\ge\frac12|x_1-x_j|$, since $y\in\Omega_1$.

For the estimate of J_0, we have for $y\in D_{\varepsilon,2\delta}\cap\Omega_1$,

$$|J_0|\le\frac{C}{(1+d(y,\Gamma_{1,\varepsilon}))^4}\sum_{j=2}^k\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{N-2}}+C\Big(\sum_{j=2}^k\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{N-2}}\Big)^{2^*-1}.\tag{5.24}$$

Using (5.23), we obtain for any $y\in\Omega_1$,

$$\begin{aligned}&\frac{1}{(1+d(y,\Gamma_{1,\varepsilon}))^4}\sum_{j=2}^k\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{N-2}}\\\le&\frac{C}{(1+d(y,\Gamma_{1,\varepsilon}))^{\frac{N+2}{2}+\tau}}\sum_{j=2}^k\frac{1}{(1+|x_j-x_1|)^{\frac{N+2}{2}-\tau}}\\\le&\frac{C}{(1+d(y,\Gamma_{1,\varepsilon}))^{\frac{N+2}{2}+\tau}}(\varepsilon k)^{\frac{N+2}{2}-\tau}\le\frac{C\varepsilon^{1+\sigma}}{(1+d(y,\Gamma_{1,\varepsilon}))^{\frac{N+2}{2}+\tau}}.\end{aligned}\tag{5.25}$$

Using the Hölder inequality, we obtain

$$
\begin{aligned}
&\Big(\sum_{j=2}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{N-2}}\Big)^{2^*-1}\\
\le&\sum_{j=2}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=2}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})}}\Big)^{\frac{4}{N-2}}\\
\le&\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N+2}{2}+\tau}}\Big(\sum_{j=2}^{k}\frac{1}{(1+|x_1-x_j|)^{\frac{N+2}{4}(\frac{N-2}{2}-\tau\frac{N-2}{N+2})}}\Big)^{\frac{4}{N-2}}\\
\le&C(\varepsilon k)^{(N+2)(\frac{1}{2}-\tau\frac{1}{N+2})}\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N+2}{2}+\tau}}\\
\le&C\varepsilon^{1+\sigma}\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{j,\varepsilon}))^{\frac{N+2}{2}+\tau}}.
\end{aligned}
\tag{5.26}
$$

So, we obtain

$$\|J_0\|_{**}\le C\varepsilon^{1+\sigma}.$$

For the estimate of J_1,

$$|J_1|\le C\varepsilon^{1+\sigma}\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{i,\varepsilon}))^{N-1-\sigma}}\le C\varepsilon^{1+\sigma}\sum_{j=1}^{k}\frac{1}{(1+d(y,\Gamma_{i,\varepsilon}))^{\frac{N+2}{2}+\tau}}, \tag{5.27}$$

which gives

$$\|J_1\|_{**}\le C\varepsilon^{1+\sigma}.$$

□

6. Proof of Theorem 2.2

In this section, we will choose f and β, such that the constants c_t in (5.8) is zero. For this purpose, we only need to solve the following problem:

$$\int_{\mathbb{R}^{N+m}}\big(\mathcal{A}\varphi-E-N(\varphi)\big)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial\beta}=0, \tag{6.1}$$

and

$$\int_{\mathbb{R}^{N+m}}\big(\mathcal{A}\varphi-E-N(\varphi)\big)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial x_i}=0;\ i=1,N. \tag{6.2}$$

Using Proposition 5.1, we can prove the following result.

Proposition 6.1. *We have*

$$
\begin{aligned}
&\varepsilon^{-2}(Q(r_0,0,z))^{\frac{N-2}{2}}\int_{\mathbb{R}^{N+m}}\big(\mathcal{A}\varphi-E-N(\varphi)\big)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial\beta}\\
=&\frac{D}{\beta^2}-\frac{B}{\beta^{N-2}}+O\big(|f|^2+\varepsilon^\sigma\big),
\end{aligned}
\tag{6.3}
$$

$$
\begin{aligned}
&\varepsilon^{-2}(Q(r_0,0,z))^{\frac{N-2}{2}}\int_{\mathbb{R}^{N+m}}\big(\mathcal{A}\varphi-E-N(\varphi)\big)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial x_1}\\
=&-\Big(\frac{Q_{rr}(r_0,d_0)}{Q(r_0,d_0)}f_1+\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}f_N\Big)\frac{1}{2^*\beta}\int_{\mathbb{R}^N}w^{2^*}+O(\varepsilon^\sigma),
\end{aligned}
\tag{6.4}
$$

and

$$
\begin{aligned}
&\varepsilon^{-2}(Q(r_0,0,z))^{\frac{N-2}{2}}\int_{\mathbb{R}^{N+m}}\big(\mathcal{A}\varphi-E-N(\varphi)\big)\frac{\partial(\xi(\varepsilon x_N)w)_{f,\beta}}{\partial x_N}\\
=&-f_1\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\beta^{-1}\int_{\mathbb{R}^N}w^{2^*}\\
&-f_N\Big(\frac{m}{Nd_0^2}-\frac{2}{N(N-2)}\frac{m^2}{d_0^2}+\frac{Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\Big)\beta^{-1}\int_{\mathbb{R}^N}w^{2^*}+O(\varepsilon^\sigma),
\end{aligned}
\tag{6.5}
$$

where B and D are the same as in Proposition 3.2.

Proof. We use ∂ to denote either ∂_{x_i}, or ∂_β. It is standard to show that

$$
\begin{aligned}
&\int_{\mathbb{R}^{N+m}}\big(\mathcal{A}\varphi-E-N(\varphi)\big)\partial(\xi(\varepsilon x_N)w)_{f,\beta}\\
=&\int_{\mathbb{R}^N}\mathbb{S}(W_m)\partial(\xi(\varepsilon x_N)w)_{f,\beta}+O\big(\varepsilon\|\phi\|_{*,\Gamma}+\|\phi\|_{*,\Gamma}^2\big)\\
=&\int_{\mathbb{R}^N}\mathbb{S}(W_m)\partial(\xi(\varepsilon x_N)w)_{f,\beta}+O(\varepsilon^{2+\sigma}).
\end{aligned}
$$

So (6.3), (6.4) and (6.5) follow from Propositions 3.2 and 5.1. We omit the details of the proof here.

□

Proof of Theorem 2.2. By Proposition 6.1, we need to solve the following equations:

$$
\frac{D}{\beta^2}-\frac{B}{\beta^{N-2}}=O\big(|f|^2+\varepsilon^\sigma\big)=0,
\tag{6.6}
$$

$$\Big(\frac{Q_{rr}(r_0,d_0)}{Q(r_0,d_0)}f_1+\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}f_N\Big)\frac{1}{2^*\beta}\int_{\mathbb{R}^N}w^{2^*}=O(\varepsilon^\sigma), \tag{6.7}$$

and

$$\begin{aligned}&f_1\frac{Q_{rd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\beta^{-1}\int_{\mathbb{R}^N}w^{2^*}\\&+f_N\Big(\frac{m}{Nd_0^2}-\frac{2}{N(N-2)}\frac{m^2}{d_0^2}+\frac{Q_{dd}(r_0,d_0)}{Q(r_0,d_0)}\frac{1}{2^*}\Big)\beta^{-1}\int_{\mathbb{R}^N}w^{2^*}=O(\varepsilon^\sigma).\end{aligned} \tag{6.8}$$

At a critical point (r_0,d_0), direct calculations show

$$\frac{\partial^2}{\partial r^2}\Big(\frac{d^m}{Q^{\frac{N-2}{2}}(r,d)}\Big)=-\frac{N-2}{2}\frac{d_0^mQ_{rr}(r_0,d_0)}{Q^{\frac{N}{2}}(r_0,d_0)},$$

$$\frac{\partial^2}{\partial r\partial d}\Big(\frac{d^m}{Q^{\frac{N-2}{2}}(r,d)}\Big)=-\frac{N-2}{2}\frac{d_0^mQ_{rd}(r_0,d_0)}{Q^{\frac{N}{2}}(r_0,d_0)},$$

and using (1.4),

$$\begin{aligned}&\frac{\partial^2}{\partial d^2}\Big(\frac{d^m}{Q^{\frac{N-2}{2}}(r,d)}\Big)\\=&\frac{m(m-1)d_0^{m-2}}{Q^{\frac{N-2}{2}}(r_0,d_0)}-(N-2)m\frac{d_0^{m-1}Q_d(r_0,d_0)}{Q^{\frac{N}{2}}(r_0,d_0)}+\frac{N(N-2)}{4}\frac{d_0^mQ_d^2(r_0,d_0)}{Q^{\frac{N}{2}+1}(r_0,d_0)}\\&-\frac{N-2}{2}\frac{d_0^mQ_{dd}(r_0,d_0)}{Q^{\frac{N}{2}}(r_0,d_0)}\\=&\Big(m(m-1)-2m^2+\frac{Nm^2}{N-2}\Big)\frac{d_0^{m-2}}{Q^{\frac{N-2}{2}}(r_0,d_0)}-\frac{N-2}{2}\frac{d_0^mQ_{dd}(r_0,d_0)}{Q^{\frac{N}{2}}(r_0,d_0)}\\=&\Big(\frac{2m^2}{N-2}-m\Big)\frac{d_0^{m-2}}{Q^{\frac{N-2}{2}}(r_0,d_0)}-\frac{N-2}{2}\frac{d_0^mQ_{dd}(r_0,d_0)}{Q^{\frac{N}{2}}(r_0,d_0)}.\end{aligned}$$

So, we find that the matrix in (6.7) and (6.8) equals

$$-\frac{Q^{\frac{N-2}{2}}(r_0,d_0)}{Nd_0^m}\frac{1}{\beta}\int_{\mathbb{R}^N}w^{2^*}\Big(D^2\frac{d^m}{Q^{\frac{N-2}{2}}(r,d)}\Big)_{(r_0,d_0)}$$

By our assumptions, we can prove easily that (6.6), (6.7) and (6.8) have solution:

$$\beta_k=\big(\frac{B}{D}\big)^{\frac{1}{N-4}}+O(\varepsilon^\sigma),\quad |f_{1,k}|\ \ |f_{N,k}|=O(\varepsilon^\sigma).$$

□

Appendix A. Some Estimates

Recall that

$$w(x)=\frac{\alpha_N}{(1+|x|^2)^{\frac{N-2}{2}}},\quad \alpha_N=(N(N-2))^{\frac{N-2}{4}}.$$

Let

$$B_2=\int_{\mathbb{R}^N}w^2.$$

Let ω_N be the area of the unit sphere S^{N-1}. We compute

$$\begin{aligned}&\int_{\mathbb{R}^N}w\big(\frac{N-2}{2}w+xDw\big)=\frac{N-2}{2}\int_{\mathbb{R}^N}w^2+\frac{1}{2}\int_{\mathbb{R}^N}xDw^2\\=&\frac{N-2}{2}\int_{\mathbb{R}^N}w^2-\frac{N}{2}\int_{\mathbb{R}^N}w^2=-\int_{\mathbb{R}^N}w^2=-B_2,\end{aligned}\tag{A.1}$$

$$\begin{aligned}&\int_{\mathbb{R}^N}xDw\big(\frac{N-2}{2}w+xDw\big)=-\frac{N(N-2)}{4}\int_{\mathbb{R}^N}w^2+\int_{\mathbb{R}^N}|x|^2|Dw|^2\\=&\omega_N\alpha_N^2\Big((N-2)^2\int_0^\infty\frac{r^{N+3}}{(1+r^2)^N}-\frac{N(N-2)}{4}\int_0^\infty\frac{r^{N-1}}{(1+r^2)^{N-2}}\Big)\\=&\frac{1}{2}\alpha_N^2\omega_N\Big((N-2)^2\int_0^\infty\frac{t^{\frac{N}{2}+1}}{(1+t)^N}-\frac{N(N-2)}{4}\int_0^\infty\frac{t^{\frac{N}{2}-1}}{(1+t)^{N-2}}\Big)\\=&\frac{3(N-2)N}{4(N-1)}\frac{1}{2}\omega_N\alpha_N^2\int_0^\infty\frac{t^{\frac{N}{2}-1}}{(1+t)^{N-2}}\\=&\frac{3(N-2)N}{4(N-1)}B_2,\end{aligned}\tag{A.2}$$

$$\begin{aligned}&\int_{\mathbb{R}^N}x_Nw_{x_N}\big(\frac{N-2}{2}w+xDw\big)=\frac{1}{N}\int_{\mathbb{R}^N}xDw\big(\frac{N-2}{2}w+xDw\big)\\=&\frac{3(N-2)}{4(N-1)}B_2.\end{aligned}\tag{A.3}$$

We also need

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |x|^2 w^{2^*-1}\big(\frac{N-2}{2}w + xDw\big) = -\frac{2}{2^*}\int_{\mathbb{R}^N} |x|^2 w^{2^*} \\
=&-\frac{N-2}{N}\alpha_N^{2^*}\int_{\mathbb{R}^N}\frac{|x|^2}{(1+|x|^2)^N} \\
=&-\frac{N-2}{N}\alpha_N^{2^*-2}\frac{1}{2}\omega_N\alpha_N^2\int_0^\infty \frac{t^{\frac{N}{2}}}{(1+t)^N} \\
=&-(N-2)^2\frac{1}{2}\omega_N\alpha_N^2\int_0^\infty\frac{t^{\frac{N}{2}}}{(1+t)^N}.
\end{aligned}
\tag{A.4}
$$

But

$$
\begin{aligned}
&\int_0^\infty \frac{t^{\frac{N}{2}}}{(1+t)^N} = \int_0^\infty \frac{t^{\frac{N}{2}}}{(1+t)^{N-1}} - \int_0^\infty \frac{t^{\frac{N}{2}+1}}{(1+t)^N} \\
=&\frac{N}{2(N-2)}\int_0^\infty \frac{t^{\frac{N}{2}-1}}{(1+t)^{N-2}} - \frac{(N+2)N}{4(N-1)(N-2)}\int_0^\infty\frac{t^{\frac{N}{2}-1}}{(1+t)^{N-2}} \\
=&\frac{N(N-4)}{4(N-2)(N-1)}\int_0^\infty \frac{t^{\frac{N}{2}-1}}{(1+t)^{N-2}}.
\end{aligned}
\tag{A.5}
$$

So, we obtain

$$
\int_{\mathbb{R}^N}|x|^2 w^{2^*-1}\big(\frac{N-2}{2}w + xDw\big) = -\frac{(N-4)N(N-2)}{4(N-1)}B_2. \tag{A.6}
$$

Appendix B. Basic Estimates

In this section, we list some lemmas, whose proof can be found in [35].

For each fixed i and j, $i \neq j$, consider the following function

$$
g_{ij}(y) = \frac{1}{(1+|y-x_j|)^\alpha}\frac{1}{(1+|y-x_i|)^\beta}, \tag{B.1}
$$

where $\alpha \geq 1$ and $\beta \geq 1$ are two constants.

Lemma B.1. *For any constant $0 < \sigma \leq \min(\alpha, \beta)$, there is a constant $C > 0$, such that*

$$
g_{ij}(y) \leq \frac{C}{|x_i - x_j|^\sigma}\Big(\frac{1}{(1+|y-x_i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|y-x_j|)^{\alpha+\beta-\sigma}}\Big).
$$

Lemma B.2. *For any constant $0 < \sigma < N-2$, there is a constant $C > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\sigma}}\, dz \le \frac{C}{(1+|y|)^{\sigma}}.$$

Let recall that W_k is defined in (2.28).

Lemma B.3. *Suppose that $N \ge 4$. Then there is a small $\theta > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_k^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-x_j|)^{\frac{N-2}{2}+\tau}}\, dz$$
$$\le C \sum_{j=1}^{k} \frac{1}{(1+|y-x_j|)^{\frac{N-2}{2}+\tau+\theta}}.$$

Proof. The proof can be found in [35].

□

Appendix C. Fundamental solutions for linear elliptic operators

In this section, we discuss the fundamental solutions for some elliptic linear operators in divergent form. Firstly, we consider the existence of fundamental solution in $\mathbb{R}^N$ for $N \ge 3$. The following result may be known. But we can not find the references. So we will give the proof of it.

Lemma C.1. *Suppose that $N \ge 3$. The following elliptic problem*

$$-\sum_{j=1}^{N} D_i(a_{ij}(x) D_j u) + c(x) u = \delta_p \quad in\ \mathbb{R}^N$$

$\lambda_0|\xi|^2 \le \sum_{j=1}^N a_{ij}(x)\xi_i\xi_j \le \lambda_1|\xi|^2$, $\lambda_1 \ge \lambda_0 > 0$, $c(x) \ge 0$, *has a solution $\Gamma(x,p)$ satisfying*

$$0 < \Gamma(x,p) \le \frac{C}{|x-p|^{N-2}}; \quad |D\Gamma(x,p)| \le \frac{C}{|x-p|^{N-1}}, \quad as\ x \to p.$$

Proof. It is well known that

$$-\sum_{j=1}^{N} D_i(a_{ij}(p) D_j u) + c(p) u = \delta_p \quad \text{in } \mathbb{R}^N$$

has a solution $\bar\Gamma(x,p)$ satisfying

$$0 < \bar\Gamma(x,p) \le \frac{C}{|x-p|^{N-2}}, \quad |D\bar\Gamma(x,p)| \le \frac{C}{|x-p|^{N-1}}.$$

where $C > 0$ is independent of p.

Consider

$$\begin{aligned} &-\sum_{j=1}^{N} D_i(a_{ij}(x)D_j v) + c(x)v = f(x,p) \\ =:& -\sum_{j=1}^{N} D_i\big((a_{ij}(x) - a_{ij}(p)\big)D_j(\xi_p\bar{\Gamma}(x,p))) + \big(c(x) - c(p)\big)(\xi_p\bar{\Gamma}(x,p)) \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{C.1}$$

where $\xi_p = 1$ in $B_1(p)$, $\xi_p = 0$ in $\mathbb{R}^N \setminus B_2(p)$ and $0 \le \xi_p \le 1$. Since

$$|f(x,p)| \in L^r(\mathbb{R}^N),$$

for $r \in (1, \frac{N}{N-1})$, (C.1) has a solution $v \in W^{2,r}_{loc}(\mathbb{R}^N)$. So, $\Gamma(x,p) = \xi_p\bar{\Gamma}(x,p) + v$ is the solution. It is easy to show that $\Gamma(x,p) > 0$.

On the other hand, let w be a the solution of

$$-\sum_{j=1}^{N} D_i(a_{ij}(x)D_j v) + c(x)v = |f(x,p)|, \quad \text{in } \mathbb{R}^N. \tag{C.2}$$

Then $|v| \le w$.

Now we estimate the blow up rate of w at the singularity $x = p$. By the Harnack inequality, we find for $x \ne p$,

$$\begin{aligned} \sup_{y\in B_{\frac{1}{2}|x-p|}(x)} w(y) \le& C\Big(\inf_{y\in B_{\frac{1}{2}|x-p|}(x)} w(y) + |x-p|^2\big(\frac{1}{|x-p|^N}\int_{B_{|x-p|}(x)} |f(x,p)|^r\big)^{\frac{1}{r}}\Big) \\ \le& C\Big(\inf_{y\in B_{\frac{1}{2}|x-p|}(x)} w(y) + \frac{1}{|x-p|^{N-3+\eta_1}}\Big), \end{aligned}$$

if $\frac{N}{N-1} - r > 0$ is small enough, where $\eta_1 > 0$ is a small constant. From $w \in W^{2,r} \subset L^{\frac{Nr}{N-2r}}$, we find that for a small $\eta > 0$,

$$\inf_{y\in B_{\frac{1}{2}|x-p|}(x)} w(y) \le \frac{C}{|x-p|^{N-2-\eta}}, \quad \forall\, x \text{ close to } p.$$

Otherwise, we have a sequence of $x_n \to p$, such that

$$\inf_{y\in B_{\frac{1}{2}|x_n-p|}(x_n)} w(y) \ge \frac{C}{|x_n-p|^{N-2-\eta}}.$$

Then for $q = \frac{Nr}{N-2r}$, $r \in (1, \frac{N}{N-1})$,

$$C \geq \int_{B_1(p)} w^q \geq \int_{B_{\frac{1}{2}|x_n-p|}(x_n)} w^q \geq \frac{c_0}{|x_n - p|^{(N-2-\eta)q-N}} \to +\infty, \quad \text{as } x_n \to p,$$

since $(N-2-\eta)q - N > 0$ if $r > 1$ and $\eta > 0$ is small. This is a contradiction. So, we have proved that

$$|v| \leq w(x) \leq \frac{C}{|x-p|^{N-2-\eta}}.$$

Using the L^q estimate, we also have

$$\begin{aligned}&\sup_{y\in B_{\frac{1}{4}|x-p|}(x)} |Dv(y)| \\ \leq & \frac{C}{|x-p|}\Big(\sup_{y\in B_{\frac{1}{2}|x-p|}(x)} |v(y)| + |x-p|^2\big(\frac{1}{|x-p|^N}\int_{B_{\frac{1}{2}|x-p|}(x)} |f(x,p)|^q\big)^{\frac{1}{q}}\Big) \\ \leq & \frac{C}{|x-p|^{N-1-\eta}}.\end{aligned}$$

Since $\Gamma(x,p) = \bar{\Gamma}(x,p) + v$, we find that for $x \in B_1(p)$,

$$0 < \Gamma(x,p) \leq \frac{C}{|x-p|^{N-2}}; \quad |D\Gamma(x,p)| \leq \frac{C}{|x-p|^{N-1}}.$$

□

Lemma C.2. *Let Ω be a bounded domain in $\mathbb{R}^N$. The following elliptic problem*

$$\begin{cases} -\sum_{j=1}^N D_i(a_{ij}(x))D_j u) + c(x))u = \delta_p, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{C.3}$$

where a_{ij} and c satisfy the same condition as in Lemma C.1 in Ω, has a solution $G(x,p)$ satisfying

$$0 < G(x,p) \leq \frac{C}{|x-p|^{N-2}}, \quad |DG(x,p)| \leq \frac{C}{|x-p|^{N-1}}.$$

Proof. Let γ be the solution of

$$\begin{cases} -\sum_{j=1}^N D_i(a_{ij}(x)D_j\gamma) + c(x)\gamma = 0, & \text{in } \Omega, \\ \gamma(x) = -\Gamma(x,p) & \text{on } \partial\Omega, \end{cases} \tag{C.4}$$

where $\Gamma(x,p)$ is the function obtained in Lemma C.2. Then $\gamma < 0$. So, $G(x,p) = \Gamma + \gamma$ is the solution satisfying the condition.

□

References

[1] N. Ackermann, M. Clapp and A. Pistoia, Boundary clustered layers near the higher critical exponents, *J. Differential Equations*, to appear.

[2] A. Ambrosetti, J. GarciaAzorero and I. Peral, Perturbation of $\Delta u - u^{\frac{N+2}{N-2}} = 0$, the scalar curvature problem in R^N and related topics, *J. Funct. Anal.* 165 (1999), 117–149.

[3] A. Bahri and J. Coron, The scalar-curvature problem on the standard three-dimensional sphere, *J. Funct. Anal.* 95 (1991), 106–172.

[4] A. Bahri and Y.Y. Li, On a min-max procedure for the existence of a positive solution for certain scalar field equations in R^N. *Rev. Mat. Iberoamericana* 6 (1990), 1–15.

[5] A. Bahri and P. L. Lions, On the existence of a positive solution of semilinear elliptic equations in unbounded domains. *Ann. Inst. H. Poincar Anal. Non Linaire* 14 (1997), 365–413.

[6] G. Bianchi, The scalar curvature equation on R^N and S^N, *Adv. Differential Equations* 1 (1996), 857–880.

[7] D.Cao, Positive solution and bifurcation from the essential spectrum of a semilinear elliptic equation on R^n. *Nonlinear Anal.* 15 (1990), 1045–1052.

[8] D. Cao, E. Noussair and S. Yan, On the scalar curvature equation $-\Delta u = (1 + \varepsilon K)u^{(N+2)/(N2)}$ in R^N, *Calc. Var. Partial Differential Equations* 15 (2002), 403–419.

[9] S.-Y.A. Chang, M. Gursky and P.C. Yang, Prescribing scalar curvature on S^2 and S^3, *Calc. Var. Partial Differential Equations* 1 (1993), 205–229.

[10] S.-Y,A. Chang and P. Yang, A perturbation result in prescribing scalar curvature on S^n, *Duke Math. J.* 64 (1991), 27–69.

[11] C.-C. Chen and C.-S. Lin, Estimate of the conformal scalar curvature equation via the method of moving planes, II, *J. Differential Geom.* 49 (1998), 115–178.

[12] C.-C. Chen and C.-S. Lin, Prescribing scalar curvature on S^N. I. A priori estimates, *J. Differential Geom.* 57 (2001), 67–171.

[13] V. Coti Zelati and P.H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on R^N. *Comm. Pure Appl. Math.* (10) 45 (1992), 1217–1269.

[14] M. Del Pino, P. Felmer and M. Musso, Two-bubble solutions in the super-critical Bahri-Coron's problem. *Calc. Var. Partial Differential Equations* 16 (2003), 113–145.

[15] M. Del Pino, P. Felmer and M. Musso, Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries. *Bull. London Math. Soc.* 35(2003), 513–521.

[16] M. del Pino, M. Musso and F. Pacard, Bubbling along boundary geodesics near the second critical exponents, *J. Eur. Math. Soc.*, J. Eur. Math. Soc. (JEMS) 12(2010), 1553–1605.

[17] M. del Pino, F. Mahmoudi and M. Musso, Bubbling on boundary submanifolds for the Lin-Ni-Takagi problem at higher critical exponents, *J. Eur. Math. Soc.* to appear.

[18] M. Del Pino, M. Kowalczyk and J. Wei, Concentration on curves for nonlinear Schrödinger equations. *Comm. Pure Appl. Math.* 60 (2007), 113–146.

[19] M. Del Pino and J. Wei, Supercritical elliptic problems in domains with small holes, *Ann. Inst. H. Poincar Anal. Non Linaire* 24 (2007), 507–520.

[20] W. Ding and W.-M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation. *Arch. Rational Mech. Anal.* 91(1986), 283–308.

[21] Y.Y. Li, On $\Delta u = K(x)u^5$ in R^3, *Comm. Pure Appl. Math.* 46 (1993), 303–340.

[22] Y.Y. Li, Prescribed scalar curvature on S^3, S^4 and related problems, *J. Funct. Anal.* 118 (1993), 43–118.

[23] Y.Y. Li, Prescribed scalar curvature on S^n and related problems, I, *J. Differential Equations* 120 (1995), 319–410.

[24] Y.Y. Li, Prescribed scalar curvature on Sn and related problems. II. Existence and compactness, *Comm. Pure Appl. Math.* 49 (1996), 541–597.

[25] Y. Li, W.M. Ni, On the conformal scalar curvature equation in R^n, *Duke Math. J.* 57 (1988), 859–924.

[26] P.L.Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincar Anal. Non Lineaire* 1 (1984), 109–145.

[27] P.L.Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincar Anal. Non Lineaire* 1 (1984) , 223–283.

[28] A. Malchiodi, Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains. *Geom. Funct. Anal.* 15 (2005), 1162–1222.

[29] A. Malchiodi and M. Montenegro, Boundary concentration phenomena for a singularly perturbed elliptic problem. *Comm. Pure Appl. Math.* 55 (2002), 1507–1568.

[30] A. Malchiodi and M. Montenegro, Multidimensional boundary layers for a singularly perturbed Neumann problem. *Duke Math. J.* 124 (2004), 105–143.

[31] D. Passaseo, New nonexistence results for elliptic equations with supercritical nonlinearity, *Differential and Integral Equations* 8, no. 3 (1995), 577–586.

[32] D. Passaseo, Nontrivial solutions of elliptic equations with supercritical exponent in contractible domains, *Duke Mathematical Journal* 92, no. 2 (1998), 429–457.

[33] L. Wang, J. Wei and S. Yan, A Neumann problem with critical exponent in nonconvex domains and Lin-Ni's conjecture. *Trans. Amer. Math. Soc.* 362 (2010), 4581–4615.

[34] L. Wang, J. Wei and S. Yan, On Lin-Ni's conjecture in convex domains. *Proc. Lond. Math. Soc.* (3) 102(2011), 1099–1126.

[35] J. Wei and S. Yan, Infinitely many solutions for the prescribed scalar curvature problem on S^N, *J. Funct. Anal.* 258(2010), 3048-3081.

[36] J. Wei and S. Yan, On a stronger Lazer-McKenna conjecture for Ambrosetti-Prodi type problems. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) 9(2010), 423–457.

[37] J. Wei and S. Yan, Infinitely many positive solutions for an elliptic problem with critical or supercritical growth. *J. Math. Pures Appl.* (9) 96 (2011), 307–333.

[38] R. Schoen and D. Zhang, Prescribed scalar curvature problem on the n-sphere, *Calc. Var. Partial Differential Equations* 4 (1996), 1–25.

[39] S. Yan, High-energy solutions for a nonlinear elliptic problem with slightly supercritical exponent. *Nonlinear Anal., Ser. A: Theory Methods* (4)38 (1999), 527–546.

[40] S. Yan, Concentration of solutions for the scalar curvature equation on R^N, *J. Differential Equations* 163 (2000) 239–264.

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