# Many droplet pattern in the cylindrical phase of diblock copolymer morphology

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**Abstract**

We study a nonlocal free boundary problem derived from the Ohta-Kawasaki density functional theory of diblock copolymers. In a proper range of the block composition parameter and the nonlocal interaction parameter, an equilibrium pattern of many droplets is prove to exist in a general planar domain. A sub-range of the parameters is identified where the multiple droplet pattern is stable. This stable droplet pattern models the cylindrical phase in the diblock copolymer morphology. Each droplet is close to a round disc. The boundaries of the droplets satisfy an equation that involves the curvature of the boundary and a quantity that depends nonlocally on the whole pattern. The locations of the droplets are determined via a Green's function of the domain. In constructing the droplet pattern we overcome three obstacles: interface oscillation, droplet coarsening, and droplet translation.

**Key words**. Cylindrical phase, diblock copolymer morphology, droplet pattern, droplet coarsening, interface oscillation, droplet translation.

**2000 AMS subject classification**. 35R35, 82B24, 82D60.

**Abbreviated title**. Many droplet pattern.

## 1 Introduction

A diblock copolymer melt is a soft material, characterized by fluid-like disorder on the molecular scale and a high degree of order at a longer length scale. A molecule in a diblock copolymer is a linear sub-chain of A-monomers grafted covalently to another sub-chain of B-monomers. Because of the repulsion between the unlike monomers, the different type sub-chains tend to segregate, but as they are chemically bonded in chain molecules, segregation of sub-chains cannot lead to a macroscopic

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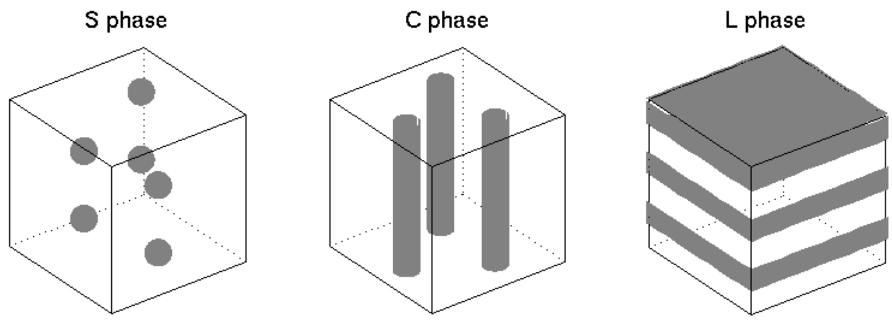


Figure 1: The spherical, cylindrical, and lamellar morphology phases commonly observed in diblock copolymer melts. The dark color indicates the concentration of type A monomers, and the white color indicates the concentration of type B monomers.

phase separation. Only a local micro-phase separation occurs: micro-domains rich in A monomers and micro-domains rich in B monomers emerge as a result. These micro-domains form patterns that are know as morphology phases. Various phases, including lamellar, cylindrical, spherical, gyroid, have been observed in experiments. See Figure 1.

This paper deals with the cylindrical phase of the block copolymer morphology (Figure 1, Plot 2). This phase occurs when $a$ is relatively close to 0 (or close to 1), and the A-monomers (or B-monomers respectively) form parallel cylinders in space. If we look at a cross section, the cylinders become droplets in a two dimensional region. We will mathematically construct a pattern with a number of droplets. In the process we achieve the following objectives.

- Identify a parameter range that produces a multiple droplet pattern.
- Find a sub-range where the multiple droplet pattern is stable.
- Determine the radius of each droplet.
- Determine the locations of the droplets.
- Find the free energy of the droplet pattern.
- Determine the optimal number of droplets.

The model we use is a nonlocal free boundary problem derived by Ren and Wei [20] from the Ohta-Kawasaki density functional theory of diblock copolymers [17]. Let $D$ be a bounded and sufficiently smooth domain in $R^2$. For any subset $E$ of $D$, denote the Lebesgue measure of $E$ by $|E|$ and denote the part of the boundary of $E$ that is in $D$ by $\partial_D E$. Let $\chi_E$ be the characteristic function of $E$, i.e. $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \in D \backslash E$. Given a fixed number $a \in (0,1)$ we look for a subset $E$ of $D$ and a number $\lambda$ such that $\partial_D E$ is a smooth curve, or a union of several smooth curves, $|E| = a|D|$, and at every point on $\partial_D E$

$$H(\partial_D E) + \gamma(-\Delta)^{-1}(\chi_E - a) = \lambda. \tag{1.1}$$

Here $H(\partial_D E)$ is the curvature of $\partial_D E$ viewed from $E$ and $\gamma$ is a given positive number. The expression $(-\Delta)^{-1}(\chi_E - a)$ is the solution $v$ of the problem

$$-\Delta v = \chi_E - a \text{ in } D, \ \partial_\nu v = 0 \text{ on the boundary of } D, \quad \overline{v} = 0$$

where the bar over a function is the average of the function over its domain, i.e.

$$\overline{v} = \frac{1}{|D|} \int_D v(x)\, dx.$$

Because $(-\Delta)^{-1}$ is a nonlocal operator, the free boundary problem (1.1) is nonlocal.

The main difficulty in (1.1) comes from the nonlocal term. Without it, i.e. if $\gamma = 0$, (1.1) would just be the equation of constant curvature. However with the nonlocal term the curvature of a solution in general is not constant. One exception occurs in the study of the lamellar phase (Figure 1, Plot 3) where interfaces are parallel planes (Ren and Wei [20, 23]). The solution we are looking for in this paper is a union of a number of disconnected sets each of which is close to a small disc. These approximate discs are called droplets and the solution is termed a droplet solution.

The equation (1.1) is the Euler-Lagrange equation of the following variational problem.

$$J(E) = |D\chi_E|(D) + \frac{\gamma}{2} \int_D |(-\Delta)^{-1/2}(\chi_E - a)|^2\, dx. \tag{1.2}$$

The admissible set $\Sigma$ of the functional $J$ is the collection of all the measurable subsets of $D$ of measure $a|D|$ and of finite perimeter, i.e.

$$\Sigma = \{E \subset D : E \text{ is Lebesgue measurable}, \ |E| = a|D|, \ \chi_E \in BV(D)\}. \tag{1.3}$$

Here $BV(D)$ is the space of functions of bounded variation on $D$. The nonlocal integral operator $(-\Delta)^{-1}$ is defined by solving

$$-\Delta v = q \text{ in } D, \ \partial_\nu v = 0 \text{ on the boundary of } D, \overline{v} = 0$$

for $q \in L^2(D)$, $\overline{q} = 0$. Then $(-\Delta)^{-1/2}$ is the positive square root of $(-\Delta)^{-1}$.

Since $\chi_E \in BV(D)$, we view $D\chi_E$ as a vector valued, signed measure, and let $|D\chi_E|$ be the positive total variation measure of $D\chi_E$. The first term in (1.2), $|D\chi_E|(D)$, is the $|D\chi_E|$ measure of the entire domain $D$. When $\partial_D E$ is a smooth curve, or a union of smooth curves, $|D\chi_E|(D)$ is just the length of $\partial_D E$. The constant $\lambda$ in (1.1) comes as a Lagrange multiplier from the constraint $|E| = a|D|$.

The functional $J$ is the free energy of a diblock copolymer system in two dimensions. The A-monomers occupy the set $E$ and the B-monomers occupy the set $D \backslash E$. The number $a$ is the block composition fraction. It is the number of the A-monomers divided by the number of all the A- and B- monomers in a polymer chain. The interface between the A-monomer regions and B-monomer regions is $\partial_D E$ whose tension is its length. The connectivity of A and B monomers in a chain molecule is described by the nonlocal term in $J$.

Nishiura and Ohnishi [15] formulated the Ohta-Kawasaki theory on a bounded domain as a singularly perturbed variational problem with a nonlocal term. Ren and Wei [20] showed that (1.2) is a $\Gamma$-limit of the singularly perturbed variational problem. See the last section for more discussion on the Ohta-Kawasaki theory and $\Gamma$-convergence.

Since then much work has been done mathematically to these problems. The lamellar phase is studied by Ren and Wei [20, 22, 23, 27, 28], Fife and Hilhorst [9], Chen and Oshita [2], and Choksi and Sternberg [6]. The result obtained by Müller [14] is also related to the lamellar phase. Radially symmetric bubble and ring patterns are studied by Ren and Wei [21, 26, 29]. The gyroid phase is numerically studied by Teramoto and Nishiura [30]. Triblock copolymers are studied by Ren and Wei [24, 25]. A diblock copolymer - homopolymer blend is studied by Choksi and Ren [5]. Also see Ohnishi and Nishiura [16], Ohnishi *et al* [16], Choksi [3], and Choksi and Ren [4].

## 2 Theorems and implications

The Green's function of $-\Delta$ is denoted by $G$. It is a sum of two parts:

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + R(x,y). \tag{2.1}$$

The regular part of $G(x,y)$ is $R(x,y)$. The Green's function satisfies the equation

$$-\Delta_x G(x,y) = \delta(x-y) - \frac{1}{|D|} \text{ in } D, \quad \partial_{\nu(x)} G(x,y) = 0 \text{ on } \partial D, \quad \overline{G(\cdot,y)} = 0 \text{ for every } y \in D. \tag{2.2}$$

Here $\Delta_x$ is the Laplacian with respect to the $x$-variable of $G$, and $\nu(x)$ is the outward normal direction at $x \in \partial D$. We set

$$F(\xi_1,\xi_2,...,\xi_k) = \sum_{k=1}^{K} R(\xi_k,\xi_k) + \sum_{k=1}^{K}\sum_{l=1,l\neq k}^{K} G(\xi_k,\xi_l), \tag{2.3}$$

for $\xi_k \in D$ and $\xi_k \neq \xi_l$ if $k \neq l$. Because $G(x,y) \to \infty$ if $|x-y| \to 0$ and $R(x,x) \to \infty$ if $x \to \partial D$, $F$ admits at least one global minimum.

The average droplet radius is

$$\rho = \sqrt{\frac{a|D|}{K\pi}}. \tag{2.4}$$

Let $U_1$ be a small neighborhood in $D^K$ of the set $\{\eta :\ F(\eta) = \min_{\kappa\in D^K} F(\kappa)\}$, and $U_2$ be the set

$$U_2 = \{(r_1,r_2,...,r_K) \in R^K :\ r_k \in ((\frac{1}{1+\epsilon})^{1/3}\rho, 2\rho),\ k=1,2,...,K,\ \sum_{k=1}^{\infty} \pi r_k^2 = a|D|\}. \tag{2.5}$$

The particular number $(\frac{1}{1+\epsilon})^{1/3}$ will be used in the proof of Lemma 8.2 and in Appendix C. Define

$$U = U_1 \times U_2. \tag{2.6}$$

Let $\xi_1,\xi_2,...,\xi_K$ be $K$ distinct points in $D$ such that $\xi = (\xi_1,\xi_2,...,\xi_k)$ is in $U_1$, and $r = (r_1,r_2,...,r_k)$ is in $U_2$. Denote the disc centered at $\xi_k$ of radius $r_k$ by $B_k$. The union of the $B_k$'s is $B$:

$$B = \bigcup_{k=1}^{K} B_k = \bigcup_{k=1}^{K} \{x \in R^2 : |x-\xi_k| < r_k\}. \tag{2.7}$$

With $U_1$ close to $\{\eta :\ F(\eta) = \min_{\kappa\in D^K} F(\kappa)\}$ and $\rho$ sufficiently small, the discs $B_k$ are all inside $D$ and disjoint.

The main result of this paper is the following existence theorem.

**Theorem 2.1** *Let $K \geq 2$ be an integer.*

1. *For every $\epsilon > 0$ there exists $\delta > 0$, depending on $\epsilon$, $K$ and $D$ only, such that if*
$$\gamma\rho^3 \log \frac{1}{\rho} > 1 + \epsilon, \tag{2.8}$$
$$|\gamma\rho^3 - 2n(n+1)| > \epsilon n^2, \quad \text{for all } n = 2, 3, 4, ..., \tag{2.9}$$
*and*
$$\rho < \delta, \tag{2.10}$$
*then there exists a solution $E$ of (1.1) with $K$ droplets.*

2. *The radius of each droplet is close to $\rho$.*

3. *Let the centers of these droplets be $\zeta_1$, $\zeta_2$, ..., $\zeta_K$. Then $\zeta = (\zeta_1, \zeta_2, ..., \zeta_K)$, is close to a global minimum of the function $F$.*

We have opted for a rather general existence theorem. The solution found in the theorem is not necessarily stable. The stability of the solution depends on how (2.9) is satisfied.

**Theorem 2.2** *If (2.9) is satisfied because*
$$\gamma\rho^3 - 2n(n+1) < -\epsilon n^2, \text{ for all } n \geq 2, \tag{2.11}$$
*then the droplet solution is stable. Otherwise if (2.9) is satisfied but*
$$\epsilon n^2 < \gamma\rho^3 - 2n(n+1), \text{ and } \gamma\rho^3 - 2(n+1)(n+2) < -\epsilon(n+1)^2 \tag{2.12}$$
*for some $n \geq 2$, then the droplet solution is unstable.*

These two theorems address a number of critical issues in the study of the cylindrical phase of diblock copolymer morphology.

**Parameter range for existence.** When we delete intervals around $2n(n+1)$, $n = 2, 3, ...$, in (2.9), the width of the intervals, $2\epsilon n^2$, grows as $n$ becomes large. At some point an interval will include nearby members in the sequence $2n(n+1)$. When this happens, $\gamma\rho^3$ can not be placed above such $2n(n+1)$. This implies that there exists $C(\epsilon) > 0$ depending on $\epsilon$ such that
$$\gamma < \frac{C(\epsilon)}{\rho^3}. \tag{2.13}$$
Combing this with (2.8) we see that $\rho$ and $\gamma$ are in a somewhat narrow parameter range
$$\rho < \delta, \quad \frac{1+\epsilon}{\rho^3 \log \frac{1}{\rho}} < \gamma < \frac{C(\epsilon)}{\rho^3}. \tag{2.14}$$
Hence $\rho$ must be small and $\gamma$ be appropriately large.

There are three main obstacles to be overcome in the proof of Theorem 2.1. They are droplet coarsening, interface oscillation and droplet translation. Droplet coarsening refers to a phenomenon that some droplets become larger and other droplets become smaller. The condition (2.8) eliminates this phenomenon. Interface oscillation refers to a phenomenon that oscillations appear to the boundary of a droplet. This is ruled out by the gap condition (2.9). Droplet translation means that arbitrarily placed droplets are in general not stable. They tend to move to a particular configuration which turns out to be a minimum of $F$.

The gap condition also suggests bifurcations to oscillating solutions. Elsewhere gap conditions have appeared in constructing layered solutions for singularly perturbed problems. See Malchiodi and Montenegro [11], M. del Pino, M. Kowalczyk and Wei [8], Pacard and Ritoré [18], and the references therein.

**Parameter range for stability.** The solution found in Theorem 2.1 may be unstable because of interface oscillation. The condition (2.11) eliminates this possibility. Under (2.11) $\rho$ and $\gamma$ must satisfy a more stringent requirement

$$\rho < \delta, \quad \frac{1+\epsilon}{\rho^3 \log \frac{1}{\rho}} < \gamma < \frac{12-4\epsilon}{\rho^3}, \quad . \tag{2.15}$$

If (2.12) holds, we have an unstable mode that tends to bring oscillations to the droplet boundaries.

**Droplet sizes and droplet locations.** The droplets in the solution we construct are all close to round discs. They all have the same approximate radius. Theorem 2.1, Part 3, asserts that the droplet centers must minimize $F$ approximately.

When the domain $D$ is the unit disc, the Green's function $G$ and hence $F$ are known explicitly:

$$G(x,y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + \frac{1}{2\pi}\left[\frac{|x|^2}{2} + \frac{|y|^2}{2} + \log \frac{1}{|x\overline{y}-1|}\right] + C \tag{2.16}$$

where the constant $C$ is chosen so that $\overline{G(\cdot, y)} = 0$. Here $\overline{y}$ is the complex conjugate of $y$. $x\overline{y}$ is the complex product of $x$ and $\overline{y}$. Figure 2 shows the droplet patterns through numerical minimization of $F$ for $K = 2$ to 10.

Physicists believe that the droplets must pack in a hexagonal (honeycomb) pattern (see Bates and Fredrickson [1], e.g.). To see a hexagonal pattern here we must have a large number of droplets so that the boundary of the domain has a limited influence. Figure 3 shows a numerical minimizer of $F$ with $K = 100$. We see an almost perfect hexagonal pattern of droplets. We claim that a mathematically rigorous justification for this particularly important pattern in block copolymer morphology is found in this paper.

**Optimal number of droplets.** Let us consider the physically most relevant case in the range (2.15) for the cylindrical phase of diblock copolymers. We assume that $a$ is small and $\gamma$ is a large number of a particular order: $\gamma \sim \frac{1}{a^{3/2} \log \frac{1}{a}}$. More precisely there exists $\mu > 0$ such that

$$\gamma = \frac{\mu}{a^{3/2} \log \frac{1}{a}} = \frac{\mu}{(\frac{K\pi}{|D|})^{3/2} \rho^3 \log \frac{|D|}{K\pi\rho^2}}. \tag{2.17}$$

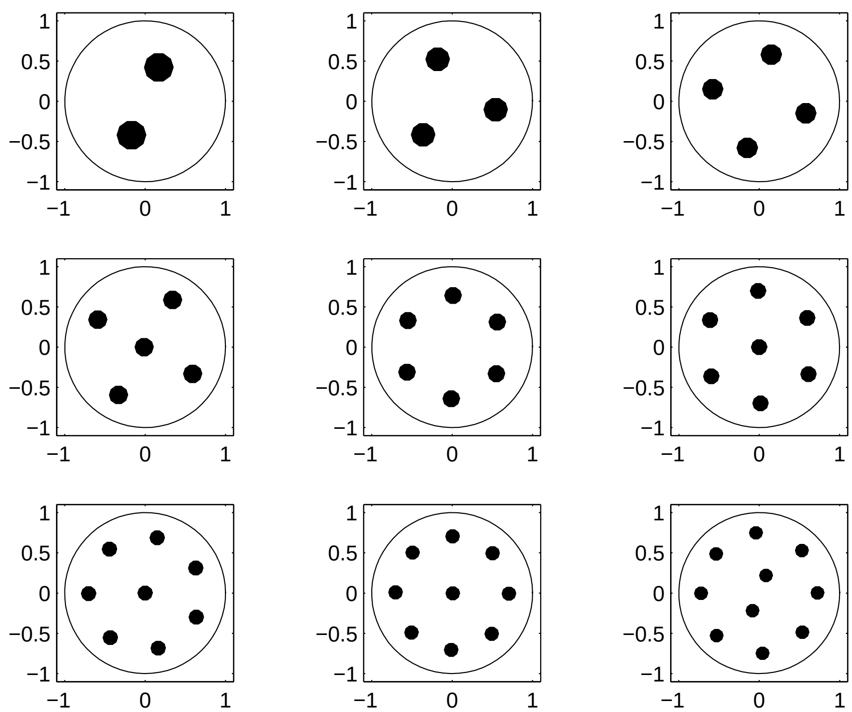


Figure 2: The droplet patterns for $K = 2$ through 10.

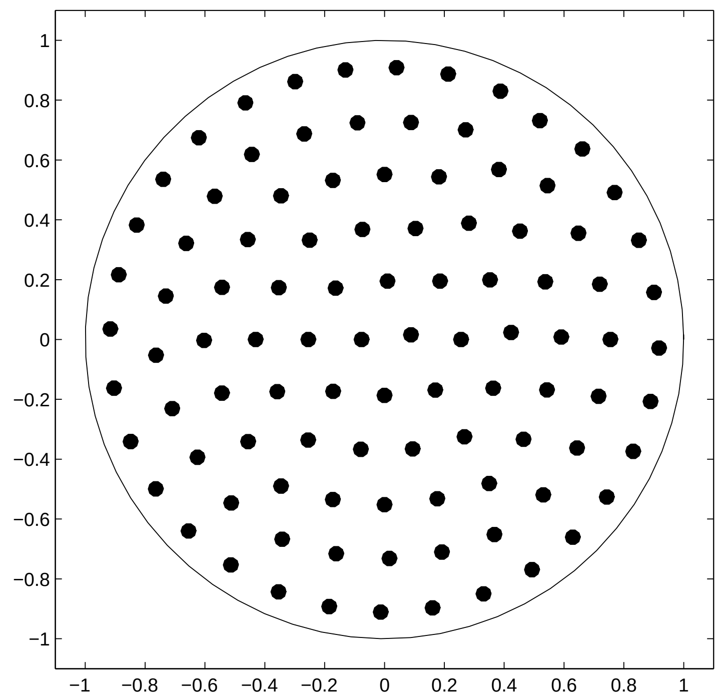


Figure 3: The 100 $\zeta^k$'s determined by numerically minimizing $F$ on a unit disk.

Now $a$ and $\mu$ are the two main parameters of the problem. We hold $\mu$ fixed and make $a$ and hence $\rho$ small. We see that (2.8) is satisfied if

$$\mu > 2(\frac{K\pi}{|D|})^{3/2}, \quad \text{and } \rho \text{ is small.} \tag{2.18}$$

The condition (2.11) is also easily satisfied when $\rho$ is small. So we have a stable droplet solution. With (2.17) and (2.4) the leading order of the free energy is calculated from formula (8.2)

$$2K\pi\rho + (\frac{\pi^2\gamma\rho^4}{2})(\frac{-K\log\rho}{2\pi}) = 2\sqrt{a|D|\pi}\sqrt{K} + \frac{\sqrt{a}|D|^2\mu}{8\pi}\frac{1}{K} + \text{smaller quantity.} \tag{2.19}$$

With respect to $K$ the last quantity is minimized at

$$K \approx \frac{|D|\mu^{3/2}}{4\pi}. \tag{2.20}$$

Note that the choice (2.20) of $K$ does not violate the condition (2.18) of $\mu$. It gives us the optimal number of droplets in a cylindrical pattern.

The theorems are proved by a reduction procedure. In Section 2 we construct a family of approximate solutions that are unions of round discs parametrized by their centers and radii. They form a $3K-1$ dimensional manifold. In Section 3 we perturb each set by perturbing its discs to find a new set in a subspace approximately normal to the manifold. The new sets better approximate a solution of (1.1). With these sets of perturbed discs we have a new manifold that consists of solutions of (1.1) modulo translation and coarsening. In this step we use a fixed point argument, for which we must analyze the linearization of (1.1) at each approximate solution and also the second Fréchet derivative. The main obstacle to the invertibility of the linearized operator is the oscillation phenomenon. We avoid this problem by using condition (2.9). In Section 4 we find a particular set of perturbed discs in the new manifold which solves (1.1) exactly. The centers and radii of the droplets in this particular pattern are found by minimizing $J$ on the new manifold. To show that the minimizer is indeed an exact solution of (1.1), we use a tricky re-parametrization argument.

The main difficulty in this approach lies in the analysis of the nonlocal part of (1.1), such as the proofs of Lemmas 5.3 and 7.1. It involves a singular integral operator similar to the Hilbert transform.

We use $S^1$ to denote the interval $[0, 2\pi]$ with 0 and $2\pi$ identified. The $L^2$ space on $S^1$ is $L^2(S^1)$. The inner product in $L^2(S^1)$ is denoted by $\langle\cdot,\cdot\rangle$. The $L^2$ norm is denoted by $\|\cdot\|_{L^2}$, and the $L^\infty$ norm by $\|\cdot\|_{L^\infty}$. The Sobolev $W^{2,k}$ space is denoted by $H^k(S^1)$ where $k \geq 1$ is an integer. The $W^{2,k}$ norm is denoted by $\|\cdot\|_{H^k}$. We also use a product of $K$ copies of $L^2(S^1)$ on which we have an inherited norm and an inner product, which we still denote by $\|\cdot\|_{L^2}$ and $\langle\cdot,\cdot\rangle$. The reader should be able to tell from the context what we refer to. The inherited norm of a product of $K$ copies of the Sobolev space $W^{2,k}(S^1)$ is also denoted by $\|\cdot\|_{H^k}$.

We use $C$ to denote a positive constant which is independent of $a$, $\rho$, $\gamma$, and the points $(\xi, r)$ in $U$, where $U$ is a given in (2.6). $C$ can only depend on $D$, $K$ and $\epsilon$. The value of $C$ may change from place to place.

We write $e^{i\theta}$ instead of $(\cos\theta, \sin\theta)$ for a simpler notation even though no complex structure is assumed on $R^2$. The reader will see things like $e^{i\theta}\cdot x$ which is simply the inner product of two real vectors $e^{i\theta}$ and $x$.

From now on we are given $\epsilon > 0$, and $\gamma$ and $\rho$ satisfy (2.8) and (2.9).

## 3 Approximate solutions

Recall $B$, the union of $K$ perfect discs with $(\xi_1, ..., \xi_k) \in U_1$ and $(r_1, ..., r_K) \in U_2$. Note that the requirement $|B| = a|D|$ is met because of (2.5). We put $B$ into the left side of (1.1) and check how accurate $B$ is as an approximate solution.

**Lemma 3.1** *When $E$ is $B$, the left side of (1.1), at each $\xi_k + r_k e^{i\theta}$, is*

$$\frac{1}{r_k} + \gamma[-\frac{r_k^2 \log r_k}{2} + \pi r_k^2 R(\xi_k, \xi_k) + \sum_{l \neq k} \pi r_l^2 G(\xi_k, \xi_l)] + O(1).$$

*Proof.* The curvature is $\frac{1}{r_k}$. We compute $v_l = (-\Delta)^{-1}(\chi_{B_l} - \frac{\pi r_l^2}{|D|})$. Define

$$P(x) = \begin{cases} -\frac{|x|^2}{4} + \frac{r_l^2}{4} - \frac{r_l^2}{2} \log r_l, & \text{if } |x| < r_l \\ -\frac{r_l^2}{2} \log |x|, & \text{if } |x| \geq r_l \end{cases}.$$

Then $-\Delta P(\cdot - \xi_l) = \chi_{B_l}$. Write $v_l(x) = P(x - \xi_l) + Q(x, \xi_l)$. Clearly

$$-\Delta Q(x, \xi_l) = -\frac{\pi r_l^2}{|D|}, \ \partial_{\nu(x)} Q(x, \xi_l) = \partial_\nu \frac{r_l^2}{2} \log |x - \xi_l| \text{ on } \partial D, \ \overline{Q(\cdot, \xi_l)} = -\overline{P(|\cdot - \xi_l|)}.$$

Here the Laplacian $\Delta$ and the outward normal derivative $\partial_{\nu(x)}$ are taken with respect to $x$. Note that the Green's function $G$ satisfies the equation (2.2). This shows that $Q(x, \xi_l)$ and $\pi r_l^2 R(x, \xi_l)$ satisfy the same equation and the same boundary condition. Recall that $R$ is the regular part of the Green function $G$. Therefore they can differ only by a constant. This constant is $\overline{Q(\cdot, \xi_l)} - \pi r_l^2 \overline{R(\cdot, \xi_l)}$. But $\overline{v_l} = \overline{G(\cdot, \xi_l)} = 0$ implies that this constant is also

$$-\frac{r_l^2}{2} \overline{\log |\cdot - \xi_l|} - \overline{P(\cdot - \xi_l)} = \frac{\pi r_l^4}{8|D|}.$$

Hence

$$Q(x, \xi_l) = \pi r_l^2 R(x, \xi_l) + \frac{\pi r_l^4}{8|D|},$$

and

$$v_l(x) = P(x - \xi_l) + \pi r_l^2 R(x, \xi_l) + \frac{\pi r_l^4}{8|D|}. \tag{3.1}$$

Let $v = (-\Delta)^{-1}(\chi_B - a) = \sum_l v_l$. Then at $\xi_k + r_k e^{i\theta_k}$

$$\begin{aligned} v(\xi_k + r_k e^{i\theta_k}) &= -\frac{r_k^2 \log r_k}{2} + \pi r_k^2 R(\xi_k + r_k e^{i\theta_k}, \xi_k) + \sum_{l \neq k} \pi r_l^2 G(\xi_k + r_k e^{i\theta_k}, \xi_l) + \sum_{l=1}^{K} \frac{K \pi r_l^4}{8|D|} \\ &= -\frac{r_k^2 \log r_k}{2} + \pi r_k^2 R(\xi_k, \xi_k) + \sum_{l \neq k} \pi r_l^2 G(\xi_k, \xi_l) + O(\rho^3). \end{aligned} \tag{3.2}$$

The lemma follows from (2.13). □

**Lemma 3.2** *The free energy of $B$ is*

$$\begin{aligned} J(B) &= \sum_{k=1}^K 2\pi r_k + \frac{\gamma\pi^2}{2}[\sum_{k=1}^K(-\frac{r_k^4\log r_k}{2\pi} + \frac{r_k^4}{8\pi} + r_k^4 R(\xi_k,\xi_k)) \\ &+ \sum_{k=1}^K\sum_{l\neq k} r_k^2 r_l^2 G(\xi_k,\xi_l) + \sum_{k=1}^K\sum_{l=1}^K(\frac{r_k^2 r_l^4}{8|D|} + \frac{r_k^4 r_l^2}{8|D|})]. \end{aligned}$$

*Proof.* Let $v = (-\Delta)^{-1}(\chi_B - a)$ as in the proof of Lemma 3.1. The local part of $J(B)$ is just the total arc length

$$\sum_{k=1}^K 2\pi r_k. \tag{3.3}$$

The nonlocal part of $J(B)$ is, with the help of (3.1),

$$\begin{aligned} &\frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_B - a)|^2\,dx \\ &= \frac{\gamma}{2}\int_D (\chi_B - a)v(x)\,dx \;=\; \frac{\gamma}{2}\int_D \chi_B v(x)\,dx \;=\; \frac{\gamma}{2}\int_B v(x)\,dx \\ &= \frac{\gamma}{2}\sum_{l=1}^K\sum_{k=1}^K\int_{B_l} v_k(x)\,dx \;=\; \frac{\gamma}{2}\sum_{l=1}^K\sum_{k=1}^K[\int_{B_l} P(x-\xi_k)\,dx + \int_{B_l} Q(x,\xi_k)\,dx]. \end{aligned}$$

There are two possibilities. When $l = k$, from the definition of $P$ we find

$$\int_{B_k} P(x-\xi_k)\,dx = \frac{\pi r_k^4}{8} - \frac{\pi r_k^4\log r_k}{2}. \tag{3.4}$$

For the integral of $Q$, note that, since $\Delta Q(\cdot,\xi_k) = \frac{\pi r_k^2}{|D|}$, $Q(x,\xi_k) - \frac{\pi r_k^2}{4|D|}|x-\xi_k|^2$ is harmonic in $x$. By the Mean Value Theorem for harmonic functions

$$\begin{aligned} \int_{B_k} Q(x,\xi_k)\,dx &= \int_{B_k}(Q(x,\xi_k) - \frac{a}{4K}|x-\xi_k|^2)\,dx + \int_{B_k}\frac{a}{4K}|x-\xi_k|^2\,dx \\ &= \pi r_k^2 Q(\xi_k,\xi_k) + \frac{\pi^2 r_k^6}{8|D|} \;=\; \pi^2 r_k^4 R(\xi_k,\xi_k) + \frac{\pi^2 r_k^6}{4|D|}. \end{aligned} \tag{3.5}$$

When $l \neq k$, for $x \in B_l$, $P(x-\xi_k) = -\frac{r_k^2}{2}\log|x-\xi_k|$ which is harmonic, without singularity, in $B_l$, and hence by the Mean Value Theorem

$$\int_{B_l} P(x-\xi_k)\,dx = -\frac{r_k^2 r_l^2}{2}\log|\xi_l - \xi_k|. \tag{3.6}$$

Also $Q(x,\xi_k) - \frac{a}{4K}|x-\xi_l|^2$ is harmonic in $B_l$, so

$$\begin{aligned} \int_{B_l} Q(x,\xi_k)\,dx &= \int_{B_l}(Q(x,\xi_k) - \frac{\pi r_k^2}{4|D|}|x-\xi_l|^2)\,dx + \int_{B_l}\frac{\pi r_k^2}{4|D|}|x-\xi_l|^2\,dx \\ &= \pi r_l^2 Q(\xi_l,\xi_k) + \frac{\pi^2 r_k^2 r_l^4}{8|D|} \;=\; \pi^2 r_k^2 r_l^2 R(\xi_l,\xi_k) + \frac{\pi^2 r_k^2 r_l^4}{8|D|} + \frac{\pi^2 r_l^2 r_k^4}{8|D|}. \end{aligned} \tag{3.7}$$

The lemma then follows from (3.3), (3.4), (3.5), (3.6), and (3.7). □

## 4 Perturbed discs

We perturb each disc $B_k$ considered in the last section. A perturbed disc denoted by $E_{\phi_k}$ is described by a $2\pi$ periodic function $\phi_k = \phi_k(\theta_k)$, $\theta_k \in [0, 2\pi]$:

$$E_{\phi_k} = \{\xi_k + te^{i\theta_k} : \ \theta_k \in [0, 2\pi], \ t \in [0, \sqrt{r_k^2 + \phi_k(\theta_k)})\} \tag{4.1}$$

Each $\phi_k$ is small compared to $r_k^2$ so that $r_k^2 + \phi_k(\theta_k)$ is positive. The $\phi_k$'s also satisfy

$$\sum_{k=1}^{K} \int_0^{2\pi} \phi_k(\theta_k)\, d\theta_k = 0 \tag{4.2}$$

so that the combined area of the perturbed discs remains $a|D|$:

$$\sum_{k=1}^{K} |E_{\phi_k}| = \sum_k \int_0^{2\pi} \int_0^{\sqrt{r_k^2+\phi_k(\theta_k)}} t\, dt d\theta_k = \sum_k \int_0^{2\pi} (\frac{r_k^2}{2} + \frac{\phi_k(\theta_k)}{2})\, d\theta_k = \sum_k \pi r_k^2 = a|D|.$$

The union of the $E_{\phi_k}$'s is $E_\phi$:

$$E_\phi = \bigcup_{k=1}^{K} E_{\phi_k}. \tag{4.3}$$

We let $\theta = (\theta_1, \theta_2, ..., \theta_K)$ and $\phi(\theta) = (\phi_1(\theta_1), \phi_2(\theta_2), ...\phi_K(\theta_K))$. Note that $\phi$ is not a function from $S^1$ to $R^K$. It is a collection of $K$ functions from $S^1$ to $R$, where each function $\phi_k$ in the collection has its own variable $\theta_k$. We could view $\phi$ as a function from $(S^1)^K$ to $R^K$ with a particular form

$$\phi(\theta_1, \theta_2, ..., \theta_K) = (\phi_1(\theta_1), \phi_2(\theta_2), ..., \phi_K(\theta_K)). \tag{4.4}$$

The arc length of $\partial_D E_\phi$ can be expressed as

$$\sum_{k=1}^{K} |D\chi_{E_{\phi_k}}|(D) = \sum_{k=1}^{K} \int_0^{2\pi} \sqrt{r_k^2 + \phi_k(\theta_k) + \frac{(\phi_k'(\theta_k))^2}{4(r_k^2 + \phi_k(\theta_k))}}\, d\theta_k \tag{4.5}$$

The nonlocal part of $J$ in (1.2) may be written in terms of $\phi$ as

$$\frac{\gamma}{2} \int_D |(-\Delta)^{-1/2}(\chi_{E_\phi} - a)|^2\, dx \quad = \quad \frac{\gamma}{2} \int_{E_\phi} \int_{E_\phi} G(x, y)\, dx dy \tag{4.6}$$

We write the equation (1.1) in terms of $\phi$. The curvature of a point on $\partial_D E_{\phi_k}$ is given by

$$\mathcal{H}_k(\phi_k)(\theta_k) = \frac{r_k^2 + \phi_k(\theta_k) + \frac{3(\phi_k'(\theta_k))^2}{4(r_k^2+\phi_k(\theta_k))} - \frac{\phi_k''(\theta_k)}{2}}{(r_k^2 + \phi_k(\theta_k) + \frac{(\phi_k'(\theta_k))^2}{4(r_k^2+\phi_k(\theta_k))})^{3/2}} \tag{4.7}$$

The nonlocal part of (1.1) is first written as

$$\begin{aligned} &\gamma(-\Delta)^{-1}(\chi_{E_\phi} - a)(\theta_k) \\ = &\sum_{l=1}^{K} \gamma \int_{E_{\phi_l}} G(\xi_k + \sqrt{r_k^2 + \phi_k(\theta_k)} e^{i\theta_k}, y)\, dy \\ = &\sum_{l=1}^{K} \gamma \int_0^{2\pi} \int_0^{\sqrt{r_l^2+\phi_l(\omega_l)}} G(\xi_k + \sqrt{r_k^2 + \phi_k(\theta_k)} e^{i\theta_k}, \xi_l + te^{i\omega_l}) t\, dt d\omega_l \end{aligned} \tag{4.8}$$

**Remark 4.1** *The expressions (4.7) and (4.8) may be obtained by calculating the variations of (4.5) and (4.6) with respect to $\phi$. Then there will be an extra $\frac{1}{2}$ in front of both (4.7) and (4.8).*

There are two cases in the sum over $l$ in (4.8), when $l = k$ we write

$$\begin{aligned}
&\gamma \int_0^{2\pi} \int_0^{\sqrt{r_k^2+\phi_k(\omega_k)}} G(\xi_k + \sqrt{r_k^2 + \phi_k(\theta_k)} e^{i\theta_k}, \xi_k + te^{i\omega_k}) t\, dt d\omega_k \\
&= -\frac{\gamma \log r_k}{2\pi} |E_{\phi_k}| - \frac{\gamma}{2\pi} \int_0^{2\pi} \int_0^{\sqrt{r_k^2+\phi_k(\omega_k)}} \log |\sqrt{1 + \frac{\phi_k(\theta_k)}{r_k^2}} e^{i\theta_k} - \frac{te^{i\omega_k}}{r_k}| t\, dt d\omega_k \\
&\quad + \gamma \int_0^{2\pi} \int_0^{\sqrt{r_k^2+\phi_k(\omega_k)}} R(\xi_k + \sqrt{r_k^2 + \phi_k(\theta_k)} e^{i\theta_k}, \xi_k + te^{i\omega_k}) t\, dt d\omega_k
\end{aligned} \tag{4.9}$$

We denote the three terms in (4.9) by

$$\mathcal{I}_k(\phi_k)(\theta_k) = -\frac{\gamma \log r_k}{2\pi} |E_{\phi_k}| = -\frac{\gamma r_k^2 \log r_k}{2} + \frac{1}{2} \int_0^{2\pi} \phi^k(\theta_k) d\theta_k \tag{4.10}$$

$$\mathcal{A}_k(\phi_k)(\theta_k) = -\frac{\gamma}{2\pi} \int_0^{2\pi} \int_0^{\sqrt{r_k^2+\phi_k(\omega_k)}} \log |\sqrt{1 + \frac{\phi_k(\theta_k)}{r_k^2}} e^{i\theta_k} - \frac{te^{i\omega_k}}{r_k}| t\, dt d\omega_k \tag{4.11}$$

$$\mathcal{B}_k(\phi_k)(\theta_k) = \gamma \int_0^{2\pi} \int_0^{\sqrt{r_k^2+\phi_k(\omega_k)}} R(\xi_k + \sqrt{r_k^2 + \phi_k(\theta_k)} e^{i\theta_k}, \xi_k + te^{i\omega_k}) t\, dt d\omega_k \tag{4.12}$$

When $l \neq k$ in (4.8) we let

$$\mathcal{C}_{kl}(\phi_k, \phi_l)(\theta_k) = \gamma \int_0^{2\pi} \int_0^{\sqrt{r_l^2+\phi_l(\omega_l)}} G(\xi_k + \sqrt{r_k^2 + \phi_k(\theta_k)} e^{i\theta_k}, \xi_l + te^{i\omega_l}) t\, dt d\omega_l \tag{4.13}$$

The left side of (1.1) now becomes

$$\mathcal{H}_k(\phi_k)(\theta_k) + \mathcal{I}_k(\phi_k)(\theta_k) + \mathcal{A}_k(\phi_k)(\theta_k) + \mathcal{B}_k(\phi_k)(\theta_k) + \sum_{l \neq k} \mathcal{C}_{kl}(\phi_k, \phi_l)(\theta_k)$$

at $\xi_k + \sqrt{r_k^2 + \phi_k(\theta_k)} e^{i\theta_k}$. Let us define $\mathcal{S}$ by

$$\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_k) \tag{4.14}$$

where

$$\mathcal{S}_k(\phi)(\theta_k) = \mathcal{H}_k(\phi_k)(\theta_k) + \mathcal{I}(\phi_k)_k(\theta_k) + \mathcal{A}_k(\phi_k)(\theta_k) + \mathcal{B}_k(\phi_k)(\theta_k) + \sum_{l \neq k} \mathcal{C}_{kl}(\phi_k, \phi_l)(\theta_k) + \lambda(\phi). \tag{4.15}$$

Here $\lambda(\phi)$ is a number, independent of $k$. It is given by

$$\lambda(\phi) = -\frac{1}{K} \sum_{k=1}^{K} \overline{[\mathcal{H}_k(\phi_k) + \mathcal{I}_k(\phi_k) + \mathcal{A}_k(\phi_k) + \mathcal{B}_k(\phi_k) + \sum_{l \neq k} \mathcal{C}_{kl}(\phi_k, \phi_l)]}. \tag{4.16}$$

The bar over the quantity here stands for the average of the quantity over $[0, 2\pi]$. With this definition of $\lambda$,

$$\sum_{k=1}^{K} \overline{\mathcal{S}_k(\phi_k)} = 0. \tag{4.17}$$

The operator $\mathcal{S}$ maps from

$$\mathcal{X} = \{\phi = (\phi_1, \phi_2, ..., \phi_k) : \ \phi_k \in H^2(S^1), \ k = 1, 2, ..., K, \ \sum_{k=1}^{K} \overline{\phi_k} = 0\} \tag{4.18}$$

to

$$\mathcal{Y} = \{q = (q_1, q_2, ..., q_k) : \ q_k \in L^2(S^1), \ k = 1, 2, ..., K, \ \sum_{k=1}^{K} \overline{q_k} = 0\} \tag{4.19}$$

The equation (1.1) now becomes

$$\mathcal{S}(\phi) = 0. \tag{4.20}$$

By defining

$$\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_K), \text{ where } \mathcal{C}_k(\phi_1, \phi_2, ..., \phi_k) = \sum_{l \neq k} \mathcal{C}_{kl}(\phi_k, \phi_l), \tag{4.21}$$

we write

$$\mathcal{S} = \mathcal{H} + \mathcal{I} + \mathcal{A} + \mathcal{B} + \mathcal{C} + \lambda. \tag{4.22}$$

In the map $\mathcal{S}$ the inputs $\phi_1$, $\phi_2$, ..., $\phi_k$ only interact in $\mathcal{C}$ and $\lambda$. The other operators can be written in the block matrix form

$$\mathcal{H} = \begin{bmatrix} \mathcal{H}_1 & 0 & ... & 0 \\ 0 & \mathcal{H}_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \mathcal{H}_K \end{bmatrix}, \ \mathcal{I} = \begin{bmatrix} \mathcal{I}_1 & 0 & ... & 0 \\ 0 & \mathcal{I}_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \mathcal{I}_K \end{bmatrix}, \tag{4.23}$$

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & 0 & ... & 0 \\ 0 & \mathcal{A}_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \mathcal{A}_K \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} \mathcal{B}_1 & 0 & ... & 0 \\ 0 & \mathcal{B}_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \mathcal{B}_K \end{bmatrix}, \tag{4.24}$$

where each entry in a matrix is an operator from $H^2(S^1)$ to $L^2(S^1)$. The scalar operator $\lambda$ gives the projection $(\lambda_\xi(\phi), \lambda_\xi(\phi), ..., \lambda_\xi(\phi))$ of $\mathcal{H}(\phi) + \mathcal{I}(\phi) + \mathcal{A}(\phi) + \mathcal{B}(\phi) + \mathcal{C}(\phi)$ to the one dimensional space spanned by $(1, 1, ..., 1)$.

Let us write down the first Fréchet derivatives of these operators. We set

$$\tilde{E}_{\phi_k} = \{\alpha e^{i\theta_k} : \ \theta_k \in [0, 2\pi], \ \alpha \in [0, \sqrt{1 + \phi_k(\theta_k)/r_k^2})\} \tag{4.25}$$

to be a shifted and re-scaled version of $E_{\phi_k}$. Denote the derivatives of $\mathcal{H}_k$ against $\phi_k$, $\phi_k'$ and $\phi_k''$ by $\mathcal{H}_{k,1}$, $\mathcal{H}_{k,2}$ and $\mathcal{H}_{k,3}$ respectively. Calculations show that

$$\mathcal{H}_k'(\phi_k)(u_k) \ = \ \mathcal{H}_{k,1}(\phi_k)u_k + \mathcal{H}_{k,2}(\phi_k)u_k' + \mathcal{H}_{k,3}(\phi_k)u_k'' \tag{4.26}$$

$$\mathcal{I}'(\phi_k)(u_k) = -\frac{\gamma \log r_k}{4\pi} \int_0^{2\pi} u_k(\theta_k)\, d\theta_k \tag{4.27}$$

$$\begin{aligned}\mathcal{A}_k'(\phi_k)(u_k)(\theta_k) &= -\frac{\gamma}{4\pi} \int_0^{2\pi} u_k(\omega_k) \log |\sqrt{1+\frac{\phi_k(\theta_k)}{r_k^2}} e^{i\theta_k} - \sqrt{1+\frac{\phi_k(\omega_k)}{r_k^2}} e^{i\omega_k}|\, d\omega_k \\ &\quad -\frac{\gamma u_k(\theta_k)}{4\pi\sqrt{1+\phi_k(\theta_k)/r_k^2}} \int_{\tilde{E}_{\phi_k}} \frac{(\sqrt{1+\frac{\phi_k(\theta_k)}{r_k^2}} e^{i\theta_k} - y)\cdot e^{i\theta_k}}{|\sqrt{1+\frac{\phi_k(\theta_k)}{r_k^2}} e^{i\theta_k} - y|^2}\, dy. \end{aligned} \tag{4.28}$$

$$\begin{aligned}\mathcal{B}_k'(\phi_k)(u_k)(\theta_k) &= \frac{\gamma}{2} \int_0^{2\pi} u_k(\omega_k) R(\xi_k + \sqrt{r_k^2+\phi_k(\theta_k)} e^{i\theta_k}, \xi_k + \sqrt{r_k^2+\phi_k(\omega_k)} e^{i\omega_k})\, d\omega_k \\ &\quad + \frac{\gamma u_k(\theta_k)}{2\sqrt{r_k^2+\phi_k(\theta_k)}} \int_{E_{\phi_k}} \nabla R(\xi_k + \sqrt{r_k^2+\phi_k(\theta_k)} e^{i\theta_k}, y)\cdot e^{i\theta_k}\, dy. \end{aligned} \tag{4.29}$$

$$\begin{aligned}\mathcal{C}_{kl}'(\phi_k,\phi_l)(u_k,u_l)(\theta_k) &= \frac{\gamma}{2} \int_0^{2\pi} u_l(\omega_l) G(\xi_k + \sqrt{r_k^2+\phi_k(\theta_k)} e^{i\theta_k}, \xi_l + \sqrt{r_k^2+\phi_l(\omega_l)} e^{i\omega_l})\, d\omega_l \\ &\quad + \frac{\gamma u_k(\theta_k)}{2\sqrt{r_k^2+\phi_k(\theta_k)}} \int_{E_{\phi_l}} \nabla G(\xi_k + \sqrt{r_k^2+\phi_k(\theta_k)} e^{i\theta_k}, y)\cdot e^{i\theta_k}\, dy. \end{aligned} \tag{4.30}$$

The derivative

$$\lambda_\xi'(\phi_1,\phi_2,...,\phi_k)(u_1,u_2,...,u_k) \tag{4.31}$$

is so chosen that

$$\sum_{k=1}^K \overline{\mathcal{S}_k'(u)} = 0. \tag{4.32}$$

# 5 A linear operator

Let $\mathcal{L}$ be the linearized operator of $\mathcal{S}$ at $\phi = 0$, i.e.

$$\mathcal{L} = \mathcal{S}'(0). \tag{5.1}$$

Going back to (4.26), (4.27), (4.28), (4.29) and (4.30) we find that

$$\begin{aligned}\mathcal{H}_k'(0)(u_k) &= -\frac{1}{2r_k^3}(u_k'' + u_k) \\ \mathcal{I}_k'(0)(u_k) &= -\frac{\gamma \log r_k}{4\pi} \int_0^{2\pi} u_k(\theta_k)\, d\theta_k \\ \mathcal{A}_k'(0)(u_k)(\theta_k) &= -\frac{\gamma}{4\pi} \int_0^{2\pi} u_k(\omega_k) \log |e^{i\theta_k} - e^{i\omega_k}|\, d\omega_k - \frac{\gamma u_k(\theta_k)}{4} \\ \mathcal{B}_k'(0)(u_k)(\theta_k) &= \frac{\gamma}{2} \int_0^{2\pi} u_k(\omega_k) R(\xi_k + r_k e^{i\theta_k}, \xi_k + r_k e^{i\omega_k})\, d\omega_k \\ &\quad + \frac{\gamma u_k(\theta_k)}{2r_k} \int_{B_k} \nabla R(\xi_k + r_k e^{i\theta_k}, y)\cdot e^{i\theta_k}\, dy \end{aligned}$$

$$\begin{aligned}\mathcal{C}'_{kl}(0,0)(u_k,u_l)(\theta_k) &= \frac{\gamma}{2}\int_0^{2\pi} u_l(\omega_l)G(\xi_k+r_ke^{i\theta_k},\xi_l+r_le^{i\omega_l})\,d\omega_l \\ &\quad +\frac{\gamma u_k(\theta_k)}{2r_k}\int_{B_l}\nabla G(\xi_k+r_ke^{i\theta_k},y)\cdot e^{i\theta_k}\,dy.\end{aligned}$$

The derivation of $\mathcal{A}'(0)$ is explained in more detail in Appendix A.

Let us separate $\mathcal{L}$ to a dominant part $\mathcal{L}_1$ and a minor part $\mathcal{L}_2$. We define $\mathcal{L}_{1,k}$, the $k$-th component of $\mathcal{L}_1$, to be

$$\begin{aligned}\mathcal{L}_{1,k}(u)(\theta_k) &= -\frac{1}{2r_k^3}(u_k''(\theta_k)+u_k(\theta_k)) - \frac{\gamma\log r_k}{4\pi}\int_0^{2\pi}u_k(\theta_k)\,d\theta_k \\ &\quad -\frac{\gamma}{4\pi}\int_0^{2\pi}u_k(\omega_k)\log|e^{i\theta_k}-e^{i\omega_k}|\,d\omega_k - \frac{\gamma u_k(\theta_k)}{4} \\ &\quad +\frac{\gamma R(\xi_k,\xi_k)}{2}\int_0^{2\pi}u_k(\theta_k)\,d\theta_k + \sum_{l\neq k}\frac{\gamma G(\xi_k,\xi_l)}{2}\int_0^{2\pi}u_l(\theta_l)\,d\theta_l \\ &\quad +l_1(u).\end{aligned}$$

The real valued linear operator $l_1$ is independent of $k$. It is so chosen that $\mathcal{L}_1$ maps from $\mathcal{X}$ to $\mathcal{Y}$. The rest of $\mathcal{L}$ is denoted by $\mathcal{L}_2$.

We are more interested in the operators $\Pi\mathcal{L}$ and $\Pi\mathcal{L}_1$ where $\Pi$ is the orthogonal projection operator from $\mathcal{Y}$ to

$$\mathcal{Y}_* = \{q=(q_1,...,q_k)\in\mathcal{Y}:\ q_k\perp\cos\theta_k,\ q_k\perp\sin\theta_k,\ q_k\perp 1,\ k=1,...,K\}. \tag{5.2}$$

The operator $\Pi\mathcal{L}_\xi$ is defined on

$$\mathcal{X}_* = \{q=(q_1,...,q_k)\in\mathcal{X}:\ q_k\perp\cos\theta_k,\ q_k\perp\sin\theta_k,\ q_k\perp 1,\ k=1,...,K\}. \tag{5.3}$$

We use the same $\Pi$ to denote the orthogonal projection from

$$L^2(S^1)\ \text{to}\ \{q_k\in L^2(S^1): q_k\perp\cos\theta_k,\ q_k\perp\sin\theta_k,\ q_k\perp 1\}. \tag{5.4}$$

**Lemma 5.1** *Consider $\Pi\mathcal{L}_1$ as an operator from $\mathcal{X}_*$ to $\mathcal{Y}_*$. The eigenvalues of $\Pi\mathcal{L}_1$ are*

$$\lambda_{k,n} = \frac{n^2-1}{2r_k^3}+\frac{\gamma}{4n}-\frac{\gamma}{4},\quad k=1,2,...,K,\ n=2,3,4,... \tag{5.5}$$

*whose multiplicity is 2. The corresponding eigenvectors are*

$$(0,0,0,...,\cos n\theta_k,...,0),\ (0,0,0,...,\sin n\theta_k,...,0).$$

*Proof.* In $\mathcal{X}_*$, $\mathcal{L}_1$ is simplified to

$$\mathcal{L}_{1,k}(u) = -\frac{1}{2r_k^3}(u_k''+u_k) - \frac{\gamma}{4\pi}\int_0^{2\pi}u_k(\omega_k)\log|e^{i\theta_k}-e^{i\omega_k}|\,d\omega_k - \frac{\gamma u_k(\theta_k)}{4}.$$

Note that $\Pi\mathcal{L}_1 = \mathcal{L}_1$ on $\mathcal{X}_*$. The spectrum of $\Pi\mathcal{L}_{\xi,1}$ is best computed using Fourier series. The Fourier space of $\mathcal{X}_*$ is

$$\widehat{\mathcal{X}_*} = \{(\{l_{1,n_1}\},\{l_{2,n_2}\},...,\{l_{K,n_K}\}) : \sum_{n_k=-\infty}^{\infty} |l_{k,n_k}|^2 < \infty, \quad l_{k,0} = l_{k,\pm 1} = 0, \quad k = 1,2,...,K\} \tag{5.6}$$

Let

$$\widehat{u_k}(n_k) = \int_0^{2\pi} u_k(\theta_k) e^{-in_k\theta_k}\, d\theta_k$$

be the $n$-th Fourier coefficient of $u_k$, then

$$\widehat{\mathcal{L}_{1,k}(u)}(n_k) = [\frac{n_k^2 - 1}{2r_k^3} + \gamma(\frac{1}{4|n_k|} - \frac{1}{4})]\widehat{u_k}(n_k). \tag{5.7}$$

Here we have used the well known formula

$$\log|1 - e^{i\theta}| = \log|2\sin(\frac{\theta}{2})| = -\sum_{n=1}^{\infty} \frac{\cos n\theta}{n}. \tag{5.8}$$

See Tolstov [31, Page 93] for instance.

The eigenvalues are easily found to be

$$\lambda_{k,n} = \frac{n^2 - 1}{2r_k^3} + \gamma(\frac{1}{4n} - \frac{1}{4}), \quad k = 1,2,...,K, \quad n = 2,3,...$$

whose multiplicity is 2. The corresponding eigenvectors are

$$(0,0,...,0,\cos n\theta_k,0,...,0) \text{ and } (0,0,...,0,\sin n\theta_k,0,...,0).$$

This proves the lemma. □

The second part $\mathcal{L}_2$ is a minor operator.

**Lemma 5.2** *There exists $C > 0$ independent of $\xi$, $r$, $\rho$ and $\gamma$ such that for all $u \in \mathcal{Y}_*$, $\|\mathcal{L}_2(u)\|_{L^2} \leq \frac{C}{\rho^2}\|u\|_{L^2}$.*

*Proof.* Let $\mathcal{L}_{2,k}$ be the $k$-th component of $\mathcal{L}_2$. Then

$$\begin{aligned}
\mathcal{L}_{2,k}(u)(\theta_k) \quad = \quad & \frac{\gamma}{2}\int_0^{2\pi} u_k(\omega_k)(R(\xi_k + \rho e^{i\theta_k}, \xi_k + \rho e^{i\omega_k}) - R(\xi_k,\xi_k))\, d\omega_k \\
& + \frac{\gamma u_k(\theta_k)}{2r_k}\int_{B_k} \nabla R(\xi_k + r_k e^{i\theta_k}, y)\cdot e^{i\theta_k}\, dy \\
& + \sum_{l\neq k} \frac{\gamma}{2}\int_0^{2\pi} u_l(\omega_l)(G(\xi_k + r_k e^{i\theta_k}, \xi_l + r_l e^{i\omega_l}) - G(\xi_k,\xi_l))\, d\omega_l \\
& + \sum_{l\neq k} \frac{\gamma u_k(\theta_k)}{2r_k}\int_{B_l} \nabla G(\xi_k + r_k e^{i\theta_k}, y)\cdot e^{i\theta_k}\, dy \\
& + l_2(u)
\end{aligned}$$

where $l_2(u)$ is real valued and independent of $k$. It is included so that $\mathcal{L}_2(u)$ is in $\mathcal{Y}_*$.

Because

$$\begin{aligned} R(\xi_k + r_k e^{i\theta_k}, \xi_k + r_k e^{i\omega_k}) - R(\xi_k, \xi_k) &= O(\rho), \\ G(\xi_k + r_k e^{i\theta_k}, \xi_l + r_l e^{i\omega_l}) - G(\xi_k, \xi_l) &= O(\rho), \end{aligned}$$

we obtain that

$$\begin{aligned} \|\frac{\gamma}{2}\int_0^{2\pi} u_k(\omega_k)(R(\xi_k + r_k e^{i\theta_k}, \xi_k + r_k e^{i\omega_k}) - R(\xi_k, \xi_k))\, d\omega_k\|_{L^2} &\leq C\gamma\rho\|u\|_{L^2} \\ \|\frac{\gamma}{2}\int_0^{2\pi} u_l(\omega_l)(G(\xi_k + r_k e^{i\theta_k}, \xi_l + r_l e^{i\omega_l}) - G(\xi_k, \xi_l))\, d\omega_l\|_{L^2} &\leq C\gamma\rho\|u_k\|_{L^2}; \end{aligned}$$

since the area of $B_k$ is $\pi\rho^2$,

$$\begin{aligned} \|\frac{\gamma u_k(\theta_k)}{2r_k}\int_{B_k} \nabla R(\xi_k + r_k e^{i\theta_k}, y)\cdot e^{i\theta_k}\, dy\|_{L^2} &\leq C\gamma\rho\|u_k\|_{L^2} \\ \|\frac{\gamma u_k(\theta_k)}{2r_k}\int_{B_l} \nabla G(\xi_k + r_k e^{i\theta_k}, y)\cdot e^{i\theta_k}\, dy\|_{L^2} &\leq C\gamma\rho\|u_k\|_{L^2}. \end{aligned}$$

The condition

$$\sum_{k=1}^{K} \overline{\mathcal{L}_{2,k}(u)(\theta_k)} = 0$$

implies that

$$|l_2(u)| \leq C\gamma\rho\|u\|_{L^2}.$$

The lemma then follows, with the help of (2.13). □

**Lemma 5.3** *1. There exists $C > 0$ such that*

$$\|u\|_{H^2} \leq C\rho^3\|\Pi\mathcal{L}(u)\|_{L^2}$$

*for all $u \in \mathcal{X}_*$.*

*2. If (2.11) is satisfied, then for $u \in \mathcal{X}_*$*

$$\|u\|_{H^1}^2 \leq C\rho^3\langle\Pi\mathcal{L}(u), u\rangle.$$

*3. The operator $\Pi L_\xi$ is invertible from $\mathcal{X}_*$ to $\mathcal{Y}_*$.*

*Proof.* The condition (2.9) implies that

$$\frac{|\lambda_{k,n}|}{n^2} > \frac{\epsilon(n-1)}{4nr_k^3} \geq \frac{C}{\rho^3}, \quad n = 2, 3, \ldots.$$

Therefore

$$\|u\|_{H^2} \leq C\rho^3\|\Pi\mathcal{L}_1(u)\|_{L^2}. \tag{5.9}$$

Lemma 5.2 then implies that when $\rho$ is small,

$$\|\Pi\mathcal{L}(u)\|_{L^2} \geq \|\Pi\mathcal{L}_1(u)\|_{L^2} - \|\Pi\mathcal{L}_2(u)\|_{L^2} \geq \frac{C}{\rho^3}\|u\|_{H^2} - \frac{C}{\rho^2}\|u\|_{L^2} \geq \frac{C}{\rho^3}\|u\|_{H^2},$$

proving Part 1 of the lemma.

If (2.11) holds,

$$\frac{\lambda_n}{n^2} \geq \frac{C}{\rho^3} \quad n = 2, 3, \ldots.$$

This implies that

$$\langle \Pi\mathcal{L}_1(u), u\rangle \geq \frac{C}{\rho^3}\|u\|_{H^1}^2.$$

By Lemma 5.2 we deduce that

$$\langle \Pi\mathcal{L}(u), u\rangle = \langle \Pi\mathcal{L}_1(u), u\rangle - \langle \Pi\mathcal{L}_2(u), u\rangle \geq \frac{C}{\rho^3}\|u\|_{H^1}^2 - \frac{C}{\rho^2}\|u\|_{L^2}^2 \geq \frac{C}{\rho^3}\|u\|_{H^1}^2,$$

proving Part 2.

The last part is proved by a weaker version of Part 1:

$$\|u\|_{L^2} \leq C\rho^3\|\Pi\mathcal{L}\|_{L^2}. \tag{5.10}$$

This ensures that $\Pi\mathcal{L}$ is one-to-one from $\mathcal{X}_*$ to $\mathcal{Y}_*$. Since $\Pi\mathcal{L}$ is self-adjoint and hence closed, (5.10) also ensures that the range of $\Pi\mathcal{L}$ is closed. The Closed Range Theorem (See Yosida [32, Page 205], e.g.) then implies that $\Pi\mathcal{L}$ is onto. □

# 6 The Second Fréchet derivative

**Lemma 6.1** *Suppose that $\|\phi\|_{H^2} \leq c\rho^2$ where $c$ is sufficiently small. The following holds*

1. $\|\mathcal{H}_k''(\phi_k)(u_k, v_k)\|_{L^2} \leq \frac{C}{\rho^5}\|u_k\|_{H^2}\|v_k\|_{H^2}$.

2. $\|\mathcal{A}_k''(\phi_k)(u_k, v_k)\|_{L^2} \leq \frac{C}{\rho^5}\|u_k\|_{H^1}\|v_k\|_{H^1}$.

3. $\|\mathcal{B}_k''(\phi_k)(u_k, v_k)\|_{L^2} \leq \frac{C}{\rho^4}\|u_k\|_{H^1}\|v_k\|_{H^1}$.

4. $\|\mathcal{C}_{kl}''(\phi_k, \phi_l)(u_k, u_l)(v_k, v_l)\|_{L^2} \leq \frac{C}{\rho^4}(\|u_k\|_{H^1} + \|u_l\|_{H^1})(\|v_k\|_{H^1} + \|v_l\|_{H^1})$.

5. $|\lambda''(\phi)(u, v)| \leq \frac{C}{\rho^5}\|u\|_{H^2}\|v\|_{H^2}$.

*Note that $\mathcal{I}'' = 0$.*

*Proof.* Note that by taking $c$ small, we keep $\rho^2 + \phi$ positive, so $E_\phi$ is a perturbed disc.

$\mathcal{H}_k$ may be better understood after re-scaling. Introduce

$$\Phi = \frac{\phi_k}{r_k^2}, \;\; \Phi' = \frac{\phi_k'}{r_k^2}, \;\; \Phi'' = \frac{\phi_k''}{r_k^2},$$

and

$$\tilde{H}(\Phi, \Phi', \Phi'') = r_k \mathcal{H}_k(\phi, \phi_k', \phi_k'').$$

Then

$$\tilde{H}(\Phi, \Phi', \Phi'') = \frac{1 + \Phi + \frac{3(\Phi')^2}{4(1+\Phi)} - \frac{\Phi''}{2}}{(1 + \Phi + \frac{(\Phi')^2}{4(1+\Phi)})^{3/2}}$$

does not involve $r_k$. The condition $\|\phi_k\|_{H^2} \le c\rho^2$ with a small $c$ means that $\|\Phi\|_{H^2}$ is small compared to 1. With $\tilde{H}_1(\Phi)$, $\tilde{H}_2(\Phi)$, and $\tilde{H}_3(\Phi)$ denoting the derivatives of $H(\Phi, \Phi', \Phi'')$ with respect to its three arguments, the second Fréchet derivative of $\tilde{H}$ is

$$\begin{aligned}
&\tilde{H}''(\Phi, \Phi', \Phi'')(u_k, v_k) \\
&= \tilde{H}_{11}(\Phi) u_k v_k + \tilde{H}_{22}(\Phi) u_k' v_k' + \tilde{H}_{12}(\Phi)(u_k' v_k + u_k v_k') \\
&\quad + \tilde{H}_{23}(\Phi)(u_k' v_k'' + u_k'' v_k') + \tilde{H}_{31}(u_k'' v_k + u_k v_k'')
\end{aligned}$$

Note that we do not have $u_k'' v_k''$ on the right side since $\tilde{H}_{33} = 0$. Because of this absence, the Sobolev Embedding Theorem implies that

$$\|\tilde{H}''(\Phi)(u_k, v_k)\|_{L^2} \le C \|u_k\|_{H^2} \|v_k\|_{H^2}$$

In terms of $\mathcal{H}_k$ and $\phi_k$,

$$\|\mathcal{H}_k''(\phi)(u_k, v_k)\|_{L^2} \le \frac{C}{\rho^5} \|u_k\|_{H^2} \|v_k\|_{H^2}. \tag{6.1}$$

This proves Part 1.

To prove Part 2, let us again set $\Phi = \frac{\phi_k}{r_k^2}$ and introduce

$$A(\Phi)(\theta) = \int_0^{2\pi} \int_0^{\sqrt{1+\Phi(\omega)}} \log |\sqrt{1 + \Phi(\theta)} e^{i\theta} - s e^{i\omega}| s \, ds d\omega. \tag{6.2}$$

In our estimation of $\mathcal{A}_k''$ and $\mathcal{B}_k''$ we write $\theta$ instead of $\theta_k$ and $\omega$ instead of $\omega_l$ for simplicity. Then

$$\mathcal{A}_k(\phi_k) = -\frac{\gamma r_k^2}{2\pi} A(\Phi) \tag{6.3}$$

The change from $\phi_k$ and $\mathcal{A}$ to $\Phi$ and $A$ scales away $r_k$. The first Fréchet derivative of $A$ is given by

$$\begin{aligned}
A'(\Phi)(u_k)(\theta) &= \frac{1}{2} \int_0^{2\pi} u_k(\omega) \log |\sqrt{1 + \Phi(\theta)} e^{i\theta} - \sqrt{1 + \Phi(\omega)} e^{i\omega}| \, d\omega \\
&\quad + \frac{u_k(\theta)}{2\sqrt{1 + \Phi(\theta)}} \int_0^{2\pi} \int_0^{\sqrt{1+\Phi(\omega)}} \frac{(\sqrt{1 + \Phi(\theta)} e^{i\theta} - s e^{i\omega}) \cdot e^{i\theta}}{|\sqrt{1 + \Phi(\theta)} e^{i\theta} - s e^{i\omega}|^2} s \, ds d\omega
\end{aligned} \tag{6.4}$$

The second Fréchet derivative of $A$ is

$$A''(\Phi)(u_k, v_k) = A_1(\Phi)(u_k, v_k)+A_2(\Phi)(u_k, v_k)+A_3(\Phi)(u_k, v_k)+A_4(\Phi)(u_k, v_k)+A_5(\Phi)(u_k, v_k) \quad (6.5)$$

where

$$\begin{aligned}
A_1(\Phi)(u_k, v_k) &= \frac{v_k(\theta)e^{i\theta}}{4\sqrt{1+\Phi(\theta)}} \cdot \int_0^{2\pi} K(\theta,\omega)u_k(\omega)\, d\theta \\
A_2(\Phi)(u_k, v_k) &= \frac{u_k(\theta)e^{i\theta}}{4\sqrt{1+\Phi(\theta)}} \int_0^{2\pi} K(\theta,\omega)v_k(\omega)\, d\omega \\
A_3(\Phi)(u_k, v_k) &= -\frac{1}{4}\int_0^{2\pi} K(\theta,\omega) \cdot \frac{u_k(\omega)v_k(\omega)e^{i\theta}}{\sqrt{1+\Phi(\omega)}}\, d\omega \\
A_4(\Phi)(u_k, v_k) &= \frac{u_k(\theta)v_k(\theta)}{4(1+\Phi(\theta))} \int_{\tilde{E}_{\phi_k}} \frac{|\sqrt{1+\Phi(\theta)}e^{i\theta} - y|^2 - 2(\sqrt{1+\Phi(\theta)} - e^{i\theta}\cdot y)^2}{|\sqrt{1+\Phi(\theta)}e^{i\theta} - y|^4}\, dy \\
A_5(\Phi)(u_k, v_k) &= -\frac{u_k(\theta)v_k(\theta)}{4(1+\Phi(\theta))^{3/2}} \int_0^{2\pi}\int_0^{\sqrt{1+\Phi(\omega)}} \frac{(\sqrt{1+\Phi(\theta)}e^{i\theta} - se^{i\omega})\cdot e^{i\theta}}{|\sqrt{1+\Phi(\theta)}e^{i\theta} - se^{i\omega}|^2} s\, ds d\omega
\end{aligned}$$

where $\tilde{E}_{\phi_k}$ is given in (4.25). The kernel $K$ is

$$K(\theta,\omega) = \frac{\sqrt{1+\Phi(\theta)}e^{i\theta} - \sqrt{1+\Phi(\omega)}e^{i\omega}}{|\sqrt{1+\Phi(\theta)}e^{i\theta} - \sqrt{1+\Phi(\omega)}e^{i\omega}|^2} \quad (6.6)$$

Here we encounter a singular integral operator

$$\mathcal{K}(u_k)(\theta) = \int_0^{2\pi} K(\theta,\omega)u_k(\omega)\, d\omega \quad (6.7)$$

since the singularity of $K(\theta,\omega)$ is of the type $\frac{\theta-\omega}{|\theta-\omega|^2}$. This operator is very much like the Hilbert transform. To define the operator properly, we first write

$$\mathcal{K}(u_k)(\theta) = \int_0^{2\pi} K(\theta,\omega)(u_k(\omega) - u_k(\theta))\, d\omega + u_k(\theta)\int_0^{2\pi} K(\theta,\omega)\, d\omega. \quad (6.8)$$

For $u_k \in H^1(S^1) \subset H^2(S^1)$, $u_k$ is Hölder continuous. Hence

$$|u_k(\omega) - u_k(\theta)| \le |\omega - \theta|^\alpha \|u_k\|_{C^\alpha}$$

for some $\alpha \in (0,1)$. Therefore

$$|K(\theta,\omega)(u_k(\omega) - u_k(\theta))| \le C|\omega-\theta|^{-1+\alpha}\|u_k\|_{C^\alpha},$$

and the first term in (6.8) is convergent. Here $\|u_k\|_\alpha$ is the $C^\alpha$ norm of $u_k$. The second term is defined by its principal part:

$$\int_0^{2\pi} K(\theta,\omega)\, d\omega = \lim_{\epsilon\to 0}\int_{|\omega-\theta|>\epsilon} K(\theta,\omega)\, d\omega.$$

The limit converges due to the cancelation effect for $\omega$ before and after $\theta$. We have derived

$$\|\mathcal{K}(u_k)\|_{L^\infty} \le C\|u_k\|_{C^\alpha} \le C\|u_k\|_{H^1}. \tag{6.9}$$

We can now estimate $A_1$, $A_2$ and $A_3$. By (6.9)

$$\|A_1(\Phi)(u_k, v_k)\|_{L^2} \le C\|u_k\|_{H^1}\|v_k\|_{L^2} \tag{6.10}$$

Similarly

$$\|A_2(\Phi)(u_k, v_k)\|_{L^2} \le C\|u_k\|_{L^2}\|v_k\|_{H^1}. \tag{6.11}$$

For $A_3$ we have

$$\|A_3(\Phi)(u_k, v_k)\|_{L^\infty} \le C\|u_k v_k\|_{C^\alpha} \le C\|u_k\|_{H^1}\|v_k\|_{H^1}. \tag{6.12}$$

We now turn to $A_4$. The integral

$$\int_{\tilde{E}_{\phi_k}} \frac{|\sqrt{1+\Phi(\theta)}e^{i\theta} - y|^2 - 2(\sqrt{1+\Phi(\theta)} - e^{i\theta}\cdot y)^2}{|\sqrt{1+\Phi(\theta)}e^{i\theta} - y|^4}\, dy$$

is a convergent improper integral defined by its principal part. It is uniformly bounded with respect to $\theta$. In the case of $\Phi$ equal to 0, it may be explicitly computed. (See Appendix B.) Therefore

$$\|A_4(\Phi)(u_k, v_k)\|_{L^\infty} \le C\|u_k\|_{H^1}\|v_k\|_{H^1} \tag{6.13}$$

For $A_5$, because of the mild singularity, we easily find

$$\|A_5(\Phi)(u_k, v_k)\|_{L^\infty} \le C\|u_k\|_{H^1}\|v_k\|_{H^1}. \tag{6.14}$$

Following (6.10), (6.11), (6.12), (6.13) and (6.14) we obtain

$$\|A''(\Phi)(u_k, v_k)\|_{L^2} \le C\|u_k\|_{H^1}\|v_k\|_{H^1}, \tag{6.15}$$

and by (6.3) we have

$$\|\mathcal{A}_k''(\phi)(u_k, v_k)\|_{L^2} \le \frac{C\gamma}{\rho^2}\|u_k\|_{H^1}\|v_k\|_{H^1} \le \frac{C}{\rho^5}\|u_k\|_{H^1}\|v_k\|_{H^1},$$

proving Part 2.

The kernel $R$ in $\mathcal{B}_k$ is a smooth function. Calculations show that

$$\begin{aligned}
&\mathcal{B}_k''(\phi)(u_k, v_k)(\theta) \\
=\ & \frac{\gamma v_k(\theta)}{4\sqrt{r_k^2+\phi(\theta)}} \int_0^{2\pi} u_k(\omega)\nabla_1 R(\xi_k + \sqrt{r_k^2+\phi(\theta)}e^{i\theta}, \xi_k + \sqrt{r_k^2+\phi(\omega)}e^{i\omega})\cdot e^{i\theta}\, d\omega \\
& + \frac{\gamma u_k(\theta)}{4\sqrt{r_k^2+\phi(\theta)}} \int_0^{2\pi} v_k(\omega)\nabla_1 R(\xi_k + \sqrt{r_k^2+\phi(\theta)}e^{i\theta}, \xi_k + \sqrt{r_k^2+\phi(\omega)}e^{i\omega})\cdot e^{i\theta}\, d\omega \\
& + \frac{\gamma}{4}\int_0^{2\pi} \frac{u_k(\omega)v_k(\omega)}{\sqrt{r_k^2+\phi(\omega)}}\nabla_2 R(\xi_k + \sqrt{r_k^2+\phi(\theta)}e^{i\theta}, \xi_k + \sqrt{r_k^2+\phi(\omega)}e^{i\omega})\cdot e^{i\theta}\, d\omega \\
& + \frac{\gamma u_k(\theta)v_k(\theta)}{4(r_k^2+\phi(\theta))}\int_B D_1^2 R(\xi + \sqrt{r_k^2+\phi(\theta)}, y)e^{i\theta}\cdot e^{i\theta}\, dy \\
& - \frac{\gamma u_k(\theta)v_k(\theta)}{4(r_k^2+\phi(\theta))^{3/2}}\int_B \nabla_1 R(\xi_k + \sqrt{r_k^2+\phi(\theta)}e^{i\theta}, y)\cdot e^{i\theta}\, dy
\end{aligned}$$

where $\nabla_1$ and $\nabla_2$ refer to the derivatives of $R$ with respect to its first and second arguments respectively. $D_1^2 R$ is the second derivative matrix of $R$ with respect to the first argument of $R$. Part 3 is now proved easily.

The function $G$ is also smooth in this context.

$$
\begin{aligned}
&\mathcal{C}''_{kl}(\phi_k,\phi_l)(u_k,u_l)(v_k,v_l)(\theta_k)\\
&= \frac{\gamma v_k(\theta_k)}{4\sqrt{r_k^2+\phi_k(\theta_k)}}\int_0^{2\pi} u_l(\omega_l)\nabla_1 G(\xi_k+\sqrt{r_k^2+\phi_k(\theta_k)}e^{i\theta_k},\xi_l+\sqrt{r_l^2+\phi_l(\omega_l)}e^{i\omega_l})\cdot e^{i\theta_k}\,d\omega_l\\
&+\frac{\gamma u_k(\theta_k)}{4\sqrt{r_k^2+\phi_k(\theta_k)}}\int_0^{2\pi} v_l(\omega_l)\nabla_1 G(\xi_k+\sqrt{r_k^2+\phi_k(\theta_k)}e^{i\theta_k},\xi_l+\sqrt{r_l^2+\phi_l(\omega_l)}e^{i\omega_l})\cdot e^{i\theta_k}\,d\omega_l\\
&+\frac{\gamma}{4}\int_0^{2\pi}\frac{u_l(\omega_l)v_l(\omega_l)}{\sqrt{r_l^2+\phi_l(\omega_l)}}\nabla_2 G(\xi_k+\sqrt{r_k^2+\phi_k(\theta_k)}e^{i\theta_k},\xi_l+\sqrt{r_l^2+\phi_l(\omega_l)}e^{i\omega_l})\cdot e^{i\omega_l}\,d\omega_l\\
&+\frac{\gamma u_k(\theta_k)v_k(\theta_k)}{4(r_k^2+\phi_k(\theta_k))}\int_{E_{\phi_l}} D_1^2 G(\xi_k+\sqrt{r_k^2+\phi_k(\theta_k)},y)e^{i\theta_k}\cdot e^{i\theta_k}\,dy\\
&-\frac{\gamma u_k(\theta_k)v_k(\theta_k)}{4(r_k^2+\phi_k(\theta_k))^{3/2}}\int_{E_{\phi_l}}\nabla_1 G(\xi_k+\sqrt{r_k^2+\phi_k(\theta_k)}e^{i\theta_k},y)\cdot e^{i\theta_k}\,dy.
\end{aligned}
$$

Part 4 then follows.

Part 5 follows from Parts 1-4 and the fact that

$$
\begin{aligned}
0 &= \sum_k \overline{\mathcal{S}''_k(\phi)(u,v)}\\
&= \sum_k \overline{\mathcal{H}''_k(\phi_k)(u_k,v_k)}+\sum_k \overline{\mathcal{A}''_k(\phi_k)(u_k,v_k)}+\sum_k \overline{\mathcal{B}''_k(\phi_k)(u_k,v_k)}\\
&\quad+\sum_k \overline{\mathcal{C}''_k(\phi)(u)}+K\lambda''_\xi(\phi)(u,v). \quad\square
\end{aligned}
$$

# 7 Reduction to $3K-1$ dimensions

We view $\mathcal{S}$ as a nonlinear operator from $\mathcal{X}$ to $\mathcal{Y}$. In this section it will be proved that, for each $(\xi,r)\in U$, a $\varphi(\cdot,\xi,r)$ exists such that $\varphi(\cdot,\xi,r)\in\mathcal{X}_*$ and

$$\mathcal{S}_k(\varphi)(\theta_k)=A_{k,1}\cos\theta_k+A_{k,2}\sin\theta_k+A_k,\quad k=1,2,...,K \tag{7.1}$$

for some numbers $A_{k,1},A_{k,2},A_k$. Note that $\varphi$ is sought in $\mathcal{X}_*$. Each $\phi\in\mathcal{X}_*$ satisfies

$$\int_0^{2\pi}\phi_k(\theta_k)\,d\theta_k = 0,\quad k=1,2,...,K \tag{7.2}$$

$$\int_0^{2\pi}\phi_k(\theta_k)\cos\theta_k\,d\theta_k = 0,\quad k=1,2,...K \tag{7.3}$$

$$\int_0^{2\pi}\phi_k(\theta_k)\sin\theta_k\,d\theta_k = 0,\quad k=1,2,...,K. \tag{7.4}$$

Write the equation (7.1) as

$$\Pi\mathcal{S}(\varphi) = 0 \tag{7.5}$$

where $\Pi$ is the orthogonal projection operator from $\mathcal{Y}$ to $\mathcal{Y}_*$. In the next section we will find a particular $(\xi, r)$, say $(\zeta, s)$ at which $A_{k,1} = A_{k,2} = A_k = 0$, i.e. $\mathcal{S}(\varphi(\cdot, \zeta, s)) = 0$. This means that by finding $\varphi$ we reduce the original problem (1.1) to a problem of finding a $(\zeta, s)$ in a $3K - 1$ dimensional set $U$.

Recall $\mathcal{L}$, the linearized operator of $\mathcal{S}$ at $\phi = 0$. Expand $\mathcal{S}(\phi)$ as

$$\mathcal{S}(\phi) = \mathcal{S}(0) + \mathcal{L}(\phi) + \mathcal{N}(\phi) \tag{7.6}$$

where $\mathcal{N}$ is a higher order term defined by (7.6). Turn (7.5) to a fixed point form:

$$\phi = -(\Pi\mathcal{L})^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{N}(\phi)) \tag{7.7}$$

**Lemma 7.1** *There exists $\varphi = \varphi(\theta, \xi, r)$ such that for every $(\xi, r) \in U$, $\varphi(\cdot, \xi, r) \in \mathcal{X}_*$ solves (7.7) and $\|\varphi\|_{H^2} \le c\rho^3$ where $c$ is a sufficiently large constant independent of $\xi$, $r$, $\rho$ and $\gamma$.*

*Proof.* To use the Contraction Mapping Principle, let

$$\mathcal{T}(\phi) = -(\Pi\mathcal{L})^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{N}(\phi)) \tag{7.8}$$

be an operator defined on

$$D(\mathcal{T}) = \{\phi \in \mathcal{X}_* : \|\phi\|_{H^2} \le c\rho^3\} \tag{7.9}$$

where the constant $c$ is sufficiently large which will be made more transparent later.

Lemma 3.1 shows that

$$\mathcal{S}_k(0)(\theta_k) - \lambda(0) = \frac{1}{r_k} + \gamma[-\frac{r_k^2 \log r_k}{2} + \pi r_k^2 R(\xi_k, \xi_k) + \sum_{l \neq k} \pi r_l^2 G(\xi_k, \xi_l)] + O(1).$$

Each $\mathcal{S}_k(0)$ is sum of a number independent of $\theta_k$ and a quantity of order $O(1)$. After we apply the projection operator $\Pi$ the number vanishes and

$$\|\Pi\mathcal{S}(0)\|_{L^2} = O(1).$$

By Lemma 5.3 we find

$$\|(\Pi\mathcal{L})^{-1}\Pi\mathcal{S}(0)\|_{H^2} \le C\rho^3. \tag{7.10}$$

For $\mathcal{N}(\phi)$ we decompose it into three parts. The first is $\mathcal{N}_1$ whose $k$-th component is

$$\mathcal{N}_{1,k}(\phi_k) = \mathcal{H}_k(\phi_k) - \frac{1}{r_k} + \frac{1}{2r_k^3}(\phi_k'' + \phi_k) = \mathcal{H}_k(\phi_k) - \mathcal{H}_k(0) - \mathcal{H}_k'(0)(\phi_k) \tag{7.11}$$

which is $\mathcal{H}_k(\phi)$ minus its linear approximation at 0. Lemma 6.1, Part 1, shows that

$$\|\mathcal{N}_1(\phi)\|_{L^2} \le \frac{C}{\rho^5}\|\phi\|_{H^2}^2. \tag{7.12}$$

The second part of $\mathcal{N}$, denoted by $\mathcal{N}_2$, is $\mathcal{A}(\phi) + \mathcal{B}(\phi) + \mathcal{C}(\phi)$ minus its linear approximation, i.e.

$$\mathcal{N}_2(\phi) = \mathcal{A}(\phi) - \mathcal{A}(0) - \mathcal{A}'(0)(\phi) + \mathcal{B}(\phi) - \mathcal{B}(0) - \mathcal{B}'(0)(\phi) + \mathcal{C}(\phi) - \mathcal{C}(0) - \mathcal{C}'(0)(\phi). \tag{7.13}$$

Lemma 6.1, Parts 2, 3, and 4, implies that

$$\|\mathcal{N}_2(\phi)\|_{L^2} \leq \frac{C}{\rho^5}\|\phi\|^2_{H^1} \tag{7.14}$$

The third part of $\mathcal{N}$, which is denoted by $\mathcal{N}_3$, merely gives a constant so that

$$\sum_k \overline{\mathcal{N}_k(\phi)} = \sum_k \overline{\mathcal{N}_{1,k}(\phi)} + \sum_k \overline{\mathcal{N}_{2,k}(\phi)} + K\mathcal{N}_3(\phi) = 0.$$

It follows that

$$|\mathcal{N}_3(\phi)| \leq \frac{C}{\rho^5}\|\phi\|^2_{H^2}. \tag{7.15}$$

Therefore we deduce, from (7.12), (7.14), (7.15) and with the help of Lemma 5.3, that

$$\|\mathcal{N}(\phi)\|_{L^2} \leq \frac{C}{\rho^5}\|\phi\|^2_{H^2} \tag{7.16}$$

$$\|(\Pi\mathcal{L})^{-1}\Pi\mathcal{N}(\phi)\|_{H^2} \leq \frac{C}{\rho^2}\|\phi\|^2_{H^2} \tag{7.17}$$

Using (2.13), (7.10), (7.9), and (7.17) we find

$$\|\mathcal{T}(\phi)\|_{H^2} \leq C\rho^3 + Cc^2\rho^4 \leq c\rho^3$$

if $c$ is sufficiently large and $\rho$ sufficiently small. Therefore $\mathcal{T}$ is a map from $D(\mathcal{T})$ into itself.

Next we show that $\mathcal{T}$ is a contraction. Let $\phi_1, \phi_2 \in D(\mathcal{T})$. To estimate $\mathcal{N}_1(\phi_1) - \mathcal{N}_1(\phi_2)$ we proceed as in the proof of Lemma 6.1, Part 1. Let $\Phi_1 = \frac{\phi_{1,k}}{r_k^2}$ and $\Phi_2 = \frac{\phi_{2,k}}{r_k^2}$. Then, writing $\tilde{H}(\Phi_1)$ for $\tilde{H}(\Phi_1, \Phi_1', \Phi_1'')$ for simplicity, we find

$$\begin{aligned}
&r_k|\mathcal{N}_{1,k}(\phi_{1,k}) - \mathcal{N}_{1,k}(\phi_{2,k})| \\
&\quad= |\tilde{H}(\Phi_1) - \tilde{H}(\Phi_2) - \tilde{H}_1(0)(\Phi_1 - \Phi_2) - \tilde{H}_2(0)(\Phi_1' - \Phi_2') - \tilde{H}_3(0)(\Phi_1'' - \Phi_2'')| \\
&\quad= |\tilde{H}_1(\Phi_2)(\Phi_1 - \Phi_2) + \tilde{H}_2(\Phi_2)(\Phi_1' - \Phi_2') + \tilde{H}_3(\Phi_2)(\Phi_1'' - \Phi_2'') \\
&\qquad+\frac{1}{2}\tilde{H}_{11}(t\Phi_1 - (1-t)\Phi_2)(\Phi_1 - \Phi_2)^2 + \frac{1}{2}\tilde{H}_{22}(t\Phi_1 - (1-t)\Phi_2)(\Phi_1' - \Phi_2')^2 \\
&\qquad+\tilde{H}_{12}(t\Phi_1 - (1-t)\Phi_2)(\Phi_1 - \Phi_2)(\Phi_1' - \Phi_2') + \tilde{H}_{23}(t\Phi_1 - (1-t)\Phi_2)(\Phi_1' - \Phi_2')(\Phi_1'' - \Phi_2'') \\
&\qquad+\tilde{H}_{31}(t\Phi_1 - (1-t)\Phi_2)(\Phi_1'' - \Phi_2'')(\Phi_1 - \Phi_2) \\
&\qquad-\tilde{H}_1(0)(\Phi_1 - \Phi_2) - \tilde{H}_2(0)(\Phi_1' - \Phi_2') - \tilde{H}_3(0)(\Phi_1'' - \Phi_2'')| \\
&\quad\leq C[(|\Phi_1| + |\Phi_2|)|\Phi_1 - \Phi_2| + (|\Phi_1'| + |\Phi_2'|)|\Phi_1' - \Phi_2'| \\
&\qquad+(|\Phi_1| + |\Phi_2|)|\Phi_1' - \Phi_2'| + (|\Phi_1'| + |\Phi_2'|)|\Phi_1 - \Phi_2| \\
&\qquad+(|\Phi_1'| + |\Phi_2'|)|\Phi_1'' - \Phi_2''| + (|\Phi_1''| + |\Phi_2''|)|\Phi_1' - \Phi_2'| \\
&\qquad+(|\Phi_1''| + |\Phi_2''|)|\Phi_1 - \Phi_2| + (|\Phi_1| + |\Phi_2|)|\Phi_1'' - \Phi_2''|].
\end{aligned}$$

Since there is no $(|\Phi_1''| + |\Phi_2''|)|\Phi_1'' - \Phi_2''|$ term, by the Sobolev Embedding Theorem we deduce, after returning to $\phi_1$ and $\phi_2$,

$$\|\mathcal{N}_1(\phi_1) - \mathcal{N}_1(\phi_2)\|_{L^2} \leq \frac{C}{\rho^5}(\|\phi_1\|_{H^2} + \|\phi\|_{H^2})\|\phi_1 - \phi_2\|_{H^2} \leq \frac{C}{\rho^2}\|\phi_1 - \phi_2\|_{H^2}. \tag{7.18}$$

For $\mathcal{N}_2$ we note that

$$\begin{aligned}\mathcal{N}_2(\phi_1)-\mathcal{N}_2(\phi_2) &= \mathcal{A}(\phi_1)-\mathcal{A}(\phi_2)-\mathcal{A}'(0)(\phi_1-\phi_2)+\mathcal{B}(\phi_1)-\mathcal{B}(\phi_2)-\mathcal{B}'(0)(\phi_1-\phi_2)\\ &\quad +\mathcal{C}(\phi_1)-\mathcal{C}(\phi_2)-\mathcal{C}'(0)(\phi_1-\phi_2). \qquad (7.19)\end{aligned}$$

Therefore using Lemma 6.1, Part 2, we obtain

$$\begin{aligned}&\|\mathcal{A}(\phi_1)-\mathcal{A}(\phi_2)-\mathcal{A}'(0)(\phi_1-\phi_2)\|_{L^2}\\ \le\ & \|\mathcal{A}'(\phi_2)(\phi_1-\phi_2)-\mathcal{A}'(0)(\phi_1-\phi_2)\|_{L^2}+\frac{C}{\rho^5}\|\phi_1-\phi_2\|^2_{H^1}\\ \le\ & \frac{C}{\rho^5}\|\phi_2\|_{H^1}\|\phi_1-\phi_2\|_{H^1}+\frac{C}{\rho^5}\|\phi_1-\phi_2\|^2_{H^1}\\ \le\ & \frac{C}{\rho^5}(\|\phi_1\|_{H^1}+\|\phi_2\|_{H^2})\|\phi_1-\phi_2\|_{H^1}.\end{aligned}$$

Similarly using Lemma 6.1, Parts 3 and 4, we deduce

$$\begin{aligned}\|\mathcal{B}(\phi_1)-\mathcal{B}(\phi_2)-\mathcal{B}'(0)(\phi_1-\phi_2)\|_{L^2} &\le \frac{C}{\rho^4}(\|\phi_1\|_{H^1}+\|\phi_2\|_{H^2})\|\phi_1-\phi_2\|_{H^1}\\ \|\mathcal{C}(\phi_1)-\mathcal{C}(\phi_2)-\mathcal{C}'(0)(\phi_1-\phi_2)\|_{L^2} &\le \frac{C}{\rho^4}(\|\phi_1\|_{H^1}+\|\phi_2\|_{H^2})\|\phi_1-\phi_2\|_{H^1}.\end{aligned}$$

From (7.19) we conclude that

$$\|\mathcal{N}_2(\phi_1)-\mathcal{N}_2(\phi_2)\|_{L^2}\le\frac{C}{\rho^5}(\|\phi_1\|_{H^1}+\|\phi_2\|_{H^1})\|\phi_1-\phi_2\|_{H^1}\le\frac{C}{\rho^2}\|\phi_1-\phi_2\|_{H^1} \qquad (7.20)$$

We also have

$$\|\mathcal{N}_3(\phi_1)-\mathcal{N}_3(\phi_2)\|_{L^2}\le\frac{C}{\rho^2}\|\phi_1-\phi_2\|_{H^2}. \qquad (7.21)$$

Hence, following (7.18), (7.20), and (7.21), we find that

$$\|\mathcal{T}(\phi_1)-\mathcal{T}(\phi_2)\|_{H^2}\le C\rho\|\phi_1-\phi_2\|_{H^2}, \qquad (7.22)$$

i.e. that $\mathcal{T}$ is a contraction map if $\rho$ is sufficiently small. A fixed point $\varphi$ exists. □

Since $\varphi$ satisfies $\|\phi\|_{H^2}\le c\rho^3$, by taking $\rho$ small we see that $r_k^2+\varphi_k$ remains positive. $E_{\varphi_k}$ is a perturbed disc.

Denote $\mathcal{S}'(\varphi)$ by $\tilde{\mathcal{L}}$. We derive a lemma for $\tilde{\mathcal{L}}$ similar to Lemma 5.3.

**Lemma 7.2** *Let $\Pi$ be the same projection operator from $\mathcal{X}$ to $\mathcal{X}_*$.*

1. *There exists $C>0$ such that for all $u\in\mathcal{X}_*$*

$$\|u\|_{H^2}\le C\rho^3\|\Pi\tilde{\mathcal{L}}(u)\|_{L^2}$$

2. *If (2.11) holds,*

$$\|u\|^2_{H^1}\le C\rho^3\langle\Pi\tilde{\mathcal{L}}(u),u\rangle.$$

*Proof.* By (2.13), Lemma 5.3, Part 1, Lemma 6.1, (2.13) and the fact $\|\varphi\|_{H^2} = O(\rho^3)$, we deduce

$$\begin{aligned}
\|\Pi\tilde{\mathcal{L}}(u)\|_{L^2} &\geq \|\Pi\mathcal{L}(u)\|_{L^2} - \|\Pi(\tilde{\mathcal{L}} - \mathcal{L})(u)\| \\
&\geq \frac{C}{\rho^3}\|u\|_{H^2} - \frac{C}{\rho^5}\|\varphi\|_{H^2}\|u\|_{H^2} \\
&\geq \frac{C}{\rho^3}\|u\|_{H^2} - \frac{C}{\rho^2}\|u\|_{H^2} \\
&\geq \frac{C}{\rho^3}\|u\|_{H^2}
\end{aligned}$$

when $\rho$ is small. This proves part 1.

Write $\tilde{\mathcal{L}} = \mathcal{H}'(\varphi) + \mathcal{A}'(\varphi) + \mathcal{B}'(\varphi) + \mathcal{C}'(\varphi) + \lambda'(\varphi)$. Let

$$Q(\varphi_k, \varphi_k') = 2\sqrt{r_k^2 + \varphi_k + \frac{(\varphi_k')^2}{4(r_k^2 + \varphi_k)}}. \tag{7.23}$$

Then

$$\langle \mathcal{H}_k'(\varphi_k)(u_k), u_k\rangle = \int_0^{2\pi} [Q_{11}(\varphi_k, \varphi_k')u_k^2 + 2Q_{12}(\varphi_k, \varphi_k')u_k u_k' + Q_{22}(\varphi_k, \varphi_k')(u_k')^2]\, d\theta.$$

and a similar expression holds for $\mathcal{L}$ if we replace $\varphi_k$ and $\varphi_k'$ by 0 in the last formula. Here $Q_{11}$ is the second derivative with respect to the first argument of $Q$, etc. With $\|\varphi\|_{H^2} = O(\rho^3)$ calculations show that

$$\begin{aligned}
|\langle(\mathcal{H}_k'(\varphi_k) - \mathcal{H}_k'(0))u_k, u_k\rangle| &\leq |\int_0^{2\pi} (Q_{11}(\varphi_k, \varphi_k') - Q_{11}(0,0))u_k^2\, d\theta_k| \\
&\quad + |\int_0^{2\pi} 2(Q_{12}(\varphi_k, \varphi_k') - Q_{12}(0,0))u_k u_k'\, d\theta_k| \\
&\quad + |\int_0^{2\pi} (Q_{22}(\varphi_k, \varphi_k') - Q_{22}(0,0))(u_k')^2\, d\theta_k| \\
&\leq \frac{C}{\rho^2}\|u\|_{L^2}^2 + \frac{C}{\rho^2}\|u\|_{L^2}\|u'\|_{L^2} + \frac{C}{\rho^2}\|u'\|_{L^2}^2 \\
&\leq \frac{C}{\rho^2}\|u\|_{H^1}^2
\end{aligned} \tag{7.24}$$

Lemma 6.1, Parts 2-4, and the fact $\|\varphi\|_{H^2} = O(\rho^3)$ show that

$$\|(\mathcal{A}'(\varphi) + \mathcal{B}'(\varphi) + \mathcal{C}'(\varphi) - \mathcal{A}'(0) - \mathcal{B}'(0) - \mathcal{C}'(0))u\|_{L^2} \leq \frac{C}{\rho^2}\|u\|_{H^1}. \tag{7.25}$$

If (2.11) holds, we combine Lemma 5.3, Part 2, (7.24), (7.25) and (2.13) to deduce that

$$\langle \Pi\tilde{\mathcal{L}}(u), u\rangle = \langle \Pi\mathcal{L}(u), u\rangle + \langle \Pi(\tilde{\mathcal{L}} - \mathcal{L})u, u\rangle \geq \frac{C}{\rho^3}\|u\|_{H^1}^2 - \frac{C}{\rho^2}\|u\|_{H^1}^2 \geq \frac{C}{\rho^3}\|u\|_{H^1}^2,$$

proving the second part. □

One consequence of Lemma 7.2 is an estimate of $\frac{\partial\varphi}{\partial\xi_{l,j}}$.

**Lemma 7.3** *The fixed point $\varphi$ satisfies $\|\frac{\partial\varphi}{\partial\xi_{l,j}}\|_{H^2} = O(\rho^2)$, $l = 1, 2, ..., K$, $j = 1, 2$.*

*Proof.* We prove this lemma by the Implicit Function Theorem. Fix $l \in \{1, 2, ..., K\}$ and $j \in \{1, 2\}$. Differentiating $\Pi\mathcal{S}_\xi(\varphi)$ with respect to $\xi_{l,j}$ finds that, for $k = 1, 2, ..., K$, if $k = l$, then

$$\begin{aligned}
\frac{\partial\Pi\mathcal{S}_l(\varphi)}{\partial\xi_{l,j}} &= \Pi\tilde{\mathcal{L}}_l(\frac{\partial\varphi}{\partial\xi_{l,j}}) \\
&+\Pi\gamma\int_{E_{\varphi_l}}[\frac{\partial R(\xi_l+\sqrt{r_l^2+\varphi_l(\theta_l)}e^{i\theta_l},y)}{\partial x_j}+\frac{\partial R(\xi_l+\sqrt{r_l^2+\varphi_l(\theta_l)}e^{i\theta_l},y)}{\partial y_j}]\,dy \\
&+\sum_{m\neq l}\Pi\gamma\int_{E_{\varphi m}}\frac{\partial G(\xi_l+\sqrt{r_l^2+\varphi_l(\theta_l)}e^{i\theta_l},y)}{\partial x_j}\,dy,
\end{aligned}$$

and if $k \neq l$,

$$\frac{\partial\Pi\mathcal{S}_k(\varphi)}{\partial\xi_{l,j}} = \Pi\tilde{\mathcal{L}}_k(\frac{\partial\varphi}{\partial\xi_{l,j}}) + \Pi\gamma\int_{E_{\varphi_l}}\frac{\partial G(\xi_k+\sqrt{r_k^2+\varphi_k(\theta_k)}e^{i\theta_k},y)}{\partial y_j}\,dy.$$

Here $R = R(x, y)$ and $G = G(x, y)$. It is clear that

$$\|\gamma\int_{E_{\varphi_l}}[\frac{\partial R(\xi_l+\sqrt{r_l^2+\varphi_l(\theta_l)}e^{i\theta_l},y)}{\partial x_j}+\frac{\partial R(\xi_l+\sqrt{r_l^2+\varphi_l(\theta_l)}e^{i\theta_l},y)}{\partial y_j}]\,dy\|_{L^2} = O(\gamma\rho^2),$$

$$\|\gamma\int_{E_{\varphi m}}\frac{\partial G(\xi_l+\sqrt{r_l^2+\varphi_l(\theta_l)}e^{i\theta_l},y)}{\partial x_j}\,dy\|_{L^2} = O(\gamma\rho^2),$$

$$\|\gamma\int_{E_{\varphi_l}}\frac{\partial G(\xi_k+\sqrt{r_k^2+\varphi_k(\theta_k)}e^{i\theta_k},y)}{\partial y_j}\,dy\|_{L^2} = O(\gamma\rho^2).$$

Therefore

$$\frac{\partial\Pi\mathcal{S}(\varphi)}{\partial\xi_{l,j}} = \Pi\tilde{\mathcal{L}}_\xi(\frac{\partial\varphi}{\partial\xi_{l,j}}) + W, \quad \text{where } \|W\|_{L^2} = O(\gamma\rho^2).$$

On the other hand

$$\frac{\partial\Pi\mathcal{S}(\varphi)}{\partial\xi_{l,j}} = 0, \quad \text{since } \Pi\mathcal{S}(\varphi) = 0.$$

By Lemma 7.2 we deduce that

$$\|\frac{\partial\varphi}{\partial\xi_{l,j}}\|_{H^2} \leq C\rho^3\gamma\rho^2 \leq C\rho^2. \quad \square$$

# 8 Solving the reduced problem

We now turn to solve $\mathcal{S}(\phi) = 0$.

**Lemma 8.1** $J(E_\varphi) = J(B) + O(\rho^3)$.

*Proof.* Expanding $J(E_\varphi)$ yields

$$J(E_\varphi) = J(B) + \frac{1}{2}\sum_k \int_0^{2\pi} \mathcal{S}_k(0)\varphi_k \, d\theta_k + \frac{1}{4}\sum_k \int_0^{2\pi} \mathcal{L}_k(\varphi)\varphi_k \, d\theta_k + O(\rho^4). \tag{8.1}$$

The error term $O(\rho^4)$ in (8.1) is obtained in the same way that (7.16) is derived.

On the other hand $\Pi\mathcal{S}(\varphi) = 0$ implies that

$$\Pi(\mathcal{S}_k(0) + \mathcal{L}_k(\varphi) + \mathcal{N}_k(\varphi)) = 0$$

where $\mathcal{N}$ is given in (7.6) and estimated in (7.16). We multiply the last equation by $\varphi_k$ and integrate to derive

$$\int_0^{2\pi} \mathcal{S}_k(0)\varphi_k \, d\theta_k + \int_0^{2\pi} \mathcal{L}(\varphi_k)\varphi_k \, d\theta_k = O(\rho^4).$$

We can now rewrite (8.1) as

$$J(E_\varphi) = J(B) + \frac{1}{4}\sum_k \int_0^{2\pi} \mathcal{S}_k(0)\varphi_k \, d\theta_k + O(\rho^4).$$

Note that $\mathcal{S}_k(0)$ is the sum of a number independent of $\theta_k$ and a quantity of order 1 by Lemma 3.1. Since $\varphi_k$ satisfies (7.2), the inner product of the number and $\varphi_k$ is zero and hence

$$\int_0^{2\pi} \mathcal{S}_k(0)\varphi_k \, d\theta = O(\rho^3).$$

Therefore

$$J(E_\varphi) = J(B) + O(\rho^3) + O(\rho^4) = J(B) + O(\rho^3). \quad \square$$

If we consider $J(\varphi(\cdot, \xi, r))$ as a function of $\xi$ and $r$, then Lemmas 3.2 and 8.1 imply that

$$J(E_{\varphi(\cdot,\xi)}) = \sum_{k=1}^K 2\pi r_k + \frac{\gamma\pi^2}{2}\Big[\sum_{k=1}^K \Big(-\frac{r_k^4 \log r_k}{2\pi} + \frac{r_k^4}{8\pi} + r_k^4 R(\xi_k, \xi_k)\Big) + \sum_{k=1}^K \sum_{l\neq k} r_k^2 r_l^2 G(\xi_k, \xi_l)\Big] + O(\rho^3). \tag{8.2}$$

**Lemma 8.2** *When $\rho$ is sufficiently small, $J(\varphi(\cdot, \xi, r))$ is minimized at some $(\xi, r) = (\zeta, s) \in U$. As $\rho \to 0$, $\frac{s}{\rho} \to (1, 1, ..., 1)$, and $\zeta \to \zeta_0$ along a subsequence where $\zeta_0 \in U_1$ is a global minimum of $F$.*

*Proof.* Let us rescale the problem with

$$R = \frac{r}{\rho}, \quad \tilde{J}(\xi, R) = \frac{2}{\pi^2\gamma\rho^4 \log\frac{1}{\rho}} J(\cdot, \xi, r), \quad (\xi, R) \in U_1 \times \tilde{U}_2$$

where

$$\tilde{U}_2 = \{(R_1, R_2, ..., R_K) : \ (\frac{1}{1+\epsilon})^{1/3} < R_k < 2, \ \sum_k R_k^2 = K\}$$

is a scaled version of $U_2$. Note that by (2.8) and (8.2),

$$\begin{aligned}\tilde{J}(\xi, R) &= \frac{4}{\pi\gamma\rho^3 \log\frac{1}{\rho}} \sum_{k=1}^{K} R_k + \sum_{k=1}^{K} \frac{R_k^4}{2\pi} \\ &+ \frac{1}{\log\frac{1}{\rho}} [\sum_{k=1}^{K} (\frac{R_k^4 \log\frac{1}{R_k}}{2\pi} + \frac{R_k^4}{8\pi} + R_k^4 R(\xi_k, \xi_k)) + \sum_{k=1}^{K} \sum_{l\neq k} R_k^2 R_l^2 G(\xi_k, \xi_l)] + O(\rho^2).\end{aligned}$$

Again by (2.8) we may assume that along a subsequence

$$\frac{4}{\pi\gamma\rho^3 \log\frac{1}{\rho}} \to b_0 \leq \frac{4}{(1+\epsilon)\pi}, \text{ as } \rho \to 0. \tag{8.3}$$

Let $(\zeta, S)$ be the global minimum of $\tilde{J}$ on the closure of $U = U_1 \times \tilde{U}_2$. Here $S = \frac{s}{\rho}$. Let $(\zeta, S) \to (\zeta_0, S_0)$ along a subsequence as $\rho$ tends to 0. First we claim that $S_0 = (1, 1, ..., 1)$. Suppose that this is false, i.e. $S_0 \neq (1, 1, ..., 1)$. Then as $\rho$ tends to 0,

$$\begin{aligned}\tilde{J}(\zeta, (1, ..., 1)) - \tilde{J}(\zeta, S) &= \sum_k \frac{4}{\pi\gamma\rho^3 \log\frac{1}{\rho}} + \sum_k \frac{1}{2\pi} - \sum_k \frac{4S_k}{\pi\gamma\rho^3 \log\frac{1}{\rho}} - \sum_k \frac{S_k^4}{2\pi} + O(\frac{1}{\log\frac{1}{\rho}}) \\ &\to \sum_k b_0 + \sum_k \frac{1}{2\pi} - \sum_k b_0 S_{0,k} - \sum_k \frac{S_{0,k}^4}{2\pi}.\end{aligned}$$

Because of (8.3) and the constraint $\sum_k S_{0,k}^2 = K$, it is easy to show that the last line is negative. See Appendix C for more details. This is a contradiction to that $(\zeta, S)$ is a minimum of $\tilde{J}$.

Next we claim that $\zeta_0$ minimizes $F$ in $U_1$. Suppose that this is false. Let $\eta$ be a minimum of $F$ in $U_1$. Then $F(\eta) < F(\zeta_0)$. Consider

$$\begin{aligned}(\log\frac{1}{\rho})(\tilde{J}(\eta, S) - \tilde{J}(\zeta_0, S)) &= \sum_{k=1}^{K} S_k^4 R(\eta_k, \eta_k) + \sum_{k=1}^{K} \sum_{l\neq k} S_k^2 S_l^2 G(\eta_k, \eta_l) \\ &- \sum_{k=1}^{K} S_k^4 R(\zeta_{0,k}, \zeta_{0,k}) - \sum_{k=1}^{K} \sum_{l\neq k} S_k^2 S_l^2 G(\zeta_{0,k}, \zeta_{0,l}) + O(\rho^2 \log\frac{1}{\rho}) \\ &\to F(\eta) - F(\zeta_0) < 0, \text{ as } \rho \to 0,\end{aligned}$$

another contradiction to that $(\zeta, S)$ minimizes $\tilde{J}$. Note that $(\zeta, S) \in U$ when $\rho$ is small, since $(\zeta_0, S_0) \in U$. The lemma is proved. □

We show that $\varphi(\cdot, \zeta, s)$ is an exact solution of (1.1) in the next two lemmas.

**Lemma 8.3** *At $\xi = \zeta$ and $r = s$, $\mathcal{S}_k(\varphi(\cdot, \zeta, s))(\theta_k) = A_{k,1} \cos\theta_k + A_{k,2} \sin\theta_k$.*

*Proof.* At each $(\xi, r) \in U$ let

$$p_k = r_k^2, \quad q_k = s_k^2. \tag{8.4}$$

Calculations show that

$$\begin{aligned}
\frac{\partial J(E_\varphi)}{\partial p_k} &= \frac{1}{2}\int_0^{2\pi}[\mathcal{H}_k(\varphi_k)+\mathcal{A}_k(\varphi_k)+\mathcal{B}_k(\varphi_k)+\sum_{l\neq k}\mathcal{C}_{kl}(\varphi)]\frac{\partial(p_k+\varphi_k)}{\partial p_k}\,d\theta_k \\
&= \frac{1}{2}\int_0^{2\pi}[\mathcal{S}(\varphi)-\lambda(\varphi)](1+\frac{\partial\varphi_k}{\partial p_k})\,d\theta_k \\
&= \frac{1}{2}\int_0^{2\pi}\mathcal{S}(\varphi)(1+\frac{\partial\varphi_k}{\partial p_k})\,d\theta_k-\frac{\lambda(\varphi)}{2}\int_0^{2\pi}(1+\frac{\partial\varphi_k}{\partial p_k})\,d\theta_k \\
&= \frac{1}{2}\int_0^{2\pi}(A_{k,1}\cos\theta_k+A_{k,2}\sin\theta_k+A_k)(1+\frac{\partial\varphi_k}{\partial p_k})\,d\theta_k-\pi\lambda(\varphi) \\
&= \pi A_k-\pi\lambda(\varphi).
\end{aligned}$$

Here we have used the facts that

$$\frac{\partial\varphi_k}{\partial p_k}\perp\cos\theta_k,\ \sin\theta_k,\ 1$$

which follow from $\varphi\in\mathcal{X}_*$.

On the other hand at the minimum $p=q$ and $\xi=\zeta$ with respect to $p$, we must have

$$\frac{\partial J(E_\varphi)}{\partial p_k}|_{\xi=\zeta,p=q}=\mu$$

for all $k=1,2,...,K$. Here $\mu$ is a Lagrange multiplier coming from the constraint

$$\sum_{k=1}^K p_k=\frac{a|D|}{\pi}.$$

Therefore we deduce that

$$A_k=\frac{\mu}{\pi}+\lambda$$

which is independent of $k$. By (4.17) we derive that $\sum_{k=1}^K A_k=0$ and then we conlude that each $A_k$ must be 0. □

Next we show that $A_{k,1}$ and $A_{k,2}$ in (7.1) are 0 at $\xi=\zeta$ and $r=s$. It uses a tricky re-parametrization technique.

**Lemma 8.4** *At $\xi=\zeta$ and $r=s$, $\mathcal{S}(\varphi(\cdot,\zeta,s))=0$.*

*Proof.* To simplify notations in this proof, we do not explicitly indicate the dependence of $\varphi$ on $r$, i.e. we write $\varphi(\cdot,\xi)$ instead of $\varphi(\cdot,\xi,r)$. For each $\xi_k=(\xi_{k,1},\xi_{k,2})$ near $\zeta_k$ we re-parametrize $\partial_D E_{\varphi_k(\cdot,\xi)}$. Let $\zeta_k$ be the center of a new polar coordinates, $r_k^2+\psi_k$ the new radius square and $\eta_k$ the new angle. A point on $\partial_D E_{\varphi_k(\cdot,\xi)}$ is described as $\zeta_k+\sqrt{r_k^2+\psi_k}e^{i\eta_k}$. It is related to the old polar coordinates via

$$\zeta_k+\sqrt{r_k^2+\psi_k}e^{i\eta_k}=\xi_k+\sqrt{r_k^2+\varphi_k}e^{i\theta_k} \tag{8.5}$$

In the new coordinates $E_{\varphi_k}$ becomes $E_{\psi_k}$. It is viewed as a perturbation of the disc centered at $\zeta_k$ with radius $r_k$. The perturbation is described by $\psi_k$ which is a function of $\eta_k$ and $\xi$.

The main effect of the new coordinates is to "freeze" the center. The center of the new polar system is $\zeta_k$ which is fixed while the center of the old polar system is $\xi_k$ which varies in $D$.

We now consider the derivative of $J(E_{\varphi(\cdot,\xi)}) = J(E_{\psi(\cdot,\xi)})$ with respect to $\xi_k$. On one hand, at $\xi = \zeta$ and $r = s$,

$$\frac{\partial J(E_{\psi(\cdot,\xi)})}{\partial \xi_{k,j}}|_{\xi=\zeta} = \frac{\partial J(E_{\varphi(\cdot,\xi)})}{\partial \xi_{k,j}}|_{\xi=\zeta} = 0, \quad j = 1, 2, \tag{8.6}$$

since $\zeta$ is a minimum.

On the other hand calculations show that

$$\frac{\partial J(E_{\psi(\cdot,\xi)})}{\partial \xi_{k,j}} = \frac{1}{2}\sum_{l=1}^{K} \int_0^{2\pi} \mathcal{S}_l(\psi(\cdot,\xi))(\eta_l) \frac{\partial \psi_l}{\partial \xi_{k,j}}\, d\eta_l. \tag{8.7}$$

We emphasize that (8.7) is obtained under the re-parametrized coordinates, in which the dependence of $J(E_{\psi(\cdot,\xi)})$ on $\xi$ is only reflected in the dependence of $\psi$ on $\xi$. Had we calculated in the original coordinates, $\xi$ would have appeared also in the nonlocal part of $J$ through $R(\xi_l + ..., \xi_l + ...)$ and $G(\xi_k + ..., \xi_l + ...)$. The result would have been very different from (8.7). See the proof of Lemma 7.3 which involves differentiation with respect to $\xi$ in the original coordinates. In the derivation of (8.7) we have used the fact that $\sum_l \int_0^{2\pi} \psi_l\, d\eta_l = 0$ which implies that $\sum_l \int_0^{2\pi} \frac{\partial \psi_l}{\partial \xi_{k,j}}\, d\eta_l = 0$, so that $\sum_l \int_0^{2\pi} \lambda(\psi) \frac{\partial \psi_l}{\partial \xi_{k,j}}\, d\eta_l = 0$ where $\lambda(\psi)$ is part of

$$\mathcal{S}_l(\psi) = \mathcal{H}_l(\psi) + \mathcal{I}_l(\psi) + \mathcal{A}_l(\psi) + \mathcal{B}_l(\psi) + \mathcal{C}_l(\psi) + \lambda(\psi),$$

and we can reach the right side of (8.7). See Remark 4.1 for the coefficient $\frac{1}{2}$ in (8.7).

The expression $\mathcal{S}(\phi)$ is invariant under re-parametrization, i.e.

$$\mathcal{S}_l(\varphi(\cdot,\xi))(\theta_l) = \mathcal{S}_l(\psi(\cdot,\xi))(\eta_l). \tag{8.8}$$

Now we return to the original coordinate system and integrate with respect to $\theta_l$ in (8.7). Then

$$\frac{\partial J(E_{\psi(\cdot,\xi)})}{\partial \xi_{k,j}} = \frac{1}{2}\sum_{l=1}^{K} \int_0^{2\pi} \mathcal{S}_l(\varphi(\cdot,\xi))(\theta_l) \frac{\partial \psi_l(\eta_l(\theta_l,\xi),\xi)}{\partial \xi_{k,j}} \frac{\partial \eta_l}{\partial \theta_l}\, d\theta_l \tag{8.9}$$

There are two cases: $l = k$ and $l \neq k$. We start with the first case. Recall that $\psi_k$ and $\eta_k$ are defined implicitly as functions of $\theta_k$ and $\xi$ by (8.5). Let us agree that $\psi_k = \psi_k(\eta_k, \xi)$ is a function of $\eta_k$ and $\xi$. Set $\Psi_k(\theta_k, \xi) = \psi_k(\eta_k(\theta_k, \xi), \xi)$. Implicit differentiation shows that, with the help of Lemmas 7.1 and 7.3,

$$\begin{bmatrix} \frac{\partial \eta_k}{\partial \theta_k} & \frac{\partial \eta_k}{\partial \xi_{k,1}} & \frac{\partial \eta_k}{\partial \xi_{k,2}} \\ \frac{\partial \Psi_k}{\partial \theta_k} & \frac{\partial \Psi_k}{\partial \xi_{k,1}} & \frac{\partial \Psi_k}{\partial \xi_{k,2}} \end{bmatrix} = -\begin{bmatrix} \sqrt{r_k^2 + \Psi_k}\sin\eta_k & -\frac{\cos\eta_k}{2\sqrt{r_k^2+\Psi_k}} \\ -\sqrt{r_k^2 + \Psi_k}\cos\eta_k & -\frac{\sin\eta_k}{2\sqrt{r_k^2+\Psi_k}} \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \frac{\cos\theta_k}{2\sqrt{r_k^2+\varphi_k}}\frac{\partial \varphi_k}{\partial \theta_k} - \sqrt{r_k^2+\varphi_k}\sin\theta_k & 1 + \frac{\cos\theta_k}{2\sqrt{r_k^2+\varphi_k}}\frac{\partial \varphi_k}{\partial \xi_{k,1}} & \frac{\cos\theta_k}{2\sqrt{r_k^2+\varphi_k}}\frac{\partial \varphi_k}{\partial \xi_{k,2}} \\ \frac{\sin\theta_k}{2\sqrt{r_k^2+\varphi_k}}\frac{\partial \varphi_k}{\partial \theta_k} + \sqrt{r_k^2+\varphi_k}\cos\theta_k & \frac{\sin\theta}{2\sqrt{r_k^2+\varphi_k}}\frac{\partial \varphi_k}{\partial \xi_{k,1}} & 1 + \frac{\sin\theta_k}{2\sqrt{r_k^2+\varphi_k}}\frac{\partial \varphi_k}{\partial \xi_{k,2}} \end{bmatrix}$$

$$= 2\begin{bmatrix} \frac{-\sin\eta_k}{2\sqrt{r_k^2+\Psi_k}} & \frac{\cos\eta_k}{2\sqrt{r_k^2+\Psi_k}} \\ \sqrt{r_k^2+\Psi_k}\cos\eta_k & \sqrt{r_k^2+\Psi_k}\sin\eta_k \end{bmatrix}$$

$$\times \begin{bmatrix} -\sqrt{r_k^2+\varphi_k}\sin\theta_k + O(\rho^2) & 1+O(\rho) & O(\rho) \\ \sqrt{r_k^2+\varphi_k}\cos\theta_k + O(\rho^2) & O(\rho) & 1+O(\rho) \end{bmatrix}$$

At $\xi = \zeta$, $\eta = \theta$, $\Psi = \varphi$ and the above becomes

$$\begin{bmatrix} \frac{\partial\eta_k}{\partial\theta_k} & \frac{\partial\eta_k}{\partial\xi_{k,1}} & \frac{\partial\eta_k}{\partial\xi_{k,2}} \\ \frac{\partial\Psi_k}{\partial\theta_k} & \frac{\partial\Psi_k}{\partial\xi_{k,1}} & \frac{\partial\Psi_k}{\partial\xi_{k,2}} \end{bmatrix}_{\xi=\zeta}$$

$$= \begin{bmatrix} 1+O(\rho) & -\frac{\sin\theta_k}{\sqrt{r_k^2+\varphi_k}} + O(1) & \frac{\cos\theta_k}{\sqrt{r_k^2+\varphi_k}} + O(1) \\ O(\rho^3) & 2\sqrt{r_k^2+\varphi_k}\cos\theta_k + O(\rho^2) & 2\sqrt{r_k^2+\varphi_k}\sin\theta_k + O(\rho^2) \end{bmatrix} \tag{8.10}$$

We have found that at $\xi = \zeta$,

$$\frac{\partial\Psi_k}{\partial\xi_{k,1}}|_{\xi=\zeta} = 2r_k\cos\theta_k + O(\rho^2), \quad \frac{\partial\Psi_k}{\partial\xi_{k,2}}|_{\xi=\zeta} = 2r_k\sin\theta_k + O(\rho^2). \tag{8.11}$$

To compute $\frac{\partial\psi_k}{\partial\xi_{k,j}}$, we invert $\eta_k = \eta_k(\xi, \theta_k)$ to express $\theta_k = \Theta_k(\eta_k, \xi)$. Then

$$\frac{\partial\psi_k}{\partial\xi_{k,j}} = \frac{\partial\Psi_k}{\partial\xi_{k,j}} + \frac{\partial\Psi_k}{\partial\theta_k}\frac{\partial\Theta_k}{\partial\xi_{k,j}}.$$

At $\xi = \zeta$, since

$$\frac{\partial\Psi_k}{\partial\theta_k}|_{\xi=\zeta} = O(\rho^3), \quad \frac{\partial\Theta_k}{\partial\xi_{k,j}}|_{\xi=\zeta} = -\frac{\frac{\partial\eta_k}{\partial\xi_{k,j}}}{\frac{\partial\eta_k}{\partial\theta_k}} = O(\frac{1}{\rho}), \tag{8.12}$$

we deduce that

$$\frac{\partial\psi_k}{\partial\xi_{k,1}}|_{\xi=\zeta} = 2\rho\cos\theta_k + O(\rho^2), \quad \frac{\partial\psi_k}{\partial\xi_{k,2}}|_{\xi=\zeta} = 2\rho\sin\theta_k + O(\rho^2). \tag{8.13}$$

The second case $l \neq k$ is slightly simpler. Implicit differentiation shows that, with the help of Lemmas 7.1 and 7.3,

$$\begin{bmatrix} \frac{\partial\eta_l}{\partial\theta_l} & \frac{\partial\eta_l}{\partial\xi_{k,1}} & \frac{\partial\eta_l}{\partial\xi_{k,2}} \\ \frac{\partial\Psi_l}{\partial\theta_l} & \frac{\partial\Psi_l}{\partial\xi_{k,1}} & \frac{\partial\Psi_l}{\partial\xi_{k,2}} \end{bmatrix} = -\begin{bmatrix} \sqrt{r_l^2+\Psi_l}\sin\eta_l & -\frac{\cos\eta_l}{2\sqrt{r_l^2+\Psi_l}} \\ -\sqrt{r_l^2+\Psi_l}\cos\eta_l & -\frac{\sin\eta_l}{2\sqrt{r_l^2+\Psi_l}} \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \frac{\cos\theta_l}{2\sqrt{r_l^2+\varphi_l}}\frac{\partial\varphi_l}{\partial\theta_l} - \sqrt{r_l^2+\varphi_l}\sin\theta_l & \frac{\cos\theta_l}{2\sqrt{r_l^2+\varphi_l}}\frac{\partial\varphi_l}{\partial\xi_{k,1}} & \frac{\cos\theta_l}{2\sqrt{r_l^2+\varphi_l}}\frac{\partial\varphi_l}{\partial\xi_{k,2}} \\ \frac{\sin\theta_l}{2\sqrt{r_l^2+\varphi_l}}\frac{\partial\varphi_l}{\partial\theta_l} + \sqrt{r_l^2+\varphi_l}\cos\theta_l & \frac{\sin\theta}{2\sqrt{r_l^2+\varphi_l}}\frac{\partial\varphi_l}{\partial\xi_{k,1}} & \frac{\sin\theta_l}{2\sqrt{r_l^2+\varphi_l}}\frac{\partial\varphi_l}{\partial\xi_{k,2}} \end{bmatrix}$$

$$= 2\begin{bmatrix} \frac{-\sin\eta_l}{2\sqrt{r_l^2+\Psi_l}} & \frac{\cos\eta_l}{2\sqrt{r_l^2+\Psi_l}} \\ \sqrt{r_l^2+\Psi_l}\cos\eta_l & \sqrt{r_l^2+\Psi_l}\sin\eta_l \end{bmatrix}$$

$$\times \begin{bmatrix} -\sqrt{r_l^2+\varphi_l}\sin\theta_l + O(\rho^2) & O(\rho) & O(\rho) \\ \sqrt{r_l^2+\varphi_l}\cos\theta_l + O(\rho^2) & O(\rho) & O(\rho) \end{bmatrix}$$

At $\xi = \zeta$, $\eta = \theta$, $\Psi = \varphi$ and the above becomes

$$\begin{bmatrix} \frac{\partial \eta_l}{\partial \theta_l} & \frac{\partial \eta_l}{\partial \xi_{k,1}} & \frac{\partial \eta_l}{\partial \xi_{k,2}} \\ \frac{\partial \Psi_l}{\partial \theta_l} & \frac{\partial \Psi_l}{\partial \xi_{k,1}} & \frac{\partial \Psi_l}{\partial \xi_{k,2}} \end{bmatrix}_{\xi=\zeta} = \begin{bmatrix} 1+O(\rho) & O(1) & O(1) \\ O(\rho^3) & O(\rho^2) & O(\rho^2) \end{bmatrix} \tag{8.14}$$

We have found that at $\xi = \zeta$,

$$\frac{\partial \Psi_l}{\partial \xi_{k,1}}|_{\xi=\zeta} = O(\rho^2), \quad \frac{\partial \Psi_l}{\partial \xi_{k,2}}|_{\xi=\zeta} = O(\rho^2). \tag{8.15}$$

To compute $\frac{\partial \psi_l}{\partial \xi_{k,j}}$, we invert $\eta_l = \eta_l(\xi, \theta_l)$ to express $\theta_l = \Theta_l(\eta_l, \xi)$. Then

$$\frac{\partial \psi_l}{\partial \xi_{k,j}} = \frac{\partial \Psi_l}{\partial \xi_{k,j}} + \frac{\partial \Psi_l}{\partial \theta_l}\frac{\partial \Theta_l}{\partial \xi_{k,j}}.$$

At $\xi = \zeta$, since

$$\frac{\partial \Psi_l}{\partial \theta_l}|_{\xi=\zeta} = O(\rho^3), \quad \frac{\partial \Theta_l}{\partial \xi_{k,j}}|_{\xi=\zeta} = -\frac{\frac{\partial \eta_l}{\partial \xi_{k,j}}}{\frac{\partial \eta_l}{\partial \theta_l}} = O(1), \tag{8.16}$$

we deduce that

$$\frac{\partial \psi_l}{\partial \xi_{k,1}}|_{\xi=\zeta} = O(\rho^2), \quad \frac{\partial \psi_l}{\partial \xi_{k,2}}|_{\xi=\zeta} = O(\rho^2). \tag{8.17}$$

Following (8.13), (8.17) and the fact that $\frac{\partial \eta_l}{\partial \theta_l}|_{\xi=\zeta} = 1 + O(\rho)$ we find that (8.9) becomes

$$\begin{aligned} 2\frac{\partial J(E_{\psi(\cdot,\xi)})}{\partial \xi_{k,1}}|_{\xi=\zeta} &= \int_0^{2\pi} \mathcal{S}_k(\varphi)(2r_k \cos\theta_k + O(\rho^2))\, d\theta_k + \sum_{l\neq k} \int_0^{2\pi} \mathcal{S}_l(\varphi) O(\rho^2)\, d\theta_l, \\ 2\frac{\partial J(E_{\psi(\cdot,\xi)})}{\partial \xi_{k,2}}|_{\xi=\zeta} &= \int_0^{2\pi} \mathcal{S}_k(\varphi)(2r_k \sin\theta_k + O(\rho^2))\, d\theta_k + \sum_{l\neq k} \int_0^{2\pi} \mathcal{S}_l(\varphi) O(\rho^2)\, d\theta_l, \end{aligned}$$

Now we combine (7.1), (8.6) and the above to derive that at $\xi = \zeta$ and $r = s$,

$$\begin{aligned} A_{k,1}\int_0^{2\pi} \cos\theta_k(2r_k\cos\theta_k + O(\rho^2))\, d\theta_k + A_{k,2}\int_0^{2\pi} \sin\theta_k(2r_k\cos\theta_k + O(\rho^2))\, d\theta_k & \\ + \sum_{l\neq k} A_{l,1}O(\rho^2) + \sum_{l\neq k} A_{l,2}O(\rho^2) &= 0 \\ A_{k,1}\int_0^{2\pi} \cos\theta_k(2r_k\sin\theta_k + O(\rho^2))\, d\theta_k + A_{k,2}\int_0^{2\pi} \sin\theta_k(2r_k\sin\theta_k + O(\rho^2))\, d\theta_k & \\ + \sum_{l\neq k} A_{l,1}O(\rho^2) + \sum_{l\neq k} A_{l,2}O(\rho^2) &= 0 \end{aligned}$$

Writing the system in matrix form

$$\left(\begin{bmatrix} 2\pi r_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2\pi r_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2\pi r_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 2\pi r_2 & \dots & 0 & 0 \\ .. & & & & & & \\ .. & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 2\pi r_K & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2\pi r_K \end{bmatrix} + O(\rho^2)\right) \begin{bmatrix} A^1_{1,\zeta} \\ A^1_{2,\zeta} \\ A^2_{1,\zeta} \\ A^2_{2,\zeta} \\ \dots \\ \dots \\ A_{K,1} \\ A_{K,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ 0 \end{bmatrix} \tag{8.18}$$

we deduce, since (8.18) is non-singular when $\rho$ is small, $A_{k,1} = A_{k,2} = 0$, proving the lemma. □

The existence part of Theorem 2.1 follows from Lemma 8.4. The centers $\zeta_k$ and radii $s_k$ of the droplets are found in Lemma 8.2. In Lemma 7.1 we see that $\|\varphi\|_{H^2} \leq C\rho^3$, which implies that the radius of a droplet is approximately

$$\sqrt{s_k^2 + \varphi_k(\theta_k)} = s_k + \frac{O(|\varphi_k(\theta_k)|)}{\rho} = s_k + O(\rho^2).$$

By Lemma 8.2, $\zeta$ is close to a minimum of $F$ and $s_k$ is close to $\rho$. The formula (8.2) gives the free energy of our solution.

To prove Theorem 2.2 we note that under the condition (2.11) Lemma 7.2 shows that each $\varphi(\cdot,\xi,r)$ we found in $\mathcal{X}$ locally minimizes $J(E_\phi)$ in the subspace $\mathcal{X}_*$ for $\phi$. On the other hand $\varphi(\cdot,\zeta,s)$ minimizes $J(E_{\varphi(\cdot,\xi,r)})$ with respect to $\xi$ and $r$. At this point we conclude that $E_{\varphi(\cdot,\zeta,s)}$ is a local minimizer of $J$ in

$$\bigcup_{(\xi,r)\in U} \bigcup_{\phi} \{E_\phi : \|\phi\|_{H^2} \leq c\rho^3\} \tag{8.19}$$

where $c$ is a large number given in Lemma 7.1. In this sense we claim that $E_{\varphi(\cdot,\zeta,s)}$ is stable.

If (2.12) holds, then there exists $\lambda_n$ for some $n \in \{2, 3, ...\}$ such that

$$\lambda_n < -\frac{C}{\rho^3}, \quad \langle \mathcal{L}_1(e_n), e_n\rangle < -\frac{C}{\rho^3}\|e_n\|^2_{L^2}$$

where $e_n$ is an eigenvector corresponding to $\lambda_n$. By Lemma 5.2, the last inequality implies that

$$\langle \mathcal{L}(e_n), e_n\rangle < -\frac{C}{\rho^3}\|e_n\|^2_{L^2}.$$

Then by Lemma 6.1, Parts 2, 3 and 4, and (7.24) in the proof of Lemma 7.2

$$\langle \tilde{\mathcal{L}}(e_n), e_n\rangle < -\frac{C}{\rho^3}\|e_n\|^2_{L^2}.$$

Therefore the solution is unstable.

# 9 Discussion

The case $K = 1$ is studied in [19]. With only one droplet to construct, the condition (2.8) is no longer needed. We proved the following result.

**Theorem 9.1 (Ren and Wei [19])** *Let $\tilde{R}(\xi) = R(\xi, \xi)$. For any $\epsilon > 0$ there exists $\delta > 0$ such that when $\rho$ and $\gamma$ satisfy*

$$|\gamma\rho^3 - 2n(n+1)| > \epsilon n^2, \quad \textit{for all } n = 2, 3, 4, ..., \quad \textit{and } \gamma\rho^4 < \delta,$$

*then (1.1) admits a solution of a single droplet pattern. Moreover*

1. *the radius of the droplet is $\rho + O(\rho^2)$;*
2. *the center of the droplet is near a global minimum of $\tilde{R}$ in $D$;*
3. *if*
$$1 - 2n(n+1) < -\epsilon n^2, \ \textit{for all } n \geq 2,$$
*then the droplet solution is stable; otherwise the droplet solution is unstable.*

To have a stable single droplet solution, because there is no coarsening to worry about, we only need to make

$$\gamma\rho < 12 - 4\epsilon, \quad \gamma\rho^4 \ll 1. \tag{9.1}$$

This is a much wider parameter range than (2.15) is for we can even achieve (9.1) by haveing a large $\rho$ and small $\gamma$. Indeed with less effort than in [19] and here, Oshita proved that for any $\rho \in (0, 1)$, there is $\gamma_0$ such that if $\gamma < \gamma_0$, (1.1) admits a single droplet solution. The bound $\gamma_0$ for $\gamma$ depends on $\rho$.

It is possible to extend Theorems 2.1 and 2.2 to a wider range. We may look for a saddle point of $J(\varphi(\cdot, \xi, r))$ in $U$ when (2.8) is not satisfied. Such a saddle point is unstable with respect to coarsening.

It is also possible to look for a droplet pattern where $F$ attains a local minimum or another type of critical point.

The functional (1.2) was derived from the Ohta-Kawasaki theory of diblock copolymers in [20]. They use a function $u$ on $D$ to describe the density of A-monomers and $1 - u$ to describe the density of B-monomers. The free energy of a diblock copolymer is

$$I(u) = \int_D [\frac{\epsilon^2}{2}|\nabla u|^2 + W(u) + \frac{\sigma}{2}|(-\Delta)^{-1/2}(u - a)|^2]\, dx \tag{9.2}$$

where $u$ is in

$$\{u \in H^1(D) : \overline{u} = a\}. \tag{9.3}$$

The function $W$ is a balanced double well potential such as $W(u) = \frac{1}{4}u^2(1-u)^2$. There are three positive parameters in (9.2): $\epsilon$, $\sigma$, and $a$, where $\epsilon$ is small and $a$ is in $(0, 1)$.

If we take $\sigma$ to be of order $\epsilon$, i.e. by setting

$$\sigma = \epsilon\gamma \tag{9.4}$$

for some $\gamma$ independent of $\epsilon$. As $\epsilon$ tends to 0, the limiting problem of $\epsilon^{-1}I$ turns out to be

$$J(E) = \tau|D\chi_E|(D) + \frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_E - a)|^2\,dx \tag{9.5}$$

which is the same as the $J$ in (1.2) except for the additional constant $\tau$ here. This constant is known as the surface tension and is given by

$$\tau = \int_0^1 \sqrt{2W(q)}\,dq. \tag{9.6}$$

The functional (9.5) is defined on the same admissible set $\Sigma$, (1.3).

The theory of $\Gamma$-convergence was developed by De Giorgi [7], Modica and Mortola [13], Modica [12], and Kohn and Sternberg [10]. It was proved that $\epsilon^{-1}I$ $\Gamma$-converges to $J$ in the following sense.

**Proposition 9.2 (Ren and Wei [20])** *1. For every family $\{u_\epsilon\}$ of functions in (9.3) satisfying $\lim_{\epsilon\to 0} u_\epsilon = \chi_E$ in $L^2(D)$,*

$$\liminf_{\epsilon\to 0} \epsilon^{-1}I(u_\epsilon) \geq J(E);$$

*2. For every $E$ in $\Sigma$, there exists a family $\{u_\epsilon\}$ of functions in (9.3) such that $\lim_{\epsilon\to 0} u_\epsilon = \chi_E$ in $L^2(D)$, and*

$$\limsup_{\epsilon\to 0} \epsilon^{-1}I(u_\epsilon) \leq J(E).$$

The relationship between $I$ and $J$ becomes more clear when a result of Kohn and Sternberg [10] was used to show the following.

**Proposition 9.3 (Ren and Wei [20])** *Let $\delta > 0$ and $E \in \Sigma$ be such that $J(E) < J(F)$ for all $\chi_F \in B_\delta(\chi_E)$ with $F \neq E$, where $B_\delta(\chi_E)$ is the open ball of radius $\delta$ centered at $\chi_E$ in $L^2(D)$. Then there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ there exists $u_\epsilon \in B_{\delta/2}(\chi_E)$ with $I(u_\epsilon) \leq I(u)$ for all $u \in B_{\delta/2}(\chi_E)$. In addition $\lim_{\epsilon\to 0}\|u_\epsilon - \chi_E\|_{L^2(D)} = 0$.*

The existence of a stable solution $E_{\varphi(\cdot,\zeta)}$ to (1.1) in the sense of Theorem 2.1 does not immediately imply the existence of a local minimizer, close to $\chi_{E_{\varphi(\cdot,\zeta)}}$ in $L^2(D)$, of $I$. One must show that $E_{\varphi(\cdot,\zeta)}$ is a strict local minimizer in the sense of Proposition 9.3.

# A Appendix

We drop the subscript $k$ in this appendix. The derivative of $\mathcal{A}$ at 0 has two terms according to (4.28). The first is

$$-\frac{\gamma}{4\pi}\int_0^{2\pi} u(\omega)\log|1 - e^{i(\theta-\omega)}|\,d\omega.$$

The second is

$$-\frac{\gamma u(\theta)}{4\pi}\int_{B_1(0)}\frac{(e^{i\theta}-y)\cdot e^{i\theta}}{|e^{i\theta}-y|^2}\,dy$$

for which we calculate the integral. Here $B_1(0)$ is the unit ball. Let $y = e^{i\theta}((1,0)-z)$, and $z = re^{i\beta}$. The disc $B_1(0)$ now becomes $B_1(1,0)$, the disc centered at $(1,0)$ of radius 1. Its boundary is parametrized in the polar coordinates by $r = 2\cos\beta$. Then we have

$$\int_{B_1(0)}\frac{(e^{i\theta}-y)\cdot e^{i\theta}}{|e^{i\theta}-y|^2}\,dy = \int_{B_1(1,0)}\frac{e^{i\theta}z\cdot e^{i\theta}}{|z|^2}\,dz = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\beta}\cos\beta\,drd\beta = \pi.$$

Then it follows that

$$\mathcal{A}'(0)(u) = -\frac{\gamma}{8\pi}\int_0^{2\pi}u(\omega)\log(1-\cos(\theta-\omega))\,d\omega - \frac{\gamma u}{4}. \tag{A.1}$$

# B Appendix

Evaluate

$$\int_{B_1(0)}\frac{|e^{i\theta}-y|^2-2(1-e^{i\theta}\cdot y)^2}{|e^{i\theta}-y|^4}\,dy \tag{B.1}$$

where $B_1(0)$ is the unit disc. Let $y = e^{i\theta}((1,0)-z)$, and $z = re^{i\beta}$. The disc $B_1(0)$ now becomes $B_1(1,0)$, the disc centered at $(1,0)$ of radius 1. Its boundary is parametrized in the polar coordinates by $r = 2\cos\beta$. Then (B.1) becomes

$$\int_{B_1(1,0)}\frac{|z|^2-2(e^{i\theta}\cdot e^{i\theta}z)^2}{|z|^4}\,dz = \int_0^2\int_{-\arccos(r/2)}^{\arccos(r/2)}\frac{1-2\cos^2\beta}{r}\,d\beta dr = \frac{\pi}{2} \tag{B.2}$$

Note that the last integral must be in the $d\beta dr$ order, otherwise it would be divergent.

# C Appendix

To show that

$$\sum_k b_0 + \sum_k\frac{1}{2\pi} - \sum_k b_0 S_{0,k} - \sum_k\frac{S_{0,k}^4}{2\pi} < 0, \tag{C.1}$$

let

$$f(x) = \frac{x^2}{2\pi} + b_0\sqrt{x}.$$

For $x \in ((\frac{1}{1+\epsilon})^{2/3}, 4)$, by (8.3),

$$f''(x) = \frac{1}{\pi} - \frac{b_0}{4}x^{-3/2} > \frac{1}{\pi} - \frac{b_0}{4}[(\frac{1}{1+\epsilon})^{2/3}]^{-3/2} \geq \frac{1}{\pi} - \frac{4}{4(1+\epsilon)\pi}(1+\epsilon) = 0.$$

Therefore $f$ is strictly convex on $((\frac{1}{1+\epsilon})^{2/3}, 4)$. If $x_1, x_2, ..., x_K$ are in this interval,

$$f(\frac{x_1+...+x_K}{K}) \leq \frac{1}{K}\sum_k f(x_k)$$

where the equality holds only if $x_1 = x_2 = ... = x_K$. One proves (C.1) by setting $x_k = S_{0,k}^2$.

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