INTRODUCTION TO LYAPUNOV SMICHDT REDUCTION METHODS FOR SOLVING PDE'S

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1. Allen Cahn Equation

Energy: Phase transition model.
Let $\Omega \subseteq \mathbb{R}^N$ of a "binary mixture": Two materials coexisting (or one material in two phases). We can take as an example of this: Water in solid phase $(+1)$, and water in liquid phase (-1). The configuration of this mixture in Ω can be described as a function

$$u^*(x) = \begin{cases} +1 & \text{in } \Lambda \\ -1 & \text{in } \Omega \setminus \Lambda \end{cases}$$

where Λ is some open subset of Ω. We will say that u^* is the phase function.

Consider the functional

$$\frac{1}{4}\int_\Omega (1-u^2)^2$$

minimizes if $u = 1$ or $u = -1$. Function u^* minimize this energy functional. More generally this well happen for

$$\int_\Omega W(u)dx$$

where $W(u)$ minimizes at 1 and -1, i.e. $W(+1) = W(-1) = 0$, $W(x) > 0$ if $x \neq 1$ or $x \neq -1$,$W''(+1), W''(-1) > 0$.

1.1. **The gradient theory of phase transitions.**

Possible configurations will try to make the boundary $\partial\Lambda$ as nice as possible: smooth and with small perimeter. In this model the step phase function u^* is replaced by a smooth function u_ε, where $\varepsilon > 0$ is a small parameter, and

$$u_\varepsilon(x) \approx \begin{cases} +1 & \text{inside } \Lambda \\ -1 & \text{inside } \Omega \setminus \Lambda \end{cases}$$

and u_ε has a sharp transition between these values across a "wall" of width roughly $O(\varepsilon)$: the interface (thin wall).

In grad theory of phase transitions we want minimizers, or more generally, critical points u_ε of the functional

$$J_\varepsilon(u) = \varepsilon \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon}\int_\Omega \frac{(1-u^2)^2}{4}$$

Let us observe that the region where $(1-u_\varepsilon^2) > \gamma > 0$ has area of order $O(\varepsilon)$ and the size of the gradient of u_ε in the same region is $O(\varepsilon^2)$ in such a way $J(u_\varepsilon) = O(1)$. We will find critical points u_ε to functionals of this type so that $J(u_\varepsilon) = O(1)$.

Let us consider more generally the case in which the container isn't homogeneous so that distinct costs are paid for parts of the interface in different locations

$$J_\varepsilon(u) = \int_\Omega \left(\varepsilon \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon}\frac{(1-u^2)^2}{4} \right) a(x) dx$$

$a(x)$ non-constant, $0 < \gamma \leq a(x) \leq \beta$ and smooth.

1.2. **Critical points of J_ε.** First variation of J_ε at u_ε is equal to zero.

$$\left.\frac{\partial}{\partial t} J_\varepsilon(u_\varepsilon + t\varphi)\right|_{t=0} = DJ_\varepsilon(u_\varepsilon)[\varphi] = 0, \quad \forall \varphi \in C_c^\infty(\Omega)$$

We have

$$J_\varepsilon(u_\varepsilon + t\varphi) =$$

i.e. $\forall \varphi \in C_c^\infty(\Omega)$

$$0 = DJ_\varepsilon(u_\varepsilon)[\varphi] = \varepsilon \int_\Omega (\nabla u_\varepsilon \nabla \varphi) a + \frac{1}{\varepsilon}\int_\Omega W'(u_\varepsilon)\phi a.$$

If $u_\varepsilon \in C^2(\Omega)$

$$\int_\Omega \left(-\varepsilon \nabla \cdot (a \nabla u_\varepsilon) + \frac{a}{\varepsilon} W'(u_\varepsilon) \right) \varphi = 0, \quad \forall \varphi \in C_c^\infty(\Omega)$$

This give us the weighted Allen Cahn equation in Ω

$$-\varepsilon \nabla \cdot (a \nabla u) + \frac{a}{\varepsilon} u(1-u^2) = 0 \text{ in } \Omega.$$

We will assume in the next lectures $\Omega = \mathbb{R}^N$, where $N = 1$ or $N = 2$. If $N = 1$ weight Allen Cahn equation is

$$\varepsilon^2 u'' + \varepsilon^2 u' \frac{a'}{a} + (1-u^2)u = 0, \text{ in } (-\infty, \infty). \tag{1.1}$$

Look for u_ε that connects the phases -1 and $+1$ from $-\infty$ to ∞. Multiplying (1.1) against u' and integrating by parts we obtain

$$\int_{-\infty}^{\infty} \frac{d}{dx}\left(\varepsilon \frac{u'^2}{2} - \frac{(1-u^2)^2}{4} \right) + \int_{-\infty}^{\infty} \frac{a'}{a} u'^2 = 0 \tag{1.2}$$

Assume that $u(-\infty) = -1$, $u(\infty) = 1$, $u'(-\infty) = u'(\infty) = 0$, $a > 0$, then (1.2) implies that

$$\frac{(1-u^2)^2}{4} + \int_{-\infty}^{\infty} \frac{a'}{a} u'^2 = 0$$

from which we conclude that unless a is constant, we need a' to change sign. So: if a is monotone and $a' \neq 0$ implies the non-existence of solutions as we look for. We need the existence (if $a' \neq 0$) of local maximum or local minimum of a. We will prove that under some general assumptions on $a(x)$, given a local max. or local min. x_0 of a non-degenerate ($a''(x_0) \neq 0$), then a solution to (1.1) exists, with transition layer.

We consider first the problem with $a \equiv 1$, $\varepsilon = 1$:

$$W'' + (1 - W^2)W = 0, \quad W(-\infty) = -1, W(\infty) = 1. \tag{1.3}$$

The solution of this problem is

$$W(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$$

This solution is called "the heteroclinic solution", and it's the unique solution of the problem (1.3)up to translations.

Observation 1.1. *This solution exists also for the problem*

$$w'' + f(w) = 0, \quad w(-\infty) = -1, w(\infty) = 1 \tag{1.4}$$

where $f(w) = -W'(w)$. *Solutions satisfies* $\frac{w'^2}{2} - W(w) = E$, *where* E *is constant, and* $w(-\infty) = -1$ *and* $w(\infty) = 1$ *if and only if* $E = 0$. *This implies*

$$\int_0^w \frac{ds}{\sqrt{2w(s)}} = t$$

$t(w) \to \infty$ *if* $w \to 1$, *and* $t(w) \to -\infty$ *if* $w \to -1$, *so the previous relation defines a solution* w *such that* $w(0) = 0$, *and* $w(-\infty) = -1$, $w(\infty) = 1$.

If we wright the Hamiltonian system associated to the problem we have:

$$p' = -f(q), \quad q' = p.$$

Trajectories lives on level curves of $H(p,q) = \frac{p^2}{2} - W(q)$, *where* $W(q) = \frac{(1-q^2)^2}{4}$.

Let $x_0 \in \mathbb{R}$ (we will make assumptions on this point). Fix a number $h \in \mathbb{R}$ and set

$$v(t) = u(x_0 + \varepsilon(t+h)), \quad v'(t) = \varepsilon u'(x_0 + \varepsilon(t+h))$$

Using (1.1), we have

$$\varepsilon^2 u''(x_0+\varepsilon(t+h)) = -\varepsilon^2 \frac{a'}{a} u'(x_0+\varepsilon(t+h)) - (1-v^2(t))v(t)$$

so we have the problem

(1.5)

$$v''(t)+\varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))v'(t)+(1-v(t)^2)v(t)^2 = 0, \quad w(-\infty)=-1,\, w(\infty)=1.$$

Let us observe that if $\varepsilon = 0$ the previous problem becomes formally in (1.3), so is natural to look for a solution $v(t) = W(t) + \phi$, with ϕ a small error in ε.

Assumptions:

(1) There exists $\beta, \gamma > 0$ such that $\gamma \le a(x) \le \beta$, $\forall x \in \mathbb{R}$
(2) $\|a'\|_{L^\infty(\mathbb{R})}$, $\|a''\|_{L^\infty(\mathbb{R})} < +\infty$
(3) x_0 is such that $a'(x_0) = 0$, $a''(x_0) \neq 0$, i.e. x_0 is a non-degenerate critical point of a.

Theorem 1.1. $\forall \varepsilon > 0$ *sufficiently small, there exists a solution* $v = v_\varepsilon$ *to (1.5) for some* $h = h_\varepsilon$*, where* $|h_\varepsilon| \le C\varepsilon$ *and* $v_\varepsilon(t) = w(t) + \phi_\varepsilon(t)$ *and*

$$\|\phi_\varepsilon\| \le C\varepsilon$$

Proof. We write in (1.5) $v(t) = w(t) + \phi(t)$. From now on we write $f(v) = v(1-v^2)$. We get

$$w''+\phi''+\varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))\phi'+\varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))w'+f(w+\phi)-f(w)-f'(w)\phi+f(w)+f'(w)\phi = 0$$

$$\phi(-\infty) = \phi(\infty) = 0.$$

It can be written in the following way

(1.6) $$\phi'' + f'(w(t))\phi + E + B(\phi) + N(\phi) = 0, \quad \phi(-\infty) = \phi(\infty) = 0$$

where

$$\begin{aligned} B(\phi) &= \varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))\phi', \\ N(\phi) &= f(w+\phi) - f(w) - f'(w) = -3w\phi^2 - \phi^3, \\ E &= \varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))w'. \end{aligned}$$

We consider the problem

(1.7) $$\phi'' + f'(w(t))\phi + g(t) = 0, \quad \phi \in L^\infty(\mathbb{R}),$$

and we want to know when (1.7) is solvable. We will assume $g \in L^\infty(\mathbb{R})$. Multiplying (1.7) against w' we get

$$\int_{-\infty}^{\infty} (w''' + f'(w)w')\phi + \int_{-\infty}^{\infty} gw' = 0$$

the first integral is zero because (1.4). We conclude that a necessary condition is

$$\int_{-\infty}^{\infty} gw' = 0.$$

This condition is actually sufficient for solvability. In fact, we write $\phi = w'\Psi$, we have

$$\phi'' + f'(w)\phi = g \Leftrightarrow w'\Psi + 2w''\Psi' = -g$$

Multiplying this last expression by w' (integration factor), we get

$$(w'^2\Psi')' = gw' \Rightarrow w'2\Psi'(t) = -\int_{-\infty}^{\infty} g(s)w'(s)ds$$

Let us choose

$$\Psi(t) = -\int_0^t \frac{d\tau}{w'^2(\tau)} \int_{-\infty}^{\tau} g(s)w'(s)ds$$

Then the function

$$\phi(t) = -w'(t)\int_0^t \frac{d\tau}{w'^2(\tau)} \int_{-\infty}^{\tau} g(s)w'(s)ds$$

Recall that

$$w'(t) \approx 2\sqrt{2}e^{-\sqrt{2}|t|}$$

Claim: if $\int_{-\infty}^{\infty} gw' = 0$ then we have

$$\|\phi\|_\infty \leq C\|g\|_\infty.$$

In fact, if $t > 0$

$$|\phi(t)| \leq |w'(t)| \int_0^t \frac{C}{e^{-2\sqrt{2}\tau}} \left| \int_{\tau}^{\infty} gw'ds \right| d\tau \leq C\|g\|_\infty e^{-\sqrt{2}t} \int_0^t e^{\sqrt{2}\tau} d\tau \leq C\|g\|_\infty.$$

For $t < 0$ a similar estimate yields, so we conclude

$$|\phi(t)| \leq C\|g\|_\infty.$$

□

The solution of (1.7) is not unique because if ϕ_1 is a solution implies that $\phi_2 = \phi_1 + Cw'(t)$ is also a solution. The solution we found is actually the only one with $\phi(0) = 0$. For $g \in L^\infty$ arbitrary we consider the problem

$$\phi'' + f'(w)\phi + (g - cw') = 0, \text{ in } \Re, \quad \phi \in L^\infty(\mathbb{R}) \tag{1.8}$$

where $C = C(g) = \dfrac{\int_{-\infty}^{\infty} gw'}{\int_{-\infty}^{\infty} w'^2}$.

Lemma 1.1. *$\forall g \in L^\infty(\mathbb{R})$ (1.8) has a solution which defines a operator $\phi = T[g]$ with*

$$\|T[g]\|_\infty \leq C\|g\|_\infty.$$

In fact if $\hat{T}[\hat{g}]$ is the solution find in the previous step then $\phi = \hat{T}[g - C(g)w']$ solves (1.8) and

$$\|\phi\|_\infty \leq C\|g\|_\infty + |C(g)|C \leq C\|g\|_\infty \tag{1.9}$$

Proof. Back to the original problem: We solve first the projected problem

$$\phi'' + f'(w)\phi + E + B(\phi) + N(\phi) = Cw', \quad \phi \in L^\infty(\mathbb{R})$$

where

$$C = \frac{\int_{\mathbb{R}}(E + B(\phi) + N(\phi))w'}{\int_{\mathbb{R}} w'^2}.$$

We solve first (1.9) and then we find $h = h_\varepsilon$ such that in (1.9) C=0 in such a way we find a solution to the original problem. We assume $|h| \leq 1$. It's sufficient to solve

$$\phi = T[E + B(\phi) + N(\phi)] := M[\phi].$$

We have the following remark

$$|E| \leq C\varepsilon^2, \quad \|B(\phi)\|_\infty \leq C\varepsilon\|\phi'\|_\infty, \quad \|N(\phi)\| \leq C(\|\phi^2\|_\infty + \|\phi^3\|_\infty)$$

where C is uniform on $|h| \leq 1$. We have

$$\|M\|_\infty + \|\frac{d}{dt}M\|_\infty \leq C(\|E\|_\infty + \|B(\phi)\|_\infty + \|N(\phi)\|_\infty \leq C(\varepsilon^2 + \varepsilon\|\phi'\|_\infty + \|\phi^2\|_\infty + \|\phi^3\|_\infty)$$

then if $\|\phi\|_\infty + \|\phi'\|_\infty \leq M\varepsilon^2$ we have

$$\|M\|_\infty + \|\frac{d}{dt}M\|_\infty \leq C^*\varepsilon^2.$$

We define the space $X = \{\phi \in C^1(\mathbb{R}) : \|\phi\|_\infty + \|\phi'\|_\infty \leq C^*\varepsilon^2\}$. Let us observe that $M(X) \subset X$. In addition

$$\|M(\phi_1) - M(\phi_2)\|_\infty + \|\frac{d}{dt}(M(\phi_1) - M(\phi_2))\|_\infty \leq C\varepsilon(\|\phi_1 - \phi_2\|_\infty + \|\phi_1' - \phi_2'\|_\infty).$$

So if ε is small M is a contraction mapping which implies that there exists a unique $\phi \in X$ such that $\phi = M[\phi]$. □

In summary: We found for each $|h| \leq 1$

$$\phi = \Phi(h), \text{ solution of1.7}$$

. We recall that

$$h \to \Phi(h)$$

is continuous (in $\|\|_{C^1}$) . Notice that from where we deduce that M is continuous in h.

The problem is reduced to finding h such that $C = 0$ in (1.7) for $\phi\Phi(h) =$. Let us observe that

$$C = 0 \Leftrightarrow \alpha_\varepsilon(h) := \int_{\mathbb{R}} (E_h + B[\Phi(h)]) + N[\Phi(h)])w' = 0.$$

Let us observe that if we call $\psi(x) = \frac{a'}{a}(x)$, then

$$\psi(x_0+\varepsilon(t+h)) = \psi(x_0)+\psi'(x_0)\varepsilon(t+h)+\int_0^1(1-s)\psi''(x_0+s\varepsilon(t+h))\varepsilon^2(t+h)^2ds$$

We add the assumption $a''' \in L^\infty(\mathbb{R})$ in order to have $\psi'' \in L^\infty(\mathbb{R})$. We deduce that

$$\int E_h w' = \varepsilon^2\psi'(x_0)\int(t+h)w'(t)^2+\varepsilon^3\int_{\mathbb{R}}(\int_0^1(1-s)\psi''(x_0+s\varepsilon(t+h))ds)(t+h)^2w'(t)dt$$

We recall that: $\int_{\mathbb{R}} tw'(t)^2$ and

$$|\int_{\mathbb{R}}(B[\phi(h)] + N[\phi(h)])w'| \leq C(\varepsilon\|\Phi(h)\|_{C^1} + \|\Phi(h)\|_{L^\infty}) \leq C\varepsilon^3.$$

So, we conclude that

$$\alpha_\varepsilon(h) = \psi'(x_0)\varepsilon^2(h + O(\varepsilon))$$

and the term inside the parenthesis change sign. This implies that $\exists h_\varepsilon : |h_\varepsilon| \leq M\varepsilon$ such that $\alpha_\varepsilon(h) = 0$, so $C = 0$.

Observe that

$$\overline{L}(\phi) = \phi''-2\phi+\varepsilon\psi+3(1-w^2)\phi+\frac{1}{2}f''(w+s\phi)\phi\phi+O(\varepsilon^2)e^{-\sqrt{2}|t|} = 0, \quad |t| > R$$

We consider $t > R$. Notice that $\frac{1}{2}f''(w + s\phi)\phi = O(\varepsilon^2)$. Then using $\hat{\phi} = \varepsilon e^{-|t|} + \delta e^{|t|}$. Then using maximum principle and after taking $\delta \to 0$, we obtain $\phi \leq \varepsilon e^{-|t|}$.

A property: We call

$$\mathcal{L}(\phi) = \phi'' + f'(w)\phi, \quad \phi \in H^2(\mathbb{R}).$$

We consider the bilinear form associated

$$B(\phi, \phi) = -\int_{\mathbb{R}} \mathcal{L}(\phi)\phi = \int_{\mathbb{R}} \phi'^2 - f'(w)^2\phi^2, \quad \phi \in H^1(\mathbb{R}).$$

Claim: $B(\phi, \phi) \geq 0, \forall\phi \in H^1(\mathbb{R})$ and $B(\phi, \phi) = 0 \Leftrightarrow \phi = cw'(t)$. In fact: $J''(w)[\phi, \phi] = B(\phi, \phi)$. We give now the proof of the claim:

Take $\phi \in C_c^\infty(\mathbb{R})$. Write $\phi = w'\Psi \implies \Psi \in C_c^\infty(\mathbb{R})$. Observe that $\mathcal{L}[w'\Psi] = \frac{1}{w'}(w'^2\Psi')'$ and

$$B(\phi,\phi) = -\int \frac{1}{w'}(w'^2\Psi')'w'\Psi = \int_{\mathbb{R}} w'^2\Psi'^2, \quad \forall \phi \in C_c^\infty(\mathbb{R})$$

Same is valid for all $\phi \in H^1(\mathbb{R})$, by density. So $B(\phi,\phi) = \int_{\mathbb{R}} |\phi'|^2 - f'(w)\phi^2 = \int_{\mathbb{R}} w'2|\Psi'|^2 \geq 0$ and $B(\phi,\phi) = 0 \Leftrightarrow \Psi' = 0$ which implies $\phi = cw'$.

Corollary 1.1. *Important for later porpuses There exists $r > 0$ such that if $\phi \in H^1(\mathbb{R})$ and $\int_{\mathbb{R}} \phi w' = 0$ then*

$$B(\phi,\phi) \geq \gamma \int_{\mathbb{R}} \phi^2$$

Proof. If not there exists $\phi_n \int H^1(\mathbb{R})$ such that $0 \leq B(\phi_n,\phi_n) < \frac{1}{n}\int_{\mathbb{R}} \phi_n^2$. We may assume without loss of generality $\int \phi_n^2 = 1$ which implies that up to subsequence

$$\phi_n \rightharpoonup \phi \in H^1(\mathbb{R})$$

and $\phi_n \to \phi$ uniformly and in L^2sense on bounded intervals. This implies that

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}} \phi_n w' = \int_{\mathbb{R}} \phi w'$$

On the other hand

$$\int |\phi_n'|^2 + 2\int \phi_n^2 - 3\int (1-w^2)\phi_n^2 \to 0$$

and also $\int |\phi_n'|^2 + 2\int \phi_n^2 - 3\int(1-w^2)\phi_n^2 \to \int |\phi'|^2 + 2\int \phi^2 - 3\int(1-w^2)\phi^2$, so $B(\phi,\phi) = 0$, and $\int w'\phi = 0$ so $\phi = 0$. But also

$$2 \leq 3\int (1-w^2)\phi_n^2 + o(1)$$

which implies that $2 \leq 3\int(1-w^2)\phi^2$ and this means that $\phi \neq 0$, so we obtain a contradiction. □

Observation 1.2. *If we choose $\delta = \frac{\gamma}{2\|f'\|_\infty}$ then*

$$\int \phi'^2 - (1+\delta)f'(w)\phi^2 \geq 0.$$

This implies in fact that

$$B(\phi,\phi) \geq \alpha \int \phi'^2.$$

2. Nonlinear Schrödinger eqution (NLS)

$$\varepsilon i\Psi_t = \varepsilon^2\Delta\Psi - W(x)\Psi + |\Psi|^{p-1}\Psi.$$

A first fact is that $\int_{\mathbb{R}^N} |\Psi|^2 = constant$. We are interested into study solutions of the form $\Psi_(x,t) = e^{-iEt}u(x)$ (we will call this solutions standing wave solution). Replacing this into the equation we obtain

$$\varepsilon Eu = \varepsilon^2\Delta u - Wu - |u|^{p-1}u$$

whose transforms into

$$\varepsilon^2\Delta u - (W-\lambda)u + |u|^{p-1}u = 0, \quad u(x) \to 0, \text{ as } |x| \to \infty$$

choosing $E = \frac{\lambda}{\varepsilon}$. We define $V(x) = (W(x) - \lambda)$

2.1. The case of dimension 1.

(2.1)
$$\varepsilon^2 u'' - V(x)u + u^p = 0, \quad x \in \mathbb{R}, \quad 0 < u(x) \to 0, \text{ as } |x| \to \infty, p > 1.$$

Assume: $V \geq \gamma > 0$, $V, V', V'', V''' \in L^\infty$, and $V \in C^3(\mathbb{R})$. Starting point

$$w'' - w + w^p = 0, \quad w > 0, \quad w(\pm\infty) = 0, p > 1 \tag{2.2}$$

There exists a homoclinic solution

$$w(t) = \frac{C_p}{\cosh\left(\frac{p-1}{2}t\right)^{\frac{2}{p-1}}}, \quad C_p = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}$$

Let us observe that $w(t) \approx 2^{2/(p-1)}C_p e^{-|t|}$ as $t \to \infty$ and also that $W(t+c)$ satisfies same equation.

Staid at x_0 with $V(x_0) = 1$ we want $u_\varepsilon(x) \approx w\left(\frac{x-x_0}{\varepsilon}\right)$ of the problem (2.1).

Observation 2.1. *Given x_0 we can assume $V(x_0) = 1$. Indeed writing*

$$u(x) = \lambda^{\frac{2}{p-1}} v(\lambda x_0 + (1-\lambda)x_0)$$

we obtain the equation

$$\varepsilon^2 v''(y) - \hat{V}(y)v + v^p = 0$$

where $y = \lambda x_0 + (1-\lambda)x_0$, and $\hat{V}(y) = V(\frac{y-(1-\lambda)x_0}{\lambda})$. Then choosing $\lambda = \sqrt{V(x_0}$, we obtain $\hat{V}(x_0) = 1$.

Theorem 2.1. *We assume $V(x_0) = 1, V'(x_0) = 0, V''(x_0) \neq 0$. Then there exists a solution to (2.1) with the form*

$$u_\varepsilon(x) \approx w\left(\frac{x-x_0}{\varepsilon}\right).$$

We define $v(t) = u(x_0 + \varepsilon(t + h))$, with $|h| \le 1$. Then v solves the problem

$$v'' - V(x_0 + \varepsilon(t + h)v + v^p = 0, \quad v(\pm\infty) = 0. \tag{2.3}$$

We define $v(t) = w(t) + \phi(t)$, so ϕ solves

(2.4)
$$\phi'' - \phi + pw^{p-1}\phi - (V(x_0 + \varepsilon(t+h)) - V(x_0))\phi + (w+\phi)^p - w^p - pw^{p-1}\phi$$

$$-(V(x_0 + \varepsilon(t + h)) - V(x_0))w(t) = 0 \tag{2.5}$$

So we want a solution of

$$\phi'' - \phi + pw^{p-1}\phi + E + N(\phi) + B(\phi) = 0, \quad \phi(\pm) = 0. \tag{2.6}$$

Observe that

$$E = \frac{1}{2}V''(x_0 + \xi\varepsilon(t + h))\varepsilon^2(t + h)^2 w(t),$$

so $|E| \le C\varepsilon^2(t^2 + 1)e^{-|t|} \le Ce^{-\sigma t}$ for $0 < \sigma < 1$.

We won't have a solution unless V' doesn't change sign and $V \neq 0$. For instance consider $V'(x) \ge 0$, and after multiplying the equation by u' and integrating by parts, we see that $\int_{\mathbb{R}} v' \frac{u^2}{2} = 0$, which by ODE implies that $u \equiv 0$, because u and u' equals 0 on some point.

2.2. **Linear projected problem.**

$$L(\phi) = \phi'' - \phi + pw^{p-1}\phi + g = 0, \quad \phi \in L^\infty(\mathbb{R})$$

For solvability we have the necessary condition $\int L(\phi)w' = 0$. Assume g such that $\int_{\mathbb{R}} gw' = 0$. We define $\phi = w'\Psi$, but we have the problem that $w'(0) = 0$. We conclude that $(w'^2\Psi')' + w'g = 0$ for $t \neq 0$. We take for $t < 0$

$$\phi(t) = w'(t) \int_t^{-1} \frac{d\tau}{w'(\tau)^2} \int_{-\infty}^{\tau} g(s)w'(s)ds$$

and for $t > 0$

$$\phi(t) = w'(t) \int_1^t \frac{d\tau}{w'(\tau)^2} \int_{\tau}^{\infty} g(s)w'(s)ds$$

In order to have a solution of the problem we need $\phi(0^-) = \phi(0^+)$.

$$\phi(0^-) = \lim_{t\to 0^-} \frac{-\int_{-1}^t \frac{d\tau}{w'(\tau)^2} \int_{-\infty}^{\tau} g(s)w'(s)ds}{\frac{1}{w'(t)}} = \lim_{t\to 0^-} \frac{-\frac{1}{w'(t)^2}\int_{-\infty}^t gw'}{-\frac{1}{w'(t)^2}w''(t)} = \frac{1}{w''(0)\int_{-\infty}^0 gw'}$$

and

$$\phi(0^+) = -\frac{1}{w''(0)\int_0^\infty gw'}$$

and the condition is satisfies because of the assumption of orthogonality condition.

We get $\|\phi\|_\infty \leq C\|g\|_\infty$. In fact we get also: $\forall\, 0 < \sigma < 1, \exists C > 0 :$

$$\|\phi e^{\sigma t}\|_{L^\infty} + \|\phi' e^{\sigma t}\|_{L^\infty} \leq C\|g e^{\sigma t}\|$$

Observation: We use $g = g - cw'$.(Correct this part!!!!)

2.3. **Method for solving.** In this section we consider a smooth radial cut-off function $\eta \in C^\infty(\mathbb{R})$, such that $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ if $s > 2$. For $\delta > 0$ small fixed, we consider $\eta_{k,\varepsilon} = \eta\left(\frac{\varepsilon|t|}{k\delta}\right)$, $k \geq 1$.

2.3.1. *The gluing procedure.* Write $\tilde{\phi} = \eta_{2,\varepsilon}\phi + \Psi$, then ϕ solves (2.5) if and only if

$$\eta_{2,\varepsilon}\left[\phi'' + (pw^{p-1} - 1)\phi + B(\phi) + 2\phi'\eta'_{2,\varepsilon}\right] \tag{2.7}$$

$$+\left[\Psi'' + (pw^{p-1} - 1)\Psi + B\Psi\right] + E + N(\eta_{2,\eta}\phi + \Psi) = 0. \tag{2.8}$$

(ϕ, Ψ) solves (2.8) if is a solution of the system

(2.9)
$$\phi'' - (1 - pw^{p-1})\phi + \eta_{1,\varepsilon}E + \eta_{3,\varepsilon}B(\phi) + \eta_{1,\varepsilon}pw^{p-1}\Psi + \eta_{1,\eta}N(\phi + \Psi) = 0$$

$$\Psi'' - (V(x_0 + \varepsilon(t + h)) - pw^{p-1}(1 - \eta_{1,\varepsilon}))\Psi \tag{2.10}$$

$$+(1 - \eta_{1,\varepsilon})E + (1 - \eta_{1,\varepsilon})N(\eta_{2,\varepsilon}\phi + \Psi) + 2\phi'\eta'_{2,\varepsilon} + \eta''_{2,\varepsilon}\phi = 0 \tag{2.11}$$

We solve first (2.11). We look first the problem

$$\Psi'' - W(x)\Psi + g = 0$$

where $0 < \alpha \leq W(x) \leq \beta$, W continuous and $g \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We claim that (2.3.1) has a unique solution $\phi \in L^\infty(\mathbb{R})$. Assume first that g has compact support and consider the well defined functional in $H^1(\mathbb{R})$

$$J(\Psi) = \frac{1}{2}\int_{\mathbb{R}} |\Psi'|^2 + \frac{1}{2}\int_{\mathbb{R}} w\Psi^2 - \int_{\mathbb{R}} g\Psi.$$

Also, this functional is convex and coercive. This implies that J has a minimizer, unique solution of (2.3.1) in $H^1(\mathbb{R})$ and it is bounded. Now we consider the problem

$$\Psi''_R - W\Psi_R + g\eta\left(\frac{|t|}{R}\right) = 0$$

Let us see that Ψ_R has a uniform bound. Take $\varphi(t) = \frac{\|g\|_\infty}{\alpha} + \rho \cosh\left(\frac{\sqrt{\alpha}}{2}|t|\right)$ for $\rho > 0$ very small. Since $\Psi_R \in L^\infty(\mathbb{R})$ we have

$$\Psi_R \leq \varphi(t), \quad \text{for } |t| > t_{\rho,R}.$$

Let us observe that in $[-t_{\rho,R}, t_{\rho,R}]$

$$\varphi'' - W\varphi + g\eta\left(\frac{|t|}{R}\right) < 0.$$

From (2.3.1), we see that $\gamma = (\Psi_R - \varphi)$ satisfies

$$\gamma'' - W\gamma > 0. \tag{2.12}$$

Claim: $\gamma \leq 0$ on $\mathbb{R}$. It's for $|t| > t_{\rho,R}$ if $\gamma(\bar{t}) > 0$ there is a global maximum positive $\gamma \in [-t_{\rho,R}, t_{\rho,R}]$. This implies that $\gamma''(t) \leq 0$ which is a contradiction with (2.12). This implies that $\Psi_R(t) \leq \frac{\|g\|_\infty}{\alpha} + \rho\cosh\left(\frac{\sqrt{\alpha}}{2}t\right)$. Taking the limit ρ going to 0 we get $\Psi_R \leq \frac{\|g\|_\infty}{\alpha}$, and similarly we can conclude that

$$\|\Psi_R\|_{L^\infty} \leq \frac{\|g\|_\infty}{\alpha}, \quad \forall R$$

Passing to a subsequence we get a solution $\Psi = \lim_{R\to\infty} \Psi_R$, and the convergence is uniform over compacts sets, to (2.3.1) with

$$\|\Psi\|_\infty \leq \frac{\|g\|_\infty}{\alpha}$$

. Also, the same argument shows that the solution is unique (in L^∞ sense). Besides: We observe that if $\|e^{\sigma|t|}g\|_\infty < \infty$, $0 < \sigma < \sqrt{\alpha}$ then

$$\|e^{\sigma|t|}\Psi\|_\infty \leq C\|e^{\sigma|t|}g\|$$

The proof of this fact is similar to the previous one. Just take as the function φ as follows

$$\varphi = M\frac{\|e^{\sigma|t|}g\|_\infty}{\alpha}e^{-\sigma|t|} + \rho\cosh\left(\frac{\sqrt{\alpha}}{2}|t|\right).$$

Observe now that Ψ satisfies (2.11) if and only if

$$\Psi = \left(-\frac{d^2}{dt^2} + W\right)^{-1}[F[\Psi,\phi]]$$

where $W(x) = V(x_0+\varepsilon(t+h)) - pw^{p-1}(1-\eta_{1,\varepsilon})$ and $F[\phi] = (1-\eta_{1,\varepsilon})E + (1-\eta_{1,\varepsilon})N(\eta_{2,\varepsilon}\phi + \Psi) + 2\phi'\eta'_{2,\varepsilon} + \eta''_{2,\varepsilon}\phi$. The previous result tell us that the inverse of the operator $\left(-\frac{d^2}{dt^2} + W\right)$ is well define. Assume that $\|\phi\|_{C^1} := \|\phi\|_\infty + \|\phi'\|_\infty \leq 1$, for some $\sigma < 1$ and $\|\Psi\|_\infty \leq \rho$, where ρ

is a very small positive number. Observe that $\|(1-\eta_{1,\varepsilon})E\|_\infty \le e^{-c\delta/\varepsilon}$. Furthermore, we have

$$|F(\Psi,\phi)| \le e^{-c\delta/\varepsilon} + c\varepsilon\|\phi\|_{C^1} + \|\phi\|_\infty^2 + \|\Psi\|_\infty^2$$

This implies that

$$\|M[\Psi]\| \le C_*[\mu + \|\Psi\|_\infty^2]$$

where $\mu = e^{-c\delta/\varepsilon} + c\varepsilon\|\phi\|_{C^1} + \|\phi\|_\infty^2$. If we assume $\mu < \frac{1}{4C_*2}$, and choosing $\rho = 2C_*\mu$, we have

$$\|M[\Psi]\| < \rho.$$

If we define $X = \{\Psi | \|\Psi\|_\infty < \rho\}$, then M is a contraction mapping in X. We conclude that

$$\|M[\Psi_1] - M[\Psi_2]\| \le C_*C\|\Psi_1 - \Psi_2\|, \quad \text{where } C_*C < 1.$$

Conclusion: There exists a unique solution of (2.11) for given ϕ (small in C^1-norm) such that

$$\|\Psi(\phi)\|_\infty \le [e^{-c\delta/\varepsilon} + \varepsilon\|\phi\|_{C^1} + \|\phi\|_\infty^2]$$

Besides: If $\|\phi\| \le \rho$, independent of ε, we have

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_\infty \le o(1)\|\phi_1 - \phi_2\|.$$

Next step: Solver for (2.9), with $\|\phi\|$ very small, the problem
(2.13)

$$\phi'' - (1-pw^{p-1})\phi + \eta_{1,\varepsilon}E + \eta_{3,\varepsilon}B(\phi) + \eta_{1,\varepsilon}pw^{p-1}\Psi + \eta_{1,\eta}N(\phi+\Psi) - cw' = 0$$

where $c = \frac{1}{\int w'^2}\int_{\mathbb{R}}(\eta_{3,\varepsilon}B(\phi) + \eta_{1,\varepsilon}pw^{p-1}\Psi + \eta_{1,\eta}N(\phi+\Psi))w'$. To solve (2.13) we write it as

$$\phi = T[\eta_{3,\varepsilon}B\phi] + T[N(\phi + \Psi(\phi)) + pw^{p-1}\Psi(\phi)] + T[E] =: Q[\phi]$$

Choosing δ sufficiently small independent of ε we conclude that $Q(x) \subseteq X$, and Q is a contraction in X for $\|\cdot\|_{C^1}$. This implies that (2.13) has a unique solution ϕ with $\|\phi\|_{C^1} < M\varepsilon^2$. Also the dependence $\phi = \Phi(h)$ is continuous. Now we only need to adjust h in such a way that $c = 0$. After some calculations we obtain

$$0 = K\varepsilon^2 V''(x_0)h + O(\varepsilon^3) + O(\delta\varepsilon^2).$$

So we can find $h = h_\varepsilon$ and $|h_\varepsilon| \le C\varepsilon$, such that $c = 0$.

3. Schrodinger equation in dimension N

$$(3.1)\qquad \begin{cases} \varepsilon^2\Delta u - V(y)u + u^p = 0 & \text{in } \mathbb{R}^N \\ 0 < u \text{ in } \mathbb{R}^n & u(x) \to 0, \text{ as}|x| \to \infty \end{cases}$$

We consider $1 < p < \infty$ if $N \leq 2$, and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$. The basic problem that we consider is

$$\begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbb{R}^N \\ 0 < u \text{ in } \mathbb{R}^n & w(x) \to 0, \text{ as}|x| \to \infty \end{cases}$$

We look for a solution $w = w(|x|)$, a radially symmetric solution. $w(r)$ satisfies the ordinary differential equation

$$(3.2)\qquad \begin{cases} w'' + \frac{N-1}{r}w' - w + w^p = 0 & r \in (0, \infty) \\ w'(0) = 0,\ 0 < w \text{ in } (0, \infty) & w(|x|) \to 0, \text{ as}|x| \to \infty \end{cases}$$

Proposition 3.1. *There exist a solution to (3.2).*

Proof. Let us consider the space

$$H_r^1 = \{u = u(|x|) : u \in H^1(\mathbb{R}^N)\}$$

with the norm $\|u\|_{H^1} = \int_0^\infty (|u'|^2 + |u|^2) r^{N-1} dr$. Let

$$S = \inf_{u \neq 0, u \in H_r^1} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2}{(\int_{\mathbb{R}^N} |u|^{p+1})^{2/(p+1)}}$$

We recall that $H^1(\mathbb{R}^N) \to L^{p+1}(\mathbb{R}^N)$ continuously, which means that $S > 0$ (the larger constant such that $c\|u\|_{L^{p+1}} \leq \|u\|_{H^1}$). Strategy: Take $u_n \geq 0$a minimizing sequence for S. We may assume $\|u_n\|_{L^{p+1}} = 1$. This means that $\|u_n\|_{H^1}^2 \to S$. This means that the sequence is bounded in $H^1 1$. We may assume $u_n \rightharpoonup u \in H^1$. We have by lower weak s.c.i.

$$\int |\nabla u|^2 + u^2 \leq \lim_n \int |\nabla u_n|^2 + u_n^2 = S.$$

We could get existence of a minimizer for S if we prove that $\|u\|_{L^{p+1}=1}$. This is indeed the case thanks to:

Strauss Lemma: There exist a constant C such that $\forall u \in H_r^1(\mathbb{R}^N)$:

$$|u(|x|)| \leq \frac{C}{|x|^{\frac{N-1}{2}}} \|u\|_{H^1}$$

The proof of this fact is the following: Let $u \in C_c^\infty(\mathbb{R}^N)$, $u = u(|x|)$.

$$u^2(r) = -2\int_r^\infty u(s)u'(s)ds \leq 2\int_r^\infty |u(s)||u'(s)| \frac{s^{N-1}}{r^{N-1}} ds$$

$$(3.3) \quad \leq \frac{2}{r^{N-1}}\Big(\int_0^\infty |u|^2 s^{n-1} ds\Big)^{1/2}\Big(\int_0^\infty |u'|^2 s^{N-1} ds\Big)^{1/2} \leq \frac{C}{r^{N-1}}\|u\|^2_{H^1}$$

By density we concludes the proof.

Let us observe that

$$\|u_n\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} = \|u_n\|^{p+1}_{L^{p+1}(B_R)} + \|u_n\|^{p+1}_{L^{p+1}(B_R^c)}$$

and

$$\|u_n\|^{p+1}_{L^{p+1}(B_R^c)} \leq \|u_n\|^{p-1}_{L^\infty(|x|>R)} \int_{\mathbb{R}^N} u_n^2 \leq \varepsilon$$

if $R \geq R_0(\varepsilon)$ (here we use the lemma of Strauss). On the other hand:

$$u_n \to u$$

strong in $L^{p+1}(B_R)$ since $H^1(B_R) \to L^{p+1}(B_R)$ compactly. This implies that $1 \leq \lim_{n\to\infty} \|u_n\|^{p+1}_{L^{p+1}(B_R)} + \varepsilon = \|u\|_{L^{p+1}(B_R)} + \varepsilon \leq \|u\|_{L^{p+1}(\mathbb{R}^N)} + \varepsilon$

This implies that $\|u\|_{L^{p+1}} \geq 1$ and we conclude $\|u\|_{L^{p+1}} = 1$.

u is a minimizer for S, $u \geq 0$, $u \neq 0$. We define $\Phi(v) = \|v\|_{H^1}/(\int |v|^{p+1})^2/p+1$. So u is a minimizer for Φ. This means that u is a weak solution of the problem

$$-\Delta u + u = \alpha u^p$$

where $\alpha = \|u\|_{H^1}$. We define $u = \alpha^{\frac{-1}{p-1}}\tilde{u}$, then $\tilde{u}$ is a solution of

$$-\Delta \tilde{u} + \tilde{u} = \tilde{u}^p$$

And, with the aid of maximum strong principle we can conclude that $\tilde{u}$ is in fact strictly positive everywhere. This concludes the proof ☐

Observation 3.1. *There no exist a solution of class C^2 for $p \geq \frac{N+2}{N-2}$. The proof of this fact is an application of Pohozaev identity.*

We claim that $w(r) \approx Cr^{-\frac{N-1}{2}}e^{-r}$. This can be proved with the change of variables $k = r^{-\frac{N-1}{2}}h$. The equation that satisfies h is like $h'' - h(1 + \frac{c}{r^2}) = 0$, and the solution of this equation is like e^{-r}.

Theorem 3.1. *Kwong, 1989 The radial solution of (3.2) is unique.*

3.1. **Linear problem.** Consequence of the proof of Kwong: We define

$$L(\phi) = \Delta\phi + pw(x)^{p-1}\phi - \phi.$$

Let us consider the problem

$$L(\phi) = 0, \quad \phi \in L^\infty(\mathbb{R}^N)$$

A known fact is that if ϕ is a solution of this problem, then ϕ is a linear combination of the functions $\frac{\partial w}{\partial x_j}(x)$, $j = 1, \ldots, N$. This is known as non degeneracy of w.

We assume as always $0 < \alpha \leq V \leq \beta$. We want to solve the problem

$$(3.4) \qquad \begin{cases} \varepsilon^2 \Delta \tilde{u} - V(y)\tilde{u} + \tilde{u}^p = 0 & \text{in } \mathbb{R}^N \\ 0 < \tilde{u} \text{ in } \mathbb{R}^n & \tilde{u}(x) \to 0, \text{ as} |x| \to \infty \end{cases}$$

We fix a point $\xi \in \mathbb{R}^N$. Observe that $U_\varepsilon(y) = V(\xi)^{\frac{1}{p-1}} \left(\sqrt{V(\xi)} \frac{y-\xi}{\varepsilon} \right)$, is a solution of the problem equation

$$\varepsilon^2 \Delta u - V(\xi)u + u^p = 0.$$

We will look for a solution of (3.4) $u_\varepsilon(x) \approx U_\varepsilon(y)$. We define $w_\lambda = \lambda^{\frac{1}{p-1}} w(\sqrt{\lambda}x)$.

Let us observe that if $\tilde{u}$ satisfies (3.4), then $u(x) = \tilde{u}(\varepsilon z)$ satisfies the problem

$$(3.5) \qquad \begin{cases} \Delta u - V(\varepsilon z)u + u^p = 0 & \text{in } \mathbb{R}^N \\ 0 < u \text{ in } \mathbb{R}^n & u(x) \to 0, \text{ as} |x| \to \infty \end{cases}$$

Let $\xi' = \frac{\xi}{\varepsilon}$. We want a solution of (3.5) with the form $u(z) = w_\lambda(z - \xi') + \tilde{\phi}(z)$, with $\lambda = V(\xi)$ and $\tilde{\phi}$ small compared with $w_\lambda(z - \xi')$.

3.2. **Equation in terms of ϕ.** $\phi(x) = \tilde{\phi}(\xi' - x)$. Then ϕ satisfies the equation $\Delta_x[w_\lambda(x)+\phi(x)] - V(\xi+\varepsilon x)[w_\lambda(x)+\phi(x)] + [w_\lambda(x)+\phi(x)]^p = 0$. We can write this equations as

$$\Delta\phi - V(\xi)\phi + pw_\lambda^{p-1}(x)\phi - E + B(\phi) + N(\phi) = 0$$

where $E = (V(\xi + \varepsilon x) - V(\xi))w_\lambda(x)$, $B(\phi) = (V(\xi) - V(\xi + \varepsilon x))\phi$ and $N(\phi) = (w_\lambda + \phi)^p - w_\lambda^p - pw_\lambda^{p-1}\phi$. We consider the linear problem for $\lambda = V(\xi)$,

$$(3.6) \qquad \begin{cases} L(\phi) = \Delta\phi - V(\xi + \varepsilon x)\phi + pw_\lambda(x)\phi = g - \sum_{i=1}^N c_i \frac{\partial w}{\partial x_i} \\ \int_{\mathbb{R}^N} \phi \frac{\partial w_\lambda}{\partial x_i} = 0, \quad i = 1, \dots, N \end{cases}$$

The $c_i's$ are defined as follows

$$\int L(\phi)(w_\lambda)_{x_i} = \int L_0(\phi)(w_\lambda)_{x_i} + \int (V(\xi) - V(\xi + \varepsilon x))\phi(w_\lambda)_{x_i}$$

$w = w(|x|)$. $(w_\lambda)_{x_i}(x) = w'_\lambda \frac{x_i}{|x|}$. This implies that

$$\int (w_\lambda)x_i(w_\lambda)x_j = \int w'_\lambda(|x|)^2 x_i x_j \frac{1}{|x|^2}$$

This integral is 0 if $i \neq j$ and equals to $\int_{\mathbb{R}^N} w'_\lambda(|x|)^2 x_i^2 \frac{1}{|x|^2} dx = 1/N \int |\nabla w_\lambda|^2 = \gamma$. Then $c_i = \int g(w_\lambda)x_i + \int_{\mathbb{R}^N} [V(\xi + \varepsilon x) - V(\xi)]\phi(w_\lambda)x_i \frac{1}{\int (w_\lambda)x_i{}^2}$.

Problem: Given $g \in L^\infty(\mathbb{R}^N)$ we want to find $\phi \in L^\infty(\mathbb{R}^N)$ solution to the problem (3.6). Assumptions: We assume $V \in C^1(\mathbb{R}^N)$, $\|V\|_{C^1} < \infty$. We assume in addition that $|\xi| \leq M_0$ and $0 < \alpha \leq V$.

Proposition 3.2. *There exists* ε_0, $C_0 > 0$ *such that* $\forall 0 < \varepsilon \le \varepsilon_0$, $\forall |\xi| \le M_0$, $\forall g \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, *there exist a unique solution* $\phi \in L^\infty(\mathbb{R}^N)$ *to (3.6),* $\phi = T[g]$ *satisfies*

$$\|\phi\|_{C^1} \le C_0 \|g\|_\infty$$

Proof. Step 1: A priori estimates on bounded domains: There exist R_0, ε_0, C_0 such that $\forall \varepsilon < \varepsilon_0$, $R > R_0$, $|\xi| \le M_0$ such that $\forall \phi, g \in L^\infty$ solving $L(\phi) = g - \sum_i c_i (w_\lambda)_{x_i}$ in B_R, $\int_{B_R} \phi (w_\lambda)_{x_i} = 0$ and $\phi = 0$ on ∂B_R, we have

$$\|\phi\|_{C^1(B_R)} \le C_0 \|g\|_\infty$$

We prove first $\|\phi\|_\infty \le C_0 \|g\|_\infty$. Assume the opposite, then there exist sequences ϕ_n, g_n, $\varepsilon \to 0$, $R_n \to \infty$, $|\xi_n| \le M_0$ such that

$$L(\phi_n) = g_n - c_i^n \frac{\partial w_\lambda}{\partial x_i}$$

. The first fact is that $c_i^n \to 0$ as $n \to \infty$. This fact follows just after multiplying the equation against $(w_\lambda)_{x_i}$ and integrating by parts.

Observation: If $\Delta \phi = g$ in B_2 then there exist C such that

$$\|\nabla \phi\|_{L^\infty(B_1)} \le C[\|g\|_{L^\infty(B_2)} + \|\phi\|_{L^\infty(B_2)}]$$

Where B_1 and B_2 are concentric balls. This implies that $\|\nabla \phi_n\|_{L^\infty(B)} \le C$ a given bounded set B, $\forall n \ge n_0$. This implies that passing to a subsequence $\phi_n \to \phi$ uniformly on compact sets, and $\phi \in L^\infty(\mathbb{R}^N)$. Observe that $\|\phi_n\|_\infty = 1$, and this implies that $\|\phi\|_\infty \le 1$. We can assume that $\xi_n \to \xi_0$.

ϕ satisfies the equation $\Delta \phi - V(\xi_0)\phi + p w_{\lambda_0}^{p-1}(x)\phi = 0$, where $\lambda_0 = V(\xi_0)$, and this implies that $\phi \in \text{Span}\left\{\frac{\partial w_{\lambda_0}}{x_1}, \dots, \frac{\partial w_{\lambda_0}}{x_N}\right\}$, but also $\int_{\mathbb{R}^N} \phi (w_{\lambda_0})_{x_i} = 0$, $i = 1, \dots, N$. Then $\phi = 0$ and this implies that $\|\phi_n\|_{L^\infty(B_M(0))} \to 0, \forall M < \infty$. Maximum principle implies that $\|\phi_n\|_{L^\infty(B_{R_n} \setminus B_{M_0}} \to 0$, just because $|\phi_n| = o(1)$ on $\partial B_{R_n} \setminus B_{M_0}$ and $\|g_n\|_\infty \to 0$. Therefore $\|\phi_n\|_\infty \to 0$, a contradiction. This implies that $\|\phi\|_{L^\infty(B_R)} \le C_0 \|g\|_{L^\infty(B_R)}$ uniformly on large R. The C^1 estimate follows from elliptic local boundary estimates for Δ.

Step 2: Existence: $g \in L^\infty$. We want to solve (3.6). We claim that solving (3.6) is equivalent to finding $\phi \in X = \{\psi \in H_0^1 : \int \psi (w_\lambda)_{x_i} = 0, i = 1, \dots, N\}$ such that

$$\int \nabla \phi \nabla \psi + \int V(\xi + \varepsilon x)\phi\psi - p w^{p-1} \phi \psi + \int g \psi = 0, \quad \forall \psi \in X.$$

Take general $\Psi \in H_0^1$, $\Psi = \psi + \sum_i \alpha_i (w_\lambda)_{x_i}$, with $\alpha_i = \frac{\int \Psi (w_\lambda)_{x_i}}{\int (w_\lambda)_{x_i}}$. We have

$$-\int \Delta(\sum_i \alpha_i (w_\lambda)_{x_i})\nabla\phi + \int V(\xi)(\sum_i \alpha_i (w_\lambda)_{x_i})\phi - pw^{p-1}(\sum_i \alpha_i (w_\lambda)_{x_i})\phi = 0$$

Which implies that

$$\int \nabla\phi\nabla\Psi + \int V(\xi)\phi\Psi - pw^{p-1}\phi\Psi$$

$$-\int (V(\xi) - V(\xi+\varepsilon x))(\Psi - \sum_i \alpha_i (w_\lambda)_{x_i}) + \int g(\Psi - \sum_i \alpha_i (w_\lambda)_{x_i})$$

Then

$$\int [(V(\xi+\varepsilon x) - V(\xi))\phi + g](\Psi - \sum_i \alpha_i (w_\lambda)_{x_i})$$

and $\Pi_X(\Psi) = \sum_i \alpha_i (w_\lambda)_{x_i}$, then the previos integral is equal to

$$\int \Pi_X([(V(\xi+\varepsilon x) - V(\xi))\phi + g]\phi)\Psi$$

□

This implies that

$$-\Delta\phi + V(\xi)\phi - pw^{p-1}\phi + \Pi_X([(V(\xi+\varepsilon x) - V(\xi))\phi + g]\phi) = 0.$$

The problem is formulated weakly as

$$\int \nabla\phi\nabla\psi + \int (V(\xi+\varepsilon x) - pw^{p-1})\phi\psi + \int g\psi = 0, \phi \in X, \forall\psi \in X$$

This can be written as $\phi = A[\phi] + \tilde{g}$, where A is a compact operator. The a priori estimate implies that the only solution when $g = 0$ of this equation is $\phi = 0$. We conclude existence by Fredholm alternative.

We look for a solution which near $x_j = \xi_j' = \xi_j/\varepsilon$, $j = 1, \ldots, k$ looks like $v(x) \approx W_{\lambda_j}(x - \xi_j')$, $\lambda_j = V(\xi_j)$, where W_λ solves

$$\Delta W_\lambda - \lambda W + W^p = 0, \quad W_\lambda \text{ radial}, \quad W_\lambda(|x|) \to 0, \text{ as } |x| \to \infty$$

Observe that $W_\lambda(y) = \lambda^{1/(p-1)} w(\sqrt{\lambda} y)$, where w solves the equation $\Delta w - w + w^p = 0$. The equation

$$\Delta v - V(\varepsilon x)v + v^p = 0$$

looks like $\Delta v - V(\xi_j)v + v^p = 0$, where $\xi_1, \xi_2, \ldots \xi_k \in \mathbb{R}^N$ and we assume also $|\xi_j' - \xi_l'| \gg 1$, if $j \neq l$. We look for a solution $v(x) \approx \sum_{j=1}^k W_{\lambda_j}(x - \xi_j')$, $\lambda_j = V(\xi_j)$. We assume $V \in C^2(R^N)$ and $\|V\|_{C^2} < \infty$, $0 < \alpha \leq V$.

We use the notation $W_j = W_{\lambda_j}(x - \xi'_j)$, $\lambda_j = V(\xi_j)$ and $W = \sum_{j=1}^n W_j$. Look for a solution $v = W + \phi$, so ϕ solves the problem

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + E + N(\phi) = 0$$

where

$$E = \Delta W - VW + W^p, \quad N(\phi) = (W+\phi)^p - W^p - pW^{p-1}\phi.$$

Observe that $\Delta W = \sum_j \Delta W_j = \sum_j \lambda_j W_j - W_j^p$. So we can write

$$E = \sum_j (\lambda_j - V(\varepsilon x))W_j + (\sum_j W_j)^p - \sum_j W_j^p.$$

3.3. **Linearized (projected) problem.** We use the following notation $Z_j^i = \frac{\partial W_j}{\partial x_i}$. The linearized projected problem is the following

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i,$$

with the orthogonality condition $\int \phi Z_j^i = 0$, $\forall i, j$. The Z_j^i's are "nearly orthogonal" if the centers ξ'_j are far away one to each other. The c_j^i's are, by definition, the solution of the linear system

$$\int_{\mathbb{R}^N} (\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g)Z_{j_0}^{i_0} = \sum_{i,j} c_j^i \int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0},$$

for $i_0 = 1, \ldots, N$, $j_0 = 1, \ldots, k$. The c_j^i's are indeed uniquely determined provided that $|\xi'_l - \xi'_j| > \mathbb{R}_0 \gg 1$, because the matrix with coefficients $\alpha_{i,j,i_0,j_0} = \int Z_j^i Z_{j_0}^{i_0}$ is "nearly diagonal", this means

$$\alpha_{i,j,i_0,j_0} = \begin{cases} \frac{1}{N}\int |\nabla W_j|^2 & \text{if } (i,j) = (i_0, j_0), \\ o(1) & \text{if not} \end{cases}$$

Moreover:

$$|c_{j_0}^{i_0}| \leq C \sum_{i,j} \int |\phi|[|\lambda_j - V| + p|W^{p-1} - W_j^{p-1}|]|Z_j^i| + \int |g||Z_j^i| \leq C(\|\phi\|_\infty + \|g\|_\infty)$$

with C uniform in large R_0. Even more, if we take $x = \xi' + y$

$$|(\lambda_j - V(\varepsilon x))Z_j^i| \leq |(V(\xi_j) - V(\xi_j + \varepsilon y))||\frac{\partial W_{\lambda_j}}{\partial y_i}| \leq C\varepsilon e^{-\frac{\sqrt{\alpha}}{2}|y|},$$

because $|\frac{\partial W_{\lambda_j}}{\partial y_i}| \leq Ce^{-|y|\sqrt{\lambda_j}}|y|^{-(N-1)/2}$. Observe also that

$$|(W^{p-1} - W_j^{p-1})Z_j^i| = |((1 - \sum_{l \neq j} \frac{W_l}{W_j})^{p-1} - 1)|W_j^{p-1} Z_j^i.$$

Observe that if $|x-\xi'_j| < \delta_0 \min_{j_1\neq j_2} |\xi'_{j_1}-\xi'_{j_2}|$, then

$$\frac{W_l(x)}{W_j(x)} \approx \frac{e^{-\sqrt{\lambda_l}|x-\xi'_l|}}{e^{-\sqrt{\lambda_j}|x-\xi'_j|}} < \frac{e^{-\sqrt{\lambda_l}|x-\xi'_l|}}{e^{-\sqrt{\lambda_j}\delta_0 \min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|}}$$

If $\delta_0 \ll 1$ but fixed, we conclude that $e^{-\sqrt{\lambda_l}|\xi'_j-\xi'_l|+\delta_0(\sqrt{\lambda_l}-\sqrt{\lambda_j})\min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|} < e^{-\rho \min_{j_1\neq j_2}|\xi'_{j_1}-xi'_{j_2}|\ll 1}$. Conclusion: if $|x-\xi'_j| < \delta_0 \min_{j_1\neq j_2}|\xi'_{j_1}-xi'_{j_2}|$ implies that

$$|(W^{p-1}-W_j^{p-1})Z_j^i| \le e^{-\rho \min_{j_1\neq j_2}|\xi'_{j_1}-xi'_{j_2}|} e^{-\frac{\alpha}{2}|x-\xi'_j|}.$$

If $|x-\xi'_j| > \delta_0 \min_{j_1\neq j_2}|\xi'_{j_1}-xi'_{j_2}|$, then

$$|(W^{p-1}-W_j^{p-1})Z_j^i| \le C|Z_j^i| \le Ce^{-\rho \min_{j_1\neq j_2}|\xi'_{j_1}-xi'_{j_2}|} e^{-\frac{\alpha}{2}|x-\xi'_j|}$$

As a conclusion we get

$$|c_{j_0}^{i_0}| \le C(\varepsilon + e^{-\rho \min_{j_1\neq j_2}|\xi'_{j_1}-\xi'_{j_2}|})\|\phi\|_\infty + \|g\|_\infty$$

Lemma 3.1. *Given $k \ge 1$, there exist R_0, C_0, ε_0 such that for all points ξ'_j with $|\xi'_{j_1}-\xi'_{j_2}| > R_0$, $j=1,\dots,k$ and all $\varepsilon < \varepsilon_0$ then exist a unique solution ϕ to the linearized projected problem with*

$$\|\phi\|_\infty \le C_0\|g\|_\infty.$$

Proof. We first prove the a priori estimate $\|\phi\|_\infty \le C_0\|g\|_\infty$. If not there exist $\varepsilon_n \to 0$, $\|\phi_n\|_\infty = 1$, $\|g_n\| \to 0$, $\xi_j'^n$ with $\min_{j_1\neq j_2}|\xi_{j_1}'^n-\xi_{j_2}'^n| \to \infty$. We denote $W_n = \sum_j W_{j_n}$, and we have

$$\Delta\phi_n - V(\varepsilon_n x)\phi_n + pW_n^{p-1}\phi_n + g_n = \sum_{i,j}(c_j^i)_n(z_j^i)_n$$

First observation: $(c_j^i)_n \to 0$ (follows from estimate for $c_{j_0}^{i_0}$). Second: $\forall R > 0$ $\|\phi_n\|_{L^\infty(B(\xi_j'^n,R))} \to 0$, $j=1,\dots,k$. If not, there exist j_0 $\|\phi_n\|_{L^\infty(B(\xi_{j_0}'^n,R))} \ge \gamma > 0$. We denote $\tilde{\phi}_n(y) := \phi_n(\xi_{j_0}'^n + y)$. We have $\|\tilde{\phi}_n\|_{L^\infty(B(0,R))} \ge \gamma > 0$. Since $|\Delta\tilde{\phi}_n| \le C$, $\|\tilde{\phi}_n\|_\infty \le 1$. This implies that $\|\nabla\tilde{\phi}_n\| \le C$. Passing to a subsequence we may assume $\tilde{\phi}_n \to \tilde{\phi}$ uniformly on compacts sets. Observe that also $V(\varepsilon_n(\xi_{j_0}'^n + y)) = V(\varepsilon_n\xi_{j_0}'^n) + O(\varepsilon_n|y|) \to \lambda_{j_0}$ over compact sets and $W_n(\xi_{j_0}'^n + y) \to W_{\lambda_{j_0}}(y)$ uniformly on compact sets. This implies that $\tilde{\phi}$ is a solution of the problem

$$\Delta\tilde{\phi} - \lambda_{j_0}\tilde{\phi} + pw_{\lambda_0}^{p-1}p\tilde{-}1 = 0, \qquad \int \tilde{\phi}\frac{\partial W_{\lambda_{j_0}}}{\partial y_i}dy = 0, i=1,\dots,N$$

Non degeneracy of $w_{\lambda_{j_0}}$ implies that $\tilde{\phi} = \sum_i \alpha_i \frac{\partial w_{\lambda_{j_0}}}{\partial y_i}$. The orthogonality condition implies that $\alpha_i = 0$, $\forall i = 1,\dots,N$. This implies that

$\tilde{\phi} = 0$ but $\|\tilde{\phi}\|_{L^\infty(B(0,R))} \geq \gamma > 0$, a contradiction. Now we prove: $\|\phi_n\|_{L^\infty}(\mathbb{R}^N \setminus \cup_n B(\xi_j'^n, R)) \to 0$, provided that $R \gg 1$ and fixed so that $\phi_n \to 0$ in the sense of $\|\phi_n\|_\infty$ (again a contradiction). We will denote $\Omega_n = \mathbb{R}^N \setminus \cup_n B(\xi_j'^n, R)$. For $R \gg 1$ the equation for ϕ_n has the form

$$\Delta\phi_n - Q_n\phi_n + g_n = 0$$

where $Q_n = V(\varepsilon x) - pW_n^{p-1} \geq \frac{\alpha}{2} > 0$ for some R sufficiently large (but fixed). Let's take for $\sigma^2 < \alpha/2$

$$\bar{\phi} = \delta \sum_j e^{\sigma|x-\xi_j'^n|} + \mu_n.$$

We denote $\varphi(y) = e^{\sigma|y|}$, $r = |y|$. Observe that $\Delta\varphi - \alpha/2\varphi = e^{\sigma|y|}(\sigma^2 + \frac{N-1}{|y|} - \alpha/2) < 0$ if $|y| > R \gg 1$. Then

$$-\Delta\bar{\phi} + Q_n\bar{\phi} - g_n > -\Delta\bar{\phi} + \frac{\alpha}{2}\bar{\phi} - \|g_n\|_\infty > \frac{\alpha}{2}\mu_n - \|g_n\|_\infty > 0$$

if we choose $\mu_n \geq \|g_n\|_\infty \frac{2}{\alpha}$. In addition we take $\mu_n = \sum_j \|\phi_n\|_{L^\infty(B(\xi_j^n,R))} + \|g_n\|_\infty \frac{2}{\alpha}$. Maximum principle implies that $\phi_n(x) \leq \bar{\phi}$ for all $x \in \Omega_n$. Taking $\delta \to 0$ this implies that $\phi_n(x) \leq \mu_n$, for all $x \in \Omega_n$. Also true that $|\phi_n(x)| \leq \mu_n$ for all $x \in \Omega_n^c$, and this implies that $\|\phi_n\|_{L^\infty(\mathbb{R}^N)} \to 0$. □

Observation 3.2. *If in addition we have* $\theta_n = \|g_n \left(\sum_j e^{-\rho|x-\xi_j'^n|}\right)^{-1}\|_\infty \to 0$ *with* $\rho < \alpha/2$. *Then we can use as a barrier*

$$\bar{\phi} = \delta \sum_j e^{\sigma|x-\xi_j'^n|} + \mu_n \sum_j e^{-\rho|x-\xi_j'^n|}$$

with $\mu_n = e^{\rho R} \sum_j \|\phi_n\|_{L^\infty(B(\xi_j'^n,R))} + \theta_n$, *then* $\bar{\phi}$ *is a super solution of the equation and we have* $|\phi_n| \leq \bar{\phi}$, *and letting* $\delta \to 0$ *we get* $|\phi_n(x)| \leq \mu_n \sum_j e^{-\rho|x-\xi_j'^n|}$. *As a conclusion we also get the a priori estimate*

$$\|\phi \left(\sum_{j=1}^k e^{-\rho|x-\xi_j'|}\right)^{-1}\|_\infty \leq C\|g\left(\sum_{j=1}^k e^{-\rho|x-\xi_j'|}\right)^{-1}\|_\infty$$

provided that $0 \leq \rho < \alpha/2$, $|\xi_{j_1}' - \xi_{j_2}'| > R_0 \gg 1$, $\varepsilon < \varepsilon_0$.

We now give the proof of existence

Proof. Take g compactly supported. The weak formulation for

$$\Delta\phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_j^i Z_j^i, \quad \int \phi Z_j^i, \forall i, j \tag{3.7}$$

is find $\phi \in X = \{\phi \in H^1(\mathbb{R}^N) : \int \phi Z_j^i = 0, \forall i, j\}$ such that

$$\int_{\mathbb{R}^N} \nabla\phi\nabla\psi + V\phi\psi - pw^{p-1}\phi\psi - g\psi = 0, \quad \forall \psi \in X. \tag{3.8}$$

Assume ϕ solves (3.7). For $g \in L^2$, write $g = \tilde{g} + \Pi[g]$ where $\int \tilde{g} Z_j^i = 0$, for all i, j. Π is the orthogonal projection of g onto the space spanned by the Z_j^i's. Take $\psi \in H^1(\mathbb{R}^N)$ arbitrary and use $\psi - \Pi[\psi]$ as a test function in (3.8). Then if $\varphi \in C_c^\infty(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} \nabla\varphi\nabla(\Pi[\psi]) = -\int_{\mathbb{R}^N} \Delta\varphi\Pi[\psi] = -\int_{\mathbb{R}^N} \Pi[\Delta\varphi]\psi.$$

But $\Pi[\Delta\varphi] = \sum_{i,j} \alpha_{i,j} Z_j^i$, where

$$\sum \alpha_{i,j} \int Z_{i,j} Z_{i_0,j_0} = \int \Delta\varphi Z_{i_0}^{j_0} = \int \varphi \Delta Z_{i_0}^{j_0}$$

Then $\|\Pi[\Delta\varphi]\|_{L^2} \leq C\|\varphi\|_{H^1}$. By density is true also for $\varphi \in H^1$ where $\Delta\varphi \in H^{-1}$. Therefore

$$\int \nabla\phi\nabla\psi + \int (V\phi - pW^{p-1}\phi - g)\psi = \int \Pi(V\phi - pW^{p-1}\phi + g)\psi$$

then ϕ solves in weak sense

$$-\Delta\phi + V\phi - pW^{p-1}\phi - g = \Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g]$$

and $\Pi[-\Delta\phi + V\phi - pW^{p-1}\phi - g] = \sum_{i,j} c_i^j Z_i j$. Therefore by definition ϕ solves (3.8) implies that ϕ solves (3.8). Classical regularity gives that this weak solution is solution of (3.7) in strong sense, in particular $\phi \in L^\infty$ so that

$$\|\phi\|_\infty \leq C\|g\|_\infty$$

. Now we give the proof of existence for (3.7). We take g compactly supported. The equation (3.8) can be written in the following way (using Riesz theorem):

$$\langle \phi, \psi \rangle_{H^1} + \langle B[\phi], \psi \rangle_{H^1} = \langle \tilde{g}, \psi \rangle_{H^1}$$

or $\phi + B[\phi] = \tilde{g}$, $\phi \in X$. We claim that B is a compact operator. Indeed if $\phi_n \rightharpoonup 0$ in X, then $\phi_n \to 0$ in L^2 over compacts.

$$|\langle B[\phi_n], \psi \rangle| \leq |\int pW^{p-1}\phi_n\psi| \leq (\int pw^{p-1}\phi_n^2)^{1/2}(\int pW^{p-1}\psi^2)^{1/2}$$

then

$$|\langle B[\phi_n], \psi \rangle| \leq c(\int pW^{p-1}\phi_n^2)^{1/2}\|\psi\|_{H^1}$$

Take $\psi = B[\phi_n]$, which implies

$$\|B[\phi_n]\|_{H^1} \leq c(\int pW^{p-1}\phi_n^2)^{1/2} \to 0.$$

This implies that B is a compact operator. Now we prove existence with the aid of fredholm alternative. Problem is solvable if for $\tilde{g} = 0$ implies that $\phi = 0$. But $\phi + B[\phi] = 0$ implies solve (3.7)(strongly) with $g = 0$. This implies $\phi \in L^\infty$, and the a priori estimate implies $\phi = 0$. Considering $g\Xi_{B_R(0)}$ we conclude that

$$\|\phi_R\|_\infty \leq \|g\|_\infty$$

Taking $R \to \infty$ then along a subsequence $\phi_R \to \phi$ uniform over compacts. □

We take $g \in L^\infty$. We have $\phi = T_{\xi'}[g]$, where $\xi' = (\xi'_1, \ldots, \xi'_k)$. We want to analyze derivatives $\partial_{\xi'_{ji}} T_{\xi'}[g]$. We know that $\|T_{\xi'}[g]\| \leq C_0\|g\|_\infty$. First we will make a formal differentiation. We denote $\Phi = \frac{\partial\phi}{\partial\xi'_{i_0j_0}}$.

We have $\Delta\phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c^i_j Z^i_j$ and $\int \phi Z^i_j = 0$, for all i, j. Formal differentiation yields

$$\Delta\Phi - V\Phi + pW^{p-1}\Phi + +\partial_{\xi_{i_0j_0}}(W^{p-1})\phi - \sum_{i,j} c^i_j \partial_{\xi_{i_0j_0}} Z^j_i = \sum_{i,j} \tilde{c}^i_j Z^i_j$$

where formally $\tilde{c}^j_i = \partial_{\xi_{i_0j_0}} c^j_i$. The orthogonality conditions traduces into

$$\int_{\mathbb{R}^N} \Phi Z^i_j = \begin{cases} 0 & \text{if } j \neq j_0 \\ -\int \phi \partial_{\xi_{i_0j_0}Z^i_{j_0}} & \text{if } j = j_0 \end{cases}$$

Let us define $\tilde{\Phi} = \Phi - \sum_{i,j} \alpha_{i,j} Z^i_j$. We want $\int \tilde{\Phi} Z^i_j = 0$, for all i, j. We need

$$\sum_{i,j} \alpha_{i,j} \int Z^i_j Z^{\bar{i}}_{\bar{j}} = \begin{cases} 0 & \text{if } \bar{j} \neq j_0 \\ -\int \phi \partial_{\xi_{i_0j_0}Z^i_{j_0}} & \text{if } \bar{j} = j_0 \end{cases}$$

The system has a unique solution and $|\alpha_{i,j}| \leq C\|\phi\|_\infty$ (since the system is almost diagonal). So we have the condition $\int \tilde{\Phi} Z^i_j = 0$, for all i, j. We add to the equation the term $\sum_{i,j} \alpha_{i,j}(\Delta - V + pW^{p-1})Z^i_j$, so $\tilde{\Phi}$ satisfies the equation$\Delta\phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c^i_j Z^i_j$

$$\Delta\tilde{\Phi} - V\tilde{\Phi} + pW^{p-1}\tilde{\Phi} + \partial_{\xi_{i_0j_0}}(W^{p-1})\phi - \sum_{i,j} c^i_j \partial_{\xi_{i_0j_0}} Z^j_i = \sum_{i,j} \tilde{c}^i_j Z^i_j - \sum_{i,j} \alpha_{i,j}(\Delta - V + pW^{p-1})Z^i_j$$

This implies $\|\tilde{\Phi}\| \leq C(\|h\| + \|g\|) \leq C\|g\|_\infty$. This implies $\|\Phi\| \leq C\|g\|_\infty$. We do this in a discrete way, and passing to the limit all these calculations are still valid. Conclusion: The map $\xi \to \partial_\xi \phi$ is well defined and continuous (into L^∞). Besides $\|\partial_\xi \phi\|_\infty \le C\|g\|_\infty$, and this implies

$$\|\partial_\xi T_\xi[\phi]\| \le C\|g\|$$

3.4. **Nonlinear projected problem.** Consider now the nonlinear projected problem

$$\Delta\phi - V\phi + pw^{p-1}\phi + E + N(\phi) = \sum_{i,j} c_i^j Z_j^i, \qquad \int \phi Z_i^j = 0, \, \forall i, j$$

We solve this by fixed point. We have $\phi = T(E + N(\phi)) =: M(\phi)$. We define $\Lambda = \{\phi \in C(\mathbb{R}^N) \cap L^\infty(R^N) : \|\phi\|_\infty \le M\|E\|_\infty\}$. Remember that $E = \sum_i (\lambda_j - V(\varepsilon x))W_j + (\sum_j W_j)^p - \sum_j W_j^p$. Observe that

$$|E| \le \varepsilon \sum_i e^{-\sigma|x-\xi_j'|} + ce^{-\delta_0 \min_{j_1 \ne j_2} |\xi_{j_1}' - \xi_{j_2}'|} \sum_j e^{-\sigma|x-\xi_j'|}$$

so, for existence we have $\|E\| \le C[\varepsilon + e^{-\delta_0 \min_{j_1 \ne j_2} |\xi_{j_1}' - \xi_{j_2}'|}] =: \rho$ (see that ρ is small). Contraction mapping implies unique existence of $\phi = \Phi(\xi)$ and $\|\Phi(\xi)\| \le M\rho$.

3.5. **Differentiability in ξ' of $\Phi(\xi')$.** We have

$$\Phi - T_\xi'(E_\xi' + N_\xi'(\phi)) = A(\Phi, \xi') = 0$$

If $(D_\Phi A)(\Phi(\xi'), \xi')$ is invertible in L^∞, then $\Phi(\xi')$ turns out to be of class C^1. This is a consequence of the fixed point characterization, i.e., $D_\Phi A(\Phi(\xi'), \xi') = I + o(1)$ (the order $o(1)$ is a direct consequence of fixed point characterization). Then is invertible. Theorem and the C^1 derivative of $A(\Phi, \xi')$ in (ϕ, ξ'). This implies $\Phi(\xi')$ is C^1. $\|D_\xi' \Phi(\xi')\| \le C\rho$ (just using the derivate given by the implicit function theorem).

3.6. **Variational reduction.** We want to find ξ' such that the $c_j^i = 0$, for all i, j, to get a solution to the original problem. We use a procedure that we call Variational Reduction in which the problem of finding ξ' with $c_j^i = 0$, for all i, j, is equivalent to finding a critical point of a functional of ξ'. Recall:

$$J(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + V(\varepsilon x)v^2 - \frac{1}{p+1} \int_{\mathbb{R}^{N+1}} v_+^{p+1}$$

is defined in $H^1(\mathbb{R}^N)$, since $1 < p < \frac{N+2}{N-2}$. v is a solution of $\Delta v - Vv + v^p = 0$, $v \to 0$ if and only if $v \in H^1(\mathbb{R}^N)$ and $J'(v) = 0$. Observe that $\langle J'(v), \varphi \rangle = \int \nabla v \nabla \varphi + Vv\varphi - v_+^p \varphi$.

The following fact happens: $v = W_{\xi_*'} + \phi(\xi')$ is a solution of the original problem (for $\rho \ll 1$) if and only if

$$\partial_{\xi'} J(W_{\xi'} + \phi(\xi'))|_{\xi' = \xi_*'} = 0.$$

Indeed, observe that $v(\xi') := W_{\xi'} + \phi(\xi')$ solves the problem $\Delta v(\xi') - V(\varepsilon x)v(\xi') + v(\xi')^p = \sum_{i,j} c_j^i Z_j^i$ and also that

$$\partial_{\xi'_{j_0 i_0}} J(v(\xi')) = \langle J'(v(\xi')), \partial_{\xi'_{j_0 i_0}} v(\xi')\rangle = -\sum_{j,i} c_j^i \int Z_j^i \partial_{\xi'_{j_0 i_0}} v = -\sum_{i,j} c_j^i \int Z_i^j (\partial_{\xi'_{j_0 i_0}} W_{\xi'} + \partial_{\xi'_{j_0 i_0}}$$

Remember that $W_{\xi'} = \sum_{j=1}^k w_{\lambda_j}(x - \xi_j')$,

$$\partial_{\xi'_{j_0 i_0}} W_\xi' = \partial_{\xi'_{j_0 i_0}} w_{\lambda_{j_0(\xi')}}(x - \xi_j') = (\partial_\lambda w_\lambda (x - \xi'_{j_0}))|_{\lambda = \lambda_{j_0}} - \partial_{x_{i_0}} w_{\lambda_{j_0}}(x - \xi'_{j_0}) = O(e^{-\delta|x-\xi_0'|})o(\varepsilon) - Z_j$$

This because $\partial_\lambda w_\lambda = O(e^{-\delta|x-\xi_0'|})$. On the other hand $|\partial_{\xi'_{j_0 i_0}} \phi| \le C\rho e^{-\delta|x-\xi'_{j_0}|}$. Finally, observe that

$$-\int Z_j^i (\partial_{\xi'_{j_0 i_0}} W_\xi' + \partial_{\xi'_{j_0 i_0}} \phi) = \int Z_j^i Z_{j_0}^{i_0} + O(\rho)$$

The matrix of these numbers is invertible provided $\rho \ll 1$.

A consequence (D, Felmer 1996): Assume $j = 1$ and that there exist an open, bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$\inf_{\partial\Lambda} V > \inf_{\Lambda} V,$$

then there exist a solution to the original problem, v_ε with $v_\varepsilon(x) = W_{V(\xi_\varepsilon)}((x - \xi_\varepsilon)/\varepsilon) + o(1)$ and $V(\xi_\varepsilon) \to \min_\Lambda V$, $\xi = \xi_\varepsilon$.

Another consequence (D, Felmer 1998): $\Lambda_1, \dots, \Lambda_k$ disjoint bounded with $\inf_{\Lambda_j} V < \inf_{\partial\Lambda_j} V$, for all j. For the problem $\varepsilon^2 \Delta u - V(x)u + u^p = 0$, $0 < u \to 0$ at ∞, there exist a solution u_ε with $u_\varepsilon(x) \approx \sum_{j=1}^k W_{V(\xi_j^\varepsilon)}(x - \xi_j^\varepsilon/\varepsilon)$, $\xi_j^\varepsilon \in \Lambda_j$ and $V(\xi_j^\varepsilon) \to \inf_{\Lambda_j} V$ (in the case of non-degeneracy minimal or more generally non-degenerate critical points the result is due to Oh (1990))

Proof. First result: $j = 1$. $v(\xi') = W_{\xi'} + \phi(\xi')$. Then

$$J(W_(\xi')) = J(W_{\xi'} + \phi(\xi')) + \langle J'(W_\xi' + \phi), -\phi\rangle + \frac{1}{2} J''(W_\xi' + (1-t)\phi)[\phi]^2$$

(Taylor expansion of the function $\alpha(t) = J(W_\xi' + (1-t)\phi)$). Observe that $\langle J'(W_\xi' + \phi), -\phi\rangle = \sum_{i,j} c_j^i \int Z_i^j \phi = 0$. Also observe that

$$J''(W_\xi' + (1-t)\phi)[\phi]^2 = \int |\nabla\phi|^2 + V(\varepsilon x)\phi^2 - p(W_\xi' + (1-t)\phi)\phi^2 = O(\varepsilon^2)$$

uniformly on ξ' because $\nabla\phi, \phi = O(\varepsilon e^{-\delta|x-\xi'|})$. We call $\Phi(\xi) := J(v(\xi')) = J(W_\xi') + O(\varepsilon^2)$, and

$$J(W_\xi') = \frac{1}{2}\int |\nabla W_\xi'|^2 + V(\xi) W_\xi'^2 - \frac{1}{p+1}\int W_\xi'^{p+1} + \int (V(\varepsilon x) - V(\xi'))W_\xi^2$$

Taking $\lambda = V(\xi)$, we have that

$$\int |\nabla w_\lambda(x)|^2 = \lambda^{-N/2} \int |\nabla w(\lambda^1/2x)|^2 \lambda^{1+2/(p-1)} \lambda^{N/2} dx = \lambda^{-N/2+p+1/p-1} |\nabla w(y)|^2 dy$$

and

$$\lambda \int w_\lambda^2(x) = \lambda^{-N/2p+1/p-1} \int w(y)^{p+1} dy$$

This implies that

$$\frac{1}{2} \int |\nabla W'_\xi|^2 + V(\xi) W'^2_\xi - \frac{1}{p+1} \int W'^{p+1}_\xi = V(\xi)^{p+1/p-1-N/2} c_{p,N}.$$

also

$$\int (V(\varepsilon x) - V(\xi')) w_\lambda (x - \xi')^2 = O(\varepsilon)$$

uniformly on ξ. In summary $\Phi(\xi) = J(v(\xi')) = V(\xi)^{p+1/p-1-N/2} c_{p,N} + O(\varepsilon)$ and $\frac{p+1}{p-1} - \frac{N}{2} > 0$. Then $\forall \varepsilon \ll 1$ we have

$$\inf_{\xi \in \Lambda} \Phi(\xi) < \inf_{\xi \in \partial\Lambda} \Phi(\xi)$$

therefore Φ has a local minimum $\xi_\varepsilon \in \Lambda$ and $V(\xi_\varepsilon) \to \min_\Lambda V$. Same thing works at a maximum.

For several spikes separated: $|\xi_{j_1} - \xi_{j_2}| > \delta$, for all $j_1 \neq j_2$. $\rho = e^{-\delta_0 \min_{j_1 \neq j_2} |\xi'_{j_1} - \xi'_{j_2}|} + \varepsilon \leq e^{-\delta_0 \delta/\varepsilon} + \varepsilon < 2\varepsilon$, so we have

$$|\nabla_x \phi(\xi')| + |\phi(\xi')| \leq C\varepsilon \sum_j e^{-\delta_0 |x - \xi'_j|}$$

Now we get

$$J(v(\xi')) = \sum_j V(\varepsilon_j)^{p+1/p-1-N/2} c_{p,N} + O(\varepsilon)$$

$\xi' = 1/\varepsilon(\xi_1, \dots, \xi_k)$ implies for several minimal on the Λ_j we have the result desired. □

Result at one non-degenerate critical point: if ξ_0 is a non-degenerate critical point of V ($V'(\xi_0) = 0$ and $V''(\xi_0)$ invertible), then there exist a solution $u_\varepsilon(x)$ such that

$$u_\varepsilon(x) \approx W_{V(\xi_\varepsilon)}(x - \xi_\varepsilon)/\varepsilon, \quad \xi_\varepsilon \to \xi_0.$$

For small δ we have that $J(v)$ has degree different from 0 in a ball centered at x_0 and of radius δ.

4. Back to Allen Cahn in $\mathbb{R}^2$

We consider the functional

$$J(u) = \int_{\mathbb{R}^2} \left(\varepsilon^2 \frac{|\nabla u|^2}{2} + \frac{(1-u^2)^2}{4} \right) a(x) dx.$$

Critical points of J are solutions of

$$\varepsilon^2 \mathrm{div}(a(x)\nabla u) + a(x)(1-u^2)u = 0,$$

where we suppose $0 < \alpha \leq a(x) \leq \beta$. This equation is equal to

$$\varepsilon^2 \Delta u + \varepsilon^2 \frac{\nabla a}{a}(x)\nabla u + (1-u^2)u = 0. \tag{4.1}$$

Using the change of variables $v(x) = u(\varepsilon x)$, we find the equation

$$\Delta v + \varepsilon \frac{\nabla a}{a}(x)\nabla v + (1-v^2)v = 0. \tag{4.2}$$

We will study the problem: Given a curve Γ in $\mathbb{R}^2$ we want to find a solution $u_\varepsilon(x)$ to (4.1) such that $u_\varepsilon(x) \approx w(\frac{z}{\varepsilon})$, for points $x = y + z\nu(y)$, $y \in \Gamma$, $|z| < \delta$, where $\nu(y)$ is a vector perpendicular to the curve and $w(t) = \tanh(\frac{t}{\sqrt{2}})$, which solves the problem

$$w'' + (1-w^2)w = 0, \quad w(\pm\infty) = \pm 1.$$

First issue: Laplacian near Γ, which we will consider as smooth as we need.

Assume: Γ is parametrized by arc-length

$$\gamma : [0,l] \to \mathbb{R}^2, \ s \to \gamma(s), \ |\dot\gamma(s)| = 1, l = |\Gamma|.$$

Convention: $\nu(s)$ inner unit normal at $\gamma(s)$. We have that $|\nu(s)|^2 = 1$, which implies that $2\nu\dot\nu = 0$, so we take $\dot\nu(s) = -k(s)\dot\gamma(s)$, where $k(s)$ is the curvature.

Coordinates: $x(s,t) = \gamma(s) + z\nu(s)$, $s \in (0,l)$ and $|z| < \delta$. If we take a compact supported function $\psi(x)$ near Γ, and we call $\tilde\psi(s,z) = \psi(\gamma(s) + z\nu(s))$, then $\frac{\partial\tilde\psi}{\partial s} = \nabla\psi \cdot [\dot\gamma + z\dot\nu] = (1-kz)\nabla\psi \cdot \dot\gamma$ and $\frac{\partial\tilde\psi}{\partial t} = \nabla\psi \cdot \nu$. Observe that $\nabla\psi = (\nabla\psi \cdot \dot\gamma)\dot\gamma(\nabla \cdot \nu)\nu$. This means that $\nabla\psi = \frac{1}{1-kz}\frac{\partial\tilde\psi}{\partial s}\dot\gamma + \frac{\partial\tilde\psi}{\partial z}\nu$, and $|\nabla\psi|^2 = \frac{1}{(1-kz)^2}|\tilde\psi_s|^2 + |\tilde\psi_z|^2$. Then

$$\int_{\mathbb{R}^2} |\nabla\psi(x)|^2 dx = \iint \left(\frac{1}{(1-kz)^2}|\tilde\psi_s|^2 + |\tilde\psi_z|^2 \right)(1-kz)dsdz$$

$\psi \to \psi + t\varphi$ and differentiating at $t = 0$ we get

$$\int \nabla\psi\nabla\varphi dx = \iint \frac{1}{(1-kz)}\tilde\psi_s\tilde\varphi_s + \tilde\psi_z\tilde\varphi_z(1-kz)dsdz$$

So

$$-\int \Delta\psi\varphi dx = -\iint \frac{1}{(1-kz)}\left(\left(\frac{1}{(1-kz)}\tilde{\psi}_s\right)_s + (\tilde{\psi}_z(1-lz))_z\right)\tilde{\varphi}(1-kz)dsdz$$

then

$$\Delta\tilde{\psi} = \frac{1}{(1-kz)}\frac{\partial}{\partial s}(\frac{1}{1-kz}\tilde{\psi}_s) + \tilde{\psi}_{zz} - \frac{k}{1-kz}\tilde{\psi}_z$$

We just say

$$\Delta\tilde{\psi} = \frac{1}{1-kz}(\frac{1}{1-kz}\psi_s)_s + \psi_{zz} - \frac{k}{1-kz}\psi_z$$

Near Γ ($x = \gamma(s) + z\nu(s)$), we have the new equation for $u \to \tilde{u}(s,z)$

$$S[u] = \varepsilon^2\frac{1}{1-kz}(\frac{1}{1-kz}u_s)_s + \varepsilon^2 u_{zz} + (1-u^2)u - \frac{\varepsilon^2 k}{1-kz}u_z + \frac{\varepsilon^2}{1-kz}\frac{a_s}{a}u_s + \frac{\varepsilon^2}{1-kz}\frac{a_z}{a}u_z = 0$$

we want a solution $u(s,z) \approx w(\frac{z}{\varepsilon})$.

$$S[w(\frac{z}{\varepsilon})] = \varepsilon[\frac{a_z}{a} - \frac{k(s)}{1-k(s)z}]w'(\frac{z}{\varepsilon})$$

The condition we ask (geodesic condition) is $\frac{a_z}{a}(s,0) = k(s)$. In v language we want

$$\Delta v + \varepsilon\frac{\nabla a}{a}(\varepsilon x)\cdot\nabla v + f(v) = 0$$

transition on $\Gamma_\varepsilon = \frac{1}{\varepsilon}\Gamma$. we use coordinates relative to Γ_ε rather than Γ

$$X_\varepsilon(s,z) = \frac{1}{\varepsilon}\gamma(\varepsilon s) + z\nu(\varepsilon s), \quad |z| < \delta/\varepsilon$$

Laplacian for coordinates relative to Γ_ε are

$$\Delta\psi = \frac{1}{(1-\varepsilon k(\varepsilon s)z)}\left(\frac{1}{(1-\varepsilon k(\varepsilon s)z)}v_s\right)_s + \psi_{zz} - \frac{\varepsilon k(\varepsilon s)}{(1-\varepsilon k(\varepsilon s)z)} + \varepsilon\frac{a_s}{a}\frac{1}{(1-\varepsilon k(\varepsilon s)z)^2}v_s + \varepsilon\frac{a_z}{a}v_z +$$

where we use the computation $\frac{\partial\gamma(\varepsilon s)}{\partial s} = -k(\varepsilon)\dot{\gamma_\varepsilon}(s)$, where $k_\varepsilon = \varepsilon k(\varepsilon s)$

Hereafter we use $\tilde{s}$ instead of s and $\tilde{z}$ instead of $\tilde{z}$. Observation: The operator is closed to the Laplacian on $(\tilde{s},\tilde{z})$ variables, at least on the curve Γ, if we assume the validity of the relation

$$a_{\tilde{z}}(\tilde{s},0) = k(\tilde{s})a(\tilde{s},0), \quad \forall\tilde{s} \in (0,l).$$

We can write this relation also like $\partial_\nu a = ka$ on Γ (Geodesic condition). This relation means that Γ is a critical point of curve length weighted by a. Let $L_a[\Gamma] = \int_\Gamma a dl$. Consider a normal perturbation of Γ, say $\Gamma_h := \{\gamma(\tilde{s}) + h(\tilde{s})\nu(\tilde{s}) | \tilde{s} \in (0,l)\}$, $\|h\|_{C^2(\Gamma)} \ll 1$. We want: first variation along this type of perturbation be equal to zero. This is

$$DL_a[\Gamma_h]|_{h=0} = 0$$

This means

$$\frac{\partial}{\partial \lambda} L[\Gamma_{\lambda h}]|_{h=0} = 0$$

or just $\langle DL(\Gamma), h\rangle = 0$ for all h. Observe that

$$L(\Gamma_{\lambda h}) = \int_0^l a(\gamma(\tilde{s}) + h(\tilde{s})\nu(\tilde{s})) \cdot |\dot{\gamma}(\tilde{s})_{\lambda h}| d\tilde{s}$$

and also $\dot{\gamma}_{\lambda h}(\tilde{s}) = \dot{\gamma}(\tilde{s}) + \lambda \dot{h}\nu + \lambda h\dot{\nu}$, and $\dot{\nu} = -k\dot{\gamma}$. With the taylor expansion

$$(1-2k\lambda h+\lambda^2k^2h^2+\lambda^2\dot{h}^2)^{1/2} = 1+\frac{1}{2}(-2k\lambda h+\lambda^2k^2h^2+\lambda^2\dot{h}^2)-\frac{1}{8}4k^2\lambda^2h^2+O(\lambda^2h^3)$$

and

$$a(\gamma(\tilde{(s)})+\lambda h(\tilde{s}\nu(\tilde{s})) = a(\tilde{s},\lambda h(\tilde{s})) = a(\tilde{s},0)+\lambda a_{\tilde{z}}(\tilde{s},0)h(\tilde{s})+\frac{1}{2}\lambda^2a_{\tilde{z}\tilde{z}}(\tilde{s},0)h(\tilde{s})^2+O(\lambda^3h^3).$$

we conclude

$$L_h[\Gamma_{\lambda h}] = L_a(\Gamma) = \lambda\int_0^l(-ka+a_{\tilde{z}})(\tilde{s},0)h(\tilde{s})d\tilde{s}+\lambda^2\int_0^l(a\frac{\dot{h}^2}{2}+a_{\tilde{z}}k^2h^2+\frac{1}{2}a_{\tilde{z}\tilde{z}}h^2)+O(\lambda^3h^3)$$

This tells us:

$$\frac{\partial}{\partial\lambda}L_h[\Gamma_{\lambda h}]|_{\lambda=0} = 0 \Leftrightarrow k(\tilde{s})a(\tilde{s},0) = a_{\tilde{z}}(\tilde{s},0),$$

the geodesic condition. Also we conclude that

$$\frac{\partial^2}{\partial\lambda^2}L(\Gamma_{\lambda h})|_{\lambda=0} = \int_0^l(a\dot{h}^2-2k^2a+a_{\tilde{z}\tilde{z}}h^2)d\tilde{s} = -\int_0^l(a(\tilde{s},0)\dot{h}\tilde{s})'h+(2a(\tilde{s},0)k^2-a_{\tilde{z}\tilde{z}}(\tilde{s},0)h)h$$

This can be expressed as $D^2L(\Gamma) = J_a$, which means $D^2L(\Gamma)[h]^2 = -\int_0^l J_a[h]h$. $J_a[h]$ is called the Jacobi operator of the geodesic Γ. Assumption: J_a is invertible.

We assume that if $h(\tilde{s})$, $\tilde{s} \in (0,l)$ is such that $h(0) = h(l)$, $\dot{h}(0) = \dot{h}(l)$ and $J_a[h] = 0$ then $h \equiv 0$. $Ker(J_a) = \{0\}$, in the space of $l-$periodic C^2 functions. This implies (exercise) that the problem

$$J_a[h] = g, g \in C(0,l), g(0) = g(l), h(0) = h(l), \dot{h}(0) = \dot{h}(l)$$

has a unique solution ϕ. Moreover $\|\phi\|_{C^{2,\alpha}(0,l)} \leq C\|g\|_{C^\alpha(0,l)}$.

Remember that the equation in coordinates (s,z) is

$$E(v) = \frac{1}{(1-\varepsilon k(\varepsilon s)z)}\left(\frac{1}{(1-\varepsilon k(\varepsilon s)z)}v_s\right)_s + v_{zz} - \frac{\varepsilon k(\varepsilon s)}{(1-\varepsilon k(\varepsilon s)z)}v_z +$$

$$\varepsilon\frac{a_{\tilde{s}}}{a}\frac{1}{(1-\varepsilon k(\varepsilon s)z)^2}v_s + \varepsilon\frac{a_{\tilde{z}}}{a}v_z + f(v) = 0$$

Change of variables: Fix a function $h \in C^{2,\alpha}(0,l)$ with $\|h\| \le 1$ and do the change of variables $z - h(\varepsilon s) = t$ and take as first approximation $v_0 \equiv w(t)$. Let us see that $v_0(s,z) = w(z - h(\varepsilon s))$ so

$$E(v_0) = \frac{1}{1-\varepsilon kz}\Big(\frac{1}{1-\varepsilon kz} w'(-\dot h(\varepsilon s, \varepsilon z))_s + w'' + f(w)$$

$$+\varepsilon\Big(\frac{a_{\tilde z}}{a}(\varepsilon s, \varepsilon z) - \frac{k(\varepsilon s)}{1-k(\varepsilon s)\varepsilon z}\Big)w' - \varepsilon \dot h \frac{\varepsilon}{(1-\varepsilon kz)^2}\frac{a_{\tilde s}}{a} w'$$

Error in terms of coordinates (s,t) $z = t + h(\varepsilon s)$:

$$E(v_0)(s,t) = \varepsilon w'(t)\left[\frac{a_{\tilde z}}{a}(\varepsilon s, \varepsilon(t+h)) - \frac{k(\varepsilon s)}{1-k(\varepsilon s)(t+h)\varepsilon}\right] - \frac{\varepsilon^2 w'}{(1-k\varepsilon(t+h))^2}h''$$

$$+\frac{1}{(1-k\varepsilon(t+h))^2}w''\dot h^2\varepsilon^2 - \frac{1}{(1-\varepsilon k(t+h))^3}\varepsilon^2 \dot k(t+h)\dot h w'(t) - \varepsilon \dot h \frac{\varepsilon}{(1-\varepsilon kz)^2}\frac{a_{\tilde s}}{a}w'$$

In fact

$$|E(v_0)(t,s)| \le C\varepsilon^2 e^{-\sigma|t|}$$

$\sigma < 1$, and

$$\|e^{\sigma|t|}E(v_0)\|_{C^{0,\alpha}(|t|<\frac{\delta}{\varepsilon})} \le C\varepsilon^2$$

Formal computation: We would like $\int_{-\delta/\varepsilon}^{\delta/\varepsilon} E(v_0)(s,y)w'(t)dt \approx 0$. Observe that

$$-\varepsilon^2 h''(\varepsilon s)\int_{|t|<\delta/\varepsilon}\frac{w'^2}{(1-k\varepsilon(t+h))} = -\varepsilon^2 h''\int_{\mathbb{R}} w'^2 dt + O(\varepsilon^3)$$

Also

$$\dot h^2\varepsilon^2\int \frac{1}{1-\varepsilon k(t+h)}w''w'dt = 0 + O(\varepsilon^3).$$

$$\varepsilon^2\dot h\int \frac{a_s}{a}(\varepsilon s, \varepsilon(t+h))w'^2/(1+k\varepsilon(t+h))^2 = \varepsilon^2 \dot h\frac{a_{\tilde s}}{a}(\varepsilon s, 0)\int w'^2 + O(\varepsilon^3)$$

and finally

$$\varepsilon\int_{|t|<\delta/\varepsilon} w'^2\Big(\frac{a_{\tilde z}}{a}(\varepsilon s, \varepsilon(t+h)) - \frac{k(\varepsilon s)}{1-k(\varepsilon s)(t+h)\varepsilon}\Big) = \varepsilon^2\int_{\mathbb{R}} w'(t)^2(\varepsilon^2)\Big(\Big(\frac{a_{\tilde z}}{a}\Big)(\varepsilon s, 0) - k^2\Big)h(\varepsilon s) + O(\varepsilon$$

Then

$$\frac{-\int Ew'dt}{\varepsilon^2\int w'^2} = h'' + h'\frac{a_{\tilde s}}{a} - \Big(\Big(\frac{a_{\tilde z}}{a}\Big)_{\tilde z}(\varepsilon s, 0) - k^2\Big)h + O(\varepsilon)$$

we call $\tilde s = \varepsilon s$, and we conclude that the right hand side of the above equality is equal to

$$\frac{1}{a(\tilde s, 0)}\big((a(\tilde s,0))h'(\tilde s)' + (2k^2 a(\tilde s, 0) - a_{\tilde z\tilde z}(\tilde s, 0))h\big) + O(\varepsilon)$$

and this is equal to

$$\frac{1}{a(\tilde{s},0)}(J_a[h]+O(\varepsilon))$$

We need the equation for $v(s,z)=\tilde{v}(s,z-h(\varepsilon s))$. We have

$$\frac{\partial v}{\partial s}=\frac{\partial \tilde{v}}{\partial s}-\frac{\partial \tilde{v}}{\partial t}\dot{h}\varepsilon$$

We write $z=t+h$, so we have

$$S(\tilde{v})=\frac{1}{(1-\varepsilon kz)}(\frac{\partial}{\partial s}-\varepsilon\dot{h}\frac{\partial}{\partial t})[\frac{1}{1-\varepsilon k(t+h)}(\frac{\partial}{\partial s}-\varepsilon\dot{h}\frac{\partial}{\partial t})]\tilde{v}+\tilde{v}_{tt}$$

$$\varepsilon[-\frac{k}{1-\varepsilon kz}+\frac{a_{\tilde{z}}}{a}]\tilde{v}_t+\varepsilon\frac{a_{\tilde{s}}}{a}\frac{1}{1-k\varepsilon z}^2[\tilde{v}_s-\varepsilon\dot{h}\tilde{v}_t]+f(\tilde{v})=0$$

The first term of this equation is equal to

$$\frac{1}{1-\varepsilon kz}\{\frac{\varepsilon(\varepsilon\dot{k}(t+h)+\varepsilon k\dot{h})}{(1-\varepsilon k(t+h))^2}(\tilde{v}_s-\varepsilon\dot{h}v_t)+\frac{1}{1-k\varepsilon(t+h)}(-\varepsilon^2h''v_t-2\varepsilon\dot{h}\tilde{v}_ts)+\frac{1}{1+\varepsilon k(t+h)}\tilde{v}_ss\}$$

$$-\varepsilon\dot{h}\{\frac{\varepsilon k}{(1-\varepsilon k(t+h))^2}(\tilde{v}_s-\varepsilon\dot{h}\tilde{v}_t)+\frac{1}{1-\varepsilon k(t+h)}(-\varepsilon\dot{h}\tilde{v}_{tt})\}+f(\tilde{v})=0$$

Let us observe that for $|t|<\delta/\varepsilon$, $\delta\ll 1$

$$S[\tilde{v}](s,t)=\tilde{v}_{ss}+\tilde{v}_{tt}+O(\varepsilon)\partial_{ts}\tilde{v}+O(\tilde{\varepsilon})\partial_{tt}\tilde{v}+O(\varepsilon k(|t|+1))\partial_{ss}\tilde{v}+O(\varepsilon)\partial_t\tilde{v}+O(\varepsilon)\partial_s\tilde{v}+f(v)=0$$

We will call the operator that appears in the equation $B[\tilde{v}]$. We look for a solution of the form $\tilde{v}(s,t)=w(t)+\phi(s,t)$. The equation for ϕ is

$$\phi_{ss}+\phi_{tt}+f'(w(t))\phi+E+B(\phi)+N(\phi)=0,\quad |t|<\delta/\varepsilon$$

where $E=S(w(t))=O(\varepsilon^2e^{-\sigma t})$, $N(\phi)=f(w+\phi)-f(w)-f'(w)\phi$, $s\in(0,l/\varepsilon)$. We use the notation $L(\phi)=\phi_{ss}+\phi_{tt}+f'(w(t))\phi$. We also need the boundary condition $\phi(0,t)=\phi(l/\varepsilon,t)$ and $\phi_s(0,t)=\phi_s(l/\varepsilon,t)$.

It is natural to study the linear operator in $\mathbb{R}^2$ and the linear projected problem

$$\phi_{ss}+\phi_{tt}+f'(w(t))\phi+g(t,s)=c(s)w'(t)$$

where $c(s)=\frac{\int_{\mathbb{R}}g(t,s)w'(t)dt}{\int_{\mathbb{R}}w'(t)^2dt}$ and under the orthogonally condition

$$\int_{-\infty}^{\infty}\phi(s,t)w'(t)dt=0,\quad \forall s\in\mathbb{R}$$

Basic ingredient: (Even more general) Consider the problem in $\mathbb{R}^m\times\mathbb{R}$, with variables (y,t):

$$\Delta_y\phi+\phi_{tt}+f'(w(t))\phi=0,\quad \phi\in L^\infty(\mathbb{R}^m\times\mathbb{R})$$

If ϕ is a solution of the above problem, then $\phi(y,t)=\alpha w'(t)$ some $\alpha\in\mathbb{R}$. Ingredient: $\exists\gamma>0:\quad \int_{\mathbb{R}}p'(t)^2-f'(w(t))p(t)^2\geq\gamma\int_{\mathbb{R}}p^2(t)dt$

for all $p \in H^1$ with $\int_{\mathbb{R}} pw' = 0$. $\psi(y) = \int_{\mathbb{R}} \phi^2(y,t)dt$. This is well defined (as we will see) Indeed: It turns out that $|\phi(y,t)| \le Ce^{-\sigma t}$, $\sigma < \sqrt{2}$, thanks to the fact that $\phi \in L^\infty$. We use $x = (y,t)$ and we obtain

$$\Delta_x \phi - (2 - 3(1 - w(t)^2))\phi = 0$$

Observe that $1 - w(t)^2$ is small if $|t| \gg 1$. Fix $0 < \sigma < \sqrt{2}$, for $|t| > R_0$ we have $2 - 3(1 - w^2(t)) > \sigma^2$. Let

$$\bar{\phi}_\rho(y,t) = \rho \sum_{i=1}^{n} \cosh(\sigma y_i) + \rho \cosh(\sigma t) + \|\phi\|_\infty e^{\sigma R_0} e^{-\sigma |t|}.$$

We have that

$$\phi(y,t) \le \bar{\phi}_\rho(y,t), \quad \text{for } |t| = R_0$$

also true that for $|t| + |y| > R_\rho \gg 1$, $\phi(y,t) \le \bar{\phi}_\rho$.

$$-\Delta_x \phi + (2 - 3(1 - w(t)^2))\bar{\phi} = (2 - \sigma^2 - 3(1 - w(t)^2)\bar{\phi}_\rho) > 0$$

for $|t| > R_0$. So is a supersolution of the operator

$$-\Delta_x \phi + (2 - 3(1 - w(t)^2))\phi$$

in D_ρ, which implies that $\phi \le \bar{\phi}_\rho$ for $|t| > R_0$. This implies that $|\phi(x)| \le C\bar{\phi}_\rho$ for all x, and we conclude the assertion taking $\rho \to 0$. If ϕ solves $-\Delta\phi + (1 - 3w^2)\phi = 0$, then $\|\phi\|_{C^{2,\alpha}}(B_1(x_0)) \le C\|\phi\|_{L^\infty(B_2(x_0))}$. This implies that also

$$|\phi_y| + |\phi_{yy}| \le Ce^{-\sigma t}.$$

Let $\tilde{\phi(y,t)} = \phi(y,t) - \frac{\int \phi(y,\tau)w'(\tau)d\tau}{\int w'^2} w'$. We call $\beta(y) = \frac{\int \phi(y,\tau)w'(\tau)d\tau}{\int w'^2}$

$$\Delta\tilde{\phi} + f'(w)\tilde{\phi} = \Delta\phi + f'(w)\phi + (\Delta_y \beta)w' + \beta(\Delta w' + f'(w))w' = 0$$

because $\Delta_y \beta = 0$ by integration by parts. Let $\psi(y) = \int_{\mathbb{R}} \tilde{\phi}^2 dt$.

$$\Delta_y \psi = \int_{\mathbb{R}} \nabla_y (2\tilde{\phi}\nabla_y \tilde{\phi})dt = 2\int |\nabla_y \tilde{\phi}|^2 dt + 2\int \tilde{\phi}\Delta_y \tilde{\phi} = 2\int |\nabla_y \tilde{\phi}|^2 - 2\int \tilde{\phi}[\tilde{\phi}_{tt} + f'(w)\tilde{\phi}]dt$$

Using $2\int |\nabla_y \tilde{\phi}|^2 dt + 2\int(\tilde{\phi}_t^2 - f'(w)\tilde{\phi}^2)$ This implies that $\Delta\psi \ge 2\gamma\psi$ which implies $-\Delta\psi + 2\gamma\psi \le 0$, $0 \le \psi \le c$.

We obtain that $\psi \equiv 0$ and this implies $\tilde{\phi} = 0$. This implies that $\phi(t) = (\int \phi w')w' = \beta(y)w'$ and $\Delta\beta = 0$, $\beta \in L^\infty$. Liouville implies that $\beta = constant$ so $\phi = constant w'$.

Lemma: L^∞ a priori estimates for the linear projected problem: $\exists C : \|\phi\|_\infty \le C\|g\|_\infty$.

Proof: If not exists $\|g_n\|_\infty \to 0$ and $\|\phi_n\|_\infty = 1$.

$$L[\phi_n] = -g_n + c_n(t)w'(t) = h_n(t)$$

and $h_n \to 0$ in L^∞. $\|\phi_n\| = 1$ which implies that $\exists (y_n, t_n)$: $|\phi(y_n, t_n)| \geq \gamma > 0$. Assume that $|t_n| \leq C$ and define $\tilde{\phi}(y,t) = \phi_n(y_n + y, t)$. Then

$$\Delta \tilde{\phi}_n + f'(w(t))\tilde{\phi}_n = \tilde{h}_n$$

but $f'(w(t))\tilde{\phi}_n$ is uniformly bounded and the right hand side goes to 0. This implies that $\|\phi\|_{C^1(\mathbb{R}^{m+|})} \leq C$ This implies that $\tilde{\phi}_n \to \tilde{\phi}$ passing to subsequence, and the convergence is uniformly on compacts, where $\Delta\tilde{\phi} + f'(w)\tilde{\phi} = 0$, $\tilde{\phi} \in L^\infty$. We conclude after a classic argument that $\tilde{\phi} = 0$. We have also that $\|e^{\sigma|t|}\phi\|_\infty \leq C\|e^{\sigma|t|}g\|_\infty$, $0 < \sigma < \sqrt{2}$. Elliptic regularity implies that $\|e^{\sigma|t|}\phi\|_{C^{2,\sigma}} \leq \|e^{\sigma|t|}g\|_{C^{0,\sigma}}$.

Existence: Assume g has compact support and take the weak formulation: Find $\phi \in H$ such that $\int_{\mathbb{R}^{m+1}} \nabla\phi\nabla\psi - f'(w)\phi\psi = \int gy$, for all $\psi \in H$, where $H = \{f \in H^1(\mathbb{R}^{m+1}) | \int_{\mathbb{R}} \psi w' dt = 0, \forall y \in \mathbb{R}^m\}$. Let us see that $a(\psi,\psi) = \int |\nabla\psi|^2 - f'(w)\psi^2 \geq \gamma \int \psi^2 + \psi^2$. So $a(\psi,\psi) \geq C\|\psi\|^2_{H^1(\mathbb{R}^{m+1})}$ This implies the unique existence solution. Observe that

$$\int (\Delta\phi + f'(w)\phi + g)\psi = 0$$

for all $\psi \in H$. Let $\psi \in H^1$ and $\psi = \tilde{\psi} - \frac{\int \tilde{\psi} w' dt}{\int w'^2} w' = \Pi(\tilde{\psi})$. We have that

$$\int dy \int g\Pi(\tilde{\psi})dt = \int \Pi(g)\psi$$

which implies that $\Pi(\Delta\phi + f'(w)\phi + g) = 0$ if and only if $\Delta\phi + f'(w) + \phi + g = \frac{\int(\Delta\phi + f'(w) + g)}{\int w'^2} w'$ Regularity implies that $\phi \in L^\infty$ and $\|\phi\|_\infty \leq C\|g\|_\infty$. Approximating $g \in L^\infty$ by $g_R \in C_c^\infty(\mathbb{R}^N)$ locally over compacts. This implies existence result.

We can bound ϕ in other norms. For example if $0 < \sigma < \sqrt{2}$, then

$$\|e^{\sigma|t|}\phi\|_\infty \leq C\|e^{\sigma|t|}g\|_\infty.$$

Indeed, $f'(w) < -\sigma^2 - \eta$ if $|t| > R$, with $\eta = (2 - \sigma^2)/2$. We set

$$\bar{\phi} = Me^{-\sigma|t|} + \rho \sum_{i=1}^{n} \cosh(\sigma y_i) + \rho\cosh(\sigma t).$$

Therefore

$$-\Delta\bar{\phi} + (-f'(w))\bar{\phi} \geq -\delta\bar{\phi} + (\sigma^2 + \eta)\bar{\phi} = \eta\bar{\phi} > \tilde{g} = -g + c(y)w'(t)$$

if $M \geq \frac{A}{\eta}\|e^{\sigma|t|}g\|_\infty$. In addition we have $\bar{\phi} \geq \phi$ on $|t| = R$ if $M \geq \|\phi\|_\infty e^{\sigma R}$. By an standard argument based on maximum principle, we conclude that $\phi \leq \bar{\phi}$. This means, letting $\rho \to 0$, $\phi \leq Me^{-\sigma|t|}$, where $M \geq C\max\{\|\phi\|_\infty, \|ge^{\sigma|t|}\|_\infty\}$. Since $\|\phi\|_\infty \leq C\|g\|_\infty \leq C\|ge^{\sigma|t|}\|_\infty$, we can take $M = C\|ge^{\sigma|t|}\|_\infty$. Finally, we conclude $\|\phi e^{\sigma|t|}\|_\infty \leq \|ge^{\sigma|t|}\|_\infty$.

Reminder: If $\Delta\phi = p$ implies that

$$\|\nabla\phi\|_{L^\infty(B_1(0))} \le C[\|\phi\|_{L^\infty B_2(0)} + \|p\|_{L^\infty(B_1(0))}].$$

Remember that

$$\|p\|_{C^{0,\alpha}(A)} = \|p\|_\infty + [\phi]_{0,\alpha,A}$$

where $[\phi]_{0,\alpha,A} = \sup_{x_1,x_2\in A, x_1\ne x_2} \frac{|p(x_1)-p(x_2)|}{|x_1-x_2|^\alpha}$. Also we have the following interior Schauder estimate: for $0<\alpha<1$

$$\|\phi\|_{C^{2,\sigma}(B_1)} \le C[\|\phi\|_{L^\infty(B_2(0))} + \|p\|_{C^{0,\alpha}(B_2(0))}].$$

Conclusion: If ϕ solves the equation in $\mathbb{R}^{n+1}$ then

$$\|\phi\|_{C^{2,\alpha}(\mathbb{R}^{n+1})} \le C\|g\|_{C^{0,\alpha}(\mathbb{R}^{n+1})}.$$

Sketch of the proof of this fact: Fix $x_0 \in \mathbb{R}^{n+1}$, then

$$C[\phi]_{0,\alpha,B_1(x_0)} \le \|\nabla\phi\|_{L^\infty(B_1(x_0))} \le C[\|\phi\|_\infty + \|g\|_\infty] \le C\|g\|_\infty$$

This implies that $\|\phi\|_{C^{0,\alpha}(B_1(x_0))} \le C\|g\|_\infty$, which implies $\|\phi\|_{C^{0,\alpha}(\mathbb{R}^n)} \le C\|g\|_\infty$. Clearly $\|p\|_{C^{0,\alpha}(B_2(x_0))} \le C\|g\|_\infty$, so $\|\phi\|_{C^{0,\alpha}(B_1(x_0))} \le C\|g\|_{C^{0,\alpha}(\mathbb{R}^{n+1})}$, from where we deduce the estimate.

We also get

$$\|e^{\sigma|t|}\phi\|_{C^{2,\alpha}(\mathbb{R}^{n+1})} \le C\|e^{\sigma|t|}g\|_{C^{0,\alpha}(\mathbb{R}^{n+1})}.$$

The proof of this fact is very similar to the previous one (use that $g \le e^{-\sigma|t_0|\|ge^{\sigma|t|}\|}$, for $|t_0| \gg 1$).

Another result is the following

$$\|(1+|y|^2)^{\mu/2}\phi\|_\infty \le C\|(1+|y|^2)^{\mu/2}g\|_\infty$$

In order to prove this result we define $\rho(y) = (1+|y|^\mu)$ and we consider $\tilde\phi = \rho(\delta y)\phi$. Observe that

$$\Delta\phi = \rho^{-1}\Delta\tilde\phi - 2\delta\nabla\tilde\phi\nabla(\rho^{-1}(\delta y)) + \tilde\phi\delta^2\Delta(\rho^{-1})(\delta y) = f'(w)\phi + g - cw'$$

We get $L[\tilde\phi] + O(\delta^2)\tilde\phi + O(\delta)\nabla\tilde\phi = \rho(g - cw')$. We get

$$\|\nabla\tilde\phi\|_\infty + \|\tilde\phi\|_\infty \le C[\delta^2\|\tilde\phi\|_\infty + \delta\|\nabla\tilde\phi\|_\infty + \|\rho g\|_\infty].$$

If δ is small we conclude that

$$\|\tilde\phi\|_\infty + \|\nabla\tilde\phi\|_\infty \le C\|\rho g\|_\infty$$

and we obtain

$$\|\rho\phi\|C \le \|\rho g\|.$$

Our setting:

$$\varepsilon^2[\delta u + \frac{\nabla a}{a}\cdot\nabla u] + f(u) = 0 \tag{4.3}$$

We want a solution to (4.3) $u_\varepsilon(x) \approx W(z/\varepsilon)$. Writing $x = y + z\gamma(y)$, $|z| < \delta$, we have

$$\Delta v + \nabla a(\varepsilon x)/a \cdot \nabla v + f(v) = 0,$$

in $\Gamma_\varepsilon = \frac{1}{\varepsilon}\Gamma$: $x = y + z\nu(\varepsilon y)$, which means $x = \frac{1}{\varepsilon}\gamma(\varepsilon s) + z\nu(\varepsilon s)$. Remember that $|\dot\gamma(\tilde s)| = 1$ which implies $\dot\nu(\tilde s) = -k(\tilde s)\dot\gamma(\tilde s)$. We also set $z = h(\varepsilon s) + t$. $x = \frac{1}{\varepsilon}\gamma(\varepsilon s) + (t + h(\varepsilon s))\nu(\varepsilon s)$. We assume $\|h\|_{\alpha,(0,l)} \leq 1$, for $0 < \alpha < 1$. We wrote Δ_x in terms of this coordinates (t, s) and the equations $S(v) = 0$ is rewritten taking as first approximation $w(t)$. We evaluated $S(w(t))$ and got that $S(w(t)) = 0$.

From the expression of Δ_x we get $(x = \frac{1}{\varepsilon}\gamma(\varepsilon s) + (t + h(\varepsilon s))\nu(\varepsilon s))$

$$\Delta_x v = \partial_{ss} + \partial_{tt} + \varepsilon[b_1^\varepsilon(t,s)\partial_{ss} + b_2^\varepsilon\partial_{tt} + b_3^\varepsilon\partial_{st} + b_4^\varepsilon\partial_t + b_5^\varepsilon\partial_s]$$

$|\varepsilon b_i| \leq C\delta$ in the region $|t| < \delta/\varepsilon$. The coefficients are periodic (same values at $s = 0$ and $s = l/\varepsilon$). Our equation reads

$$\partial_{ss}v + \partial_{tt}v + B_\varepsilon[v] + f(v) = 0, \quad \text{for } s \in (0, l/\varepsilon), |t| < \delta/\varepsilon.$$

This expression does not make sense globally. We consider $\delta \ll 1$. We define

$$H(x) = \begin{cases} -1 & \text{in } \Omega_-^\varepsilon \\ +1 & \text{in } \Omega_+^\varepsilon \end{cases}$$

where Ω_+^ε is a bounded component of $\mathbb{R}^2 \setminus \Gamma$, and Ω_-^ε the other. For the equation

$$\Delta v + \varepsilon\frac{\nabla a}{a} \cdot \nabla v + f(v) = 0$$

we take as first (global) approximation

$$v_0(x) = w(t)\eta_3 + (1 - \eta_4)H(x)$$

where

$$\eta_l(x) = \begin{cases} \eta\left(\frac{\varepsilon|t|}{l\delta}\right) & \text{if } |t| < 2\delta l/\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Look for a solution of the form $v = v_0 + \tilde\phi$, so

$$\Delta_x\tilde\phi + \varepsilon\frac{\nabla a}{a} \cdot \nabla\tilde\phi + f'(v_o)\tilde\phi + E + N(\tilde\phi) = 0$$

where $E = S(v_0)$ and $N(\tilde\phi) = f(v_0 + \tilde\phi) - f(v_0) - f'(v_0)\tilde\phi$.

We write $\tilde\phi = \eta_3\phi + \psi$. We require that ϕ and ψ solve the system

$$\Delta_x\psi - 2\psi + (2 + f'(v_0))(1 - \eta_1)\psi + \varepsilon\frac{\nabla a}{a}\nabla\psi + (1 - \eta_1)E + (1 - \eta_1)N(\eta_3\phi + \psi) + \nabla\eta_3\nabla\phi + \nabla\eta_3\nabla\phi + \varepsilon\frac{\nabla}{}$$

$$\eta_3\left[\Delta_x\phi + f'(w(t))\phi + \eta_1(2 + f'(w(t)))\psi + \eta_1 E + \eta_1 N(\phi + \psi) + \varepsilon\frac{\nabla a}{a} \cdot \nabla\phi\right] = 0.$$

We need that the ϕ above satisfies the equation just for $|t| < 6\delta/\varepsilon$. We assume that $\phi(s,t)$ is defined for all s and t (and it is l/ε- periodic in s). We require that ϕ satisfies globally

$$\phi_{tt} + \phi_{ss} + \eta_6 B_\varepsilon[\phi] + f'(w(t))\phi + \eta_1 E + \eta_1 N(\phi + \psi) + \eta_1(2 + f'(w))\psi = 0$$

and $\phi \in L^\infty(\mathbb{R}n + 1)$ and periodic in s. Notice that $\phi_{tt} + \phi_{ss} + \eta_6 B_\varepsilon[\phi] = \Delta_x \phi$ inside the support of η_3. Rather than solving this problem directly we solve the projected problem

(4.4)

$$\phi_{tt} + \phi_{ss} + \eta_6 B_\varepsilon[\phi] + f'(w(t))\phi + \eta_1 E + \eta_1 N(\phi + \psi) + \eta_1(2 + f'(w))\psi = c(s)w'(t)$$

and $\int_{\mathbb{R}} \phi w'(t)dt = 0$. We solve (4)-(4.4) first, then we find h such that $c(s) \equiv 0$. We consider ϕ with $\|\phi\|_\infty + \|\nabla\phi\|_\infty \leq \varepsilon$. The operator $-\Delta\psi + 2\psi$ is invertible $L^\infty(\mathbb{R}^3) \to C^1(\mathbb{R}^2)$. We conclude that if $g \in L^\infty$ the exist a unique solution $\psi = T[g] \in C^1(\mathbb{R}^2)$ with $\|\phi\|_{C^1} \leq C\|g\|_\infty$ of equation $-\Delta\psi + 2\psi = g$ in $\mathbb{R}^2$. Observe that (4) is equivalent to

$$\psi = T[(2 + f'(v_0))(1 - \eta_1)\psi + \varepsilon\frac{\nabla a}{a}\nabla\psi + (1 - \eta_1)E + (1 - \eta_1)N(\eta_3\phi + \psi) + \nabla\eta_3\nabla\phi + \nabla\eta_3\nabla\phi + \varepsilon\frac{\nabla a}{a}\nabla$$

Using contraction mapping in C^1 on $\|\psi\|_{C^1} \leq C\varepsilon$, we conclude that there exist a unique solution of the this problem $\psi = \psi(\phi, h)$ such that

$$\|\psi\| \leq C[\varepsilon^2 + \varepsilon\|\phi\|_{C^1}].$$

Even more, $\|\psi(\phi_1, h) - \psi(\phi_2, h)\|_{C^1} \leq C\varepsilon\|\phi_1 - \phi_2\|_{C^1}$.