EXCITED STATES OF BOSE-EINSTEIN CONDENSATES WITH DEGENERATE ATTRACTIVE INTERACTIONS

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Abstract. We study the Bose-Einstein condensates (BEC) in two or three dimensions with attractive interactions, described by L^2 constraint Gross-Pitaevskii energy functional. First, we give the precise description of the chemical potential of the condensate μ and the attractive interaction a. Next, for a class of degenerate trapping potential with non-isolated critical points, we obtain the existence and the local uniqueness of the excited states by accurately analyzing the location of the concentrated points and the Lagrange multiplier. Our results on degenerate trapping potential with non-isolated critical points are new for ground states of BEC and singularly perturbed nonlinear Schrödinger equations.

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1. Introduction

The idea of Bose-Einstein condensates (BEC) originated in 1924-1925, when Einstein predicted that, below a critical temperature, part of the bosons would occupy the same quantum state to form a condensate. Over the last two decades, remarkable experiments on BEC in dilute gases of alkali atoms [2, 4, 14] have revealed various interesting quantum phenomena. These new experimental advances make many mathematicians study again the following of Gross-Pitaevskii (GP) equations proposed by Gross [18] and Pitaevskii [31] in the 1960s:

$$i\partial_t\psi(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t) - a|\psi(x,t)|^2\psi(x,t), \quad x \in \mathbb{R}^N, \tag{1.1}$$

with the constraint

$$\int_{\mathbb{R}^N} |\psi(x,t)|^2\,dx = 1,$$

where $N = 2,3$, $V(x) \geq 0$ is a real-valued potential and $a \in \mathbb{R}$ is treated as an arbitrary dimensionless parameter. For better understanding of the long history and further results on Bose-Einstein condensates, we refer to [11, 27–29] and the references therein.

If we want to find a solution for (1.1) of the form $\psi(x,t) = u(x)e^{-i\mu t}$, where μ represents the chemical potential of the condensate and $u(x)$ is a function independent of time, then the unknown pair (μ,u) satisfies the following nonlinear eigenvalue equation

$$-\Delta u + V(x)u = au^3 + \mu u, \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

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and the following constraint

$$\int_{\mathbb{R}^N} u^2 = 1. \tag{1.3}$$

The energy functional corresponding to (1.2) is given by

$$J(u) = \int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big) - \frac{a}{4}\int_{\mathbb{R}^N} u^4. \tag{1.4}$$

A ground state solution of (1.2) is a minimizer of the following minimization problem:

$$I_a := \inf\Big\{\frac{1}{2}\int_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)u^2\big) - \frac{a}{4}\int_{\mathbb{R}^N} u^4 :\ u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} u^2 = 1\Big\}. \tag{1.5}$$

Any eigenfunction of (1.2) whose energy is larger than that of the ground state is usually called the excited states in the physics literatures in [3].

Let us first recall the existence result for the ground state. Denote by $Q(x)$ the unique positive solution of $-\Delta u + u = u^3,\ u \in H^1(\mathbb{R}^N)$ with $N = 2, 3$. Let $a_* = \displaystyle\int_{\mathbb{R}^N} Q^2$. We have

Theorem A.(c.f.[3]) *Suppose $V(x) \geq 0$ $(x \in \mathbb{R}^N)$ satisfies*

$$\lim_{|x|\to+\infty} V(x) = +\infty.$$

If (i) $a < a_*$ and $N = 2$; or (ii) $a \leq 0$ and $N = 3$, then (1.2)–(1.3) has a ground state. On the other hand, (1.2)–(1.3) has no ground state if (i') $a \geq a_*$ and $N = 2$; or (ii') $a > 0$ and $N = 3$.

In the last few years, lots of efforts have been made to the study of asymptotic behaviors of the minimizers of (1.5) as $a \nearrow a_*$ when $N = 2$. See for example [23–25] and the references therein, where the main tools used are the energy comparison. The main results on the asymptotic behaviors of the minimizer u_a of (1.5) with $N = 2$ as $a \nearrow a_*$ are that u_a concentrates at a minimum point x_0 of $V(x)$. That is, $u_a \to 0$ uniformly in $\mathbb{R}^N \setminus B_\theta(x_0)$ for any $\theta > 0$, while $\max_{x\in B_\theta(x_0)} u_a(x) \to +\infty$. However, if $N = 3$, as $a \nearrow 0$, the minimizer u_a of (1.5) approaches to a minimizer of u_0 of I_0. Therefore, it is not obvious that (1.2)–(1.3) has solutions u_a concentrating at some points if $N = 3$.

The aim of this paper is to investigate the excited states for (1.2)–(1.3), especially those which exhibit the concentration phenomena. For this purpose, we need to consider (1.2) from different point of views as follows.

We first consider the following problem without constraint:

$$\begin{cases} -\Delta w + (\lambda + V(x))w = w^3, & \text{in } \mathbb{R}^N; \\ w \in H^1(\mathbb{R}^N), \end{cases} \tag{1.6}$$

where $\lambda > 0$ is a large parameter. It is well known that for large $\lambda > 0$, we can construct various positive solutions concentrating at some stable critical points of $V(x)$. In particular, we can construct positive k-peak solutions for (1.6) in the sense that

$$w_\lambda(x) = \sqrt{\lambda}\Big(\sum_{i=1}^k Q\big(\sqrt{\lambda}(x - x_{\lambda,i})\big) + \omega_\lambda(x)\Big), \tag{1.7}$$

with $\int_{\mathbb{R}^N}\big[\frac{1}{\lambda}|\nabla\omega_\lambda|^2+\omega_\lambda^2\big]=o\big(\lambda^{-\frac{N}{2}}\big)$. Let $u_\lambda=\frac{w_\lambda}{\left(\int_{\mathbb{R}^N}w_\lambda^2\right)^{1/2}}$. Then $\int_{\mathbb{R}^N}u_\lambda^2=1$, and

$$\begin{cases}-\Delta u_\lambda+(\lambda+V(x))u_\lambda=a_\lambda u_\lambda^3, & \text{in } \mathbb{R}^N;\\ u_\lambda\in H^1(\mathbb{R}^N),\end{cases}\tag{1.8}$$

with

$$a_\lambda=\int_{\mathbb{R}^N}w_\lambda^2=k\lambda^{1-\frac{N}{2}}\Big(\int_{\mathbb{R}^N}Q^2+o(1)\Big)=k\lambda^{1-\frac{N}{2}}\Big(a_*+o(1)\Big).$$

Note that $a_\lambda>0$, and as $\lambda\to+\infty$, $a_\lambda\to ka_*$ if $N=2$, while $a_\lambda\to 0$ if $N=3$. Therefore, we obtain a concentrated solution with k peaks for (1.2)–(1.3) with $\mu=-\lambda$ and suitable a_λ. Now the crucial question is for any a close to ka_* if $N=2$, or for any $a>0$ small if $N=3$, whether we can choose a suitable large $\lambda_a>0$, such that (1.2)–(1.3) holds with

$$\mu=-\lambda_a,\quad u_a=\frac{w_{\lambda_a}}{\big(\int_{\mathbb{R}^N}w_{\lambda_a}^2\big)^{1/2}}.\tag{1.9}$$

The above discussions show that the existence of concentrated solutions for (1.2)–(1.3) is closely related to the existence of peaked solutions for the nonlinear Schrödinger equations (1.6). In this paper, we will mainly investigate concentrated solutions u_a of (1.2)–(1.3) in the sense that

$$\max_{x\in B_\theta(b_i)}u(x)\to+\infty,\ \text{while } u_a(x)\to 0 \text{ uniformly in } \mathbb{R}^N\setminus\bigcup_{i=1}^k B_\theta(b_i),\ \text{for any } \theta>0,$$

as $a\to a_0$, where k is a positive integer and $b_1,\cdots,b_k$ are some points in $\mathbb{R}^N$. Here, we will study the following basic issues concerning the concentration of the solutions for (1.2)–(1.3):

(I) *The possible values for a_0 and μ_a, and the exact location of the concentrated points, if u_a concentrates.*

(II) *The existence of the concentrated solutions.*

(III) *The local uniqueness of the concentrated solutions.*

Our first result of this paper is the following.

Theorem 1.1. *Suppose u_a is a solution of (1.2)–(1.3) concentrated at some points as $a\to a_0$. Then it holds*

$$a_0\geq 0 \text{ and } \mu_a\to-\infty,\ \text{as } a\to a_0.$$

Moreover, if $N=2$, then $a_0=ka_*$ for some integer $k>0$, and u_a satisfies

$$u_a(x)=\sqrt{\frac{-\mu_a}{a}}\Big(\sum_{i=1}^k Q\big(\sqrt{-\mu_a}(x-x_{a,i})\big)+\omega_a(x)\Big),\tag{1.10}$$

with $\int_{\mathbb{R}^2}\big[-\frac{1}{\mu_a}|\nabla\omega_a|^2+\omega_a^2\big]=o\big(-\frac{1}{\mu_a}\big)$ and some points $x_{a,i}\in\mathbb{R}^2$, $i=1,\cdots,k$, satisfying $\sqrt{-\mu_a}|x_{a,j}-x_{a,i}|\to\infty$ as $a\to a_0$, $i\neq j$.

Throughout this paper, we call u_a a k-peak solution of (1.2)–(1.3) if u_a satisfies (1.10). Although the nonlinear Schrödinger equations (1.6) have been extensively studied in the last three decades (see for example [1, 5, 10, 16, 32] and the references therein), not much is known for the exact location of the concentrated point, nor for the local uniqueness of the solutions, if the critical points of $V(x)$ is not isolated. In the paper, we assume that $V(x)$ obtains its local minimum or local maximum at Γ_i $(i=1,\cdots,k)$ and Γ_i is a closed $N-1$ dimensional hyper-surface satisfying $\Gamma_i\bigcap\Gamma_j=\emptyset$ for $i\neq j$. More precisely, we assume that the following conditions hold.

(V). *There exist $\delta>0$ and some C^2 compact hypersurfaces Γ_i $(i=1,\cdots,k)$ without boundary, satisfying*

$$V(x)=V_i,\ \ \frac{\partial V(x)}{\partial \nu_i}=0,\ \ \frac{\partial^2 V(x)}{\partial \nu_i^2}\neq 0,\ \textit{for any } x\in\Gamma_i \textit{ and } i=1,\cdots,k,$$

where $V_i\in\mathbb{R}$, ν_i is the unit outward normal of Γ_i at $x\in\Gamma_i$. Moreover, $V\in C^4\big(\bigcup_{i=1}^k W_{\delta,i}\big)$ and $V(x)=O\big(e^{\alpha|x|}\big)$ for some $\alpha\in(0,2)$. Here we denote $W_{\delta,i}:=\{x\in\mathbb{R}^N, dist(x,\Gamma_i)<\delta\}$.

Note that condition (V) implies that $V(x)$ obtains its local minimum or local maximum on the hypersurface Γ_i for $i=1,\cdots,k$. It is also easy to see that if $\delta>0$ is small, the set $\Gamma_{t,i}=\big\{x:V(x)=t\big\}\bigcap W_{\delta,i}$ consists of two compact hypersurfaces in $\mathbb{R}^N$ without boundary for $t\in[V_i,V_i+\theta]$ (or $t\in[V_i-\theta,V_i]$) provided $\theta>0$ is small. Moreover, the outward unit normal vector $\nu_{t,i}(x)$ and the j-th principal tangential unit vector $\tau_{t,i,j}(x)$ $(j=1,\cdots,N-1)$ of $\Gamma_{t,i}$ at x are Lip-continuous in $W_{\delta,i}$.

Using the local Pohozaev identities, we can easily prove that a k-peak solution of (1.2)–(1.3) must concentrate at some critical points of $V(x)$, and we can also refer to [19]. If $V(x)$ satisfies (V) and the concentrated points belong to $\Gamma:=\bigcup_{i=1}^k\Gamma_i$, it is natural to ask where the concentrated points locate on Γ. And the following result gives the further answer of this question.

Theorem 1.2. *Under the condition (V), if u_a is a k-peak solution of (1.2)–(1.3), concentrating at $\{b_1,\cdots,b_k\}$ with $b_i\in\Gamma$ and $b_i\neq b_j$ if $i\neq j$, as $a\to ka_*$ and $N=2$, or $a\searrow 0$ and $N=3$, then*

$$(D_{\tau_{i,j}}\Delta V)(b_i)=0,\ \textit{with } i=1,\cdots,k \textit{ and } j=1,\cdots,N-1. \tag{1.11}$$

where $\tau_{i,j}$ is the j-th principal tangential unit vector of Γ at b_i.

It is proved in [23] that if $V(x)$ has finite minimal points with the same minimal value, then the the minimizers of (1.5) concentrate at the "flattest" minimal points of $V(x)$ along a subsequence a_l which approaches a_* from left as $l\to\infty$. On the other hand, for $V(x)=(|x|-1)^2$, in [24], it is proved that the minimizers of (1.5) concentrate at some point in $\{x:|x|=1\}$, while for

$$V(x)=\Big(\big(\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}\big)^{\frac12}-1\Big)^2,\ \text{with } a>b>0,$$

it is proved in [21] that the minimizers of (1.5) concentrate at either $(-a,0)$, or $(a,0)$ up to a subsequence. Note that in all those cases, the concentration point is a minimum point of the function ΔV on the relevant set. We should mention that the role of ΔV in the case of degenerate minimum points was first found in [30] for a related but similar singularly perturbed nonlinear Schrodinger equation.

Theorem 1.2 shows that not every $\{b_1,\cdots,b_k\}$ with $b_j\in\Gamma$ can generate a k-peak solution for (1.2)–(1.3). To study the converse of Theorem 1.2, we need the following non-degenerate condition on the critical point of $V(x)$. We say that $x_0\in\Gamma_i$ is non-degenerate on Γ_i if the following condition holds:

$$\frac{\partial^2 V(x_0)}{\partial\nu_i^2}\neq 0 \text{ and } det\Big(\Big(\frac{\partial^2\Delta V(x_0)}{\partial\tau_{i,l}\partial\tau_{i,j}}\Big)_{1\leq l,j\leq N-1}\Big)\neq 0.$$

Theorem 1.3. *Assume that (V) holds. If $b_i \in \Gamma$ is non-degenerate critical point of $V(x)$ on Γ for $i = 1, \cdots, k$ satisfying (1.11) and $b_i \neq b_j$ if $i \neq j$, then there exists a small constant $\theta > 0$, such that (1.2)–(1.3) has a k-peak solution u_a concentrating at $b_1, \cdots, b_k$ as $a \to ka_*$ if $N = 2$, provided*

$$(i).\ a \in [ka_* - \theta, ka_*),\ \text{if} \sum_{i=1}^{k} \Delta V(b_i) > 0,\ (ii).\ a \in (ka_*, ka_* + \theta],\ \text{if} \sum_{i=1}^{k} \Delta V(b_i) < 0,$$

or as $a \in (0, \theta]$ if $N = 3$.

The existence result in Theorem 1.3 is new even when $k = 1$ because it reveals that (1.2)–(1.3) still has single peak solutions for $a > a_*$ if $V(x)$ has a local maximum point or local maximum set. Let us point out that if $V(x)$ does not achieve its global minimum on Γ, any solution concentrating at a point on Γ is not a ground state. Also, if $k > 1$, then the solutions in Theorem 1.3 are not ground states. So Theorem 1.3 gives us the existence of excited states for BEC problem as $a \to ka_*$ if $N = 2$, or $a \searrow 0$ if $N = 3$. To our best knowledge, this is the first result concerning the existence of excited states for (1.2)–(1.3). Furthermore, we can prove that for any integer $k > 0$ and some a near ka^* in $N = 2$, or a near 0 in $N = 3$, (1.2)–(1.3) has an excited state solution which has k-peaks concentrated at one point (see Theorem 4.6 in Section 4).

Another main result of this paper is the following local uniqueness result.

Theorem 1.4. *Suppose (V) holds. Let $u_a^{(1)}(x)$ and $u_a^{(2)}(x)$ be two k-peak solutions of (1.2)–(1.3) concentrating at $b_1, \cdots, b_k$ with $b_i \in \Gamma$, and $b_i \neq b_j$ if $i \neq j$. If b_i is non-degenerate, $i = 1, \cdots, k$, $\sum_{i=1}^{k} \Delta V(b_i) \neq 0$, in $N = 2$, and*

$$\Big(\frac{\partial^2 \Delta V(b_i)}{\partial \tau_{i,l} \partial \tau_{i,j}}\Big)_{1 \leq l,j \leq N-1} + \frac{\partial \Delta V(b_i)}{\partial \nu_i} diag\big(\kappa_{i,1}, \cdots, \kappa_{i,N-1}\big),\ \text{for } i = 1, \cdots, k$$

is non-singular, where $\kappa_{i,j}$ is the j-th principal curvature of Γ at b_i for $j = 1, \cdots, N-1$, then there exists a small positive number θ, such that $u_a^{(1)}(x) \equiv u_a^{(2)}(x)$ for all a with $0 < |a - ka_*| \leq \theta$ if $N = 2$, or $0 < a \leq \theta$ if $N = 3$.

As far as we know, local uniqueness results for peak (or bubbling) solutions are available only for the case where the solutions blow up at x_0, which is an isolated critical point of the potential $V(x)$. If x_0 is a non-degenerate critical point of $V(x)$, that is, $(D^2 V)$ is non-singular at x_0, one can prove the local uniqueness of the peak solution concentrating at x_0 either by counting the local degree of the corresponding reduced finite dimensional problem as in [6, 8, 17], or by using Pohozaev type identities as in [7, 15, 19, 22, 26]. One of the advantage in using the Pohazaev identities to prove the local uniqueness is that it can deal with the degenerate case. See [7, 15, 26]. Let us pointing out that in [7, 15, 26], though the critical point x_0 is degenerate, the rate of degeneracy along each direction is the same. On the other hand, an example given in [19] shows that local uniqueness may not be true at a degenerate critical point x_0 of $V(x)$. Thus, it is a very subtle problem to discuss the local uniqueness of peak solutions concentrating at a degenerate critical point. Under the condition (V), the function $V(x)$ is non-degenerate along the normal direction ν_i of Γ_i. But along each tangential direction of Γ_i, $V(x)$ is degenerate. Such non-uniform degeneracy makes the estimates more sophisticated.

Here we point out that the existence and local uniqueness of excited states to (1.2)–(1.3) are also true for the following type of potential $V(x)$:

$$V(x) = \prod_{i=1}^{m} |x - x_i|^{p_i}, \text{ for all } x \in \mathbb{R}^N \text{ with } p_i > 0 \text{ and } N = 2, 3. \tag{1.12}$$

Also our arguments in this paper show that it is much more effective to use the Pohozaev identities to study the asymptotic behaviors for all kinds of concentrated solutions, not just for the minimizer. For example, using various Pohozaev identities, we can easily derive the relation between a and the Lagrange multiplier μ_a (see the proof of Proposition 3.5).

This paper is organized as follows. In section 2, we will prove Theorem 1.1, while in section 3, we estimate the Lagrange multiplier μ_a in terms of a. The results for the location of the peaks and for the existence of peak solutions are proved in section 4, and the local uniqueness of peak solutions are investigated in section 5.

For simplicity in using the notations, in this paper, we always assume that $b_j \in \Gamma_j$, $j = 1, \cdots, k$. The results for other cases can be proved without any changes.

2. A non-existence result and the Proof of Theorem 1.1

First, we study the following problem:

$$-\Delta u = V_1(x)u, \ u > 0, \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

where the function $V_1(x)$ satisfies $V_1 > 1$ in $B_R(0) \setminus B_t(0)$ for some fixed $t > 0$ and large $R > 0$.

Proposition 2.1. *Problem (2.1) has no solution.*

Proof. Suppose that (2.1) has a solution u. Consider the following problem

$$-\Delta v = v. \tag{2.2}$$

By a standard comparison argument, (2.2) has a radial solution $v(r)$, which has infinitely many zeros points $0 < r_1 < r_2 < \cdots < r_k < \cdots$. Denote $\Omega_k = B_{r_{k+1}}(0) \setminus B_{r_k}(0)$. Let $k_0 > 0$ be such that $t < r_{k_0}$. We now assume that $R >> r_{k_0}$. We take $k \geq k_0$, such that $v > 0$ in Ω_k, then, we have

$$\int_{\Omega_k} \Big(-v\Delta u + u\Delta v\Big) = \int_{\Omega_k} \Big(V_1(x)uv - vu\Big) > 0. \tag{2.3}$$

Noting that $v(r_k) = v(r_{k+1}) = 0$, we obtain from (2.3) that

$$\int_{\partial\Omega_k} u\frac{\partial v}{\partial \nu} > 0, \tag{2.4}$$

where ν is the outward unit normal of $\partial\Omega_k$. But on $\partial\Omega_k$, $\frac{\partial v}{\partial \nu} < 0$. Thus, we obtain a contradiction from (2.4). □

Proof of Theorem 1.1. First, we prove that $\mu_a \to -\infty$. We argue by contradiction. Suppose that $|\mu_a| \leq M$. Since $\int_{\mathbb{R}^N} u_a^2 = 1$, Moser iteration implies that u_a is uniformly bounded. That is, u_a does not blow up.

Suppose that $\mu_a \to +\infty$. We let $V_1(x) = \mu_a - V(x) + au_a^2$. Since u_a concentrates at some points, we may assume that $au_a^2 \geq -1$ in $\mathbb{R}^N \setminus B_t(0)$ for some $t > 0$. Therefore, for any fixed $R > 0$, we always have

$$V_1(x) = \mu_a - V(x) + au_a^2 > 1, \quad x \in B_R(0) \setminus B_t(0).$$

By Proposition 2.1, we obtain a contradiction. Therefore, we have proved that $\mu_a \to -\infty$. Let $\lambda_a = -\mu_a$. Let x_a be the maximum point of u_a. From the equation (1.2), we find

$$au_a^3(x_a) \geq \big(\lambda_a + V(x_a)\big)u_a(x_a) > 0.$$

This gives $a > 0$.

Suppose now $N = 2$. Let $\bar{u}_a(x) = \frac{1}{\sqrt{\lambda_a}} u_a\big(\frac{x}{\sqrt{\lambda_a}}\big)$. Then

$$-\Delta \bar{u}_a + \Big(1 + \frac{1}{\lambda_a} V\big(\frac{x}{\sqrt{\lambda_a}}\big)\Big)\bar{u}_a = a\bar{u}_a^3, \quad \text{in } \mathbb{R}^2, \tag{2.5}$$

and

$$\int_{\mathbb{R}^2} \bar{u}_a^2 = 1. \tag{2.6}$$

From (2.5) and (2.6), using Moser iteration, we can prove that $|u_a| \leq C$ for some constant independent of a. Let $\bar{x}_a$ be a maximum point of $\bar{u}_a$. Then

$$au_a^3(\bar{x}_a) \geq \Big(1 + \frac{1}{\lambda_a} V\big(\frac{\bar{x}_a}{\sqrt{\lambda_a}}\big)\Big)\bar{u}_a(\bar{x}_a),$$

which gives $a \geq u_a^{-2}(\bar{x}_a) \geq c_0 > 0$. Using the standard blow-up argument, in view of (2.6), we can prove that there exists an integer $k > 0$, such that

$$\bar{u}_a = \sum_{i=1}^{k} Q_{a_0}(x - \bar{x}_{a,i}) + \bar{\omega}_a(x), \tag{2.7}$$

for some $\bar{x}_{a,i} \in \mathbb{R}^2$ with

$$|\bar{x}_{a,i} - \bar{x}_{a,j}| \to +\infty, \text{ if } i \neq j, \int_{\mathbb{R}^2} \big[|\nabla \bar{\omega}_a|^2 + \bar{\omega}_a^2\big] = o(1),$$

and Q_{a_0} is the unique positive solution of

$$-\Delta u + u = a_0 u^3, \ \ u \in H^1(\mathbb{R}^2), \ \ u(0) = \max_{x \in \mathbb{R}^2} u(x).$$

Noting that $Q_{a_0} = \frac{1}{\sqrt{a_0}} Q$, we obtain from (2.6) and (2.7) that $a_0 = ka_*$.

□

3. Some estimates for general potentials

In this section, we shall estimate μ_a in terms of a.

Let $\varepsilon = \frac{1}{\sqrt{-\mu_a}}$ and $u(x) \mapsto \sqrt{\frac{-\mu_a}{a}} u(x)$, then (1.2) can be changed to the following problem:

$$-\varepsilon^2 \Delta u + \big(1 + \varepsilon^2 V(x)\big)u = u^3, \ u \in H^1(\mathbb{R}^N). \tag{3.1}$$

For any $a \in \mathbb{R}^+$, we define $\|u\|_a := \int_{\mathbb{R}^N} \big(\varepsilon^2 |\nabla u|^2 + u^2\big)$.

From (1.10), we find that a k-peak solution of (3.1) has the following form

$$\tilde{u}_a(x) = \sum_{i=1}^{k} Q_{\varepsilon, x_{a,i}}(x) + v_a(x), \text{ with } \|v_a\|_a = o(\varepsilon^{\frac{N}{2}}),$$

where $Q_{\varepsilon,x_{a,i}}(x) := Q\Big(\frac{\sqrt{1+\varepsilon^2 V_i}(x-x_{a,i})}{\varepsilon}\Big)$. Then, it holds

$$-\varepsilon^2\Delta v_a + \Big(1+\varepsilon^2 V(x) - 3\sum_{i=1}^k Q^2_{\varepsilon,x_{a,i}}(x)\Big)v_a = N\big(v_a\big) + l_a(x), \tag{3.2}$$

where

$$N_a\big(v_a\big) = \Big(\sum_{i=1}^k Q_{\varepsilon,x_{a,i}}(x) + v_a\Big)^3 - \Big(\sum_{i=1}^k Q_{\varepsilon,x_{a,i}}(x)\Big)^3 - 3\sum_{i=1}^k Q^2_{\varepsilon,x_{a,i}}(x)v_a, \tag{3.3}$$

and

$$l_a(x) = -\varepsilon^2\sum_{i=1}^k \big(V(x)-V_i\big)Q_{\varepsilon,x_{a,i}}(x) + \Big(\sum_{i=1}^k Q_{\varepsilon,x_{a,i}}(x)\Big)^3 - \sum_{i=1}^k Q^3_{\varepsilon,x_{a,i}}(x).$$

We can move $x_{a,i}$ a bit(still denoted by $x_{a,i}$), so that the error term $v_a \in \bigcap_{i=1}^k E_{a,x_{a,i}}$, where

$$E_{a,x_{a,i}} := \Big\{u(x)\in H^1(\mathbb{R}^N) : \Big\langle u, \frac{\partial Q_{\varepsilon,x_{a,i}}(x)}{\partial x_j}\Big\rangle_a = 0,\ j=1,\cdots,N\Big\}. \tag{3.4}$$

Let L_a be the bounded linear operator from $H^1(\mathbb{R}^N)$ to itself, defined by

$$\big\langle L_a u, v\big\rangle_a = \int_{\mathbb{R}^N}\Big(\varepsilon^2\nabla u\nabla v + \big(1+\varepsilon^2 V(x)\big)uv - 3\sum_{i=1}^k Q^2_{\varepsilon,x_{a,i}}(x)uv\Big). \tag{3.5}$$

Then, it is standard to prove the following lemma.

Lemma 3.1. *There exist constants $\rho > 0$ and small $\theta > 0$, such that for all a with $0 < |a - ka_*| \le \theta$ in $N = 2$ or $0 < a \le \theta$ in $N = 3$, it holds*

$$\|L_a u\|_a \ge \rho\|u\|_a, \text{ for all } u \in \bigcap_{i=1}^k E_{a,x_{a,i}}. \tag{3.6}$$

Lemma 3.2. *A k-peak solution $\tilde{u}_a$ for (3.1) concentrating at $b_1,\cdots,b_k$ has the following form*

$$\tilde{u}_a(x) = \sum_{i=1}^k Q_{\varepsilon,x_{a,i}}(x) + v_a(x), \tag{3.7}$$

with $v_a \in \bigcap_{i=1}^k E_{a,x_{a,i}}$ and

$$\|v_a\|_a = O\Big(\big|\sum_{i=1}^k\big(V(x_{a,i})-V_i\big)\big|\varepsilon^{\frac{N}{2}+2} + \big|\sum_{i=1}^k\nabla V(x_{a,i})\big|\varepsilon^{\frac{N}{2}+3} + \varepsilon^{\frac{N}{2}+4}\Big). \tag{3.8}$$

Proof. By standard calculations, we find

$$\|l_a\|_a = O\Big(\big|\sum_{i=1}^k\big(V(x_{a,i})-V_i\big)\big|\varepsilon^{\frac{N}{2}+2} + \big|\sum_{i=1}^k\nabla V(x_{a,i})\big|\varepsilon^{\frac{N}{2}+3} + \varepsilon^{\frac{N}{2}+4}\Big), \tag{3.9}$$

and

$$\|N\big(v_a\big)\|_a = o(1)\|v_a\|_a. \tag{3.10}$$

Then from (3.2), (3.6), (3.9) and (3.10), we get (3.7) and (3.8).

□

Let $\tilde{v}_a(x) = v_a(\varepsilon x + x_{a,i})$. Then, $\tilde{v}_a$ satisfies $\|\tilde{v}_a\|_a = O(\varepsilon^2)$. Using the Moser iteration, we can prove $\|\tilde{v}_a\|_{L^\infty(\mathbb{R}^N)} = o(1)$. From this and the comparison theorem, we can prove the following estimates for $\tilde{u}_a(x)$ away from the concentrated points $b_1, \cdots, b_k$.

Proposition 3.3. *Suppose that $\tilde{u}_a(x)$ is a k-peak solution of (3.1) concentrated at $b_1, \cdots, b_k$. Then for any fixed $R \gg 1$, there exist some $\theta > 0$ and $C > 0$, such that*

$$|\tilde{u}_a(x)| + |\nabla \tilde{u}_a(x)| \le C \sum_{i=1}^{k} e^{-\theta |x - x_{a,i}|/\varepsilon}, \text{ for } x \in \mathbb{R}^N \backslash \bigcup_{i=1}^{k} B_{R\varepsilon}(x_{a,i}). \tag{3.11}$$

Lemma 3.4. *It holds*

$$(4 - N) \int_{\mathbb{R}^N} Q^4 = 4 \int_{\mathbb{R}^N} Q^2, \text{ for } N = 2, 3. \tag{3.12}$$

Proof. It is direct by the following Pohozaev identities:

$$\int_{\mathbb{R}^N} |\nabla Q|^2 + \int_{\mathbb{R}^N} Q^2 = \int_{\mathbb{R}^N} Q^4, \ (N-2) \int_{\mathbb{R}^N} |\nabla Q|^2 + N \int_{\mathbb{R}^N} Q^2 = \frac{N}{2} \int_{\mathbb{R}^N} Q^4. \tag{3.13}$$

□

Proposition 3.5. *Let $N = 2$ and $a \to ka_*$, it holds*

$$(ka_* - a)\mu_a^2 = \frac{1}{2} \sum_{i=1}^{k} \Delta V(b_i) \int_{\mathbb{R}^2} |x|^2 Q^2(x) + o(1). \tag{3.14}$$

Proof. Let u_a be a solution of (1.2)–(1.3), multiplying $\langle x - x_{a,i}, \nabla u_a \rangle$ on both sides of (1.2) and integrating on $B_d(x_{a,i})$, we find

$$\int_{B_d(x_{a,i})} \left[\big(2V(x) + \langle \nabla V(x), x - x_{a,i} \rangle \big) u_a^2 - 2\mu_a u_a^2 - a u_a^4 \right] = \int_{\partial B_d(x_{a,i})} W(x) d\sigma, \tag{3.15}$$

where

$$W(x) := -2 \frac{\partial u_a}{\partial \nu} \langle x - x_{a,i}, \nabla u_a \rangle + \langle x - x_{a,i}, \nu \rangle \left[|\nabla u_a|^2 + V(x) u_a^2 - \mu_a u_a^2 - \frac{a}{2} u_a^4 \right].$$

Also, from (3.7), (3.8) and (1.10), we can write $u_a(x)$ as follows:

$$u_a(x) = \sum_{i=1}^{k} \sqrt{\frac{-\mu_a + V_i}{a}} \Big(\bar{Q}_{a,x_{a,i}}(x) + v_a(x) \Big). \tag{3.16}$$

where $\bar{Q}_{a,x_{a,i}}(x) := Q\Big(\sqrt{-\mu_a + V_i}(x - x_{a,i})\Big)$ and $\|v_a\|_a = O\Big(\frac{1}{(\sqrt{-\mu_a})^3}\Big)$.

Hence, using (3.16), we get

$$\begin{aligned}
&\int_{B_d(x_{a,i})}\big(2V(x)+\langle\nabla V(x),x-x_{a,i}\rangle\big)u_a^2\\
&=\frac{-\mu_a+V_i}{a}\int_{B_d(x_{a,i})}\Big(\langle\nabla V(x),x-x_{a,i}\rangle+2\big(V(x)-V_i\big)\Big)\bar{Q}^2_{a,x_{a,i}}(x)\\
&\quad+2V_i\int_{B_d(x_{a,i})}u_a^2+O\Big(-\frac{1}{\mu_a^3}\Big)\\
&=\frac{1}{a(-\mu_a+V_i)}\Delta V(x_{a,i})\int_{\mathbb{R}^2}|x|^2Q^2(x)+2V_i\frac{a^*}{a}+o\Big(-\frac{1}{\mu_a}\Big).
\end{aligned}\tag{3.17}$$

Also, we have

$$\Delta V(x_{a,i})=\Delta V(b_i)+O(|x_{a,i}-b_i|)=\Delta V(b_i)+o(1).\tag{3.18}$$

Then from (3.17) and (3.18), we get

$$\begin{aligned}
&\int_{B_d(x_{a,i})}\big(2V(x)+\langle\nabla V(x),x-x_{a,i}\rangle\big)u_a^2\\
&=\frac{1}{a(-\mu_a+V_i)}\Delta V(b_i)\int_{\mathbb{R}^2}|x|^2Q^2(x)+2V_i\int_{B_d(x_{a,i})}u_a^2+o\big(-\frac{1}{\mu_a}\big).
\end{aligned}\tag{3.19}$$

So summing (3.15) from $i=1$ to $i=k$ and using (3.19), we find

$$2\mu_a+a\int_{\mathbb{R}^2}u_a^4=\sum_{i=1}^k\frac{1}{a(-\mu_a+V_i)}\Delta V(b_i)\int_{\mathbb{R}^2}|x|^2Q^2(x)+\frac{2a^*}{a}\sum_{i=1}^kV_i+o\big(-\frac{1}{\mu_a}\big).\tag{3.20}$$

On the other hand, we can obtain

$$\begin{aligned}
\int_{\mathbb{R}^2}u_a^4=&\sum_{i=1}^k\frac{(-\mu_a+V_i)^2}{a^2}\Big[\int_{\mathbb{R}^2}\bar{Q}^4_{a,x_{a,i}}(x)+4\int_{\mathbb{R}^2}\bar{Q}^3_{a,x_{a,i}}(x)v_a\\
&+O\Big(\int_{\mathbb{R}^2}\bar{Q}^2_{a,x_{a,i}}(x)v_a^2+O\Big(\int_{\mathbb{R}^2}v_a^4\Big)\Big]+o\Big(-\frac{1}{\mu_a}\Big)\\
=&\frac{2a_*}{a^2}\sum_{i=1}^k(-\mu_a+V_i)+O\Big(\mu_a^2\|v_a\|_a^2\Big)=\frac{2a_*}{a^2}\sum_{i=1}^k(-\mu_a+V_i)+o\Big(-\frac{1}{\mu_a}\Big).
\end{aligned}\tag{3.21}$$

Here we use the fact that $\int_{\mathbb{R}^2}\bar{Q}^3_{a,x_{a,i}}(x)v_a=0$ for $i=1,\cdots,k$ by orthogonality.

Then (3.20) and (3.21) imply

$$\mu_a(a-ka_*)=\frac{1}{2}\sum_{i=1}^k\frac{1}{-\mu_a+V_i}\Delta V(b_i)\int_{\mathbb{R}^2}|x|^2Q^2(x)+o\big(-\frac{1}{\mu_a}\big),$$

which gives (3.14). □

Proposition 3.6. *Let $N=3$ and $a\searrow 0$, it holds*

$$a\sqrt{-\mu_a}=ka_*+\sum_{i=1}^k\frac{V_i}{2\mu_a}+O\Big(\big|\sum_{i=1}^kV(x_{a,i})\big|\frac{1}{(-\mu_a)}+\big|\sum_{i=1}^k\nabla V(x_{a,i})\big|\frac{1}{(\sqrt{-\mu_a})^3}+\frac{1}{\mu_a^2}\Big).\tag{3.22}$$

Proof. From (1.2) and (1.3), we have

$$1=\int_{\mathbb{R}^3}\big(u_a(x)\big)^2=\int_{\mathbb{R}^3}\Big(\sum_{i=1}^k\sqrt{\frac{-\mu_a+V_i}{a}}\Big(\bar{Q}_{\varepsilon,x_{a,i}}(x)+v_a(x)\Big)^2, \tag{3.23}$$

which gives

$$a=\sum_{i=1}^k\frac{a_*}{\sqrt{-\mu_a+V_i}}+O\Big(\big|\sum_{i=1}^k(V(x_{a,i})-V_i)\big|\frac{1}{(\sqrt{-\mu_a})^3}+\big|\sum_{i=1}^k\nabla V(x_{a,i})\big|\frac{1}{\mu_a^2}+\frac{1}{(\sqrt{-\mu_a})^5}\Big).$$

Then we can find (3.22). □

4. Locating the peaks and the Existence of peak solutions

First, we locate the peaks for a k-peak solution. Let $\tilde{u}_a$ be a k-peak solution of (3.1), then for any small fixed $d>0$, we find

$$\begin{aligned}\varepsilon^2&\int_{B_d(x_{a,i})}\frac{\partial V(x)}{\partial x_j}\big(\tilde{u}_a\big)^2\\ =&-2\varepsilon^2\int_{\partial B_d(x_{a,i})}\frac{\partial \tilde{u}_a)}{\partial\bar{\nu}}\frac{\partial\tilde{u}_a}{\partial x_j}+\varepsilon^2\int_{\partial B_d(x_{a,i})}|\nabla\tilde{u}_a|^2\bar{\nu}_j(x)\\ &+\int_{\partial B_d(x_{a,i})}\big(1+\varepsilon^2V(x)\big)\big(\tilde{u}_a\big)^2\bar{\nu}_j(x)\mathrm{d}\sigma-\frac{1}{2}\int_{\partial B_d(x_{a,i})}|\tilde{u}_a|^4\bar{\nu}_j(x)\\ =&O(e^{-\frac{\gamma}{\varepsilon}}),\ \text{with some }\gamma>0,\end{aligned} \tag{4.1}$$

where $i=1,\cdots,k$, $j=1,\cdots,N$ and $\bar{\nu}(x)=\big(\bar{\nu}_1(x),\cdots,\bar{\nu}_N(x)\big)$ is the outward unit normal of $\partial B_d(x_{a,i})$. And then (4.1) implies the first necessary condition for the concentrated points b_i:

$$\nabla V(b_i)=0,\ \text{for } i=1,\cdots,k.$$

Proof of Theorem 1.2: Since $x_{a,i}\to b_i\in\Gamma_i$, we find that there is a $t_a\in[V_i,V_i+\theta]$ if Γ_i is a local minimum set of $V(x)$, or $t_a\in[V_i-\theta,V_i]$ if Γ_i is a local maximum set of $V(x)$, such that $x_{a,i}\in\Gamma_{t_a,i}$. Let $\tau_{a,i}$ be the unit tangential vector of $\Gamma_{t_a,i}$ at $x_{a,i}$. Then

$$G(x_{a,i})=0,\ \text{where } G(x)=\big\langle\nabla V(x),\tau_{a,i}\big\rangle.$$

Also we have the following expansion:

$$G(x)=\langle\nabla G(x_{a,i}),x-x_{a,i}\rangle+\frac{1}{2}\big\langle\langle\nabla^2G(x_{a,i}),x-x_{a,i}\rangle,x-x_{a,i}\big\rangle+o\big(|x-x_{a,i}|^2\big),\ \text{for } x\in B_d(x_{a,i}).$$

Then it follows from (4.1) and the above expansion that

$$\begin{aligned}&\int_{B_d(x_{a,i})}G(x)Q^2_{\varepsilon,x_{a,i}}(x)\\ &\quad=-2\int_{B_d(x_{a,i})}G(x)Q_{\varepsilon,x_{a,i}}(x)v_a-\int_{B_d(x_{a,i})}G(x)v_a^2+O\big(e^{-\frac{\gamma}{\varepsilon}}\big)\\ &\quad=O\Big([\varepsilon^{\frac{N}{2}+1}|\nabla G(x_{a,i})|+\varepsilon^{\frac{N}{2}+2}]\|v_a\|_a+\varepsilon|\nabla G(x_{a,i})|\cdot\|v_a\|_a^2\Big)+O\big(e^{-\frac{\gamma}{\varepsilon}}\big)\\ &\quad=O\big(\varepsilon^{N+3}\big).\end{aligned} \tag{4.2}$$

Also, in view of $G(x_{a,i})=0$, it is easy to show

$$\int_{\mathbb{R}^N} G(x)Q^2_{\varepsilon,x_{a,i}}(x)=\frac{1}{2N}\varepsilon^{N+2}\Delta G(x_{a,i})\int_{\mathbb{R}^N}|x|^2Q^2+O\big(\varepsilon^{N+4}\big). \tag{4.3}$$

Then (4.2) and (4.3) give $(\Delta G)(x_{a,i})=O\big(\varepsilon\big)$. Thus by the condition (V), we obtain (1.11). □

Now, we consider the existence of peak solutions for (1.6) with $\lambda>0$ a large parameter. Let $\eta=\frac{1}{\sqrt{\lambda}}$ and $w(x)\mapsto\sqrt{\lambda}w(x)$, then (1.6) can be changed to following problem:

$$-\eta^2\Delta w+\big(1+\eta^2V(x)\big)w=w^3,\ w\in H^1(\mathbb{R}^N). \tag{4.4}$$

In the following, we denote $\langle u,v\rangle_\eta=\int_{\mathbb{R}^N}\big(\eta^2\nabla u\nabla v+uv\big)$ and $\|u\|_\eta=\langle u,u\rangle_\eta^{\frac12}$. Now for $\eta>0$ small, we construct a k-peak solution u_η of (4.4) concentrating at $b_1,\cdots,b_k$. Here we can prove the following result in a standard way.

Proposition 4.1. *There is an $\eta_0>0$, such that for any $\eta\in(0,\eta_0]$, and z_i close to b_i, there exists $v_{\eta,z}\in F_{\eta,z}$ with $z=(z_1,\cdots,z_k)$, such that*

$$\int_{\mathbb{R}^N}\big(\eta^2\nabla w_\eta\nabla\psi+\big(1+\eta^2V(x)\big)w_\eta\psi=\int_{\mathbb{R}^N}w_\eta^3\psi,\ \text{for all }\psi\in F_{\eta,z},$$

where $w_\eta(x)=\sum_{i=1}^k Q_{\eta,z_i}(x)+v_{\eta,z}(x)$, *and*

$$F_{\eta,z}=\Big\{u(x)\in H^1(\mathbb{R}^N):\Big\langle u,\frac{\partial Q_{\eta,z_i}(x)}{\partial x_j}\Big\rangle_\eta=0,\ j=1,\cdots,N,\ i=1,\cdots,k\Big\}.$$

Moreover, it holds

$$\|v_{\eta,z}\|_\eta=O\big(\sum_{i=1}^k\big|V(z_i)-V_i\big|\eta^{\frac N2+2}+\sum_{i=1}^k\big|\nabla V(z_i)\big|\eta^{\frac N2+3}+\eta^{\frac N2+4}\big).$$

To obtain a true solution for (4.4), we need to choose z, such that

$$\int_{B_d(x_{a,i})}\Big(-\eta^2\Delta w_\eta\frac{\partial w_\eta}{\partial x_j}+\big(1+\eta^2V(x)\big)w_\eta\frac{\partial w_\eta}{\partial x_j}-w_\eta^3\frac{\partial w_\eta}{\partial x_j}\Big)=0,\quad\forall\, i=1,\cdots,k,\ j=1,\cdots,N.$$

It is easy to check that the above identities are equivalent to

$$\int_{B_d(x_{a,i})}\frac{\partial V(x)}{\partial x_j}w_\eta^2=O\big(e^{-\frac{\gamma}{\eta}}\big),\quad\forall\, i=1,\cdots,k,\ j=1,\cdots,N. \tag{4.5}$$

For z_i close to b_i, $z_i\in\Gamma_{t,i}$ for some t close to V_i and $z=(z_1,\cdots,z_k)$. In the following, we use ν_i to denote the unit normal vector of $\Gamma_{t,i}$ at z_i, while we use $\tau_{i,j}$ $(j=1,\cdots,N-1)$, to denote the principal directions of $\Gamma_{t,i}$ at $x_{a,i}$. Then, at z_i, it holds

$$D_{\tau_{i,j}}V(z_i)=0,\ \text{for } j=1,\cdots,N-1,\ \text{and } |\nabla V(z_i)|=|D_{\nu_i}V(z_i)|.$$

We first prove the following results.

Lemma 4.2. *Under the condition (V), $\int_{B_d(x_{a,i})}D_{\nu_i}V(x)u_\eta^2=O\big(e^{-\frac{\gamma}{\eta}}\big)$ is equivalent to*

$$D_{\nu_i}V(z_i)=O\big(\eta^2\big). \tag{4.6}$$

Proof. First, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} D_{\nu_i}V(x)Q^2_{\eta,z_i}(x)\\
&=-2\int_{\mathbb{R}^N} D_{\nu_i}V(x)Q_{\eta,z_i}(x)v_{\eta,z}-\int_{\mathbb{R}^N} D_{\nu_i}V(x)v^2_{\eta,z}\\
&=O\big(|D_{\nu_i}V(z_i)|\eta^{\frac{N}{2}}\cdot\|v_{\eta,z}\|_\eta+\eta^{\frac{N}{2}+1}\|v_{\eta,z}\|_\eta+\|v_{\eta,z}\|^2_\eta\big)=O\big(\eta^{N+2}\big).
\end{aligned}\tag{4.7}$$

On the other hand, we have

$$\int_{\mathbb{R}^N} D_{\nu_i}V(x)Q^2_{\eta,z_i}(x)=a_*\eta^N D_{\nu_i}V(z_i)+O\big(\eta^{N+2}\big),\tag{4.8}$$

Then we get (4.6) by combining (4.7) and (4.8).

□

Lemma 4.3. *Under the condition* (V), $\int_{B_d(x_{a,i})} D_{\tau_i}V(x)u^2_\eta=O\big(e^{-\frac{\gamma}{\eta}}\big)$ *is equivalent to*

$$(D_{\tau_i}\Delta V)(z_i)=O\Big(\big(\sum_{l=1}^{k}\big|V(z_l)-V_l\big|\big)\eta+\eta^2\Big).\tag{4.9}$$

Proof. Let $G(x)=\big\langle\nabla V(x),\tau_i\big\rangle$. Then, similar to the estimate (4.2), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} G(x)Q^2_{\eta,z_i}(x)=&-2\int_{\mathbb{R}^N} G(x)Q_{\eta,z_i}(x)v_{\eta,z}-\int_{\mathbb{R}^N} G(x)v^2_{\eta,z}\\
=&O\Big(\big(\sum_{l=1}^{k}\big|V(z_l)-V_l\big|\big)\eta^{N+3}+\eta^{N+4}\Big).
\end{aligned}\tag{4.10}$$

On the other hand, in view of $G(z_i)=0$, it is easy to show

$$\int_{\mathbb{R}^N} G(x)Q^2_{\eta,z_i}(x)=\frac{1}{2}\eta^{N+2}\Delta G(z_i)B+O\big(\eta^{N+4}\big),\tag{4.11}$$

where

$$B=\frac{1}{N}\int_{\mathbb{R}^N}|x|^2Q^2.\tag{4.12}$$

Thus, (4.9) follows from (4.10) and (4.11).

□

Theorem 4.4. *For* $\lambda>0$ *large,* (1.6) *has a solution* u_λ*, satisfying*

$$u_\lambda(x)=\sum_{i=1}^{k}\sqrt{\lambda}Q\big(\sqrt{\lambda}(x-x_{\lambda,i})\big)+\omega_\lambda,$$

where $x_{\lambda,i}\to b_i$*, and* $\int_{\mathbb{R}^N}\big(|\nabla\omega_\lambda|^2+\omega^2_\lambda\big)\to 0$ *as* $\lambda\to+\infty$.

Proof. As pointed out earlier, we need to solve (4.5). By Lemmas 4.2 and 4.3, the equation (4.5) is equivalent to

$$D_{\nu_i}V(z_i)=O\big(\eta^2\big),\quad (D_{\tau_i}\Delta V)(z_i)=O\Big(\big(\sum_{l=1}^{k}\big|V(z_l)-V_l\big|\big)\eta+\eta^2\Big).$$

Let $\bar z_i \in \Gamma_i$ be the point such that $z_i - \bar z_i = \alpha_i \nu_i$ for some $\alpha_i \in \mathbb{R}$. Then, we have $D_{\nu_i} V(\bar z_i) = 0$. As a result,

$$D_{\nu_i} V(z_i) = D_{\nu_i} V(z_i) - D_{\nu_i} V(\bar z_i) = D^2_{\nu_i\nu_i} V(\bar z_i)\langle z_i - \bar z_i, \nu_i\rangle + O(|z_i - \bar z_i|^2).$$

By the non-degenerate assumption, we find that $D_{\nu_i} V(z_i) = O\big(\eta^2\big)$ is equivalent to $\langle z_i - \bar z_i, \nu_i\rangle = O\big(\eta^2 + |z_i - \bar z_i|^2\big)$. This means that $D_{\nu_i} V(z_i) = O\big(\eta^2\big)$ can be written as

$$|z_i - \bar z_i| = O\big(\eta^2\big). \tag{4.13}$$

Let $\bar\tau_{i,j}$ be the j-th tangential unit vector of Γ_i at $\bar z_i$. Now by the condition (V), we have

$$(D_{\tau_{i,j}}\Delta V)(z_i) = (D_{\bar\tau_{i,j}}\Delta V)(\bar z_i) + O(|z_i - \bar z_i|) = (D_{\bar\tau_{i,j}}\Delta V)(\bar z_i) + O(\varepsilon^2),$$

and

$$(D_{\bar\tau_{i,j}}\Delta V)(\bar z_i) =(D_{\bar\tau_{i,j}}\Delta V)(\bar z_i) - (D_{\tau_{i,j,0}}\Delta V)(b_i) = \big\langle (\nabla_T D_{\tau_{i,j,0}}\Delta V)(x_0), \bar z_i - b_i\big\rangle + O(|\bar z_i - b_i|^2),$$

where ∇_{T_i} is the tangential gradient on Γ_i at $b_i \in \Gamma_i$, and $\tau_{i,j,0}$ is the j-th tangential unit vector of Γ_i at b_i. Therefore, $(D_{\tau_i}\Delta V)(z_i) = O\Big(\big(\sum_{l=1}^k |V(z_l) - V_l|\big)\eta + \eta^2\Big)$ can be rewritten as

$$\big\langle (\nabla_T D_{\tau_{i,j,0}}\Delta V)(x_0), \bar z_i - b_i\big\rangle = O(\eta^2 + |\bar z_i - b_i|^2). \tag{4.14}$$

So we can solve (4.13) and (4.14) to obtain $z_i = x_{\eta,i}$ with $x_{\eta,i} \to b_i$ as $\eta \to 0$. □

Proof of Theorem 1.3. Let w_λ is a k-peak solution as in Theorem 4.4, and we define

$$u_\lambda = \frac{w_\lambda}{\Big(\displaystyle\int_{\mathbb{R}^N} w_\lambda^2\Big)^{\frac12}}.$$

Then $\displaystyle\int_{\mathbb{R}^N} u_\lambda^2 = 1$, and

$$-\Delta u_\lambda + V(x) u_\lambda = a_\lambda u_\lambda^3 - \lambda u_\lambda, \quad \text{in } \mathbb{R}^N, \tag{4.15}$$

with $a_\lambda = \displaystyle\int_{\mathbb{R}^N} w_\lambda^2$.

For $N = 2$, similar to (3.14), under the condition (V), we can prove that

$$\Big(ka_* - \int_{\mathbb{R}^2} w_\lambda^2\Big)\lambda^2 = \frac12 \sum_{i=1}^k \Delta V(b_i) \int_{\mathbb{R}^2} |x|^2 Q^2(x) + o\big(1\big).$$

This shows that if $\displaystyle\sum_{i=1}^k \Delta V(b_i) \neq 0$, $\displaystyle\int_{\mathbb{R}^N} w_\lambda^2 \neq ka_*$. Take $\lambda_0 > 0$ large and let $b_0 = \displaystyle\int_{\mathbb{R}^2} w_{\lambda_0}^2$. Then,

$$b_0 < ka_*, \text{ if } \sum_{i=1}^k \Delta V(b_i) > 0 \text{ and } b_0 > ka_* \text{ if } \sum_{i=1}^k \Delta V(b_i) < 0.$$

By the mean value theorem, for any a between b_0 and ka_*, there exists $\lambda_a > 0$ large, such that the solution w_a of (1.6) with $\lambda = \lambda_a$ satisfies $\displaystyle\int_{\mathbb{R}^2} w_\lambda^2 = a$. Thus, for such a, we obtain a k-peak solution for (1.2), where $\mu_a = -\lambda_a$.

For $N=3$, similar to (3.22), we can prove

$$\sqrt{\lambda}\int_{\mathbb{R}^3} w_\lambda^2 = ka_* + o(1), \text{ as } a\searrow 0.$$

Take $\lambda_0>0$ large and let $b_0=\displaystyle\int_{\mathbb{R}^3} w_{\lambda_0}^2$. Then, by the mean value theorem, for any a between 0 and b_0, there exists $\lambda_a>0$ large, such that the solution u_a of (1.6) with $\lambda=\lambda_a$ satisfies $\displaystyle\int_{\mathbb{R}^3} w_\lambda^2 = a$. Thus, for such a, we obtain a k-peak solution for (1.2), where $\mu_a=-\lambda_a$.

□

Before we end this section, we discuss briefly the existence of clustering k-peak solutions for (1.2)–(1.3). The function $\Delta V(x)|_{x\in\Gamma_i}$ has a minimum point and a maximum point. Let us assume that $\Delta V(x)|_{x\in\Gamma_i}$ has an isolated maximum point $b_i\in\Gamma_i$. That is, we assume that $\Delta V(x)<\Delta V(b_i)$ for all $x\in\Gamma_i\bigcap(B_\delta(b_i)\setminus\{b_i\})$. We now use $\sum_{j=1}^k Q_{\eta,x_{\eta,j}}$ as an approximate solution of (4.4), where $x_{\eta,j}$ satisfies $x_{\eta,j}\to b_i$, $j=1,\cdots,k$, $\frac{|x_{\eta,j}-x_{\eta,l}|}{\eta}\to+\infty$, $l\neq j$, as $\eta\to 0$. We have the following existence result for (4.4).

Proposition 4.5. *Assume that* (V) *holds, and* $\frac{\partial^2 V(\bar x)}{\partial\nu_i^2}\neq 0$ *for any* $\bar x\in\Gamma_i$ *with some* $i\in\{1,\cdots,k\}$. *If* $b_i\in\Gamma_i$ *is an isolated maximum point of* $\Delta V(x)|_{x\in\Gamma_i}$ *on* Γ_i, *then for any integer* $k>0$, *there exists an* $\eta_0>0$, *such that for any* $\eta\in(0,\eta_0]$, *problem* (4.4) *has a solution* u_η, *satisfying*

$$u_\eta(x)=\sum_{j=1}^k Q_{\eta,x_{\eta,j}}+\omega_\eta,$$

where $x_{\eta,j}\to b_i$, $j=1,\cdots,k$, $\frac{|x_{\eta,j}-x_{\eta,l}|}{\eta}\to+\infty$, $l\neq j$, *and* $\displaystyle\int_{\mathbb{R}^N}\big(\eta^2|\nabla\omega_\eta|^2+\omega_\eta^2\big)=o(\eta^{\frac N2})$ *as* $\eta\to 0$.

Proof. Define

$$I(u)=\frac12\int_{\mathbb{R}^N}\big(\eta^2|\nabla u|^2+(1+\eta^2V(x))u^2\big)-\frac14\int_{\mathbb{R}^N}u^4.$$

We have the following energy expansion:

$$\begin{aligned}
&I\Big(\sum_{j=1}^k Q_{\eta,x_{\eta,j}}\Big)\\
=&kE\eta^N+E\eta^N\sum_{j=1}^k\frac{\partial^2V(\bar x_{\eta,j})}{\partial\nu_{i,j}^2}r_{\eta,j}^2+F\eta^{N+4}\sum_{j=1}^k\Delta V(\bar x_{\eta,j})\\
&-\sum_{j>m}(a_0+o(1))\eta^Ne^{-\frac{|x_{\eta,m}-x_{\eta,j}|}{\eta}}\Big(\frac{\eta}{|x_{\eta,m}-x_{\eta,j}|}\Big)^{\frac{N-1}{2}}+O(\eta^{N+5}+\eta^{N+2}r_{\eta,j}^3),
\end{aligned}\tag{4.16}$$

where $a_0 > 0$, $E = \frac{1}{4}\int_{\mathbb{R}^N} Q^4 > 0$, $F > 0$ and $r_{\eta,j} = |x_{\eta,j} - \bar{x}_{\eta,j}|$, $\bar{x}_{\eta,j} \in \Gamma_i$ is the point such that $|x_{\eta,j} - \bar{x}_{\eta,j}| = d(x_{\eta,j}, \Gamma_i)$. In fact,

$$\begin{aligned} I\big(\sum_{j=1}^{k} Q_{\eta,x_{\eta,j}}\big) =& E\sum_{j=1}^{k} V(x_{\eta,j})\eta^N + F\eta^{N+4}\sum_{j=1}^{k}\Delta V(x_{\eta,j}) \\ & - \sum_{j>m}(a_0+o(1))\eta^N e^{-\frac{|x_{\eta,m}-x_{\eta,j}|}{\eta}}\Big(\frac{\eta}{|x_{\eta,m}-x_{\eta,j}|}\Big)^{\frac{N-1}{2}} + O(\eta^{N+5}). \end{aligned} \tag{4.17}$$

Also, we have

$$V(x_{\eta,j}) = 1 + \frac{\partial^2 V(\bar{x}_{\eta,j})}{\partial \nu_{i,j}^2} r_{\eta,j}^2 + O(r_{\eta,j}^3) \text{ and } \Delta V(x_{\eta,j}) = \Delta V(\bar{x}_{\eta,j}) + O(r_{\eta,j}). \tag{4.18}$$

So, (4.16) follows from (4.17) and (4.18).

To obtain a solution u_η of the form $\sum_{j=1}^{k} Q_{\eta,x_{\eta,j}} + \omega_\eta$, we can first carry out the reduction argument as in Proposition 4.1 to obtain ω_η, satisfying

$$\|\omega_\eta\|_\eta = \eta^{N+2} O\big(\sum_{j=1}^{k}|\nabla V(x_{\eta,j})|\eta + \eta^2 + \sum_{j\neq m} e^{-\frac{(1+\sigma)|x_{\eta,m}-x_{\eta,j}|}{2\eta}}\big), \tag{4.19}$$

for some $\sigma > 0$. Define

$$K(x_{\eta,1}, \cdots, x_{\eta,k}) = I\big(Q_{\eta,x_{\eta,j}} + \omega_\eta\big).$$

Then, it follows from (4.19) that we can obtain the same expansion (4.16) for $K(x_{\eta,1}, \cdots, x_{\eta,k})$. Now we set

$$M = \big\{(r, \bar{x}) : r \in (-\delta\eta^2, \delta\eta^2),\ \bar{x} \in B_\delta(b_i) \cap \Gamma_i\big\}.$$

If Γ_i is a local maximum set of $V(x)$ and $\frac{\partial^2 V(\bar{x})}{\partial \nu_i^2} < 0$ for any $\bar{x} \in \Gamma_i$, then it is easy to prove that $K(x_{\eta,1}, \cdots, x_{\eta,k})$ has a critical point, which is a maximum point of K in

$$S_{\eta,k} := \big\{(x_{\eta,1}, \cdots, x_{\eta,k}) : x_{\eta,j} \in M, |x_{\eta,j} - x_{\eta,m}| > \theta\eta|\ln\eta|,\ m \neq j\big\},$$

where $\theta > 0$ is some constant.

If Γ_i is a local minimum set of $V(x)$ and $\frac{\partial^2 V(\bar{x})}{\partial \nu_i^2} > 0$ for any $\bar{x} \in \Gamma_i$, then

$$E\frac{\partial^2 V(\bar{x})}{\partial \nu_i^2} r^2 + F\eta^2 \Delta V(\bar{x})$$

has a saddle point $(0, b_i)$ in M. We can use a topological argument as in [12, 13] to prove that $K(x_{\eta,1}, \cdots, x_{\eta,k})$ has a critical point in $S_{\eta,k}$. □

Similar to the proofs of Theorem 1.3, from Proposition 4.5, we have the following result:

Theorem 4.6. *Assume that* (V) *holds, and* $\frac{\partial^2 V(\bar{x})}{\partial \nu_i^2} \neq 0$ *for any* $\bar{x} \in \Gamma_i$ *and some* $i \in \{1, \cdots, k\}$. *If* $b_i \in \Gamma_i$ *is an isolated maximum point of* $\Delta V(x)|_{x\in\Gamma_i}$ *on* Γ_i, *then for any integer* $k > 0$, (1.2)–(1.3) *has a solution satisfying*

$$u_a = \sqrt{\frac{-\mu_a}{a}}\Big(\sum_{j=1}^{k} Q\big(\sqrt{-\mu_a}(x - x_{a,j})\big) + \omega_a(x)\Big),$$

with

$$\mu_a \to -\infty,\ x_{a,j} \to b_i,\ j = 1, \cdots, k,\ \frac{|x_{a,j} - x_{a,l}|}{\sqrt{-\mu_a}} \to +\infty,\ l \neq j,$$

and $\int_{\mathbb{R}^N} \big[-\frac{1}{\mu_a} |\nabla \omega_a|^2 + \omega_a^2 \big] = o\big((\frac{1}{\sqrt{-\mu_a}})^N \big)$ *as* $a \to ka_*$ *if* $N = 2$, *or* $a \searrow 0$ *if* $N = 3$.

5. Local uniqueness

From Lemma 3.2, a k-peak solution $\tilde{u}_a$ can be written as

$$\tilde{u}_a(x) = \sum_{i=1}^{k} Q_{\varepsilon, x_{a,i}} + v_a(x), \tag{5.1}$$

with $|x_{a,i} - b_i| = o(1)$, $\varepsilon = \frac{1}{\sqrt{-\mu_a}}$, $v_a \in \bigcap_{i=1}^{k} E_{a, x_{a,i}}$ and

$$\|v_a\|_a = O\Big(\sum_{i=1}^{k} \big| V(x_{a,i}) - V_i \big| \varepsilon^{\frac{N}{2}+2} + \sum_{i=1}^{k} \big| \nabla V(x_{a,i}) \big| \varepsilon^{\frac{N}{2}+3} + \varepsilon^{\frac{N}{2}+4} \Big). \tag{5.2}$$

Also we know $x_{a,i} \in \Gamma_{t_a,i}$ for some $t_a \to V_i$. Similar to the last section, we use $\nu_{a,i}$ to denote the unit normal vector of $\Gamma_{t_a,i}$ at $x_{a,i}$, while we use $\tau_{a,i,j}$ to denote the principal direction of $\Gamma_{t_a,i}$ at $x_{a,i}$. Then, at $x_{a,i}$, it holds

$$D_{\tau_{a,i,j}} V(x_{a,i}) = 0, \quad \big| \nabla V(x_{a,i}) \big| = \big| D_{\nu_{a,i}} V(x_{a,i}) \big|. \tag{5.3}$$

We first prove the following result.

Lemma 5.1. *Under the condition* (V), *we have*

$$D_{\nu_{a,i}} V(x_{a,i}) = O\big(\varepsilon^2\big), \text{ for } i = 1, \cdots, k. \tag{5.4}$$

Proof. We use (4.1) to obtain

$$\int_{B_d(x_{a,i})} D_{\nu_{a,i}} V(x) \tilde{u}_a^2 = O(e^{-\frac{\gamma}{\varepsilon}}). \tag{5.5}$$

Then by (5.1)–(5.3) and (5.5), we get

$$\begin{aligned} &\int_{B_d(x_{a,i})} D_{\nu_{a,i}} V(x) Q^2_{\varepsilon, x_{a,i}} \\ &\qquad = -2 \int_{B_d(x_{a,i})} D_{\nu_{a,i}} V(x) Q_{\varepsilon, x_{a,i}} v_a - \int_{B_d(x_a)} D_{\nu_a} V(x) v_a^2 + O\big(e^{-\frac{\gamma}{\varepsilon}}\big) \\ &\qquad = O\big(|D_{\nu_{a,i}} V(x_{a,i})| \varepsilon^{\frac{N}{2}} \cdot \|v_a\|_a + \varepsilon^{\frac{N}{2}+1} \|v_a\|_a + \|v_a\|_a^2 \big) + O\big(e^{-\frac{\gamma}{\varepsilon}}\big) \\ &\qquad = O\big(\sum_{i=1}^{k} \big| V(x_{a,i}) - V_i \big| \varepsilon^{\frac{N}{2}+2} + |D_{\nu_{a,i}} V(x_{a,i})| \varepsilon^{\frac{N}{2}+3} + \varepsilon^{\frac{N}{2}+4} \big). \end{aligned} \tag{5.6}$$

On the other hand, by Taylor's expansion, we have

$$\int_{B_d(x_{a,i})} D_{\nu_{a,i}} V(x) Q^2_{\varepsilon, x_{a,i}} = \varepsilon^N \big[(1 + V_i \varepsilon^2) a_* D_{\nu_{a,i}} V(x_{a,i}) + \frac{B \varepsilon^2}{2} \Delta D_{\nu_{a,i}} V(x_{a,i}) + O\big(\varepsilon^4\big) \big], \tag{5.7}$$

where B is the constant in (4.12). And then (5.4) follows from (5.6) and (5.7).

□

Let $\bar{x}_{a,i} \in \Gamma_i$ be the point such that $x_{a,i} - \bar{x}_{a,i} = \beta_{a,i}\nu_{a,i}$ for some $\beta_{a,i} \in \mathbb{R}$ and $i = 1, \cdots, k$. Then we can prove

Lemma 5.2. *Under the condition* (V)*, we have*

$$\begin{cases} \bar{x}_{a,i} - b_i = L_i\varepsilon^2 + O(\varepsilon^4), \\ x_{a,i} - \bar{x}_{a,i} = -\dfrac{B}{2a_*}\dfrac{\partial \Delta V(b_i)}{\partial \nu_i}\Big(\dfrac{\partial^2 V(b_i)}{\partial \nu_i^2}\Big)^{-1}\varepsilon^2\nu_{a,i} + O(\varepsilon^4), \end{cases} \tag{5.8}$$

where B *is the constant in* (4.12) *and* L_i *is a vector depending on* b_i *and* $i = 1, \cdots, k$.

Proof. It follows from (5.6) and (5.7) that

$$\begin{aligned} &\big(a_* + O(\varepsilon^2)\big)D_{\nu_{a,i}}V(x_{a,i}) + \frac{B\varepsilon^2}{2}\Delta D_{\nu_{a,i}}V(x_{a,i}) \\ &\quad = O\big(\varepsilon^4 + \varepsilon^2\sum_{i=1}^{k}|V(x_{a,i}) - V_i|\big) = O\big(\varepsilon^4 + \varepsilon^2\sum_{i=1}^{k}|x_{a,i} - \bar{x}_{a,i}|^2\big). \end{aligned} \tag{5.9}$$

Since $\frac{\partial^2 V(b_i)}{\partial \nu_i^2} \neq 0$, the outward unit normal vector $\nu_{a,i}(x)$ and the tangential unit vector $\tau_{a,i}(x)$ of $\Gamma_{t_a,i}$ at $x_{a,i}$ are Lip-continuous in $W_{\delta,i}$, then from (5.9), we find

$$x_{a,i} - \bar{x}_{a,i} = -\frac{B}{2a_*}\big(\Delta D_{\nu_i}V(b_i)\big)\Big(\frac{\partial^2 V(b_i)}{\partial \nu_i^2}\Big)^{-1}\varepsilon^2 + O\big(\varepsilon^4 + \varepsilon^2\big|\bar{x}_{a,i} - b_i\big|\big). \tag{5.10}$$

Then (5.2) and (5.10) implies

$$\|v_a\|_a = O\Big(\sum_{i=1}^{k}|x_{a,i} - \bar{x}_{a,i}|\varepsilon^{\frac{N}{2}+2} + \varepsilon^{\frac{N}{2}+4}\Big) = O\big(\varepsilon^{\frac{N}{2}+4}\big). \tag{5.11}$$

Recall $G(x) = \big\langle \nabla V(x), \tau_{a,i,j}\big\rangle$. Then $G(x_{a,i}) = 0$. Similar to (4.2) and (5.6), we have

$$\begin{aligned} &\int_{B_d(x_{a,i})} G(x)Q_{\varepsilon,x_{a,i}}^2 \\ &\qquad = -2\int_{B_d(x_{a,i})} G(x)Q_{\varepsilon,x_{a,i}}v_a - \int_{B_d(x_{a,i})} G(x)v_a^2 + O\big(e^{-\frac{\gamma}{\varepsilon}}\big) \\ &\qquad = -2\int_{B_d(x_{a,i})} G(x)Q_{\varepsilon,x_{a,i}}v_a + O\big(\|v_a\|_a^2\big) + O\big(e^{-\frac{\gamma}{\varepsilon}}\big) \\ &\qquad = -2\int_{B_d(x_{a,i})} \langle \nabla G(x_{a,i}), x - x_{a,i}\rangle Q_{\varepsilon,x_{a,i}}v_a + O\big(\varepsilon^{N+6}\big). \end{aligned} \tag{5.12}$$

On the other hand, in view of $\nabla V(x) = 0$, $x \in \Gamma_i$, we find

$$\nabla G(x_{a,i}) = \big\langle \nabla^2 V(x_{a,i}), \tau_{a,i,j}\big\rangle = \big\langle \nabla^2 V(\bar{x}_{a,i}), \bar{\tau}_{a,i,j}\big\rangle + O\big(|x_{a,i} - \bar{x}_{a,i}|\big) = O\big(|x_{a,i} - \bar{x}_{a,i}|\big), \tag{5.13}$$

where $\bar{x}_{a,i} \in \Gamma_i$ is the point such that $x_{a,i} - \bar{x}_{a,i} = \beta_{a,i}\nu_{a,i}$ for some $\beta_{a,i} \in \mathbb{R}$, and $\bar{\tau}_{a,i,j}$ is the tangential vector of Γ_i at $\bar{x}_{a,i} \in \Gamma_i$. Therefore, from (5.10), (5.11) and (5.13), we know

$$\begin{aligned}&\int_{B_d(x_{a,i})} \langle \nabla G(x_{a,i}), x - x_{a,i}\rangle Q\Big(\frac{x - x_{a,i}}{\varepsilon}\Big) v_a \\ &\qquad =O\big(\varepsilon^{\frac{N}{2}+1}|\nabla G(x_{a,i})|\|v_a\|_a\big) = O\big(|x_{a,i} - \bar{x}_{a,i}|\varepsilon^{N+5}\big) = O\big(\varepsilon^{N+7}\big).\end{aligned} \tag{5.14}$$

Then by (5.12) and (5.14), we find

$$\int_{B_d(x_{a,i})} G(x) Q^2_{\varepsilon,x_{a,i}} = O\big(\varepsilon^{N+6}\big). \tag{5.15}$$

On the other hand, by the Taylor's expansion, we can prove

$$\int_{B_d(x_{a,i})} G(x) Q^2_{\varepsilon,x_{a,i}} = \big[\frac{B\varepsilon^{N+2}}{2}(1 + V_i\varepsilon^2)\big](D_{\tau_{a,i,j}}\Delta V)(x_{a,i}) + \frac{H_{i,j}\varepsilon^{N+4}}{24} + O\big(\varepsilon^{N+6}\big), \tag{5.16}$$

where

$$H_{i,j} = \sum_{l=1}^{N}\sum_{m=1}^{N} \frac{\partial^4 G(b_i)}{\partial x_l^2 \partial x_m^2} \int_{\mathbb{R}^N} x_l^2 x_m^2 Q^2.$$

So (5.15) and (5.16) give

$$(D_{\tau_{a,i,j}}\Delta V)(x_{a,i}) = -\frac{H_{i,j}\varepsilon^2}{12B} + O\big(\varepsilon^4\big). \tag{5.17}$$

We denote by $\bar{\tau}_{a,i,j}$ the j^{th} principal tangential vector of Γ_i at $\bar{x}_{a,i}$. Then by (5.10), we get

$$\begin{aligned}(D_{\tau_{a,i,j}}\Delta V)(x_{a,i}) =&(D_{\bar{\tau}_{a,i,j}}\Delta V)(\bar{x}_{a,i}) + \langle A_{i,j}, x_{a,i} - \bar{x}_{a,i}\rangle + O(|x_{a,i} - \bar{x}_{a,i}|^2) \\ =&(D_{\bar{\tau}_{a,i,j}}\Delta V)(\bar{x}_{a,i}) + B_{i,j}\varepsilon^2 + O(\varepsilon^4),\end{aligned} \tag{5.18}$$

where $A_{i,j}$ is a vector depending on b_i and $B_{i,j}$ is a constant depending on b_i. Moreover,

$$(D_{\bar{\tau}_{a,i,j}}\Delta V)(\bar{x}_{a,i}) = \big\langle D_{\tau_i}D_{\tau_{i,j}}(\Delta V)(b_i), \bar{x}_{a,i} - b_i\big\rangle + O(|\bar{x}_{a,i} - b_i|^2). \tag{5.19}$$

Therefore, from (5.17)–(5.19), we find

$$\big\langle D_{\tau_i}D_{\tau_{i,j}}(\Delta V)(b_i), \bar{x}_{a,i} - b_i\big\rangle = -(\frac{H_{i,j}}{12B} + B_{i,j})\varepsilon^2 + O\big(\varepsilon^4\big) + O(|\bar{x}_{a,i} - b_i|^2). \tag{5.20}$$

Since $D^2_{\tau_{i,j}\tau_{i,l}}(\Delta V)(b_i)$ is non-singular, we can complete the proofs of (5.8) from (5.10) and (5.20). □

Let

$$\delta_a := \begin{cases} |ka_* - a|^{\frac{1}{4}}|\beta_1|^{-\frac{1}{4}}B^{\frac{1}{4}}, & \text{for } N = 2, \\ a(ka_*)^{-1}, & \text{for } N = 3,\end{cases}$$

where $\beta_1 = \sum_{i=1}^{k} \Delta V(b_i)$ and B is the constant in (4.12).

Proposition 5.3. *Under the condition (V), for $N = 2, 3$, it holds*

$$-\mu_a\delta_a^2 = 1 + \gamma_1\delta_a^2 + O\big(\delta_a^4\big), \tag{5.21}$$

and

$$x_{a,i} - b_i = \bar{L}_i\delta_a^2 + O(\delta_a^4), \ \text{for } i = 1, \cdots, k. \tag{5.22}$$

where γ_1 and the vector $\bar{L}_i$ are constants.

Proof. First, (3.22) shows that (5.21) holds for the case $N = 3$.

For $N = 2$, from (5.8), we get $x_{a,i} - b_i = -\tilde{L}_i \frac{1}{\mu_a} + O(\frac{1}{\mu_a^2})$ for some vector $\tilde{L}_i$. Also from (3.8), we know $\|v_a\|_a = O\big(\varepsilon^5\big)$. we can calculate (3.18)–(3.21) more precise, which will gives us

$$\int_{\mathbb{R}^2} u_a^4 = -\frac{2ka_*\mu_a}{a^2} + \frac{b}{\mu_a^2} + O\Big(-\frac{1}{\mu_a^3}\Big), \tag{5.23}$$

and

$$\text{LHS of (3.15)} = -\frac{1}{a\mu_a}\Delta V(b_j)\int_{\mathbb{R}^2}|x|^2Q^2(x) + \frac{\tilde{b}}{\mu_a^2} + O\big(-\frac{1}{\mu_a^3}\big), \tag{5.24}$$

where b and $\tilde{b}$ are some constants. Then from (3.15), (3.21), (5.23) and (5.24), we get (5.21).

Finally, we can find (5.22) by (5.8) and (5.21). $\square$

By a change of variable, problem (1.2)–(1.3) can be changed into the following problem

$$-\delta_a^2\Delta u + \big(-\mu_a\delta_a^2 + \delta_a^2V(x)\big)u = u^3,\ \ u \in H^1(\mathbb{R}^N), \tag{5.25}$$

and

$$\int_{\mathbb{R}^N} u^2 = a\delta_a^2. \tag{5.26}$$

Then similar to Lemma 3.2, the k-peak solution of (5.25)–(5.26) concentrating at $b_1, \cdots, b_k$ can be written as $\sum\limits_{i=1}^{k}\tilde{Q}_{\delta_a,x_{a,i}} + \bar{v}_a(x)$, with $|x_{a,i} - b_i| = o(1)$, $\|\bar{v}_a\|_{\delta_a} = o(\delta_a^{\frac{N}{2}})$, and

$$\bar{v}_a \in \bigcap_{i=1}^{k}\tilde{E}_{a,x_{a,i}} := \left\{v \in H^1(\mathbb{R}^N) : \Big\langle v, \frac{\partial \tilde{Q}_{\delta_a,x_{a,i}}}{\partial x_j}\Big\rangle_a = 0,\ i = 1, \cdots, k,\ j = 1, \cdots, N\right\},$$

where $\tilde{Q}_{\delta_a,x_{a,i}} := Q\Big(\frac{\sqrt{1+(\gamma_1+V_i)\delta_a^2}(x-x_{a,i})}{\delta_a}\Big)$, $\|v\|_{\delta_a}^2 := \int_{\mathbb{R}^N}\big(\delta_a^2|\nabla v|^2 + v^2\big)$ and γ_1 is the constant in (5.21). Then we can write the equation (5.25) as follows:

$$L_a(\bar{v}_a) = N_a\big(\bar{v}_a\big) + \bar{l}_a(x),$$

where N_a, L_a are defined by (3.3) and (3.5). And

$$\bar{l}_a = \big(-\mu_a\delta_a^2 - 1 + \delta_a^2V(x)\big)\sum_{i=1}^{k}\tilde{Q}_{\delta_a,x_{a,i}} + \Big(\sum_{i=1}^{k}\tilde{Q}_{\delta_a,x_{a,i}}\Big)^3 - \sum_{i=1}^{k}\tilde{Q}_{\delta_a,x_{a,i}}^3. \tag{5.27}$$

Lemma 5.4. *It holds*

$$\|\bar{v}_a\|_{\delta_a} = O\big(\delta_a^{\frac{N}{2}+4}\big). \tag{5.28}$$

Proof. The proofs are similar to that of Lemma 3.2. The difference is

$$\|\bar{l}_a\|_{\delta_a} = O\Big(\big(\sum_{i=1}^{k}\big|V(x_{a,i}) - V_i\big|\big)\delta_a^{\frac{N}{2}+2} + \sum_{i=1}^{k}\big|\nabla V(x_{a,i})\big|\delta_a^{\frac{N}{2}+3} + \delta_a^{\frac{N}{2}+4}\Big) = O\Big(\delta_a^{\frac{N}{2}+4}\Big). \tag{5.29}$$

Here the definition of $\bar{l}_a$ is given in (5.27). Finally, (5.29) and (3.6) imply (5.28). $\square$

Let $u_a^{(1)}$ and $u_a^{(2)}$ be two k-peak solutions of (5.25)–(5.26) concentrating at k points $b_1,\cdots,b_k$, which can be written as

$$u_a^{(l)} = \sum_{i=1}^{k} \tilde{Q}_{\delta_a, x_{a,i}^{(l)}} + v_a^{(l)}(x), \text{ for } l=1,2, \text{ and } v_a^{(l)} \in \bigcap_{i=1}^{k} \tilde{E}_{a, x_{a,i}^{(l)}}. \tag{5.30}$$

Now we set $\xi_a(x) = \frac{u_a^{(1)}(x)-u_a^{(2)}(x)}{\|u_a^{(1)}-u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}}$. Then $\xi_a(x)$ satisfies $\|\xi_a\|_{L^\infty(\mathbb{R}^N)}=1$. And from (5.25), we find that ξ_a satisfies

$$-\delta_a^2 \Delta\xi_a(x) + C_a(x)\xi_a = g_a(x),$$

where

$$C_a(x) = \delta_a^2 V(x) - \delta_a^2 \mu_a^{(1)} - \big(\sum_{l=1}^{2}(u_a^{(l)})^2 + u_a^{(1)}u_a^{(2)}\big), \; g_a(x) = \frac{\delta_a^2(\mu_a^{(1)}-\mu_a^{(2)})}{\|u_a^{(1)}-u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}} u_a^{(2)}.$$

Also, similar to (3.11), for any fixed $R \gg 1$, there exist some $\theta>0$ and $C>0$, such that

$$|\xi_a(x)| + |\nabla\xi_a(x)| \le C\sum_{i=1}^{k} e^{-\theta|x-x_{a,i}|/\delta_a}, \text{ for } x\in\mathbb{R}^N\backslash \bigcup_{i=1}^{k} B_{R\delta_a}(x_{a,i}). \tag{5.31}$$

Now let $\bar{\xi}_{a,i}(x) = \xi_a\big(\frac{\delta_a x + x_{a,i}^{(1)}}{\sqrt{1+(\gamma_1+V_i)\delta_a^2}}\big)$, for $i=1,\cdots,k$, we have

$$-\Delta\bar{\xi}_{a,i}(x) + \frac{C_a(\delta_a x + x_{a,i}^{(1)})}{1+(\gamma_1+V_i)\delta_a^2}\bar{\xi}_{a,i} = \frac{g_a(\delta_a x + x_{a,i}^{(1)})}{1+(\gamma_1+V_i)\delta_a^2}. \tag{5.32}$$

Lemma 5.5. *For $x\in B_{d\delta_a^{-1}}(0)$, it holds*

$$\frac{C_a(\delta_a x + x_{a,i}^{(1)})}{1+(\gamma_1+V_i)\delta_a^2} = 1-3Q^2(x) + O\Big(\delta_a^4 + \sum_{l=1}^{2} v_a^{(l)}(\delta_a x + x_{a,i}^{(1)})\Big), \tag{5.33}$$

and

$$\frac{g_a(\delta_a x + x_{a,i}^{(1)})}{1+(\gamma_1+V_i)\delta_a^2} = -\frac{2}{ka_*}Q(x)\sum_{l=1}^{k}\int_{\mathbb{R}^N} Q^3(x)\bar{\xi}_{a,l}(x) + O\Big(\delta_a^4 + \sum_{l=1}^{2} v_a^{(l)}(\delta_a x + x_{a,i}^{(1)})\Big). \tag{5.34}$$

Proof. First, (5.33) can be deduced by (5.21) and (5.22) directly. Now we prove (5.34).

From (5.25) and (5.26), for $l=1,2$, we find

$$a\mu_a^{(l)}\delta_a^4 = \delta_a^2\int_{\mathbb{R}^N}\big(|\nabla u_a^{(l)}|^2 + V(x)(u_a^{(l)})^2\big) - \int_{\mathbb{R}^N}(u_a^{(l)})^4,$$

which gives

$$
\begin{aligned}
&\frac{a\delta_a^4(\mu_a^{(1)}-\mu_a^{(2)})}{\|u_a^{(1)}-u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}}\\
=&\mu_a^{(2)}\delta_a^2\int_{\mathbb{R}^N}(u_a^{(1)}+u_a^{(2)})\xi_a+\delta_a^2\int_{\mathbb{R}^N}\big(\nabla(u_a^{(1)}+u_a^{(2)})\cdot\nabla\xi_a+V(x)(u_a^{(1)}+u_a^{(2)})\xi_a\big)\\
&-\int_{\mathbb{R}^N}(u_a^{(1)}+u_a^{(2)})\big((u_a^{(1)})^2+(u_a^{(2)})^2\big)\xi_a\\
=&\big(\mu_a^{(2)}-\mu_a^{(1)}\big)\delta_a^2\int_{\mathbb{R}^N}u_a^{(1)}\xi_a-\int_{\mathbb{R}^N}(u_a^{(1)}+u_a^{(2)})u_a^{(1)}u_a^{(2)}\xi_a,
\end{aligned}
\tag{5.35}
$$

here we use the following identity:

$$
\int_{\mathbb{R}^N}\big(u_a^{(1)}+u_a^{(2)}\big)\xi_a=\frac{1}{\|u_a^{(1)}-u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}}\Big(\int_{\mathbb{R}^N}(u_a^{(1)})^2-\int_{\mathbb{R}^N}(u_a^{(2)})^2\Big)=0.
$$

Then from (5.21), (5.22), (5.28) and (5.35), we know

$$
\frac{\delta_a^2(\mu_a^{(1)}-\mu_a^{(2)})}{\|u_a^{(1)}-u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}}=-\frac{1}{ka_*\delta_a^N}\int_{\mathbb{R}^N}(u_a^{(1)}+u_a^{(2)})u_a^{(1)}u_a^{(2)}\xi_a+O\Big(\delta_a^4\Big). \tag{5.36}
$$

So we can find (5.34) by (5.36). □

Then from Lemma 5.5, we have the following result:

Lemma 5.6. *From $|\bar{\xi}_{a,i}|\le 1$, we suppose that $\bar{\xi}_{a,i}(x)\to\xi_i(x)$ in $C^1_{loc}(\mathbb{R}^N)$. Then $\xi_i(x)$ satisfies following system:*

$$
-\Delta\xi_i(x)+\big(1-3Q^2(x)\big)\xi_i(x)=-\frac{2}{ka_*}Q(x)\Big(\sum_{l=1}^k\int_{\mathbb{R}^N}Q^3(x)\xi_l(x)\Big),\ \textit{for } i=1,\cdots,k.
$$

To prove $\xi_i=0$, we write

$$
\bar{\xi}_{a,i}(x)=\sum_{j=0}^N\gamma_{a,i,j}\psi_j+\tilde{\xi}_{a,i}(x),\ \text{ in } H^1(\mathbb{R}^N), \tag{5.37}
$$

where $\psi_j(j=0,1,\cdots,N)$ are the functions in (A.3) and $\tilde{\xi}_{a,i}(x)\in\tilde{E}$ with

$$
\tilde{E}=\{u\in H^1(\mathbb{R}^N),\langle u,\psi_j\rangle=0,\ \text{for } j=0,1,\cdots,N\}.
$$

It is standard to prove the following result:

Lemma 5.7. *For any $u\in\tilde{E}$, there exists $\bar{\gamma}>0$ such that*

$$
\|\tilde{L}(u)\|\ge\bar{\gamma}\|u\|,
$$

where $\tilde{L}$ is defined by

$$
\tilde{L}(u):=-\Delta u(x)+u(x)-3Q^2(x)u(x)+\frac{2}{a_*}Q(x)\int_{\mathbb{R}^N}Q^3(x)u(x).
$$

Proposition 5.8. *Let $\tilde{\xi}_{a,i}(x)$ be as in (5.37), then*

$$
\|\tilde{\xi}_{a,i}\|=O(\delta_a^4),\ \textit{for } i=1,\cdots,k. \tag{5.38}
$$

Proof. First, Lemma 5.7 gives

$$\|\tilde{\xi}_{a,i}\| \leq C\|\tilde{L}(\tilde{\xi}_{a,i})\|. \tag{5.39}$$

On the other hand, from (5.32)–(5.37), we can prove

$$\tilde{L}(\tilde{\xi}_{a,i}) = O\big(\delta_a^4\big) + O\Big(\sum_{l=1}^{2} Q(x) v_a^{(l)}(\delta_a x + x_{a,i}^{(1)})\Big). \tag{5.40}$$

So from (5.28), (5.39) and (5.40), we prove (5.38).

□

Lemma 5.9. *For $N = 2, 3$, we have the following estimate on ξ_a:*

$$\begin{aligned}
&\delta_a^2 \sum_{i=1}^{k} \int_{B_d(x_{a,i}^{(1)})} \big(2V(x) + \langle \nabla V(x), x - x_{a,i}^{(1)}\rangle\big)(u_a^{(1)} + u_a^{(2)})\xi_a \\
=&2\big(\mu_a^{(2)} - \mu_a^{(1)}\big)\delta_a^2 \int_{\mathbb{R}^N} u_a^{(1)}\xi_a + \int_{\mathbb{R}^N} (u_a^{(1)} + u_a^{(2)})\big(u_a^{(1)} - u_a^{(2)}\big)^2\xi_a \\
&+ \big(1 - \frac{N}{2}\big) \sum_{i=1}^{k} \int_{B_d(x_{a,i}^{(1)})} (u_a^{(1)} + u_a^{(2)})\big((u_a^{(1)})^2 + (u_a^{(2)})^2\big)\xi_a + O\Big(e^{-\frac{\gamma}{\delta_a}}\Big).
\end{aligned} \tag{5.41}$$

Proof. Since $u_a^{(1)}$ and $u_a^{(2)}$ are two k-peak solutions of (5.25)–(5.26), then similar to (3.15), we have following local Pohozaev identities:

$$\begin{aligned}
&\int_{B_d(x_{a,i}^{(1)})} \delta_a^2\big(2V(x) + \langle \nabla V(x), x - x_{a,i}^{(1)}\rangle\big)(u_a^{(l)})^2 \\
&= \int_{B_d(x_{a,i}^{(1)})} \big[2\mu_a^{(l)}\delta_a^2(u_a^{(l)})^2 + (2 - \frac{N}{2})(u_a^{(l)})^4\big] + \delta_a^2 \int_{\partial B_d(x_{a,i}^{(1)})} W^{(l)}(x) d\sigma,
\end{aligned} \tag{5.42}$$

where

$$\begin{aligned}
W^{(l)}(x) =& - N\frac{\partial u_a^{(l)}}{\partial \nu_i}\langle x - x_{a,i}^{(1)}, \nabla u_a^{(l)}\rangle + \langle x - x_{a,i}^{(1)}, \nu_i\rangle|\nabla u_a^{(l)}|^2 \\
&+ \langle x - x_{a,i}^{(1)}, \nu_i\rangle\big[\big(V(x) - \mu_a^{(l)}\big)\big(u_a^{(l)}\big)^2 - \frac{1}{2\delta_a^2}\big(u_a^{(l)}\big)^4\big].
\end{aligned}$$

Then (5.42) implies

$$\begin{aligned}
&\int_{B_d(x_{a,i}^{(1)})} \delta_a^2\big(2V(x) + \langle \nabla V(x), x - x_{a,i}^{(1)}\rangle\big)(u_a^{(1)} + u_a^{(2)})\xi_a \\
=&\frac{2\delta_a^2(\mu_a^{(1)} - \mu_a^{(2)})}{\|u_a^{(1)} - u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}} \int_{B_d(x_{a,i}^{(1)})} \big(u_a^{(1)}\big)^2 - 2\mu_a^{(2)}\delta_a^2 \int_{B_d(x_{a,i}^{(1)})} (u_a^{(1)} + u_a^{(2)})\xi_a \\
&+ \big(2 - \frac{N}{2}\big) \int_{B_d(x_{a,i}^{(1)})} (u_a^{(1)} + u_a^{(2)})\big((u_a^{(1)})^2 + (u_a^{(2)})^2\big)\xi_a + J_{a,i},
\end{aligned} \tag{5.43}$$

where $J_{a,i} := \frac{\delta_a^2}{\|u_a^{(1)} - u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}} \int_{\partial B_d(x_{a,i}^{(1)})} \big(W^{(1)}(x) - W^{(2)}(x)\big) d\sigma.$

Next, we calculate the term $J_{a,i}$.

$$
\begin{aligned}
J_{a,i} =& -2\delta_a^2 \int_{\partial B_d(x_{a,i}^{(1)})} \big[\frac{\partial u_a^{(1)}}{\partial \nu_i}\langle x - x_{a,i}^{(1)}, \nabla \xi_a\rangle + \frac{\partial \xi_a}{\partial \nu_i}\langle x - x_{a,i}^{(1)}, \nabla u_a^{(1)}\rangle\big] \\
&+ 2\delta_a^2 \int_{\partial B_d(x_{a,i}^{(1)})} \langle x - x_{a,i}^{(1)}, \nu_i\rangle \big[\nabla(u_a^{(1)} + u_a^{(2)}) \cdot \nabla \xi_a + \big(V(x) - \mu_a^{(1)}\big)(u_a^{(1)} + u_a^{(2)})\xi_a\big] \\
&+ \frac{\delta_a^2(\mu_a^{(1)} - \mu_a^{(2)})}{\|u_a^{(1)} - u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}} \int_{\partial B_d(x_{a,i}^{(1)})} \langle x - x_{a,i}^{(1)}, \nabla u_a^{(1)}\rangle \big(u_a^{(2)}\big)^2 \\
&- \frac{1}{2}\int_{\partial B_d(x_{a,i}^{(1)})} \langle x - x_{a,i}^{(1)}, \nabla u_a^{(1)}\rangle (u_a^{(1)} + u_a^{(2)})\big((u_a^{(1)})^2 + (u_a^{(2)})^2\big)\xi_a \\
=& O\Big(e^{-\frac{\gamma}{\delta_a}}\Big).
\end{aligned}
$$

Summing (5.43) from $i = 1$ to $i = k$ and using (3.11), (5.31), we find

$$
\begin{aligned}
&\sum_{i=1}^{k} \int_{B_d(x_{a,i}^{(1)})} \delta_a^2\big(2V(x) + \langle \nabla V(x), x - x_{a,i}^{(1)}\rangle\big)(u_a^{(1)} + u_a^{(2)})\xi_a \\
=& \big(2 - \frac{N}{2}\big)\int_{\mathbb{R}^N} (u_a^{(1)} + u_a^{(2)})\big((u_a^{(1)})^2 + (u_a^{(2)})^2\big)\xi_a \\
&+ \frac{2\delta_a^2(\mu_a^{(1)} - \mu_a^{(2)})}{\|u_a^{(1)} - u_a^{(2)}\|_{L^\infty(\mathbb{R}^N)}} \int_{\mathbb{R}^N} \big(u_a^{(1)}\big)^2 + O\Big(e^{-\frac{\gamma}{\delta_a}}\Big).
\end{aligned} \tag{5.44}
$$

Then from (5.35) and (5.44), we deduce (5.41). □

Let $\gamma_{a,i,j}$ be as in (5.37), using $|\bar{\xi}_{a,i}| \leq 1$, we find

$$
\gamma_{a,i,j} = \frac{\big\langle \bar{\xi}_{a,i}, \varphi_j \big\rangle}{\|\varphi_j\|^2} = O\big(\|\bar{\xi}_{a,i}\|\big) = O(1), \;\; j = 0, 1, \cdots, N. \tag{5.45}
$$

Lemma 5.10. *For $N = 2, 3$, it holds*

$$
\gamma_{a,i,0} = o(1), \;\; \textit{for } i = 1, \cdots, k. \tag{5.46}
$$

Proof. First, we have

$$
u_a^{(2)}(\delta_a x + x_{a,i}^{(1)}) = Q(x) + O\Big(\frac{|x_{a,i}^{(1)} - x_{a,i}^{(2)}|}{\delta_a}|\nabla Q(x)|\Big) + v_a^{(2)}(\delta_a x + x_{a,i}^{(1)}). \tag{5.47}
$$

Then from (5.22), (5.28), (5.30) and (5.47), we know

$$
\begin{aligned}
&\int_{B_d(x_{a,i}^{(1)})} \big(2V(x) + \langle \nabla V(x), x - x_{a,i}^{(1)}\rangle\big)(u_a^{(1)} + u_a^{(2)})\xi_a \\
=& 2\int_{B_d(x_{a,i}^{(1)})} \big(2(V(x) - V_i) + \langle \nabla V(x), x - x_{a,i}^{(1)}\rangle\big) Q_{\delta_a, x_{a,i}^{(1)}}\xi_a \\
&+ 4V_i \int_{B_d(x_{a,i}^{(1)})} (u_a^{(1)} + u_a^{(2)})\xi_a + O\big(\delta_a^{N+4}\big).
\end{aligned} \tag{5.48}
$$

Also, from (5.22), (5.37) and (5.38), we find

$$\begin{aligned}&\int_{B_d(x^{(1)}_{a,i})}\big[V(x)-V_i\big]Q_{\delta_a,x^{(1)}_{a,i}}\xi_a\\ &\qquad=\delta_a^N\int_{\mathbb{R}^N}\big[V(\delta_a x+x^{(1)}_{a,i})-V_i\big]Q(x)\Big(\sum_{j=0}^N\gamma_{a,i,j}\psi_j\Big)+O\big(\delta_a^{N+4}\big)\\ &\qquad=-\frac{B}{2}\Delta V(b_i)\gamma_{a,i,0}\delta_a^{N+2}+O\big(\delta_a^{N+3}\big),\end{aligned}\tag{5.49}$$

where B is the constant in (4.12). Similar to the estimate of (5.49), we can find

$$\int_{B_d(x^{(1)}_{a,i})}\langle\nabla V(x),x-x^{(1)}_{a,i}\rangle Q_{\delta_a,x^{(1)}_{a,i}}\xi_a=-\frac{B}{2}\Delta V(b_i)\gamma_{a,i,0}\delta_a^{N+2}+O(\delta_a^{N+3}).\tag{5.50}$$

So from (5.49) and (5.50), we get

$$\text{LHS of (5.41)}=-\frac{3B}{2}\Delta V(b_i)\gamma_{a,i,0}\delta_a^{N+4}+O(\delta_a^{N+5})+4V_i\delta_a^2\int_{B_d(x^{(1)}_{a,i})}(u_a^{(1)}+u_a^{(2)})\xi_a.\tag{5.51}$$

Next we know

$$\begin{aligned}&\int_{B_d(x^{(1)}_{a,i})}(u_a^{(1)}+u_a^{(2)})\big((u_a^{(1)})^2+(u_a^{(2)})^2\big)\xi_a\\ &\qquad=\Big(\frac{4-N}{4}+o(1)\Big)\gamma_{a,i,0}\delta_a^N\int_{\mathbb{R}^N}Q^4=\big(4a_*+o(1)\big)\gamma_{a,i,0}\delta_a^N.\end{aligned}\tag{5.52}$$

Also from (5.47), we get

$$\begin{aligned}&\int_{B_d(x^{(1)}_{a,i})}(u_a^{(1)}+u_a^{(2)})\xi_a\\ &\qquad=2\gamma_{a,i,0}\delta_a^N\int_{\mathbb{R}^N}Q(Q+x\cdot\nabla Q)+O\big(|x_a^{(1)}-x_a^{(2)}|\delta_a^{N-1}+\delta_a^{\frac{N}{2}}\|v_a^{(2)}\|_{\delta_a}\big)\\ &\qquad=\big(2-N\big)a_*\gamma_{a,i,0}\delta_a^N+O\big(\delta_a^{N+3}\big),\end{aligned}$$

which, together with (5.52), gives

$$\text{LHS of (5.41)}=2\big(2-N\big)\big(a_*+o(1)\big)\gamma_{a,i,0}\delta_a^N-\frac{3B}{2}\Delta V(b_i)\gamma_{a,i,0}\delta_a^{N+4}+O(\delta_a^{N+5}).\tag{5.53}$$

Also by (5.21), (5.22) and (5.28), we find

$$(\mu_a^{(2)}-\mu_a^{(1)})\int_{\mathbb{R}^N}u_a^{(2)}\xi_a=O\big(\delta_a^{N+4}\big).\tag{5.54}$$

On the other hand, by (5.28), (5.30), (5.31), (5.37), (5.38), (5.47) and (5.48), we find

$$\begin{aligned}&\int_{\mathbb{R}^N}(u_a^{(1)}+u_a^{(2)})(u_a^{(1)}-u_a^{(2)})^2\xi_a\\ &\qquad=\delta_a^N O\big(\delta_a^{-2}|x_a^{(1)}-x_a^{(2)}|^2\big)+O\big(\|v_a^{(1)}-v_a^{(2)}\|^2_{\delta_a}\big)=O\big(\delta_a^{N+6}\big).\end{aligned}\tag{5.55}$$

Then (5.41), (5.53)–(5.55) and Lemma A.1 imply

$$\Big(4\big(2-N\big)\big(a_*+o(1)\big)-3B\delta_a^4\big(\sum_{l=1}^{k}\Delta V(b_l)\big)\Big)\gamma_{a,i,0}=O(\delta_a^5),\ \text{for } i=1,\cdots,k,$$

which gives (5.46). □

Proposition 5.11. *It holds*

$$\gamma_{a,i,j}=o(1),\quad i=1,\cdots,k,\ j=0,1,\cdots,N. \tag{5.56}$$

Proof. **Step 1:** To prove $\gamma_{a,i,N}=O(\delta_a)$ for $i=1,\cdots,k$.

Using (4.1), we deduce

$$\int_{B_d(x^{(1)}_{a,i})}\frac{\partial V(x)}{\partial\nu_{a,i}}B_a(x)\xi_a=O\big(e^{-\frac{\gamma}{\delta_a}}\big), \tag{5.57}$$

where $\nu_{a,i}$ is the outward unit vector of $\partial B_d(x^{(1)}_{a,i})$ at x, $B_a(x)=\sum\limits_{l=1}^{2}u_a^{(l)}(x)$.

On the other hand, by (5.22), we have

$$B_a(x)=\Big(2+O(\delta_a^2)\Big)\sum_{i=1}^{k}Q_{\delta_a,x^{(1)}_{a,i}}(x)+O\Big(\sum_{l=1}^{2}|v_a^{(l)}(x)|\Big). \tag{5.58}$$

Also, from (5.21), we find

$$\frac{\partial V(x^{(1)}_{a,i})}{\partial\nu_{a,i}}=\frac{\partial V(x^{(1)}_{a,i})}{\partial\nu_{a,i}}-\frac{\partial V(\bar{x}^{(1)}_{a,i})}{\partial\nu_{a,i}}=O\Big(\big|x^{(1)}_{a,i}-\bar{x}^{(1)}_{a,i}\big|\Big)=O(\delta_a^2),$$

and

$$\frac{\partial^2 V(x^{(1)}_{a,i})}{\partial\nu_{a,i}\partial\tau_{a,i,j}}=\frac{\partial^2 V(x^{(1)}_{a,i})}{\partial\nu_{a,i}\partial\tau_{a,i,j}}-\frac{\partial^2 V(\bar{x}^{(1)}_{a,i})}{\partial\nu_{a,i}\partial\tau_{a,i,j}}=O\Big(\big|x^{(1)}_{a,i}-\bar{x}^{(1)}_{a,i}\big|\Big)=O(\delta_a^2),\ \text{for } j=1,\cdots,N-1.$$

From (3.11), (4.1) and (5.58), we get

$$\begin{aligned}&\int_{B_d(x^{(1)}_{a,i})}\frac{\partial V(x)}{\partial\nu_{a,i}}B_a(x)\xi_a\\&=\int_{\mathbb{R}^N}\frac{\partial V(x^{(1)}_{a,i})}{\partial\nu_{a,i}}B_a(x)\xi_a+\int_{\mathbb{R}^N}\Big\langle\nabla\frac{\partial V(x^{(1)}_{a,i})}{\partial\nu_{a,i}},x-x^{(1)}_{a,i}\Big\rangle B_a(x)\xi_a+O\big(\delta_a^{N+2}\big)\\&=-\frac{\partial^2 V(x^{(1)}_{a,i})}{\partial\nu_{a,i}^2}a_*\gamma_{a,i,N}\delta_a^{N+1}+O\big(\delta_a^{N+2}\big).\end{aligned} \tag{5.59}$$

Then (5.57) and (5.59) imply $\gamma_{a,i,N}=O(\delta_a)$.

Step 2: To prove $\gamma_{a,i,j}=o(1)$ for $i=1,\cdots,k$ and $j=1,\cdots,N-1$.

Similar to (5.57), we have

$$\int_{B_d(x^{(1)}_{a,i})}\frac{\partial V(y)}{\partial\tau_{a,i,j}}B_a(y)\xi_a=O\big(e^{-\frac{\gamma}{\delta_a}}\big),\ \text{for } i=1,\cdots,k,\ j=1,\cdots,N-1. \tag{5.60}$$

Using suitable rotation, we assume that $\tau_{a,i,1} = (1,0,\cdots,0),\cdots,\ \tau_{a,i,N-1} = (0,\cdots,0,1,0)$ and $\nu_{a,i} = (0,\cdots,0,1)$. Under the condition (V), we know

$$\begin{aligned}\frac{\partial V(\delta_a y + x_{a,i}^{(1)})}{\partial \tau_{a,i,j}} =&\delta_a \sum_{l=1}^{N} \frac{\partial^2 V(x_{a,i}^{(1)})}{\partial y_l \partial \tau_{a,i,j}} y_l + \frac{\delta_a^2}{2} \sum_{m=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 V(x_{a,i}^{(1)})}{\partial y_m \partial y_l \partial \tau_{a,i,j}} y_m y_l \\ &+ \frac{\delta_a^3}{6} \sum_{s=1}^{N}\sum_{m=1}^{N}\sum_{l=1}^{N} \frac{\partial^4 V(x_{a,i}^{(1)})}{\partial y_s \partial y_l \partial y_m \partial \tau_{a,i,j}} y_s y_l y_m + o\big(\delta_a^3 |y|^3\big),\ \text{in } B_{d\delta_a^{-1}}(0).\end{aligned} \tag{5.61}$$

By (1.11), (5.8), (5.28), (5.37), (5.58) and the symmetry of $\varphi_j(x)$, we find, for $j = 1,\cdots,N-1$,

$$\begin{aligned}&\sum_{m=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 V(x_{a,i}^{(1)})}{\partial y_m \partial y_l \partial \tau_{a,i,j}} \int_{B_{d\delta_a^{-1}}(0)} B_a(\delta_a y + x_{a,i}^{(1)}) \bar{\xi}_{a,i} y_m y_l \\ &\quad = 2\sum_{m=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 V(x_{a,i}^{(1)})}{\partial y_m \partial y_l \partial \tau_{a,i,j}} \int_{B_{d\delta_a^{-1}}(0)} Q\big(\sqrt{1+V_i\delta_a^2}y\big)\bar{\xi}_{a,i} y_m y_l + O(\delta_a^2) \\ &\quad = B\gamma_{a,i,0} \frac{\partial \Delta V(x_{a,i}^{(1)})}{\partial \tau_{a,i,j}} + O(\delta_a^2) = O\big(|x_{a,i}^{(1)} - b_i|\big) + O(\delta_a^2) = O(\delta_a^2).\end{aligned} \tag{5.62}$$

Also from (5.56) and (5.58), we get

$$\begin{aligned}&\sum_{s=1}^{N}\sum_{m=1}^{N}\sum_{l=1}^{N} \frac{\partial^4 V(x_{a,i}^{(1)})}{\partial y_s \partial y_l \partial y_m \partial \tau_{a,i,j}} \int_{B_{d\delta_a^{-1}}(0)} B_a(\delta_a y + x_{a,i}^{(1)}) \bar{\xi}_{a,i} y_s y_l y_m \\ =&2\sum_{s=1}^{N}\sum_{m=1}^{N}\sum_{l=1}^{N} \frac{\partial^4 V(x_{a,i}^{(1)})}{\partial y_s \partial y_l \partial y_m \partial \tau_{a,i,j}} \int_{B_{d\delta_a^{-1}}(0)} Q(y)\Big(\sum_{q=1}^{N-1}\gamma_{a,i,q}\Big)\varphi_q(y) y_s y_l y_m + o(1) \\ =&2\sum_{q=1}^{N-1}\gamma_{a,i,q}\int_{B_{d\delta_a^{-1}}(0)} Q(y)\varphi_q(y) y_q \Big[\frac{\partial^4 V(x_{a,i}^{(1)})}{\partial y_q^3 \partial \tau_{a,i,j}} y_q^2 + 3\sum_{l=1,l\neq q}^{N} \frac{\partial^4 V(x_{a,i}^{(1)})}{\partial y_q \partial y_l^2 \partial \tau_{a,i,j}} y_l^2\Big] + o(1) \\ =&-3B\Big(\sum_{q=1}^{N-1} \frac{\partial^2 \Delta V(x_{a,i}^{(1)})}{\partial \tau_{a,i,q}\partial \tau_{a,i,j}}\gamma_{a,i,q}\Big) + o(1) \\ =&-3B\Big(\sum_{q=1}^{N-1} \frac{\partial^2 \Delta V(b_i)}{\partial \tau_{a,i,q}\partial \tau_{a,i,j}}\gamma_{a,i,q}\Big) + o(1).\end{aligned} \tag{5.63}$$

By (B.2), we estimate

$$\frac{\partial^2 V(x_{a,i}^{(1)})}{\partial y_l \partial \tau_{a,i,j}} = -\frac{\partial V(x_{a,i}^{(1)})}{\partial \nu_{a,i}} \kappa_{i,l}(x_{a,i}^{(1)})\delta_{lj}, \quad l,j = 1,\cdots,N-1.$$

Since $\frac{\partial V(\bar{x}_{a,i}^{(1)})}{\partial \nu_{a,i}} = 0$, from (5.8), we find

$$\begin{aligned}\frac{\partial^2 V(x_{a,i}^{(1)})}{\partial y_l \partial \tau_{a,i,j}} =& -\Big(\frac{\partial V(x_{a,i}^{(1)})}{\partial \nu_{a,i}} - \frac{\partial V(\bar{x}_{a,i}^{(1)})}{\partial \nu_{a,i}}\Big)\kappa_{i,l}(x_{a,i}^{(1)})\delta_{lj} \\ =& -\frac{\partial^2 V(\bar{x}_{a,i}^{(1)})}{\partial \nu_{a,i}^2}(x_{a,i}^{(1)} - \bar{x}_{a,i}^{(1)})\cdot \nu_{a,i}\kappa_{i,l}(x_{a,i}^{(1)})\delta_{lj} + o(\delta_a^2) \\ =& -\frac{\partial^2 V(b_i)}{\partial \nu_i^2}(x_{a,i}^{(1)} - \bar{x}_{a,i}^{(1)})\cdot \nu_{a,i}\kappa_{i,l}(b_i)\delta_{lj} + o(\delta_a^2) \\ =& \frac{B}{2a_*}\frac{\partial \Delta V(b_i)}{\partial \nu_i}\delta_a^2\kappa_{i,l}(b_i)\delta_{lj} + o(\delta_a^2).\end{aligned} \tag{5.64}$$

Therefore from (5.28), (5.37), (5.58) and (5.64), we get

$$\begin{aligned}&\sum_{l=1}^N \frac{\partial^2 V(x_{a,i}^{(1)})}{\partial y_l \partial \tau_{a,i,j}}\int_{B_{d\delta_a^{-1}}(0)} B_a(\delta_a y + x_{a,i}^{(1)})\bar{\xi}_{a,i} y_l \\ =& \frac{B}{2a_*}\frac{\partial \Delta V(b_i)}{\partial \nu_i}\delta_a^2\kappa_{i,j}(b_i)\int_{B_{d\delta_a^{-1}}(0)} B_a(\delta_a y + x_{a,i}^{(1)})\bar{\xi}_{a,i} y_j + o(\delta_a^2) \\ =& \frac{B}{a_*}\frac{\partial \Delta V(b_i)}{\partial \nu_i}\delta_a^2\kappa_{i,j}(b_i)\gamma_{a,i,j}\int_{\mathbb{R}^N} Q(y)\frac{Q(y)}{\partial y_j} y_j + o(\delta_a^2) \\ =& -\frac{B}{2}\frac{\partial \Delta V(b_i)}{\partial \nu_i}\delta_a^2\kappa_{i,j}(b_i)\gamma_{a,i,j} + o(\delta_a^2).\end{aligned} \tag{5.65}$$

Combining (5.61)–(5.65), we obtain

$$\begin{aligned}&\int_{B_d(x_{a,i}^{(1)})}\frac{\partial V(y)}{\partial \tau_{a,i,j}}B_a(y)\xi_a \\ &\quad = -\frac{B}{2}\frac{\partial \Delta V(b_i)}{\partial \nu_i}\delta_a^{N+3}\kappa_{i,j}(b_i)\gamma_{a,i,j} - \frac{B}{2}\Big(\sum_{l=1}^{N-1}\frac{\partial^2 \Delta V(b_i)}{\partial \tau_{i,l}\partial \tau_{i,j}}\gamma_{a,i,j}\Big)\delta_a^{N+3} + o(\delta_a^{N+3}).\end{aligned} \tag{5.66}$$

From (5.60) and (5.66), we find

$$\frac{\partial \Delta V(b_i)}{\partial \nu_i}\kappa_{i,j}(b_i)\gamma_{a,i,j} + \Big(\sum_{l=1}^{N-1}\frac{\partial^2 \Delta V(b_i)}{\partial \tau_{i,l}\partial \tau_{i,j}}\gamma_{a,i,l}\Big) = o(1),$$

which implies $\gamma_{a,i,j} = o(1)$ for $i = 1, \cdots, k$ and $j = 1, \cdots, N-1$.

□

Proof of Theorem 1.4: First, for large fixed R, (5.33) and (5.34) give

$$C_a(x) \geq \frac{1}{2}, \quad |g_a(x)| \leq C\sum_{i=1}^k e^{\frac{|x - x_{a,i}^{(1)}|}{\delta_a}}, \quad x \in \mathbb{R}^N \setminus \bigcup_{i=1}^k B_{R\delta_a}(x_{a,i}^{(1)}).$$

Using the comparison principle, we get

$$\xi_a(x) = o(1), \text{ in } \mathbb{R}^N \setminus \bigcup_{i=1}^{k} B_{R\delta_a}(x_{a,i}^{(1)}).$$

On the other hand, it follows from (5.38), (5.46) and (5.56) that

$$\xi_a(x) = o(1), \text{ in } \bigcup_{i=1}^{k} B_{R\delta_a}(x_{a,i}^{(1)}).$$

This is in contradiction with $\|\xi_a\|_{L^\infty(\mathbb{R}^N)} = 1$. So $u_a^{(1)}(x) \equiv u_a^{(2)}(x)$ for $a \to ka_*$ in $N = 2$ or $a \searrow 0$ in $N = 3$. □

Appendix

A. The kernel of a linear operator

Lemma A.1. *Let $\xi_0 := (\xi_1, \cdots, \xi_k)$ be a bounded solution of following system:*

$$-\Delta\xi_i(x) + \big(1 - 3Q^2(x)\big)\xi_i(x) = -\frac{2}{ka_*}Q(x)\Big(\sum_{l=1}^{k}\int_{\mathbb{R}^N} Q^3(x)\xi_l(x)\Big), \text{ for } i = 1, \cdots, k. \tag{A.1}$$

Then for $N = 2, 3$, it holds

$$\xi_i(x) = \sum_{j=0}^{N} \gamma_{i,j}\psi_j, \tag{A.2}$$

where $\gamma_{i,j}$ are some constants,

$$\psi_0 = Q + x \cdot \nabla Q, \ \psi_j = \frac{\partial Q}{\partial x_j}, \text{ for } j = 1, \cdots, N. \tag{A.3}$$

Moreover, $\gamma_{i,0} = \gamma_{l,0}$ for all $i, l = 1, \cdots, k$.

Proof. Noting that $Q(x)$ is a radial function, using the technique of the separation of variables, we can prove

$$\xi_i(x) = \sum_{j=1}^{N} \gamma_{i,j}\psi_j + \xi_{i,0},$$

where $\gamma_{i,j}$ is some constant, and $\xi_{i,0}$ is a radial function, satisfying

$$-\Delta\xi_{i,0}(x) + \big(1 - 3Q^2(x)\big)\xi_{i,0}(x) = -\frac{2}{ka_*}Q(x)\Big(\sum_{l=1}^{k}\int_{\mathbb{R}^N} Q^3(x)\xi_l(x)\Big).$$

We set $\bar{L}(u) := -\Delta u(x) + \big(1 - 3Q^2(x)\big)u(x)$. Then

$$\bar{L}\psi_0 = -2Q.$$

Since $\bar{L}$ has no non-trivial bounded radially symmetric kernel, it holds

$$\xi_{i,0} = \gamma_{i,0}\psi_0.$$

Using (3.12), we find that $\gamma_{i,0}$ satisfies

$$-2Q(x)\gamma_{i,0} = -\frac{2}{ka_*}Q(x)\frac{4-N}{4}\int_{\mathbb{R}^N} Q^4 = -\frac{2}{k}Q(x)\sum_{l=1}^{k}\gamma_{l,0},$$

which gives $\gamma_{i,0} = \gamma_{l,0}$ for all $i, l = 1, \cdots, k$. □

B. Calculations involving curvatures

Now let $\Gamma \in C^2$ be a closed hypersurface in $\mathbb{R}^N$. For $y \in \Gamma$, let $\nu(y)$ and $T(y)$ denote respectively the outward unit normal to Γ at y and the tangent hyperplane to Γ at y. The curvatures of Γ at a fixed point $y_0 \in \Gamma$ are determined as follows. By a rotation of coordinates, we can assume that $y_0 = 0$ and $\nu(0)$ is the x_N-direction, and x_j-direction is the j-th principal direction.

In some neighborhood $\mathcal{N} = \mathcal{N}(0)$ of 0, we have

$$\Gamma = \big\{x : x_N = \varphi(x')\big\},$$

where $x' = (x_1, \cdots, x_{N-1})$,

$$\varphi(x') = \frac{1}{2}\sum_{j=1}^{N-1}\kappa_j x_j^2 + O(|x'|^3),$$

where κ_j, is the j-th principal curvature of Γ at 0. The Hessian matrix $[D^2\varphi(0)]$ is given by

$$[D^2\varphi(0)] = diag[\kappa_1, \cdots, \kappa_{N-1}].$$

Suppose that W is a smooth function, such that $W(x) = a$ for all $x \in \Gamma$.

Lemma B.1. *We have*

$$\frac{\partial W(x)}{\partial x_l}\Big|_{x=0} = 0, \quad l = 1, \cdots, N-1, \tag{B.1}$$

$$\frac{\partial^2 W(x)}{\partial x_m \partial x_l}\Big|_{x=0} = -\frac{\partial W(x)}{\partial x_N}\Big|_{x=0}\kappa_i\delta_{ml}, \text{ for } m, l = 1, \cdots, N-1, \tag{B.2}$$

where $\kappa_1, \cdots, \kappa_{N-1}$, are the principal curvatures of Γ at 0.

Proof. First, we have $W\big(x', \varphi(x')\big) = 0$. And then we find

$$\frac{\partial W\big(x', \varphi(x')\big)}{\partial x_m} + \frac{\partial W\big(x', \varphi(x')\big)}{\partial x_N}\frac{\partial \varphi(x')}{\partial x_m} = 0, \text{ for } m = 1, \cdots, N-1. \tag{B.3}$$

Letting $x' = 0$ in (B.3), we obtain (B.1).

Differentiating (B.3) with respect to x_l for $l = 1, \cdots, N-1$, we get

$$\begin{aligned}\frac{\partial^2 W\big(x', \varphi(x')\big)}{\partial x_m \partial x_l} + \frac{\partial^2 W\big(x', \varphi(x')\big)}{\partial x_m \partial x_N}\frac{\partial \varphi(x')}{\partial x_l} + \frac{\partial W\big(x', \varphi(x')\big)}{\partial x_N}\frac{\partial^2 \varphi(x')}{\partial x_m x_l} \\ + \Big(\frac{\partial^2 W\big(x', \varphi(x')\big)}{\partial x_N \partial x_l} + \frac{\partial^2 W\big(x', \varphi(x')\big)}{\partial x_N \partial x_N}\frac{\partial \varphi(x')}{\partial x_l}\Big)\frac{\partial \varphi(x')}{\partial x_m} = 0.\end{aligned} \tag{B.4}$$

Let $x' = 0$ in (B.4), then we get (B.2). □

C. An example

In this section, we use the above results to the following potential $V(x)$. Let

$$F_1(x)=\sum_{j=1}^{N}\frac{x_j^2}{a_j^2}-1,\ F_2(x)=\sum_{j=1}^{N}\big(\frac{x_j}{a_j}-3\big)^2-1,$$

where $a_j>0$, $a_j\neq a_l$ for $j\neq l$. Let Γ_i is defined by $F_i(x)=0$ with $i=1,2$. Take

$$V(x)=\begin{cases}F_1^2+1, \text{ in } W_1,\\ F_2^2+1, \text{ in } W_2,\\ \text{else}, \text{ in } \mathbb{R}^N\backslash\bigcup_{i=1}^2 W_i.\end{cases}$$

and

$$W_i=\Big\{x\in\mathbb{R}^N;|F_i|\le\delta_0\Big\},\ \text{with some small fixed } \delta_0>0 \text{ and } i=1,2.$$

Lemma C.1. *All critical points ΔV on Γ_1 are $(\pm a_1,0,\cdots,0),\cdots,(0,\cdots,0,\pm a_N)$.*

Proof. First, we find

$$\nabla V(x)=2F\nabla F,\ \Delta V=2F\Delta F+2|\nabla F|^2=\Big(\sum_{l=1}^{N}\frac{x_l^2}{a_l^2}-1\Big)\sum_{l=1}^{N}\frac{4}{a_i^2}+\sum_{l=1}^{N}\frac{8x_l^2}{a_l^4}.$$

To find a critical point ΔV on Γ, we need to study the following equation

$$\nabla(\Delta V)=\lambda\nabla F,$$

for some unknown constant λ. That is,

$$\frac{8x_l}{a_l^2}\sum_{k=1}^{N}\frac{1}{a_k^2}+\frac{16x_i}{a_l^4}=\frac{2\lambda x_i}{a_l^2},\quad l=1,\cdots,N.$$

Thus, either $x_l=0$, or $\lambda=4\sum_{k=1}^{N}\frac{1}{a_k^2}+\frac{8}{a_l^2}$. If $\lambda=4\sum_{k=1}^{N}\frac{1}{a_k^2}+\frac{8}{a_l^2}$, then $x_j=0$ for all $j\neq l$. This shows that all critical points ΔV on Γ are $(\pm a_1,0,\cdots,0),\cdots,(0,\cdots,0,\pm a_N)$. □

Without loss of generality, we consider the point $b_1=(0,\cdots,0,a_N)$. In this case, τ_j is the x_j direction, $j=1,\cdots,N-1$, and ν is the x_N direction.

Lemma C.2. *If $a_l\neq a_j$ for $l\neq j$. Thus, b_1 is non-degenerate on Γ_1.*

Proof. We have

$$\frac{\partial^2 V(b_1)}{\partial x_N^2}=\frac{8}{a_N^2}>0,\ \frac{\partial\Delta V(b_1)}{\partial x_N}=\frac{8}{a_N}\sum_{l=1}^{N}\frac{1}{a_l^2}+\frac{16}{a_N^3}>0.$$

On Γ_1, it holds

$$\Delta V(x)=\sum_{l=1}^{N}\frac{8x_l^2}{a_l^4}=8\Big(\sum_{l=1}^{N-1}\frac{x_l^2}{a_l^4}+\frac{1}{a_N^2}\Big(1-\sum_{l=1}^{N-1}\frac{x_l^2}{a_l^2}\Big)\Big).$$

This gives

$$\Big(\frac{\partial^2\Delta V(b_1)}{\partial x_l x_j}\Big)_{1\le l,j\le N-1}=\ diag\ \Big(\frac{16}{a_1^2}\Big(\frac{1}{a_1^2}-\frac{1}{a_N^2}\Big),\cdots,\frac{16}{a_{N-1}^2}\Big(\frac{1}{a_{N-1}^2}-\frac{1}{a_N^2}\Big)\Big),$$

which is non-singular since $a_l \neq a_j$ for $l \neq j$. Thus, b_1 is also non-degenerate on Γ_1. □

Lemma C.3. *The matrix*

$$\Big(\frac{\partial^2 \Delta V(b_1)}{\partial x_i x_j}\Big)_{1\le i,j\le N-1} + \frac{\partial \Delta V(b_1)}{\partial x_N} diag(\kappa_1, \cdots, \kappa_{N-1}) \tag{C.1}$$

is non singular if one of the following conditions holds

(1) $a_N < a_l$, $l = 1, \cdots, N-1$.

(2) $a_N > a_l$, $l = 1, \cdots, N-1$ *and all the* a_i *are close to a constant.*

Proof. Near $b_1 = (0, \cdots, 0, a_N)$, Γ is given by

$$x_N = a_N \sqrt{1 - \sum_{l=1}^{N-1} \frac{x_l^2}{a_l^2}} = a_N - \frac{1}{2}\sum_{l=1}^{N-1} \frac{a_N x_l^2}{a_l^2} + O(|x'|^3).$$

Thus, $\kappa_l = -\frac{a_N}{a_l^2}, \quad l = 1, \cdots, N-1$. So we have

$$\frac{\partial \Delta V(b_1)}{\partial x_N}\kappa_j(b_1) = -\Big(\frac{8}{a_j^2}\sum_{l=1}^{N}\frac{1}{a_l^2} + \frac{16}{a_j^2 a_N^2}\Big).$$

If x_0 is a maximum point of ΔV on Γ, that is $a_N < a_l$, $l = 1, \cdots, N-1$, then $\Big(\frac{\partial^2 \Delta V(x_0)}{\partial x_l x_j}\Big)_{1\le l,j\le N-1}$ is negative. Thus, (C.1) is also a negative matrix. On the other hand, if b_1 is a minimum point of ΔV on Γ_1 and all the a_j are close to a constant, that is $a_N > a_l$, $l = 1, \cdots, N-1$, then (C.1) is negative. □

Finally, for $b_2 = (0, \cdots, 0, 4a_N) \in \Gamma_2$, we have the similar results.

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