ON INTERACTING BUMPS OF SEMI-CLASSICAL STATES OF NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We study concentrated positive bound states of the following nonlinear Schrödinger equation:

$$h^2\Delta u - V(x)u + u^p = 0, \quad u > 0, \quad x \in R^N$$

, where p is subcritical. We prove that, at a local maximum point x_0 of the potential function $V(x)$ and for arbitrary positive integer $K(K > 1)$, there always exist solutions with K interacting bumps concentrating near x_0. We also prove that at a nondegenerate local minimum point of $V(x)$ such solutions do not exist.

1. Introduction. Of concern are standing wave solutions of the following nonlinear Schrödinger equation:

$$ih\frac{\partial\psi}{\partial t} = \frac{-h^2}{2m}\triangle\psi + V(x)\psi - \gamma\,|\psi|^{p-1}\,\psi \qquad \text{with } x \in R^N, \tag{1.1}$$

i.e., solutions of the form $\psi\,(x,t) = \exp\,(i\,E\,t/h)\,u(x)$, where h, m, γ and p are positive constants, $p > 1, E \in R$, V is real and belongs to $C^2\left(R^N\right)$ and u is real. Assuming without loss of generality that $2m = 1, \gamma = 1$ and $E = 0$, it is easy to see that u satisfies

$$h^2\triangle u - V(x)u + |u|^{p-1}\,u = 0,\ x \in R^N. \tag{1.2}$$

In this paper, we always assume that

$$\inf_{x\in R^N} V(x) > 0, V(x) \in C^2\left(R^N\right), 1 < p < (\frac{N+2}{N-2})_+, \tag{1.3}$$

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where $(\frac{N+2}{N-2})_+ = \frac{N+2}{N-2}$ if $N \geq 3$; $= +\infty$ if $N = 1, 2$.

Floer and Weinstein in [8] proved for small $h > 0$ (and for $p = 3, N = 1$) the existence of a single-bump solution of (1.2) concentrating at each given nondegenerate critical point of the potential function V, under the condition that V is bounded. In [19] and [20], Oh generalized this result and obtained for small $h > 0$ the existence of multi-bump solutions with u in (1.2) being positive and concentrating at given finite collection of nondegenerate critical points of V, under the condition $N \geq 1$, $1 < p < (\frac{N+2}{N-2})_+$ and $V \in (V)_a$ (namely, either $V \equiv a$ or $V(x) > a$ and $(V-a)^{-\frac{1}{2}} \in Lip(R^N)$).

The existence of solutions of (1.2) and its various generalizations has long been studied extensively (mostly by variational methods). The interested reader may consult, in addition to the papers mentioned below, the survey articles [15] and [17] and references therein. Most of the results provide existence of solutions for arbitrary $h > 0$. Several papers deal with existence of "ground states," i.e., in case of (1.2), solutions with least "energy,"

$$\frac{1}{2}\int_{R^N} (h^2 |\nabla u|^2 + V u^2)\, dx - \frac{1}{p+1}\int_{R^N} |u|^{p+1}\, dx \tag{1.4}$$

among all nontrivial $H^1(R^N)$ solutions of (1.2).

In [21], Rabinowitz showed that (1.2) has a positive ground state for "every $h > 0$" if $\limsup_{|x|\to\infty} V(x) = \sup_{x\in R^N} V(x)$ or if $\liminf_{|x|\to\infty} V(x) > \inf_{x\in R^N} V(x)$. See [2] and [3] for more results on existence and [23] on asymptotic behavior of ground state solutions.

There are many interesting results concerning higher energy solutions. Del Pino and Felmer in [4] studied the case when $V(x)$ has a local minimum point (may be degenerate) and constructed single-bump positive solutions. Both Del Pino, Felmer [5] and Gui [9] glued the single bump positive solutions and obtained multi-bump positive solutions at *separate* local minimum points of V. Del Pino and Felmer in [6] were able to construct single-bump positive solutions at *any topologically nontrivial* critical points of $V(x)$. Related results are obtained by Ambrosetti, Badiale and Cingolani [1]; Li [14]; and Lu and the second author [16]. As far as we know, N. Thandi [22] obtained the existence of infinite-bump solutions under some further hypothesis on the potential function V.

In all the above papers multi-bump solutions are obtained at *"separate"* local maximum or local minimum points of V. These bumps are well *separated,* and hence the interactions of these bumps are neglected. (Here "interaction" means the effect of one bump on other bumps. Mathematically, it can be computed as in Lemma 2.4 below.) The main purpose of this paper is to study the effect of such interactions. We show that at a local maximum point of $V(x)$ the interactions can contribute to the existence of multi-bump solutions while at a nondegenerate local minimum point of $V(x)$ multi-bump solutions do not exist.

Our first result of this paper is the following.

Theorem 1.1. *Assume that $V(x)$ and p satisfy assumption* (1.3). *Let P_0 be a local maximum point of the potential $V(x)$; i.e., there exists a bounded open set Γ such that*

$$P_0 \in \Gamma, V(P_0) = \max_{x\in\Gamma} V(x) > V(P), \forall P \in \Gamma\backslash\{P_0\}. \tag{1.5}$$

Then for any positive integer $K \in \mathbb{Z}$, there exists $h_0 > 0$ such that for any $h < h_0$ there exists a positive solution u_h of (1.2) *with the following properties:*

(1) *The solution u_h has exactly K local maximum points $Q_1^h, \ldots, Q_K^h$ and $Q_i^h \to P_0$ as $h \to 0$. Moreover,*

$$\frac{|Q_i^h - Q_j^h|}{h} \geq V(P_0)^{-\frac{1}{2}} \log\frac{C}{h} \to \infty,\ i \neq j,\ i,j = 1,\ldots,K$$

for some $C > 0$, as $h \to 0$.

(2) *Thesolution$u_h(x) \leq Ce^{-\beta\frac{\min_{i=1,\ldots,K}|x-Q_i^h|}{h}}$ for some $\beta > 0$, $C > 0$ and $u_h(Q_i^h) \to \alpha$, $\alpha > 0$, $i = 1,\ldots,K$ as $h \to 0$; i.e., u_h concentrates at $Q_1^h, \ldots, Q_K^h$.*

Remark. In [14], Li proved that if $V(x)$ has K (different) local maximum points $Q_1, \ldots, Q_K, Q_i \neq Q_j$ for $i \neq j$, then for h sufficiently small there exists a positive solution u_h of (1.2) such that u_h has K local maximum points $Q_1^h, \ldots, Q_K^h$ with $Q_i^h \to Q_i$, $i = 1, \ldots, K$, as $h \to 0$. Since the K bumps are separated in the limit, the interactions between bumps are of the order $e^{-\delta_0/h}$ for some constant $\delta_0 > 0$, which are exponentially small and are essentially neglected in [14]. Our theorem here is quite different from his. In fact we construct multi-bump solutions at *one* local maximum point of V. The distance between the bumps are of the order $O(h \log h)$, and thus the interactions between the bumps are of algebraic order $O(h^m)$ for some $m > 0$, which can't be neglected. So the interactions between bumps *do* play a very important role. This is a new and interesting phenomenon. The next result shows that this phenomenon does not occur at a nondegenerate local minimum point of $V(x)$.

Theorem 1.2. *Fix any positive integer $K > 1$. Let P_0 be a local minimum point of $V(x)$ such that $\det(\nabla^2 V(P_0)) \neq 0$. Then there is $h_0 > 0$ such that for $h < h_0$ equation (1.2) cannot have a positive solution u_h with the following properties:*

(1) The solution u_h has exactly K local maximum points $Q_1^h, Q_2^h, \ldots, Q_K^h$ and $Q_i^h \to P_0$, $\frac{|Q_i^h - Q_j^h|}{h} \to \infty$ as $h \to 0$, where $i, j = 1, \ldots, K$, $i \neq j$.

(2) The solution $u_h(x) \le Ce^{-\beta \frac{\min_{i=1,\ldots,K} |x - Q_i^h|}{h}}$ for some $\beta > 0$, $C > 0$ and $u_h(Q_i^h) \to \alpha$, $\alpha > 0$, $i = 1, \ldots, K$ as $h \to 0$.

In the rest of this section, we briefly outline the proof of Theorem 1.1. The main idea is to reduce the problem on $H^2(R^N)$ into a finite-dimensional problem on the space of bumps. To this end, we use the classical Liapunov-Schmidt reduction method (a similar method has been used in [10], [11], [26], [27], etc.) We shall follow the ideas in [10].

Without loss of generality, we can assume that P_0 in Theorem 1.1 is the origin and that $V(P_0) = 1$. Since we are looking for positive solutions, equation (1.2) becomes

$$h^2 \Delta u - V(x) u + u^p = 0, \ u > 0, \ x \in R^N. \tag{1.6}$$

(Recall that $V(x)$ satisfies (1.3) and p is subcritical.)

To introduce the main idea of the proofs of Theorems 1.1, we need to give some necessary notations and definitions first.

Let w be the unique solution of the following problem:

$$\begin{cases} \Delta w - w + w^p = 0 \text{ in } R^N \\ w > 0, \ w(0) = \max_{y \in R^N} w(y) \\ w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{1.7}$$

The solution of (1.7) is radial ([12]) and unique ([13]). Moreover, w is radially symmetric, decreasing and

$$\lim_{|y| \to \infty} w(y) e^{|y|} |y|^{\frac{N-1}{2}} = \lambda_0 > 0, \quad \lim_{|y| \to \infty} \frac{w'(y)}{w(y)} = -1 \tag{1.8}$$

for some constant $\lambda_0 > 0$. Let

$$I(w) = \frac{1}{2} \int_{R^N} |\nabla w|^2 + \frac{1}{2} \int_{R^N} w^2 - \frac{1}{p+1} \int_{R^N} w^{p+1}$$

be the ground energy of w.

Note that for fixed $a > 0$, $w_a(y) := a^{\frac{1}{p-1}} w(a^{\frac{1}{2}} y)$ is the unique solution of the following problem:

$$\begin{cases} \Delta v - av + v^p = 0 \text{ in } R^N \\ v > 0, \ v(0) = \max_{y \in R^N} v(y) \\ v(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{1.9}$$

Associated with problem (1.2) is the following energy functional:

$$J_h(u) = h^{-N}\Big(\frac{1}{2}\int_{R^N}(h^2|\nabla u|^2 + Vu^2) - \int_{R^N} F(u)\Big),$$

where $F(u) = \int_0^u f(s)\, ds$, $f(s) = |s|^{p-1}s$ and $u \in \mathcal{E}$, where the space $\mathcal{E}$ is defined by

$$\mathcal{E} = \{u : \int_{R^N}(h^2|\nabla u|^2 + Vu^2) < \infty\}. \tag{1.10}$$

(The factor h^{-N} comes from scaling.) Let Γ be as in Theorem 1.1 and $c_0 > 0$ be a small number. Set

$$\Lambda_h = \Big\{\mathbf{P} = (P_1, \dots, P_K) \in \Gamma \times \cdots \times \Gamma,\ w(\frac{|P_k - P_l|}{h}) < c_0 h, \\ k, l = 1, \dots, K,\ k \neq l\Big\}.$$

Let $\chi(x)$ be a cut-off function such that $\chi(x) = 1$ for x such that $d(x, \Gamma) < \frac{1}{2}$ and $\chi(x) = 0$ for x such that $d(x, \Gamma) \geq 1$.

Fix $\mathbf{P} = (P_1, P_2, \dots, P_K) \in \overline{\Lambda}_h$. We set

$$\hat{w}_{P_i}(x) = w_{V(P_i)}(\frac{x - P_i}{h}),\ w_{P_i}(x) = \hat{w}_{P_i}(x)\chi(x). \tag{1.11}$$

Since we look for solutions of (1.6) of the K-bump type $\sum_{i=1}^K w_{P_i}$, we set

$$u(x) = \sum_{i=1}^K w_{P_i} + \Phi_{h,\mathbf{P}}.$$

However, since the linearized operator at $\sum_{i=1}^K w_{P_i}$ is not uniformly invertible with respect to h, we introduce the approximate *kernel*

$$\mathcal{K}_{h,\mathbf{P}} = span\{h\frac{\partial w_{P_i}}{\partial P_{i,j}},\ i = 1, \dots, K,\ j = 1, \dots, N\} \subset H^2(R^N)$$

and the approximate *cokernel*

$$\mathcal{C}_{h,\mathbf{P}} = span\ \{h\frac{\partial w_{P_i}}{\partial P_{i,j}},\ i=1,\ldots,K,\ j=1,\ldots,N\} \subset L^2(R^N),$$

where $P_{i,j}$ is the j-th component of P_i, $i=1,\ldots,K$.

We first solve for $\Phi_{h,\mathbf{P}}$ in $\mathcal{K}^{\perp}_{h,\mathbf{P}}$ up to $\mathcal{C}^{\perp}_{h,\mathbf{P}}$ by using the Liapunov-Schmidt reduction method. (This method has been used by [8], [16], [19], [20] and [14].) Then we show that $\Phi_{h,\mathbf{P}}$ is C^1 in $\mathbf{P}$. After that, we define a new function

$$M_h(\mathbf{P}) = J_h(\sum_{i=1}^K w_{P_i} + \Phi_{h,\mathbf{P}}) : \bar{\Lambda}_h \to R. \tag{1.12}$$

We compute M_h and obtain the following asymptotic behavior:

$$M_h(\mathbf{P}) = \sum_{i=1}^K (c+o(1))V(P_i)^{\frac{p+1}{p-1}-\frac{N}{2}} - \sum_{k\neq l}(d+o(1))w(\frac{P_k-P_l}{h}) \tag{1.13}$$

for some positive constants c, d, where $o(1)$ means $|o(1)| \to 0$ as $h \to 0$.

We maximize $M_h(\mathbf{P})$ over $\overline{\Lambda}_h$. Condition (1.5) ensures that $M_h(\mathbf{P})$ attains its maximum inside Λ_h, say $\mathbf{P}^h \in \Lambda_h$. Then the corresponding function $u_h = \sum_{i=1}^K w_{P_i^h} + \Phi_{h,\mathbf{P}^h}$ is a solution of (1.6). We show that u_h has the properties of Theorem 1.1.

Theorem 1.2 is proved by asymptotic analysis.

This paper is organized as follows. In Section 2, we state some preliminary estimates leading to (1.13). Section 3 contains the standard Liapunov-Schmidt procedure. In Section 4, we apply a maximizing procedure to solve the reduced problem and thus complete the proof of Theorem 1.1 in Section 5. Section 6 contains the proof of Theorem 1.2: We first obtain a system of equations on the locations of the bumps, and then we reach a contradiction by using the fact that V has a nondegenerate local minimum at P_0. Finally we make some remarks on possible generalizations of Theorems 1.1 and 1.2 to more general problems.

Throughout this paper, the constant C denotes various generic constants independent of h. $O(A)$ means $|O(A)| \leq C|A|$ and $o(a)$ means $|o(a)|/|a| \to 0$ as $h \to 0$. We will always denote by $0 < \delta < 1$ a very small number and $f(u) = |u|^{p-1}u$. "$\sum$" always means summation from 1 to K.

2. Preliminary analysis. In this section, we first compute some integrals which will be useful in later sections. Then we obtain the energy expansion of K-bumps in $\bar{\Lambda}_h$.

First we state a useful lemma about the interactions of two w's.

Lemma 2.1 ((Lemma 2.1 of [2])). *Let $\phi \in C(R^N) \cap L^\infty(R^N)$, $\psi \in C(R^N)$ be radially symmetric and satisfy for some $\alpha \geq 0$, $\beta \geq 0$, $\gamma_0 \in R$*

$$\phi(x) \exp(\alpha|x|)|x|^\beta \to \gamma_0 \text{ as } |x| \to \infty,$$
$$\int_{R^N} |\psi(x)| \exp(\alpha|x|)(1 + |x|^\beta) < \infty.$$

Then

$$\exp(\alpha|y|)|y|^\beta \int_{R^N} \phi(x+y)\psi(x)\, dx \to \gamma_0 \int_{R^N} \psi(x) \exp(-\alpha x_1)\, dx \text{ as } |y| \to \infty.$$

Using Lemma 2.1 and the decay estimate (1.8), we then have the following estimate.

Lemma 2.2. *For h sufficiently small and $\mathbf{P} = (P_1, \ldots, P_K) \in \bar{\Lambda}_h$, we have*

$$h^{-N} \int_{R^N} w_{P_i}^p w_{P_j}\, dx = (\gamma + o(1))w(\frac{P_i - P_j}{h}),\ i \neq j,$$

where $\bar{\Lambda}_h$ is defined in Section 1, w_{P_i} is defined by (1.11) and

$$\gamma = \int_{R^N} w^p(y) e^{-y_1} dy > 0. \tag{2.1}$$

Another direct application of Lemma 2.1 is the following useful corollary.

Corollary 2.3. *Let $\beta_1 \geq 1$, $\beta_2 \geq 1$ be two positive numbers. Then we have*

$$h^{-N} \int_{R^N} w_{P_i}^{\beta_1} w_{P_j}^{\beta_2} = O(w^{\min(\beta_1,\beta_2)-\delta}(\frac{P_i - P_j}{h})),\ i \neq j, \tag{2.2}$$

where $\delta > 0$ is any small number. In particular, if $\beta_1 > \beta_2$ we can take $\delta = 0$.

The next lemma is the main result in this section.

Lemma 2.4. *For any $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}_h$, where $\bar{\Lambda}_h$ is defined in Section* 1, *and h sufficiently small, we have*

$$
\begin{aligned}
J_h(\sum_{i=1}^K w_{P_i}) &= \sum_{i=1}^K V(P_i)^{\frac{p+1}{p-1}-\frac{N}{2}} I(w) \\
-(\frac{\gamma}{2} + o(1)) & \sum_{i,l=1, i\neq l}^K w(\frac{P_i - P_l}{h}) + O(h \sum_{i=1}^K |\nabla V(P_i)| + h^2) \quad (2.3)
\end{aligned}
$$

where γ is defined by (2.1).

Remark. Roughly speaking, $w(\frac{P_i - P_l}{h})$ measures the interaction between the bump at P_i and the bump at P_l.

Proof. We shall only prove the case when $K = 1, 2$. The other cases are similar. By (1.11) and (1.8), $\hat{w}_{P_i} := w_{V(P_i)}(\frac{x-P_i}{h})$ is exponentially decaying outside any neighborhood $B_\delta(P_i)$ of P_i, where $\delta > 0$ is a small fixed number. Thus we obtain

$$
w_{P_i} = \hat{w}_{P_i} + O(e^{-\frac{\delta}{h}}) = w_{V(P_i)}(\frac{x - P_i}{h}) + O(e^{-\frac{\delta}{h}}).
$$

First for $K = 1$, if we set $P_1 + hy = x$, then we have

$$
\begin{aligned}
& J_h(w_{P_1}) = h^{-N}(\frac{1}{2}\int_{R^N} (|\nabla w_{P_1}(x)|^2 + V(x) w_{P_1}^2(x)) - \frac{1}{p+1}\int_{R^N} w_{P_1}^{p+1}(x)) \\
& = \frac{1}{2}\int_{R^N} (|\nabla w_{V(P_1)}|^2 + V(P_1 + hy) w_{V(P_1)}^2) - \frac{1}{p+1}\int_{R^N} w_{V(P_1)}^{p+1} + O(e^{-\frac{\delta}{h}}) \\
& = \frac{1}{2}\int_{R^N} (|\nabla w_{V(P_1)}|^2 + V(P_1) w_{V(P_1)}^2) - \frac{1}{p+1}\int_{R^N} w_{V(P_1)}^{p+1} \\
& + \int_{R^N} (V(P_1 + hy) - V(P_1)) w_{V(P_1)}^2 + O(e^{-\frac{\delta}{h}}) \\
& = V(P_1)^{\frac{p+1}{p-1}-\frac{N}{2}} I(w) + O(e^{-\frac{\delta}{h}}) + O(h|\nabla V(P_1)|),
\end{aligned}
$$

since

$$
V(x) - V(P_1) = O(|x - P_1||\nabla V(P_1)|) = O(h|y||\nabla V(P_1)|)
$$

and $\int_{R^N} |y| w^2(y)\, dy < \infty$.

Next we consider the case when $K = 2$. We first obtain for $i \neq j$

$$
\begin{aligned}
& h^{-N} \int_{R^N} (\nabla w_{P_i} \nabla w_{P_j} + V(x) w_{P_i} w_{P_j}) \\
& = h^{-N} [\int_{R^N} \nabla \hat{w}_{P_i} \nabla \hat{w}_{P_j} + \int_{R^N} V \hat{w}_{P_i} \hat{w}_{P_j} + O(e^{-\frac{\delta}{h}})] \\
& = \int_{R^N} w^p_{V(P_i)} w_{V(P_j)} + \int_{R^N} (V(x) - V(P_j)) w_{V(P_i)} w_{V(P_j)} + O(e^{-\frac{\delta}{h}}) \\
& = (\gamma + o(1)) w(\frac{P_i - P_j}{h}) + O(h|\nabla V(P_j)|) + O(e^{-\frac{\delta}{h}})
\end{aligned}
$$

by Lemma 2.2, where γ is defined by (2.1). Let $\Omega_1 := \{y \in R^N : |hy - P_1| \leq \frac{1-\delta}{2}|P_1 - P_2|\}$, $\Omega_2 := \{y \in R^N : |hy - P_2| \leq \frac{1-\delta}{2}|P_1 - P_2|\}$ and $\Omega_3 := R^N \backslash (\Omega_1 \cup \Omega_2)$. Then

$$
\begin{aligned}
h^{-N} \int_{\Omega_3} |(w_{P_1} + w_{P_2})^{p+1} - w^{p+1}_{P_1} - w^{p+1}_{P_2}| & \leq C h^{-N} \int_{\Omega_3} (w^{p+1}_{P_1} + w^{p+1}_{P_2}) \\
& \leq C e^{-\frac{(p+1)(1-\delta)|P_1 - P_2|}{2h}} = o(w(\frac{P_1 - P_2}{h}))
\end{aligned}
$$

by (1.8), if we choose δ such that $(p+1)(1-\delta) > 2$ (note that $\mathbf{P} \in \bar{\Lambda}_h$).

On Ω_1, we have

$$
\begin{aligned}
& h^{-N} \int_{\Omega_1} ((w_{P_1} + w_{P_2})^{p+1} - w^{p+1}_{P_1} - w^{p+1}_{P_2}) \\
& = (p+1) h^{-N} \int_{\Omega_1} w^p_{P_1} w_{P_2} + h^{-N} \int_{\Omega_1} O(w^{p+1}_{P_2} + w^{p-\delta}_{P_1} w^{1+\delta}_{P_2}) \\
& = (\gamma(p+1) + o(1)) w(\frac{P_1 - P_2}{h})
\end{aligned}
$$

by Lemma 2.2 and Corollary 2.3. Similarly, on Ω_2 we have

$$
h^{-N} \int_{\Omega_2} ((w_{P_1} + w_{P_2})^{p+1} - w^{p+1}_{P_1} - w^{p+1}_{P_2}) = (\gamma(p+1) + o(1)) w(\frac{P_1 - P_2}{h}).
$$

Hence

$$
\begin{aligned}
& J_h(w_{P_1} + w_{P_2}) = J_h(w_{P_1}) + J_h(w_{P_2}) + h^{-N} \int_{R^N} (\nabla w_{P_1} \nabla w_{P_2} + V w_{P_1} w_{P_2}) \\
& - \frac{1}{p+1} h^{-N} \int_{R^N} ((w_{P_1} + w_{P_2})^{p+1} - w^{p+1}_{P_1} - w^{p+1}_{P_2})
\end{aligned}
$$

$$= [V(P_1)^{\frac{p+1}{p-1}-\frac{N}{2}} + V(P_2)^{\frac{p+1}{p-1}-\frac{N}{2}}]I(w) + (\gamma + o(1))w(\frac{P_1 - P_2}{h})$$
$$+ O(h|\nabla V(P_1)| + h|\nabla V(P_2)|) - 2(\gamma + o(1))w(\frac{P_1 - P_2}{h})$$
$$= [V(P_1)^{\frac{p+1}{p-1}-\frac{N}{2}} + V(P_2)^{\frac{p+1}{p-1}-\frac{N}{2}}]I(w)$$
$$- (\gamma + o(1))w(\frac{P_1 - P_2}{h}) + O(h|\nabla V(P_1)| + h|\nabla V(P_2)|).$$

Lemma 2.4 is thus proved.

3. Liapunov-Schmidt reduction. In this section, we solve problem (1.2) with an appropriate kernel and cokernel, respectively. Since the procedure has now become standard, we shall only give a sketch of the proof. For more details, please see [20] and [14].

We first introduce some notations. Let $S_h(u) = \Delta u - V(hy)u + f(u)$, where $f(u) = |u|^{p-1}u$, for $u \in H^2(R^N) \cap \mathcal{E}$. Then solving equation (1.2) is equivalent to solving $S_h(u) = 0$, $u \in H^2(R^N)\cap\mathcal{E}$. Fix $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}_h$. To study (1.2) we first consider the linearized operator

$$\tilde{L}_h : \Phi(z) \mapsto \Delta\Phi(z) - V(hz)\Phi(z) + f'(\sum_{i=1}^{K} w_{P_i})\Phi(z),$$

$H^2(R^N) \cap \mathcal{E} \to L^2(R^N)$. It is easy to see (integration by parts) that the cokernel of $\tilde{L}_h$ coincides with its kernel. Choose the approximate cokernel and kernel as

$$\mathcal{C}_{h,\mathbf{P}} = \text{span}\{h\frac{\partial w_{P_i}}{\partial P_{i,j}}|i = 1, \dots, K,\ j = 1, \dots, N\} \subset L^2(R^N),$$
$$\mathcal{K}_{h,\mathbf{P}} = \text{span}\{h\frac{\partial w_{P_i}}{\partial P_{i,j}}|i = 1, \dots, K,\ j = 1, \dots, N\} \subset H^2(R^N).$$

Remark. Note that $h\frac{\partial \hat{w}_{P_i}}{\partial P_{i,j}}$ satisfies the following equation:

$$\Delta v(y) - V(P_i)v(y) + pw_{V(P_i)}^{p-1}v(y) - h\frac{\partial V(P_i)}{\partial x_j}w_{V(P_i)} = 0,\ y \in R^N,$$

and hence it is easy to see that

$$h\frac{\partial w_{P_i}}{\partial P_{i,j}} = -\frac{\partial w_{V(P_i)}(y)}{\partial y_j} + O(h|\nabla V(P_i)|w_{V(P_i)} + e^{-\frac{\delta}{h}}w_{V(P_i)}). \tag{3.1}$$

Let $\pi_{h,\mathbf{P}}$ denote the projection from $L^2(R^N)$ onto $\mathcal{C}^{\perp}_{h,\mathbf{P}}$. Our goal in this section is to show that the equation

$$\pi_{h,\mathbf{P}} \circ S_h(\sum_{i=1}^{K} w_{P_i} + \Phi) = 0$$

has a unique solution $\Phi = \Phi_{h,\mathbf{P}} \in \mathcal{K}^{\perp}_{h,\mathbf{P}}$ if h is small enough. Moreover $\Phi_{h,\mathbf{P}}$ is C^1 in $\mathbf{P} = (P_1, \dots, P_K)$.

As a preparation, in the following proposition we show the invertibility of the corresponding linearized operator. The proofs are standard and thus are omitted. See [19] and [20]. (Note that here $\bar{\Lambda}_h$ depends on h. But the same proof goes through since we have that $|P_i - P_j|/h \to +\infty$, $i \neq j$, as $h \to 0$, where $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}_h$.)

Proposition 3.1. *Let $L_{h,\mathbf{P}} := \pi_{h,\mathbf{P}} \circ \tilde{L}_h$. Then there exist positive constants $\overline{h}$ such that for all $h \in (0, \overline{h})$ and $\mathbf{P} = (P_1, \dots, P_K) \in \bar{\Lambda}_h$ the map*

$$L_{h,\mathbf{P}} = \pi_{h,\mathbf{P}} \circ \tilde{L}_h : \mathcal{K}^{\perp}_{h,\mathbf{P}} \to \mathcal{C}^{\perp}_{h,\mathbf{P}}$$

is both injective and surjective. Moreover

$$\|L_{h,\mathbf{P}}\Phi\|_{L^2(R^N)} \geq C\|\Phi\|_{H^2(R^N)} \tag{3.2}$$

for all $\Phi \in \mathcal{K}^{\perp}_{h,\mathbf{P}}$.

We are now in a position to solve the equation

$$\pi_{h,\mathbf{P}} \circ S_h(\sum_{i=1}^{K} w_{P_i} + \Phi) = 0, \Phi \in \mathcal{K}^{\perp}_{h,\mathbf{P}}. \tag{3.3}$$

Note that simple computations show that

$$S_h(\sum_{i=1}^{K} w_{P_i} + \Phi) = \tilde{L}_h(\Phi) + N_{h,\mathbf{P}}(\Phi) + \sum_{j=1}^{3} M^j_{h,\mathbf{P}}, \tag{3.4}$$

where

$$N_{h,\mathbf{P}}(\Phi) = f(\sum_{i=1}^{K} w_{P_i} + \Phi) - f(\sum_{i=1}^{K} w_{P_i}) - f^{'}(\sum_{i=1}^{K} w_{P_i})\Phi$$

$$M^1_{h,\mathbf{P}} = -\sum_{i=1}^{K}(V - V(P_i))w_{P_i}, M^2_{h,\mathbf{P}} = f(\sum_{i=1}^{K} w_{P_i}) - \sum_{i=1}^{K} f(w_{P_i})$$

$$M^3_{h,\mathbf{P}} = \sum_{i=1}^{K} [\Delta w_{P_i} - V(P_i) w_{P_i} + f(w_{P_i})].$$

Before we move on, we need the following error estimates.

Lemma 3.2. *For h sufficiently small, we have*

$$|N_{h,\mathbf{P}}(\Phi)| \leq C|\Phi|^{1+\sigma} \tag{3.5}$$

$$\|M^1_{h,\mathbf{P}}\|_{L^2(R^N)} \leq C(h \sum_{i=1}^{K} |\nabla V(P_i)|) \tag{3.6}$$

$$\|M^2_{h,\mathbf{P}}\|_{L^2(R^N)} \leq C\delta_h^{(1+\sigma)/2} \tag{3.7}$$

$$\|M^3_{h,\mathbf{P}}\|_{L^2(R^N)} \leq Ce^{-\frac{\delta}{h}}, \tag{3.8}$$

where

$$\sigma = \min(1, p-1) - \delta, \delta_h = \max_{i \neq j} \gamma w(\frac{P_i - P_j}{h}). \tag{3.9}$$

Proof. It is easy to derive (3.5) from the mean value theorem. Inequality (3.8) follows from the definition and (1.8). For (3.6), we note that $V(x) - V(P_i) = V(P_i + hy) - V(P_i) = O(h|\nabla V(P_i)||y|)$ and $\int_{R^N} |y|^2 w^2(y) < \infty$.

It remains to prove (3.7). To this end, we divide the domain R^N into $(K+1)$ parts: Let $R^N = \cup_{i=1}^{K+1} \Omega_i$, where $\Omega_i := \{y : |hy - P_i| \leq \frac{1}{2} \min_{k \neq l} |P_k - P_l|\}$, $i = 1, \ldots, K$, $\Omega_{K+1} = R^N \setminus \cup_{i=1}^{K} \Omega_i$.

We now estimate $M^2_{h,\mathbf{P}}$ in each domain. In Ω_{K+1}, we have

$$|M^2_{h,P_1,\ldots,P_K}| \leq C(w_{P_1} + \cdots + w_{P_K})^p \leq O(e^{-\frac{1+\sigma}{2}\frac{1}{h}\min_{k \neq l}|P_k - P_l|}).$$

Hence $\|M^2_{h,\mathbf{P}}\|_{L^2(\Omega_{K+1})} \leq O(\delta_h^{\frac{1+\sigma}{2}})$. In $\Omega_i, i = 1, \ldots, K$, we have

$$|M^2_{h,\mathbf{P}}| \leq \sum_{j \neq i} |f'(w_{P_i}) w_{P_j}| + O(\sum_{j \neq i} |w_{P_j}|^{1+\sigma}).$$

Note that in Ω_i, $i = 1, \ldots, K$, we have $w_{P_j} \leq w_{P_i}$ for $j \neq i$, and hence

$$h^{-N} \int_{\Omega_2} |f'(w_{P_i}) w_{P_j}|^2 \leq Ch^{-N} \int_{\Omega_2} w_{P_i}^{2(p-1)} w_{P_j}^2 \leq \delta_h^{2-\delta}$$

if $p \geq 2$, by Corollary 2.3. When $1 < p < 2$, we have

$$h^{-N}\int_{\Omega_2} |f'(w_{P_i})w_{P_j}|^2 \leq Ch^{-N}\int_{\Omega_2} w_{P_i}^{2(p-1)+2-p-\delta} w_{P_j}^{p+\delta} \leq \delta_h^{p-\delta}$$

by Corollary 2.3 (here we need $2-p-\delta > 0$). Hence we obtain

$$\|M^2_{h,\mathbf{P}}\|_{L^2(\Omega_i)} \leq O(\delta_h^{\frac{1+\sigma}{2}}).$$

Combining the estimates for $i = 1, \dots, K+1$, we obtain (3.7). □

Next we solve (3.3). Since $L_{h,\mathbf{P}}|_{\mathcal{K}^{\perp}_{h,\mathbf{P}}}$ is invertible (call the inverse $L^{-1}_{h,\mathbf{P}}$) we can rewrite (3.3) as

$$\Phi = -(L^{-1}_{h,\mathbf{P}} \circ \pi_{h,\mathbf{P}})(\sum_{j=1}^{3} M^j_{h,\mathbf{P}}) - (L^{-1}_{h,\mathbf{P}} \circ \pi_{h,\mathbf{P}})N_{h,\mathbf{P}}(\Phi) \equiv G_{h,\mathbf{P}}(\Phi), \quad (3.10)$$

where the operator $G_{h,\mathbf{P}}$ is defined by the last equation for $\Phi \in H^2(R^N)$. We are going to show that the operator $G_{h,\mathbf{P}}$ is a contraction on

$$B_{h,\eta} \equiv \{\Phi \in H^2(R^N) : \|\Phi\|_{H^2(R^N)} < \eta\}$$

if $\eta = C_0(\delta_h^{\frac{1+\sigma}{2}} + h\sum_{i=1}^K |\nabla V(P_i)|)$ and $C_0 > 0$ is large enough. In fact, we have

$$\begin{aligned}\|G_{h,\mathbf{P}}(\Phi)\|_{H^2(R^N)} &\leq C(\|\pi_{h,\mathbf{P}} \circ N_{h,\mathbf{P}}(\Phi)\|_{L^2(R^N)} + \|\pi_{h,\mathbf{P}} \circ (\sum_{j=1}^{3} M^j_{h,\mathbf{P}})\|_{L^2(R^N)}) \\ &\leq C(c(\eta)\eta + \delta_h^{\frac{1+\sigma}{2}} + h\sum_{i=1}^{K} |\nabla V(P_i)|) < \eta,\end{aligned}$$

where $C > 0$ is independent of $\eta > 0$, δ_h is defined by (3.9) and $c(\eta) \to 0$ as $\eta \to 0$. If we choose C_0 large enough, then $G_{h,\mathbf{P}}$ is a map from $B_{h,\eta}$ to $B_{h,\eta}$. Similarly we can show

$$\|G_{h,\mathbf{P}}(\Phi) - G_{h,\mathbf{P}}(\Phi')\|_{H^2(R^N)} \leq Cc(\eta)\|\Phi - \Phi'\|_{H^2(R^N)},$$

where $c(\eta) \to 0$ as $\eta \to 0$. Therefore $G_{h,\mathbf{P}}$ is a contraction on $B_{h,\eta}$. The existence of a fixed point $\Phi = \Phi_{h,\mathbf{P}}$ now follows from the contraction mapping principle, and hence $\Phi_{h,\mathbf{P}}$ is a solution of (3.10).

Because of the fact that

$$\|\Phi_{h,\mathbf{P}}\|_{H^2(R^N)} \le C(\|N_{h,\mathbf{P}}(\Phi_{h,\mathbf{P}})\|_{L^2(R^N)} + \|\sum_{j=1}^{3} M^j_{h,\mathbf{P}}\|_{L^2(R^N)})$$
$$\le C(C\delta_h^{\frac{1+\sigma}{2}} + h\sum_{i=1}^{K}|\nabla V(P_i)| + c(\eta)\|\Phi_{h,\mathbf{P}}\|_{H^2(R^N)}),$$

we have

$$(1 - Cc(\eta))\|\Phi_{h,\mathbf{P}}\|_{H^2(R^N)} \le C(\delta_h^{\frac{1+\sigma}{2}} + h\sum_{i=1}^{K}|\nabla V(P_i)|).$$

We have thus proved the following:

Lemma 3.3. *There exists $\overline{h} > 0$ such that for any $0 < h < \overline{h}$ and $\mathbf{P} \in \bar{\Lambda}_h$ there exists a unique $\Phi_{h,\mathbf{P}} \in \mathcal{K}^\perp_{h,\mathbf{P}}$ satisfying $S_h(\sum_{i=1}^K w_{P_i} + \Phi_{h,\mathbf{P}}) \in \mathcal{C}^\perp_{h,\mathbf{P}}$ and*

$$\|\Phi_{h,\mathbf{P}}\|_{H^2(R^N)} \le C\delta_h^{\frac{1+\sigma}{2}} + Ch\sum_{i=1}^{K}|\nabla V(P_i)|. \tag{3.11}$$

Finally we show that $\Phi_{h,\mathbf{P}}$ is actually smooth in $\mathbf{P}$.

Lemma 3.4. *Let $\Phi_{h,\mathbf{P}}$ be defined by Lemma 3.3. Then $\Phi_{h,\mathbf{P}} \in C^1$ in $\mathbf{P}$.*

Proof. Recall that $\Phi_{h,\mathbf{P}}$ is a solution of the equation

$$\pi_{h,\mathbf{P}} \circ S_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}}) = 0 \tag{3.12}$$

such that

$$\Phi_{h,\mathbf{P}} \in \mathcal{K}^\perp_{h,\mathbf{P}}. \tag{3.13}$$

Notice that it is easy to see that the functions w_{P_i} and $\partial^2 w_{P_i}/(\partial P_{i,j}\partial P_{i,k})$ are C^1 in $\mathbf{P}$. This implies that the projection $\pi_{h,\mathbf{P}}$ is C^1 in $\mathbf{P}$.

Applying $\partial/\partial P_{i,j}$ to (3.12) gives

$$\pi_{h,\mathbf{P}} \circ DS_h\Big(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}}\Big)\Big(\sum_{i=1}^{K}\frac{\partial w_{P_i}}{\partial P_{i,j}} + \frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}}\Big) \tag{3.14}$$
$$+ \frac{\partial \pi_{h,\mathbf{P}}}{\partial P_{i,j}} \circ S_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}}) = 0,$$

where

$$DS_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}}) = \Delta - V + f^{'}(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}}).$$

We decompose $\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}}$ into two parts:

$$\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}} = \Big(\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}}\Big)_1 + \Big(\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}}\Big)_2,$$

where $(\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}})_1 \in \mathcal{K}_{h,\mathbf{P}}$ and $(\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}})_2 \in \mathcal{K}_{h,\mathbf{P}}^{\perp}$.

We can easily see that $(\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}})_1$ is continuous in $\mathbf{P}$ since

$$\int_{R^N} \Phi_{h,\mathbf{P}} \frac{\partial w_{P_k}}{\partial P_{k,l}} = 0, \quad k, l = 1, \ldots, N,$$

and hence

$$\int_{R^N} \frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}} \frac{\partial w_{P_k}}{\partial P_{k,l}} + \int_{R^N} \Phi_{h,\mathbf{P}} \frac{\partial^2 w_{P_k}}{\partial P_{i,j} \partial P_{k,l}} = 0$$

where i, j, k, l are indices running from 1 to K.

Now we can write equation (3.14) as

$$\begin{aligned} &\pi_{h,\mathbf{P}} \circ DS_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}})\Big((\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}})_2\Big) \\ &+ \pi_{h,\mathbf{P}} \circ DS_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}})\Big(\sum_{i=1}^{K} \frac{\partial w_{P_i}}{\partial P_{i,j}} + (\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}})_1\Big) \\ &+ \frac{\partial \pi_{h,\mathbf{P}}}{\partial P_{i,j}} \circ S_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}}) = 0. \end{aligned} \tag{3.15}$$

As in the proof of Proposition 3.1 we can show that the operator $\pi_{h,\mathbf{P}} \circ DS_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}})$ is invertible from $\mathcal{K}_{h,\mathbf{P}}^{\perp}$ to $\mathcal{C}_{h,\mathbf{P}}^{\perp}$. Then we can take the inverse of $\pi_{h,\mathbf{P}} \circ DS_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}})$ in the above equation, and the inverse is continuous in $\mathbf{P}$.

Since $\frac{\partial w_{P_i}}{\partial P_{i,j}}, (\frac{\partial \Phi_{h,\mathbf{P}}}{\partial P_{i,j}})_1 \in \mathcal{K}_{h,\mathbf{P}}$ are continuous in $\mathbf{P} \in \bar{\Lambda}_h$ and so is $\frac{\partial \pi_{h,\mathbf{P}}}{\partial P_{i,j}}$, we conclude that $(\partial \Phi_{h,\mathbf{P}}/(\partial P_{i,j}))_2$ is also continuous in $\mathbf{P}$. This is the same as the C^1 dependence of $\Phi_{h,\mathbf{P}}$ in $\mathbf{P}$. The proof is finished. □

4. A maximizing procedure. In this section, we study a maximizing problem. Fix $\mathbf{P} \in \overline{\Lambda}_h$. Let $\Phi_{h,\mathbf{P}}$ be the solution given by Lemma 3.3. We define a new functional

$$M_h(\mathbf{P}) = J_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,\mathbf{P}}) : \overline{\Lambda}_h \to R. \tag{4.1}$$

We first have the following asymptotic expansion of $M_h(\mathbf{P})$.

Lemma 4.1. *For* $\mathbf{P} \in \bar{\Lambda}_h$, *we have*

$$M_h(\mathbf{P}) = \sum_{i=1}^{K} V(P_i)^{\frac{p+1}{p-1}-\frac{N}{2}} I(w) \tag{4.2}$$

$$- (\frac{\gamma}{2} + o(1)) \sum_{k \neq l} w(\frac{P_k - P_l}{h}) + O(h \sum_{i=1}^{K} |\nabla V(P_i)| + h^2).$$

Proof. For any $\mathbf{P} \in \overline{\Lambda}_h$, we have

$$M_h(\mathbf{P}) = J_h(\sum_{i=1}^{K} w_{P_i}) + g_{h,\mathbf{P}}(\Phi_{h,\mathbf{P}}) + O(\|\Phi_{h,\mathbf{P}}\|^2_{H^1(R^N)}),$$

where

$$g_{h,\mathbf{P}}(\Phi_{h,\mathbf{P}}) = h^{-N}[\int_{R^N} \sum_{i=1}^{K} (\nabla w_{P_i} \nabla \Phi_{h,\mathbf{P}} + V w_{P_i} \Phi_{h,\mathbf{P}}) - \int_{R^N} f(\sum_{i=1}^{K} w_{P_i}) \Phi_{h,\mathbf{P}}]$$

$$= h^{-N} \int_{R^N} [\sum_{i=1}^{K} (V - V(P_i)) \hat{w}_{P_i} \Phi_{h,\mathbf{P}} + (\sum_{i=1}^{K} f(\hat{w}_{P_i})$$

$$- f(\sum_{i=1}^{K} \hat{w}_{P_i})) \Phi_{h,\mathbf{P}} + O(e^{-\frac{\delta}{h}} |\Phi_{h,\mathbf{P}}|)]$$

$$\leq \| \sum_{i=1}^{K} |(V - V(P_i)) \hat{w}_{P_i}| \|_{L^2(R^N)} \|\Phi_{h,\mathbf{P}}\|_{L^2(R^N)}$$

$$+ \|| \sum_{i=1}^{K} f(\hat{w}_{P_i}) - f(\sum_{i=1}^{K} \hat{w}_{P_i})| \|_{L^2(R^N)} \|\Phi_{h,\mathbf{P}}\|_{L^2(R^N)} + Ce^{-\frac{\delta}{h}}$$

$$\leq O(h \sum_{i=1}^{K} |\nabla V(P_i)| + e^{-\frac{\delta}{h}} + \delta_h^{1+\sigma}).$$

By Lemma 2.4 and Lemma 3.3, we obtain (4.2). □

From Lemma 4.1, we have the following:

Proposition 4.2. *For h sufficiently small, the following maximizing problem,*

$$\max\{M_h(\mathbf{P}) : \mathbf{P} \in \overline{\Lambda}_h\}, \tag{4.3}$$

has a solution $\mathbf{P}^h \in \Lambda_h$.

Proof. Since $J_h(\sum_{i=1}^K w_{P_i} + \Phi_{h,\mathbf{P}})$ is continuous in $\mathbf{P}$, the maximizing problem has a solution. Let $M_h(\mathbf{P}^h)$ be the maximum of J_h, where $\mathbf{P}^h \in \overline{\Lambda}_h$.

We claim that $\mathbf{P}^h \in \Lambda_h$. We prove this by energy comparison. We first obtain a lower bound for $M_h(\mathbf{P}^h)$. Let us choose $P_j^0 = P_0 + h^{3/4} X_j$ where X_j, $j = 1, \ldots, K$ are the K vortices of K-polygon centered at 0 with $|X_i - X_j| = 1$ for $i \neq j$. Then certainly $P_j^0 \in \Gamma$. Moreover, $w(\frac{|P_j^0 - P_i^0|}{h}) = w(h^{-1/4}) = o(h^{3/2}) < c_0 h$ for h small. So $\mathbf{P}^0 = (P_1^0, \ldots, P_K^0) \in \Lambda_h$. We have by Taylor's expansion

$$V(P_i^0) = V(0) + O(h^{3/2}), |\nabla V(P_i^0)| = O(h^{3/4}), \quad i = 1, \ldots, K.$$

Hence by (4.2) we obtain

$$M_h(\mathbf{P}^h) = \max_{\mathbf{P} \in \bar{\Lambda}_h} M_h(\mathbf{P}) \geq M_h(\mathbf{P}^0) \geq KI(w) - Ch^{3/2},$$

which implies that (by Lemma 4.1)

$$\sum_{i=1}^K V(P_i^h)^{\frac{p+1}{p-1} - \frac{N}{2}} I(w) \tag{4.4}$$
$$- (\frac{\gamma}{2} + o(1)) \sum_{k \neq l} w(\frac{P_k^h - P_l^h}{h}) + O(h \sum_{i=1}^K |\nabla V(P_i^h)|) \geq KI(w) - Ch^{3/2}.$$

From (4.4), we can deduce that $\mathbf{P}^h \in \Lambda_h$. In fact, suppose not; then by the definition of Λ_h there are two possibilities. Either one of the P_i^h is on $\partial\Gamma$. In this case, we have by condition (1.5) (noting that $V(P_i^h) < V(0) - \mu_1$ for some $\mu_1 > 0$ if $P_i^h \in \partial\Gamma$)

$$\sum_{i=1}^K V(P_i^h)^{\frac{p+1}{p-1} - \frac{N}{2}} I(w) \leq KI(w) - \mu_2$$

for some $\mu_2 > 0$, which is impossible by (4.4). Or $w(\frac{P_k^h - P_l^h}{h}) = c_0 h$ for some $k \neq l$. In this case $w(\frac{P_k^h - P_l^h}{h}) = c_0 h$ and

$$\sum_{i=1}^{K} V(P_i^h)^{\frac{p+1}{p-1}-\frac{N}{2}} I(w) - (\frac{\gamma}{2} + o(1)) \sum_{i \neq j} w(\frac{P_i^h - P_j^h}{h}) \leq K I(w) - (\frac{\gamma}{2} + o(1)) c_0 h,$$

which is impossible by (4.4). Hence $\mathbf{P}^h \in \Lambda_h$, which completes the proof of Proposition 4.2. □

Remark. From the proof of Proposition 4.2 and (4.2) we can obtain

$$V(P_i^h) - \max_{P \in \Gamma} V(P) = V(P_i^h) - V(P_0) = o(1),$$
$$w(\frac{P_i^h - P_j^h}{h}) = o(h), \ \forall i \neq j,$$

which means that $P_i^h - P_0 = o(1)$, $|P_i^h - P_j^h|/h \geq \log \frac{C}{h}$ for some $C > 0$.

5. Proof of Theorem 1.1. In this section, we apply results in Section 3 and Section 4 to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.3 and Lemma 3.4, there exists h_0 such that for $h < h_0$ we have a C^1 map which, to any $\mathbf{P} \in \overline{\Lambda}_h$, associates $\Phi_{h,P_1,\ldots,P_K} \in \mathcal{K}_{h,\mathbf{P}}^{\perp}$ such that

$$S_h(\sum_{i=1}^{K} w_{P_i} + \Phi_{h,P_1,\ldots,P_K}) = \sum_{k=1,\ldots,K; l=1,\ldots,N} \alpha_{kl} \frac{\partial w_{P_k}}{\partial P_{k,l}} \tag{5.1}$$

for some constants $\alpha_{kl} \in R^{KN}$. By Proposition 4.2, we have $\mathbf{P}^h \in \Lambda_h$, achieving the maximum of the maximization problem in Proposition 4.2. Let $\Phi = \Phi_{h,\mathbf{P}^h}$ and $u_h = \sum_{i=1}^{K} w_{P_i^h} + \Phi_{h,P_1^h,\ldots,P_K^h}$. Then we have

$$D_{P_{i,j}}|_{P_i = P_i^h} M_h(\mathbf{P}^h) = 0, \quad i = 1, \ldots, K, j = 1, \ldots, N.$$

Hence, we have

$$\int_{R^N} [\nabla u_h \nabla \frac{\partial (w_{P_i} + \Phi_{h,P_1,\ldots,P_K})}{\partial P_{i,j}}|_{P_i = P_i^h} + V u_h \frac{\partial (w_{P_i} + \Phi_{h,P_1,\ldots,P_K})}{\partial P_{i,j}}|_{P_i = P_i^h}$$
$$- |u_h|^{p-1} u_h \frac{\partial (w_{P_i} + \Phi_{h,P_1,\ldots,P_K})}{\partial P_{i,j}}|_{P_i = P_i^h}] = 0$$

for $i = 1, \dots, K$ and $j = 1, \dots, N$. Therefore, we obtain

$$\sum_{k=1,\dots,K;l=1,\dots,N} \alpha_{kl} \int_{R^N} \frac{\partial w_{P_k^h}}{\partial P_{k,l}} \frac{\partial (w_{P_i^h} + \Phi_{h,P_1^h,\dots,P_K^h})}{\partial P_{i,j}^h} = 0, \tag{5.2}$$

$\forall i = 1, \dots, K$, $j = 1, \dots, N$. Since $\Phi_{h,P_1^h,\dots,P_K^h} \in \mathcal{K}_{h,\mathbf{P}^h}^{\perp}$, we have that

$$h^{-N} \int_{R^N} \frac{\partial w_{P_k^h}}{\partial P_{k,l}^h} \frac{\partial \Phi_{h,P_1^h,\dots,P_K^h}}{\partial P_{i,j}^h} = -h^{-N} \int_{R^N} \frac{\partial^2 w_{P_i^h}}{\partial P_{k,l}^h \partial P_{i,j}^h} \Phi_{h,P_1^h,\dots,P_K^h}$$

$$= \| \frac{\partial^2 w_{P_i^h}}{\partial P_{k,l}^h \partial P_{i,j}^h} \|_{L^2(R^N)} \| \Phi_{h,P_1^h,\dots,P_K^h} \|_{L^2(R^N)}$$

$$= O(\frac{1}{h^2} (\sum_{i \neq j} w^{\frac{1+\sigma}{2}} (\frac{P_i^h - P_j^h}{h}) + h | \sum_{i=1}^K \nabla V(P_i^h) |)) = o(h^{-2})$$

by Lemma 3.3. Note that

$$h^{-N} \int_{R^N} \frac{\partial w_{P_k^h}}{\partial P_{k,l}^h} \frac{\partial w_{P_i^h}}{\partial P_{i,j}^h} = \begin{cases} h^{-2} \int_{R^N} (\frac{\partial w}{\partial y_j})^2 + o(h^{-2}) \text{ if } i = k, j = l \\ o(h^{-2}) \text{ otherwise.} \end{cases}$$

Thus, equation (5.2) becomes a system of homogeneous equations for α_{kl}, and the matrix of the system is nonsingular since it is diagonally dominant. So $\alpha_{kl} \equiv 0$, $k = 1, \dots, K$, $l = 1, \dots, N$. Hence, $u_h = \sum_{i=1}^K w_{P_i^h} + \Phi_{h,P_1^h,\dots,P_K^h}$ is a critical point of J_h and $u_h(x)$ satisfies (1.2). It remains to prove that $u_h > 0$.

Multiplying equation (1.2) by $u_h^- = \min(u_h, 0)$ and integrating by parts, we have

$$\int_{R^N} [|\nabla u_h^-(hy)|^2 + V(u_h^-(hy))^2] = \int_{R^N} (u_h^-(hy))^{p+1}.$$

By Sobolev's imbedding theorem we obtain either $\int_{R^N} (u_h^-(hy))^{p+1} \geq C$ or $u_h^- \equiv 0$. By our construction, we have that

$$\int_{R^N} (u_h^-(hy))^{p+1} = o(1).$$

Hence $u_h^- \equiv 0$; i.e., $u_h \geq 0$. It is easy to see that by the Maximum Principle $u_h > 0$ in R^N. Moreover $J_h(u_h) \to KI(w)$ and u_h has only K local maximum points $Q_1^h, \ldots, Q_K^h$. By the structure of u_h we see that (up to a permutation) $Q_i^h - P_i^h = o(h)$. Since $P_i^h - P_0 = o(1)$, we obtain that $Q_i^h - P_0 = o(1)$ and $V(Q_i^h) \to V(P_0)$ as $h \to 0$. Moreover, $w(\frac{|Q_i^h - Q_j^h|}{h}) \leq 2c_0 h$, which implies that $\frac{|Q_i^h - Q_j^h|}{h} \geq \log \frac{C}{h}$ for $i \neq j$. This proves Theorem 1.1. $\square$

6. Proof of Theorem 1.2. In this section, we prove Theorem 1.2. Let P_0 be a local minimum point of $V(x)$ such that $det(\nabla^2 V(P_0)) \neq 0$. Without loss of generality, we may assume that $P_0 = 0$ and $V(P_0) = 1$. Let $K > 1$ be an integer.

Suppose Theorem 1.2 is not true. Namely there exists a sequence of solutions u_h such that for h sufficiently small

(i) u_h has only K local maximum points $Q_1^h, \ldots, Q_K^h$ with $Q_i^h \to 0$ and $|Q_i^h - Q_j^h|/h \to \infty$ as $h \to 0$, $\forall i, j = 1, \ldots, K$, $i \neq j$, and

(ii) $u_h \leq Ce^{-\beta \frac{\min_{i=1,\ldots,K} |x - Q_i^h|}{h}}$, $u_h(Q_i^h) \to \alpha > 0$ for some $\alpha > 0$, $\beta > 0$, $\forall i = 1, \ldots, K$.

Recall that

$$\hat{w}_{Q_i^h}(x) = w_{V(Q_i^h)}(\frac{x - Q_i^h}{h}), w_{Q_i^h}(x) = \hat{w}_{Q_i^h}(x)\chi(x). \tag{6.1}$$

To avoid clumsy notations, in this section we use $\hat{w}_i$ to denote $\hat{w}_{Q_i^h}$ and w_i to denote $w_{Q_i^h}$. Furthermore, we set

$$x = hy, \quad \delta_{ij}^h = \gamma w(\frac{Q_i^h - Q_j^h}{h}), \quad i \neq j, \ \delta_h = \max_{i \neq j} \delta_{ij}^h, \quad \text{and} \tag{6.2}$$
$$u_h(hy) = \sum_{i=1}^K w_i(hy) + \phi_h(y).$$

It is easy to see that ϕ_h satisfies

$$\Delta\phi_h - V\phi_h + p(\sum_{i=1}^K w_i)^{p-1}\phi_h + \sum_{i=1}^K (V(Q_i^h) - V)w_i + (\sum_{i=1}^K w_i)^p \tag{6.3}$$
$$- \sum_{i=1}^K w_i^p + (\sum_{i=1}^K w_i + \phi_h)^p - (\sum_{i=1}^K w_i)^p - p(\sum_{i=1}^K w_i)^{p-1}\phi_h$$
$$+ \sum_{i=1}^K [\Delta(w_i - \hat{w}_i) - V(Q_i^h)(w_i - \hat{w}_i) + w_i^p - (\hat{w}_i)^p] = 0.$$

Then we have

Lemma 6.1. *For h sufficiently small, we have*

$$\int_{R^N}(|\nabla\phi_h|^2+V\phi_h^2)=O(h^2+\delta_h^{1+\sigma}), \tag{6.4}$$

where $\sigma=\min(1,p-1)-\delta$, $\delta>0$ is small and δ_h is defined by (6.2).

From Lemma 6.1, we deduce the following important result on the location of the K-bumps.

Lemma 6.2. *For h sufficiently small, we have*

$$-h\frac{\partial V(Q_i^h)}{\partial x_j}+c\sum_{l\neq i}\delta_{il}^h(\frac{Q_l^h-Q_i^h}{|Q_l^h-Q_i^h|})_j+o(h|Q_i^h|+\sum_{l\neq i}\delta_{il}^h)=0, \tag{6.5}$$

for $i=1,\ldots,K$, $j=1,\ldots,N$, where $c>0$ is a positive number and b_j means the j-th component of a vector $\vec{b}\in R^N$.

We postpone the proofs of Lemma 6.1 and Lemma 6.2 until the end of this section. Let us now use them to prove Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality, suppose $|Q_1^h-Q_2^h|=\min_{i\neq j}|Q_i^h-Q_j^h|:=d_h$. So $\delta_{12}^h=\max_{i\neq j}\delta_{ij}^h:=\delta_h$.

We first claim that $\delta_h=O(h)$. In fact, suppose not. Consider a subset S_h of $\{Q_1^h,\ldots,Q_K^h\}$ such that $Q_\eta^h\in S_h$ if and only if $Q_\eta^h=Q_1^h$ or there exists $Q_{\eta_1}^h,\ldots,Q_{\eta_l}^h$ such that $\lim_{h\to 0}|Q_{\eta_j}^h-Q_{\eta_1}^h|/d_h=1$, $j=2,\ldots,l$. It is easy to see that there is a point, say $Q_i^h\in S_h$, and a hyperplane H such that $Q_i^h\in H$, and all the other points of S_h belong to the same halfspace of R^N divided by H. We divide (6.5) by δ_h (noting that $\frac{h}{\delta_h}\to 0$); then we have

$$c\sum_{l\neq i}\frac{\delta_{il}^h}{\delta_{12}^h}(\frac{Q_l^h-Q_i^h}{|Q_l^h-Q_i^h|})=o(1),\qquad \sum_{l\neq i,Q_l^h\in S_h}\frac{\delta_{il}^h}{\delta_h}(\frac{Q_l^h-Q_i^h}{|Q_l^h-Q_i^h|})=o(1). \tag{6.6}$$

But there is $l\neq i$ such that $\lim_{h\to 0}\delta_{il}^h/\delta_h>0$ (since $Q_i^h\in S_h$), and all Q_j^h, $j\neq i$, lie in the same halfspace of R^N divided by H; this is impossible by (6.6)! So $\delta_h=O(h)$. Next we choose a point Q_0 of $\{Q_1^h,\ldots,Q_K^h\}$ such that $d(0,Q_0)=\max_{j=1,\ldots,k}d(0,Q_j^h):=l_h$. Without loss of generality we can suppose $Q_0=Q_1^h$. Through a rotation, we can suppose $Q_1^h=(-l_h,0,\ldots,0)$;

i.e., the direction from Q_1^h to 0 is the positive x_1-axis. It is easy to check for other points Q_j^h, $j \neq 1$, $(Q_j^h - Q_1^h)_1 > 0$.

From (6.5) with $i = 1$, $j = 1$, we have

$$-\frac{\partial V(Q_1^h)}{\partial x_1} + c \sum_{l \neq 1} \frac{\delta_{l1}^h}{h} (\frac{Q_l^h - Q_1^h}{|Q_l^h - Q_1^h|})_1 + o(l_h + \frac{\delta_h}{h}) = 0.$$

We now claim that

$$\frac{\delta_h}{h} = O(l_h). \tag{6.7}$$

If (6.7) holds, then we have

$$V_{11}(0) l_h + c \sum_{l \neq 1} \frac{\delta_{l1}^h}{h l_h} (\frac{Q_l^h - Q_1^h}{|Q_l^h - Q_1^h|})_1 l_h + o(l_h) = 0,$$

which is impossible since $V_{11}(0) + c \sum_{l \neq 1} \frac{\delta_{l1}^h}{h l_h} (\frac{Q_l^h - Q_1^h}{|Q_l^h - Q_1^h|})_1 > V_{11}(0) > 0$. Theorem 1.2 is thus proved.

It remains to prove (6.7). If $\frac{\delta_h}{h} \neq O(l_h)$, then $l_h = o(\frac{\delta_h}{h}), |Q_l^h| = o(\frac{\delta_h}{h})$. As before, there is a point, say $Q_i^h \in S_h$, and a hyperplane H such that $Q_i^h \in H$, and all the other points of S_h belong to the same halfspace of R^N divided by H. Going back to (6.5) we obtain

$$c \sum_{l \neq i} \frac{\delta_{li}^h}{h} (\frac{Q_l^h - Q_i^h}{|Q_l^h - Q_i^h|}) + o(\frac{\delta_h}{h}) = 0. \tag{6.8}$$

But there exists $l \neq i$ such that $\lim_{h \to 0} \frac{\delta_{li}^h}{\delta_h} = 1$ and for all Q_l^h, $\lim_{h \to 0} |Q_l^h - Q_i^h|/d_h = 1$; $Q_l^h - Q_i^h$ are vectors lying on the same halfspace. It is impossible by (6.8). Hence $\frac{\delta_h}{h} = O(l_h)$. The proof is completed. □

Finally in this section, we prove Lemma 6.1 and Lemma 6.2. We first prove Lemma 6.2, assuming that Lemma 6.1 holds.

Proof of Lemma 6.2. We only prove the case for $i = 1$. The other cases are similar. Multiplying both sides of (6.3) by $\frac{\partial w_1}{\partial y_j}$ (here we set $hy + Q_1^h = x$), we have

$$
\begin{aligned}
&\int_{R^N}(\Delta\phi_h - V\phi_h + p(\sum_{i=1}^K w_i)^{p-1}\phi_h)\frac{\partial w_1}{\partial y_j} + \int_{R^N}(V(Q_1^h) - V)w_1\frac{\partial w_1}{\partial y_j} \\
&+ \int_{R^N}\sum_{i\neq 1}(V(Q_i^h) - V)w_i\frac{\partial w_1}{\partial y_j} + \int_{R^N}[(\sum_{i=1}^K w_i)^p - \sum_{i=1}^K w_i^p]\frac{\partial w_1}{\partial y_j} \\
&+ \int_{R^N}[(\sum_{i=1}^K w_i + \phi_h)^p - (\sum_{i=1}^K w_i)^p - p(\sum_{i=1}^K w_i)^{p-1}\phi_h]\frac{\partial w_1}{\partial y_j} \\
&+ \int_{R^N}\sum_{i=1}^K[\Delta(w_i - \hat{w}_i) - V(Q_1^h)(w_i - \hat{w}_i) + w_i^p - (\hat{w}_i)^p]\frac{\partial w_1}{\partial y_j} \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0
\end{aligned}
$$

where the I_i's, $i = 1, \ldots, 6$ are defined in the last equality.

We now compute each term. First for I_1, we have by using the equation for $\frac{\partial w_1}{\partial y_j}$

$$
\begin{aligned}
I_1 &= \int_{R^N}(V(Q_1^h) - V)\phi_h\frac{\partial w_1}{\partial y_j} + p((\sum_{i=1}^K w_i)^{p-1} - w_1^{p-1})\frac{\partial w_1}{\partial y_j}\phi_h \\
&= O(\|(V(Q_1^h) - V)\frac{\partial w_1}{\partial y_j}\|_{L^2(R^N)}\|\phi_h\|_{L^2(R^N)} \\
&+ \|((\sum_{i=1}^K w_i)^{p-1} - w_1^{p-1})\frac{\partial w_1}{\partial y_j}\|_{L^2(R^N)}\|\phi_h\|_{L^2(R^N)}) = O(h^2 + \sum_{i\neq 1}(\delta_{1i}^h)^{1+\sigma})
\end{aligned}
$$

since

$$
\begin{aligned}
&\|(V(Q_1^h) - V)\frac{\partial w_1}{\partial y_j}\|_{L^2(R^N)}^2 \leq Ch^2\int_{R^N}|y|^2(\frac{\partial w_1}{\partial y_j})^2 \leq Ch^2, \\
&\|((\sum_{i=1}^K w_i)^{p-1} - w_1^{p-1})\frac{\partial w_1}{\partial y_j}\|_{L^2(R^N)} \leq C\sum_{i\neq 1}(\delta_{1i}^h)^{\frac{1+\sigma}{2}}.
\end{aligned}
$$

(For the proof of the last inequality, please see the proof of Lemma 3.2.)

For I_2, we obtain

$$
\begin{aligned}
I_2 &= \int_{R^N} (V(Q_1^h) - V) w_1 \frac{\partial w_1}{\partial y_j} = \int_{R^N} (-\frac{\partial V}{\partial x_k}(Q_1^h) h y_k w_1 \frac{\partial w_1}{\partial y_j}) + O(h^2) \\
&= -h \frac{\partial V}{\partial x_j}(Q_1^h) \int_{R^N} y_j \hat{w}_1 \frac{\partial \hat{w}_1}{\partial y_j} + O(h^2) \\
&= -h \frac{\partial V}{\partial x_j}(Q_1^h) V(Q_1^h)^{\frac{2}{p-1} - \frac{N}{2}} \gamma_1 + O(h^2),
\end{aligned}
$$

where

$$
\gamma_1 = \int_{R^N} y_j w \frac{\partial w}{\partial y_j} = \int_{R^N} y_j w w' \frac{y_j}{|y|} = \frac{1}{N} \int_{R^N} w w' |y| < 0.
$$

For I_3, we have

$$
\begin{aligned}
I_3 &= \sum_{i \neq 1} \int_{R^N} (V(Q_i^h) - V) w_i \frac{\partial w_1}{\partial y_j} \\
&= \sum_{i \neq 1} \int_{R^N} (V(Q_i^h) - V(Q_i^h + hz)) w(z) \frac{\partial w_1}{\partial y_j}(z + \frac{Q_i^h - Q_1^h}{h}) \\
&= \sum_{i \neq 1} \int_{R^N} [(V(Q_i^h) - V(Q_1^h)) w(z) \frac{\partial w_1}{\partial y_j}(z + \frac{Q_i^h - Q_1^h}{h}) \\
&\quad + (V(Q_1^h) - V) w(z) \frac{\partial w_1}{\partial y_j}(z + \frac{Q_i^h - Q_1^h}{h})] \\
&= o(\sum_{i \neq 1} \delta_{1i}^h) + O(h |Q_1^h| \int_{R^N} |z| w(z) |\frac{\partial w_1}{\partial y_j}(z + \frac{Q_2^h - Q_1^h}{h})|) \\
&= o(h |Q_1^h|) + o(\sum_{i \neq 1} \delta_{1i}^h)
\end{aligned}
$$

since $V(Q_i^h) - V(Q_1^h) = o(1)$, $\int_{R^N} w \frac{\partial w}{\partial y_j}(z + \frac{Q_i^h - Q_1^h}{h}) = O(\delta_{1i}^h)$ and

$$
\int_{R^N} |z| w \frac{\partial w}{\partial y_j}(z + \frac{Q_i^h - Q_1^h}{h}) = O(w^{1-\delta}(\frac{Q_1^h - Q_i^h}{h}))
$$

by Lemma 2.1.

For I_4, we set $B := \{y \in R^N : |hy - Q_1^h| \leq (1-\delta)\frac{\min_{i\neq 1}|Q_1^h - Q_i^h|}{2}\}$. On B, we have

$$\int_{R^N}((\sum_{i=1}^K w_i)^p - \sum_{i=1}^K w_i^p)\frac{\partial w_1}{\partial y_j} = p\int_{R^N}\sum_{i\neq 1} w_1^{p-1}w_i\frac{\partial w_1}{\partial y_j} + o(\sum_{i\neq 1}\delta_{1i}^h)$$

$$= p\int_{R^N}\sum_{i\neq 1} w^{p-1}\frac{\partial w}{\partial y_j}w(y + \frac{Q_1^h - Q_i^h}{h}) + o(\sum_{i\neq 1}\delta_{1i}^h)$$

$$= \sum_{i\neq 1} w(\frac{Q_1^h - Q_i^h}{h})p\int_{R^N} w^{p-1}w'\frac{y_j}{|y|}\frac{w(y + \frac{Q_1^h - Q_i^h}{h})}{w(\frac{Q_1^h - Q_i^h}{h})} + o(\sum_{i\neq 1}\delta_{1i}^h)$$

$$= \sum_{i\neq 1}\delta_{1i}^h p\int_{R^N} w^{p-1}w'\int_{|\theta|=1}\theta_j e^{-\langle y, \frac{Q_1^h - Q_i^h}{|Q_1^h - Q_i^h|}\rangle}d\theta + o(\sum_{i\neq 1}\delta_{1i}^h)$$

$$= \gamma_2\sum_{i\neq 1}\delta_{1i}^h(\frac{Q_i^h - Q_1^h}{|Q_i^h - Q_1^h|})_j + o(\sum_{i\neq 1}\delta_{1i}^h),$$

where

$$\gamma_2 = p\int_{R^N} w^{p-1}w'\int_{|\theta|=1}\theta_1 e^{-y_1}d\theta = p\int_{R^N} w^{p-1}w'\frac{\partial u_0^*}{\partial r} < 0$$

and u_0^* is the unique solution of the following problem:

$$\Delta v - v = 0,\ v(0) = 1,\ v > 0,\ v = v(|y|),\ y \in R^N. \tag{6.9}$$

(Hence $\frac{\partial u_0^*}{\partial r} > 0$. See Lemma 4.7 in [18].) Outside B, we have

$$\int_{B^c}((\sum_{i=1}^K w_i)^p - \sum_{i=1}^K w_i^p)\frac{\partial w_1}{\partial y_j} = O(\int_{B^c}\sum_{i\neq 1} w_i^{p-1+\delta}w_1^{1-\delta}|\frac{\partial w_1}{\partial y_j}|) = o(\sum_{i\neq 1}\delta_{1i}^h).$$

For I_5, we have

$$|I_5| \leq C\|\phi_h\|_{L^2(R^N)}^2 \leq C(\sum_{i\neq 1}(\delta_{1i}^h)^{1+\sigma}) = o(\sum_{i\neq 1}\delta_{1i}^h).$$

Finally, due to the exponential decay of $\hat{w}_i - w_i$, we have

$$I_6 = O(e^{-\frac{\delta}{h}}) = o(\sum_{i\neq 1}\delta_{1i}^h).$$

Combining the estimates for I_i, $i=1,\dots,6$, we have

$$-h\frac{\partial V}{\partial x_j}(Q_1^h)V(Q_1^h)^{\frac{2}{p-1}-\frac{N}{2}}\gamma_1+\gamma_2\sum_{i\neq 1}\delta_{1i}^h(\frac{Q_i^h-Q_1^h}{|Q_i^h-Q_1^h|})_j+o(h|Q_1^h|+\sum_{i\neq 1}\delta_{1i}^h)=0 \tag{6.10}$$

for $j=1,\dots,N$, which proves (6.5) for $i=1$ with $c=\frac{\gamma_2}{\gamma_1}>0$. □

Proof of Lemma 6.1. We follow closely the arguments of the proof of Lemma A in [25] with slight changes.

Set $\tilde\phi_h=\phi_h/\tilde h$ where $\tilde h=h+\sum_{j\neq i}(\delta_{ij}^h)^{\frac{1+\sigma}{2}}$. All we need to prove is that

$$\int_{R^N}[|\nabla\tilde\phi_h|^2+V(\tilde\phi_h)^2]=O(1).$$

To this end, we note that $\tilde\phi_h$ satisfies

$$\begin{aligned}&\Delta\tilde\phi_h-V\tilde\phi_h+p(\sum_{i=1}^K w_i)^{p-1}\tilde\phi_h+\frac{1}{\tilde h}\sum_{i=1}^K(V(Q_i^h)-V)w_i+\frac{1}{\tilde h}[(\sum_{i=1}^K w_i)^p-\sum_{i=1}^K w_i^p]\\&+\frac{1}{\tilde h}[(\sum_{i=1}^K w_i+h\tilde\phi_h)^p-(\sum_{i=1}^K w_i)^p-p(\sum_{i=1}^K w_i)^{p-1}h\tilde\phi_h]\\&+\frac{1}{\tilde h}\sum_{i=1}^K[\Delta(w_i-\hat w_i)-V(P_i)(w_i-\hat w_i)+w_i^p-\hat w_i^p]=0\text{ in }R^N.\end{aligned}$$

Since

$$|\frac{1}{h}(V(Q_i^h)-V)w_i|\le C|y|w(y)\in L^2(R^N)\cap L^\infty(R^N),\text{ and}$$
$$\frac{1}{\sum_{j\neq i}(\delta_{ij}^h)^{\frac{1+\sigma}{2}}}[(\sum_{i=1}^K w_i)^p-\sum_{i=1}^K w_i^p]\in L^2(R^N)\cap L^\infty(R^N)$$

(see the proof of Lemma 3.2), we have that $\tilde\phi_h$ satisfies

$$\Delta\tilde\phi_h-V\tilde\phi_h+p(\sum_{i=1}^K w_i)^{p-1}\tilde\phi_h+o(1)\tilde\phi_h+F_h=0\text{ in }R^N, \tag{6.11}$$

where $\|F_h\|_{L^\infty(R^N)} \le C$, $\|F_h\|_{L^2(R^N)} \le C$. By elliptic regularity theory, all we need to prove is that $\|\tilde{\phi}_h\|_{L^\infty(R^N)} = O(1)$. (In fact, multiplying (6.11) by $\tilde{\phi}_h$ and integrating by parts we have

$$\int_{R^N} (|\nabla\tilde{\phi}_h|^2 + (V + o(1))\tilde{\phi}_h^2) \le C$$

since $F_h \in L^2(R^N)$ and $(\sum_{i=1}^K w_i)^{p-1} \in L^2(R^N)$.)

Suppose not. Let $|\tilde{\phi}_h(y_h)| = \max_{y\in R^N} |\tilde{\phi}_h(y)|$. By the equation (6.11) for $\tilde{\phi}_h$ (since $V(x)$ satisfies (1.3) and w decays at $+\infty$), it is easy to see that $y_h \in \cup_{j=1}^K B_R(\frac{Q_j^h}{h})$ for some $R > 0$ independent of h. Without loss of generality, we can assume that $y_h \in B_R(\frac{Q_1^h}{h})$. Set $\tilde{\tilde{\phi}}_h(y) = \frac{\tilde{\phi}_h(y+\frac{Q_1^h}{h})}{|\tilde{\phi}_h(y_h)|}$. Then $\|\tilde{\tilde{\phi}}_h\|_{H^1(R^N)} \le C$, $\|\tilde{\tilde{\phi}}_h\|_{L^\infty(R^N)} \le 1$. As $h \to 0$, the limit of $\tilde{\tilde{\phi}}_h(y)$ (by taking a subsequence) exists and is denoted by $\phi_0(y)$. Moreover $\tilde{\tilde{\phi}}_h \to \phi_0(y)$ in $C^1_{loc}(R^N)$, where $\phi_0(y)$ satisfies

$$\Delta\phi_0 - \phi_0 + pw^{p-1}\phi_0 = 0, \quad \phi_0 \in H^1(R^N).$$

It is well known that (see Lemma 6.5 of [18]) $\phi_0(y) = \sum_{j=1}^N a_j \frac{\partial w}{\partial y_j}$ for some constants a_j, $j = 1, \dots, N$. On the other hand, since

$$\tilde{\tilde{\phi}}_h = \frac{u_h - \sum_{i=1}^K w_i}{\tilde{h}|\tilde{\phi}_h(y_h)|},$$

we have that

$$\nabla\tilde{\tilde{\phi}}_h(0) = \frac{0 - 0 - \sum_{i\ne 1} w'(\frac{Q_1^h - Q_i^h}{h})}{\tilde{h}|\tilde{\phi}_h(y_h)|} = \frac{(\gamma^{-1} + o(1))\sum_{i\ne 1}\delta_{1i}^h}{\tilde{h}|\tilde{\phi}_h(y_h)|}$$

(by (1.8)), and hence $\nabla\tilde{\tilde{\phi}}_h(0) \to 0$ as $h \to 0$. Since $\nabla\frac{\partial w}{\partial y_j}(0)$, $j = 1, \dots, N$, are linearly independent, we have $a_j = 0$, $j = 1, \dots, N$, and hence $\phi_0(0) \equiv 0$. But $\tilde{\tilde{\phi}}_h(\tilde{y}_h) = 1$, where $\tilde{y}_h = y_h - \frac{Q_1^h}{h}$ and $|\tilde{y}_h| \le R$ (since $y_h \in B_R(\frac{Q_1^h}{h})$), which is a contradiction to the fact that $\tilde{\tilde{\phi}}_h(\tilde{y}_h) - \phi_0(\tilde{y}_h) \to 0$. Lemma 6.1 is thus proved. □

7. Concluding remarks. In this section, we make some remarks on possible generalizations of Theorems 1.1 and 1.2.

1. If V has K local maximum points, then we can glue the multiple peaks together. In fact we can prove the following more general theorem.

Theorem 7.1. *Let P_j, $j = 1, \dots, K$, be a local maximum point of the potential $V(x)$; i.e., there exists an bounded open set Γ_i such that*

$$P_i \in \Gamma_i, \quad V(P_i) = \max_{x \in \Gamma_i} V(x) > V(P), \forall P \in \Gamma_i \backslash \{P_i\}.$$

Then for any positive integer $K \in \mathbb{Z}$, there exists $h_0 > 0$ such that for any $h < h_0$ there exists a solution u_h of (1.6) *with the following properties:*

(1) *u_h has exactly K local maximum points $Q_1^h, \dots, Q_K^h$ and $Q_i^h \to P_i$, $\frac{|Q_i^h - Q_j^h|}{h} \to \infty$, $i \neq j$, $i, j = 1, \dots, K$, as $h \to 0$, and*

(2) *$u_h(x) \le Ce^{-\beta \frac{\min_{i=1,\dots,K} |x - Q_i^h|}{h}}$ for some $\beta > 0, C > 0$ and $u_h(Q_i^h) \to w(0)$, $i = 1, \dots, K$, as $h \to 0$; i.e., u_h concentrates at $Q_1^h, \dots, Q_K^h$.*

Note. We can allow $P_i = P_j$ for $i \neq j$. By taking $\Gamma_i = \Gamma$, $P_i = P_0$, $i = 1, \dots, K$, we obtain Theorem 1.1.

The proof of the above theorem is very similar to that of Theorem 1.1. In fact, we just need to take $\Lambda_h = \{\mathbf{P} = (P_1, \dots, P_K) \in \Gamma_1 \times \cdots \times \Gamma_K,\ w(\frac{|P_k - P_l|}{h}) < c_0 h,\ k, l = 1, \dots, K,\ k \neq l\}$, where c_0 is a small number and Γ_i is given in Theorem 7.1.

2. It is possible to generalize Theorem 1.1 to more general nonlinearities. In particular, Theorem 1.1 still holds for the following problem:

$$h^2 \Delta u + f(x, u) = 0,\ x \in R^N, \tag{7.1}$$

where $f(x,u) = -V(x)u + K(x)u^p - Q(x)u^q$, where $V(x), K(x), Q(x) > 0$ and $1 < q < p < (\frac{N+2}{N-2})_+$. We note that single-bump solutions with such nonlinearities have been treated in [24] and [28]. In this case, the role of $V(x)$ is replaced by the parametrized energy which was introduced in [24].

3. Theorems 1.1 and 7.1 still hold if we replace the domain R^N by any smooth domain (bounded or unbounded) $\Omega \subset R^N$ and if we impose a Dirichlet condition on the boundary. The proofs are essentially the same. We omit the details. Theorem 1.2 can also be generalized accordingly.

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