AMS/IP Studies in Advanced Mathematics
Volume 51, 2012

Geometrization Program of Semilinear Elliptic Equations

Juncheng Wei

Abstract. Understanding the entire solutions of nonlinear elliptic equations in $\mathbb{R}^N$ such as

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \tag{0.1}$$

is a basic problem in PDE research. This is the context of various classical results in literature like the Gidas-Ni-Nirenberg theorems on radial symmetry, Liouville type theorems, or the achievements around De Giorgi's conjecture. In those results, the geometry of level sets of the solutions turns out to be a posteriori very simple (planes or spheres). On the other hand, problems of the form (0.1) do have solutions with more interesting patterns, and the structure of their solution sets has remained mostly a mystery. A major aspect of our research program is to bring ideas from Differential Geometry into the analysis and construction of entire solutions for two important equations: (1) the Allen-Cahn equation and (2) the nonlinear Schrodinger equation (NLS). Though simple-looking, they are typical representatives of two classes of semilinear elliptic problems. The structure of entire solutions is quite rich. In this survey, we shall establish an intricate correspondence between the study of entire solutions of some scalar equations and the theories of minimal surfaces and constant mean curvature surfaces (CMC).

1. Part I: Geometrization Program of Allen-Cahn Equation

In this section, we survey the studies on entire solutions of Allen-Cahn equation.

1.1. Background. The Allen-Cahn equation in $\mathbb{R}^N$ is the semilinear elliptic problem

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

Originally formulated in the description of bi-phase separation in fluids [**16**] and ordering in binary alloys [**2**], Equation (1.1) has received extensive mathematical study. It is a prototype for the modeling of phase transition phenomena in a variety of contexts.

Introducing a small positive parameter ε and writing $v(x) := u(\varepsilon^{-1}x)$, we get the scaled version of (1.1),

$$\varepsilon^2 \Delta v + v - v^3 = 0 \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

On every bounded domain $\Omega \subset \mathbb{R}^N$, (1.1) is the Euler-Lagrange equation for the action functional

$$J_\varepsilon(v) = \int_\Omega \frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{4\varepsilon}(1-v^2)^2.$$

We observe that the constant functions $v = \pm 1$ minimize J_ε. They are idealized as two *stable phases* of a material in Ω. It is of interest to analyze configurations in which the two phases coexist. These states are represented by stationary points of J_ε, or solutions v_ε of Equation (1.2), that take values close to $+1$ in a subregion of Ω of and -1 in its complement. The theory of Γ-convergence developed in the 70s and 80s, showed a deep connection between this problem and the theory of minimal surfaces, see Modica, Mortola, Kohn, Sternberg, [**38, 57, 58, 59, 86**]. In fact, it is known that for a family u_ε of local minimizers of u_ε with uniformly bounded energy must converge, up to subsequences, in L^1-sense to a function of the form $\chi_E - \chi_{E^c}$ where χ denotes characteristic function, and ∂E has minimal perimeter. Thus the interface between the stable *phases* $u = 1$ and $u = -1$, represented by the sets $[u_\varepsilon = \lambda]$ with $|\lambda| < 1$ approach a minimal hypersurface, see Caffarelli and Córdoba [**12, 13**], Hutchinson and Tonegawa [**37**], Röger and Tonegawa [**65**] for stronger convergence and uniform regularity results on these level surfaces.

1.2. Formal asymptotic behavior of v_ε. Let us argue formally to obtain an idea on how a solution v_ε of Equation (1.2) with uniformly bounded energy should look like near a limiting interface Γ. Let us assume that Γ is a smooth hypersurface and let ν designate a choice of its unit normal. Points δ-close to Γ can be uniquely represented as

$$x = y + z\nu(y), \quad y \in \Gamma, \ |z| < \delta. \tag{1.3}$$

A well known formula for the Laplacian in these coordinates reads as follows:

$$\Delta_x = \partial_{zz} + \Delta_{\Gamma^z} - H_{\Gamma_z}\partial_z. \tag{1.4}$$

Here

$$\Gamma^z := \{y + z\nu(y) \ / \ y \in \Gamma\}.$$

Δ_{Γ^z} is the Laplace-Beltrami operator on Γ^z acting on functions of the variable y, and H_{Γ^z} is its mean curvature. Let $k_1, \ldots, k_N$ denote the principal curvatures of Γ. Then we have the validity of the expression

$$H_{\Gamma^z} = \sum_{i=1}^N \frac{k_i}{1 - zk_i}. \tag{1.5}$$

It is reasonable to assume that the solution is a smooth function of the variables (y, ζ), where $\zeta = \varepsilon^{-1}z$, and the equation for $v_\varepsilon(y,\zeta)$ reads

$$\begin{aligned} &\varepsilon^2 \Delta_{\Gamma^{\varepsilon\zeta}} v_\varepsilon - \varepsilon H_{\Gamma^{\varepsilon\zeta}}(y)\,\partial_\zeta v_\varepsilon + \\ &\qquad \partial_\zeta^2 v_\varepsilon + v_\varepsilon - v_\varepsilon^3 = 0, \quad y \in \Gamma, \quad |\zeta| < \delta\varepsilon^{-1}. \end{aligned} \tag{1.6}$$

We shall make two assumptions:

1. The zero-level set of v_ε lies within a $O(\varepsilon^2)$-neighborhood of Γ, that is in the region $|\zeta| = O(\varepsilon)$ and $\partial_\zeta v_\varepsilon > 0$ along this nodal set, and
2. $v_\varepsilon(y,\zeta)$ can be expanded in powers of ε as

$$v_\varepsilon(y,\zeta) = v_0(y,\zeta) + \varepsilon v_1(y,\zeta) + \varepsilon^2 v_2(y,\zeta) + \cdots, \tag{1.7}$$

where v_j are smooth and uniformly bounded together with their derivatives. We observe also that

$$\int_\Gamma \int_{-\delta/\varepsilon}^{\delta/\varepsilon} [\,\frac{1}{2}|\partial_\zeta v_\varepsilon|^2 + \frac{1}{4}(1-v_\varepsilon^2)^2\,]\, d\zeta\, d\sigma(y) \;\le\; J_\varepsilon(v_\varepsilon) \le C. \tag{1.8}$$

Substituting Expression (1.7) in Equation (1.6), using the first assumption, and letting $\varepsilon \to 0$, we get

$$\begin{aligned} \partial_\zeta^2 v_0 + v_0 - v_0^3 = 0, &\quad (y,\zeta) \in \Gamma\times\mathbb{R},\\ v_0(0,y) = 0, \quad \partial_\zeta(0,y) \ge 0, &\quad y\in\Gamma, \end{aligned} \tag{1.9}$$

while from (1.8) we get

$$\int_{\mathbb{R}} [\,\frac{1}{2}|\partial_\zeta v_0|^2 + \frac{1}{4}(1-v_0^2)^2\,]\, d\zeta \;<\; +\infty. \tag{1.10}$$

Conditions (1.10) and (1.9) force $v_0(y,\zeta) = w(\zeta)$ where w is the unique solution of the ordinary differential equation

$$w'' + w - w^3 = 0, \quad w(0) = 0, \quad w(\pm\infty) = \pm 1,$$

which is given explicitly by

$$w(\zeta) \;:=\; \tanh(\zeta/\sqrt{2}). \tag{1.11}$$

On the other hand, substitution yields that $v_1(y,\zeta)$ satisfies

$$\partial_\zeta^2 v_1 + (1-3w(\zeta)^2)v_1 \;=\; H_\Gamma(y)\, w'(\zeta), \quad \zeta\in(-\infty,\infty). \tag{1.12}$$

Testing this equation against $w'(\zeta)$ and integrating by parts in ζ we get the relation

$$H_\Gamma(y) = 0 \quad \text{for all} \quad y\in\Gamma$$

which tells us precisely that Γ must be a minimal surface, as expected. Hence, we get $v_1 = -h_0(y)w'(\zeta)$ for a certain function $h_0(y)$. As a conclusion, from (1.7) and a Taylor expansion, we can write

$$v_\varepsilon(y,\zeta) = w(\zeta - \varepsilon h_0(y)) + \varepsilon^2 v_2 + \cdots$$

It is convenient to write this expansion in terms of the variable $t = \zeta - \varepsilon h_0(y)$ in the form

$$v_\varepsilon(y,\zeta) = w(t) + \varepsilon^2 v_2(t,y) + \varepsilon^3 v_3(t,y) + \cdots \tag{1.13}$$

Using expression (1.5) and the fact that Γ is a minimal surface, we expand

$$H_{\Gamma^{\varepsilon\zeta}}(y) = \varepsilon^2\zeta\,|A_\Gamma(y)|^2 + \varepsilon^3\zeta^2\, H_3(y) + \cdots$$

where

$$|A_\Gamma|^2 = \sum_{i=1}^8 k_i^2, \quad H_3 = \sum_{i=1}^8 k_i^3.$$

Thus setting $t = \zeta - \varepsilon h_0(y)$ and using (1.13), we compute

$$\begin{aligned} 0 = \Delta v_\varepsilon + v_\varepsilon + v_\varepsilon^3 \;=\; &[\partial_t^2 + (1-3w(t)^2)]\,(\varepsilon^2 v_2 + \varepsilon^3 v_3)\\ -w'(t)\,[\,\varepsilon^3\Delta_\Gamma h_0 \;+\; &\varepsilon^3 H_3\, t^2 + \varepsilon^2|A_\Gamma|^2\,(t+\varepsilon h_0)\,] + O(\varepsilon^4). \end{aligned}$$

And then letting $\varepsilon \to 0$ we arrive to the equations

$$\partial_t^2 v_2 + (1-3w^2)v_2 = |A_\Gamma|^2\, tw', \tag{1.14}$$

$$\partial_t^2 v_3 + (1-3w^2)v_3 = [\Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 + H_3\, t^2]\, w'. \tag{1.15}$$

Equation (1.14) has a bounded solution since $\int_{\mathbb{R}} tw'(t)^2\, dt = 0$. Instead the bounded solvability of (1.15) is obtained if and only if h_0 solves the following elliptic equation in Γ

$$\mathcal{J}_\Gamma[h_0](y) \;:=\; \Delta_\Gamma h_0 + |A_\Gamma|^2 h_0 \;=\; c_0 \sum_{i=1}^{8} k_i^3 \quad \text{in } \Gamma, \tag{1.16}$$

where $c_0 = -\int_{\mathbb{R}} t^2 {w'}^2\, dt / \int_{\mathbb{R}} {w'}^2\, dt$. $\mathcal{J}_\Gamma$ is by definition the *Jacobi operator* of the minimal surface Γ.

Conversely, an interesting problem is to construct entire solutions of Equation (1.2), that exhibit the asymptotic behavior described above, around a given, fixed minimal hypersurface Γ that splits the space $\mathbb{R}^N$ into two components, and for which the coordinates (1.3) are defined for some uniform $\delta > 0$. A key element for such a construction is precisely the question of solvability of Equation (1.16), that determines at main order the deviation of the nodal set of the solution from Γ.

To put the above in terms of the original problem (1.1), we consider a fixed minimal surface $\Gamma \in \mathbb{R}^N$ together with its image by a dilation:

$$\Gamma_\varepsilon \;:=\; \varepsilon^{-1}\Gamma.$$

We want to find an entire solution u_ε to problem (1.1) such that for a function h_ε defined on Γ with

$$\sup_{\varepsilon>0} \|h_\varepsilon\|_{L^\infty(\Gamma)} \;<\; +\infty, \tag{1.17}$$

we have

$$u_\varepsilon(x) = w(\zeta - \varepsilon h_\varepsilon(\varepsilon y)) + O(\varepsilon^2), \tag{1.18}$$

uniformly for

$$x = y + \zeta\nu(\varepsilon y), \quad |\zeta| \le \frac{\delta}{\varepsilon}, \quad y \in \Gamma_\varepsilon,$$

while

$$|u_\varepsilon(x)| \to 1 \quad \text{as dist}\,(x, \Gamma_\varepsilon) \to +\infty. \tag{1.19}$$

In what remains of this section, we shall answer affirmatively this question for some important examples of minimal surfaces. One of them is a non-hyperplanar minimal graph in $\mathbb{R}^9$. In this case the solution of (1.1) is a counterexample to a famous conjecture due to Ennio De Giorgi [**19**]. The second example, in $\mathbb{R}^3$ we find entire solutions of (1.1) with finite Morse index. The last example, in $\mathbb{R}^8$ we find stable entire solutions of (1.1) concentrating on foliations of Simons' cone. Our results suggest extensions of De Giorgi's conjecture for solutions of (1.1) which parallel known classification results for minimal surfaces.

1.3. From Bernstein's to De Giorgi's conjecture. Ennio De Giorgi [**19**] formulated in 1978 the following celebrated conjecture concerning entire solutions of equation (1.1).

De Giorgi's Conjecture: *Let u be a bounded solution of equation* (1.1) *such that $\partial_{x_N} u > 0$. Then the level sets $[u = \lambda]$ are all hyperplanes,* **at least** *for dimension $N \le 8$.*

Equivalently, u must depend only on one Euclidean variable so that it must have the form $u(x) = w((x-p)\cdot\nu)$ for some $p \in \mathbb{R}^N$ and some ν with $|\nu| = 1$ and $\nu_N > 0$.

The condition $\partial_{x_N} u > 0$ implies that the level sets of u are all graphs of functions of the first $N-1$ variables. As we have discussed in the previous section, level sets of non-constant solutions are closely connected to minimal hypersurfaces. De Giorgi's conjecture is in fact a parallel to the following classical statement.

Bernstein's conjecture: *A minimal hypersurface in $\mathbb{R}^N$, which is also the graph of a smooth entire function of $N-1$ variables, must be a hyperplane.*

In other words, if Γ is an *entire minimal graph*, namely

$$\Gamma = \{(x', x_N) \mid x' \in \mathbb{R}^{N-1},\ x_N = F(x')\} \tag{1.20}$$

where F solves the minimal surface equation

$$H_\Gamma \equiv \nabla \cdot \left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}} \right) = 0 \quad \text{in } \mathbb{R}^{N-1}, \tag{1.21}$$

then Γ must be a hyperplane, hence F must be a linear affine function.

Bernstein's conjecture is known to be true up to dimension $N = 8$, see Simons [**85**] and references therein, while it is *false* for $N \ge 9$, as proven by Bombieri, De Giorgi and Giusti [**9**], who found a nontrivial solution to Equation (1.21). To explain the idea of their construction, let us write $x' \in \mathbb{R}^8$ as $x' = (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^4 \times \mathbb{R}^4$ and consider the set

$$T \ := \ \{ (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^8 \mid \ |\mathbf{v}| > |\mathbf{u}| > 0 \ \}. \tag{1.22}$$

The set $\{|\mathbf{u}| = |\mathbf{v}|\} \in \mathbb{R}^8$ is Simons' minimal cone [**85**]. The solution found in [**9**] is radially symmetric in both variables, namely $F = F(|\mathbf{u}|, |\mathbf{v}|)$. In addition, F is positive in T and it vanishes along Simons' cone. Moreover, it satisfies

$$F(|\mathbf{u}|, |\mathbf{v}|) = -F(|\mathbf{v}|, |\mathbf{u}|)\ . \tag{1.23}$$

Let us write $(|\mathbf{u}|, |\mathbf{v}|) = (r\cos\theta, r\sin\theta)$. In [**24**] it is found that there is a function $g(\theta)$ with

$$g(\theta) > 0, \quad \text{in } (\pi/4, \pi/2), \quad g'(\pi/2) = 0 = g(\pi/4), \quad g'(\pi/4) > 0,$$

such that for some $\sigma > 0$,

$$F(|\mathbf{u}|, |\mathbf{v}|) \ = \ g(\theta)\, r^3 + O(r^{-\sigma}) \quad \text{in } T. \tag{1.24}$$

More importantly this asymptotic formula is correct (with obvious adjustments) for the derivatives of F. This nontrivial refinement of the result in [**9**] relies on a theorem of Simon [**84**] and a construction of suitable sub/sup-solutions for the mean curvature operator (1.21).

De Giorgi's conjecture has been established for $N = 2$ by Ghoussoub and Gui [**31**] and for $N = 3$ by Ambrosio and Cabré [**4**]. Savin [**82**] proved its validity for $4 \le N \le 8$ under the additional assumption

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1 \quad \text{for all} \quad x' \in \mathbb{R}^{N-1}. \tag{1.25}$$

The following result shows that De Giorgi's caveat was justified since the conjecture fails for $N \ge 9$.

THEOREM 1. *(del Pino-Kowalczyk-Wei [**24**]) Let $N \ge 9$. Then there is an entire minimal graph Γ which is not a hyperplane, such that all $\varepsilon > 0$ sufficiently small there exists a bounded solution $u_\varepsilon(x)$ of equation* (1.1) *that satisfies properties* (1.17)-(1.19). *Besides, $\partial_{x_N} u_\varepsilon > 0$ and u_ε satisfies condition* (1.25).

A counterexample to De Giorgi's conjecture in dimension $N \ge 9$ was believed to exist for a long time. Partial progress in this direction was made by Jerison and Monneau [**40**] and by Cabré and Terra [**11**]. See also the survey article by Farina and Valdinoci [**29**].

1.4. Outline of the proof. To begin with we observe that a counterexample in dimension $N = 9$ automatically provides one in all dimensions. Thus in what follows we will assume $N = 9$. For a small $\varepsilon > 0$ we look for a solution u_ε of the form (near Γ_ε),

$$u_\varepsilon(x) = w(\zeta - \varepsilon h(\varepsilon y)) + \phi(\zeta - \varepsilon h(\varepsilon y), y), \quad x = y + \zeta \nu(\varepsilon y), \tag{1.26}$$

where $y \in \Gamma_\varepsilon$, ν is a unit normal to Γ with $\nu_N > 0$, h is a function defined on Γ, which is left as a parameter to be adjusted. Setting $r(y', y_9) = |y'|$, we assume a priori in h that

$$\|(1 + r^2) D_\Gamma h\|_{L^\infty(\Gamma)} + \|(1 + r) h\|_{L^\infty(\Gamma)} \ \le \ M \tag{1.27}$$

for some large, fixed number M, also with a uniform control on $(1 + r^3) D^2_\Gamma h$. In addition, because of (1.23) it is natural to assume that u_ε and h satisfy similar symmetries, consistent with those of the minimal graph, namely:

$$u_\varepsilon(\mathbf{u}, \mathbf{v}, x_9) = -u_\varepsilon(P\mathbf{v}, Q\mathbf{u}, -x_9), \quad h(\mathbf{u}, \mathbf{v}) = -h(P\mathbf{v}, Q\mathbf{u}), \tag{1.28}$$

where P and Q are orthogonal transformations of $\mathbb{R}^4$. Most of our argument does not in fact depend on (1.28) and the significance of this assumption becomes apparent only at the end of the construction.

Letting $f(u) = u - u^3$ and using Expression (1.4) for the Laplacian, the equation becomes

$$\begin{aligned} S(u_\varepsilon) := \Delta u_\varepsilon + f(u_\varepsilon) \ &= \\ \Delta_{\Gamma^\zeta_\varepsilon} u_\varepsilon - \varepsilon H_{\Gamma^\zeta_\varepsilon}(\varepsilon y)\, \partial_\zeta u_\varepsilon \ &+ \\ \partial^2_\zeta u_\varepsilon + f(u_\varepsilon) \ &= 0, \quad y \in \Gamma_\varepsilon, \ |\zeta| < \delta/\varepsilon. \end{aligned} \tag{1.29}$$

Letting $t = \zeta - \varepsilon h(\varepsilon y)$, we look for u_ε of the form

$$u_\varepsilon(t, y) = w(t) + \phi(t, y)$$

for a small function ϕ. The equation in terms of ϕ becomes

$$\partial^2_t \phi + \Delta_{\Gamma_\varepsilon} \phi + B\phi + f'(w(t))\phi + N(\phi) + E \ = \ 0 \tag{1.30}$$

where B is a small linear second order differential operator, and

$$E = S(w(t)), \quad N(\phi) = f(w+\phi) - f(w) - f'(w)\phi \approx f''(w)\phi^2.$$

While the expression (1.30) makes sense only for $|t| < \delta\varepsilon^{-1}$, it turns out that the equation in the entire space can be reduced to one similar to (1.30) in entire $\mathbb{R}\times\Gamma_\varepsilon$, where E and the undefined coefficients in B are just cut-off far away, while the operator N is slightly modified by the addition of a small nonlinear, nonlocal operator of ϕ. Rather than solving this problem directly we carry out an infinite dimensional form of Lyapunov-Schmidt reduction, considering a projected version of it,

$$\begin{aligned} \partial_t^2\phi + \Delta_{\Gamma_\varepsilon}\phi + B\phi + f'(w(t))\phi + N(\phi) + E \ &= \\ c(y)w'(t) \quad \text{in } \mathbb{R}\times\Gamma_\varepsilon&, \\ \int_{\mathbb{R}} \phi(t,y)w'(t)\,dt \ = \ 0 \quad \text{for all} \quad y \in \Gamma_\varepsilon&. \end{aligned} \tag{1.31}$$

The error of approximation E has roughly speaking a bound $O(\varepsilon^2 r(\varepsilon y)^{-2}e^{-\sigma|t|})$, and it turns out that one can find a solution $\phi = \Phi(h)$ to problem (1.31) with the same bound. We then get a solution to our original problem if h is such that $c(y)\equiv 0$. Thus the problem is reduced to finding h such that

$$c(y)\int_{\mathbb{R}} w'^2 = \int_{\mathbb{R}} (E + B\Phi(h) + N(\Phi(h)))\,w'dt \equiv 0.$$

A computation similar to that in the formal derivation yields that this problem is equivalent to a small perturbation of Equation (1.16)

$$\mathcal{J}_\Gamma(h) := \Delta_\Gamma h + |A_\Gamma|^2 h \ = \ c_0\sum_{i=1}^{8} k_i^3 + \mathcal{N}(h) \quad \text{in } \Gamma, \tag{1.32}$$

where $\mathcal{N}(h)$ is a small operator. ¿From an estimate by Simon [**84**] we know that $k_i = O(r^{-1})$. Hence $H_3 := \sum_{i=1}^{8} k_i^3 = O(r^{-3})$. A central point is to show that the unperturbed equation (1.16) has a solution $h = O(r^{-1})$, which justifies a posteriori the assumption (1.27) made originally on h. This step uses the asymptotic expression (1.24). The symmetries of the solution (1.28) allow to reduce the domain of the problem and we end up solving it in the sector T (1.22) with zero Dirichlet boundary conditions on Simons' cone. From (1.24) we have that $H_3 = O(g(\theta)r^{-3})$ and we get a priori estimates for the equation $\mathcal{J}_\Gamma(h) = O(g(\theta)r^{-3})$ by constructing a positive barrier of size $O(r^{-1})$. The operator $\mathcal{J}_\Gamma$ satisfies maximum principle and existence thus follows. The full nonlinear equation is then solved with the aid of contraction mapping principle. The detailed proof of this theorem is contained in [**24**].

The program towards the counterexample in [**40**] and [**4**] mimics the classical program that lead to the proof of Bernstein's conjecture: the existence of the counterexample is reduced to establishing the minimizing character of a *saddle solution* in $\mathbb{R}^8$ that vanishes on Simon's cone. Our approach of direct construction is actually applicable to build solutions, which may be in principle unstable, associated to general minimal surfaces, as we illustrate in the next section. We should mention that method of infinite dimensional reduction for the Allen Cahn equation in compact settings has precedents with similar flavor in [**63**], [**39**], [**23**]. Using variational approach, local minimizers were built in [**38**].

1.5. Generalized De Giorgi Conjecture: Stable Solutions. The assumption of monotonicity in one direction for the solution u in De Giorgi's conjecture implies a form of stability, locally minimizing character for u when compactly supported perturbations are considered in the energy. Indeed, the linearized operator $L = \Delta + (1-3u^2)$, satisfies maximum principle since $L(Z) = 0$ for $Z = \partial_{x_N} u > 0$. This implies stability of u, in the sense that its associated quadratic form, namely the second variation of the corresponding energy,

$$\mathcal{Q}(\psi,\psi) := \int_{\mathbb{R}^3} |\nabla \psi|^2 + (3u^2-1)\,\psi^2 \tag{1.33}$$

satisfies $\mathcal{Q}(\psi,\psi) > 0$ for all $\psi \neq 0$ smooth and compactly supported.

Stability of u is sufficient for De Giorgi's statement to hold in dimension $N = 2$, as observed by Dancer [**20**] while it remains an open problem for $3 \leq N \leq 8$. In fact, the monotonicity assumption, together with (1.25), actually implies that u is a *global minimizer*, in the following sense, for any bounded domain Ω

$$\int_\Omega (\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2) \geq \int_\Omega (\frac{1}{2}|\nabla v|^2 + \frac{1}{4}(1-v^2)^2) \tag{1.34}$$

for any function $v \in H^1(\Omega)$ with $v = u$ on $\partial\Omega$. This fact was observed by Alberti-Ambrosio-Cabre [**1**].

Naturally, one would ask the following generalized De Giorgi Conjecture.

Generalized De Giorgi's Conjecture: *Let u be a bounded and* **stable** *solution of equation* (1.1). *Then the level sets $[u = \lambda]$ are all hyperplanes,* **at least** *for dimension $N \leq 7$*

The dimension 7 is again motivated by the study of minimal surface. The generalized De Giorgi's conjecture is in fact a parallel to the following classical statement.

Generalized Bernstein Theorem: *A stable minimal hypersurface must be a hyperplane.*

The stability conjecture for minimal surfaces is known to be true in dimension $N = 3$ by do Carmo and Peng [**28**], Fischer-Colbrie and Schoen [**30**], it is *false* for $N \geq 9$, as proven by Bombieri, De Giorgi and Giusti [**9**], who proved that there is a foliation of Simons's cone in dimension eight or higher. Yau [**91**] asked whether one can prove that a complete minimal hypersurface in $\mathbb{R}^{n+1}$ $(n \leq 7)$ is a hyperplane. Although much hard work on this problem has been done, it remains still open in dimensions $3 \leq n \leq 7$.

Using the foliation of the Simon's cone, the following theorem shows that the generalized De Giorgi Conjecture is not true in dimension 8 (and hence higher).

THEOREM 2. *(Pacard-Wei [**64**]). Let $N = 8$. Then there exists a stable and bounded solution to (1.1) whose level sets approach one of the foliations of the Simons cone.*

1.6. Finite Morse index solutions in $\mathbb{R}^3$. The *Morse index* $m(u)$ is defined as the maximal dimension of a vector space E of compactly supported functions such that

$$\mathcal{Q}(\psi,\psi) < 0 \quad \text{for all} \quad \psi \in E \setminus \{0\}.$$

In view of the discussion so far, it seems natural to associate complete, embedded minimal surfaces Γ with finite Morse index, and solutions of (1.1). The

Morse index of the minimal surface Γ, $i(\Gamma)$, has a similar definition relative to the quadratic form for its Jacobi operator $\mathcal{J}_\Gamma := \Delta_\Gamma + |A_\Gamma|^2$. The number $i(\Gamma)$ is the largest dimension for a vector spaced E of compactly supported smooth functions in Γ with

$$\int_\Gamma |\nabla k|^2\, dV \; - \; \int_\Gamma |A|^2 k^2\, dV \; < 0 \quad \text{for all} \quad k \in E \setminus \{0\}.$$

We point out that for complete, embedded surfaces in $\mathbb{R}^3$, finite index is equivalent to *finite total curvature,* namely

$$\int_\Gamma |K|\, dV \; < \; +\infty$$

where K denotes Gauss curvature of the minimal surface, see §7 of [**34**] and references therein.

1.7. Embedded minimal surfaces of finite total curvature. The theory of embedded, minimal surfaces of finite total curvature in $\mathbb{R}^3$, has reached a notable development in the last 25 years. For more than a century, only two examples of such surfaces were known: the plane and the catenoid. The first nontrivial example was found in 1981 by C. Costa, [**14, 15**]. The *Costa surface* is a genus one complete and properly embedded minimal surface, which outside a large ball has exactly three components (its *ends*). The upper and the lower end are asymptotic to a catenoid, while the middle end is asymptotic to a plane perpendicular to the axis of the catenoid. The complete proof of embeddedness is due to Hoffman and Meeks [**35**]. In [**36**] these authors generalized notably Costa's example by exhibiting a class of three-end, embedded minimal surface, with the same asymptotic behavior the Costa surface far away, but with an array of tunnels connecting the upper and the lower end resulting in a surface with arbitrary genus $\ell \geq 1$. This is known as the Costa-Hoffman-Meeks surface with genus ℓ.

As a special case of the main results of [**25**] we have the following:

THEOREM 3. *(del Pino-Kowalcyzk-Wei* [**25**]*) Let $\Gamma \subset \mathbb{R}^3$ be either a catenoid or a Costa-Hoffman-Meeks surface with genus $\ell \geq 1$. Then for all sufficiently small $\varepsilon > 0$ there exists a solution u_ε of Problem* (1.1) *with the properties* (1.17)-(1.19). *In the case of the catenoid, the solution found is radially symmetric in two of its variables and $m(u_\varepsilon) = 1$. For the Costa-Hoffman-Meeks surface with genus $\ell \geq 1$, we have $m(u_\varepsilon) = 2\ell + 3$.*

1.8. A general case. In what follows Γ is a complete, embedded minimal surface in $\mathbb{R}^3$ with finite total curvature. Then Γ is orientable and the set $\mathbb{R}^3 \setminus \Gamma$ has exactly two components S_+, S_-, see [**34**]. In what follows we fix a continuous choice of unit normal field $\nu(y)$, which conventionally we take it to point towards S_+.

For $x = (x', x_3) \in \mathbb{R}^3$, we denote as before, $r = r(x) = |x'|$. It is known that after a suitable rotation of the coordinate axes, outside the infinite cylinder $r < R_0$ with sufficiently large radius R_0, Γ decomposes into a finite number m of unbounded components $\Gamma_1, \ldots, \Gamma_m$, its *ends*. From a result in [**83**], we know that asymptotically each end of Γ_k either resembles a plane or a catenoid. More precisely, Γ_k can be represented as the graph of a function F_k of the first two variables,

$$\Gamma_k = \{ \, y \in \mathbb{R}^3 \ / \ r(y) > R_0, \ y_3 = F_k(y') \, \}$$

where F_k is a smooth function which can be expanded as

$$F_k(y') = a_k \log r + b_k + \sum_{i=1}^{3} b_{ik} \frac{y_i}{r^2} + O(r^{-3}) \quad \text{as } r \to +\infty, \tag{1.35}$$

for certain constants a_k, b_k, b_{ik}, and this relation can also be differentiated. Here

$$a_1 \le a_2 \le \ldots \le a_m \ , \qquad \sum_{k=1}^{m} a_k \ = \ 0\,. \tag{1.36}$$

We say that Γ has *non-parallel ends* if all the above inequalities are strict.

Let us consider the Jacobi operator of Γ

$$\mathcal{J}_\Gamma(h) \ := \ \Delta_\Gamma h + |A_\Gamma|^2 h \tag{1.37}$$

where $|A_\Gamma|^2 = k_1^2 + k_2^2 = -2K$. A smooth function $z(y)$ defined on Γ is called a *Jacobi field* if $\mathcal{J}_\Gamma(z) = 0$. Rigid motions of the surface induce naturally some bounded Jacobi fields. For example there 4 obvious Jacobi fields associated, respectively, to translations along coordinates axes and rotation around the x_3-axis:

$$z_1(y) = \nu(y) \cdot e_i, \quad y \in \Gamma, \quad i = 1, 2, 3,$$

$$z_4(y) = (-y_2, y_1, 0) \cdot \nu(y), \quad y \in \Gamma. \tag{1.38}$$

We assume that Γ is *non-degenerate* in the sense that these functions are actually *all* the bounded Jacobi fields, namely

$$\{ \ z \in L^\infty(\Gamma) \ / \ \mathcal{J}_\Gamma(z) = 0 \, \} \ = \ \text{span}\, \{ \, z_1, z_2, z_3, z_4 \, \}\,. \tag{1.39}$$

This property is known in some important cases, most notably the catenoid and the Costa-Hoffmann-Meeks surface of any order $\ell \ge 1$. See Nayatani [**61, 62**] and Morabito [**60**].

THEOREM 4. *(del Pino-Kowalcyzk-Wei* [**25**]*) Let $N = 3$ and Γ be a minimal surface embedded, complete with finite total curvature and non-parallel ends, which is in addition nondegenerate. Then for all sufficiently small $\varepsilon > 0$ there exists a solution u_ε of Problem* (1.1) *with the properties* (1.17)-(1.19)*. Moreover, we have*

$$m(u_\varepsilon) \ = \ i(\Gamma).$$

Besides, the solution is non-degenerate, in the sense that any bounded solution of

$$\Delta\phi \, + \, (1 - 3u_\varepsilon^2)\, \phi \, = \, 0 \quad \text{in } \mathbb{R}^3$$

must be a linear combination of the functions Z_i, $i = 1, 2, 3, 4$ defined as

$$Z_i = \partial_i u_\varepsilon, \quad i = 1, 2, 3, \quad Z_4 = -x_2 \partial_1 u_\varepsilon + x_1 \partial_2 u_\varepsilon.$$

It is well-known that if Γ is a catenoid then $i(\Gamma) = 1$. Moreover, in the Costa-Hoffmann-Meeks surface it is known that $i(\Gamma) = 2\ell + 3$ where ℓ is the genus of Γ. See [**61, 62, 60**].

1.9. Further comments. In analogy with De Giorgi's conjecture, it seems plausible that qualitative properties of embedded minimal surfaces with finite Morse index should hold for the level sets of finite Morse index solutions of Equation (1.1), provided that these sets are embedded manifolds outside a compact set. As a sample, one may ask if the following two statements are valid:

- *The level sets of any finite Morse index solution u of* (1.1) *in* $\mathbb{R}^3$*, such that* $\nabla u \neq 0$ *outside a compact set should have a finite, even number of catenoidal or planar ends with a common axis.*

The above fact does hold for minimal surfaces with finite total curvature and embedded ends as established by Ossermann and Schoen. On the other hand, the above statement should not hold true if the condition $\nabla u \neq 0$ outside a large ball is violated. For instance, let us consider the octant $\{x_1, x_2, x_3 \geq 0\}$. Problem (1.1) in the octant with zero boundary data can be solved by a super-subsolution scheme (similar to that in [**41**]) yielding a positive solution. Extending by successive odd reflections to the remaining octants, one generates an entire solution (likely to have finite Morse index), whose zero level set does not have the characteristics above: the condition $\nabla u \neq 0$ far away corresponds to *embeddedness of the ends of the level sets.*

An analog of De Giorgi's conjecture for the solutions that follow in complexity the stable ones, namely those with Morse index one, may be the following:

- *A bounded solution u of* (1.1) *in* $\mathbb{R}^3$ *with* $i(u) = 1$*, and* $\nabla u \neq 0$ *outside a bounded set, must be axially symmetric, namely radially symmetric in two variables.*

The solution we found, with transition on a dilated catenoid has this property. This statement would be in correspondence with results by Schoen [**83**] and López and Ros [**56**]: if $i(\Gamma) = 1$ and Γ has embedded ends, then it must be a catenoid.

1.10. Finite Morse Index Solutions in $\mathbb{R}^2$. The only minimal surface Γ that we can consider in this case is a straight line, to which the planar solution depending on its normal variable can be associated.

A class of solutions to (1.1) with a *finite number of transition lines*, likely to have finite Morse index, has been recently built in [**26**]. The location and shape of these lines is governed by the *Toda system*, a classical integrable model for particles moving on a line with exponential forces between any two closest neighbors:

$$\frac{\sqrt{2}}{24} f_j'' = e^{-\sqrt{2}(f_j - f_{j-1})} - e^{-\sqrt{2}(f_{j+1} - f_j)}, \quad j = 1, \dots k. \tag{1.40}$$

For definiteness we take $f_0 \equiv -\infty$, $f_{k+1} \equiv +\infty$. It is known that for any given solution there exist numbers $a_j^\pm, b_j^\pm$ such that

$$f_j(z) = a_j^\pm |z| + b_j^\pm + O(e^{-|z|}) \quad \text{as } z \to \pm\infty, \tag{1.41}$$

where $a_j^\pm < a_{j+1}^\pm$, $j = 1, \dots, k-1$ (long-time scattering).

The role of this system in the construction of solutions with multiple transition lines in the Allen-Cahn equation in bounded domains was discovered in [**23**]. In entire space the following result holds.

THEOREM 5. *(del Pino-Kowalczyk-Pacard-Wei* [**26**]*) Given a solution f of* (1.40) *if we scale*

$$f_{\varepsilon,j}(z) := \sqrt{2}\,(j - \frac{k+1}{2}) \log \frac{1}{\varepsilon} + f_j(\varepsilon z),$$

then for all small ε there is a solution u_ε with k transitions layers $\Gamma_{\varepsilon,j}$ near the lines $x_2 = f_{\varepsilon,j}(x_1)$. More precisely $\Gamma_{\varepsilon,j}$ are graphs of functions:

$$x_1 = f_{\varepsilon,j}(x_2) + h_{\varepsilon,j}(\varepsilon x_2),$$

where $h_{\varepsilon,j}(z) = O(\varepsilon^\alpha)(|z|+1)$, with some $\alpha > 0$. In addition

$$u_\varepsilon(x_1,x_2) = \sum_{j=1}^{k}(-1)^{j-1} w(x_1 - f_{\varepsilon,j}(x_2) - h_{\varepsilon,j}(\varepsilon x_2)\,) + \sigma_k + O(\varepsilon^\alpha), \tag{1.42}$$

where $\sigma_k = -\frac{1}{2}(1 + (-1)^k)$.

The transition lines are therefore nearly parallel and asymptotically straight, see (1.41). In particular, if $k = 2$ and f solves the ODE

$$\frac{\sqrt{2}}{24} f''(z) = e^{-2\sqrt{2} f(z)}, \quad f'(0) = 0,$$

and $f_\varepsilon(z) := \sqrt{2} \log \frac{1}{\varepsilon} + f(\varepsilon z)$, then there exists a solution u_ε to (1.1) in $\mathbb{R}^2$ with

$$u_\varepsilon(x_1,x_2) = w(x_1 + f_\varepsilon(x_2)\,) + w(x_1 - f_\varepsilon(x_2)) - 1 + O(\varepsilon^\alpha). \tag{1.43}$$

In general in the case of even solutions to the Toda system the deficiency functions $h_{\varepsilon,j}(z)$ decay exponentially as $|z| \to \infty$, c.f. [**26**].

1.11. Remarks. The solutions (1.42) show a major difference between the theory of minimal surfaces and the Allen-Cahn equation, as it is the fact that two separate interfaces *interact*, leading to a major deformation in their asymptotic shapes. We believe that these examples should be prototypical of bounded finite Morse index solutions of (1.1). A finite Morse index solution u should be stable outside a bounded set. If we follow a component of its nodal set along a unbounded sequence, translation and a standard compactness argument leads in the limit to a stable solution. Hence from the result in [**20**] its profile must be one-dimensional and hence its nodal set is a straight line. This makes it plausible that asymptotically the nodal set of u consists of a *finite, even number of straight lines, the ends.* If this is the case, those lines are not distributed in arbitrarily: Gui [**32**] proved that if $e_1, \dots e_{2k}$ are unit vectors in the direction of the ends of the nodal set of a solution of (1.1) in $\mathbb{R}^2$, then the balancing formula $\sum_{j=1}^{2k} e_j = 0$ holds.

As we have mentioned, another (possibly finite Morse index) solution is known, [**41**]. This is the so-called saddle solution. It is built by positive barriers with zero boundary data in a quadrant, and then extended by odd reflections to the rest of the plane, so that its nodal set is an infinite cross, hence having 4 straight ends.

An interesting question is whether one can find a 4-end family of solutions (1.43) depending continuously on the parameter $\varepsilon \in (0, \frac{\pi}{2})$ in such a way that when $\varepsilon \searrow 0$ the ends of the nodal set become parallel while when $\varepsilon \nearrow \frac{\pi}{2}$ they become orthogonal, as in the case of the saddle solution. Similarly, a saddle solutions with

$2k$ ends with consecutive angles $\frac{\pi}{k}$ has been built in [**3**]. One may similarly ask whether this solution is in some way connected to the $2k$-end family (1.42).

1.12. Final Summary. We summary the above discussions on Geometrization Program of Allen-Cahn Equation as follows:

De Giorgi Conjecture	$\longleftrightarrow$	Bernstein Theorem
Ghoussoub-Gui's Proof of De Giorgi Conjecture	$\longleftrightarrow$	Bernstein Theorem in Dimension Two (Bernstein)
Ambrosio-Cabre's Proof of De Giorgi Conjecture	$\longleftrightarrow$	Almgren's Proof of Bernstein Theorem
Savin's Proof of De Giorgi Conjecture	$\longleftrightarrow$	Simons's Proof of Bernstein Theorem
DKW Counterexample of De Giorgi Conjecture	$\longleftrightarrow$	BDG Counterexample of Bernstein Theorem
Cabre-Terra's Saddle Solutions	$\longleftrightarrow$	Simons' Cone
Pacard-Wei's Stable Solutions	$\longleftrightarrow$	Foliations of Simons' Cone

2. Part II: Geometrization Program of Nonlinear Schrodinger Equation

In the second part of this paper, we survey recent results on the study and construction of entire solutions to the nonlinear Schroddinger equation.

2.1. Background. The second part deals with another classical semilinear elliptic problem

$$\Delta u - u + u^p = 0, \quad u > 0, \quad \text{in } \mathbb{R}^N \tag{2.1}$$

where $p > 1$. Equation (2.1) arises for instance as the standing-wave problem for the standard nonlinear Schrödinger equation

$$i\psi_t = \Delta_y \psi + |\psi|^{p-1}\psi,$$

typically $p = 3$, corresponding to that of solutions of the form $\psi(y,t) = u(y)e^{-it}$. It also arises in nonlinear models in Turing's theory biological theory of pattern formation [**90**] such as the Gray-Scott or Gierer-Meinhardt systems, [**46, 45**].

The solutions of (2.1) which decay to zero at infinity are well understood. Problem (2.1) has a radially symmetric solution $w_N(y)$ which approaches 0 at infinity provided that

$$1 < p < \begin{cases} \frac{N+2}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2, \end{cases}$$

see [**89, 7, 8**]. This solution is unique [**50**], and actually any positive solution to (2.1) which vanishes at infinity must be radially symmetric around some point ([**44**]). This solution will be called *Type I Solution*.

Problem (2.1) and its variations have been broadly treated in the PDE literature in the last two decades. These variations are mostly of one of the two types: (2.1) is changed to a non-autonomous problem with a potential depending on the space variable; or (2.1) is considered in a bounded domain under suitable boundary conditions. Typically, in both versions a small parameter is introduced rendering (2.1) a singular perturbation problem. We refer the reader to the works [**43, 47, 48, 53, 67, 68, 77**] and references therein. Many constructions in the literature refer to "multi-bump solutions", built by perturbation of a sum of copies of the basic radial bump w_N suitably scaled, with centers adjusted in equilibrium under appropriate constraints on the potential or the geometry of the underlying domain.

2.2. Type II Solutions: Dancer's Solution. Much less is known about solutions to this equation in entire space which do not vanish at infinity (while they are all known to be bounded, see [**80**]). For example, the solution w_N of (2.1) in $\mathbb{R}^N$ induces a solution in $\mathbb{R}^{N+1}$ which only depends on N variables. This solution vanishes asymptotically in all but one variable. For simplicity, we restrict ourselves to the case $N = 2$, and consider positive solutions $u(x, z)$ to problem (2.1) which are even in z and vanish as $|x| \to +\infty$, namely

$$u(x, z) = u(x, -z) \quad \text{for all } (x, z) \in \mathbb{R}^2, \tag{2.2}$$

and

$$\lim_{|x|\to+\infty} u(x, z) = 0 \quad \text{for all } z \in \mathbb{R}. \tag{2.3}$$

A canonical example is thus built from the one-dimensional bump w_1, which we denote in the sequel just by w, namely the unique solution of the ODE

$$w'' - w + w^p = 0, \quad w > 0, \quad \text{in } \mathbb{R}, \tag{2.4}$$

$$w'(0) = 0, \quad w(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{2.5}$$

corresponding in phase plane to a homoclinic orbit for the equilibrium 0. Using this function we can define a family of solutions u of equation (2.1) with the properties (2.2)-(2.3) setting $u(x, z) := w(x - a)$, $a \in \mathbb{R}$. By analogy with the above terminology, we may call these solutions "single bump-lines". A natural question is whether a solution that satisfies (2.2)-(2.3) and which is in addition even in x must equal $w(x)$. The solution w of (2.1) was found to be isolated by Busca and Felmer in [**10**] in a uniform topology which avoids oscillations at infinity. On the other hand, a second class of solutions which are even both in z and x was discovered by Dancer in [**20**] via local bifurcation arguments. They constitute a one-parameter family of solutions which are periodic in the z variable and originate from $w(x)$.

Let us briefly review Dancer's construction: we consider (2.1) with T-periodic conditions in z,

$$u(x, z + T) = u(x, z) \quad \text{for all } (x, z) \in \mathbb{R}^2, \tag{2.6}$$

and regard $\lambda > 0$ as a bifurcation parameter. The linearized operator around the single bump line is

$$L(\phi) = \phi_{zz} + \phi_{xx} + (pw^{p-1} - 1)\phi.$$

It is well known that the eigenvalue problem

$$\phi_{xx} + (pw^{p-1} - 1)\phi = \lambda\phi, \tag{2.7}$$

has a unique positive eigenvalue λ_1 with $Z(x)$ a positive eigenfunction. We observe that the operator L has a bounded element of its kernel given by

$$Z(x)\cos(\sqrt{\lambda_1}z),$$

which turns out to be the only one which is even, both in x and z variable, and in addition $T = \frac{2\pi}{\sqrt{\lambda_1}}$-periodic in z. Crandall-Rabinowitz bifurcation theorem can then be adapted to yield existence of a continuum of solutions bifurcating at this value of T, periodic in z with period $T_\delta = \frac{2\pi}{\sqrt{\lambda_1}} + O(\delta)$. They are uniformly close to $w(x)$ and their asymptotic formula is:

$$w_\delta(x,z) = w(x) + \delta Z(x)\cos(\sqrt{\lambda_1}z) + O(\delta^2)e^{-|x|}.$$

We refer to the functions w_δ in what follows as *Dancer solutions* or *Type II Solution.*

2.3. Type III Solutions: Multi-bump Line Solutions. Using Dancer's solutions, in [**27**], del Pino, Kowalczyk, Pacard and Wei constructed a new type of solutions of (2.1) in $\mathbb{R}^2$ that have multiple ends in the form of multiple bump-lines, and satisfy in addition properties (2.2)-(2.3). What they actually constructed is a solution $u(x,z)$ which is close, up to lower order terms, to a multi bump-line of the form

$$w_*(x,z) = \sum_{j=0}^{k} w_{\delta_j}(x - f_j(z), z), \tag{2.8}$$

for suitable small numbers δ_j and even functions

$$f_1(z) \ll f_2(z) \ll \cdots \ll f_k(z),$$

which have uniformly small derivatives. The functions f_j cannot be arbitrary and they turn out to satisfy (asymptotically) a second order system of differential equations, the *Toda system*, given by

$$c_p^2 f_j'' = e^{f_{j-1}-f_j} - e^{f_j - f_{j+1}} \quad \text{in } \mathbb{R}, \quad j = 1,\dots,k, \tag{2.9}$$

with the conventions $f_0 = -\infty$, $f_{k+1} = +\infty$, where c_p is an explicit positive constant.

Observe that for an even solution $\mathbf{f} = (f_1,\dots,f_k)$ of this system, function $\mathbf{f}_\alpha$ defined by

$$\mathbf{f}_\alpha = (f_{\alpha 1},\dots,f_{\alpha k}), \quad f_{\alpha j}(z) := f_{1j}(\alpha z) + (j - \frac{k+1}{2})\log\frac{1}{\alpha} \tag{2.10}$$

is also an even solution of the system.

Fix numbers

$$a_1 < a_2 < \cdots < a_k, \qquad \sum_{i=1}^{k} a_i = 0, \tag{2.11}$$

and consider the unique solution $\mathbf{f}$ of system (2.9) for which

$$f_j(0) = a_j, \quad f_j'(0) = 0, \quad j = 1,\dots,k, \tag{2.12}$$

and their associated scalings $\mathbf{f}_\varepsilon$. It is easy to see that (globally) for ε small

$$f_{\varepsilon 1}(z) \ll f_{\varepsilon 2}(z) \ll \cdots \ll f_{\varepsilon k}(z), \qquad f'_{\varepsilon j}(+\infty) = \nu_j \varepsilon,$$

and

$$f_{\varepsilon j}(z) = \nu_j \varepsilon\,|z| + b_j + (j - \frac{k+1}{2}) \log \frac{1}{\alpha} + O(e^{-\vartheta\varepsilon|z|}), \quad \text{as } |z| \to +\infty,$$

for certain scalars b_j and $\vartheta > 0$. These are standard facts about the Toda system that can be found for instance in [**51**]. Thus, each of the multiple ends of $u(x,z)$ is a bump-line that is nearly straight but bent, with an angle slightly distinct than the angles o f the other ends. The Toda system is a classical model describing scattering of k particles distributed on a straight line, which interact only with their closest neighbors with a forces given by a potential depending on the exponentials of their mutual distances. Here the z variable is interpreted as time. In this context, $\mathbf{f}_\alpha$ corresponds to a setting in which the particles starting from the rest, scatter at slightly different, nearly constant small velocities whose average is zero. The latter fact follows the identity $\sum_{j=1}^k f''_{\varepsilon j}(z) = 0$, which also implies conservation of the center of mass used in (2.11).

The main result in [**27**] is:

THEOREM 6. *Assume that $N = 2$ and $p \geq 2$. Given $k \geq 2$ and numbers a_j as in* (2.11)*, for any sufficiently small number $\alpha > 0$, there exists a solution u_α of equation* (2.1) *which satisfies conditions* (2.2)-(2.3)*, and that has the form*

$$u_\varepsilon(x,z) = \sum_{j=1}^{k} w_{\delta_j}(x - f_{\alpha j}(z), z)\,(1 + o(1)).$$

Here $\mathbf{f}_\alpha$ is the scaling (2.10) *of $\mathbf{f}$, the unique solution of* (2.9) *satisfying* (2.12)*, and $o(1) \to 0$, $\delta_j \to 0$ as $\alpha \to 0$.*

2.4. Type IV Solution: Malchiodi's Three-Ray Solutions. On the other hand, in [**66**], Malchiodi constructed another new kind of solutions with three rays of bumps. More precisely, the solutions constructed in [**66**] have the form

$$u(x,z) \approx \sum_{j=1}^{3} \sum_{i=1}^{+\infty} w_2((x,z) - iL\vec{l}_j) \tag{2.13}$$

where $\vec{l}_j, j = 1,2,3$ are three unit vectors satisfying some balancing conditions (**Y**-shaped solutions, see Figure 2). Here w_2 is the unique ground state solution (Type I Solutions) in $\mathbb{R}^2$.

2.5. Solution Type V: Front-Spike Solutions. In [**81**], Santra and Wei constructed solutions with the coexistence of *both fronts* and *bumps*. More precisely we look for positive solutions of the form

$$u_\sharp(x,z) = w(x - f(z)) + \sum_{i=1}^{\infty} w_2((x,z) - \xi_i \vec{e}_1) \tag{2.14}$$

for suitable large $L > 0$ and ξ_i's are such that $\xi_1 - f(0) = L$ and

$$\xi_1 < \xi_2 < \cdots < \xi_i < \cdots$$

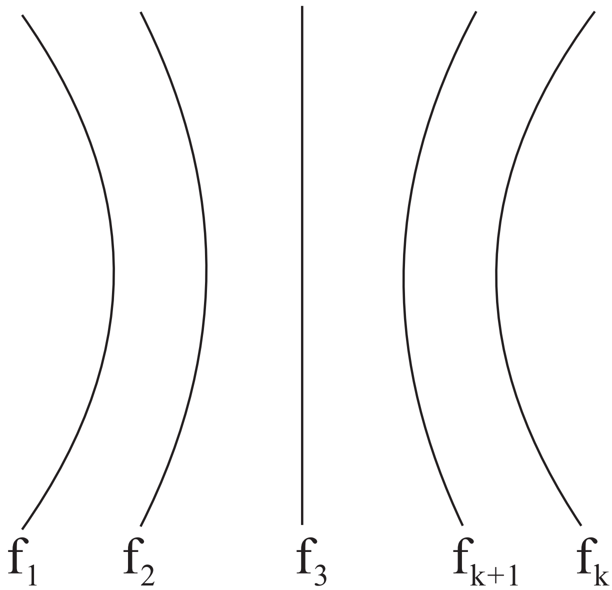

FIGURE 1. Multi-front solutions with even-ends.

and satisfy

$$\xi_j = jL + \mathcal{O}(1) \tag{2.15}$$

for all $j \geq 1$; w is the unique one-dimensional solution and w_2 is the unique positive solution of (2.1) and $\vec{e}_1 = (1, 0)$.

Because of the new interaction between the fronts and bumps, we are led to considering the following second order ODE:

$$\begin{cases} f''(z) = \Psi_L(f, z) & \text{in } \mathbb{R} \\ f(0) = 0, \quad f'(0) = 0, \end{cases} \tag{2.16}$$

where $\Psi_L(f, z)$ is a function measuring the interactions between bumps and fronts. Asymptotically $\Psi_L(f, z) \sim ((f - L)^2 + z^2)^{-\frac{1}{2}} e^{-\sqrt{(f-L)^2+z^2}}$. Let $\alpha = \int_0^{+\infty} \Psi(\sqrt{L^2 + z^2}) dz$.

The main result of [**81**] can be summarized as

THEOREM 7. *Let $N = 2$. For $p > 2$ and sufficiently large $L > 0$, (1.12) admits a one parameter family of positive solution satisfying*

$$\begin{cases} u_L(x, z) = u_L(x, -z) & \text{for all } (x, z) \in \mathbb{R}^2 \\ u_L(x, z) = \left(w_\delta(x - f(z) - h_L(z), z) + \displaystyle\sum_{i=1}^{\infty} w_2((x, z) - \xi_i \vec{e}_1) \right)(1 + o_L(1)) \end{cases} \tag{2.17}$$

where $\delta = \delta_L$ is a small constant, ω_δ is the Dancer's solution, f is the unique solution of (2.16), ξ_j satisfy (2.15) and $o_L(1) \to 0$ as $L \to +\infty$, and the function $\|h_L\|_{C^{2,\mu}_\theta(\mathbb{R}) \oplus \mathcal{E}} \leq C\alpha^{1+\gamma}$ for some constant $\theta > 0, \gamma > 0$. Moreover, the solution has three ends.

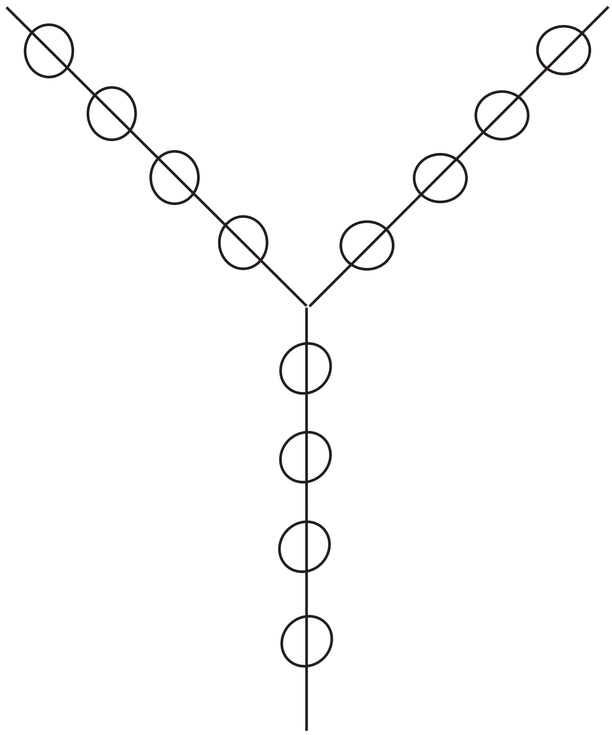

FIGURE 2. Multi-bump solutions with Y shape.

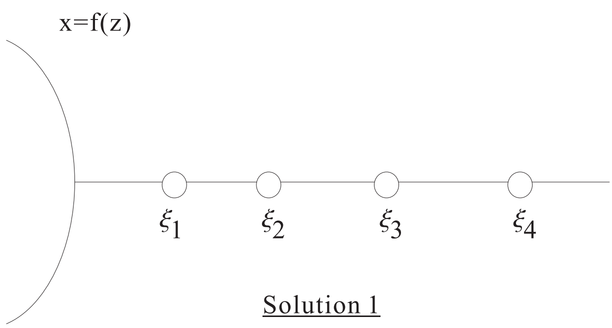

FIGURE 3. Isosceles triunduloid

Figure 3 shows graphly how the solution constructed in Theorem 7 looks like.

A modification of our technique can be used to construct the following two new types of solutions: the first one is a combination of positive bump and infinitely many sign-changing bumps–we call it Solution 2. The second one is two fronts with one (or finitely many) bumps–we call it Solution 3.

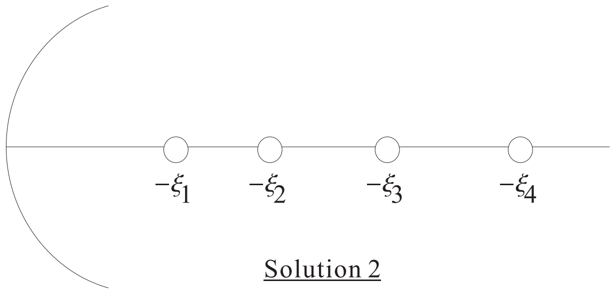

FIGURE 4. Isosceles triunduloid

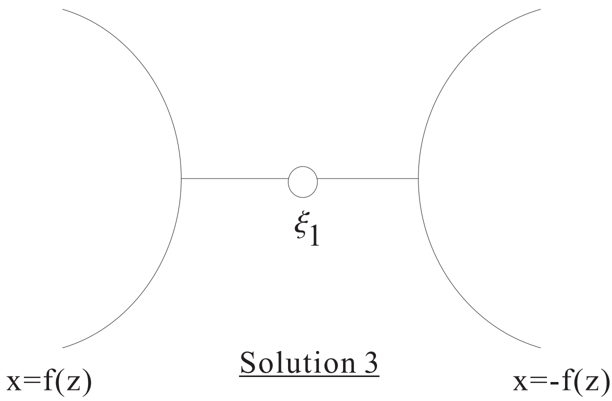

FIGURE 5. End to end gluing construction

2.6. Solution Type VI: Finite-Energy Sign-Changing Solutions. Obviously (2.1) is equivariant with respect to the action of the group of isometries of $\mathbb{R}^N$, it is henceforth natural to ask whether all solutions of (2.1) are radially symmetric. In that regard, the classical result of Gidas, Ni and Nirenberg [**44**] asserts that all *positive* solutions of (2.1) are indeed radially symmetric. Therefore, nonradial solutions, if they exist, are necessarily sign-changing solutions. Berestycki and Lions [**7**], [**8**] and Struwe [**87**] have obtained the existence of infinitely many *radially symmetric* sign-changing solutions to (2.1) in the subcritical case (we also refer to the work of Bartsch and Willem [**6**], Conti, Merizzi and Terracini [**18**] for different approaches and weaker assumptions on the nonlinearity).

The existence of *nonradial* sign-changing solutions to (2.1) was first proved by Bartsch and Willem [**5**] in dimension $N = 4$ and $N \geq 6$. The key idea is to look for solutions invariant under the action of $O(2) \times O(N-2) \subset O(N)$ to recover some compactness property. Later on, this result was generalized by Lorca and Ubilla [**54**] to handle the $N = 5$ dimensional case. The proofs of both results rely on variational methods and the oddness of the nonlinearity. The question of the existence of nonradial solutions remained open in dimensions $N = 2, 3$.

In a recent paper [**76**], Musso, Pacard and Wei constructed unbounded sequences of solutions of (2.1) in any dimensions $N \geq 2$. The solutions they obtained are nonradial, have finite energy and are invariant under the action of $D_k \times O(N-2)$, for some given $k \geq 7$, where $D_k \subset O(2)$ is the dihedral group of rotations and reflections leaving a regular polygon with k sides invariant. Moreover, these solutions are not invariant under the action of $O(2) \times O(N-2)$ and hence they are different from the solutions constructed in [**5**] and [**54**].

The energy functional associated to (2.1) is given by

$$\mathcal{E}(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1}\, dx. \tag{2.18}$$

Denote

$$\mathcal{E} =:= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w_N|^2 + w_N^2)\, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |w_N|^{p+1}\, dx. \tag{2.19}$$

The main result of [**76**] can be summarized as follows

THEOREM 8. *Assume that $k \geq 7$ is a fixed integer. Then, there exist two sequences of integers, $(m_i)_{i\geq 0}$ and $(n_i)_{i\geq 0}$, tending to $+\infty$, and $(u_i)_{i\geq 0}$, a sequence of nonradial sign-changing solutions of (2.1), whose energy $\mathcal{E}(u_i)$ is equal to*

$$\mathcal{E}(u_i) = k\,((m_i + 2n_i)\mathcal{E}) + o(1).$$

Moreover, the solutions u_i are invariant under the action of $D_k \times O(N-2)$ but are not invariant under the action of $O(2) \times O(N-2)$.

2.7. Geometric Analogues with CMC Theory. One of the striking features of all the *six types of solutions*, which are purely PDE results, is that their counterparts and origins can be found in geometric framework. Indeed, there are many examples where correspondence between solutions of (2.1) and those of some geometric problem can be drawn. To illustrate this, we will concentrate on what is perhaps the most spectacular one: the analogy between the theory of complete constant mean curvature surfaces in Euclidean 3-space and the study of entire solutions of (2.1). For simplicity we will restrict ourselves to constant mean curvature surfaces in $\mathbb{R}^3$ which have embedded coplanar ends. In the following we will draw parallels between these geometric objects and families of solutions of (2.1).

Embedded constant mean curvature surfaces of revolution were found by Delaunay in the mid 19th century [**22**]. They constitute a smooth one-parameter family of singly periodic surfaces D_τ, for $\tau \in (0, 1]$, which interpolate between the cylinder $D_1 = S^1(1) \times \mathbb{R}$ and the singular surface $D_0 := \lim_{\tau\to 0} D_\tau$, which is the union of an infinitely many spheres of radius $1/2$ centered at each of the points $(0, 0, n)$ as $n \in \mathbb{Z}$. The Delaunay surface D_τ can be parametrized by

$$X_\tau(x, z) = (\varphi(z) \cos x, \varphi(z) \sin x, \psi(z)) \in D_\tau \subset \mathbb{R}^3,$$

for $(x, z) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. Here the function φ is smooth solution of

$$(\varphi')^2 + \left(\frac{\varphi^2 + \tau}{2}\right)^2 = \varphi^2,$$

and the function ψ is defined by

$$\psi' = \frac{\varphi^2 + \tau}{2}.$$

As already mentioned, when $\tau = 1$, the Delaunay surface is nothing but a right circular cylinder $D_1 = S^1(1) \times \mathbb{R}$, with the unit circle as the cross section. This cylinder is clearly invariant under the continuous group of vertical translations, in the same way that the single bump-line solution of (2.1) is invariant under a one parameter group of translations. It is then natural to agree on the correspondence between

$$\boxed{\begin{array}{c}\text{The cylinder}\\ D_1 = S^1 \times \mathbb{R}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The single bump-line}\\ (x,z) \longmapsto w(x)\end{array}}.$$

Recall that w_2 the unique radially symmetric, decaying solution of (2.1). Inspection of the other end of the Delaunay family, namely when the parameter τ tends to 0, suggests the correspondence between

$$\boxed{\begin{array}{c}\text{The sphere}\\ S^1(1/2)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The radially symmetric solution}\\ (x,z) \longmapsto w_2(\sqrt{x^2+z^2})\end{array}}.$$

To justify this correspondence, let us observe that on the one hand, as the parameter τ tends to 0, the surfaces D_τ resemble a sequence of spheres of radius $1/2$ arranged along the x_3-axis which are connected together by small catenoidal necks. On the other hand an analogous solution of (2.1) can be built as follows. Let $S_R = \mathbb{R}\times(0,R)$ and consider a least energy (mountain pass) solution in $H^1(S_R)$ for the the energy

$$\frac{1}{2}\int_{S_R} |\nabla u|^2 + \frac{1}{2}\int_{S_R} u^2 - \frac{1}{p+1}\int_{S_R} u^{p+1},$$

for large $R > 0$, which we may assume to be even in x and with maximum located at the origin. For R very large, this solution, which satisfies zero Neumann boundary conditions, resembles half of the unique radial, decaying solution w_2 of (2.1). Extension by successive even reflections in z variable yields a solution to (2.1) which resembles a periodic array of radially symmetric solutions of (2.1), with a very large period, along the z-axis. While this is not known, these solutions may be understood as a limit of the branch solutions constructed by Dancer.

More generally, there is a natural correspondence between

$$\boxed{\begin{array}{c}\text{Delaunay surfaces}\\ D_\tau\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Dancer solutions}\\ (x,z) \longmapsto w_\delta(x,z)\end{array}}.$$

To give further credit to this correspondence, let us recall that the Jacobi operator about the cylinder D_1 corresponds to the linearized mean curvature operator when nearby surfaces are considered as normal graphs over D_1. In the above parameter ization, the Jacobi operator reads $J_1 = \frac{1}{\varphi^2}\left(\partial_x^2 + \partial_z^2 + 1\right)$. In this geometric context, it plays the role of the linear operator which is the linearization of (2.1) about the single bump-line solution w. Hence we have the correspondence

$$\boxed{\begin{array}{c}\text{The Jacobi operator}\\ J_1 = \frac{1}{\varphi^2}\left(\partial_x^2 + \partial_z^2 + 1\right)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The linearized operator}\\ L = \partial_x^2 + \partial_z^2 - 1 + p\, w^{p-1}\end{array}}.$$

In the construction of del Pino-Kowalczyk-Pacard-Wei [**27**], the polynomially bounded kernel of the linearized operator L plays a crucial role. Similarly, the polynomially bounded kernel of the Jacobi operator J_1 has some geometric interpretation. Let us recall that we only consider surfaces whose ends are coplanar, the Jacobi fields associated to the action of rigid motions are then given by

$$(x,z) \longmapsto \cos x \qquad \text{and} \qquad (x,z) \longmapsto z\, \cos x,$$

which correspond respectively to the action of translation and the action of the rotation of the axis of the Delaunay surface D_1. Clearly, these Jacobi fields are the counterpart of the elements of the kernel of L which are given by

$$(x,z) \longmapsto \partial_x w(x) \qquad \text{and} \qquad (x,z) \longmapsto z\,\partial_x w(x),$$

since the latter are also generated using the invariance of the problem with respect to the same kind of rigid motions.

Two additional Jacobi fields associated to J_1 are given by

$$(x,z) \longmapsto \cos z \qquad \text{and} \qquad (x,z) \longmapsto \sin z,$$

which are associated to the existence of the family D_τ as τ is close to 1, as can be easily seen using a bifurcation analysis, in a similar way that the functions

$$(x,z) \longmapsto Z(x)\,\cos(\sqrt{\lambda_1} z) \qquad \text{and} \qquad (x,z) \longmapsto Z(x)\,\sin(\sqrt{\lambda_1} z),$$

are associated to the existence of Dancer solutions when the parameter δ is close to 0. These two bifurcation results have their origin in the fact that we have the correspondence between

$$\boxed{\begin{array}{c}\text{The ground state 1 of}\\ \partial_x^2+1\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The first eigenfunction } Z(x) \text{ of}\\ \partial_x^2-1+p\,w^{p-1}\end{array}},$$

both of them associated to negative eigenvalues. The fact that the least eigenvalue of these operators is negative is precisely the reason why a bifurcation analysis can be performed and gives rise to the existence of Delaunay surfaces close to D_1 or Dancer's solutions close to the bump-line w.

A CMC surface $\mathcal{S}$ of finite topology is *Alexandrov-embedded*; if $\mathcal{S}$ is properly immersed, and if each end of $\mathcal{S}$ is embedded; there exist a compact manifold M with boundary of dimension three and a proper immersion $F : M\backslash\{q_1, q_2, \cdots, q_m\} \to \mathbb{R}^3$ such that $F\,|_{\partial M\backslash\{q_1,q_2,\cdots,q_m\}}$ parametrizes M. Moreover, the mean curvature normal of $\mathcal{S}$ points into M. Then we define *tridulonoid* as an Alexandrov embedded CMC surface having zero genus and three ends. Triunduloids are a basic building block for Alexandrov embedded CMC surface with any number of ends. Nonexistence of one end Alexandrov embedded CMC surface was proved by Meeks [**69**]. Kapouleas [**49**], G-Brauckmann [**55**] and Mazzeo-Pacard [**71**] established existence of tridunduloid with small necksize or high symmetry. In fact G-Brauckmann [**55**] used conjugate surface theory construction to obtain families of symmetric embedded complete CMC surfaces. The geometry of moduli space plays an very important role for the understanding of the structure of CMC's.

With these analogies in mind, we can now *translate* the six types of solutions above into the constant mean curvature surface framework.

The result of Theorem 6 corresponds to the connected sum of finitely many copies of the cylinder $S^1(1) \times \mathbb{R}$ which have a common plane of symmetry. The connected sum construction is performed by inserting small catenoidal necks between two consecutive cylinders and this can be done in such a way that the ends of the resulting surface are coplanar. Such a result, in the context of constant mean curvature surfaces, follows at once from [**72**]. It is observed that, once the connected sum is performed the ends of the cylinder have to be slightly bent and moreover, the ends cannot be kept asymptotic to the ends of right cylinders but have to be asymptotic to Delaunay ends with parameters close to 1, in agreement with the result of Theorem 6.

However there is a major difference. The Toda system which governs the level sets has found no analogy in the constant mean curvature surfaces. This is mainly due to the strong interactions in the elliptic equations.

Another (older) construction of complete noncompact constant mean curvature surfaces was performed by N. Kapouleas [**49**] (see also [**71**]) starting with finitely many halves of Delaunay surfaces with parameter τ close to 0 which are connected to a central sphere. The corresponding solutions of (2.1) are the Type IV solutions constructed by A. Malchiodi in [**66**].

It is well known that the story of complete constant mean curvature surfaces in $\mathbb{R}^3$ parallels that of complete locally conformally flat metrics with constant, positive scalar curvature. Therefore, it is not surprising that there should be a correspondence between these objects in conformal geometry and solutions of (2.1). For example, Delaunay surfaces and Dancer solutions should now be replaced by Fowler solutions which correspond to constant scalar curvature metrics on the cylinder $\mathbb{R} \times S^{n-1}$ which are conformal to the product metric $dz^2 + g_{S^{n-1}}$, when $n \geq 3$. These are given by

$$v^{\frac{4}{n-2}}(dz^2 + g_{S^{n-1}}),$$

where $z \longmapsto v(z)$ is a smooth positive solution of

$$(v')^2 - v^2 + \tfrac{n-2}{n}\, v^{\frac{2n}{n-2}} = -\tfrac{2}{n}\,\tau^2.$$

When $\tau = 1$ and $v \equiv 1$ the solution is a straight cylinder while as τ tends to 0 the metrics converge on compacts to the round metric on the unit sphere. The connected sum construction for such Fowler type metrics was performed by R. Mazzeo, D. Pollack and K. Uhlenberk [**74**] (where it is called the dipole construction). N. Kapouleas' construction mentioned above was initially performed by R. Schoen [**88**] (see also R. Mazzeo and F. Pacard [**71**]).

Type IV and Type V solutions belong to the same *tridunduloid* type of solution for (2.1) in $\mathbb{R}^2$ i.e. a solution having three ends.

Finally we comment on the relation between the result Theorem 8 and the corresponding construction for constant mean curvature surfaces in Euclidean 3-space. The construction in Theorem 8 follows very closely a similar construction of compact constant mean curvature surfaces given in [**49**]. In this framework one tries to construct compact constant mean curvature surfaces in Euclidean 3-space by connecting together spheres of radius 1 which are tangent. In the initial configuration, the center of the spheres can be arranged along the edges of a very large regular polygon and also along the rays joining the center to the vertices of the polygon. It is proven in [**49**] that a perturbation argument can be applied and, as a result, a compact constant mean curvature surface is obtained (provided the size of the polygon is large enough). This surface can be constructed in such a way that the pieces which are close to the rays joining the origin to the vertices are embedded and close to embedded constant mean curvature surfaces which are known as *unduloïds*, while the pieces which are close to the edges of the regular polygon are immersed constant mean surfaces which are close to *nodoïds* (in our framework, this corresponds to the fact that we arrange solutions with the same sign along the rays joining the origin to the vertices of the polygon and solutions with alternative sign along the edges of the polygon). A similar construction has also been obtained by Jleli and Pacard in [**42**].

2.8. Geometrization Program of Stationary Nonlinear Schrodinger Equations. We summarize the results on the *six types of* entire solutions of nonlinear scalar equations with analogues of CMC

$\boxed{\text{Ground States}} \longleftrightarrow \boxed{\text{Sphere}}$

$\boxed{\text{Dancer's Solution}} \longleftrightarrow \boxed{\text{Delaunay Surfaces}}$

DKPW's Multiple-bump Solutions	$\longleftrightarrow$	Connected Sums of CMC (Mazzeo, Pacard, Pollack, ...)
Malchiodi's Three Ray Solution	$\longleftrightarrow$	Kapouleas' Noncompact CMC
Santra-Wei's Front-Spike Solutions	$\longleftrightarrow$	End-to-end Gluing of CMC (Mazzeo, Pacard, Ratzin, ...)
Musso-Pacard-Wei's Sign-Changing Solutions	$\longleftrightarrow$	Compact Immersed CMC (Kapouleas, ...)

Acknowledgments: This work has been partly supported an Earmarked Grant from RGC of Hong Kong. I would like to take this opportunity to thank my collaborators Professors Manuel del Pino, M. Kowalczyk, M. Musso, Frank Pacard for sharing their ideas, insights, and for their patience and help.

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Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong.
E-mail address: wei@math.cuhk.edu.hk