SINGULARITY FORMATION FOR THE TWO-DIMENSIONAL HARMONIC MAP FLOW INTO S^2

JUAN DAVILA, MANUEL DEL PINO, AND JUNCHENG WEI

1. Introduction and main result

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We denote by S^2 the standard 2-sphere. We consider the *harmonic map flow* for maps from Ω into S^2, given by the semilinear parabolic equation

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0,T) \tag{1.1}$$

$$u = \varphi \quad \text{on } \partial\Omega \times (0,T) \tag{1.2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega \tag{1.3}$$

for a function $u : \Omega \times [0,T) \to S^2$. Here $u_0 : \bar{\Omega} \to S^2$ is a given smooth map and $\varphi = u_0\big|_{\partial\Omega}$. Local existence and uniqueness of a classical solution follows from the works [5,14,31]. Equation (1.1) formally corresponds to the negative L^2-gradient flow for the Dirichlet energy $\int_\Omega |\nabla u|^2 dx$. This energy is decreasing along smooth solutions $u(x,t)$:

$$\frac{\partial}{\partial t}\int_\Omega |\nabla u(\cdot,t)|^2 = -\int_\Omega |u_t(\cdot,t)|^2.$$

Struwe [31] established the existence of an H^1-weak solution, where just for a finite number of points in space-time loss of regularity occurs. This solution is unique within the class of weak solutions with decreasing energy, see Freire [15].

If $T > 0$ designates the first instant at which smoothness is lost, we must have

$$\|\nabla u(\cdot,t)\|_\infty \to +\infty \quad \text{as} \quad t \uparrow T.$$

Several works have clarified the possible blow-up profiles as $t \uparrow T$. The following fact follows from results by Ding-Tian [13], Lin-Wang [18], Qing [23], Qing-Tian [25], Struwe [31], Topping [33] and Wang [36]:

Along a sequence $t_n \to T$ and points $q_1, \dots, q_k \in \Omega$, not necessarily distinct, $u(x,t_n)$ blows-up occurs at exactly those k points in the form of *bubbling*. Precisely, we have

$$u(x,t_n) \;-\; u_*(x) - \sum_{i=1}^k \left[\, U_i\left(\frac{x-q_i^n}{\lambda_i^n}\right) - U_i(\infty)\,\right] \to 0 \quad \text{in } H^1(\Omega) \tag{1.4}$$

where $u_* \in H^1(\Omega)$, $q_i^n \to q_i$, $0 < \lambda_i^n \to 0$, satisfy for $i \neq j$,

$$\frac{\lambda_i^n}{\lambda_j^n} + \frac{\lambda_j^n}{\lambda_i^n} + \frac{|q_i^n - q_j^n|^2}{\lambda_i^n \lambda_j^n} \to +\infty.$$

The U_i's are entire, finite energy harmonic maps, namely solutions $U : \mathbb{R}^2 \to S^2$ of the equation

$$\Delta U + |\nabla U|^2 U = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty.$$

After stereographic projection, U lifts to a smooth map in S^2, so that its value $U(\infty)$ is well-defined. It is known that U is in correspondence with a complex rational function or its conjugate. Its energy corresponds to the absolute value of the degree of that map times the area of the unit sphere, and hence

$$\int_{\mathbb{R}^2} |\nabla U|^2 = 4\pi m, \quad m \in \mathbb{N}, \tag{1.5}$$

Date: 2019/02/20.

see Topping [33].

In particular, $u(\cdot,t_n) \rightharpoonup u_*$ in $H^1(\Omega)$ and for some positive integers m_i, we have

$$|\nabla u(\cdot,t_n)|^2 \ \rightharpoonup \ |\nabla u_*|^2 + \sum_{i=1}^k 4\pi m_i\, \delta_{q_i} \tag{1.6}$$

in the measures sense, were δ_q denotes the unit Dirac mass at q.

Topping [34] estimated the blow-up rates as $\lambda_i^n = o(T-t_n)^{\frac12}$ (also valid for more general targets), a fact that tells that the blow-up is of "type II", namely it does not occur at a self-similar rate.

A decomposition similar to (1.4) holds if blow-up occurs in infinite time, $T=+\infty$. In such a case one has the additional information that u_* is a harmonic map, and the convergence in (1.4) also holds uniformly in Ω (the latter is called the "no-neck property"), see Qing and Tian [25]. Finer properties of the bubble-decomposition have been found by Topping [33].

A *least energy* entire, non-trivial harmonic map is given by

$$W(x) = \frac{1}{1+|x|^2}\begin{pmatrix} 2x \\ |x|^2-1\end{pmatrix}, \quad x\in\mathbb{R}^2, \tag{1.7}$$

which satisfies

$$\int_{\mathbb{R}^2} |\nabla W|^2 = 4\pi, \quad W(\infty) = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Very few examples are known of solutions, which exhibit the singularity formation phenomenon (1.6), and all of them concern single-point blow-up in radially symmetric *corrotational* classes. When Ω is a disk or the entire space, a 1-corrotational solution of (1.1) is one of the form

$$u(x,t) = \begin{pmatrix} e^{i\theta}\sin v(r,t)\\ \cos v(r,t)\end{pmatrix}, \quad x = re^{i\theta}.$$

Within this class, (1.1) reduces to the scalar, radially symmetric problem

$$v_t = v_{rr} + \frac{v_r}{r} - \frac{\sin v\cos v}{r^2}. \tag{1.8}$$

We observe that the function

$$w(r) = \pi - 2\arctan(r)$$

is a steady state of (1.8) which corresponds precisely to the harmonic map W in (1.7). Indeed,

$$W(x) = \begin{pmatrix} e^{i\theta}\sin w(r)\\ \cos w(r)\end{pmatrix}.$$

Chang, Ding and Ye [6] found the first example of a blow-up solution of problem (1.1)-(1.3) (which was previously conjectured not to exist). It is a 1-corrotational solution in a disk with the blow-up profile

$$u(x,t) = W\left(\frac{x}{\lambda(t)}\right) + O(1), \tag{1.9}$$

with $O(1)$ bounded in H^1-norm and $0<\lambda(t)\to 0$ as $t\to T$. No information on the blow-up rate $\lambda(t)$ is obtained. Angenent, Hulshof and Matano [1] estimated the blow-up rate of 1-corrotational maps as $\lambda(t) = o(T-t)$. Using matched asymptotics formal analysis for problem (1.8), van den Berg, Hulshof and King [3] demonstrated that this rate for 1-corrotational maps should generically be given by

$$\lambda(t) \approx \kappa\frac{T-t}{|\log(T-t)|^2}, \tag{1.10}$$

for some $\kappa>0$. Raphael and Schweyer [28] succeeded to rigorously construct an entire 1-corrotational solution with this blow-up rate.

In this paper we deal with the general, nonsymmetric case in (1.1)-(1.3). Our first result asserts that for any given finite set of points of Ω and suitable initial and boundary values, a solution with a simultaneous blow-up at those points exists, with a profile resembling a translation and rotation of that in (1.9) around each bubbling point.

To state our result, we observe that the functions

$$U_{\lambda,q,Q}(x) := QW\left(\frac{x-q}{\lambda}\right)$$

with $\lambda > 0$, $q \in \mathbb{R}^2$ and Q an orthogonal matrix in $\mathbb{R}^3$ do solve problem (1.5), and all share the least energy property:

$$\int_{\mathbb{R}^2} |\nabla U_{\lambda,q,Q}|^2 = 4\pi.$$

Let us consider the α-rotation matrix around the third axis given by

$$e^{J\alpha} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In all what follows, we consider problem (1.1)-(1.3) with the boundary condition (1.2) given by the constant

$$\varphi(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.11}$$

This constant precisely corresponds to $W(\infty)$. In the radial 1-corrotational equation (1.8), this boundary condition in the disk $\Omega = D(0,R)$ simply corresponds to $v(R,t) = 0$. All results below do apply to a boundary condition which slightly perturbs (1.11), or in the case of entire space $\mathbb{R}^2$ where this value is set as a condition at infinity.

Theorem 1. *Given points $q = (q_1, \dots, q_k) \in \Omega^k$ and any sufficiently small $T > 0$, there exist u_0 such the solution $u_q(x,t)$ of problem (1.1)-(1.3), for φ given by (1.11), blows-up at exactly those k points as $t \uparrow T$. More precisely, there exist numbers $\kappa_i^* > 0$, α_i^* and a function $u_* \in H^1(\Omega) \cap C(\bar\Omega)$ such that*

$$u_q(x,t) - u_*(x) - \sum_{j=1}^{k} e^{J\alpha_i^*}\left[\, W\left(\frac{x-q_i}{\lambda_i}\right) - W(\infty)\,\right] \;\to\; 0 \quad \text{as} \quad t \uparrow T, \tag{1.12}$$

in the H^1 and uniform senses in Ω where

$$\lambda_i(t) = \kappa_i^* \frac{T-t}{|\log(T-t)|^2}\big(1 + o(1)\big) \quad \text{as} \quad t \uparrow T. \tag{1.13}$$

In particular, we have

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 4\pi \sum_{j=1}^{k} \delta_{q_j} \quad \text{as} \quad t \uparrow T.$$

In the next result we analyze the stability of the solutions constructed in Theorem 1. We recall that in the 1-corrotational class in a disc, Chang-Ding-Ye [6] provided robust conditions on initial and boundary data that guarantee finite time blow-up. Raphael-Schweyer [28] established stability *within* the 1-corrotational class in entire space for a solution blowing-up with the rate (1.10). Merle-Raphael-Rodnianski [22] and Raphael-Schweyer [28] conjectured instability outside the 1-corrotational class. Van der Berg and Williams [4] provided formal and numerical evidence that blow-up may indeed be destroyed by small non-radial perturbations of a 1-corrotational singularity.

Our proof of Theorem 1 yields *codimension-one stability* of the predicted blow-up phenomenon in the case of a single blow-up point when no symmetries are assumed. The meaning of this form of stability is as follows:

Theorem 2. *Let $u(x,t)$ be the solution predicted in Theorem 1 of the problem (1.1)-(1.3) that blows-up at a point $q \in \Omega$ and a time $T > 0$. Then there exists a C^1 manifold $\mathcal{M}$ in $C^1(\bar\Omega, S^2)$ with codimension one that contains u_0 such that for any $\tilde u_0 \in \mathcal{M}$ close to u_0, the solution $\tilde u(x,t)$ of problem (1.1)-(1.3) with initial datum $\tilde u_0$ blows-up at a point $\tilde q \in \Omega$ and a time $\tilde T$ which are close respectively to q and T.*

The solutions in Theorems 1 are classical in $[0,T)$. Our next result concerns the continuation of the solution after blow-up. As we have mentioned Struwe [31] defined a global H^1-weak solution of (1.1)-(1.3). Struwe's solution is obtained by just dropping the bubbles appearing at the blow-up time and then restarting the flow. The energy has jumps at each blow-up time generated by this procedure and it is decreasing. Decreasing energy suffices for uniqueness of the weak solution, as proven in [15]. On the other hand the bubble-dropping procedure modifies in time the topology of the image of the solution map. Topping [34] showed a different way to construct a continuation after blow up in the symmetric 1-corrotational class. The solution in [6] is continued after blow-up by attaching a bubble with opposite orientation, which unfolds continuously the energy. The solution referred to is a *reverse bubbling solution.* As emphasized in [34], this continuation has the advantage that, unlike Struwe's solution, it preserves the homotopy class of the map after blow-up. Formal asymptotic rates for 1-corrotational reverse bubbling were found in [3]. In [2] other forms of continuation of radial solutions were found.

We establish that Topping's continuation can be made without symmetry assumptions, with exact asymptotics, for the solution in Theorem 1. We define the bubble $\bar w$ with reverse orientation to that of W as

$$\bar W(x) \ = \ e^{J\pi} W(x) = \frac{1}{1+|x|^2}\begin{pmatrix} -2x \\ |x|^2-1 \end{pmatrix} \ = \ \begin{pmatrix} -e^{i\theta}\sin w(r) \\ \cos w(r) \end{pmatrix}.$$

Theorem 3. *Let $u_q(x,t)$ be the solution in Theorem 1. Then u_q can be continued as an H^1-weak solution in $\Omega\times(0,T+\delta)$, which is continuous except at the points (q_i,T), with the property that, besides expansion (1.12), we have $u_q(x,T) = u_*(x)$*

$$u_q(x,t) - u_*(x) - \sum_{j=1}^k e^{J\alpha_i^*}\left[\, \bar W\left(\frac{x-q_i}{\lambda_i(t)}\right) - \bar W(\infty)\,\right] \ \to \ 0 \quad \text{as} \quad t\downarrow T,$$

in the H^1 and uniform senses in Ω, where

$$\lambda_i(t) = \kappa_i^* \frac{t-T}{|\log(t-T)|^2}. \tag{1.14}$$

We observe that the energy in this continuation fails to be decreasing: it has a jump exactly at time T and it goes back to its previous level immediately after.

We consider a question related to Theorem 3 treated in the 1-corrotational symmetric class in [34] and in [2]: the occurrence of perfectly smooth solutions which spontaneously develop a singularity in finite time by the addition of an infinitely concentrated bubble which instantaneously raises the energy in a multiple of 4π. We find that the typical rate for this backward bubbling is $\dot\lambda(t)$ of order $\frac{t-T}{|\log(t-T)|}$ rather than (1.14). This was formally derived in [3].

Theorem 4. *Given points $q_1,\dots,q_k$ in Ω and any sufficiently small $T>0$ there exists an H^1-weak solution $u(x,t)$ of problem (1.1)-(1.3) in $\Omega\times(0,T+\delta)$ which is continuous except at the points (q_i,T), it is smooth in $\Omega\times(0,T]$ and has spontaneous reverse bubbling at the points q_i in the form*

$$u(x,t) - u(x,T) - \sum_{j=1}^k \left[\, W\left(\frac{x-q_i}{\lambda_i(t)}\right) - W(\infty)\,\right] \ \to \ 0 \quad \text{as} \quad t\downarrow T,$$

in the H^1 and uniform senses in Ω, where for some positive numbers κ_i

$$\lambda_i(t) = \kappa_i \frac{t-T}{|\log(t-T)|}.$$

Before proceeding into the proof we make some further comments. It is plausible that the solutions of the form described in Theorem 1 represent a form of "generic" bubbling phenomena for the two-dimensional harmonic map flow, still too general in the form (1.4). For instance, it is reasonable to think, yet unknown, that the limits along any sequence should have the same elements in the bubble decomposition. On the other hand, is it possible to have bubbles other than those induced by W or $\bar{W}$, and or decomposition in several bubbles at the same point? Some evidence is already present in the literature. It is known that in the more general symmetry class of the d-corrotational ones, $d \geq 1$,

$$u(x,t) = \begin{pmatrix} e^{di\theta}\sin v(r,t) \\ \cos v(r,t) \end{pmatrix}, \quad x = re^{i\theta}$$

are steady states $v = w_d(r) = \pi - 2\arctan(r^d)$ which **do not lead** to blow-up, at least for $d \geq 4$ (conjectured for $d = 2, 3$). See Guan-Gustafson-Tsai [16]. On the other hand, no *bubble trees* in finite time exist in the 1-corrotational class. See Van der Hout [35]. Infinite time multiple bubbling was found by Topping [33] in a target different from S^2. On the other hand, bubbling rates faster than (1.13) do exist in the 1-corrotational case, but they are not stable, see Raphaël and Schweyer [29]. Many other results on bubbling phenomena, and regularity for harmonic maps and the harmonic map flow are available in the literature, we refer the reader to the book the book by Lin and Wang [19].

In bubbling phenomena in this and related problems very little is known in nonradial situations. The method in [28, 29], was successfully applied to very related blow-up phenomena in dispersive equations in symmetric classes. See for instance Rodnianski-Sterbenz [30] Merle-Raphaël-Rodnianski [22], Raphaël [26], Raphaël-Rodnianski [27]. Our results share a flavor with finite time multiple blow-up in the subcritical semilinear heat equation, as in the results by Merle and Zaag [21]. Bubbling associated to the critical exponent has been recently studied in [9, 10]. Our approach is parabolic in nature. It is based on the construction of a good approximation and then linearizing inner and outer problems. An appropriate inverse for the inner equation is then found (which works well if the parameters of the problems are suitably adjusted) which makes it possible the application of fixed point arguments. The general approach, which we call inner-outer gluing, has already been applied to various singular perturbation elliptic problems, see for instance [11, 12]. A major difficulty we have to overcome is the coupled nonlocal ODE satisfied by the scaling and rotation parameter.

2. The 1-corrotational harmonic maps and their linearized operator

The harmonic map equation for functions $U : \mathbb{R}^2 \to S^2$ is the elliptic problem

$$\Delta U + |\nabla U|^2 U \quad \text{in } \mathbb{R}^2, \quad |U| = 1. \tag{2.1}$$

For $\xi \in \mathbb{R}^2$, $\omega \in \mathbb{R}$, $\lambda > 0$, we consider the family of solutions of (2.1) given by the following 1-corrotational harmonic maps

$$U_{\lambda,\xi,\omega}(x) := Q_\omega \, W\Big(\frac{x-\xi}{\lambda}\Big),$$

where W is the canonical least energy harmonic map

$$W(y) \;=\; \frac{1}{1+|y|^2}\begin{pmatrix} 2y \\ |y|^2 - 1\end{pmatrix}, \quad y \in \mathbb{R}^2,$$

and Q_ω is the ω-rotation matrix

$$Q_\omega := \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The linearized operator for (2.1) around $U = U_{\lambda,\xi,\omega}$ is the elliptic operator

$$L_U[\varphi] = \Delta\varphi + |\nabla U|^2\varphi + 2(\nabla\varphi\cdot\nabla U)U.$$

Differentiating U with respect to each of its parameters we obtain functions that annihilate this operator, namely solutions of $L_U[\varphi]=0$. Setting $y=\frac{x-\xi}{\lambda}$, these functions are

$$\begin{aligned}\partial_\lambda U_{\lambda,\xi,\omega}(x)=&\frac{1}{\lambda}Q_\omega\nabla W(y)\cdot y,\\ \partial_\omega U_{\lambda,\xi,\omega}(x)=&(\partial_\omega Q_\omega)W(y)\\ \partial_{\xi_j}U_{\lambda,\xi,\omega}(x)=&\frac{1}{\lambda}Q_\omega\partial_{y_j}W(y).\end{aligned}$$

We observe that

$$(\partial_\omega Q_\omega)=Q_\omega J_0,\quad J_0=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}.$$

We can represent $W(y)$ in polar coordinates,

$$W(y)=\begin{pmatrix}e^{i\theta}\sin w(\rho)\\ \cos w(\rho)\end{pmatrix},\quad w(\rho)=\pi-2\arctan(\rho),\quad y=\rho e^{i\theta}.$$

We notice that

$$w_\rho=-\frac{2}{1+\rho^2},\quad \sin w=-\rho w_\rho=\frac{2\rho}{1+\rho^2},\quad \cos w=\frac{\rho^2-1}{1+\rho^2},$$

and derive the alternative expressions

$$\begin{aligned}\partial_\lambda U_{\lambda,\xi,\omega}(x)&=\frac{1}{\lambda}Q_\omega Z_{01}(y), & Z_{01}(y)&=\rho w_\rho(\rho)\,E_1(y)\\ \partial_\omega U_{\lambda,\xi,\omega}(x)&=\quad Q_\omega Z_{02}(y), & Z_{02}(y)&=\rho w_\rho(\rho)\,E_2(y)\\ \partial_{\xi_j}U_{\lambda,\xi,\omega}(x)&=\frac{1}{\lambda}Q_\omega Z_{11}(y), & Z_{11}(y)&=w_\rho(\rho)\,[\cos\theta\,E_1(y)+\sin\theta\,E_2(y)]\\ \partial_{\xi_j}U_{\lambda,\xi,\omega}(x)&=\frac{1}{\lambda}Q_\omega Z_{12}(y), & Z_{12}(y)&=w_\rho(\rho)\,[\sin\theta\,E_1(y)-\cos\theta\,E_2(y)],\end{aligned}\tag{2.2}$$

where

$$E_1(y)=\begin{pmatrix}e^{i\theta}\cos w(\rho)\\ -\sin w(\rho)\end{pmatrix},\quad E_2(y)=\begin{pmatrix}ie^{i\theta}\\ 0\end{pmatrix}.$$

The relation $|U_{\lambda,\xi,\omega}|=1$ implies that all the functions Z_{ij} are pointwise orthogonal to $U_{\lambda,\xi,\omega}$. In fact the vectors $E_1(y)$, $E_2(y)$ constitute an orthonormal basis of the tangent space to S^2 at the point $W(y)$.

We have $L_W[Z_{lj}]=0$ where for a function $\phi(y)$ we define

$$L_W[\phi]=\Delta_y\phi+|\nabla W(y)|^2\phi+2(\nabla W(y)\cdot\nabla\phi)W(y).$$

In addition to the elements (2.2) in the kernel of L_W there are also two other relevant functions in the kernel, namely

$$\begin{aligned}Z_{-1,1}&=\rho^2w_\rho(\rho)(\cos\theta E_1-\sin\theta E_2)\\ Z_{-1,2}&=\rho^2w_\rho(\rho)(\sin\theta E_1+\cos\theta E_2).\end{aligned}\tag{2.3}$$

It is worth noticing the connection between this operator and L_U which is given by

$$L_U[\varphi]=\frac{1}{\lambda^2}Q_\omega L_W[\phi],\quad \varphi(x)=\phi(y),\quad y=\frac{x-\xi}{\lambda}.$$

The linearized operator at functions orthogonal to U. It will be especially significant to compute the action of L_U on functions with values pointwise orthogonal to U. In what remains of this section we will derive various formulas that will be very useful later on.

For an arbitrary function $\Phi(x)$ with values in $\mathbb{R}^3$ we denote

$$\Pi_{U^\perp}\Phi := \Phi - (\Phi\cdot U)U$$

Then the following formula holds:

$$L_U[\Pi_{U^\perp}\Phi] = \Pi_{U^\perp}\Delta\Phi + \tilde L_U[\Phi] \tag{2.4}$$

where

$$\tilde L_U[\Phi] := |\nabla U|^2\Pi_{U^\perp}\Phi - 2\nabla(\Phi\cdot U)\nabla U,$$

and

$$\nabla(\Phi\cdot U)\nabla U = \partial_{x_j}(\Phi\cdot U)\,\partial_{x_j}U.$$

A very convenient expression for $\tilde L_U[\Phi]$ is obtained if we use polar coordinates. Writing in complex notation

$$\Phi(x) = \Phi(r,\theta), \quad x = \xi + re^{i\theta},$$

we have

$$\tilde L_U[\Phi] = -\frac{2}{\lambda}w_\rho(\rho)\,[(\Phi_r\cdot U)Q_\omega E_1 - \frac{1}{r}(\Phi_\theta\cdot U)Q_\omega E_2], \quad \rho = \frac{r}{\lambda}. \tag{2.5}$$

Proof of formula (2.4). We have that

$$\Delta(\Phi\cdot U) = (\Delta\Phi)\cdot U + 2\nabla\Phi\cdot\nabla U - (\Phi\cdot U)|\nabla U|^2$$

so that

$$\Delta\Pi_{U^\perp}\Phi = \Pi_{U^\perp}\Delta\Phi - 2(\nabla\Phi\cdot\nabla U)U + 2\Phi\cdot U|\nabla U|^2U + 2\nabla(\Phi\cdot U)\nabla U$$

Now,

$$\nabla[(\Phi\cdot U)U]\cdot\nabla U = (\Phi\cdot U)|\nabla U|^2$$

hence

$$\nabla\Pi_{U^\perp}\Phi\cdot\nabla U = \nabla\Phi\cdot\nabla U - (\Phi\cdot U)|\nabla U|^2.$$

It follows that

$$L_U[\Pi_{U^\perp}\Phi] = \Pi_{U^\perp}\Delta\Phi + |\nabla U|^2\Pi_{U^\perp}\Phi - 2\nabla(\Phi\cdot U)\nabla U$$

as desired. □

Proof of formula (2.5). We have that

$$\begin{aligned}\nabla(\Phi\cdot U)\nabla U &= \partial_r(\Phi\cdot U)\partial_r U + \frac{1}{r^2}\partial_\theta(\Phi\cdot U)\partial_\theta U\\ &= (\Phi_r\cdot U)\partial_r U + \frac{1}{r^2}(\Phi_\theta\cdot U)\partial_\theta U\\ &\quad + \frac{1}{r^2}(\Phi\cdot\partial_\theta U)\partial_\theta U.\end{aligned}$$

We see that

$$\partial_r U = \frac{1}{\lambda}w_\rho(\rho)E_1, \quad \frac{1}{r}\partial_\theta U = \frac{1}{\lambda}\frac{\sin w(\rho)}{\rho}E_2 = -\frac{1}{\lambda}w_\rho(\rho)E_2.$$

Hence

$$\begin{aligned}2\nabla(\Phi\cdot U)\nabla U &= \frac{2}{\lambda}w_\rho(\rho)[(\Phi_r\cdot U)E_1 - \frac{1}{r}(\Phi_\theta\cdot U)E_2]\\ &\quad + \frac{2}{\lambda}w_\rho(\rho)^2[(\Phi\cdot E_1)E_1 + (\Phi\cdot E_2)E_2].\end{aligned}$$

On the other hand, $|\nabla U|^2 = 2w_\rho^2$ and

$$\Pi_{U^\perp}\Phi = (\Phi\cdot E_1)Q_\omega E_1 + (\Phi\cdot E_2)Q_\omega E_2$$

hence

$$\tilde L_U[\Phi] = |\nabla U|^2 \Pi_{U^\perp}\Phi - 2\nabla(\Phi\cdot U)\nabla U \;=\; -\frac{2}{\lambda} w_\rho(\rho)[(\Phi_r\cdot U)E_1 - \frac{1}{r}(\Phi_\theta\cdot U)E_2]$$

and the proof is concluded. □

Next we single out two consequences of formula (2.5) which will be crucial for later purposes. Let us assume that $\Phi(x)$ is a C^1 function $\Phi:\Omega\to\mathbb{C}\times\mathbb{R}$, which we express in the form

$$\Phi(x) \;=\; \begin{pmatrix}\varphi_1(x)+i\varphi_2(x)\\ \varphi_3(x)\end{pmatrix}. \tag{2.6}$$

We also denote

$$\varphi = \varphi_1 + i\varphi_2, \quad \bar\varphi = \varphi_1 - i\varphi_2$$

and define the operators

$$\operatorname{div}\varphi = \partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2, \quad \operatorname{curl}\varphi = \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1.$$

We have the validity of the following formula

$$\tilde L_U[\Phi] = \tilde L_U[\Phi]_0 + \tilde L_U[\Phi]_1 + \tilde L_U[\Phi]_2\ , \tag{2.7}$$

where

$$\begin{cases} \tilde L_U[\Phi]_0 = \lambda^{-1}\rho w_\rho^2\,\big[\,\operatorname{div}(e^{-i\omega}\varphi)\,Q_\omega E_1 + \operatorname{curl}(e^{-i\omega}\varphi)\,Q_\omega E_2\,\big]\\ \tilde L_U[\Phi]_1 = -\,2\lambda^{-1} w_\rho\cos w\,\big[\,(\partial_{x_1}\varphi_3)\cos\theta + (\partial_{x_2}\varphi_3)\sin\theta\,\big]\,Q_\omega E_1\\ \qquad\qquad -\,2\lambda^{-1} w_\rho\cos w\,\big[\,(\partial_{x_1}\varphi_3)\sin\theta - (\partial_{x_2}\varphi_3)\cos\theta\,\big]\,Q_\omega E_2\ ,\\ \tilde L_U[\Phi]_2 = \quad\lambda^{-1}\rho w_\rho^2\,\big[\,\operatorname{div}(e^{i\omega}\bar\varphi)\,\cos 2\theta - \operatorname{curl}(e^{i\omega}\bar\varphi)\,\sin 2\theta\,\big]\,Q_\omega E_1\\ \qquad\qquad +\,\lambda^{-1}\rho w_\rho^2\,\big[\,\operatorname{div}(e^{i\omega}\bar\varphi)\,\sin 2\theta + \operatorname{curl}(e^{i\omega}\bar\varphi)\,\cos 2\theta\,\big]\,Q_\omega E_2. \end{cases} \tag{2.8}$$

Proof of formula (2.5). Let us assume first $\omega=0$. We notice that

$$\Phi_r = \cos\theta\,\partial_{x_1}\Phi \;+\; \sin\theta\,\partial_{x_2}\Phi$$
$$\frac{1}{r}\Phi_\theta = -\sin\theta\,\partial_{x_1}\Phi \;+\; \cos\theta\,\partial_{x_2}\Phi\,.$$

Then

$$\begin{aligned} &\Phi_r\cdot U\\ &= \sin w\,\big[\,\partial_{x_1}\varphi_1\cos^2\theta + \partial_{x_2}\varphi_1\cos\theta\sin\theta + \partial_{x_1}\varphi_2\sin\theta\cos\theta \;+\; \partial_{x_2}\varphi_2\sin\theta\sin\theta\,\big]\\ &\quad+\;\cos w\,\big[\,\partial_{x_1}\varphi_3\cos\theta + \partial_{x_2}\varphi_3\sin\theta\,\big]\\ &= \frac{1}{2}\sin w\,[\,[\partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2] + \cos 2\theta[\partial_{x_1}\varphi_1 - \partial_{x_2}\varphi_2] + [\partial_{x_2}\varphi_1 + \partial_{x_1}\varphi_2]\sin 2\theta\,]\\ &\quad+\;\cos w\,\big[\,\partial_{x_1}\varphi_3\cos\theta + \partial_{x_2}\varphi_3\sin\theta\,\big]\,, \end{aligned}$$

while

$$\begin{aligned} &\frac{1}{r}\Phi_\theta\cdot U\\ &= \sin w\,\big[\,-\partial_{x_1}\varphi_1\cos\theta\sin\theta + \partial_{x_2}\varphi_1\cos^2\theta - \partial_{x_1}\varphi_2\sin^2\theta + \partial_{x_2}\varphi_2\cos\theta\sin\theta\,\big]\\ &\quad+\;\cos w\,\big[\,-\partial_{x_1}\varphi_3\sin\theta + \partial_{x_2}\varphi_3\cos\theta\,\big]\\ &= \frac{1}{2}\sin w\,[\,[\partial_{x_2}\varphi_1 - \partial_{x_1}\varphi_2] + \cos 2\theta[\partial_{x_1}\varphi_2 + \partial_{x_2}\varphi_1] + [\partial_{x_2}\varphi_2 - \partial_{x_1}\varphi_1]\sin 2\theta\,]\\ &\quad+\;\cos w\,\big[\,-\partial_{x_1}\varphi_3\sin\theta + \partial_{x_2}\varphi_3\cos\theta\,\big]. \end{aligned}$$

Since $\sin w = -\rho w_\rho$, we obtain from formula (2.5)

$$\begin{aligned}\tilde L_U[\Phi] \ =& \lambda^{-1}\rho w_\rho^2\,[\operatorname{div}\varphi E_1 + \operatorname{curl}\varphi E_2]\\ &+\ \lambda^{-1}\rho w_\rho^2\,[\operatorname{div}\bar\varphi\cos 2\theta - \operatorname{curl}\bar\varphi\sin 2\theta]\,E_1\\ &+\ \lambda^{-1}\rho w_\rho^2\,[\operatorname{div}\bar\varphi\sin 2\theta + \operatorname{curl}\bar\varphi\cos 2\theta]\,E_2\\ &-\ 2\lambda^{-1}w_\rho\cos w\,[\partial_{x_1}\varphi_3\cos\theta + \partial_{x_2}\varphi_3\sin\theta]E_1\\ &-\ 2\lambda^{-1}w_\rho\cos w\,[\partial_{x_1}\varphi_3\sin\theta - \partial_{x_2}\varphi_3\cos\theta]E_2.\end{aligned}$$

For the case of a general ω, we observe that we have the identity

$$\tilde L_U[\Phi] = Q_\omega \tilde L_{Q_{-\omega}U}[Q_{-\omega}\Phi]$$

hence we obtain the desired result by substituting in the above formula φ by $e^{-i\omega}\varphi$. The proof is complete. □

Another corollary of formula (2.5) that we single out is the following: assume that

$$\Phi(x) = \begin{pmatrix}\phi(r)e^{i\theta}\\ 0\end{pmatrix}, \quad x = \xi + re^{i\theta}, \quad \rho = \frac{r}{\lambda}$$

where $\phi(r)$ is complex valued. Then

$$\tilde L_U[\Phi] = \frac{2}{\lambda}w_\rho(\rho)^2\left[\operatorname{Re}\,(e^{-i\omega}\partial_r\phi(r))Q_\omega E_1 + \frac{1}{r}\operatorname{Im}\,(e^{-i\omega}\phi(r))Q_\omega E_2\right]. \tag{2.9}$$

Proof of Formula (2.9). We have

$$\begin{aligned}\Phi_r\cdot U \ &=\ \begin{bmatrix}\phi_r e^{i\theta}\\ 0\end{bmatrix}\cdot\begin{bmatrix}e^{i(\theta+\omega)}\sin w\\ \cos w\end{bmatrix} \ =\ \operatorname{Re}\,(\phi_r e^{-i\omega})\sin w\\ \frac{1}{r}\Phi_\theta\cdot U \ &=\ \frac{1}{r}\begin{bmatrix}i\phi\, e^{i\theta}\\ 0\end{bmatrix}\cdot\begin{bmatrix}e^{i(\theta+\omega)}\sin w\\ \cos w\end{bmatrix} \ =\ \frac{1}{r}\operatorname{Re}\,(i\phi\, e^{-i\omega})\sin w.\end{aligned}$$

Since $\sin w = -\rho w_\rho$, formula (2.5) then yields the validity of (2.9). □

A final result in this section is a computation (in polar coordinates) of the operator L_U acting on a function of the form

$$\Phi(x) = \varphi_1(\rho,\theta)Q_\omega E_1 + \varphi_2(\rho,\theta)Q_\omega E_2, \quad x = \xi + \lambda\rho e^{i\theta}.$$

We have:

$$\begin{aligned}L_U[\Phi] = \lambda^{-2}&\left(\partial_\rho^2\varphi_1 + \frac{\partial_\rho\varphi_1}{\rho} + \frac{\partial_\theta^2\varphi_1}{\rho^2} + (2w_\rho^2 - \frac{1}{\rho^2})\varphi_1 - \frac{2}{\rho^2}\partial_\theta\varphi_2\cos w\right)Q_\omega E_1\\ &+\lambda^{-2}\left(\partial_\rho^2\varphi_2 + \frac{\partial_\rho\varphi_2}{\rho} + \frac{\partial_\theta^2\varphi_2}{\rho^2} + (2w_\rho^2 - \frac{1}{\rho^2})\varphi_2 + \frac{2}{\rho^2}\partial_\theta\varphi_1\cos w\right)Q_\omega E_2.\end{aligned} \tag{2.10}$$

Proof of Formula (2.10). Let us assume that

$$\Phi(\rho,\theta) = \varphi_1(\rho,\theta)Q_\omega E_1 + \varphi_2(\rho,\theta)Q_\omega E_1.$$

We notice that

$$\Delta_x\Phi = \lambda^{-2}\left(\partial_\rho^2\Phi + \frac{1}{\rho}\partial_\rho\Phi + \frac{1}{\rho^2}\partial_\theta^2\Phi\right).$$

Since $\Phi\cdot U = 0$ we get

$$L_U[\Phi] = \Pi_{U^\perp}\Delta_x\Phi + |\nabla U|^2\Phi.$$

Then

$$\Delta_x(\varphi_1 Q_\omega E_1) = (\Delta_x\varphi_1)Q_\omega E_1 + 2\lambda^{-2}\partial_\rho\varphi_1\partial_\rho Q_\omega E_1 + \varphi_1 Q_\omega\Delta_x E_1$$

We have that

$$Q_\omega E_{1\rho} = -U w_\rho,$$
$$Q_\omega E_{1\rho\rho} = -w_{\rho\rho} U - Q_\omega E_1 w_\rho^2,$$
$$Q_\omega E_{1\theta\theta} = -\cos w(\sin w\, U + \cos w\, Q_\omega E_1).$$

Thus

$$\lambda^2 \Delta_x (Q_\omega E_1) = -Q_\omega E_1 \left(w_\rho^2 + \frac{\cos^2 w}{\rho^2} \right) + U \left(w_{\rho\rho} + \frac{w_\rho}{\rho} + \frac{\sin w \cos w}{\rho^2} \right).$$

By definition of $w(\rho)$ we have

$$w_{\rho\rho} + \frac{w_\rho}{\rho} - \frac{\sin w \cos w}{\rho^2} = 0.$$

Hence

$$\lambda^2 \Delta_x (Q_\omega E_1) = -Q_\omega E_1 \left(w_\rho^2 + \frac{\cos^2 w}{\rho^2} \right) = -\frac{1}{\rho^2} Q_\omega E_1 - 2\frac{\sin w \cos w}{\rho^2} U.$$

Thus we have

$$\lambda^2 \Delta_x (\varphi_1 Q_\omega E_1)$$
$$= \lambda^2 (\Delta_x \varphi_1) Q_\omega E_1 - 2\varphi_{1\rho} w_\rho U + \frac{2}{\rho^2} \varphi_{1\theta} \cos w\, Q_\omega E_2 - \frac{\varphi_1 E_1}{\rho^2} - 2\frac{\sin w \cos w}{\rho^2} U.$$

Using this and (3.9) we find after a direct computation

$$L_U[\varphi_1 Q_\omega E_1] = \left(\Delta_x \varphi_1 + (2w_\rho^2 - \frac{1}{\rho^2})\varphi_1 \right) Q_\omega E_1 + \frac{2}{\rho^2} \varphi_{1\theta} \cos w\, Q_\omega E_2.$$

On the other hand, we find similarly

$$\lambda^2 \Delta_x (\varphi_2 Q_\omega E_2) = \lambda^2 (\Delta_x \varphi_2) Q_\omega E_2 - \frac{2}{\rho^2} \varphi_{2\theta} (\sin w U + \cos w Q_\omega E_1)$$

and hence

$$L_U[\varphi_2 E_2] = \left(\Delta_x \varphi_2 + \lambda^{-2} (2w_\rho^2 - \frac{1}{\rho^2})\, \varphi_2 \right) Q_\omega E_2 - \lambda^{-2} \frac{2}{\rho^2} \varphi_{2\theta} \cos w\, Q_\omega E_1.$$

The proof is concluded. □

3. The ansatz for a blowing-up solution

In what follows we shall closely follow notation and computational formulas derived in the previous sections, here applied in a time-dependent framework. Thus we consider the semilinear parabolic equation

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0, T) \tag{3.1}$$
$$u = u_{\partial\Omega} \quad \text{on } \partial\Omega \times (0, T) \tag{3.2}$$
$$u(\cdot, 0) = u_0 \quad \text{in } \Omega \tag{3.3}$$

for a function $u : \bar\Omega \times [0, T) \to S^2$. Here $u_0 : \bar\Omega \to S^2$ is a given smooth map and

$$u_{\partial\Omega} = u_0\big|_{\partial\Omega} \equiv \mathbf{e_3} \quad \text{on } \partial\Omega. \tag{3.4}$$

Here and in what follows we denote

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.5}$$

The constant boundary value $\mathbf{e_3}$ precisely corresponds to $W(\infty)$ where W is the standard 1-corotational harmonic map (1.7). This choice of $u_{\partial\Omega}$ as a constant is made for convenience, in fact sufficiently small non-constant perturbations of it are also admissible in all arguments below.

In order to keep the notation to a minimum, we shall do this in the case $k = 1$ of a single bubbling point. We will later indicate the necessary changes in the general case. Given a fixed point $q \in \Omega$, and any sufficiently small number $T>0$ we look for a solution $u(x,t)$ of problem (3.1)-(3.3) which at main order looks like

$$U(x,t) := U_{\lambda(t),\xi(t),\omega(t)}(x) = Q_{\omega(t)}\, W\Big(\frac{x-\xi(t)}{\lambda(t)}\Big)$$

for certain functions $\xi(t)$, $\lambda(t)$ and $\omega(t)$ of class $C^1([0,T])$ such that

$$\xi(T)=q, \quad \lambda(T)=0,$$

so that $u(x,t)$ blows-up at time T and the point q. We shall find values for these functions so that for a small remainder $v(x,t)$ we have that $u=U+v$ solves (3.1)-(3.3) for $u_0(x)=U(x,0)+v(x,0)$. Let us denote

$$S(u) := -u_t + \Delta u + |\nabla u|^2 u$$

A useful observation that we make is that as long as the constraint $|u|=1$ is kept at all times and $u=U+v$ with $|v|\le \frac12$ uniformly, then for u to solve equation (3.1) it suffices that

$$S(U+v) = b(x,t)U \tag{3.6}$$

for some scalar function b. Indeed, we observe that since $|u|\equiv 1$ we have

$$b\,(U\cdot u) = S(u)\cdot u = -\frac12\frac{d}{dt}|u|^2 + \frac12\Delta|u|^2 = 0,$$

and since $U\cdot u \ge \frac12$, we find that $b\equiv 0$.

We can parametrize all small functions $v(x,t)$ such that $|U+v|=1$ in the form

$$v = \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U, \tag{3.7}$$

where φ is an arbitrary small function with values into $\mathbb{R}^3$, and

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi\cdot U)U, \quad a(\zeta) := \sqrt{1-|\zeta|^2}-1.$$

Using that

$$\Delta U + |\nabla U|^2 U = 0$$

we find the following expansion for $S(U+v)$ with v given by (3.7):

$$S(U+\Pi_{U^\perp}\varphi + aU) = -U_t - \partial_t \Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi) + c(\Pi_{U^\perp}\varphi)U$$

where for $\zeta = \Pi_{U^\perp}\varphi$, $a=a(\zeta)$,

$$\begin{aligned} L_U(\zeta) &= \Delta\zeta + |\nabla U|^2\zeta + 2(\nabla U\cdot\zeta)U \\ N_U(\zeta) &= \big[2\nabla(aU)\cdot\nabla(U+\zeta) + 2\nabla U\cdot\nabla\zeta + |\nabla\zeta|^2 + |\nabla(aU)|^2\big]\zeta - aU_t \\ &\quad + 2\nabla a\nabla U, \\ c(\zeta) &= \Delta a - a_t + (|\nabla(U+\zeta+aU)|^2 - |\nabla U|^2)(1+a) - 2\nabla U\cdot\nabla\zeta \end{aligned} \tag{3.8}$$

Since we just need to have an equation of the form (3.6) satisfied, we find that

$$u = U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U$$

solves (3.1) if and only if φ satisfies

$$0 = -U_t - \partial_t\Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi) + b(x,t)U, \tag{3.9}$$

for some scalar function b. The logic of the construction goes like this: We decompose φ into the sum of two functions $\varphi = \varphi^i + \varphi^o$, the "inner" and "outer" solutions and reduce equation (3.9) to solving a system of two equations in (φ^i,φ^o) that we call the inner and outer problems.

The inner function $\varphi^i(x,t)$ will be assumed supported only near $x=\xi(t)$ and better read as a function of the scaled space variable $y = \frac{x-\xi(t)}{\lambda(t)}$ with zero initial condition and such that $\varphi^i\cdot U = 0$, so that $\Pi_{U^\perp}\varphi^i = \varphi^i$. The outer function $\varphi^o(x,t)$ will be made out of several pieces and its role is essentially to satisfy (3.9) far away from the concentration point $x=\xi(t)$.

We write equation (3.9) in the following way:

$$0 = -\partial_t\varphi^i + L_U[\varphi^i] + \tilde L_U[\varphi^o] - \Pi_{U^\perp}[\partial_t\varphi^o - \Delta\varphi^o + U_t] \\ + N_U(\varphi^i + \Pi_{U^\perp}\varphi^o) + (\varphi^o\cdot U)U_t + bU. \tag{3.10}$$

For the outer problem, we consider a function Φ^0 that depends explicitly on the parameter functions chosen in such a way that $\Pi_{U^\perp}[\partial_t\Phi^0 - \Delta\Phi^0 + U_t]$ gets concentrated near $x = \xi(t)$ by elimination of the terms in the first error U_t associated to dilation and rotation. Then we write

$$\varphi^o(x,t) = \Phi^0(x,t) + \Psi^*(x,t). \tag{3.11}$$

For the inner solution, we consider a smooth smooth cut-off function $\eta_0(s)$ with $\eta_0(s) = 1$ for $s < 1$ and $= 0$ for $s > \frac{3}{2}$. We also consider a positive, large smooth function $R(t) \to +\infty$ as $t \to T$ that we will later specify. We define

$$\eta(x,t) := \eta_0\left(R(t)^{-1}|y|\right), \quad y = \frac{x-\xi(t)}{\lambda(t)}$$

and let

$$\varphi^i(x,t) = \eta(x,t)Q_\omega\phi(y,t), \quad y = \frac{x-\xi(t)}{\lambda(t)}$$

for a function $\phi(y,t)$ with initial condition $\phi(\cdot,0) = 0$ that satisfies $\phi(\cdot,t)\cdot W \equiv 0$, defined for $|y| \le 2R(t)$ and that vanishes as $t \to T$. Then we have

$$Q_{-\omega}L_U[\varphi^i] = \lambda^{-2}\eta L_W[\phi] + (\Delta_x\eta)\phi + 2\lambda^{-1}\nabla_x\eta\nabla_y\phi$$
$$Q_{-\omega}\varphi^i_t = \eta\big(\phi_t - \lambda^{-1}\dot\lambda y\cdot\nabla_y\phi - \lambda^{-1}\dot\xi\cdot\nabla_y\phi + \dot\omega Q_{-\omega}\partial_\omega Q_\omega\phi\big) + \eta_t\phi.$$

Equation (3.10) then becomes

$$\begin{aligned} 0 = {} & \lambda^{-2}\eta Q_\omega[-\lambda^2\phi_t + L_W[\phi] + \lambda^2 Q_{-\omega}\tilde L_U[\Psi^*]] \\ & + \eta Q_\omega(\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi + \lambda^{-1}\dot\xi\cdot\nabla_y\phi - \dot\omega J\phi) \\ & + \tilde L_U[\Phi^0] + \Pi_{U^\perp}[\partial_t\Phi^0 - \Delta_x\Phi^0 + U_t] \\ & - \partial_t\Psi^* + \Delta\Psi^* + (1-\eta)\tilde L_U[\Psi^*] + Q_\omega[(\Delta_x\eta)\phi + 2\nabla_x\eta\nabla_x\phi - \eta_t\phi] \\ & + N_U(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0 + \Psi^*)) + ((\Psi^* + \Phi^0)\cdot U)U_t + bU. \end{aligned} \tag{3.12}$$

Next we will define precisely the operator Φ^0 and estimate the quantity

$$\tilde L_U[\Phi^0] + \Pi_{U^\perp}[\partial_t\Phi^0 - \Delta_x\Phi^0 + U_t]. \tag{3.13}$$

The idea is to choose Φ^0 such that $\partial_t\Phi^0 - \Delta_x\Phi^0 + U_t \approx 0$ whenever $|x-\xi| \gg \lambda$, so that in particular the last error term in the outer equation (3.11) is of smaller order.

Invoking formulas (2.2) to compute U_t we get

$$U_t = \dot\lambda\partial_\lambda U_{\lambda,\xi,\omega} + \dot\omega\partial_\omega U_{\lambda,\xi,\omega} + \partial_\xi U_{\lambda,\xi,\omega}\cdot\dot\xi = \mathcal{E}_0 + \mathcal{E}_1,$$

where, setting $y = \frac{x-\xi}{\lambda} = \rho e^{i\theta}$, we have

$$\mathcal{E}_0(x,t) = -Q_\omega[\frac{\dot\lambda}{\lambda}\rho w_\rho(\rho)\, E_1(y) \,+\, \dot\omega\rho w_\rho(\rho)\, E_2(y)\,]$$

$$\begin{aligned} \mathcal{E}_1(x,t) = {} & -\frac{\dot\xi_1}{\lambda}\, w_\rho(\rho)\, Q_\omega[\,\cos\theta\, E_1(y) + \sin\theta\, E_2(y)] \\ & - \frac{\dot\xi_2}{\lambda}\, w_\rho(\rho)\, Q_\omega[\sin\theta\, E_1(y) - \cos\theta\, E_2(y)\,]. \end{aligned}$$

Since $\mathcal{E}_1$ has faster space decay in ρ than $\mathcal{E}_0$ we will choose Φ^0 to be an approximate solution of

$$\Phi^0_t - \Delta_x\Phi^0 + \mathcal{E}_0 = 0. \tag{3.14}$$

For $x = \xi + re^{i\theta}$ and $r \gg \lambda$ we have

$$\mathcal{E}_0(x,t) = -\frac{2r}{r^2+\lambda^2}\left[\dot\lambda Q_\omega E_1 + \lambda\dot\omega Q_\omega E_2\right] \approx -\frac{2r}{r^2+\lambda^2}\begin{bmatrix}(\dot\lambda + i\lambda\dot\omega)e^{i(\theta+\omega)}\\ 0\end{bmatrix}.$$

Here and in what follows we let

$$p(t) = \lambda(t)e^{i\omega(t)}.$$

Then

$$-\frac{2r}{r^2+\lambda^2}\begin{bmatrix}(\dot\lambda + i\lambda\dot\omega)e^{i(\theta+\omega)}\\ 0\end{bmatrix} = -\frac{2r}{r^2+\lambda^2}\begin{bmatrix}\dot p(t)e^{i\theta}\\ 0\end{bmatrix} =: \tilde{\mathcal{E}}_0(x,t).$$

With the aid of Duhamel's formula for the standard heat equation, we find that the following function is a good approximate solution of $\Phi^0_t - \Delta_x\Phi^0 + \tilde{\mathcal{E}}_0 = 0$ and hence of (3.14). We define

$$\Phi^0[\omega,\lambda,\xi] := \begin{bmatrix}\varphi^0(r,t)e^{i\theta}\\ 0\end{bmatrix} \tag{3.15}$$

$$\varphi^0(r,t) = -\int_{-T}^t \dot p(s) r k(z(r), t-s)\,ds \tag{3.16}$$

$$z(r) = \sqrt{r^2+\lambda^2}, \quad k(z,t) = 2\frac{1-e^{-\frac{z^2}{4t}}}{z^2},$$

where for technical reasons that will be made clear later on, $p(t)$ is also assumed to be defined for negative values of t. See section 17 for a derivation of the formula (3.16).

A direct computations yields

$$\Phi^0_t + \Delta_x\Phi^0 + \tilde{\mathcal{E}}_0 = \tilde{\mathcal{R}}_0 + \tilde{\mathcal{R}}_1, \quad \tilde{\mathcal{R}}_0 = \begin{pmatrix}\mathcal{R}_0\\ 0\end{pmatrix}, \quad \tilde{\mathcal{R}}_1 = \begin{pmatrix}\mathcal{R}_1\\ 0\end{pmatrix}$$

where

$$\mathcal{R}_0 := -re^{i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t \dot p(s)(zk_z - z^2k_{zz})(z(r),t-s)\,ds$$

and

$$\begin{aligned}\mathcal{R}_1 := &-e^{i\theta}\mathrm{Re}\,(e^{-i\theta}\dot\xi(t))\int_{-T}^t \dot p(s)\,k(z(r),t-s)\,ds\\ &+\frac{r}{z^2}e^{i\theta}\,(\lambda\dot\lambda(t) - \mathrm{Re}\,(re^{i\theta}\dot\xi(t)))\int_{-T}^t \dot p(s)\;zk_z(z(r),t-s)\,ds.\end{aligned}$$

We observe that $\mathcal{R}_1$ is actually a term of smaller order. Using formulas (2.7), (2.9) and the facts

$$\frac{\lambda^2 r}{z^4} = \frac{1}{4\lambda}\rho w_\rho^2, \quad \frac{r}{z^2}(1-\cos w) = \frac{1}{2\lambda}\rho w_\rho^2,$$

we derive an expression for the quantity (3.13):

$$\begin{aligned}&\tilde L_U[\Phi^0] + \Pi_{U^\perp}[-U_t + \Delta\Phi^0 - \Phi^0_t]\\ &= \tilde L_U[\Phi^0] - \mathcal{E}_1 + \Pi_{U^\perp}[\tilde{\mathcal{E}}_0] - \mathcal{E}_0 + \Pi_{U^\perp}[\tilde{\mathcal{R}}_0] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1]\\ &= \mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1]\end{aligned}$$

where

$$\mathcal{K}_0[p,\xi] = \mathcal{K}_{01}[p,\xi] + \mathcal{K}_{02}[p,\xi]$$

with

$$\begin{aligned}\mathcal{K}_{01}[p,\xi] := -\frac{2}{\lambda}\rho w_\rho^2\int_{-T}^t &\left[\mathrm{Re}\,(\dot p(s)e^{-i\omega(t)})Q_\omega E_1 + \mathrm{Im}\,(\dot p(s)e^{-i\omega(t)})Q_\omega E_2\right]\\ &\cdot k(z,t-s)\,ds\end{aligned} \tag{3.17}$$

$$
\begin{aligned}
\mathcal{K}_{02}[p,\xi] :=& \frac{1}{\lambda}\rho w_\rho^2 \left[\dot\lambda - \int_{-T}^t \operatorname{Re}(\dot p(s)e^{-i\omega(t)}) r k_z(z,t-s) z_r\, ds\right] Q_\omega E_1 \\
&- \frac{1}{4\lambda}\rho w_\rho^2 \cos w \left[\int_{-T}^t \operatorname{Re}(\dot p(s)e^{-i\omega(t)})\,(z k_z - z^2 k_{zz})(z,t-s)\, ds\right] Q_\omega E_1 \\
&- \frac{1}{4\lambda}\rho w_\rho^2 \left[\int_{-T}^t \operatorname{Im}(\dot p(s)e^{-i\omega(t)})\,(z k_z - z^2 k_{zz})(z,t-s)\, ds\right] Q_\omega E_2, \qquad (3.18)
\end{aligned}
$$

$$
\mathcal{K}_1[p,\xi] := \frac{1}{\lambda} w_\rho \left[\operatorname{Re}\big((\dot\xi_1 - i\dot\xi_2)e^{i\theta}\big) Q_\omega E_1 + \operatorname{Im}\big((\dot\xi_1 - i\dot\xi_2)e^{i\theta}\big) Q_\omega E_2\right]. \qquad (3.19)
$$

We insert this decomposition in equation (3.12) and see that we will have a solution to the equation if the pair (ϕ, Ψ^*) solves the *inner-outer gluing system*

$$
\begin{cases}
\lambda^2 \phi_t = L_W[\phi] + \lambda^2 Q_{-\omega}\left[\tilde L_U[\Psi^*] + \mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]\right] \quad \text{in } \mathcal{D}_{2R} \\
\phi \cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\
\phi(\cdot, 0) = 0 = \phi(\cdot, T),
\end{cases} \qquad (3.20)
$$

$$
\Psi^*_t = \Delta_x \Psi^* + g[p,\xi,\Psi^*,\phi] \quad \text{in } \Omega \times (0,T) \qquad (3.21)
$$

where

$$
\begin{aligned}
g[p,\xi,\Psi^*,\phi] :=& (1-\eta)\tilde L_U[\Psi^*] + (\Psi^* \cdot U) U_t \qquad (3.22) \\
&+ Q_\omega\big((\Delta_x \eta)\phi + 2\nabla_x \eta \nabla_x \phi - \eta_t \phi\big) \\
&+ \eta Q_\omega\big(-\dot\omega J\phi + \lambda^{-1}\dot\lambda y \cdot \nabla_y \phi + \lambda^{-1}\dot\xi \cdot \nabla_y \phi\big) \\
&+ (1-\eta)[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1] + (\Phi^0 \cdot U) U_t \\
&+ N_U(\eta Q_\omega \phi + \Pi_{U^\perp}(\Phi^0 + \Psi^*)),
\end{aligned}
$$

and we denote

$$
\mathcal{D}_{\gamma R} = \{(y,t) \in \mathbb{R}^2 \times (0,T) \ / \ |y| < \gamma R(t)\}.
$$

Indeed if (ϕ, Ψ^*) solves this system, then we have that

$$
u(x,t) = U + \Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega \phi] + a(\Pi_{U^\perp}[\Phi^0 + \Psi^* + \eta Q_\omega \phi])U \qquad (3.23)
$$

solves equation (3.1). The boundary condition (3.4) $u = \mathbf{e}_3$ amounts to

$$
\Pi_{U^\perp}[\Phi^0 + \Psi^*] + a(\Pi_{U^\perp}[U + \Phi^0 + \Psi^*])U = (\mathbf{e}_3 - U)
$$

and then it suffices that we take the boundary condition for (3.21)

$$
\Psi^*\big|_{\partial\Omega} = \mathbf{e}_3 - U - \Phi^0. \qquad (3.24)
$$

Since we want that $u(x,t)$ be a small perturbation of $U(x,t)$ when we stand close to (q,T), it is natural to require that Ψ^* satisfies the final condition

$$
\Psi^*(q,T) = 0.
$$

This constraint amounts to three Lagrange multipliers when we solve the problem, which we choose to put in the initial condition. Then we assume

$$
\Psi^*(x,0) = Z_0^*(x) + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3,
$$

where c_1, c_2, c_3 are undetermined constants and $Z_0^*(x)$ is a small function for which specific assumptions will later be made.

4. The reduced equations

In this section we will informally discuss the procedure to achieve our purpose in particular deriving the order of vanishing of the scaling parameter $\lambda(t)$ as $t \to T$.

The main term that couples equations (3.20) and (3.21) inside the second equation is the linear expression

$$Q_\omega[(\Delta_x \eta)\phi + 2\nabla_x \eta \nabla_x \phi + \eta_t \phi],$$

which is supported in $|y| = O(R)$. This motivates the fact that we want ϕ to exhibit some type of space decay in $|y|$ since in that way Ψ^* will eventually be smaller and in turn that would make the two equations at main order *uncoupled*. Equation (3.20) has the form

$$\begin{aligned} \lambda^2 \phi_t &= L_W[\phi] + h[p, \xi, \Psi^*](y,t) \quad \text{in } \mathcal{D}_{2R} \\ \phi \cdot W &= 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi(\cdot, 0) &= 0 \quad \text{in } B_{2R(0)}, \end{aligned}$$

where, for convenience we assume that $h(y,t)$ is defined for all $y \in \mathbb{R}^2$ extending outside $\mathcal{D}_{2R}$ as

$$h[p, \xi, \Psi^*] = \lambda^2 Q_{-\omega} \tilde{L}_U[\Psi^*]\chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi] + \lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]\chi_{\mathcal{D}_{2R}}, \tag{4.1}$$

where χ_A designates characteristic function of a set A, $\mathcal{K}_0$ is defined in (3.17), (3.18) and $\mathcal{K}_1$ in (3.19). If $\lambda(t)$ has a relatively smooth vanishing as $t \to T$ it seems natural that the term $\lambda^2 \phi_t$ be of smaller order and then the equation is approximately represented by the elliptic problem

$$L_W[\phi] + h[p, \xi, \Psi^*] = 0, \quad \phi \cdot W = 0 \quad \text{in } \mathbb{R}^2. \tag{4.2}$$

Let us consider the decaying functions $Z_{lj}(y)$ defined in formula (2.2), which satisfy $L_W[Z_{lj}] = 0$. If $\phi(y,t)$ is a solution of (4.2) with sufficient decay, then necessarily

$$\int_{\mathbb{R}^2} h[p, \xi, \Psi^*](y,t) \cdot Z_{lj}(y)\, dy = 0 \qquad \text{for all} \quad t \in (0,T), \tag{4.3}$$

for $l = 0,1$, $j = 1,2$. These relations amount to an integro-differential system of equations for $p(t)$, $\xi(t)$, which, as a matter of fact, *detemine* the correct values of the parameters so that the solution (ϕ, Ψ^*) with appropriate asymptotics exists.

We derive next useful expressions for relations (4.3). Let us first compute the quantities

$$\mathcal{B}_{0j}[p](t) := \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} Q_{-\omega}[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] \cdot Z_{0j}(y)\, dy. \tag{4.4}$$

Using (3.17), (3.18) the following expressions for $\mathcal{B}_{01}$, $\mathcal{B}_{02}$ are readily obtained:

$$\begin{aligned} \mathcal{B}_{01}[p](t) &= 2\int_{-T}^{t} \operatorname{Re}\left(\dot{p}(s) e^{-i\omega(t)}\right) \Gamma_1\left(\frac{\lambda(t)^2}{t-s}\right) \frac{ds}{t-s} - 2\dot{\lambda}(t) \\ \mathcal{B}_{02}[p](t) &= 2\int_{-T}^{t} \operatorname{Im}\left(\dot{p}(s) e^{-i\omega(t)}\right) \Gamma_2\left(\frac{\lambda(t)^2}{t-s}\right) \frac{ds}{t-s} \end{aligned}$$

where $\Gamma_j(\tau)$, $j = 1,2$ are the smooth functions defined as follows:

$$\begin{aligned} \Gamma_1(\tau) &= -\int_0^\infty \rho^3 w_\rho^3 \left[K(\zeta) + 2\zeta K_\zeta(\zeta) \frac{\rho^2}{1+\rho^2} - 4\cos(w)\zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta = \tau(1+\rho^2)} d\rho \\ \Gamma_2(\tau) &= -\int_0^\infty \rho^3 w_\rho^3 \left[K(\zeta) - \zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta = \tau(1+\rho^2)} d\rho \end{aligned}$$

where

$$K(\zeta) = 2\frac{1 - e^{-\frac{\zeta}{4}}}{\zeta},$$

and we have used that $\int_0^\infty \rho^3 w_\rho^3 d\rho = -2$. Using these expressions we find that

$$|\Gamma_l(\tau) - 1| \le C\tau(1 + |\log\tau|) \quad \text{for } \tau < 1, \tag{4.5}$$
$$|\Gamma_l(\tau)| \le \frac{C}{\tau} \quad \text{for } \tau > 1.$$

Let us define

$$\mathcal{B}_0[p] := \frac{1}{2} e^{i\omega(t)} \left(\mathcal{B}_{01}[p] + i\mathcal{B}_{02}[p]\right) \tag{4.6}$$

and

$$a_{0j}[p,\xi,\Psi^*] := -\frac{\lambda}{2\pi} \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*] \cdot Z_{0j}(y)\, dy$$
$$a_0[p,\xi,\Psi^*] := \frac{1}{2} e^{i\omega(t)} \left(a_{01}[p,\xi,\Psi^*] + i a_{02}[p,\xi,\Psi^*]\right). \tag{4.7}$$

Similarly, we let

$$\mathcal{B}_{1j}[\xi](t) := \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} Q_{-\omega}[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] \cdot Z_{1j}(y)\, dy,$$
$$\mathcal{B}_1[\xi](t) := \mathcal{B}_{11}[\xi](t) + i\mathcal{B}_{12}[\xi](t).$$

Using (3.19), (2.2) and the fact that $\int_0^\infty \rho w_\rho^2 d\rho = 2$ we get

$$\mathcal{B}_1[\xi](t) = 2[\,\dot\xi_1(t) + i\dot\xi_2(t)\,].$$

At last, we set

$$a_{1j}[p,\xi,\Psi^*] := \frac{\lambda}{2\pi} \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*] \cdot Z_{1j}(y)\, dy$$
$$a_1[p,\xi,\Psi^*] := -e^{i\omega(t)} (a_{11}[p,\xi,\Psi^*] + i a_{12}[p,\xi,\Psi^*]).$$

We get that the four conditions (4.3) reduce to the system of two complex equations

$$\mathcal{B}_0[p] = a_0[p,\xi,\Psi^*], \tag{4.8}$$
$$\mathcal{B}_1[\xi] = a_1[p,\xi,\Psi^*]. \tag{4.9}$$

At this point we will make some preliminary considerations on this system that will allow us to find a first guess of the parameters $p(t)$ and $\xi(t)$. First, we observe that

$$\mathcal{B}_0[p] = \int_{-T}^{t-\lambda^2} \frac{\dot p(s)}{t-s} ds + O\big(\|\dot p\|_\infty\big).$$

To get an approximation for a_0, we analyze the operator $\tilde{L}_U$ in a_0. For this let us write

$$\Psi^* = \begin{bmatrix} \psi^* \\ \psi_3^* \end{bmatrix}, \quad \psi^* = \psi_1^* + i\psi_2^*.$$

From formula (2.7) we find that

$$\tilde{L}_U[\Psi^*](y) = [\tilde{L}_U]_0[\Psi^*] + [\tilde{L}_U]_1[\Psi^*] + [\tilde{L}_U]_2[\Psi^*],$$

where

$$\begin{aligned}
\lambda Q_{-\omega}[\tilde{L}_U]_0[\Psi^*] &= \rho w_\rho^2 \left[\, \operatorname{div}(e^{-i\omega}\psi^*)\, E_1 + \operatorname{curl}(e^{-i\omega}\psi^*)\, E_2 \,\right] \\
\lambda Q_{-\omega}[\tilde{L}_U]_1[\Psi^*] &= -2w_\rho \cos w \left[\, (\partial_{x_1}\psi_3^*)\cos\theta + (\partial_{x_2}\psi_3^*)\sin\theta \,\right] E_1 \\
&\quad - 2w_\rho \cos w \left[\, (\partial_{x_1}\psi_3^*)\sin\theta - (\partial_{x_2}\psi_3^*)\cos\theta \,\right] E_2 \,, \\
\lambda Q_{-\omega}[\tilde{L}_U]_2[\Psi^*] &= \rho w_\rho^2 \left[\, \operatorname{div}(e^{i\omega}\bar\psi^*)\, \cos 2\theta - \operatorname{curl}(e^{i\omega}\bar\psi^*)\, \sin 2\theta \,\right] E_1 \\
&\quad + \rho w_\rho^2 \left[\, \operatorname{div}(e^{i\omega}\bar\psi^*)\, \sin 2\theta + \operatorname{curl}(e^{i\omega}\bar\psi^*)\, \cos 2\theta \,\right] E_2,
\end{aligned}$$

and the differential operators in Ψ^* on the right hand sides are evaluated at (x,t) with $x = \xi(t) + \lambda(t)y$, $y = \rho e^{i\theta}$ while $E_l = E_l(y)$, $l = 1, 2$.

From the above decomposition, assuming that Ψ^* is of class C^1 in space variable, we find that

$$a_0[p,\xi,\Psi^*] = [\operatorname{div}\psi^* + i\operatorname{curl}\psi^*](\xi,t) + o(1),$$

where $o(1) \to 0$ as $t \to T$.

Similarly, we have that

$$\begin{aligned} a_1(p,\xi) &= 2(\partial_{x_1}\psi_3^* + i\partial_{x_2}\psi_3^*)(\xi,t)\int_0^\infty \cos w\, w_\rho^2 \rho\, d\rho + o(1) \\ &= o(1) \quad \text{as} \quad t \to T, \end{aligned}$$

since $\int_0^\infty w_\rho^2 \cos w \rho\, d\rho = 0$.

Let us discuss informally how to handle (4.8)-(4.9). For this we simplify this system in the form

$$\begin{aligned} \int_{-T}^{t-\lambda^2} \frac{\dot p(s)}{t-s} ds &= [\operatorname{div}\psi^* + i\operatorname{curl}\psi^*](\xi(t),t) + o(1) + O(\|\dot p\|_\infty) \\ \dot\xi(t) &= o(1) \quad \text{as} \quad t \to T. \end{aligned} \tag{4.10}$$

We assume for the moment that the function $\Psi^*(x,t)$ is fixed, sufficiently regular, and we regard T as a parameter that will always be taken smaller if necessary. We recall that we want $\xi(T) = q$ where $q \in \Omega$ is given, and $\lambda(T) = 0$. Equation (4.10) immediately suggests us to take $\xi(t) \equiv q$ as a first approximation. Neglecting lower order terms, we arrive at the "clean" equation for $p(t) = \lambda(t)e^{i\omega(t)}$,

$$\int_{-T}^{t-\lambda(t)^2} \frac{\dot p(s)}{t-s} ds = \operatorname{div}\psi^*(q,0) + i\operatorname{curl}\psi^*(q,0) =: a_0^* \tag{4.11}$$

At this point we make the following assumption:

$$\operatorname{div}\psi^*(q,0) < 0. \tag{4.12}$$

This implies that $a_0^* = -|a_0^*|e^{i\omega_0}$ for a unique $\omega_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let us take $\omega(t) \equiv \omega_0$. Then equation (4.11) becomes

$$\int_{-T}^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s} ds = -|a_0^*|. \tag{4.13}$$

We claim that a good approximate solution of (4.13) as $t \to T$ is given by

$$\dot\lambda(t) = -\frac{\kappa}{\log^2(T-t)}$$

for a suitable $\kappa > 0$. In fact, substituting, we have

$$\begin{aligned} \int_{-T}^{t-\lambda^2} \frac{\dot\lambda(s)}{t-s}\, ds \;&=\; \int_{-T}^{t-(T-t)} \frac{\dot\lambda(s)}{t-s}\, ds + \dot\lambda(t)\,[\log(T-t) - 2\log(\lambda(t))] \\ &\quad + \int_{t-(T-t)}^{t-\lambda(t)^2} \frac{\dot\lambda(s) - \dot\lambda(t)}{t-s} ds \\ &\approx \int_{-T}^{t} \frac{\dot\lambda(s)}{T-s}\, ds \;-\; \dot\lambda(t)\log(T-t) \;=:\beta(t) \end{aligned} \tag{4.14}$$

as $t \to T$. We see that

$$\log(T-t)\frac{d\beta}{dt}(t) = \frac{d}{dt}(\log^2(T-t)\,\dot\lambda(t)) = 0$$

from the explicit form of $\dot\lambda(t)$. Hence $\beta(t)$ is constant. As a conclusion, equation (4.13) is approximately satisfied if κ is such that

$$\kappa\int_{-T}^{T} \frac{\dot\lambda(s)}{T-s} \;=\; -|a_0^*|.$$

And this finally gives us the approximate expression

$$\dot\lambda(t) = -|\operatorname{div}\psi^*(q,0) + i\operatorname{curl}\psi^*(q,0)|\,\dot\lambda_*(t),$$

where

$$\dot{\lambda}_*(t) = -\frac{|\log T|}{\log^2(T-t)}.$$

Naturally imposing $\lambda_*(T) = 0$ we then have

$$\lambda_*(t) = \frac{|\log T|}{\log^2(T-t)}(T-t)\,(1+o(1)) \quad \text{as} \quad t \to T.$$

5. Solving the inner-outer gluing system

Our purpose is to determine, for a given $q \in \Omega$ and a sufficiently small $T > 0$, a solution (ϕ, Ψ^*) of system (3.20)-(3.21) with a boundary condition of the form (3.24) such that $u(x,t)$ given by (3.23) blows up with $U(x,t)$ as its main order profile. This will only be possible for adequate choices of the parameter functions $\xi(t)$ and $p(t) = \lambda(t)e^{i\omega(t)}$. These functions will eventually be found by fixed point arguments, but a priori we need to make some assumptions regarding their behavior. For some positive numbers a_1, a_2, σ independent of T we will assume that

$$a_1|\dot{\lambda}_*(t)| \le |\dot{p}(t)| \le a_2|\dot{\lambda}_*(t)| \quad \text{for all} \quad t \in (0,T), \tag{5.1}$$

$$|\dot{\xi}(t)| \le \lambda_*(t)^\sigma \qquad \text{for all} \quad t \in (0,T). \tag{5.2}$$

We also take

$$R(t) = \lambda_*(t)^{-\beta}, \tag{5.3}$$

where $\beta \in (0, \frac{1}{2})$.

To solve the outer equation (3.21) we will decompose Ψ^* in the form

$$\Psi^* = Z^* + \psi$$

where we let $Z^* : \Omega \times (0,\infty) \to \mathbb{R}^3$ satisfy

$$\begin{cases} Z^*_t = \Delta Z^* \quad \text{in } \Omega \times (0,\infty), \\ Z^*(\cdot,t) = 0 \quad \text{in } \partial\Omega \times (0,\infty), \\ Z^*(\cdot,0) = Z^*_0 \quad \text{in } \Omega, \end{cases} \tag{5.4}$$

with $Z^*_0(x)$ a function satisfying certain conditions to be described below. Since we would like that $u(x,t)$ given by (3.23) has a blow-up behavior given at main order by that of $U(x,t)$, we will require

$$\Psi^*(q,T) = 0.$$

This constraint has three parameters. Therefore we need three "Lagrange multipliers" which we include in the initial datum.

5.1. Assumptions on Z^*_0.

To describe the assumptions on Z^*_0, let us write

$$Z^*_0(x) = \begin{bmatrix} z^*_0(x) \\ z^*_{03}(x) \end{bmatrix}, \quad z^*_0(x) = z^*_{01}(x) + iz^*_{02}(x). \tag{5.5}$$

A first condition that we require, consistent with (4.12), is $\operatorname{div} z^*_0(q) < 0$. In addition we require that $Z^*_0(q) \approx 0$ in a non-degenerate way.

We want also Z^* to be sufficiently small, but independently of T, so that the heat equation (5.4) is a good approximation of the linearized harmonic map flow far from the singularity. In order to achieve later the desired stability property, it is convenient split Z^*_0 into two parts

$$Z^*_0 = Z^{*0}_0 + Z^{*1}_0,$$

where Z_0^{*0} is sufficiently smooth and Z_0^{*1} allows more irregular perturbations. More precisely, for Z_0^{*0} we assume that for some $\alpha_0 > 0$ small and some $\alpha_1, \alpha_2 > 0$, all independent of T, we have

$$\begin{cases} \|Z_0^{*0}\|_{C^3(\overline{\Omega})} \le \alpha_0, \\ |Z_0^{*0}(q)| \le 5T, \\ |(Dz_0^{*0}(q))^{-1}| \le \alpha_1, \\ -\alpha_1 \le \operatorname{div} z_0^{*0}(q) \le -\alpha_2. \end{cases} \tag{5.6}$$

(The notation here is analogous to (5.5).)

To describe Z_0^{*1} we introduce the following norm

$$\begin{aligned} \|Z_0^{*1}\|_* = \sup_\Omega |Z_0^{*1}(x)| + \frac{1}{|\log \varepsilon_*|} \sup_\Omega |\nabla_x Z_0^{*1}(x)| \\ + \frac{1}{|\log \varepsilon_*|^{1/2}} \sup_\Omega \left(|x - q_0| + \varepsilon_*\right) |D_x^2 Z_0^{*1}(x)|, \end{aligned} \tag{5.7}$$

where

$$\varepsilon_* = \frac{T}{|\log T|}.$$

Then we assume that for some $\sigma > 0$ fixed we have

$$\|Z_0^{*1}\|_* \le T^\sigma. \tag{5.8}$$

In summary, the conditions on Z_0^* are the following:

$$Z_0^* = Z_0^{*0} + Z_0^{*1} \text{ with } Z_0^{*0},\ Z_0^{*1} \text{ satisfying (5.6) and (5.8).} \tag{5.9}$$

5.2. **Linear theory for the inner problem.** The inner problem (3.20) is written as

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h[p, \xi, \Psi^*] \quad \text{in } \mathcal{D}_{2R} \\ \phi \cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi(\cdot, 0) = 0 \quad \text{in } B_{2R(0)} \end{cases}$$

where $h[p, \xi, \Psi^*]$ is given by (4.1). To find a good solution to this problem we would like that $h[p, \xi, \Psi^*]$ satisfies the orthogonality conditions (4.3).

We split the right hand side $h[p, \xi, \Psi^*]$ and the inner solution into components with different roles regarding these orthogonality conditions.

Recall that

$$h[p, \xi, \Psi^*] = \lambda^2 Q_{-\omega} \tilde{L}_U[\Psi^*] \chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi] + \lambda^2 Q_{-\omega} \mathcal{K}_1[p, \xi] \chi_{\mathcal{D}_{2R}},$$

the decomposition of $\tilde{L}_U$ given in (2.7):

$$\tilde{L}_U[\Psi^*] = \tilde{L}_U[\Psi^*]_0 + \tilde{L}_U[\Psi^*]_1 + \tilde{L}_U[\Psi^*]_2 \ ,$$

and the explicit formula:

$$\begin{aligned} \tilde{L}_U[\Phi]_1 = & -2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1} \varphi_3) \cos\theta + (\partial_{x_2} \varphi_3) \sin\theta \right] Q_\omega E_1 \\ & -2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1} \varphi_3) \sin\theta - (\partial_{x_2} \varphi_3) \cos\theta \right] Q_\omega E_2 \ . \end{aligned}$$

Using the notation (2.6), we then define

$$\begin{aligned} \tilde{L}_U[\Phi]_1^{(0)} = & -2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1} \varphi_3(\xi(t), t)) \cos\theta + (\partial_{x_2} \varphi_3(\xi(t), t))) \sin\theta \right] Q_\omega E_1 \\ & -2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1} \varphi_3(\xi(t), t))) \sin\theta - (\partial_{x_2} \varphi_3(\xi(t), t))) \cos\theta \right] Q_\omega E_2 \ . \end{aligned} \tag{5.10}$$

That is, we freeze the derivatives of ϕ_3 in the definition of the operator.

We then decompose

$$h = h_1 + h_2 + h_3$$

where

$$h_1[p,\xi,\Psi^*] = \lambda^2 Q_{-\omega}(\tilde L_U[\Psi^*]_0 + \tilde L_U[\Psi^*]_2)\chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi], \tag{5.11}$$

$$h_2[p,\xi,\Psi^*] = \lambda^2 Q_{-\omega}\tilde L_U[\Psi^*]_1^{(0)}\chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]\chi_{\mathcal{D}_{2R}} \tag{5.12}$$

$$h_3[p,\xi,\Psi^*] = \lambda^2 Q_{-\omega}(\tilde L_U[\Psi^*]_1 - \tilde L_U[\Psi^*]_1^{(0)})\chi_{\mathcal{D}_{2R}}. \tag{5.13}$$

Next we decompose $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$. The function ϕ_1 will solve the inner problem with right hand side $h_1[p,\xi,\Psi^*]$ projected so that it satisfies essentially (4.3). The advantage of doing this is that h_1 has faster spatial decay, which gives better bounds for the solution. For this we let, for any function $h(y,t)$ defined in $\mathbb{R}^2\times(0,T)$ with sufficient decay,

$$c_{lj}[h](t) := \frac{1}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{lj}|^2}\int_{\mathbb{R}^2} h(y,t)\cdot Z_{lj}(y)\,dy. \tag{5.14}$$

Note that $h[p,\xi,\Psi^*]$ is defined in $\mathbb{R}^2\times(0,T)$, and for simplicity we will assume that the right hand sides appearing in the different linear equations are always defined in $\mathbb{R}^2\times(0,T)$.

We would like that ϕ_1 solves

$$\lambda^2\partial_t\phi_1 = L_W[\phi_1] + h_1[p,\xi,\Psi^*] - \sum_{l=-1}^{1}\sum_{j=1}^{2} c_{lj}[h_1(p,\xi,\Psi^*)]w_\rho^2 Z_{lj} \quad \text{in } \mathcal{D}_{2R},$$

but the estimates for ϕ_1 are better if the projections $c_{0j}[h(p,\xi,\Psi^*)]$ are modified slightly.

Here is the precise result that we will use later. We define the norms

$$\|h\|_{\nu,a} = \sup_{\mathbb{R}^2\times(0,T)} \frac{|h(y,t)|}{\lambda_*^\nu(1+|y|)^{-a}}, \tag{5.15}$$

and

$$\|\phi\|_{*,\nu,a,\delta} = \sup_{\mathcal{D}_{2R}} \frac{|\phi(y,\tau)| + (1+|y|)|\nabla_y\phi(y,\tau)|}{\lambda_*^\nu \max(\frac{R^{\delta(5-a)}}{(1+|y|)^3}, \frac{1}{(1+|y|)^{a-2}})}. \tag{5.16}$$

Proposition 5.1. *Let $a\in(2,3)$, $\delta\in(0,1)$, $\nu>0$. Assume $\|h\|_{\nu,a}<\infty$. Then there is a solution $\phi = \mathcal{T}_{\lambda,1}[h]$, $\tilde c_{0j}[h]$ of*

$$\begin{cases} \lambda^2\partial_t\phi = L_W[\phi] + h - \sum_{j=1,2}\tilde c_{0j}[h]Z_{0j}\chi_{B_1} - \sum_{\substack{l=-1,1\\ j=1,2}} c_{lj}[h]Z_{lj}\chi_{B_1} & \text{in } \mathcal{D}_{2R} \\ \phi\cdot W = 0 & \text{in } \mathcal{D}_{2R} \\ \phi(\cdot,0) = 0 & \text{in } B_{2R(0)} \end{cases} \tag{5.17}$$

where c_{lj} is defined in (5.14), which is linear in h, such that

$$\|\phi\|_{*,\nu,a,\delta} \le C\|h\|_{\nu,a}$$

and such that

$$|c_{0j}[h] - \tilde c_{0j}[h]| \le C\lambda_*^\nu R^{-\frac12\delta(a-2)}\|h\|_{\nu,a}. \tag{5.18}$$

The function ϕ_2 solves the equation with right hand side $h_2[p,\xi,\Psi^*]$, which is in *mode 1*, a notion that we define next (this is basically motivated by the analysis of section 6, where we consider the linearized parabolic equation and use a Fourier decomposition of the right hand side and the solution).

Let $h(y,t)\in\mathbb{R}^3$, be defined in $\mathbb{R}^2\times(0,T)$ or $\mathcal{D}_{2R}$ with $h\cdot W = 0$. We say that h is un mode $k\in\mathbb{Z}$ if h has the form

$$h(y,t) = \mathrm{Re}(\tilde h_k(|y|,t)e^{ik\theta})E_1 + \mathrm{Re}(\tilde h_k(|y|,t)e^{ik\theta})E_2,$$

for some complex valued function $\tilde h_k(\rho,t)$.

Consider then

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h - \sum_{j=1,2} c_{1j}[h] w_\rho^2 Z_{1j} & \text{in } \mathcal{D}_{2R} \\ \phi \cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \\ \phi(\cdot, 0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{5.19}$$

Proposition 5.2. *Let $a \in (2,3)$, $\delta \in (0,1)$, $\nu > 0$. Assume that h is in mode 1 and $\|h\|_{\nu,a} < \infty$. Then there is a solution $\phi = \mathcal{T}_{\lambda,2}[h]$ of (5.19), which is linear in h, such that*

$$\|\phi\|_{\nu,a-2} \le C\|h\|_{\nu,a}.$$

In the above statemen the norm $\|\phi\|_{\nu,a-2}$ analogous to the one in (5.15), but the supremum is taken in $\mathcal{D}_{2R}$.

Another piece of the inner solution, ϕ_3, will handle $h_3[p,\xi,\Psi^*]$, which does not satisfy orthogonality conditions in mode 0. We will still project it to satisfy the orthogonality condition in mode 1. Let us consider then (5.19) without any orthogonality conditions on h in mode 0. We define

$$\|\phi\|_{**,\nu} = \sup_{\mathcal{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)\,|\nabla_y \phi(y,t)|}{\lambda_*(t)^\nu R(t)^2 (1+|y|)^{-1}}. \tag{5.20}$$

Proposition 5.3. *Let $1 < a < 3$ and $\nu > 0$. There exists a $C > 0$ such that if $\|h\|_{a,\nu} < +\infty$ there is a solution $\phi = \mathcal{T}_{\lambda,3}[h]$ of (5.19), which is linear in h and satisfies the estimate*

$$\|\phi\|_{**,\nu} \ \le\ C\|h\|_{a,\nu}.$$

Note that we allow a to be less than 2 in the previous proposition.

Next we have a variant of Proposition 5.3 when h is in mode -1.

Proposition 5.4. *Let $2 < a < 3$ and $\nu > 0$. There exists a $C > 0$ such that for any h in mode -1 with $\|h\|_{a,\nu} < +\infty$, there is a solution $\phi = \mathcal{T}_{\lambda,4}[h]$ of problem (5.19), which is linear in h and satisfies the estimate*

$$\|\phi\|_{***,\nu} \le C\|h\|_{a,\nu},$$

where

$$\|\phi\|_{***,\nu} = \sup_{\mathcal{D}_{2R}} \frac{|\phi(y,t)| + (1+|y|)\,|\nabla_y \phi(y,t)|}{\lambda_*(t)^\nu \log(R(t))}. \tag{5.21}$$

All propositions stated here are proved in section 6.

5.3. **The equations for $p = \lambda e^{i\omega}$.** We need to choose the free parameters p, ξ so that $c_{lj}[h(p,\xi,\Psi^*)] = 0$ for $l = -1, 0, 1$, $j = 1, 2$. This will be easy to do for $l = 1$ (mode 1), but mode $l = 0$ is more complicated.

To handle c_{0j} we note that by definitions (4.1), (4.4), (4.7)

$$c_{0,j}[h(p,\xi,\Psi^*)] = \frac{2\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} \left(\mathcal{B}_{0j}[p] - a_{0j}[p,\xi,\Psi^*]\right)$$

where B_0, a_0 are defined in (4.6), (4.7) and we recall that $p = \lambda e^{i\omega}$.

So to achieve $c_{0j}[h(p,\xi,\Psi^*)] = 0$ we should solve

$$\mathcal{B}_0[p](t) = a_0[p,\xi,\Psi^*](t), \quad t \in [0,T], \tag{5.22}$$

adjusting the parameters $\lambda(t)$ and $\omega(t)$. This equation is delicate and we will instead solve it up to an error.

To make this precise we define the following norms. Let I denote either the interval $[0,T]$ or $[-T,T]$. For $\Theta \in (0,1)$, $l \in \mathbb{R}$ and a continuous function $g : I \to \mathbb{C}$ we let

$$\|g\|_{\Theta,l} = \sup_{t \in I} (T-t)^{-\Theta} |\log(T-t)|^l |g(t)|, \tag{5.23}$$

and for $\gamma \in (0,1)$, $m \in (0,\infty)$, and $l \in \mathbb{R}$ we let

$$[g]_{\gamma,m,l} = \sup\, (T-t)^{-m} |\log(T-t)|^l \frac{|g(t)-g(s)|}{(t-s)^\gamma}, \tag{5.24}$$

where the supremum is taken over $s \le t$ in I such that $t-s \le \frac{1}{10}(T-t)$.

We have then the following result, whose proof is in section 13.

Proposition 5.5. *Let $\alpha, \gamma \in (0,\frac12)$, $l \in \mathbb{R}$, $C_1 > 1$. There is $\alpha_0 > 0$ such that if $\Theta \in (0,\alpha_0)$, $m \le \Theta - \gamma$ and $T > 0$ is small, then there are two operators $\mathcal{P}$ and $\mathcal{R}_0$ with the following properties.*

Assume that $a : [0,T] \to \mathbb{C}$ satisfies

$$\begin{cases} \dfrac{1}{C_1} \le |a(T)| \le C_1 \\ T^\Theta |\log T|^{1+\sigma-l} \|a(\cdot)-a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \le C_1, \end{cases} \tag{5.25}$$

for some $\sigma > 0$. Then $p = \mathcal{P}[a] : [-T,T] \to \mathbb{C}$ satisfies

$$\mathcal{B}_0[p](t) = a(t) + \mathcal{R}_0[a](t), \quad t \in [0,T], \tag{5.26}$$

with

$$\begin{aligned} &|\mathcal{R}_0[a](t)| \\ &\le C\Big(T^\sigma + C\frac{T^\Theta}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|a(\cdot)-a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}\Big) \\ &\quad \cdot \frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}, \end{aligned} \tag{5.27}$$

for some $\sigma > 0$.

Roughly speaking, to obtain the modified equation (5.26) we notice that the main term in p in $\mathcal{B}_0[p]$ is the integral operator

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{t-s} ds.$$

Thus we define

$$\tilde{\mathcal{B}}_0[p] = \mathcal{B}_0[p] - \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{t-s} ds.$$

It will be sufficient to solve approximately equations (4.3) replacing in part this integral operator by a "regularized" version of it following the logic of the formal derivation of the rate (4.14). For $\alpha > 0$ let us write

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{t-s} ds = S_\alpha[\dot p] + R_\alpha[\dot p]$$

where

$$S_\alpha[g] := g(t)[-2\log\lambda_*(t) + (1+\alpha)\log(T-t)] + \int_{-T}^{t-(T-t)^{1+\alpha}} \frac{g(s)}{t-s} ds, \tag{5.28}$$

$$R_\alpha[g] := -\int_{t-(T-t)^{1+\alpha}}^{t-\lambda_*^2} \frac{g(t)-g(s)}{t-s} ds. \tag{5.29}$$

Thus equation (5.22) can be written in the form

$$S_\alpha[\dot p] + R_\alpha[\dot p] + \tilde{\mathcal{B}}_0[p] = a(t), \quad \text{in } [0,T],$$

for some function $a(t)$. The modified equation is

$$S_\alpha[\dot p] + \tilde{\mathcal{B}}_0[p] = a(t) \quad \text{in } [0,T],$$

and the remainder $\mathcal{R}_0$ is essentially $R_\alpha[\dot p]$. This is a sketch of how we obtain the modified equation and remainder. For more details see section 13.

Another modification to equations (5.22) that we introduce is to replace $a_0[p,\xi,\Psi^*]$ by its main term. To do this we write

$$a_0[p,\xi,\Psi] = a_0^{(0)}[p,\xi,\Psi] + a_0^{(1)}[p,\xi,\Psi] + a_0^{(2)}[p,\xi,\Psi]$$

where

$$a_0^{(l)}[p,\xi,\Psi] = -\frac{\lambda}{4\pi} e^{i\omega} \int_{B_{2R}} \Big(Q_{-\omega}\tilde{L}_U[\Psi]_l \cdot Z_{01} + iQ_{-\omega}\tilde{L}_U[\Psi]_l \cdot Z_{02} \Big)\, dy \tag{5.30}$$

for $l = 0,1,2$.

We define

$$\begin{aligned} c_0^*[p,\xi,\Psi^*](t) := \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{01}|^2} e^{-i\omega} \Big(&\mathcal{R}_0 \left[a_0^{(0)}[p,\xi,\Psi^*] \right](t) + a_0^{(1)}[p,\xi,\Psi^*](t) \\ &+ a_0^{(2)}[p,\xi,\Psi^*](t) \Big) - (c_0[h[p,\xi,\Psi^*]] - \tilde{c}_0[h_1[p,\xi,\Psi^*]]), \end{aligned} \tag{5.31}$$

and

$$c_{01}^* := \mathrm{Re}(c_0^*), \quad c_{02}^* := \mathrm{Im}(c_0^*),$$

where $\mathcal{R}_0$ is the operator given Proposition 5.5 and $\tilde{c}_0 = \tilde{c}_{01} + i\tilde{c}_{02}$ are the operators defined in Proposition 5.1.

Let us explain the formula for c_0^*. We have

$$c_{0j}[h[p,\xi,\Psi^*]] = \frac{2\pi\lambda}{\int_{B_1} |Z_{0j}|^2} \left(\mathcal{B}_{0j}[p] - a_{0j}[p,\xi,\Psi^*] \right)$$

Then by (4.6), (4.7)

$$\frac{1}{2} e^{i\omega} c_0[h[p,\xi,\Psi^*]] = \frac{2\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} \left(\mathcal{B}_0[p] - a_0[p,\xi,\Psi^*] \right),$$

where $c_0 = c_{01} + ic_{02}$. What we will really solve is

$$\mathcal{B}_0[p] = a_0^{(0)}[p,\xi,\Psi^*](t) + \mathcal{R}_0 \left[a_0^{(0)}[p,\xi,\Psi^*] \right] \tag{5.32}$$

which is equivalent to

$$\frac{1}{2} e^{i\omega} c_0[h] = \frac{2\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} \Big(\mathcal{R}_0 \left[a_0^{(0)}[p,\xi,\Psi^*] \right] + a_0^{(1)}[p,\xi,\Psi^*] + a_0^{(2)}[p,\xi,\Psi^*] \Big),$$

that is, the equation to be solved is

$$\begin{aligned} c_0[h[p,\xi,\Psi^*]] = \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} e^{-i\omega} \Big(&\mathcal{R}_0 \left[a_0^{(0)}[p,\xi,\Psi^*] \right](t) + a_0^{(1)}[p,\xi,\Psi^*](t) \\ &+ a_0^{(2)}[p,\xi,\Psi^*](t) \Big). \end{aligned} \tag{5.33}$$

The reduced equation that we will consider is

$$\tilde{c}_0[h_1[p,\xi,\Psi^*]] = c_0^*[p,\xi,\Psi^*].$$

and we want this to be equivalent to (5.33). We rewrite (5.33) as

$$\begin{aligned} &\frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} e^{-i\omega} \Big(\mathcal{R}_0 \left[a_0^{(0)}[p,\xi,\Psi^*] \right](t) + a_0^{(1)}[p,\xi,\Psi^*](t) + a_0^{(2)}[p,\xi,\Psi^*](t) \Big) \\ &= c_0[h[p,\xi,\Psi^*]] \\ &= \tilde{c}_0[h_1[p,\xi,\Psi^*]] + (c_0[h[p,\xi,\Psi^*]] - \tilde{c}_0[h_1[p,\xi,\Psi^*]]). \end{aligned}$$

and then define

$$\begin{aligned} c_0^*[p,\xi,\Psi^*] = \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} e^{-i\omega} \Big(&\mathcal{R}_0 \left[a_0^{(0)}[p,\xi,\Psi^*] \right](t) + a_0^{(1)}[p,\xi,\Psi^*](t) \Big) \\ &+ a_0^{(2)}[p,\xi,\Psi^*](t) - (c_0[h[p,\xi,\Psi^*]] - \tilde{c}_0[h_1[p,\xi,\Psi^*]]). \end{aligned}$$

This is (5.31).

We mention here how to adapt the parameters Θ, m, l of Proposition 5.5 to the context of the main result, and what is the advantage of having a remainder $\mathcal{R}_0$ satisfying (5.27).

For this we note that from (5.32) and the norm (5.44) the natural assumptions on the function $a : [0,T] \to \mathbb{C}$ in Proposition 5.5 are given by

$$
\begin{aligned}
|a(t)-a(T)| &\le C\lambda_*(t)^{\Theta}, \quad t\in[0,T]\\
|a(t)-a(s)| &\le C\lambda_*(t)^{\Theta}\frac{(t-s)^{\gamma}}{\lambda_*(t)^{2\gamma}R(t)^{2\gamma}}
\end{aligned}
$$

for $0\le s\le t\le T$, such that $t-s\le \frac{1}{10}(T-t)$, where $\Theta>0$ and $\gamma\in(0,\frac12)$ are for the moment arbitrary.

It is therefore natural to select

$$
m := \Theta - 2\gamma(1-\beta). \tag{5.34}
$$

In order for $\|a(\cdot)-a(T)\|_{\Theta,l}$ to be finite we need $l<1+2\Theta$ and in this case we get

$$
\|a(\cdot)-a(T)\|_{\Theta,l-1} \le C|\log T|^{l-1-\Theta}.
$$

Similarly, in order for $[a]_{\gamma,m,l-1}$ to be finite we need $l<1+2m$ and in this case we get

$$
[a]_{\gamma,m,l-1} \le C|\log T|^{l-1-m}.
$$

Next we note that $m<\Theta-\gamma$ is equivalent to $\beta<\frac12$, which is true.

Let us rewrite the conclusion of Proposition 5.5, namely the estimate (5.27). We have $m=\Theta-2\gamma(1-\beta)$ and so

$$
m+(1+\alpha)\gamma = \Theta+\gamma(\alpha-1+2\beta).
$$

We want this constant to be greater than Θ, and this happens provided

$$
\alpha-1+2\beta>0. \tag{5.35}
$$

But we have the restriction that $\alpha<\frac12$. We see that it is possible to find $\alpha<\frac12$ such that (5.35) holds if

$$
\beta > \frac14 .
$$

The conclusion is that with the above choice of parameters m, γ, l, we obtain from (5.27) that

$$
\begin{aligned}
|\mathcal{R}_0[a](t)| &\le C\Big(T^{\sigma} + C\frac{T^{\Theta}}{|\log T|}\left(T^{-\gamma-m}+\log|\log T|\right)\|a(\cdot)-a(T)\|_{\Theta,l-1}+[a]_{\gamma,m,l-1}\Big)\\
&\quad\cdot\frac{(T-t)^{\Theta+\gamma(\alpha-1+2\beta)}}{|\log(T-t)|^{l}}\\
&\le C\lambda_*(t)^{\Theta+\sigma_1},
\end{aligned} \tag{5.36}
$$

where σ_1 is any fixed number such that

$$
\sigma_1\in(0,\gamma(\alpha-1+2\beta))
$$

and σ is some positive constant. This gain in the rate of vanishing of the remainder $\mathcal{R}_0[a](t)$ will be crucial to obtain the contraction property of the system.

5.4. **The system of equations.** Summarizing the above considerations, we transform the system (3.20)-(3.21) in the problem of finding functions $\psi(x,t)$, $\phi_1,\ldots,\phi_4$, parameters $p(t)=\lambda(t)e^{i\omega(t)}$, $\xi(t)$ and constants c_1,c_2,c_3 such that the following system is satisfied:

$$\begin{cases} \psi_t = \Delta_x \psi + g(p,\xi,Z^*+\psi,\phi_1+\phi_2+\phi_3+\phi_4) & \text{in } \Omega\times(0,T)\\ \psi = (\mathbf{e}_3 - U) - \Phi^0 & \text{on } \partial\Omega\times(0,T)\\ \psi(\cdot,0) = (c_1\,\mathbf{e_1}+c_2\,\mathbf{e_2}+c_3\,\mathbf{e_3})\chi + \ (1-\chi)(\mathbf{e}_3-U-\Phi^0) & \text{in } \Omega\\ \psi(q,T) = -Z^*(q,T) \end{cases} \tag{5.37}$$

$$\begin{cases} \lambda^2\partial_t\phi_1 = L_W[\phi_1] + h_1[p,\xi,\Psi^*] - \sum_{j=1,2}\tilde c_{0j}[h_1[p,\xi,\Psi^*]]w_\rho^2 Z_{0j}\\ \qquad\qquad - \sum_{\substack{l=-1,1\\ j=1,2}} c_{lj}[h_1[p,\xi,\Psi^*]]w_\rho^2 Z_{lj} \quad \text{in } \mathcal{D}_{2R}\\ \phi_1\cdot W = 0 \quad \text{in } \mathcal{D}_{2R}\\ \phi_1(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{5.38}$$

$$\begin{cases} \lambda^2\partial_t\phi_2 = L_W[\phi_2] + h_2[p,\xi,\Psi^*] - \sum_{j=1,2} c_{1j}[h_2[p,\xi,\Psi^*]]w_\rho^2 Z_{1j} \quad \text{in } \mathcal{D}_{2R}\\ \phi_2\cdot W = 0 \quad \text{in } \mathcal{D}_{2R}\\ \phi_2(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{5.39}$$

$$\begin{cases} \lambda^2\partial_t\phi_3 = L_W[\phi_3] + h_3 - \sum_{j=1,2} c_{1j}[h_3[p,\xi,\Psi^*]]w_\rho^2 Z_{1j}\\ \qquad\qquad + \sum_{j=1,2} c^*_{0j}[p,\xi,\Psi^*]w_\rho^2 Z_{0j} \quad \text{in } \mathcal{D}_{2R}\\ \phi_3\cdot W = 0 \quad \text{in } \mathcal{D}_{2R}\\ \phi_3(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{5.40}$$

$$\begin{cases} \lambda^2\partial_t\phi_4 = L_W[\phi_4] + \sum_{j=1,2} c_{-1,j}[h_1[p,\xi,\Psi^*]]w_\rho^2 Z_{-1j}\\ \phi_4\cdot W = 0 \quad \text{in } \mathcal{D}_{2R}\\ \phi_4(\cdot,t) = 0 \quad \text{on } \partial B_{2R(t)}\\ \phi_4(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{5.41}$$

$$c_{0j}[h(p,\xi,\Psi^*)](t) - \tilde c_{0j}[p,\xi,\Psi^*](t) = 0 \quad \text{for all} \quad t\in(0,T), \quad j=1,2, \tag{5.42}$$

$$c_{1j}[h(p,\xi,\Psi^*)](t) = 0 \quad \text{for all} \quad t\in(0,T), \quad j=1,2. \tag{5.43}$$

In (5.37) χ is a smooth cut-off function with compact support in Ω which is identically 1 on a fixed neighborhood of q independent of T and the function $g(p,\xi,\Psi^*,\phi)$ is given by (3.22).

We see that if $(\phi_1,\phi_2,\phi_3,\phi_4,\psi,p,\xi)$ satisfies system (5.37)–(5.43) then the functions

$$\phi = \phi_1+\phi_2+\phi_3+\phi_4, \quad \Psi^* = Z^*+\psi$$

solve the outer-inner gluing system (3.20)–(3.21).

The way in which we will proceed to solve the full problem (5.37)–(5.43) is the following. For given functions $\phi_1,\ldots,\phi_4$ and parameters p, ξ in a suitable class, we solve first the outer problem (5.37) in the form of an operator $\psi = \Psi[\phi_1+\phi_2+\phi_3+\phi_4,p,\xi]$ and denote $\Psi^*[\phi_1+\phi_2+\phi_3,p,\xi] = Z^*+\Psi[\phi_1+\phi_2+\phi_3+\phi_4,p,\xi]$. Then we substitute $\Psi^*[\phi_1+\phi_2+\phi_3+\phi_4,p,\xi]$ in (5.38)–(5.41) and solve for ϕ_1, ϕ_2, ϕ_3,

ϕ_4 as operators of the pair (p, ξ). Finally, we solve for p and ξ the remaining equations. All this will be done by suitable control on the linear parts of the equation and contraction mapping principle.

5.5. **Choice of spaces.** We will work with the following space for the inner solutions ϕ_1–ϕ_4:

$$E_1 = \{\phi_1 \in L^\infty(\mathcal{D}_{2R}) : \nabla_y \phi_1 \in L^\infty(\mathcal{D}_{2R}),\ \|\phi_1\|_{*,\nu_1,a_1,\delta} < \infty\}$$
$$E_2 = \{\phi_2 \in L^\infty(\mathcal{D}_{2R}) : \nabla_y \phi_2 \in L^\infty(\mathcal{D}_{2R}),\ \|\phi_2\|_{\nu_2,a_2} < \infty\}$$
$$E_3 = \{\phi_3 \in L^\infty(\mathcal{D}_{2R}) : \nabla_y \phi_3 \in L^\infty(\mathcal{D}_{2R}),\ \|\phi_3\|_{**,\nu_3} < \infty\}$$
$$E_4 = \{\phi_4 \in L^\infty(\mathcal{D}_{2R}) : \nabla_y \phi_4 \in L^\infty(\mathcal{D}_{2R}),\ \|\phi_4\|_{***,\nu_4} < \infty\}$$

and use the notation

$$E = E_1 \times E_2 \times E_3 \times E_4,$$

$$\Phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in E$$
$$\|\Phi\|_E = \|\phi_1\|_{*,\nu_1,a_1,\delta} + \|\phi_2\|_{\nu_2,a_2-2} + \|\phi_3\|_{**,\nu_3} + \|\phi_4\|_{***,\nu_4}$$

We define the closed ball

$$\mathcal{B} = \{\Phi \in X : \|\Phi\|_E \le 1\}.$$

For the outer problem we will work with the following norm. Given $\Theta > 0$, $\gamma \in (0, \frac{1}{2})$ we define

$$\begin{aligned} \|\psi\|_{\sharp,\Theta,\gamma} := {} & \lambda_*(0)^{-\Theta} \frac{1}{|\log T| \lambda_*(0) R(0)} \|\psi\|_{L^\infty(\Omega\times(0,T))} + \lambda_*(0)^{-\Theta} \|\nabla_x \psi\|_{L^\infty(\Omega\times(0,T))} \\ & + \sup_{\Omega\times(0,T)} \lambda_*(t)^{-\Theta-1} R(t)^{-1} \frac{1}{|\log(T-t)|} |\psi(x,t) - \psi(x,T)| \\ & + \sup_{\Omega\times(0,T)} \lambda_*(t)^{-\Theta} |\nabla_x \psi(x,t) - \nabla_x \psi(x,T)| \\ & + \sup \lambda_*(t)^{-\Theta} (\lambda_*(t) R(t))^{2\gamma} \frac{|\nabla_x \psi(x,t) - \nabla_x \psi(x',t')|}{(|x-x'|^2 + |t-t'|)^\gamma} \end{aligned} \tag{5.44}$$

where the last supremum in taken in the region

$$x, x' \in \Omega, \quad t, t' \in (0,T), \quad |x - x'| \le 2\lambda_* R(t), \quad |t - t'| < \frac{1}{4}(T-t).$$

5.6. **Choice of constants.** The spaces chosen before depend on some constants, which we would like to summarize here.

- $\beta \in (0, \frac{1}{2})$ is so that $R(t) = \lambda_*(t)^{-\beta}$.
- $\alpha \in (0, \frac{1}{2})$ appears in Proposition 5.5. It is the parameter used to define the remainder $\mathcal{R}_\alpha$ in (5.29).
- We use the norm $\|\ \|_{*,\nu_1,a_1,\delta}$ (5.16) to measure the solution ϕ_1 in (5.38). Here we will ask that $\nu_1 \in (0,1)$, $a_1 \in (2,3)$, and $\delta > 0$ small and fixed.
- We use the norm $\|\ \|_{\nu_2,a_2-2}$ (5.15) to measure the solution ϕ_2 in (5.39), with $\nu_2 \in (0,1)$, $a_2 \in (2,3)$.
- We use the norm $\|\ \|_{**,\nu_3}$ (5.20) for the solution ϕ_3 of (5.40), with $\nu_3 > 0$.
- We use the norm $\|\ \|_{***,\nu_4}$ for the solution ϕ_4 of (5.41), with $\nu_4 > 0$.
- We are going to use the norm $\|\ \|_{\sharp,\Theta,\gamma}$ with a parameters Θ, γ satisfying some restrictions given below.
- We have parameters m, l in Proposition 5.5. We work with m given by (5.34) and l satisfying $l < 1 + 2m$.

To get the estimates for the outer problem (5.37), see the computation that lead to (10.4), we need (8.3), (10.5), (10.6), and (10.13):

$$\begin{cases} \Theta < \min\Big(\beta, \dfrac{1}{2} - \beta, \nu_1 - 1 + \beta(a_1 - 1), \nu_2 - 1 + \beta(a_2 - 1), \nu_3 - 1, \nu_4 - 1 + \beta\Big) \\ \Theta < \min\Big(\nu_1 - \delta\beta(5 - a_1) - \beta, \nu_2 - \beta, \nu_3 - 3\beta, \nu_4 - \beta\Big) \\ \Theta > 0. \end{cases} \tag{5.45}$$

Also to control the nonlinear terms in (5.37) we need $\delta > 0$ in $\| \ \|_{*,\nu_1,a_1,\delta}$ to be small.

To find Θ in the range above we need

$$\begin{aligned}
\nu_1 &> \max\Big(1-\beta(a_1-1), \delta\beta(5-a_1)-\beta\Big)\\
\nu_2 &> \max\Big(1-\beta(a_2-1), \beta\Big)\\
\nu_3 &> \max(1, 3\beta)\\
\nu_4 &> \max(1-\beta, \beta).
\end{aligned}$$

To solve the inner system given by equations (5.38), (5.39), (5.40), and (5.41) we will need

$$\begin{aligned}
&\nu_1 < 1,\\
&\nu_2 < 1-\beta(a_2-2),\\
&\nu_3 < \min\Big(1+\Theta+\sigma_1, 1+\Theta+2\gamma\beta, \nu_1+\frac{1}{2}\delta\beta(a_1-2)\Big),\\
&\nu_4 < 1,
\end{aligned}$$

where σ_1 is the constant in (5.36) (see (11.1), (11.9), and (11.15), (11.48)).

5.7. **The outer problem.** The next proposition gives a solution to the outer problem (5.37).

Proposition 5.6. *Assume Z_0^* satisfies (5.9). Let $p(t) = \lambda(t)e^{i\omega(t)}$ and $\xi(t)$ satisfy estimates (5.1), (5.2), $\Phi \in \mathcal{B}$. Then there exists $C > 0$ such that if $T > 0$ is sufficiently small then there exists a solution $\psi = \Psi(p, \xi, \Phi, Z_0^*)$ to equation (5.37) such that*

$$\|\Psi(p,\xi,\Phi,Z_0^*)\|_{\sharp,\Theta,\gamma} \le CT^\sigma(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*).$$

The proof of this proposition is given in section 10.

The operator $\Psi(p, \xi, \Phi, Z_0^*)$ satisfies Lipschitz properties, which are consequence of its construction.

Corollary 5.1. *Let $\Psi(p, \xi, \Phi, Z_0^*)$ be the solution to equation (5.37) constructed in Proposition 5.6. Let p_l, ξ satisfy (5.1), (5.2) and $p_l = \lambda e^{i\omega_l}$, $\|\Phi_l\|_E \le 1$, and $\|Z_{0l}^*\|_* < \infty$, $l = 1, 2$. Then*

$$\begin{aligned}
&\|\Psi(p_1,\xi,\Phi_1,Z_{01}^*) - \Psi(p_2,\xi,\Phi_2,Z_{02}^*)\|_{\sharp,\Theta,\gamma}\\
&\quad \le CT^\sigma(\|\Phi_1-\Phi_2\|_E + \|\lambda_*(\dot\omega_1-\dot\omega_2)\|_\infty + \|Z_{01}^*-Z_{02}^*\|_*).
\end{aligned}$$

Corollary 5.1 gives a partial Lipschitz property of the exterior solution $\Psi(p, \xi, \phi)$ of (5.37) with respect to p, namely it only considers variations of $p = \lambda e^{i\omega}$ with respect to ω. We will need Lipschitz estimates for variations of $p = \lambda e^{i\omega}$ in λ and also variations with respect to ξ. These estimates are obtained for $\Psi(p, \xi, \phi)$ when considered as a function of the inner variable $(y, t) \in \mathcal{D}_{2R}$.

For this let us introduce some notation. Suppose that $\psi(x, t)$ is defined in $\Omega \times (0, T)$. We let

$$\tilde\psi(y,t) = \psi(\xi(t)+\lambda(t)y, t), \quad (y,t) \in \mathcal{D}_{2R}.$$

The following expression is $\|\psi\|_{\sharp,\Theta,\gamma}$ expressed in terms of $\tilde\psi$ (and restricted to $\mathcal{D}_{2R}$):

$$\begin{aligned}
\|\tilde\psi\|_{\sharp\sharp,\Theta,\gamma} :=\ & \lambda_*(0)^{-\Theta}\frac{1}{|\log T|\lambda_*(0)R(0)}\|\tilde\psi\|_{L^\infty(\mathcal{D}_{2R})} + \lambda_*(0)^{-\Theta-1}\|\nabla_y\psi\|_{L^\infty(\mathcal{D}_{2R})}\\
&+ \sup_{\mathcal{D}_{2R}} \lambda_*(t)^{-\Theta-1}R(t)^{-1}\frac{1}{|\log(T-t)|}|\tilde\psi(y,t)-\tilde\psi(y,T)|\\
&+ \sup_{(y,t)\in\mathcal{D}_{2R}} \lambda_*(t)^{-\Theta-1}|\nabla_y\tilde\psi(y,t)-\nabla_y\tilde\psi(y,T)|\\
&+ \sup_{(y,t),(y',t)\in\mathcal{D}_{2R}} \lambda_*(t)^{-\Theta-1}R(t)^{2\gamma}\frac{|\nabla_y\tilde\psi(y,t)-\nabla_y\tilde\psi(y',t)|}{|y-y'|^{2\gamma}}\\
&+ \sup \lambda_*(t)^{-\Theta-1}(\lambda_*(t)R(t))^{2\gamma}\frac{|\nabla_y\tilde\psi(y,t)-\nabla_y\tilde\psi(x',t')|}{|t-t'|^\gamma},
\end{aligned}$$

where the last supremum is taken in the region

$$(y,t),(y,t'),\in \mathcal{D}_{2R}, \quad |t-t'|\le \frac{1}{10}(T-t).$$

Corollary 5.2. *Let $\Psi(p,\xi,\phi)$ be the solution to equation (5.37) in Proposition 5.6. Let $p_l=\lambda_l e^{i\omega}$, ξ_l satisfy (5.1), (5.2) and $\|\phi\|_{*,a,\nu}\le 1$. Then for $\tilde\Theta\in(0,\Theta)$ we have*

$$\|\tilde\Psi(p_1,\xi_1,\phi)-\tilde\Psi(p_2,\xi_2,\phi)\|_{\sharp\sharp,\tilde\Theta,\gamma} \le C\Big[\Big\|\frac{\lambda_1-\lambda_2}{\lambda_*}\Big\|_{L^\infty}+\|\dot\lambda_1-\dot\lambda_2\|_{L^\infty}+\Big\|\frac{\xi_1-\xi_2}{\lambda_* R}\Big\|_{L^\infty}+\Big\|\frac{\dot\xi_1-\dot\xi_2}{R}\Big\|_{L^\infty}\Big].$$

The proof of this is in Section 16.

What we do next is to take $\Phi\in E$ with $\|\Phi\|_E\le 1$ and substitute $\Psi^*(p,\xi,\Phi,Z_0^*)=Z^*+\Psi(p,\xi,\Phi,Z_0^*)$ into (5.38)–(5.41). We can then write equations (5.37)–(5.41) as the fixed point problem

$$\Phi=\mathcal{F}(\Phi) \tag{5.46}$$

where

$$\mathcal{F}(\Phi)=(\mathcal{F}_1(\Phi),\mathcal{F}_2(\Phi),\mathcal{F}_3(\Phi),\mathcal{F}_4(\Phi)), \quad \mathcal{F}:\bar{\mathcal{B}}_1\subset E\to E$$

with

$$\begin{aligned}
\mathcal{F}_1(\Phi)&=\mathcal{T}_{\lambda,1}(h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)])\\
\mathcal{F}_2(\Phi)&=\mathcal{T}_{\lambda,2}(h_2[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)])\\
\mathcal{F}_3(\Phi)&=\mathcal{T}_{\lambda,3}\Big(h_3[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]+\sum_{j=1}^2 c_{0j}^*[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]w_\rho^2 Z_{0j}\Big)\\
\mathcal{F}_4(\Phi)&=\mathcal{T}_{\lambda,4}\Big(\sum_{j=1}^2 c_{-1,j}[h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]]w_\rho^2 Z_{-1,j}\Big).
\end{aligned}$$

Although $\mathcal{F}$ also depends on p, ξ, Z_0^* we will omit this dependence from the notation for the moment.

Our next step is to solve problem (5.46).

5.8. The inner problem.

Proposition 5.7. *Assume that p and ξ satisfy estimates (5.1) and that Z_0^* satisfies (5.9). Then the system of equations (5.46) for $\Phi=(\phi_1,\phi_2,\phi_3,\phi_4)$ has a solution $\Phi(p,\xi,Z_0^*)$ in $\bar{\mathcal{B}}_1\subset E$.*

The proof is in Section 11.

Let $\Phi(p,\xi,Z_0^*)$ be the solution of (5.46) constructed in Proposition 5.7. Next we show that the solution $\Phi(p,\xi,Z_0^*)$ is Lipschitz in the parameters p,ξ,Z_0^*.

Proposition 5.8. *Assume that p_1,p_2 and ξ_1,ξ_2 satisfy estimates (5.1) and that $Z_{0,1}^*$, $Z_{0,2}^*$ have the form*

$$Z_{0,l}^*=Z_0^{*0}+Z_{0,l}^{*1}, \quad l=1,2,$$

with Z_0^{*0} satisfying (5.6) and

$$\|Z_{0,l}^{*1}\|_*\le T^\sigma,$$

Let us write $p_j=\lambda_j e^{i\omega_j}$ for $j=1,2$. for some $\sigma>0$. Then

$$\begin{aligned}
\|\Phi(p_1,\xi_1,Z_{0,1}^*)-\Phi(p_2,\xi_2,Z_{0,2}^*)\|_E \le \lambda_*(0)^\sigma\Big[&\|\lambda_*(\dot\omega_1-\dot\omega_2)\|_\infty+\Big\|\frac{\lambda_1-\lambda_2}{\lambda_*}\Big\|_{L^\infty}\\
&+\|\dot\lambda_1-\dot\lambda_2\|_{L^\infty}+\Big\|\frac{\xi_1-\xi_2}{\lambda_* R}\Big\|_{L^\infty}+\Big\|\frac{\dot\xi_1-\dot\xi_2}{R}\Big\|_{L^\infty}\\
&+\|Z_{0,1}^{*1}-Z_{0,2}^{*1}\|_*\Big],
\end{aligned}$$

for some possibly smaller $\sigma>0$.

With this we can now state the following result. Let $\Phi(p, \xi, Z_0^*)$ denote the solution of (5.46) constructed in Proposition 5.7.

Proposition 5.9. *Given Z_0^* of the form (5.9) there exists $p = \lambda e^{i\omega}$ and ξ such that (5.42) and (5.43) are satisfied.*

The proposition above yields the existence of a blow-up solution. The proof is given in Section 12.

6. Linear theory for the inner problem

At the very heart of capturing the bubbling structure is the construction of an inverse for the linearized heat operator around the basic harmonic map. We consider the linear equation

$$\begin{aligned} &\lambda^2\partial_t\phi = L_W[\phi] + h(y,t) \quad \text{in } \mathcal{D}_{2R} \\ &\phi(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \\ &\phi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \end{aligned} \tag{6.1}$$

where

$$\mathcal{D}_{2R} = \{(y,t)\ /\ t\in(0,T),\ y\in B_{2R(t)}(0)\}.$$

We assume that $h(y,t)$ is defined for all $(y,t)\in\mathbb{R}^2\times(0,T)$ and satisfies

$$h\cdot W = 0, \quad |h(y,t)| \le C\frac{\lambda_*^\nu}{(1+|y|)^a},$$

where $\nu > 0$ and $a\in(2,3)$ (so that $\|h\|_{a,\nu} < \infty$ with the norm defined in (5.15)).

The parameter R is given by (5.3), that is $R(t) = \lambda_*(t)^{-\beta}$, $\beta\in(\frac14,\frac12)$. Also, we assume that the parameter function $\lambda(t)$ satisfies we have that

$$a\lambda_*(t) \le \lambda(t) \le b\lambda_*(t) \quad \text{for all} \quad t\in(0,T)$$

for some positive numbers a, b, c independent of T.

We observe that a priori we are not imposing boundary conditions in problem (6.1). Our purpose is to construct a solution ϕ that defines a linear operator of h and satisfies uniform bounds in terms of suitable norms.

All functions $h(y,t)$ with $h(y,t)\cdot W(y)\equiv 0$ can be expressed in polar form as

$$h(y,t) \ = \ h^1(\rho,\theta,t)E_1(y) + h^2(\rho,\theta,t)E_2(y), \quad y = \rho e^{i\theta}. \tag{6.2}$$

We can also expand in Fourier series

$$\tilde h(\rho,\theta,t) := h^1 + ih^2 \ = \ \sum_{k=-\infty}^{\infty}\tilde h_k(\rho,t)e^{ik\theta}, \quad \tilde h_k = \tilde h_{k1} + i\tilde h_{k2} \tag{6.3}$$

so that

$$h(y,t) = \sum_{k=-\infty}^{\infty} h_k(y,t) =: h_0(y,t) + h_1(y,t) + h_{-1}(y,t) + h^\perp(y,t), \tag{6.4}$$

where

$$h_k(y,t) = \mathrm{Re}\,(\tilde h_k(\rho,t)e^{ik\theta})\,E_1 + \mathrm{Im}\,(\tilde h_k(\rho,t)e^{ik\theta})\,E_2. \tag{6.5}$$

We consider the functions $Z_{kj}(y)$ defined in (2.2) and (2.3) and define for $k = -1, 0, 1$,

$$\bar h_k(y,t) := \sum_{j=1}^{2}\frac{\chi Z_{kj}(y)}{\int_{\mathbb{R}^2}\chi|Z_{kj}|^2}\int_{\mathbb{R}^2} h(x,t)\cdot Z_{kj}(z)\,dz,$$

where

$$\chi(y,t) = \begin{cases} w_\rho^2(|y|) & \text{if } |y| < 2R(t), \\ 0 & \text{if } |y| \ge 2R(t). \end{cases}$$

The main result in this section is the following, where we use the norm $\|h\|_{a,\nu}$ defined in (5.15).

Proposition 6.1. *Let $2<a<3$, $\nu>0$ and let h with $\|h\|_{a,\nu}<+\infty$. Let us write $h=h_0+h_1+h_{-1}+h^\perp$ with $h^\perp=\sum_{k\neq 0,\pm1}h_k$. Then there exists a solution $\phi[h]$ of problem (6.1), which defines a linear operator of h, and satisfies the following estimate in $\mathcal{D}_{2R}$:*

$$
\begin{aligned}
(1+|y|)\,|\nabla_y\phi(y,t)|+|\phi(y,t)| \lesssim\ & \frac{\lambda_*(t)^\nu}{1+|y|^{a-2}}\,\|h^\perp\|_{a,\nu}\\
&+\frac{\lambda_*(t)^\nu R(t)^{\frac{5-a}{2}}}{1+|y|}\ \min\{1,R^{\frac{5-a}{2}}|y|^{-2}\}\,\|h_0-\bar h_0\|_{a,\nu}\\
&+\frac{\lambda_*(t)^\nu R(t)^2}{1+|y|}\,\|\bar h_0\|_{a,\nu}\\
&+\frac{\lambda_*(t)^\nu}{1+|y|^{a-2}}\ \big\|h_1-\bar h_1\big\|_{a,\nu}+\frac{\lambda_*(t)^\nu R(t)^4}{1+|y|^2}\ \big\|\bar h_1\big\|_{a,\nu}\\
&+\frac{\lambda_*(t)^\nu R(t)^{\frac{5-a}{2}}}{1+|y|}\ \min\{1,R^{\frac{5-a}{2}}|y|^{-2}\}\,\|h_{-1}-\bar h_{-1}\|_{a,\nu}\\
&+\lambda_*(t)^\nu\log R(t)\,\|\bar h_{-1}\|_{a,\nu}.
\end{aligned}
$$

The construction of the operator $\phi[h]$ as stated in the proposition will be carried out mode by mode in the Fourier series expansion. We shall use the convention that $h(y,t)=0$ for $|y|>2R(t)$. Let us write

$$\phi=\sum_{k=-\infty}^{\infty}\phi_k,\qquad \phi_k(y,t)=\mathrm{Re}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_1+\mathrm{Im}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_2.$$

We shall build a solution of (6.1) by solving separately each of the equations

$$
\begin{aligned}
\lambda^2\partial_t\phi_k\ &=L_W[\phi_k]+h_k(y,t)=0\quad\text{in } \mathcal{D}_{4R},\\
\phi_k(y,0)\ &=0\quad \text{in } B_{4R(0)}(0),
\end{aligned}
\tag{6.6}
$$

which, are equivalent to the problems

$$
\begin{aligned}
\lambda^2\partial_t\varphi_k\ &=\mathcal{L}_k[\varphi_k]+\tilde h_k(\rho,t)\quad\text{in } \tilde D_{4R},\\
\varphi_k(\rho,0)\ &=0\quad\text{in } (0,4R_0)
\end{aligned}
$$

with

$$\tilde D_{4R}=\{(\rho,t)\ /\ t\in(0,T),\ \rho\in(0,4R(t))\}$$

and we recall

$$\mathcal{L}_k[\varphi_k]:=\partial_\rho^2\varphi_k+\frac{\partial_\rho\varphi_k}{\rho}-(k^2+2k\cos w+\cos(2w))\frac{\varphi_k}{\rho^2}$$

We have the validity of the following result.

Lemma 6.1. *Let $\nu>0$ and $0<a<3$, $a\neq 1,2$. Assume that*

$$\|h_k(y,t)\|_{a,\nu}<+\infty.$$

Then problem (6.6) has a unique bounded solution $\phi_k(y,t)$ of the form

$$\phi_k(y,t)\ =\ \mathrm{Re}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_1+\mathrm{Im}\,(\varphi_k(\rho,t)e^{ik\theta})\,E_2$$

which in addition satisfies the boundary condition

$$\phi_k(y,t)=0\quad\textit{for all } t\in(0,T),\quad y\in\partial B_{R(t)}(0).\tag{6.7}$$

These solutions satisfy the estimates

$$|\phi_k(y,t)|\ \le\ C\|h\|_{a,\nu}\,\lambda_*^\nu k^{-2}\begin{cases}R^{2-a} & \textit{if}\quad a<2,\\ (1+\rho)^{2-a} & \textit{if}\quad a>2,\end{cases}\qquad \textit{if } k\ge 2.$$

$$|\phi_{-1}(y,t)|\ \le\ C\|h\|_{a,\nu}\,\lambda_*^\nu\begin{cases}R^{2-a} & \textit{if}\quad a<2,\\ \log R & \textit{if}\quad a>2,\end{cases}$$

$$|\phi_0(y,t)| \ \le\ C\|h\|_{a,\nu}\lambda_*^{\nu}(1+\rho)^{-1}\left\{\begin{array}{lll} R^2 & \text{if} & a>1,\\ R^{3-a} & \text{if} & a<1,\end{array}\right.$$

$$|\phi_1(y,t)| \ \le\ C\|h\|_{a,\nu}\lambda_*^{\nu}(1+\rho)^{-2}R^4.$$

with C independent of R and k.

Proof. Standard parabolic theory yields existence of a unique solution to equation (6.6) that satisfies the boundary condition (6.7), for each k. Equivalently, the problem

$$\begin{aligned}\lambda^2\partial_t\varphi_k &= \mathcal{L}_k[\varphi_k]+\tilde h_k(\rho,t) \quad \text{in } \tilde D_{4R},\\ \varphi_k(t,4R) &= 0 \quad \text{for all} \quad t\in(0,T)\\ \varphi_k(0,\rho) &= 0 \quad \text{in } (0,4R(0)),\end{aligned} \tag{6.8}$$

$$\mathcal{L}_k[\varphi_k] = \partial_\rho^2\varphi_k + \frac{\partial_\rho\varphi_k}{\rho} - (k^2+2k\cos w+\cos(2w))\frac{\varphi_k}{\rho^2}$$

has a unique solution $\varphi_k(\rho,t)$ which is bounded in ρ for each t.

We use barriers to derive the desired estimates. A first observation we make is that for mode $k=-1$ the elliptic equation $\mathcal{L}_{-1}[\varphi]+g(\rho)=0$ in $(0,4R)$ with $\varphi(4R)=0$ has a unique bounded solution given by the variation of parameters formula

$$\begin{aligned}\varphi(\rho) &:= Z_{-1}(\rho)\int_\rho^{4R}\frac{dr}{\rho Z_{-1}(r)^2}\int_0^r g(s)Z_{-1}(s)s\,ds,\\ Z_{-1}(\rho) &= -\rho^2 w_\rho = \frac{2\rho^2}{\rho^2+1}.\end{aligned} \tag{6.9}$$

Here we have used that $\mathcal{L}_{-1}[Z_{-1}]=0$. Let us call $\varphi_0(\rho)$ the function in (6.9) with $g(\rho):=2(1+\rho)^{-a}$. We readily estimate

$$|\varphi_0(\rho)| \ \le\ \begin{cases} R^{2-a} & \text{if } a<2,\\ (1+\rho)^{2-a} & \text{if } a>2.\end{cases}.$$

Let us call $\bar\varphi(\rho,t)=\lambda_*(t)^\nu\varphi_0(\rho)$. Then we see that

$$\begin{aligned}-\lambda^2\bar\varphi_t(\rho,t)+\mathcal{L}_{-1}[\bar\varphi(\rho,t)]+\frac{\lambda_*^\nu}{(1+\rho)^a} &\le c\,\lambda_*^{\nu+1}|\dot\lambda_*|\varphi_0(\rho)-\frac{\lambda_*^\nu}{(1+\rho)^a}\\ &\le -\lambda_*^\nu(1+\rho)^{-a}\left[1-C\lambda_*R^{2-a}(1+\rho)^a\right]\\ &<0\end{aligned}$$

in $\tilde D_{4R}$. Indeed, since $R(t)\ll\lambda_*^{-\frac12}$, the inequality holds provided that T was chosen sufficiently small. Thus for $k=-1$ the barrier $\|h\|_{a,\nu}\,\bar\varphi(\rho,t)$ dominates both, real and imaginary parts of $\varphi_{-1}(\rho,t)$. As a conclusion, we find

$$|\phi_{-1}(y,t)| \ \le\ C\|h\|_{a,\nu}\lambda_*^\nu\begin{cases} R^{2-a} & \text{if } a<2,\\ (1+\rho)^{2-a} & \text{if } a>2,\end{cases} \quad \text{in } \mathcal{D}_{4R}.$$

The cases $k=0,1,-2$ can be dealt with in exactly the same manner, by replacing Z_{-1} in Formula (6.9) respectively by the functions

$$Z_0(\rho)=\frac{\rho}{\rho^2+1},\quad Z_1(\rho)=\frac{1}{\rho^2+1},\quad Z_{-2}(\rho)=\frac{\rho^3}{\rho^2+1}. \tag{6.10}$$

The estimates for ϕ_k predicted in the Lemma then readily follow for $k=-2,-1,0,1$. Finally, let us now consider k with $|k|\geq 2$ and $k\neq -2$ and the function $\bar\varphi(\rho,t)$ as above. Now we find

$$\begin{aligned} -\lambda^2\bar\varphi_t(\rho,t)+\mathcal{L}_k[\bar\varphi(\rho,t)] &\leq (\mathcal{L}_k-\mathcal{L}_{-1})[\bar\varphi(\rho,t)] \\ &\leq -C\lambda_*^{\nu}(k^2-1+2(k-1))\frac{1}{\rho^2}(1+\rho)^{2-a} \\ &< -C(k^2-1+2(k-1))\frac{\lambda_*^{\nu}}{(1+\rho)^a} \quad \text{in } \tilde{\mathcal{D}}_{4R}. \end{aligned}$$

The latter quantity is negative provided that $|k|\geq 2$ and $k\neq -2$ and hence we get the estimate

$$|\phi_k(y,t)| \ \leq \ \frac{C}{k^2}\|h\|_{a,\nu}\lambda_*^{-\nu}\begin{cases} R^{2-a} & \text{if } a<2, \\ (1+\rho)^{2-a} & \text{if } a>2, \end{cases} \quad \text{in } \mathcal{D}_{4R}.$$

The proof is concluded. □

We can get gradient estimates for the solutions built in the above lemma by means of the following result.

Lemma 6.2. *Let ϕ be a solution of the equation*

$$\begin{aligned} &\lambda^2\partial_t\phi = L_W[\phi]+h(y,t) \quad \text{in } \mathcal{D}_{4\gamma R} \\ &\phi(\cdot,0)=0 \quad \text{in } B_{4\gamma R(0)}. \end{aligned} \tag{6.11}$$

Given numbers a,b,γ, there exists a C such that if for some $M>0$ we have

$$|\phi(y,t)|+(1+|y|)^2|h(y,t)|\leq M\,\lambda_*(t)^b(1+|y|)^{-a} \quad \text{in } \mathcal{D}_{4\gamma R}, \tag{6.12}$$

then

$$(1+|y|)|\nabla_y\phi(y,t)| \ \leq \ C\,M\lambda_*(t)^b(1+|y|)^{-a} \quad \text{in } \mathcal{D}_{3\gamma R} \tag{6.13}$$

and we recall

$$\mathcal{D}_{\gamma R}=\{(y,t)\ /\ |y|<\gamma R(t), \quad t\in(0,T)\}.$$

If in addition we know that ϕ satisfies the boundary condition $\phi(\cdot,t)=0$ on $\partial B_{4\gamma R(t)}$ for all $t\in(0,T)$ then estimate (6.13) holds in the entire region $\mathcal{D}_{4\gamma R}$.

Proof. To prove the gradient estimates, we change the time variable, defining

$$\tau(t)=\int_0^t\frac{ds}{\lambda(s)^2}, \tag{6.14}$$

so that (6.11) becomes in the variables (y,τ)

$$\begin{aligned} &\partial_\tau\phi=L_W[\phi]+h(y,\tau) \quad \text{in } \mathcal{D}_{4\gamma R} \\ &\phi(\cdot,0)=0 \quad \text{in } B_{4R(0)} \end{aligned}$$

Let $\tau_1>0$ and $y_1\in B_{3\gamma R(\tau_1)}(0)$. Let $\rho=\frac{|y_1|}{5}+1$ so that $B_\rho(y_1)\subset B_{4\gamma R(\tau_1)}(0)$. Let us define

$$\tilde\phi(z,t):=\phi(y_1+\rho z,\tau_1+\rho^2 s), \quad z\in B_1(0), \quad s>-\frac{\tau_1}{\rho^2}.$$

We distinguish two cases. First, when $\tau_1\geq\rho^2$, we use interior estimates for parabolic equations, while for the case $\tau_1<\rho^2$, we use estimates for a parabolic equation with initial condition.

Assume $\tau_1\geq\rho^2$. Then $\tilde\phi(z,s)$ satisfies an equation of the form

$$\tilde\phi_s=\Delta_z\tilde\phi+A\nabla_z\tilde\phi+B\tilde\phi+\tilde h(z,s) \quad \text{in } B_1(0)\times(-1,0]$$

with coefficients $A(z,s)$ and $B(z,s)$ uniformly bounded by $O((1+\rho)^{-2})$ in $B_1(0)\times(-1,0]$ and

$$\tilde h(z,s)=\rho^2 h(y_1+\rho z,\tau_1+\rho^2 s).$$

Since $\rho\leq CR(\tau_1)$ and $R(\tau_1)^2\ll\tau_1$ for τ_1 large we get

$$\lambda_*(\tau_1)^b\lesssim\lambda_*(\tau_1+\rho^2 s)^b\lesssim\ \lambda_*(\tau_1)^b, \quad s\in(-1,0].$$

Standard parabolic estimates and assumption (6.12) yield

$$\|\nabla_z\tilde\phi\|_{L^\infty(B_{\frac14}(0)\times(1,2))} \lesssim \|\tilde\phi\|_{L^\infty B_{\frac12}(0)\times(0,2)} + \|\tilde h\|_{L^\infty(B_{\frac12}(0)\times(0,2))}$$
$$\lesssim M\,\lambda_*(\tau_1)^b\rho^{2-a},$$

so that in particular

$$\rho|\nabla_y\phi(y_1,\tau_1)| \;=\; |\nabla_z\tilde\phi(0,1)| \lesssim M\,\lambda_*(\tau_1)^b\rho^{2-a}.$$

In the case $\tau_1 \ge \rho^2$ the argument is similar, but the equation for $\tilde\phi$ holds in $B_1(0)\times(-\frac{\tau_1}{\rho^2},0]$ and has initial condition 0 at $s=-\frac{\tau_1}{\rho^2}$. Finally, for the last assertion we argue in similar way but using boundary rather that interior gradient estimates. □

In addition to estimate (6.13) we have a Hölder gradient estimate which is more natural to express using the variable τ in (6.14) as follows. We denote

$$\mathcal{B}_\ell(y,\tau) = \{(y',\tau') \;/\; |y-y'|^2 + |\tau'-\tau| < \ell^2\}.$$

For a function $g(y,\tau)$, a number $0<\alpha<1$, and a set A we let

$$[g]_{\alpha,A} := \sup\Big\{ \frac{|f(y,\tau)-f(y',\tau')|}{(|y-y'|^2+|\tau'-\tau|)^{\frac{\alpha}{2}}} \;/\; (y,\tau),\,(y',\tau')\in A\Big\}.$$

Corollary 6.1. *Let ϕ be a solution of the equation (6.11) with $h(y,\tau)=\operatorname{div} H(y,\tau)$. Given $\alpha\in(0,1)$ and constants a, b, γ there is C such that if*

$$|\phi(y,\tau)| + (1+|y|)|H(y,\tau)| + (1+|y|)^{1+\alpha}[H]_{\mathcal{B}_{\ell(y)}(y,\tau)\cap\mathcal{D}_{4\gamma R}} \le M\,\lambda_*(\tau)^b(1+|y|)^{-a}$$

in $\mathcal{D}_{4\gamma R}$, where $\ell(y)=1+\frac{|y|}{4}$, then

$$(1+|y|)|\nabla_y\phi(y,\tau)| + (1+|y|)^{1+\alpha}[\nabla_y\phi]_{\mathcal{B}_{\ell(y)}(y,\tau)\cap\mathcal{D}_{4\gamma R}} \le C\,M\lambda_*(t)^b(1+|y|)^{-a} \tag{6.15}$$

in $\mathcal{D}_{3\gamma R}$. If in addition we know that ϕ satisfies the boundary condition $\phi(\cdot,t)=0$ on $\partial B_{4\gamma R(t)}$ for all $t\in(0,T)$ then estimate (6.15) holds in the entire region $\mathcal{D}_{4\gamma R}$.

Our next goal is to construct an inverse for modes $k=-1,0,1$ with a better control when subject to a certain solvability condition.

6.1. **Mode $k=0$.** Let us consider again equation (6.6) for $k=0$ and the functions $Z_{0j}(y)$ defined in (2.2) . We have the following result.

Lemma 6.3. *Let assume that $2<a<3$, $k=0$ and*

$$\int_{\mathbb{R}^2} h_0(y,t)\cdot Z_{0j}(y)\,dy \;=\; 0 \quad \textit{for all} \quad t\in[0,T) \tag{6.16}$$

for $j=1,2$. Then there exist a solution ϕ_0 to equation (6.6) for $k=0$ that defines a linear operator of h_0 and satisfies the estimate in $\mathcal{D}_{3R}$,

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu} R^{\frac{5-a}{2}}\lambda_*^\nu(1+|y|)^{-1}\,\min\{1, R^{\frac{5-a}{2}}|y|^{-2}\}\;. \tag{6.17}$$

A central feature of estimate (6.17) is that it matches the size of the solutions obtained in Lemma 6.1 for $k\ne 0,1$ when $|y|\sim R$.

Proof. We observe that conditions (6.16) can be written as

$$\int_0^{2R} \tilde h_0(\rho,t)\,Z_0(\rho)\rho\,d\rho \;=\; 0 \quad \text{for all} \quad \tau\in(0,T). \tag{6.18}$$

Let us consider the complex valued functions

$$\tilde H_0(\rho,t) := -Z_0(\rho)\int_\rho^\infty \frac{1}{sZ_0(s)^2}\int_s^\infty \tilde h_0(\zeta,t)Z_0(\zeta)\zeta\,d\zeta, \quad k=0,1.$$

They are well-defined thanks to (6.18). Then the function

$$H_0(y,t) := \mathrm{Re}\,(\tilde H_0(\rho,\tau))E_1(y) + \mathrm{Re}\,(\tilde H_0(\rho,t))E_2(y)$$

solves

$$L_W[H_0(y,\tau)] = h_0(y,\tau) \quad \text{in } \mathcal{D}_{4R}$$

and satisfies

$$|H_0(y,t)| \ \lesssim \ \lambda_*(t)^\nu (1+|y|)^{2-a}\|h_0\|_{a,\nu} \quad \text{in } \mathcal{D}_{4R}.$$

Moreover, elliptic gradient estimates yield

$$|\nabla_y H_0(y,\tau)| \lesssim \lambda_*(t)^\nu(1+|y|)^{1-a}\|h_0\|_{a,\nu} \quad \text{in } \mathcal{D}_{3R}.$$

Let us consider the problem

$$\begin{aligned} \lambda^2\Phi_t &= L_W[\Phi] + H_0(y,t) \quad \text{in } \mathcal{D}_{4R}, \\ \Phi(y,0) &= 0 \quad \text{in } B_{4R}(0) \\ \Phi(y,t) &= 0 \quad \text{for all } t\in(0,T), \quad y\in\partial B_{4R(0)}(0) \end{aligned} \tag{6.19}$$

According to Lemma 6.1, this problem has unique solution $\Phi = \Phi_0$ that satisfies the estimates

$$|\Phi_0(y,t)| \ \le C\|H_0\|_{a-2,\nu}\lambda_*(\tau)^\nu(1+|y|)^{-1} \ \ R^{5-a} \quad \text{in } \mathcal{D}_{4R}.$$

Applying Lemma 6.2 we deduce that, also,

$$|\nabla_y\Phi_0(y,t)| \lesssim \|H_0\|_{a-2,\nu}\lambda_*(\tau)^\nu(1+|y|)^{-2} \ \ R^{5-a} \quad \text{in } \mathcal{D}_{3R}$$

Let us write

$$\Phi_{0j} := \partial_{y_j}\Phi_0, \quad H_{0j} := \partial_{y_j}H_0$$

Then we have

$$\begin{aligned} \lambda^2\partial_t\Phi_{0j} &= L_W[\Phi_{0j}] + \partial_{y_j}|\nabla W|^2\Phi_0 + 2\nabla\partial_{y_j}W\nabla\Phi_0 + H_{0j}(y,\tau) \\ &\quad + 2(\nabla\Phi_0\partial_{y_j}\nabla W)W + 2(\nabla\Phi_0\nabla W)\partial_{y_j}W \quad \text{in } \mathcal{D}_{3R}, \\ \Phi_{0j}(y,0) &= 0 \quad \text{for all } y\in B_{3R(0)}(0) \end{aligned}$$

According to Lemma 6.2 and the above estimates we obtain that

$$\begin{aligned} (1+|y|)|\nabla\Phi_{0j}(y,t)| &\lesssim \|h_0\|_{a,\nu}\lambda_*(t)^\nu(1+|y|)^{-2}R^{5-a} \\ &\quad + \|h_0\|_{a,\nu}\lambda_*(t)^\nu(1+|y|)^{4-a} \quad \text{in } \mathcal{D}_{3R}. \end{aligned}$$

Then we define

$$\phi_0 := L_W[\Phi_0]$$

so that $\phi = \phi_0$ solves

$$\begin{aligned} \lambda^2\phi_t &= L_W[\phi] + h_0(y,t) \quad \text{in } \mathcal{D}_{3R}, \\ \phi(y,0) &= 0 \quad \text{for all} \quad y\in B_{3R(0)}(0) \end{aligned}$$

and defines a linear operator of the function h_0. Moreover, observing that

$$|L_W[\Phi_0]| \lesssim \left|D_y^2\Phi_0\right| + O(\rho^{-4})\,|\Phi_0| + O(\rho^{-2})\,|D_y\Phi_0|$$

we then get the estimate

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu}R^{5-a}\lambda_*(t)^\nu(1+|y|)^{-3}\,. \tag{6.20}$$

To complete the proof of estimate (6.17), we let φ_0 be the complex valued function defined as

$$\phi_0(y,t) \ = \ \mathrm{Re}\,(\varphi_0(\rho,t))\,E_1 + \mathrm{Im}\,(\varphi_0(\rho,t))\,E_2$$

so that letting $R' = R^{\frac{5-a}{4}} \ll R$, using the notation in (6.8), φ_0 satisfies the equation

$$\begin{aligned} \lambda^2\partial_t\varphi_0 &= \mathcal{L}_0[\varphi_0] + \tilde h_0(\rho,t) \quad \text{in } \tilde D_{R'}, \\ \varphi_0(0,\rho) &= 0 \quad \text{in } (0,R'), \end{aligned} \tag{6.21}$$

and from (6.20), we can find an explicit supersolution for the real and imaginary parts of equation (6.21), which also dominates their boundary values at R', which yields

$$|\varphi_0(y,t)| \lesssim \|h_0\|_{a,\nu}\lambda_*^\nu\,|R'|^2(1+|y|)^{-1}\,, \quad |y|<R'.$$

Combining this estimate and (6.20) yields the validity of (6.17). □

We mention next a variant of Lemma 6.3, in which we weaken the hypothesis on the right hand side, allowing it to be a divergence of Hölder continuous function. This will be needed when analyzing estimates of the derivative with respect to λ of operator $\mathcal{T}_{\lambda,2}$ (Proposition 5.3).

Lemma 6.4. *Let assume that $2<a<3$, $\nu>0$, and $k=0$. Let h_0 have the form*

$$h_0(y,\tau) = \operatorname{div} H_0(y,\tau)$$

such that

$$(1+|y|)|H_0(y,\tau)| + (1+|y|)^{1+\alpha}[H_0]_{\mathcal{B}_{\ell(y)}(y,\tau)\cap\mathcal{D}_{4R}} \le \lambda_*(\tau)^\nu(1+|y|)^{-a},$$

in $\mathcal{D}_{4R}$, where $\alpha\in(0,1)$ and $\ell(y)=1+\frac{|y|}{4}$. Assume also that

$$\int_{\mathbb{R}^2} h_0(y,t)\cdot Z_{0j}(y)\,dy \;=\; 0 \quad \text{for all} \quad t\in[0,T)$$

for $j=1,2$. Then there exist a solution ϕ_0 to equation (6.6) for $k=0$ that defines a linear operator of h_0 and satisfies

$$|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu} R^{\frac{5-a}{2}}\lambda_0^\nu(1+|y|)^{-1}\,\min\{1,R^{\frac{5-a}{2}}|y|^{-2}\},$$

in $\mathcal{D}_{3R}$.

6.2. **Mode $k=-1$.** Let us consider equation (6.6) for $k=-1$ and the functions $Z_{-1j}(y)$ defined in (2.3) . We have the following result.

Lemma 6.5. *Let assume that $2<a<3$, $k=0$ and*

$$\int_{\mathbb{R}^2} h_{-1}(y,t)\cdot Z_{-1j}(y)\,dy = 0 \quad \text{for all} \quad t\in[0,T) \tag{6.22}$$

for $j=1,2$. Then there exist a solution ϕ_{-1} to equation (6.6) for $k=-1$ that defines a linear operator of h_0 and satisfies the estimate in $\mathcal{D}_{3R}$,

$$|\phi_{-1}(y,t)| \lesssim \|h_{-1}\|_{a,\nu}\lambda_*^\nu\,\min\{\log R, R^{4-a}|y|^{-2}\}. \tag{6.23}$$

Proof. The proof is essentially the same as that of Lemma 6.3. The differences are as follows.

From (6.22) we can find a function H_{-1} defined in $\mathbb{R}^2$ such that

$$L_W[H_{-1}] = h_{-1} \quad \text{in } \mathbb{R}^2,$$

and satisfying the estimates

$$|H_{-1}| \lesssim \lambda_*(t)^\nu(1+|y|)^{2-a}\|h_0\|_{a,\nu} \quad \text{in } \mathcal{D}_{4R},$$
$$|\nabla_y H_{-1}(y,\tau)| \lesssim \lambda_*(t)^\nu(1+|y|)^{1-a}\|h_0\|_{a,\nu} \quad \text{in } \mathcal{D}_{3R}.$$

Let us consider the problem

$$\lambda^2\Phi_t = L_W[\Phi] + H_{-1}(y,t) \quad \text{in } \mathcal{D}_{4R},$$
$$\Phi(y,0)=0 \quad \text{in } B_{4R}(0)$$
$$\Phi(y,t)=0 \quad \text{for all } t\in(0,T), \quad y\in\partial B_{4R(0)}(0).$$

This problem has unique solution $\Phi=\Phi_{-1}$, and applying the same proof as in Lemma 6.1, we get the estimate

$$|\Phi_{-1}(y,t)| \le C\|H_{-1}\|_{a-2,\nu}\lambda_*(\tau)^\nu R^{4-a} \quad \text{in } \mathcal{D}_{4R}.$$

Applying Lemma 6.2 we deduce that, also,

$$|\nabla_y \Phi_{-1}(y,t)| \lesssim \|H_{-1}\|_{a-2,\nu}\lambda_*(\tau)^\nu (1+|y|)^{-1} R^{4-a} \quad \text{in } \mathcal{D}_{3R}. \tag{6.24}$$

Arguing as in Lemma 6.3 we get that

$$\phi_{-1} = L_W[\Phi_{-1}]$$

satisfies

$$\lambda^2 \phi_t = L_W[\phi] + h_{-1}(y,t) \quad \text{in } \mathcal{D}_{3R},$$
$$\phi(y,0) = 0 \quad \text{for all} \quad y \in B_{3R(0)}(0)$$

and defines a linear operator of h_{-1}. Moreover we get estimate

$$|\phi_{-1}(y,t)| \lesssim \|h_{-1}\|_{a,\nu}\lambda_*(t)^\nu R^{4-a}(1+|y|)^{-2}. \tag{6.25}$$

As in Lemma 6.3, using a super solution in $\mathcal{D}_{R'}$ with $R' = R^{\frac{4-a}{2}} \log^{-1/2}(R)$ we find that

$$|\phi_{-1}(y,t)| \lesssim \|h_{-1}\|_{a,\nu} \log R, \quad \text{in } \mathcal{D}_{R'}. \tag{6.26}$$

Then combining (6.25) with (6.26) we obtain the desired estimate (6.23). □

6.3. **Mode $k=1$.** Now we deal with (6.6) for $k=1$. For convenience we give the result for a right hand side more general than strictly need for the proof of Proposition 6.1. Let us assume that h_1 is defined in entire $\mathbb{R}^2 \times (0,T)$ and that

$$h_1(y,t) = \operatorname{div}_y G(y,t) \tag{6.27}$$

where

$$|G(y,t)| \le \frac{\lambda_*(t)^\nu}{1+|y|^{a-1}}, \quad y \in \mathbb{R}^2,\ t \in (0,T), \tag{6.28}$$

for some $\nu > 0$, $a \in (2,3)$. Then the following result holds.

Lemma 6.6. *Let assume that $2 < a < 3$, $k=1$, h_1 has the form (6.27) so that (6.28) holds and*

$$\int_{\mathbb{R}^2} h_1(y,t) \cdot Z_1^j(y)\, dy \;=\; 0 \quad \textit{for all} \quad t \in (0,T)$$

for $j = 1,2$. Then there exist a solution ϕ_1 to equation (6.6) for $k=1$ that defines a linear operator of h_1 and satisfies the estimate in $\mathcal{D}_{3R}$,

$$|\phi_1(y,t)| \lesssim \lambda_*(t)^\nu (1+|y|)^{2-a}.$$

From this we get directly the next result.

Corollary 6.2. *Let assume that $2 < a < 3$, $k=1$ and*

$$\int_{B_{2R}} h_1(y,t) \cdot Z_1^j(y)\, dy \;=\; 0 \quad \textit{for all} \quad t \in (0,T)$$

for $j=1,2$. Then there exist a solution ϕ_1 to equation (6.6) for $k=1$ that defines a linear operator of h_1 and satisfies the estimate in $\mathcal{D}_{3R}$,

$$|\phi_1(y,t)| \;\lesssim\; \|h_1\|_{a,\nu}\lambda_*(t)^\nu (1+|y|)^{2-a} \;.$$

Let us do the same change of the time variable as in (6.14) so that (6.6) for $k=1$ in entire $\mathbb{R}^2$ becomes in the variables (y,τ)

$$\partial_\tau \phi \;=\; L_W[\phi] + h \quad \text{in } \mathbb{R}^2 \times (0,\infty), \tag{6.29}$$
$$\phi(\cdot,0) \;=\; 0 \quad \text{in } \mathbb{R}^2.$$

Thus, we consider a function $h(y,\tau)$ defined in entire $\mathbb{R}^2 \times (0,+\infty)$ of the form

$$h = \operatorname{Re}(\tilde h e^{i\theta})\, E_1 + \operatorname{Im}(\tilde h e^{i\theta})\, E_2, \tag{6.30}$$

that satisfies the orthogonality conditions for $j = 1, 2$

$$\int_{\mathbb{R}^2} h(\cdot,\tau)\cdot Z_{1j} \ = \ 0 \quad \text{for all} \quad \tau\in(0,\infty) \tag{6.31}$$

and such that $h(y,\tau) = 0$ for $|y| \ge 2R(\tau)$.

By standard parabolic theory, this problem has a unique solution, which is therefore of the form

$$\phi = \operatorname{Re}(\varphi e^{i\theta})\, E_1 + \operatorname{Im}(\varphi e^{i\theta})\, E_2, \tag{6.32}$$

where the complex valued function $\varphi(\rho,\tau)$ solves the initial value problem

$$\begin{aligned} \partial_\tau\varphi \ &= \mathcal{L}_1[\varphi] + \tilde h(\rho,\tau) \quad \text{in } (0,\infty)\times(0,\infty), \\ \varphi(\rho,0) \ &= 0 \quad \text{in } (0,\infty), \end{aligned} \tag{6.33}$$

$$\mathcal{L}_1[\varphi] = \partial_\rho^2\varphi + \frac{\partial_\rho\varphi}{\rho} - (1+2\cos w + \cos(2w))\frac{\varphi}{\rho^2}.$$

We have the validity of the following result.

Lemma 6.7. *Let $0<\sigma<1$, $\nu>0$. Assume that h is mode 1, that is, has the form (6.30), satisfies the orthogonality conditions (6.31), and can be written as in (6.27) with g_j satisfying (6.28) where $b = 1+\sigma$. Then there exists a constant $C>0$ such that the solution ϕ of problem (6.29) satisfies the estimate*

$$|\phi(y,t)| \le C\frac{\lambda_*(t)^\nu}{1+|y|^\sigma}. \tag{6.34}$$

For the proof of this result we will use the following Liouville type result.

Lemma 6.8. *Let $0<\sigma<1$. Suppose $\tilde\phi$ satisfies*

$$\tilde\phi_\tau = L_W[\tilde\phi] \quad \text{in } \mathbb{R}^2\times(-\infty,0],$$

$$\int_{\mathbb{R}^2}\tilde\phi(\cdot,\tau)\cdot Z_{1j} = 0 \quad \text{for all} \quad \tau\in(-\infty,0],$$

$$|\tilde\phi(y,\tau)| \le \frac{1}{1+|y|^\sigma} \quad \text{in } \mathbb{R}^2\times(-\infty,0], \quad j=1,2,$$

$$\tilde\phi(y,\tau) = \operatorname{Re}(\tilde\varphi(\rho,\tau)e^{i\theta})\, E_1 + \operatorname{Im}(\tilde\varphi(\rho,\tau)e^{i\theta})\, E_2.$$

Then $\tilde\phi = 0$.

Proof. By standard parabolic regularity $\tilde\phi(y,\tau)$ is a smooth function. A scaling argument shows that

$$(1+|y|)^{-1}|D_y\tilde\phi| + |\tilde\phi_\tau| + |D_y^2\tilde\phi| \le C(1+|y|)^{-2-\sigma}.$$

Differentiating the equation in τ, we also get $\partial_\tau\phi_\tau = L_W[\phi_\tau]$ and we find the estimates

$$(1+|y|)^{-1}|D_y\tilde\phi_\tau| \ + \ |\tilde\phi_{\tau\tau}| \ + \ |D_y^2\tilde\phi_\tau| \ \le \ C(1+|y|)^{-3-\sigma}.$$

Testing suitably the equations (taking into account the asymptotic behaviors in y in integrations by parts) we find

$$\frac12\partial_\tau\int_{\mathbb{R}^2}|\tilde\phi_\tau|^2 + B(\tilde\phi_\tau,\tilde\phi_\tau) = 0,$$

where

$$B(\tilde\phi,\tilde\phi) = -\int_{\mathbb{R}^2} L_W[\tilde\phi]\cdot\tilde\phi = \int_{\mathbb{R}^2}|\nabla\tilde\phi|^2 - |\nabla W|^2|\tilde\phi|^2.$$

It is useful to observe the following: since

$$\tilde\phi(y,\tau) \ = \ \operatorname{Re}(\tilde\varphi(\rho,\tau)e^{i\theta})\, E_1 + \operatorname{Im}(\tilde\varphi(\rho,\tau)e^{i\theta})\, E_2$$

then we compute, using that $\mathcal{L}_1[w_\rho] = 0$,

$$B(\tilde\phi,\tilde\phi) = -\int_0^\infty \mathcal{L}_1[\varphi]\bar\varphi\rho d\rho \ = \ \int_0^\infty |(w_\rho^{-1}\tilde\varphi)_\rho|^2 w_\rho^2\rho\, d\rho \ \ge \ 0.$$

We also get

$$\int_{\mathbb{R}^2} |\tilde\phi_\tau|^2 = -\frac{1}{2}\partial_\tau B(\tilde\phi,\tilde\phi).$$

From these relations we find

$$\partial_\tau \int_{\mathbb{R}^2} |\tilde\phi_\tau|^2 \le 0, \qquad \int_{-\infty}^0 d\tau \int_{\mathbb{R}^2} |\tilde\phi_\tau|^2 \ < \ +\infty$$

and hence $\tilde\phi_\tau = 0$. Thus $\tilde\phi$ is independent of τ and therefore $L_W[\tilde\phi] = 0$. Since $\tilde\phi$ is at mode 1, this implies that $\tilde\phi$ is a linear combination of Z_{1j}, $j = 1, 2$. Since $\int_{\mathbb{R}^2} \tilde\phi \cdot Z_{1j} = 0$, $j = 1, 2$ we conclude that $\tilde\phi = 0$, a contradiction. □

Proof of Lemma 6.7. Let us write

$$\|\phi\|_{b,\tau_1} := \sup_{\tau\in(0,\tau_1)} \lambda_*(\tau)^{-\nu}\|(1+|y|^b)\phi\|_{L^\infty(\mathbb{R}^2)}.$$

We claim that for any $\tau_1 > 0$ we have that

$$\|\phi\|_{2+\sigma,\tau_1} < +\infty. \tag{6.35}$$

Let us recall that with the transformations (6.32) we have that the complex valued function $\varphi(y,\tau)$ is radial in y and solves the initial value problem

$$\begin{aligned} &\partial_\tau\varphi = \Delta_{\mathbb{R}^2}\varphi - (1 + 2\cos w + \cos(2w))\frac{\varphi}{\rho^2} + \tilde h(\rho,\tau) \quad \text{in } \mathbb{R}^2\times(0,\infty),\\ &\varphi(\cdot,0) = 0 \quad \text{in } \mathbb{R}^2 \end{aligned}$$

where $\rho = |y|$, $y \in \mathbb{R}^2$ and $\tilde h$ is related to h by (6.30). Let us write $\varphi = \varphi_a + \varphi_b$ where φ_a is the unique solution to

$$\begin{aligned} &\partial_\tau\varphi_a = \Delta_{\mathbb{R}^2}\varphi_a + \tilde h(\rho,\tau) \quad \text{in } \mathbb{R}^2\times(0,\infty),\\ &\varphi_a(\cdot,0) = 0 \quad \text{in } \mathbb{R}^2 \end{aligned}$$

given by Duhamel's formula. Using the heat kernel in $\mathbb{R}^2$ one readily shows that $\|\varphi_a\|_{2+\sigma,\tau_1} < +\infty$. Let

$$\begin{aligned} &\partial_\tau\varphi_b = \Delta_{\mathbb{R}^2}\varphi_b - (1 + 2\cos w + \cos(2w))\frac{1}{\rho^2}(\varphi_a+\varphi_b) \quad \text{in } \mathbb{R}^2\times(0,\infty),\\ &\varphi_b(\cdot,0) = 0 \quad \text{in } \mathbb{R}^2. \end{aligned}$$

By standard linear parabolic theory $\phi_b(y,\tau)$ is locally bounded in time and space. More precisely, given $R > 0$ there is a $K = K(R,\tau_1)$ such that

$$|\phi_b(y,\tau)| \le K \quad \text{in } B_R(0)\times(0,\tau_1].$$

If we fix R large and take K_1 sufficiently large, we see that $K_1\rho^{-\sigma}$ is a supersolution for the real and imaginary parts of the equivalent complex valued equation (6.33) in the region $\rho > R$. As a conclusion, we find that $|\phi_b| \le 2K_1\rho^{-\sigma}$, and therefore $\|\phi_b\|_{\sigma,\tau_1} < +\infty$ for any $\tau_1 > 0$. This proves (6.35).

Next we claim that

$$\int_{\mathbb{R}^2} \phi(\cdot,\tau)\cdot Z_{1j} = 0 \quad \text{for all} \quad \tau\in(1,\tau_1), \quad j = 1,2. \tag{6.36}$$

Indeed, let us test the equation against

$$Z_{1j}\eta, \quad \eta(y) = \eta_0(R^{-1}|y|)$$

where η_0 is a smooth cut-off function with $\eta_0(r) = 1$ for $r < 1$ and $= 0$ for $r > 2$ and R is an arbitrary large constant. We find that

$$\int_{\mathbb{R}^2} \phi(\cdot,\tau)\cdot Z_{1j}\eta \ = \ \int_0^\tau ds \int_{\mathbb{R}^2} \phi(\cdot,s)\cdot(L_W[\eta Z_{1j}] + h\cdot Z_{1j}\eta). \tag{6.37}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^2}\phi\cdot(L_W[\eta Z_{1j}]+h\cdot Z_{1j}\eta_R) &= \int_{\mathbb{R}^2}\phi\cdot(Z_{1j}\Delta\eta+2\nabla\eta\cdot\nabla Z_{1j})-h\cdot Z_{1j}(1-\eta_R)\\ &= O(R^{-2-\sigma})\end{aligned}$$

uniformly on $\tau\in(0,\tau_1)$. Letting $R\to+\infty$ in (6.37) we get that (6.36) holds.

Now we claim that there exists a constant C such that for all $\tau_1>0$ we have the validity of the estimate

$$\|\phi\|_{\sigma,\tau_1}\le C, \tag{6.38}$$

so that in particular estimate (6.34) holds.

To prove (6.38) we assume by contradiction the existence of sequences $\tau_1^n\to+\infty$ and ϕ_n, h_n of the form (6.30), (6.32) satisfying

$$\begin{aligned}\partial_\tau\phi_n &= L_W[\phi_n]+h_n \quad \text{in } \mathbb{R}^2\times(1,\tau_1^n),\\ \int_{\mathbb{R}^2}\phi_n(\cdot,\tau)\cdot Z_{1j} &= 0 \quad \text{for all} \quad \tau\in(1,\tau_1^n),\\ \phi_n(\cdot,1) &= 0 \quad \text{in } \mathbb{R}^2,\end{aligned}$$

so that

$$\|\phi_n\|_{\sigma,\tau_1^n}=1 \tag{6.39}$$

but

$$h_n=\sum_{j=1}^2\partial_{y_j}g_{j,n},\quad \|g_{j,n}\|_{1+\sigma,\tau_1^n}\to 0,\quad \text{as } n\to\infty.$$

We claim first that

$$\sup_{1<\tau<\tau_1^n}\tau^\nu|\phi_n(y,\tau)|\to 0 \tag{6.40}$$

uniformly on compact subsets of $y\in\mathbb{R}^2$. If not, for some $M>0$ there are $|y_n|\le M$ and $1<\tau_2^n<\tau_1^n$ so that

$$(\tau_2^n)^\nu(1+|y_n|^\sigma)|\phi(y_n,\tau_2^n)|\ge\frac12.$$

Clearly we must have $\tau_2^n\to+\infty$. Let us define

$$\tilde\phi_n(y,\tau)=(\tau_2^n)^\nu\phi_n(y,\tau_2^n+\tau).$$

Then

$$\partial_\tau\tilde\phi_n=L_W[\tilde\phi_n]+\tilde h_n \quad \text{in } \mathbb{R}^2\times(1-\tau_2^n,0]$$

where $\tilde h_n\to 0$ has the form

$$\tilde h_n=\sum_{j=1}^2\partial_{y_j}\tilde g_{j,n},\quad |\tilde g_{j,n}(y,\tau)|\le o(1)\frac{(\tau_2^n)^\nu}{(\tau_2^n+\tau)^\nu}\frac{1}{1+|y|^{1+\sigma}}$$

and

$$|\tilde\phi_n(y,\tau)|\le\frac{1}{1+|y|^\sigma} \quad \text{in } \mathbb{R}^2\times(1-\tau_2^n,0].$$

From standard parabolic estimates, we find that passing to a subsequence, $\tilde\phi_n \to \tilde\phi$ uniformly on compact subsets of $\mathbb{R}^2 \times (-\infty, 0]$ where $\tilde\phi \neq 0$ and

$$\tilde\phi_\tau = L_W[\tilde\phi] \quad \text{in } \mathbb{R}^2 \times (-\infty, 0],$$
$$\int_{\mathbb{R}^2} \tilde\phi(\cdot, \tau) \cdot Z_{1j} = 0 \quad \text{for all} \quad \tau \in (-\infty, 0],$$
$$|\tilde\phi(y, \tau)| \leq \frac{1}{1 + |y|^\sigma} \quad \text{in } \mathbb{R}^2 \times (-\infty, 0], \quad j = 1, 2,$$
$$\tilde\phi(y, \tau) = \mathrm{Re}\,(\tilde\varphi(\rho, \tau)e^{i\theta})\, E_1 + \mathrm{Im}\,(\tilde\varphi(\rho, \tau)e^{i\theta})\, E_2.$$

But then Lemma 6.8 implies that $\tilde\phi \equiv 0$, which is a contradiction, and we conclude that (6.40) indeed holds.

From (6.39), we have that for a certain y_n with $|y_n| \to \infty$ and $\tau_2^n > 0$,

$$(\tau_2^n)^\nu |y_n|^\sigma |\phi_n(y_n, \tau_2^n)| \geq \frac{1}{2}.$$

Now we let

$$\tilde\phi_n(z, \tau) := (\tau_2^n)^\nu |y_n|^\sigma\, \phi_n(|y_n|^{-1} z, |y_n|^{-2}\tau + \tau_2^n)$$

so that

$$\partial_\tau \tilde\phi_n = \Delta_z \tilde\phi_n + a_n \cdot \nabla_z \tilde\phi_n + b_n \tilde\phi_n + \tilde h_n(z, \tau)$$

where

$$\tilde h_n(z, \tau) = (\tau_2^n)^\nu |y_n|^{2+\sigma}\, h_n(|y_n|^{-1} z, |y_n|^{-2}\tau + \tau_2^n),$$

and $|a_n| + |b_n| \to 0$ uniformly on compact sets of $\mathbb{R}^2 \setminus \{0\}$.

Note that

$$\tilde h_n = \sum_{j=1}^{2} \partial_{z_j} \tilde g_{j,n}$$

where

$$\tilde g_{j,n}(z, \tau) = (\tau_2^n)^\nu |y_n|^{1+\sigma}\, g_{j,n}(|y_n|^{-1} z, |y_n|^{-2}\tau + \tau_2^n),$$

By assumption on $g_{j,n}$ we find that $\tilde g_{j,n} \to 0$ uniformly on compact sets of $(\mathbb{R}^2 \setminus \{0\}) \times (-\infty, 0]$. Besides $|\tilde\phi_n(\frac{y_n}{|y_n|}, 0)| \geq \frac{1}{2}$. and

$$|\tilde\phi_n(z, \tau)| \leq |z|^{-\sigma}((\tau_2^n)^{-1}|y_n|^{-2}\tau + 1)^{-\nu}.$$

As a conclusion, we may assume that $\tilde\phi_n \to \tilde\phi \neq 0$ uniformly over compact subsets of $\mathbb{R}^2 \setminus \{0\} \times (-\infty, 0]$ where

$$\tilde\phi_\tau = \Delta_z \tilde\phi \quad \text{in } \mathbb{R}^2 \setminus \{0\} \times (-\infty, 0].$$

and

$$|\tilde\phi(z, \tau)| \leq |z|^{-\sigma} \quad \text{in } \mathbb{R}^2 \setminus \{0\} \times (-\infty, 0].$$

Moreover, the mode 1 assumption for ϕ_n translates for $\tilde\phi$ into

$$\tilde\phi(z, \tau) = \begin{bmatrix} \varphi(\rho, \tau)e^{2i\theta} \\ 0 \end{bmatrix}, \quad z = \rho e^{i\theta}$$

for a complex valued function φ that solves

$$\varphi_\tau = \varphi_{\rho\rho} + \frac{\varphi_\rho}{\rho} - \frac{4\varphi}{\rho^2} \quad \text{in } (0, \infty) \times (-\infty, 0], \tag{6.41}$$
$$|\varphi(\rho, \tau)| \leq \rho^{-\sigma} \quad \text{in } (0, \infty) \times (-\infty, 0].$$

Let us set

$$u(\rho, t) = (\rho^2 + t)^{-\sigma/2} + \frac{\varepsilon}{\rho^2}$$

Then

$$-u_t + \Delta u - \frac{4u}{r^2} < (\rho^2 + t)^{-\sigma/2-1}[\sigma(\sigma + 2) - 4 + \frac{\sigma}{2}] < 0.$$

It follows that the function $u(x,\tau+M)$ is a positive supersolution for the real and imaginary parts of equation (6.41) in $(0,\infty)\times[-M,0]$. We find then that $|\varphi(\rho,\tau)|\le 2u(\rho,\tau+M)$. Letting $M\to+\infty$ we find

$$|\varphi(\rho,\tau)|\le\frac{2\varepsilon}{\rho^2}$$

and since ε is arbitrary we conclude $\varphi=0$. Hence $\tilde{\phi}=0$, a contradiction that concludes the proof of the lemma. □

Proof of Lemma 6.6. We take h to be the extension as zero of the function h_1 as in the statement of the lemma. Then we let ϕ be the unique solution of the initial value problem (6.29), which clearly defines a linear operator of h_1. From Lemma 6.7, expressing the resulting estimate in the variables (y,t), we have that for any $t_1\in(0,T)$

$$|\phi(y,t)|\le C\lambda_*(t)^\nu(1+|y|)^{-\sigma}\|h\|_{2+\sigma,t_1}\quad\text{for all}\quad t\in(0,t_1),\quad y\in\mathbb{R}^2.$$

Then letting $\phi_1:=\phi\big|_{\mathcal{D}_{3R}}$ and letting $t_1\uparrow T$ the result follows. □

6.4. **Proof of Proposition 6.1.** We let h be defined in $\mathcal{D}_{2R}$ with $\|h\|_{a,\nu}<+\infty$, with $a\in(2,3)$, $\nu>0$. We consider the problem

$$\begin{aligned}&\lambda^2\partial_t\phi=L_W[\phi]+h\quad\text{in }\mathcal{D}_{4R}\\&\phi(\cdot,0)=0\quad\text{in }B_{4R(0)},\end{aligned}$$

(recall that h is assumed to be defined in $\mathbb{R}^2\times(0,T)$. Let ϕ_k be the solution estimated in Lemma 6.1 of

$$\begin{aligned}\lambda^2\partial_t\phi_k&=L_W[\phi_k]+h_k\quad\text{in }\mathcal{D}_{4R}\\\phi(\cdot,t)&=0\quad\text{on }\partial B_{4R}\quad\text{for all}\quad t\in(0,T),\\\phi(\cdot,0)&=0\quad\text{in }B_{4R(0)}.\end{aligned}$$

In addition we let ϕ_{01}, ϕ_{11}, ϕ_{-11} solve

$$\begin{aligned}\lambda^2\partial_t\phi_{k1}&=L_W[\phi_{k1}]+\bar{h}_k\quad\text{in }\mathcal{D}_{4R}\\\phi_{k1}(\cdot,t)&=0\quad\text{on }\partial B_{4R}\quad\text{for all}\quad t\in(0,T),\\\phi_{k1}(\cdot,0)&=0\quad\text{in }B_{4R(0)}\end{aligned}$$

for $k=0,1,-1$. Let us consider the functions ϕ_{02} constructed in Lemma 6.3, $\phi_{-1,2}$ constructed in Lemma 6.5, and ϕ_{12} constructed in Lemma 6.6, that solve for $k=0,1,-1$

$$\begin{aligned}\lambda^2\partial_t\phi_{k2}&=L_W[\phi_{k2}]+h_k-\bar{h}_k\quad\text{in }\mathcal{D}_{3R}\\\phi_{k2}(\cdot,0)&=0\quad\text{in }B_{3R(0)}.\end{aligned}$$

We define

$$\phi:=\sum_{k=0,1,-1}(\phi_{k1}+\phi_{k2})+\sum_{k\neq0,1,-1}\phi_k$$

which is a bounded solution of the equation

$$\lambda^2\phi_t=L_W[\phi]+h(y,t)\quad\text{in }\mathcal{D}_{3R}$$

that defines a linear operator of h. Applying the estimates for the components in Lemmas 6.1, 6.3, 6.5, and 6.6 we obtain

$$\begin{aligned}|\phi(y,t)| &\lesssim \frac{\lambda_*(t)^\nu \log R(t)}{1+|y|^{a-2}}\,\|h^\perp\|_{a,\nu}\\ &+\frac{\lambda_*(t)^\nu}{1+|y|^{a-2}}\,\big\|h_1-\bar h_1\big\|_{a,\nu}+\frac{\lambda_*(t)^\nu R^4}{1+|y|^2}\,\big\|\bar h_1\big\|_{\nu,a}\\ &+\frac{\lambda_*(t)^\nu R^{\frac{5-a}{2}}}{1+|y|}\,\min\{1,R^{\frac{5-a}{2}}|y|^{-2}\}\,\|h_0-\bar h_0\|_{a,\nu}+\frac{\lambda_*(t)^\nu R^2}{1+|y|}\|\bar h_0\|_{a,\nu}\\ &+\lambda_*^\nu\min\{\log R, R^{4-a}|y|^{-2}\}\,\|h_{-1}-\bar h_{-1}\|_{a,\nu}+\lambda_*(t)^\nu\log R\|\bar h_{-1}\|_{a,\nu},\end{aligned}$$

in $\mathcal{D}_{3R}$. Finally, Lemma 6.2 yields that the same bound is valid for $(1+|y|)|\nabla_y\phi|$ in $\mathcal{D}_{2R}$. The function $\phi\big|_{\mathcal{D}_{2R}}$ solves (6.1), it defines a linear operator of h and satisfies the required estimates. $\square$

6.5. **Modified theory for mode 0.** Let us consider the problem

$$\begin{cases}\lambda^2\varphi_t = L_W\varphi + h(y,t)+\sum_{j=1,2}\tilde c_{0j}Z_{0j}w_\rho^2 \quad \text{in } \mathcal{D}_{2R}\\ \varphi\cdot W = 0 \quad \text{in } \mathcal{D}_{2R}\\ \varphi = 0 \quad \text{on } \partial B_{2R}\times(0,T)\\ \varphi(\cdot,0)=0 \quad \text{in } B_{2R(0)},\end{cases} \tag{6.42}$$

in mode 0. The result here is the following.

Proposition 6.2. *Let $\sigma\in(0,1)$, $\delta\in(0,1)$, $\nu>0$. Assume $\|h\|_{\nu,2+\sigma}<\infty$. Then there is a solution ϕ, $\tilde c_{0j}$ of (6.42), which is linear in h, such that*

$$|\varphi(y,t)|+(1+|y|)|\nabla_y\varphi(y,t)|\le C\lambda_*^\nu\|h\|_{\nu,2+\sigma}\begin{cases}\frac{R^{\delta(3-\sigma)}}{(1+|y|)^3} & |y|\le 2R^\delta\\ \frac{1}{(1+|y|)^\sigma} & 2R^\delta\le|y|\le R,\end{cases}$$

and such that

$$\tilde c_{0j}[h] = -\frac{\int_{B_{\mathbb{R}^2}} h\cdot Z_{0j}}{\int_{\mathbb{R}^2} w_\rho^2|Z_{0j}|^2}-G[h]$$

where G is a linear operator of h satisfying the estimate

$$|G[h]|\le C\lambda_*^\nu R^{-\delta\sigma'}\|h\|_{\nu,2+\sigma}, \tag{6.43}$$

with $0<\sigma'<\sigma$.

We are using the terminology *mode 0* from §6, which means that φ has the form

$$\varphi = \mathrm{Re}(\tilde\varphi e^{i\theta})E_1+\mathrm{Im}(\tilde\varphi e^{i\theta})E_2$$

where $\tilde\varphi$ is a complex valued function of ρ and t. The equation $\lambda^2\varphi_t = L_W\varphi + h(y,t)$ (wit h also in mode 0) becomes

$$\lambda^2\partial_t\tilde\varphi = \mathcal{L}_0\tilde\varphi+\tilde h,\quad \text{where}\quad \mathcal{L}_0[\tilde\varphi]:=\partial_\rho^2\tilde\varphi+\frac{1}{\rho}\partial_\rho\tilde\varphi-\frac{\cos(2w)}{\rho^2}\tilde\varphi,$$

and we have a similar definition for $\tilde h$. Note that the operator $\mathcal{L}_0$ at $\rho=0$ and $\rho=\infty$ is given by $\partial_\rho^2\tilde\varphi+\frac{1}{\rho}\partial_\rho\tilde\varphi-\frac{1}{\rho^2}\tilde\varphi$. The last equation can be written as a regular parabolic PDE by setting $\hat\varphi(y,t)=\tilde\varphi(\rho,t)e^{-i\theta}$, $y=\rho e^{i\theta}$,

$$\lambda^2\partial_t\hat\phi = \Delta_y\hat\varphi+\frac{16\hat\varphi}{(1+|y|^2)^2}+\hat h(y,t).$$

Thus, instead of (6.42) we will construct a solution to (changing the notation to φ and h)

$$\begin{cases} \lambda^2\varphi_t = \Delta_y\varphi + \dfrac{16}{(1+|y|^2)^2}\varphi + h(y,t) + \tilde c_0\rho w_\rho^3 & \text{in } \mathcal{D}_{2R} \\ \varphi = 0 \quad \text{on } \partial B_{2R}\times(0,T) \\ \varphi(\cdot,0) = 0 \quad \text{in } B_{2R(0)}, \end{cases} \tag{6.44}$$

with φ complex valued of the form $\varphi(y) = e^{i\theta}\tilde\varphi(\rho,t)$ (and the same for h). Here $\tilde c_0$ is complex and related to $\tilde c_{0j}$ in (6.42) by $\tilde c_0 = \tilde c_{01} + i\tilde c_{02}$.

We will construct φ solving (6.44) of the form

$$\varphi = \eta\phi + \psi$$

where

$$\eta(y,t) = \eta_1\Big(\frac{|y|}{R_1}\Big)$$

and $\eta_1(r) = 1$ for $r \le 1$, $\eta_1(r) = 0$ for $r \ge 2$. Here $R_1 = R^\delta$. We find a solution to (6.44) if we get ϕ, ψ solving the system

$$\begin{cases} \lambda^2\partial_t\phi = \Delta\phi + B\phi + B\psi + h(y,t) + c_0\rho w_\rho^3 & \text{in } \mathcal{D}_{2R_1} \\ \phi(\cdot,0) = 0 \quad \text{in } B_{2R_1(0)}, \end{cases} \tag{6.45}$$

$$\begin{cases} \lambda^2\partial_t\psi = \Delta\psi + (1-\eta)B\psi + A\phi + (1-\eta)h(y,t) & \text{in } \mathcal{D}_{2R} \\ \qquad \psi = 0 \quad \text{on } \partial B_{2R}\times(0,T) \\ \psi(\cdot,0) = 0 \quad \text{on } B_{2R(0)}, \end{cases} \tag{6.46}$$

where

$$B = \frac{16}{(1+|y|^2)^2}, \qquad A\phi = \phi\Delta\eta + 2\nabla\phi\nabla\eta - \phi\eta_t.$$

Consider

$$\begin{cases} \lambda^2\partial_t\psi = \Delta\psi + (1-\eta)B\psi + h(y,t) & \text{in } \mathcal{D}_{2R} \\ \qquad \psi = 0 \quad \text{on } \partial B_{2R}\times(0,T), \\ \psi(y,0) = 0 \quad \forall y\in B_{2R(0)}, \end{cases} \tag{6.47}$$

with ψ and h of the form $\psi = \tilde\psi(\rho,t)e^{i\theta}$. Let

$$\|\psi\|^{(1)}_{\nu,\sigma} = \sup_{\mathcal{D}_{2R}}\big\{\lambda_*^{-\nu}(t)(1+|y|)^\sigma\,\big[\,|\psi(y,t)| + (1+|y|)|\nabla_y\psi(y,t)|\,\big]\big\}.$$

Lemma 6.9. *Let $\sigma\in(0,1)$, $\nu>0$ and let ψ solve (6.47). If R_1 is sufficiently large, then*

$$\|\psi\|^{(1)}_{\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}. \tag{6.48}$$

If in (6.47) h is replaced by $(1-\eta)h$ we get the additional estimate

$$|\psi(y,t)| + R_1|\nabla\psi(y,t)| \le C\lambda_*^\nu\frac{1}{R_1^\sigma}, \qquad |y|\le 2R_1.$$

Proof. To prove this lemma, we first claim that for the equation

$$\begin{cases} \lambda^2\partial_t\psi = \Delta\psi + h(y,t) & \text{in } \mathcal{D}_{2R} \\ \qquad \psi = 0 \quad \text{on } \partial B_{2R}\times(0,T), \\ \psi(y,0) = 0 \quad \forall y\in B_{2R(0)}, \end{cases}$$

with ψ and h of the form $\psi = \tilde\psi(\rho,t)e^{i\theta}$. we have

$$\|\psi\|^{(1)}_{\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}. \tag{6.49}$$

This is obtained using a barrier for the real and imaginary parts of $\tilde\psi$, which satisfies

$$\lambda^2\partial_t\tilde\psi = \partial_{\rho\rho}\tilde\psi + \frac{1}{\rho}\partial_\rho\tilde\psi - \frac{1}{\rho^2}\tilde\psi + \tilde h.$$

To find the estimate for the solution of (6.47) we need to estimate $\|(1-\eta)B\psi\|_{\nu,2+\sigma}$. We have that

$$\begin{aligned}(1-\eta)B\,|\psi| &\le (1-\eta)\lambda_*^\nu(1+|y|)^{-4-\sigma}\|\psi\|_{\nu,\sigma}^{(1)}\\ &\le R_1(0)^{-2}\lambda_*^\nu(1+|y|)^{-2-\sigma}\|\psi\|_{\nu,\sigma}^{(1)},\end{aligned}$$

and therefore

$$\|(1-\eta)B\psi\|_{\nu,2+\sigma} \le CR_1(0)^{-2}\|\psi\|_{\nu,\sigma}^{(1)}.$$

Then, if ψ satisfies (6.47), using (6.49) we get

$$\|\psi\|_{\nu,\sigma}^{(1)} \le C\|(1-\eta)B\psi + h\|_{\nu,2+\sigma} \le CR_1(0)^{-1}\|\psi\|_{\nu,\sigma}^{(1)} + C\|h\|_{\nu,2+\sigma}.$$

If $R_1(0)$ is large enough, we obtain (6.48). □

Proof of Proposition 6.2. We use Lemma 6.9 to find a solution $\psi[\phi]$ of (6.47) with h replaced by $A\phi$, and a solution $\psi[h]$ of (6.47) with h replaced by $(1-\eta)h$, so that $\psi[\phi]+\psi[h]$ is the solution of (6.46).

Let $\sigma_1\in(0,1)$. We also get the estimate

$$\|\psi[\phi]\|_{\nu,\sigma_1}^{(1)} \le C\|A\phi\|_{\nu,2+\sigma_1}. \tag{6.50}$$

We take $R_1 = R^\delta$ and construct a solution of the system (6.45), (6.46). For this it suffices to find ϕ such that

$$\begin{cases}\lambda^2\partial_t\phi = \Delta\phi + B\phi + B\psi[\phi] + B\psi[h] + h(y,t) + c_0\rho w_\rho^3 & \text{in } \mathcal{D}_{2R_1}\\ \phi(\cdot,0) = 0 \quad \text{in } B_{2R_1(0)}.\end{cases} \tag{6.51}$$

Let $\mathcal{T}$ denote the linear operator given by Lemma 6.3, Applied in $\mathcal{D}_{2R_1}$. Then to solve (6.51) we consider the fixed point problem

$$\phi = \mathcal{T}[B\psi[\phi] + B\psi[h] + h].$$

Let $\sigma\in(0,1)$. By Lemma 6.3,

$$\|\mathcal{T}[g]\|_{*,\nu,2+\sigma} \le \|g\|_{\nu,2+\sigma}, \tag{6.52}$$

where

$$\|\phi\|_{*,\nu,\sigma} = \sup\, \frac{\lambda_*^{-\nu}(1+|y|)^3}{R_1^{3-\sigma}}\left[|\phi(y,t)| + (1+|y|)|\nabla_y\phi(y,t)|\right].$$

We claim that if $\sigma_1<\sigma$ then

$$\|A\phi\|_{\nu,2+\sigma_1} \le CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}. \tag{6.53}$$

Indeed, we have

$$\begin{aligned}|\phi\Delta\eta| &\le \frac{1}{R_1^2}\lambda_*^\nu\frac{R_1^{3-\sigma}}{(1+|y|)^3}|\Delta\eta_1|\|\phi\|_{*,\nu,\sigma} \le C\lambda_*^\nu\frac{R_1^{\sigma_1-\sigma}}{(1+|y|)^{2+\sigma_1}}\|\phi\|_{*,\nu,\sigma}\\ &\le CR_1(0)^{\sigma_1-\sigma}\lambda_*^\nu\frac{1}{(1+|y|)^{2+\sigma_1}}\|\phi\|_{*,\nu,\sigma}.\end{aligned}$$

Similarly

$$|\nabla\phi\nabla\eta| \le \frac{1}{R_1}\lambda_*^\nu\frac{R_1^{3-\sigma}}{(1+|y|)^4}|\nabla\eta_1|\|\phi\|_{*,\nu,\sigma} \le C\lambda_*^\nu\frac{R_1^{\sigma_1-\sigma}}{(1+|y|)^{2+\sigma_1}}\|\phi\|_{*,\nu,\sigma}.$$

Similar estimates for the remaining terms in A prove (6.53).

From (6.50) and (6.53) we find

$$\|\psi[\phi]\|_{\nu,\sigma_1}^{(1)} \le CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}. \tag{6.54}$$

Now we claim that

$$\|B\psi\|_{\nu,2+\sigma} \le C\|\psi\|^{(1)}_{\nu,\sigma_1}. \tag{6.55}$$

Indeed,

$$B\,|\psi| \le C\frac{\lambda_*^\nu}{(1+|y|)^{4+\sigma_1}}\|\psi\|^{(1)}_{\nu,\sigma_1} \le C\frac{\lambda_*^\nu}{(1+|y|)^{2+\sigma}}\|\psi\|^{(1)}_{\nu,\sigma_1}$$

so (6.55) follows. Combining (6.55) and (6.54) we get

$$\|B\psi[\phi\|_{\nu,2+\sigma} \le C\|\psi[\phi]\|^{(1)}_{\nu,\sigma_1} \le CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}.$$

From the above inequality and (6.52) we then get

$$\|\mathcal{T}[B\psi[\phi]]\|_{*,\nu,\sigma} \le CR_1(0)^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma},$$

which shows that the operator $\phi \mapsto \mathcal{T}[B\psi[\phi]+B\psi[h]+h]$ is a contraction if $R_1(0)$ is sufficiently large, and we find a unique fixed point, which satisfies the estimate

$$\|\phi\|_{*,\nu,\sigma} \le C\|\mathcal{T}[B\psi[h]+h]\|_{*,\nu,\sigma}.$$

Next we estimate $\|\mathcal{T}[B\psi[h]+h]\|_{*,\nu,\sigma}$. We have by (6.52)

$$\begin{aligned}\|\mathcal{T}[B\psi[h]+h]\|_{*,\nu,\sigma} &\le C\|B\psi[h]+h\|_{\nu,2+\sigma}\\ &\le C\|\psi[h]\|^{(1)}_{\nu,\sigma} + \|h\|_{\nu,2+\sigma} \le C\|h\|_{\nu,2+\sigma},\end{aligned}$$

and hence

$$\|\phi\|_{*,\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}. \tag{6.56}$$

Similar to (6.54) we have

$$\|\psi[\phi]\|^{(1)}_{\nu,\sigma} \le C\|\phi\|_{*,\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}.$$

and

$$\|\psi[h]\|^{(1)}_{\nu,\sigma} \le C\|h\|_{\nu,2+\sigma}.$$

Recalling that $\varphi = \eta\phi + \psi$ and $R_1 = R^\delta$, we get

$$|\varphi(y,t)| + (1+|y|)|\nabla_y\varphi(y,t)| \le C\lambda_*^\nu\|h\|_{\nu,2+\sigma}\begin{cases}\frac{R^{\delta(3-\sigma)}}{(1+|y|)^3} & |y|\le 2R^\delta\\ \frac{1}{(1+|y|)^\sigma} & 2R^\delta \le |y| \le R.\end{cases}$$

Finally, thanks to Lemma 6.3, we have that

$$c_{0j}[h] = -\frac{1}{\int_{B_1}|Z_{0j}|^2}\left[\int_{B_{2R_1}} h\rho w_\rho + \int_{B_{2R_1}}(B\psi[\phi]+B\psi[h])\rho w_\rho\right]$$

The last term is a linear operator of h, which we estimate next. A similar computation as in (6.53) shows that

$$\|A\phi\|_{\nu+\delta(\sigma-\sigma_1),2+\sigma_1} \le C\|\phi\|_{*,\nu,\sigma}.$$

This implies

$$\|\psi[\phi]\|_{\nu+\delta(\sigma-\sigma_1),\sigma_1} \le C\|\phi\|_{*,\nu,\sigma}$$

and therefore

$$\left|\int_{B_{2R_1}} B\psi[\phi]\cdot Z_{0j}\right| \le C\lambda_*^\nu R^{\sigma_1-\sigma}\|\phi\|_{*,\nu,\sigma}$$

and using (6.56)

$$\left|\int_{B_{2R_1}} B\psi[\phi]\cdot Z_{0j}\right| C\lambda_*^\nu R^{\sigma_1-\sigma}\|h\|_{\nu,2+\sigma}.$$

We have for $|y| \le 2R^\delta$

$$|\psi[h](y,t)| + (1+|y|)|\nabla_y \psi[h](y,t)| \le C R_1^{-\sigma} \|h\|_{\nu,2+\sigma}.$$

Then for $|y| \le 2R^\delta$ we have

$$|B\,|\psi[h]| \le C\lambda_*^\nu (1+|y|)^{-4} R_1^{-\sigma} \|h\|_{\nu,2+\sigma},$$

and hence

$$\left| \int_{B_{2R_1}} B\psi[h]\rho w_\rho \right| \le C\lambda_*^\nu R_1^{-\sigma} \|h\|_{\nu,2+\sigma}.$$

We would like to have the orthogonality condition defined as an integral in $\mathbb{R}^2$. Note that

$$\begin{aligned} \left| \int_{(B_{2R^\delta})^c} h\rho w_\rho \right| &\le C\|h\|_{\nu,2+\sigma} \lambda_*^\nu \int_{(B_{2R^\delta})^c} \frac{1}{(1+|y|)^{3+\sigma}}\, dy \\ &\le C\|h\|_{\nu,2+\sigma} \lambda_*^\nu R^{-\delta(1+\sigma)}. \end{aligned}$$

Then, going back to the original notation, we get

$$c_{0j}[h] = -\frac{\int_{\mathbb{R}^2} h \cdot Z_{0j}}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} - G[h]$$

where G satisfies (6.43). □

7. Lipschitz bounds with respect to λ in the linear theory for the inner problem

Let us consider the linear operator we constructed in Proposition 6.1 as a solution $\phi[h] = \mathcal{T}_{\lambda,1}[h]$ of problem (6.1),

$$\begin{aligned} &\lambda^2 \partial_t \phi = L_W[\phi] + h(y,t) \quad \text{in } \mathcal{D}_{2R} \\ &\phi(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \\ &\phi \cdot W = 0 \quad \text{in } \mathcal{D}_{2R} \end{aligned}$$

where $\mathcal{D}_{2R} = \{(y,t) \ / \ t \in (0,T),\ y \in B_{2R(t)}(0)\}$, and we assume $h \cdot W = 0$ in $\mathcal{D}_{2R}$. The purpose in this section is find estimates for directional derivatives of the operator $\mathcal{T}_{\lambda,1}[h]$ with respect to the parameter function λ. Examining the construction of $\mathcal{T}_{\lambda,1}[h]$ as the superposition of the unique solutions of different problems, it is not hard to see that the directional derivative

$$\phi_\lambda := (\partial_\lambda \mathcal{T}_{\lambda,1})[h][\lambda_1] = \frac{d}{ds} \mathcal{T}_{\lambda+s\lambda_1,1}[h]\big|_{s=0}$$

satisfies the equation

$$\begin{aligned} &\lambda^2 \partial_t \phi_\lambda = L_W[\phi_\lambda] - 2\frac{\lambda_1}{\lambda}(L_W[\phi] + h(y,t)) \quad \text{in } \mathcal{D}_{2R} \\ &\phi_\lambda(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{aligned}$$

with $\phi = \mathcal{T}_{\lambda,1}[h]$. We will find estimates for this quantity inherited from those we have already established for ϕ. We assume that for some positive numbers a, b, c independent of T we have that

$$a\lambda_*(t) \le \lambda(t) \le b\lambda_*(t), \quad |\lambda_1(t)| \le c\lambda_*(t) \quad \text{for all} \quad t \in (0,T).$$

The following estimate holds.

Proposition 7.1. *The function ϕ_λ is well defined and satisfies the estimate*

$$\begin{aligned} &(1+|y|)\,|\nabla_y \phi_\lambda(y,t)| + |\phi_\lambda(y,t)| \\ &\lesssim\ \lambda_*^\nu \frac{R^{1+\frac{5-a}{2}} \log R}{1+|y|} \min\Big\{ \frac{R^{1+\frac{5-a}{2}}}{|y|^2}, 1 \Big\} \, \|h\|_{a,\nu} \left\| \frac{\lambda_1}{\lambda} \right\|_\infty \quad \text{in } \mathcal{D}_{2R}. \end{aligned}$$

Proof of Proposition 7.1. We recall that $\phi[h] = \mathcal{T}_{\lambda,1}[h]$ was constructed mode by mode. According to the decomposition (6.2), (6.3), (6.4), (6.5), we can write

$$\phi = \phi_0 + \phi_1 + \phi_{-1} + \phi^{\perp}, \quad h = h_0 + h_1 + h_{-1} + h^{\perp}, \tag{7.1}$$

where we can assume for $k = 0, 1,\ j = 1, 2$,

$$\int_{B_{2R}} h_k(y,t) \cdot Z_{kj}(y)dy \ = \ 0.$$

We will give the estimates for ϕ_λ in each mode separately, writing

$$\phi_\lambda = \phi_{0\lambda} + \phi_{1\lambda} + \phi_{-1\lambda} + \phi_\lambda^{\perp}.$$

We will estimate each of the terms $\phi_{0\lambda}$, $\phi_{1\lambda}$, $\phi_{-1\lambda}$, $\phi_\lambda^{\perp}$ separately.

First we give some estimates for the equation in entire space with some suitable right hand side.

7.1. Estimates for a heat equation.

Lemma 7.1. *Let ϕ be the solution of*

$$\partial_\tau \phi = \Delta_y \phi + \mathrm{div}_y G(y,\tau) \quad \text{in } \mathbb{R}^2 \times (0,\infty)$$
$$\phi(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^2$$

given by Duhamel's formula, where

$$|G(y,\tau)| \le \frac{1}{(1+\tau^\nu)(1+|y|^{a-1})}$$

and $\nu \in (0,1)$, $a \in (2,3)$. Then

$$|\phi(y,\tau)| \le \frac{C(1+\log_+ \tau))}{(1+\tau^\nu)(1+|y|^{a-2})}$$

where $\log_+ \tau = \max(0, \log \tau)$.

Proof. The solution ϕ is given by Duhamel's formula

$$\begin{aligned}\phi(y,\tau) &= \frac{1}{4\pi}\int_0^\tau \frac{1}{\tau - s}\int_{\mathbb{R}^2} e^{-\frac{|y-z|^2}{4(\tau-s)}} \mathrm{div}_z G(z,s)\, dz\, ds \\ &= C\int_0^\tau \frac{1}{\tau-s}\int_{\mathbb{R}^2} e^{-\frac{|y-z|^2}{4(\tau-s)}} \frac{y-z}{\tau-s}\cdot G(z,s)\, dz\, ds\end{aligned}$$

and so

$$|\phi(y,\tau)| \le C\int_0^\tau \frac{1}{(\tau-s)^{3/2}}\int_{\mathbb{R}^2} e^{-\frac{|y-z|^2}{4(\tau-s)}} \frac{|y-z|}{\sqrt{\tau-s}} \frac{1}{(1+s^\nu)(1+|z|^{a-1})}\, dz\, ds.$$

Letting $z = y - \sqrt{\tau-s}\,\zeta$, we get

$$|\phi(y,\tau)| \le C\int_0^\tau \frac{1}{(1+s^\nu)}\frac{1}{(\tau-s)^{1/2}}\int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{|\zeta|}{1+|y-\sqrt{\tau-s}\,\zeta|^{a-1}}\, d\zeta\, ds$$

We claim that for $y \in \mathbb{R}^2$ and $b > 0$

$$\int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{|\zeta|}{1+|y-b\zeta|^{a-1}}\, d\zeta \le \frac{C}{(1+b)(1+|y|^{a-2})}. \tag{7.2}$$

We will consider first $|y| \ge 1$. Let us consider the case $1 \le |y| \le b$. Then, by the Hardy-Littlewood inequality we have

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{\zeta}{1+|y-b\zeta|^{a-1}}\, d\zeta &\le \int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{C+|\zeta|}{1+|y-b\zeta|^{a-1}}\, d\zeta \\ &\le \int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{C+|\zeta|}{1+|b\zeta|^{a-1}}\, d\zeta.\end{aligned}$$

Letting $B = B_{\frac{1}{b}}(0)$ we get

$$\int_B e^{-\frac{|\zeta|^2}{4}} \frac{C+|\zeta|}{1+|b\zeta|^{a-1}}\,d\zeta \le \int_B e^{-\frac{|\zeta|^2}{4}} (C+|\zeta|)\,dz \le \frac{C}{b^2}$$

and

$$\int_{B^c} e^{-\frac{|\zeta|^2}{4}} \frac{C+|\zeta|}{1+|b\zeta|^{a-1}}\,d\zeta \le \int_{B^c} e^{-\frac{|\zeta|^2}{4}} \frac{C+|\zeta|}{|b\zeta|^{a-1}}\,d\zeta \le C\frac{1}{b^{a-1}}.$$

Thus in the case $1 \le |y| \le b$ we deduce that

$$\int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{|\zeta|}{1+|y-b\zeta|^{a-1}}\,d\zeta \le Cb^{1-a} \le \frac{C}{b|y|^{a-2}}. \tag{7.3}$$

Next assume that $|y| \ge b > 0$. Note that

$$\int_{|\zeta|\le\frac{|y|}{2b}} e^{-\frac{|\zeta|^2}{4}} \frac{|\zeta|}{(1+|y-b\zeta|^{a-1})}\,d\zeta \le \frac{C}{|y|^{a-1}} \le \frac{C}{(1+b)|y|^{a-2}}.$$

Next consider

$$\begin{aligned}\int_{\frac{|y|}{2b}\le|\zeta|\le 2\frac{|y|}{b}} e^{-\frac{|\zeta|^2}{4}} \frac{|\zeta|}{(1+|y-b\zeta|^{a-1})}\,d\zeta &\le e^{-\frac{|y|^2}{16b^2}} \int_{\frac{|y|}{2b}\le|\zeta|\le 2\frac{|y|}{b}} \frac{1}{(1+|y-b\zeta|^{a-1})}\,d\zeta \\ &\le e^{-\frac{|y|^2}{16b^2}} \int_0^{4\frac{|y|}{b}} \frac{1}{1+(br)^{a-1}} r\,dr \\ &\le Ce^{-\frac{|y|^2}{16b^2}} \frac{|y|^{3-a}}{b^2} \\ &\le \frac{C}{(1+b)|y|^{a-2}}.\end{aligned}$$

Finally, if $b \ge 1$

$$\begin{aligned}\int_{|\zeta|\ge 2\frac{|y|}{b}} e^{-\frac{|\zeta|^2}{4}} \frac{|\zeta|}{(1+|y-b\zeta|^{a-1})}\,d\zeta &\le C\int_{|\zeta|\ge 2\frac{|y|}{b}} e^{-\frac{|\zeta|^2}{4}} \frac{|\zeta|}{(1+|b\zeta|^{a-1})}\,d\zeta \\ &\le C\int_{|\zeta|\ge 2\frac{|y|}{b}} \frac{1}{|\zeta|}\frac{1}{|b\zeta|^{a-1}}\,d\zeta \\ &= \frac{C}{b|y|^{a-2}}.\end{aligned}$$

If $b \le 1$,

$$\begin{aligned}\int_{|\zeta|\ge 2\frac{|y|}{b}} e^{-\frac{|\zeta|^2}{4}} \frac{|\zeta|}{(1+|y-b\zeta|^{a-1})}\,d\zeta &\le C\int_{|\zeta|\ge 2\frac{|y|}{b}} e^{-\frac{|\zeta|^2}{4}} |\zeta|\,d\zeta \\ &\le C\frac{b^{a-2}}{|y|^{a-2}} \le C\frac{1}{(1+b)|y|^{a-2}}.\end{aligned}$$

Therefore for $|y| \ge \max(1,b)$

$$\int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{|\zeta|}{1+|y-b\zeta|^{a-1}}\,d\zeta \le \frac{C}{(1+b)|y|^{a-2}}. \tag{7.4}$$

From (7.3) and (7.4) we deduce (7.2) in the case $|y| \ge 1$. The case $|y| \le 1$ is proved similarly.

Using (7.2) we get

$$\begin{aligned}|\phi(y,\tau)| &\le \frac{C}{1+|y|^{a-2}} \int_0^\tau \frac{1}{(1+s^\nu)} \frac{1}{\sqrt{\tau-s}(1+\sqrt{\tau-s})}\,ds \\ &\le \frac{C(1+\log_+\tau)}{(1+\tau^\nu)(1+|y|^{a-2})}.\end{aligned}$$

□

Lemma 7.2. *Let ϕ be the solution of*

$$\partial_\tau \phi = \Delta_y \phi + G(y,\tau) \quad \text{in } \mathbb{R}^2 \times (0,\infty)$$
$$\phi(\cdot,0) = 0 \quad \text{in } \mathbb{R}^2$$

given by Duhamel's formula, where

$$|G(y,\tau)| \le \frac{1}{(1+\tau^\nu)(1+|y|^a)}$$

and $\nu \in (0,1)$, $a \in (2,3)$. Then

$$|\phi(y,\tau)| \le \frac{C(1+\log_+ \tau))}{1+\tau^\nu}$$

where $\log_+ \tau = \max(0, \log \tau)$.

Proof. We have

$$|\phi(y,\tau)| \le \frac{1}{4\pi} \int_0^\tau \frac{1}{\tau - s} \int_{\mathbb{R}^2} e^{-\frac{|y-z|^2}{4(\tau-s)}} \frac{1}{(1+s)^\nu} \frac{1}{1+|z|^a} \, dz \, ds$$

Let $z = y - \sqrt{t-s}\zeta$. Then

$$|\phi(y,\tau)| \le \frac{1}{4\pi} \int_0^\tau \frac{1}{(1+s)^\nu} \int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{1}{1+|y-\sqrt{t-s}\zeta|^a} \, d\zeta \, ds$$

We claim that for any $y \in \mathbb{R}^2$ and $b > 0$.

$$\int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{1}{1+|y-b\zeta|^a} \, d\zeta \le \frac{C}{1+b^2}.$$

Indeed, if $b \le 1$, then

$$\int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{1}{1+|y-b\zeta|^a} \, d\zeta \le C.$$

If $b \ge 1$, then

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-\frac{|\zeta|^2}{4}} \frac{1}{1+|y-b\zeta|^a} \, d\zeta &\le \int_{\mathbb{R}^2} \frac{1}{1+|b\zeta|^a} \, d\zeta \\ &= \frac{1}{b^2} \int_{\mathbb{R}^2} \frac{1}{1+|\zeta|^a} \, d\zeta.\end{aligned}$$

Then

$$|\phi(y,\tau)| \le \frac{1}{4\pi} \int_0^\tau \frac{1}{(1+s)^\nu} \frac{1}{1+\tau-s} \, ds. \le \frac{C(1+\log_+ \tau)}{1+\tau^\nu}.$$

□

7.2. **Estimate of $\phi_\lambda^\perp$ and $\phi_{-1\lambda}$.** We claim that for any $\sigma \in (0,1)$ we have

$$\begin{aligned}&(1+|y|)|\nabla \phi_\lambda^\perp(y,t)| + |\phi_\lambda^\perp(y,t)| \\ &\quad \lesssim \lambda_*(t)^\nu R^{a-2} \log R (1+|y|)^{2-a} \|h^\perp\|_{a,\nu} \left\| \frac{\lambda_1}{\lambda} \right\|_\infty \end{aligned} \tag{7.5}$$

$$\begin{aligned}&(1+|y|)|\nabla \phi_{-1\lambda}(y,t)| + |\phi_{-1\lambda}^\perp(y,t)| \\ &\quad \lesssim \lambda_*(t)^\nu R^{a-2+\sigma} (1+|y|)^{2-a} \|h_{-1}\|_{a,\nu} \left\| \frac{\lambda_1}{\lambda} \right\|_\infty. \end{aligned} \tag{7.6}$$

Proof. Let us recall that $\phi^\perp$ is the restriction to $\mathcal{D}_{2R}$ of the unique solution to the problem

$$\begin{aligned} &\lambda^2\partial_t\phi^\perp = L_W[\phi^\perp] + h^\perp \quad \text{in } \mathcal{D}_{4R}\\ &\phi^\perp(\cdot,t) = 0 \quad \text{on } \partial B_{4R} \quad \text{for all} \quad t\in(0,T),\\ &\phi^\perp(\cdot,0) = 0 \quad \text{in } B_{4R(0)}, \end{aligned}$$

and $h^\perp$ is understood as zero outside $\mathcal{D}_{2R}$. It is clear that $\phi_\lambda^\perp$ corresponds to the unique solution to

$$\begin{aligned} &\lambda^2\partial_t\phi_\lambda^\perp = L_W[\phi_\lambda^\perp] - 2\frac{\lambda_1}{\lambda}(L_W[\phi^\perp] + h^\perp) \quad \text{in } \mathcal{D}_{4R}\\ &\phi_\lambda^\perp(\cdot,t) = 0 \quad \text{on } \partial B_{4R} \quad \text{for all} \quad t\in(0,T),\\ &\phi_\lambda^\perp(\cdot,0) = 0 \quad \text{in } B_{4R(0)}, \end{aligned}$$

restricted to $\mathcal{D}_{2R}$. We will prove that

$$|\phi_\lambda^\perp(y,t)| \lesssim \lambda_*(t)^\nu R^{a-2}\log R(1+|y|)^{2-a}\|h^\perp\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty. \tag{7.7}$$

To see this, we decompose ϕ_λ in the form

$$\phi_\lambda = \phi_b + \phi_c$$

where ϕ_b is the unique solution to the Cauchy problem

$$\begin{cases} \lambda^2\partial_t\phi_b = \Delta_y\phi_b + g(y,t) \quad \text{in } \mathbb{R}^2\times(0,T)\\ \phi_b(\cdot,0) = 0 \quad \text{in } \mathbb{R}^2 \end{cases} \tag{7.8}$$

where

$$g := -2\frac{\lambda_1}{\lambda}[L_W[\phi^\perp] + h^\perp]\chi_{\mathcal{D}_{4R}},$$

represented by Duhamel's formula

$$\phi_b(y,t) = \int_0^\tau \frac{1}{4\pi(\tau-s)}ds \int_{\mathbb{R}^2} e^{-\frac{|y-z|^2}{4(\tau-s)}} g(z,t_\lambda(s))\,dz \tag{7.9}$$

and

$$\tau = \tau_\lambda(t) := \int_0^t \frac{d\theta}{\lambda^2(\theta)}, \quad t_\lambda(\tau) := \tau_\lambda^{-1}(\tau).$$

We have used the notation χ_A for the characteristic function of the set A. Using Lemmas 6.1 and 6.2 we obtain that $g(y,t) = \operatorname{div}_y G_0(y,t) + G_1(y,t)$ in $\mathcal{D}_{4R}$ with

$$\begin{aligned} &(1+|y|)|G_1(y,t)| + (1+|y|)^\alpha[G_0]_{B_\ell(y,\tau)\cap\mathcal{D}_{4R}} + |G_0(y,t)|\\ &\lesssim \lambda_*(t)^\nu(1+|y|)^{1-a}\|h^\perp\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty, \end{aligned}$$

$\ell(y) = 1 + \frac{|y|}{4}$, and $\tau = \tau_\lambda(t)$ is given by (6.14). Using Lemmas 7.1, 7.2 and Schauder estimates we obtain

$$|\phi_b(y,t)| + (1+|y|)|\nabla\phi_b(y,t)| \lesssim \lambda_*(t)^\nu\log R\|h^\perp\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty, \tag{7.10}$$

for $|y|\le 5R$. On the other hand ϕ_c solves

$$\begin{cases} \lambda^2\partial_t\phi_c = L_W[\phi_c] + |\nabla W|^2\phi_b + 2(\nabla W\cdot\nabla\phi_b)W \quad \text{in } \mathcal{D}_{4R}\\ \phi_c(\cdot,t) = -\phi_b \quad \text{on } \partial B_{4R} \quad \text{for all} \quad t\in(0,T),\\ \phi_c(\cdot,0) = 0 \quad \text{in } B_{4R(0)}. \end{cases} \tag{7.11}$$

Observe that by (7.10)

$$|\nabla W|^2|\phi_b| + 2\,|(\nabla W\cdot\nabla\phi_b)W| \lesssim \lambda_*(t)^\nu\log R(1+|y|)^{-3}\|h^\perp\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty.$$

Then, using the same supersolutions as in Lemma 6.1 but on $\mathcal{D}_{5R}$ we get

$$|\phi_c(y,t)| \lesssim \lambda_*(t)^\nu \log R(1+|y|)^{2-a}\|h^\perp\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty.$$

This and (7.10) yield (7.7). We note that by Corollary 6.1 the function $\phi^\perp$ has a gradient satisfying the Hölder condition

$$\begin{aligned}&(1+|y|)|\nabla_y\phi^\perp(y,\tau)| + (1+|y|)^{1+\alpha}[\nabla_y\phi^\perp]_{\mathcal{B}_\ell(y,\tau)\cap\mathcal{D}_{4\gamma R}}\\ &\quad\lesssim \lambda_*(\tau)^\nu(1+|y|)^{2-a}\|h^\perp\|_{a,\nu},\end{aligned}$$

expressed in the variables y, τ. Then Corollary 6.1 gives the desired estimate (7.5).

The proof of (7.6) is similar, with the following modifications. We have again that ϕ_{-1} is the restriction to $\mathcal{D}_{2R}$ of the unique solution to the problem

$$\begin{aligned}&\lambda^2\partial_t\phi_{-1} = L_W[\phi_{-1}] + h_{-1} \quad \text{in } \mathcal{D}_{4R}\\ &\phi_{-1}(\cdot,t) = 0 \quad \text{on } \partial B_{4R} \quad \text{for all} \quad t\in(0,T),\\ &\phi_{-1}(\cdot,0) = 0 \quad \text{in } B_{4R(0)},\end{aligned}$$

and h_{-1} is understood as zero outside $\mathcal{D}_{2R}$. The derivattive $\phi^\perp_{-1\lambda}$ is the unique solution to

$$\begin{aligned}&\lambda^2\partial_t\phi_{-1\lambda} = L_W[\phi_{-1\lambda}] - 2\frac{\lambda_1}{\lambda}(L_W[\phi_{-1}] + h_{-1}) \quad \text{in } \mathcal{D}_{4R}\\ &\phi_{-1\lambda}(\cdot,t) = 0 \quad \text{on } \partial B_{4R} \quad \text{for all} \quad t\in(0,T),\\ &\phi_{-1\lambda}(\cdot,0) = 0 \quad \text{in } B_{4R(0)},\end{aligned}$$

The solutions ϕ_{-1} satisfies the estimate (see Lemma 6.1)

$$|\phi_{-1}| + (1+|y|)|\nabla\phi_{-1}| + (1+|y|)^{1+\alpha}[\nabla\phi_{-1}]_{B_\ell(y,\tau)\cap\mathcal{D}_{4R}} \le C\|h\|_{a,\nu}\lambda_*^\nu\log R. \tag{7.12}$$

We again decompose

$$\phi_{-1\lambda} = \phi_b + \phi_c$$

where ϕ_b solves the problem (7.8) with

$$g = -2\frac{\lambda_1}{\lambda}[L_W[\phi_{-1}] + h_{-1}]\chi_{\mathcal{D}_{4R}}.$$

Estimate (7.12) allows us to write $g(y,t) = \operatorname{div}_y G_0(y,t) + G_1(y,t)$ in $\mathcal{D}_{4R}$ with

$$\begin{aligned}&(1+|y|)|G_1(y,t)| + (1+|y|)^\alpha[G_0]_{B_\ell(y,\tau)\cap\mathcal{D}_{4R}} + |G_0(y,t)|\\ &\quad\lesssim \lambda_*(t)^\nu\frac{\log R}{1+|y|}\|h_{-1}\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty,\end{aligned}$$

$\ell(y) = 1 + \frac{|y|}{4}$, and $\tau = \tau_\lambda(t)$ is given by (6.14). In $\mathcal{D}_{4R}$ we then have the estimate

$$\begin{aligned}&(1+|y|)|G_1(y,t)| + (1+|y|)^\alpha[G_0]_{B_\ell(y,\tau)\cap\mathcal{D}_{4R}} + |G_0(y,t)|\\ &\quad\lesssim \lambda_*(t)^\nu\frac{R^\sigma}{(1+|y|)^{1+\sigma/2}}\|h_{-1}\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty,\end{aligned}$$

for any $\sigma\in(0,1)$. Using Lemmas 7.1, 7.2 and Schauder estimates we obtain

$$|\phi_b(y,t)| + (1+|y|)|\nabla\phi_b(y,t)| \lesssim \lambda_*(t)^\nu R^\sigma\|h_{-1}\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty,$$

for $|y|\le 5R$. Then arguing as for $\phi^\perp$ we obtain (7.6).

□

7.3. **Mode 0. Estimate of $\phi_{0\lambda}$.** We claim that

$$
\begin{aligned}
&(1+|y|)|\nabla\phi_{0\lambda}(y,t)| + |\phi_{0\lambda}(y,t)| \\
&\lesssim \lambda_*^\nu \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu} \frac{R^{1+\frac{5-a}{2}}\log R}{|y|+1} \begin{cases} 1 & \text{if } |y|<R^{\frac12} \\ \frac{R}{|y|^2} & \text{if } |y|>R^{\frac12}. \end{cases}
\end{aligned}
\tag{7.13}
$$

Proof. We refer to the notation in the proof of Lemma 6.3 on the construction of ϕ_0. We recall that $\phi_0 = L_W[\Phi_0]$ where Φ_0 is the unique solution of the problem (6.19),

$$
\begin{aligned}
\lambda^2\Phi_t &= L_W[\Phi] + H_0(y,t) \quad \text{in } \mathcal{D}_{4R}, \\
\Phi(y,0) &= 0 \quad \text{in } B_{4R}(0) \\
\Phi(y,\tau) &= 0 \quad \text{for all } t\in(0,T), \quad y\in\partial B_{4R(0)}(0).
\end{aligned}
$$

Then $\phi_{0\lambda} = L_W[\Phi_{0\lambda}]$ where $\Phi_{0\lambda}$ solves

$$
\begin{aligned}
\lambda^2\partial_t\Phi_{0\lambda} &= L_W[\Phi_{0\lambda}] - 2\frac{\lambda_1}{\lambda}(\phi_0 + H_0(y,t)) \quad \text{in } \mathcal{D}_{4R}, \\
\Phi_{0\lambda}(y,0) &= 0 \quad \text{in } B_{4R}(0) \\
\Phi_{0\lambda}(y,\tau) &= 0 \quad \text{for all } t\in(0,T), \quad y\in\partial B_{4R(0)}(0).
\end{aligned}
\tag{7.14}
$$

We recall that we obtained

$$
|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu} R^{5-a}\lambda_*(t)^\nu(1+|y|)^{-3},
$$

and a posteriori the better estimate

$$
|\phi_0(y,t)| \lesssim \|h_0\|_{a,\nu}\frac{R^{\frac{5-a}{2}}\lambda_*^\nu}{1+|y|} \begin{cases} 1 & \text{if } |y|\le R^{\frac{5-a}{4}}, \\ \frac{R^{\frac{5-a}{2}}}{|y|^2} & \text{if } |y|>R^{\frac{5-a}{4}}. \end{cases}
$$

The use of an explicit barrier in (7.14) then yields

$$
|\Phi_{0\lambda}| \lesssim \lambda_*^\nu\|h_0\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty \frac{R^{\frac{5-a}{2}+2}\log R}{1+|y|}
$$

and then, arguing similarly as in the construction of ϕ_0 we obtain the estimate for $\phi_{0\lambda} = L_W[\Phi_{0\lambda}]$,

$$
|\phi_{0\lambda}(y,t)| \lesssim \lambda_*^\nu\|h_0\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty \frac{R^{\frac{5-a}{2}+2}\log R}{1+|y|^3}. \tag{7.15}
$$

Next we want to improve this estimate, as was done in Lemma 6.3. We have that $\phi_{0\lambda}$ satisfies the equation

$$
\lambda^2\partial_t\phi_{0\lambda} = L_W[\phi_{0\lambda}] + g(y,t)
$$

where

$$
g = -2\frac{\lambda_1}{\lambda}(L_W[\phi_0] + h_0(y,t)). \tag{7.16}
$$

We have that $g(y,t) = \operatorname{div}_y G_0(y,t) + G_1(y,t)$ in $\mathcal{D}_{4R}$, where

$$
\begin{aligned}
&(1+|y|)|G_1(y,t)| + (1+|y|)^\alpha[G_0]_{B_\ell(y,\tau)\cap\mathcal{D}_{4R}} + |G_0(y,t)| \\
&\lesssim \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\frac{R^{\frac{5-a}{2}}\lambda_*^\nu}{1+|y|^2} \begin{cases} 1 & \text{if } |y|\le R^{\frac{5-a}{4}}, \\ \frac{R^{\frac{5-a}{2}}}{|y|^2} & \text{if } |y|>R^{\frac{5-a}{4}}. \end{cases}
\end{aligned}
\tag{7.17}
$$

Now we argue as in the previous case. We write

$$
\phi_{0\lambda} = \phi_b + \phi_c
$$

where ϕ_b is given by the Duhamel formula (7.9) with g given by (7.16) and let ϕ_c solve (7.11). Using Lemmas 7.1 and 7.2 we find that

$$
|\phi_b(y,t)| + (1+|y|)|\nabla\phi_b(y,t)| \lesssim \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\lambda_*^\nu R^{\frac{5-a}{2}}\log R \tag{7.18}
$$

for $|y| \le 5R$. The above estimate implies that

$$|\nabla W|^2|\phi_b| + 2\,|(\nabla W\cdot\nabla\phi_b)W| \lesssim \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\,\lambda_*^\nu R^{\frac{5-a}{2}}\log R(1+|y|)^{-3} \tag{7.19}$$

Let φ_c be the complex valued function defined by

$$\phi_c(y,t) \;=\; \mathrm{Re}\,(\varphi_c(\rho,t))\,E_1 + \mathrm{Im}\,(\varphi_c(\rho,t))\,E_2$$

so that using the notation in (6.8), φ_c satisfies the equation

$$\begin{cases} \lambda^2\partial_t\varphi_c = \mathcal{L}_0[\varphi_c] + \tilde g_c(\rho,t) \quad \text{in } \tilde D_{4R}, \\ \varphi_c(0,\rho) = 0 \quad \text{in } (0,4R), \end{cases} \tag{7.20}$$

where by (7.19) $\tilde g_c$ satisfies

$$|\tilde g_c| \lesssim \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\,\lambda_*^\nu R^{\frac{5-a}{2}}\log R(1+|y|)^{-3}.$$

We can find an explicit supersolution for the real and imaginary parts of equation (7.20) in $\tilde D_{R^{1/2}}$ of the form

$$\bar\varphi_c = d(t)Z_0(\rho)\int_\rho^{2R^{1/2}} \frac{1}{Z_0(r)^2 r}\int_0^r Z_0(s)(1+s)^{-3}s\,dsdr.$$

where $d(t) = \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\,\lambda_*^\nu R^{\frac{5-a}{2}}\log R$ and Z_0 is defined in (6.10). We note that at $\rho = R^{1/2}$ the value of ϕ_c satisfies, by (7.15) and (7.18)

$$\begin{aligned} |\phi_c(R^{1/2},t)| &\le |\phi_{0\lambda}(R^{1/2},t)| + |\phi_b(R^{1/2},t)| \\ &\lesssim \lambda_*^\nu\|h_0\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty R^{\frac{6-a}{2}}\log R. \end{aligned}$$

and other hand

$$|\bar\phi_c(R^{1/2},t)| \ge c\left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\,\lambda_*^\nu R^{\frac{5-a}{2}}\log RR^{1/2}$$

for some $c > 0$. This yields

$$|\phi_c(y,t)| \lesssim \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\,\lambda_*^\nu R^{\frac{5-a}{2}+1}\log R(1+|y|)^{-1} \quad |y| < R^{1/2}.$$

and combining with (7.18) we get

$$|\phi_{0\lambda}(y,t)| \lesssim \left\|\frac{\lambda_1}{\lambda}\right\|_\infty \|h_0\|_{a,\nu}\,\lambda_*^\nu R^{\frac{5-a}{2}+1}\log R(1+|y|)^{-1} \quad |y| < R^{1/2}.$$

Using Schauder estimates together with (7.17) we obtain (7.13).

□

7.4. **Mode 1. Estimate of $\phi_{1\lambda}$.** We claim that

$$(1+|y|)|\nabla_y\phi_{1\lambda}(y,t)| + |\phi_{1\lambda}(y,t)| \le C\lambda_*(t)^\nu(1+|y|)^{2-a}\|h\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty \quad \text{in } \mathcal{D}_R.$$

Proof. We recall that ϕ_1 is the restriction to $\mathcal{D}_{2R}$ of the unique solution to the problem

$$\begin{aligned} &\lambda^2\partial_t\phi_{1\lambda} = L_W[\phi_{1\lambda}] + h_1\chi_{\mathcal{D}_{2R}} \quad \text{in } \mathbb{R}^2\times(0,T), \\ &\phi_{1\lambda}(\cdot,0) = 0 \quad \text{in } \mathbb{R}^2\times(0,T), \end{aligned}$$

where by assumption $\int_{B_{2R}} h_1(\cdot,t)\cdot Z_{1j} = 0$, $j = 1,2$, for all $t\in(0,T)$. The function $\phi_{1\lambda}$ then corresponds to the restriction to $\mathcal{D}_{2R}$ of the unique solution to

$$\begin{aligned} &\lambda^2\partial_t\phi_{1\lambda} = L_W[\phi_{1\lambda}] - g \quad \text{in } \mathbb{R}^2\times(0,T), \\ &\phi_{1\lambda}(\cdot,0) = 0 \quad \text{in } \mathbb{R}^2\times(0,T), \end{aligned}$$

were

$$g = -2\frac{\lambda_1}{\lambda}(L_W[\phi_1] + h_1(y,t)\chi_{\mathcal{D}_{2R}})$$

and ϕ_1, h_1 are is as in (7.1). The function g satisfies for $j = 1, 2$

$$\int_{\mathbb{R}^2} g(\cdot,t)\cdot Z_{1j} = 0.$$

We note that $g = \operatorname{div}_y G$, as in (6.27) with G satisfying

$$|G(y,t)| \lesssim \frac{\lambda_*(t)^\nu}{1+|y|^{a-1}}\|h\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty, \quad y\in\mathbb{R}^2, t\in(0,T).$$

Then Lemma 6.7 implies that

$$|\phi_{1\lambda}(y,t)| \lesssim \frac{\lambda_*(t)^\nu}{1+|y|^{a-2}}\|h\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty.$$

By the same argument as in section 7.2, using Corollary 6.1, we deduce also

$$(1+|y|)|\nabla_y\phi_{1\lambda}(y,t)| \le C\lambda_*(t)^\nu(1+|y|)^{2-a}\|h\|_{a,\nu}\left\|\frac{\lambda_1}{\lambda}\right\|_\infty.$$

□

8. The heat equation with right hand side

Given $q\in\Omega$ and $T>0$ sufficiently small we consider the problem

$$\begin{cases} \psi_t = \Delta_x\psi + f(x,t) \quad \text{in } \Omega\times(0,T) \\ \psi = 0 \quad \text{on } \partial\Omega\times(0,T) \\ \psi(q,T) = 0 \\ \psi(x,0) = (c_1\,\mathbf{e_1} + c_2\,\mathbf{e_2} + c_3\,\mathbf{e_3})\eta_1 \quad \text{in } \Omega \end{cases} \tag{8.1}$$

for suitable constants c_1, c_2, c_3, where $\mathbf{e_1}$, $\mathbf{e_1}$, $\mathbf{e_1}$ are defined in (3.5), and η_1 is a smooth cut-off function with compact support, such that $\eta_1 \equiv 1$ in a neighborhood of q.

The right hand side of (8.1) is assumed to be bounded with respect to some weights that appear in the exterior problem (5.37). Thus we define the weights

$$\begin{cases} \varrho_1 := \lambda_*^\Theta(\lambda_* R)^{-1}\chi_{\{r\le 3R\lambda_*\}} \\ \varrho_2 := T^{-\sigma_0}\dfrac{\lambda_*^{1-\sigma_0}}{r^2}\chi_{\{r\ge R\lambda_*\}} \\ \varrho_3 := T^{-\sigma_0}, \end{cases}$$

where $r = |x-q|$, $\Theta>0$ and $\sigma_0>0$ is small. For a function $f(x,t)$ we consider the L^∞-weighted norm

$$\|f\|_{**} := \sup_{\Omega\times(0,T)}\Big(1+\sum_{i=1}^3\varrho_i(x,t)\Big)^{-1}|f(x,t)|. \tag{8.2}$$

The factor T^{σ_0} in front of ϱ_2 and ϱ_3 is a simple way to have parts of the error small in the outer problem.

We are going to measure the solution to (8.1) in the norm $\|\ \|_{\sharp,\Theta,\gamma}$, (c.f. (5.44)) with Θ and β (recall that $R = \lambda_*^{-\beta}$) satisfying:

$$\beta\in\Big(0,\frac12\Big), \quad \Theta\in(0,\beta) \tag{8.3}$$

Our main result in this section is the following, where we use the norm $\|\ \|_{\sharp,\Theta,\gamma}$ defined in (5.44).

Proposition 8.1. *Assume* (8.3). *For* T, $\varepsilon > 0$ *small there is a linear operator that maps a function* $f : \Omega \times (0, T) \to \mathbb{R}^3$ *with* $\|f\|_{**} < \infty$ *into* ψ, c_1, c_2, c_3 *so that* (8.1) *is satisfied. Moreover the following estimate holds*

$$\|\psi\|_{\sharp,\Theta,\gamma} + \frac{\lambda_*(0)^{-\Theta}(\lambda_*(0)R(0))^{-1}}{|\log T|}(|c_1| + |c_2| + |c_3|) \le C\|f\|_{**}, \tag{8.4}$$

where $\gamma \in (0, \frac{1}{2})$.

Remark 8.1. *The condition* $\beta \in (0, \frac{1}{2})$ *is a basic assumption to have the singularity appear inside the self-similar region. The condition* $\Theta > 0$ *is needed for Lemma 8.1. The assumption* $\Theta < \beta$ *is so that the estimates provided by Lemma 8.2 are stronger than the ones of Lemma 8.1.*

To prove Proposition 8.1 we consider

$$\begin{cases} \psi_t = \Delta\psi + f \quad \text{in } \Omega \times (0, T) \\ \psi(x, 0) = 0, \quad x \in \Omega \\ \psi(x, t) = 0, \quad x \in \partial\Omega, t \in (0, T), \end{cases} \tag{8.5}$$

and let q be a point in Ω.

We always assume that R is given by (5.3).

Lemma 8.1. *Assume* $\beta \in (0, \frac{1}{2})$ *and* $\Theta > 0$. *Let* ψ *solve* (8.5) *with* f *such that*

$$|f(x,t)| \le \lambda_*(t)^{\Theta}(\lambda_*(t)R(t))^{-1}\chi_{\{|x-q| \le 3\lambda_*(t)R(t)\}}.$$

Then

$$|\psi(x,t)| \le C\lambda_*(0)^{\Theta}\lambda_*(0)R(0)|\log T|, \tag{8.6}$$

$$|\psi(x,t) - \psi(x,T)| \le \lambda_*(t)^{\Theta}\lambda_*(t)R(t)|\log(T-t)|, \tag{8.7}$$

$$|\nabla\psi(x,t)| \le C\lambda_*(0)^{\Theta}, \tag{8.8}$$

$$|\nabla\psi(x,t) - \psi(x,T)| \le C\lambda_*(t)^{\Theta}, \tag{8.9}$$

and for any $\gamma \in (0, \frac{1}{2})$,

$$\frac{|\nabla\psi(x,t) - \nabla\psi(x,t')|}{|t-t'|^{\gamma}} \le C\frac{\lambda_*(t)^{\Theta}}{(\lambda_*(t)R(t))^{2\gamma}} \tag{8.10}$$

for any x, *and* $0 \le t' \le t \le T$ *such that* $t - t' \le \frac{1}{10}(T-t)$, *and*

$$\frac{|\nabla\psi(x,t) - \nabla\psi(x',t')|}{|x-x'|^{2\gamma}} \le C\frac{\lambda_*(t)^{\Theta}}{(\lambda_*(t)R(t))^{2\gamma}} \tag{8.11}$$

for any $|x - x'| \le 2\lambda_*(t)R(t)$ *and* $0 \le t \le T$.

The proof is in section 15.1.

Lemma 8.2. *Assume* $\beta \in (0, \frac{1}{2})$ *and* $m \in (\frac{1}{2}, 1)$. *Let* ψ *solve* (8.5) *with* f *such that*

$$|f(x,t)| \le \frac{\lambda_*(t)^m}{|z-q|^2}\chi_{\{|x-q| \ge \lambda_*(t)R(t)\}}.$$

Then

$$|\psi(x,t)| \le CT^m|\log T|^{2-m}, \tag{8.12}$$

$$|\psi(x,t) - \psi(x,T)| \le C|\log T|^m(T-t)^m|\log(T-t)|^{2-2m}, \tag{8.13}$$

$$|\nabla\psi(x,t)| \le C\frac{T^{m-1}|\log T|^{2-m}}{R(T)}. \tag{8.14}$$

$$|\nabla\psi(x,t) - \nabla\psi(x,T)| \le C\frac{\lambda_*(t)^{m-1}|\log(T-t)|}{R(t)} \tag{8.15}$$

and for any $\gamma \in (0, \frac{1}{2})$*:*

$$\frac{|\nabla\psi(x,t)-\nabla\psi(x',t')|}{(|x-x'|^2+|t-t'|)^\gamma} \le C \frac{1}{(\lambda_*(t)R(t))^{2\gamma}} \frac{\lambda_*(t)^{m-1}|\log(T-t)|}{R(t)}$$

for any $|x-x'|\le 2\lambda_*(t)R(t)$ *and* $0\le t'\le t\le T$ *such that* $t-t'\le \frac{1}{10}(T-t)$*.*

The proof is in section 15.2.
Faltan las demostraciones de la cota holder

Lemma 8.3. *Let* ψ *solve* (8.5) *with* f *such that*

$$|f(x,t)|\le 1,$$

Then

$$|\psi(x,t)|\le Ct. \tag{8.16}$$

$$|\psi(x,t)-\psi(x,T)|\le C(T-t)|\log(T-t)|. \tag{8.17}$$

$$|\nabla\psi(x,t)|\le T^{1/2} \tag{8.18}$$

$$|\nabla\psi(x,t)-\nabla\psi(x,T)|\le C(T-t)^{1/2} \tag{8.19}$$

$$|\nabla\psi(x,t_2)-\nabla\psi(x,t_1)|\le C|t_2-t_1|^{1/2}. \tag{8.20}$$

$$|\nabla\psi(x_1,t)-\nabla\psi(x_2,t)|\le C|x_1-x_2||\log(|x_1-x_2|).$$

The proof is in section 15.3.
Falta la demostracion de la cota holder en x

Proof of Proposition 8.1. Let $\psi_0[f]$ denote the solution of (8.5) where f satisfies $\|f\|_{**}<\infty$.

We claim that $\|\psi_0[f]\|_*\le C\|f\|_{**}$. Indeed, given f with $\|f\|_{**}<\infty$ we decompose $f=\sum_{i=1}^3 f_i$ with $|f_i|\le C\|f\|_{**}\varrho_i$. By linearity it is sufficient to prove that when f is each of the ϱ_i, the corresponding ψ has finite $\| \ \|_{**}$ norm.

The case $f=\varrho_1$ is direct from Lemma 8.1. Using the hypothesis $\Theta<\beta$ we can find σ_0 small so that the case $f=\varrho_2$ follows from Lemma 8.2. The case $f=\varrho_3$ follows from Lemma 8.3.

Finally, let us show that in problem (8.1) we can choose c_i so that that $\psi(q,T)=0$. To do this we let ψ_i the solution

$$\begin{cases} \partial_t\psi_i=\Delta_x\psi_i \quad \text{in } \Omega\times(0,T)\\ \psi_i=0 \quad \text{on } \partial\Omega\times(0,T)\\ \psi_i(x,0)=\mathbf{e_i}\eta_1 \quad \text{in } \Omega\end{cases}$$

Let

$$\psi=\psi_0+\sum_{i=1}^3 c_i\psi_i.$$

Then for $T>0$ small there is unique choice of c_i such that $\psi(q,T)=0$. Moreover $|c_i|\le C\lambda_*(0)^\nu R(0)^{2-a}|\log T|\|f\|_{**}$ and hence ψ satisfies (8.4). $\square$

9. The heat equation with initial condition

In this section we consider the heat equation

$$\begin{cases}\partial_t\tilde Z_1(x,t)=\Delta\tilde Z_1(x,t) \quad \text{in } \Omega\times(0,T)\\ \tilde Z_1(x,0)=Z_1^*(x) \quad x\in\Omega\\ \tilde Z_1(x,t)=0 \quad (x,t)\in\partial\Omega\times(0,T),\end{cases} \tag{9.1}$$

and derive estimates assuming, roughly speaking, that Z_1^* behaves like $(r+\varepsilon)|\log(r+\varepsilon)|$.

Lemma 9.1. *Suppose $Z_1^* \in C^2(\overline{\Omega})$ satisfies*

$$|D_x^2 Z_1^*(x)| \le \frac{1}{|x-q_0|+\varepsilon} \quad x\in\Omega.$$

Then the solution $\tilde Z_1$ of (9.1) satisfies

$$|\nabla_x \tilde Z_1(x,t) - \nabla_x \tilde Z_1(x,T)| \le C\frac{T-t}{T}\Big(1+\log(\frac{T}{t})\Big) \quad \text{if } \varepsilon^2\le t\le T. \tag{9.2}$$

Proof. We do the computation when Ω is $\mathbb{R}^2$ and we deal with the solution given by Duhamel's formula. The general case follows by the decomposing the solution as a sum of the one in $\mathbb{R}^2$ and a smooth one in Ω. Then

$$\nabla_x \tilde Z_1(x,t) = \frac{1}{4\pi t}\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4t}} \nabla_x Z_1^*(x-y)\,dy. \tag{9.3}$$

Assume $\varepsilon^2\le t\le T$. Then, using (9.3), we have

$$\begin{aligned}
&|\nabla_x \tilde Z_1(0,t) - \nabla_x \tilde Z_1(0,T)|\\
&= \frac{1}{4\pi}\left|\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4}} \int_0^1 \nabla_x \tilde Z_1^*\big(-s\sqrt{T}y + (1-s)\sqrt{t}y\big)\,ds dy\right|\\
&\le C\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4}} \int_0^1 |D^2 Z_1^*\big(-s\sqrt{T}y+(1-s)\sqrt{t}y\big)|(\sqrt{T}-\sqrt{t})|y|\,dsdy\\
&\le C\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4}} \int_0^1 \frac{(\sqrt{T}-\sqrt{t})|y|}{s(\sqrt{T}-\sqrt{t})|y|+\sqrt{t}|y|+\varepsilon}\,dsdy\\
&\le C\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4}} \log\Big(\frac{\sqrt{T}|y|+\varepsilon}{\sqrt{t}|y|+\varepsilon}\Big)\,dy\\
&= C\int_0^\infty e^{-\frac{\rho^2}{4}} \log\Big(\frac{\sqrt{T}\rho+\varepsilon}{\sqrt{t}\rho+\varepsilon}\Big)\rho\,d\rho\\
&= C(\sqrt{T}-\sqrt{t})\varepsilon\int_0^\infty e^{-\frac{\rho^2}{4}} \frac{1}{(\sqrt{T}\rho+\varepsilon)(\sqrt{t}\rho+\varepsilon)}\,d\rho.
\end{aligned}$$

We claim that

$$\varepsilon\int_0^\infty e^{-\frac{\rho^2}{4}}\frac{1}{(\sqrt{T}\rho+\varepsilon)(\sqrt{t}\rho+\varepsilon)}\,d\rho \le \frac{C}{\sqrt{T}}\left(1+\log\Big(\frac{T}{t}\Big)\right).$$

Indeed,

$$\begin{aligned}
\int_{\frac{\varepsilon}{\sqrt{t}}}^\infty e^{-\frac{\rho^2}{4}}\frac{1}{(\sqrt{T}\rho+\varepsilon)(\sqrt{t}\rho+\varepsilon)}\,d\rho &\le C\frac{1}{\sqrt{Tt}}\int_{\frac{\varepsilon}{\sqrt{t}}}^\infty e^{-\frac{\rho^2}{4}}\frac{1}{\rho^2}\,d\rho\\
&\le \frac{C}{\varepsilon\sqrt{T}}
\end{aligned}$$

$$\begin{aligned}
\int_{\frac{\varepsilon}{\sqrt{T}}}^{\frac{\varepsilon}{\sqrt{t}}} e^{-\frac{\rho^2}{4}}\frac{1}{(\sqrt{T}\rho+\varepsilon)(\sqrt{t}\rho+\varepsilon)}\,d\rho &\le \frac{C}{\varepsilon\sqrt{T}}\int_{\frac{\varepsilon}{\sqrt{T}}}^{\frac{\varepsilon}{\sqrt{t}}} e^{-\frac{\rho^2}{4}}\frac{1}{\rho}\,d\rho\\
&\le \frac{C}{\varepsilon\sqrt{T}}\log\Big(\frac{T}{t}\Big).
\end{aligned}$$

$$\begin{aligned}
\int_0^{\frac{\varepsilon}{\sqrt{T}}} e^{-\frac{\rho^2}{4}}\frac{1}{(\sqrt{T}\rho+\varepsilon)(\sqrt{t}\rho+\varepsilon)}\,d\rho &\le \frac{1}{\varepsilon^2}\int_0^{\frac{\varepsilon}{\sqrt{T}}} e^{-\frac{\rho^2}{4}}\,d\rho\\
&\le \frac{1}{\varepsilon\sqrt{T}}.
\end{aligned}$$

Therefore we get

$$|\nabla_x \tilde Z_1(0,t) - \nabla_x \tilde Z_1(0,T)| \le C \frac{\sqrt{T}-\sqrt{t}}{\sqrt{T}} \left(1+\log\Big(\frac{T}{t}\Big)\right).$$

This implies (9.2). ☐

Lemma 9.2. *Suppose $Z_1^* \in C^2(\overline{\Omega})$ satisfies*

$$|D_x^2 Z_1^*(x)| \le \frac{1}{|x-q_0|+\varepsilon} \quad x\in\Omega.$$

Then the solution $\tilde Z_1$ of (9.1) satisfies

$$|D_x^2 \tilde Z_1(x,t) \le \frac{C}{\varepsilon+\sqrt{t}} \tag{9.4}$$

Proof. We use Duhamel's formula to find

$$\begin{aligned} |D_x^2 \tilde Z_1(x,t)| &= \Big|\frac{1}{4\pi t}\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4t}} D_x^2 Z_1^*(x-y)\,dy\Big| \\ &\le \frac{C}{t}\int_{\mathbb{R}^2} e^{-\frac{|y|^2}{4t}} \frac{1}{|x-y|+\varepsilon}\,dy \\ &\le \frac{C}{\sqrt{t}}\int_{\mathbb{R}^2} e^{-\frac{|z|^2}{4}} \frac{1}{|\tilde x - z|+\frac{\varepsilon}{\sqrt{t}}}\,dz \end{aligned}$$

where $\tilde x = \frac{x}{\sqrt{t}}$. By the Hardy-Littlewood inequality

$$|D_x^2 \tilde Z_1(x,t)| \le \frac{C}{\sqrt{t}}\int_{\mathbb{R}^2} e^{-\frac{|z|^2}{4}} \frac{1}{|z|+\frac{\varepsilon}{\sqrt{t}}}\,dz. \tag{9.5}$$

We claim that for $a>0$

$$\int_{\mathbb{R}^2} e^{-\frac{|z|^2}{4}} \frac{1}{|z|+a}\,dz \le C \begin{cases} 1 & \text{if } a\le 1 \\ \frac{1}{a} & \text{if } a \ge 1. \end{cases} \tag{9.6}$$

Indeed,

$$\begin{aligned} \int_{|z|\le a} e^{-\frac{|z|^2}{4}} \frac{1}{|z|+a}\,dz &\le \frac{1}{a}\int_{|z|\le a} e^{-\frac{|z|^2}{4}}\,dz \\ &\le C\frac{1}{a}\begin{cases} a^2 & \text{if } a\le 1 \\ 1 & \text{if } a\ge 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \int_{|z|\ge a} e^{-\frac{|z|^2}{4}} \frac{1}{|z|+a}\,dz &\le \int_{|z|\ge a} e^{-\frac{|z|^2}{4}} \frac{1}{|z|}\,dz \\ &\le \begin{cases} C & \text{if } a\le 1 \\ e^{-a} & \text{if } a\ge 1. \end{cases} \end{aligned}$$

This proves (9.6). Combining (9.5) and (9.6) we obtain the desired estimate (9.4). ☐

Lemma 9.3. *Suppose $Z_1^* \in C^2(\overline{\Omega})$ satisfies*

$$|D_x^2 Z_1^*(x)| \le \frac{1}{|x-q_0|+\varepsilon} \quad x\in\Omega.$$

Then the solution $\tilde Z_1$ of (9.1) *satisfies for $0\le t_0\le t_1$:*

$$|\nabla_x \tilde Z_1(x,t_1)-\nabla_x\tilde Z_1(x,t_0)|\le C\begin{cases}\frac{\sqrt{t_1}-\sqrt{t_0}}{\sqrt{t_1}}\log(2\frac{t_1}{t_0}) & \text{if } t_0\ge\varepsilon^2\\ \frac{\sqrt{t_1}-\sqrt{t_0}}{\sqrt{t_1}}\log(2\frac{t_1}{\varepsilon^2}) & \text{if } t_0\le\varepsilon^2,\ t_1\ge\varepsilon^2\\ \frac{\sqrt{t_1}-\sqrt{t_0}}{\varepsilon} & \text{if } t_1\le\varepsilon^2\end{cases}$$

Proof. Assume $0<t_0<t_1$. Then, using (9.3), we have

$$\begin{aligned}&|\nabla_x \tilde Z_1(x,t_1)-\nabla_x\tilde Z_1(x,t_0)|\\ &=\frac{1}{4\pi}\left|\int_{\mathbb{R}^2}e^{-\frac{|y|^2}{4}}\int_0^1\frac{d}{ds}\nabla_x\tilde Z_1^*\big(x-s\sqrt{t_1}y-(1-s)\sqrt{t_0}y\big)\,dsdy\right|\\ &\le C(\sqrt{t_1}-\sqrt{t_0})\int_{\mathbb{R}^2}e^{-\frac{|y|^2}{4}}\int_0^1|D^2Z_1^*\big(x-s\sqrt{t_1}y-(1-s)\sqrt{t_0}y\big)\,|y|\,dsdy\\ &\le C(\sqrt{t_1}-\sqrt{t_0})\int_{\mathbb{R}^2}e^{-\frac{|y|^2}{4}}\int_0^1\frac{|y|}{|x-s\sqrt{t_1}y-(1-s)\sqrt{t_0}y|+\varepsilon}\,dsdy\\ &=C(\sqrt{t_1}-\sqrt{t_0})\int_0^1\int_{\mathbb{R}^2}e^{-\frac{|y|^2}{4}}\frac{|y|}{|x-s\sqrt{t_1}y-(1-s)\sqrt{t_0}y|+\varepsilon}\,dy\,ds\\ &=C(\sqrt{t_1}-\sqrt{t_0})\int_0^1\int_{\mathbb{R}^2}e^{-\frac{|y|^2}{4}}\frac{1+|y|^2}{|x-s\sqrt{t_1}y-(1-s)\sqrt{t_0}y|+\varepsilon}\,dy\,ds\end{aligned}$$

We use now the Hardy-Littlewood inequality and obtain

$$\begin{aligned}&|\nabla_x \tilde Z_1(x,t_1)-\nabla_x\tilde Z_1(x,t_0)|\\ &\le C(\sqrt{t_1}-\sqrt{t_0})\int_0^1\int_{\mathbb{R}^2}e^{-\frac{|y|^2}{4}}\frac{1+|y|^2}{|s\sqrt{t_1}y+(1-s)\sqrt{t_0}y|+\varepsilon}\,dy\,ds\\ &=C(\sqrt{t_1}-\sqrt{t_0})\int_{\mathbb{R}^2}e^{-\frac{|y|^2}{4}}\int_0^1\frac{1+|y|^2}{s(\sqrt{t_1}-\sqrt{t_0})|y|+\sqrt{t_0}|y|+\varepsilon}\,ds\,dy\\ &=C\int_0^\infty e^{-\frac{\rho^2}{4}}(1+\rho^2)\log\Big(\frac{\sqrt{t_1}\rho+\varepsilon}{\sqrt{t_0}\rho+\varepsilon}\Big)\,d\rho\\ &=C\varepsilon(\sqrt{t_1}-\sqrt{t_0})\int_0^\infty g(\rho)\frac{1}{(\sqrt{t_1}\rho+\varepsilon)(\sqrt{t_0}\rho+\varepsilon)}\,d\rho,\end{aligned}$$

where

$$g(\rho)=\int_\rho^\infty e^{-\frac{s^2}{4}}(1+s^2)\,ds.$$

We claim that

$$\int_0^\infty g(\rho)\frac{1}{(\sqrt{t_1}\rho+\varepsilon)(\sqrt{t_0}\rho+\varepsilon)}\,d\rho\le\begin{cases}\frac{1}{\varepsilon\sqrt{t_1}}\log(\frac{t_1}{t_0}) & \text{if } t_0\ge 2\varepsilon^2\\ \frac{1}{\varepsilon\sqrt{t_1}}\log(\frac{t_1}{\varepsilon^2}) & \text{if } t_0\le 2\varepsilon^2,\ t_1\ge 2\varepsilon^2\\ \frac{1}{\varepsilon^2} & \text{if } t_1\le 2\varepsilon^2\end{cases}$$

Indeed,

$$\begin{aligned}\int_{\frac{\varepsilon}{\sqrt{t_0}}}^\infty g(\rho)\frac{1}{(\sqrt{t_1}\rho+\varepsilon)(\sqrt{t_0}\rho+\varepsilon)}\,d\rho&\le C\frac{1}{\sqrt{t_0t_1}}\int_{\frac{\varepsilon}{\sqrt{t_0}}}^\infty g(\rho)\frac{1}{\rho^2}\,d\rho\\ &\le C\frac{1}{\sqrt{t_0t_1}}\begin{cases}\frac{\sqrt{t_0}}{\varepsilon} & \text{if } t_0\ge 2\varepsilon^2,\\ (\frac{\sqrt{t_0}}{\varepsilon})^k & \text{if } t_0\le 2\varepsilon^2,\end{cases}\end{aligned}$$

$$\int_{\frac{\varepsilon}{\sqrt{t_1}}}^{\frac{\varepsilon}{\sqrt{t_0}}} g(\rho)\frac{1}{(\sqrt{t_1}\rho+\varepsilon)(\sqrt{t_0}\rho+\varepsilon)}\,d\rho \le \frac{C}{\varepsilon\sqrt{t_1}}\int_{\frac{\varepsilon}{\sqrt{t_1}}}^{\frac{\varepsilon}{\sqrt{t_0}}} g(\rho)\frac{1}{\rho}\,d\rho$$
$$\le \frac{C}{\varepsilon\sqrt{t_1}}\begin{cases}\log(\frac{t_1}{t_0}) & \text{if } t_0\ge 2\varepsilon^2\\ \log(\frac{t_1}{\varepsilon^2}) & \text{if } t_0\le 2\varepsilon^2,\ t_1\ge 2\varepsilon^2\\ (\frac{t_1}{\varepsilon^2})^k & \text{if } t_1\le 2\varepsilon^2\end{cases}$$

$$\int_0^{\frac{\varepsilon}{\sqrt{t_1}}} g(\rho)\frac{1}{(\sqrt{t_1}\rho+\varepsilon)(\sqrt{t_0}\rho+\varepsilon)}\,d\rho \le \frac{1}{\varepsilon^2}\int_0^{\frac{\varepsilon}{\sqrt{t_1}}} g(\rho)\,d\rho$$
$$\le \frac{C}{\varepsilon^2}\begin{cases}\frac{\varepsilon}{\sqrt{t_1}} & \text{if } t_1\ge 2\varepsilon^2,\\ 1 & \text{if } t_1\le 2\varepsilon^2,\end{cases}$$

Therefore we get

$$|\nabla_x\tilde Z_1(x,t_1)-\nabla_x\tilde Z_1(x,t_0)|\le C\begin{cases}\frac{\sqrt{t_1}-\sqrt{t_0}}{\sqrt{t_1}}\log(\frac{t_1}{t_0}) & \text{if } t_0\ge 2\varepsilon^2\\ \frac{\sqrt{t_1}-\sqrt{t_0}}{\sqrt{t_1}}\log(\frac{t_1}{\varepsilon^2}) & \text{if } t_0\le 2\varepsilon^2,\ t_1\ge 2\varepsilon^2\\ \frac{\sqrt{t_1}-\sqrt{t_0}}{\varepsilon} & \text{if } t_1\le 2\varepsilon^2\end{cases}$$

□

Let us recall the norm $\|\ \|_*$ defined in (5.7). As a corollary of the previous estimates we have.

Lemma 9.4. *Suppose $Z_0^*\in C^2(\overline{\Omega})$. Then the solution $\tilde Z^*$ of (9.1) satisfies*

$$|\nabla_x\tilde Z^*(x,t)|\le|\log\varepsilon|\,\|Z_0^*\|_*,\quad t\ge 0,$$
$$|Z^*(x,t)-Z^*(x,T)|\le C|\log T|\frac{T-t}{\sqrt{T}}\|Z_0^*\|_*,$$
$$|\nabla_x\tilde Z_1(x,t)-\nabla_x\tilde Z_1(x,T)|\le C\|Z_0^*\|_*\begin{cases}|\log\varepsilon| & \textit{if } 0\le t\le\varepsilon^2\\ |\log\varepsilon|^{1/2}\frac{T-t}{T}(1+\log(\frac{T}{t})) & \textit{if } \varepsilon^2\le t\le T.\end{cases}$$

10. The exterior problem

In this section we prove Proposition 5.6.

Proof of Proposition 5.6. We use the norms $\|\ \|_{**}$ defined in (8.2) and $\|\ \|_{\sharp,\Theta,\gamma}$ defined in (5.44), where Θ satisfies (5.45) and $\gamma\in(0,\frac12)$.

Let $\mathcal{H}$ be the operator constructed in Proposition 8.1 that maps f to the solution $\psi,c_1,\dots,c_3$ of (8.1) (here we need $\Theta>0$).

To find a solution to (5.37) we set up a fixed point problem in the space

$$Y=\{\psi:\Omega\times(0,T)\to\mathbb{R}^3:\ \|\psi\|_{\sharp,\Theta,\gamma}<\infty\}.$$

Let us define the nonlinear operator

$$\mathcal{A}[\psi]=\mathcal{H}[g(p,\xi,Z^*+\psi,\phi)],\quad \psi\in\bar B_1(0)\subset Y,$$

where g is defined in (3.22), $\phi=\phi_1+\phi_2+\phi_3+\phi_4$ and we assume $\|\Phi\|_E\le 1$, and consider the fixed point problem

$$\psi=\mathcal{A}[\psi]. \tag{10.1}$$

We claim that

$$\|g(p,\xi,Z^*+\psi,\phi)\|_{**}\le CT^\sigma\big(\|\Phi\|_E+\|\dot p\|_{L^\infty(-T,T)}+\|\dot\xi\|_{L^\infty(0,T)}+\|Z^*\|_{C^1}$$
$$+\|\psi\|_{\sharp,\Theta,\gamma}\big), \tag{10.2}$$

and

$$\|g(p,\xi,Z^*+\psi_1,\phi)-g(p,\xi,Z^*+\psi_2,\phi)\|_{**}\le CT^\sigma(\|\psi_1-\psi_2\|_{\sharp,\Theta,\gamma}), \tag{10.3}$$

for some $\sigma > 0$. Using these estimates and the contraction mapping principle we obtain ψ satisfying (10.1) and this gives a solution as stated in Proposition 5.6.

To prove (10.2) and (10.3) we will be using constantly the inclusion

$$B_{2\lambda_* R}(\xi) \subset B_{3\lambda_* R}(q),$$

which holds provided that

$$|\xi - q| \le \lambda_* R,$$

which in turn holds because of (5.2). Indeed, because of (5.2) and $\xi(T) = q$ we have $|\xi - q| \le (T-t)^{1+\sigma}$, and so

$$|\xi - q| \ll \lambda_* R = \lambda_*^{1-\beta} \sim \left(\frac{|\log T|(T-t)}{|\log(T-t)|^2} \right)^{1-\beta}.$$

Let us write

$$g(\psi) = g_1 + g_2 + g_3 + g_4$$

where

$$\begin{aligned} g_1 &= Q_\omega\big((\Delta_x \eta)\phi + 2\nabla_x\eta\nabla_x\phi - \eta_t\phi\big) \\ &\quad + \eta Q_\omega\big(-\dot\omega J\phi + \lambda^{-1}\dot\lambda y \cdot \nabla_y\phi + \lambda^{-1}\dot\xi \cdot \nabla_y \phi\big) \\ g_2 &= (1-\eta)\tilde L_U[\Psi^*] + (\Psi^* \cdot U)U_t \\ g_3 &= (1-\eta)[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1] + (\Phi^0 \cdot U)U_t, \\ g_4 &= N_U(\eta Q_\omega \phi + \Pi_{U^\perp}(\Phi^0 + \Psi)^*). \end{aligned}$$

Estimate of g_1. We claim that

$$\|g_1\|_{**} \le CT^\sigma \|\Phi\|_E, \tag{10.4}$$

for some $\sigma > 0$.

We note that

$$\begin{aligned} |\Delta_x \eta \phi_1| &\le C\lambda_*^{\nu_1 - 2} R^{-a_1} \chi_{[|x-q| \le 3\lambda_* R]} \|\phi_1\|_{*,\nu_1,a_1,\delta} \\ |\Delta_x \eta \phi_2| &\le C\lambda_*^{\nu_2 - 2} R^{-a_2} \chi_{[|x-q| \le 3\lambda_* R]} \|\phi_2\|_{\nu_2,a_2-2} \\ |\Delta_x \eta \phi_3| &\le C\lambda_*^{\nu_3 - 2} R^{-1} \chi_{[|x-q| \le 3\lambda_* R]} \|\phi_3\|_{**,\nu_3} \\ |\Delta_x \eta \phi_4| &\le C\lambda_*^{\nu_4 - 2} R^{-2} \log R \chi_{[|x-q| \le 3\lambda_* R]} \|\phi_4\|_{***,\nu_4}. \end{aligned}$$

If

$$\Theta < \min(\nu_1 - 1 + \beta(a_1 - 1), \nu_2 - 1 + \beta(a_2 - 1), \nu_3 - 1, \nu_4 - 1 + \beta), \tag{10.5}$$

we find that for any $j = 1,2,3,4$:

$$|\Delta_x \eta \phi_j| \le CT^\sigma \lambda_*^{\Theta - 1 + \beta} \chi_{[|x-q| \le 3\lambda_* R]} \|\Phi\|_E,$$

for some $\sigma > 0$.

Then we have

$$\|Q_\omega (\Delta_x \eta)\phi\|_{**} \le CT^\sigma \|\Phi\|_E$$

and similarly

$$\|(\partial_t \eta) Q_\omega \phi\|_{**} + \|Q_\omega \lambda^{-1} \nabla_x \eta \nabla_y \phi\|_{**} \le CT^\sigma \|\Phi\|_E.$$

Let us analyze $\dot\omega \eta Q_\omega J\phi$. We have

$$|\dot\omega \eta Q_\omega J\phi| \le |\dot\omega|\, |\phi| \chi_{\{|x-\xi| \le 2\lambda_*(t)R(t)\}}.$$

By (5.1), $|\dot\omega| \le C|\dot\lambda_*|\lambda_*^{-1} \le C\lambda_*^{-1}$ and hence

$$
\begin{aligned}
&|\dot\omega\eta Q_\omega J\phi_1| \le C\lambda_*^{\nu_1-1}R^{\delta(5-a_1)}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_1\|_{*,\nu_1,a_1,\delta}\\
&|\dot\omega\eta Q_\omega J\phi_2| \le C\lambda_*^{\nu_2-1}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_2\|_{\nu_2,a_2-2}\\
&|\dot\omega\eta Q_\omega J\phi_3| \le C\lambda_*^{\nu_3-1}R^{2}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_3\|_{**,\nu_3}\\
&|\dot\omega\eta Q_\omega J\phi_4| \le C\lambda_*^{\nu_4-1}\log R\,\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_4\|_{***,\nu_4}.
\end{aligned}
$$

If

$$
\Theta < \min\Big(\nu_1-\delta\beta(5-a_1)-\beta, \nu_2-\beta, \nu_3-3\beta, \nu_4-\beta\Big) \tag{10.6}
$$

we get for any $j=1,2,3,4$:

$$
|\dot\omega\eta Q_\omega J\phi_j| \le C\lambda_*(0)^\sigma\lambda_*(t)^{\Theta-1+\beta}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\Phi\|_E,
$$

for some $\sigma>0$ In particular we get for any $j=1,2,3,4$:

$$
\|\dot\omega\eta Q_\omega J\phi_j\|_{**} \le CT^\sigma\|\Phi\|_E.
$$

Let us analyze $\eta Q_\omega\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi$ and $\eta Q_\omega\lambda^{-1}\dot\xi\cdot\nabla_y\phi$. Using (5.1) we have $|\dot\lambda|\le C$ and then

$$
|\eta Q_\omega\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi_1| \le C\lambda_*^{\nu_1-1}R^{\delta(5-a_1)}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_1\|_{*,\nu_1,a_1,\delta},
$$

and similarly, because $|\dot\xi|\le C$ by (5.2),

$$
|\eta Q_\omega\lambda^{-1}\dot\xi\cdot\nabla_y\phi_1| \le C\lambda_*^{\nu_1-1}R^{\delta(5-a_1)}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_1\|_{*,\nu_1,a_1,\delta}.
$$

As before we get

$$
\begin{aligned}
&|\eta Q_\omega\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi_1| + |\eta Q_\omega\lambda^{-1}\dot\xi\cdot\nabla_y\phi_1|\\
&\le C\lambda_*(0)^\sigma\lambda_*(t)^{\Theta-1+\beta}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_1\|_{*,\nu_1,a_1,\delta}.
\end{aligned}
$$

In a similar way we obtain

$$
\begin{aligned}
&|\eta Q_\omega\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi_2| + |\eta Q_\omega\lambda^{-1}\dot\xi\cdot\nabla_y\phi_2|\\
&\le C\lambda_*(0)^\sigma\lambda_*(t)^{\Theta-1+\beta}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_2\|_{\nu_2,a_2-2}
\end{aligned}
$$

$$
\begin{aligned}
&|\eta Q_\omega\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi_3| + |\eta Q_\omega\lambda^{-1}\dot\xi\cdot\nabla_y\phi_3|\\
&\le C\lambda_*(0)^\sigma\lambda_*(t)^{\Theta-1+\beta}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_3\|_{**,\nu_3}
\end{aligned}
$$

and

$$
\begin{aligned}
&|\eta Q_\omega\lambda^{-1}\dot\lambda y\cdot\nabla_y\phi_4| + |\eta Q_\omega\lambda^{-1}\dot\xi\cdot\nabla_y\phi_4|\\
&\le C\lambda_*(0)^\sigma\lambda_*(t)^{\Theta-1+\beta}\chi_{\{|x-q|\le 3\lambda_*(t)R(t)\}}\|\phi_4\|_{***,\nu_4}.
\end{aligned}
$$

From the above estimates we deduce (10.4).

Estimate of g_2. Recall that $g_2=(1-\eta)\tilde L_U[\Psi^*]+(\Psi^*\cdot U)U_t$. We claim that

$$
\|g_2\|_{**} \le CT^\sigma(\|\psi\|_{\sharp,\Theta,\gamma}+\|\nabla Z_0^*\|_{L^\infty}). \tag{10.7}
$$

Thanks to formula (2.5):

$$
\tilde L_U[\Phi] = -\frac{2}{\lambda}w_\rho(\rho)\,[(\Phi_r\cdot U)Q_\omega E_1-\frac{1}{r}(\Phi_\theta\cdot U)Q_\omega E_2],\quad \rho=\frac{r}{\lambda},
$$

where $r=|x-\xi(t)|$, which implies that

$$
|\tilde L_U[\psi]| \le \frac{\lambda_*}{r^2+\lambda_*^2}|\nabla\psi|. \tag{10.8}
$$

But $|\nabla\psi|\le\lambda_*(0)^\Theta\|\psi\|_{\sharp,\Theta,\gamma}$ and hence

$$
(1-\eta)|\tilde L_U[\psi]| \le \varrho_2T^{\sigma_0}\lambda_*(0)^\Theta\|\psi\|_{\sharp,\Theta,\gamma}.
$$

Since $\Theta > 0$,

$$(1-\eta)|\tilde{L}_U[\psi]| \le T^{\sigma} \varrho_2 \|\psi\|_{\sharp,\Theta,\gamma}.$$

As in (10.8) we have

$$|(1-\eta)\tilde{L}_U[Z^*]| \le T^{\sigma_0} \varrho_2 \|\nabla Z^*\|_{L^\infty(\Omega\times(0,T))},$$

and hence

$$\|(1-\eta)\tilde{L}_U[\Psi^*]\|_{**} \le T^{\sigma_0}(\|\psi\|_{\sharp,\Theta,\gamma} + \|\nabla Z^*\|_{L^\infty(\Omega\times(0,T))}). \tag{10.9}$$

Next we estimate the term $(\Psi^* \cdot U)U_t$. Because $\Psi^*(q,T) = 0$ we have

$$|\Psi^*(x,t)| \le (|\Psi^*(x,t) - \Psi^*(x,T)| + |\Psi^*(x,T) - \Psi^*(q,T)|.$$

Then we estimate

$$|\psi(x,t) - \psi(x,T)| \le \lambda_*(t)^{\Theta+1} R(t) |\log(T-t)| \|\psi\|_{\sharp,\Theta,\gamma}$$
$$|\psi(x,T) - \psi(q,T)| \le |x-q|\, \|\nabla\psi\|_{L^\infty(\Omega\times(0,T)} \le |x-q|\, \|\psi\|_{\sharp,\Theta,\gamma}.$$

For the function Z^* we estimate, using Lemma 9.4,

$$\begin{aligned} |Z^*(x,t) - Z^*(x,T)| &\le C|\log T| \frac{T-t}{\sqrt{T}} \|Z_0^*\|_*, \\ &\le C|\log T| \sqrt{T-t} \|Z_0^*\|_*, \\ |Z^*(x,T) - Z^*(q,T)| &\le |x-q|\, \|\nabla Z^*\|_{L^\infty(\Omega\times(0,T)} \\ &\le C|\log T|\, |x-q|\, \|Z_0^*\|_*. \end{aligned}$$

By (8.3) we have $\Theta + 1 - \beta > \frac{1}{2}$. The above inequalities imply that

$$|\Psi^*(x,t)| \le C|\log T|(r + \sqrt{T-t})(\|\psi\|_{\sharp,\Theta,\gamma} + \|Z_0^*\|_*) \tag{10.10}$$

with $r = |x-q|$. Note that because of assumptions (5.1), (5.2) we have

$$|U_t| \le \frac{|\dot\lambda|}{\lambda}|\rho w_\rho| + |\dot\alpha||\rho w_\rho| + \frac{|\dot\xi|}{\lambda}|w_\rho| \le C\frac{|\dot\lambda_*|}{r+\lambda_*}. \tag{10.11}$$

Then thanks to (10.10) we find

$$\begin{aligned} |(\Psi^* \cdot U)U_t| &\le C|\log T|(r+\sqrt{T-t}) C \frac{|\dot\lambda_*|}{r+\lambda_*}(\|\psi\|_{\sharp,\Theta,\gamma} + \|Z_0^*\|_*) \\ &\le C\left(1 + \frac{\sqrt{T-t}}{r+\lambda_*}\right)(\|\psi\|_{\sharp,\Theta,\gamma} + \|Z_0^*\|_*). \end{aligned}$$

We estimate $\frac{\sqrt{T-t}}{r+\lambda_*}$ in the regions $r \ge \lambda_* R$ and $r \le \lambda_* R$. We have

$$\chi_{\{r\ge\lambda_* R\}} \frac{\sqrt{T-t}}{r+\lambda_*} \le \chi_{\{r\ge\lambda_* R\}}\left(1 + \frac{T-t}{r^2+\lambda_*^2}\right) \le T^{\sigma}(\varrho_2 + \varrho_3),$$

for some $\sigma > 0$. This implies

$$\chi_{\{r\ge\lambda_* R\}}|(\Psi^* \cdot U)U_t| \le T^{\sigma}(\varrho_2 + \varrho_3)(\|\psi\|_{\sharp,\Theta,\gamma} + \|Z_0^*\|_*). \tag{10.12}$$

In the other region

$$\chi_{\{r\le\lambda_* R\}} \frac{\sqrt{T-t}}{r+\lambda_*} \le C\chi_{\{r\le\lambda_* R\}} \frac{|\log(T-t)|}{|\log T|^{1/2}} \lambda_*^{-1/2}.$$

But assuming

$$\Theta < \frac{1}{2} - \beta \tag{10.13}$$

we have $\frac{|\log(T-t)|}{|\log T|^{1/2}}\lambda_*^{-1/2} \le T^{\sigma}\lambda_*^{\Theta-1+\beta}$ for some $\sigma > 0$, and hence

$$\chi_{\{r\le\lambda_* R\}}|(\Psi^* \cdot U)U_t| \le T^{\sigma}\varrho_1(\|\psi\|_{\sharp,\Theta,\gamma} + \|Z_0^*\|_*). \tag{10.14}$$

Combining (10.12) and (10.14) we get

$$\|(\Psi^* \cdot U)U_t\|_{**} \le CT^\sigma(\|\psi\|_{\sharp,\Theta,\gamma} + \|Z_0^*\|_*).$$

Combining this with (10.9) we obtain (10.7).

Estimate of g_3. Let us estimate $g_3 = (1-\eta)[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1] + (\Phi^0 \cdot U)U_t$. Here we claim that

$$\|g_3\|_{**} \le CT^\sigma(\|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)}). \tag{10.15}$$

First let us consider the term $(U \cdot \Phi^0)U_t$. By (5.1)

$$|\Phi^0(r,t)| \le Cr\big(|\log(r+\lambda_*(t))| + 1\big).$$

This in combination with (10.11) gives

$$|(U\cdot\Phi^0)U_t| \le C\big(|\log(r+\lambda_*(t))| + 1\big) \le CT^\sigma(\varrho_1 + \varrho_2 + \varrho_3).$$

Next, using (5.1) and (5.2), we find that

$$\|(1-\eta)[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]]\|_{**} \le CT^\sigma(\|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)}).$$

Indeed, let us consider $\mathcal{K}_{01}$, see (3.17). We have

$$|(1-\eta)\mathcal{K}_{01}| \le C\chi_{\{r\ge\lambda_* R\}}\frac{\lambda_*^2}{(r+\lambda_*)^3}\int_{-T}^t |\dot p(s)| k(z,t-s)\,ds.$$

where we recall that $k(z,t) = 2\frac{1-e^{-\frac{z^2}{4t}}}{z^2}$ and $z(r) = \sqrt{r^2+\lambda^2}$. Then

$$\begin{aligned}\int_{-T}^t |\dot p(s)| k(z,t-s)\,ds &\le C\int_{-T}^{t-z^2}\frac{|\dot p(s)|}{t-s}\,ds + C\int_{t-z^2}^t |\dot p(s)|\,ds\\ &\le C|\log(r^2+\lambda(t)^2)|\|\dot p\|_{L^\infty(-T,T)}.\end{aligned}$$

Using (5.1), we obtain

$$\begin{aligned}|(1-\eta)\mathcal{K}_{01}| &\le C\chi_{\{r\ge\lambda_*R\}}\frac{\lambda_* R^{-1}}{r^2}|\log r|\|\dot p\|_{L^\infty(-T,T)}\\ &\le CT^\sigma\varrho_2\|\dot p\|_{L^\infty(-T,T)}.\end{aligned}$$

The same estimate holds for $\mathcal{K}_{02}$ and hence

$$\|(1-\eta)\mathcal{K}_0\|_{**} \le CT^\sigma\|\dot p\|_{L^\infty(-T,T)}. \tag{10.16}$$

The estimate for $\mathcal{K}_1$ is similar, gives

$$\|(1-\eta)\mathcal{K}_1\|_{**} \le CT^\sigma\|\dot\xi\|_{L^\infty(0,T)}. \tag{10.17}$$

The term $\Pi_{U^\perp}[\tilde{\mathcal{R}}_1]$ is quadratic in the parameters p, ξ and by (5.1), (5.2) we can bound

$$\|\Pi_{U^\perp}[\tilde{\mathcal{R}}_1]\|_{**} \le CT^\sigma(\|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)}).$$

From the above estimate and (10.16) and (10.17) we obtain (10.15).

Estimate of g_4. Finally, we estimate $g_4(\psi) = N_U(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0 + Z^* + \psi))$ where N_U is given in (3.8). We claim that

$$\|g_4(\psi)\|_{**} \le CT^\sigma(\|\Phi\|_E + \|Z_0^*\|_* + \|p\|_{L^\infty} + \|\dot\xi\|_{L^\infty} + \|\psi\|_{\sharp,\Theta,\gamma}), \tag{10.18}$$

and

$$\|g_4(\psi_1) - g_4(\psi)\|_{**} \le CT^\sigma\|\psi_1 - \psi_2\|_{\sharp,\Theta,\gamma}.$$

The computations are similar to the ones before. We omit the details. □

11. The interior problem

In this section we prove Proposition 5.7. We use the notation of Section 5.

We need to show that for $T>0$ small, $\mathcal{F}:\bar{\mathcal{B}}_1\subset E\to\bar{\mathcal{B}}_1$ and that it is a contraction.

Estimate of $\mathcal{F}_1$. Assume

$$\nu_1<1, \tag{11.1}$$

Recall that we have decomposed $Z_0^*=Z_0^{*0}+Z_0^{*1}$ (c.f. (5.9)). We claim that for $\|\Phi\|_E\le 1$ we have

$$\|\mathcal{F}_1(\Phi)\|_{*,a_1,\nu_1}\le C\lambda_*(0)^{\Theta}T^\sigma(\|\Phi\|_E+\|\dot p\|_{L^\infty(-T,T)}+\|\dot\xi\|_{L^\infty(0,T)})+CT^\sigma\|Z_0^{*0}\|_*, \tag{11.2}$$

and for $\|\Phi_1\|_E,\|\Phi_2\|_E\le 1$

$$\|\mathcal{F}_1(\Phi_1)-\mathcal{F}_1(\Phi_2)\|_{*,a_1,\nu_1}\le CT^\sigma\lambda_*(0)^\Theta\|\Phi_1-\Phi_2\|_E. \tag{11.3}$$

We start with (11.2). By Proposition 5.1 we have

$$\|\mathcal{F}_1(\Phi)\|_{*,\nu_1,a_1,\delta}\le C\|h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_1,a_1}.$$

We estimate from the definition of h_1 in (5.11) and recalling that

$$\Psi^*(p,\xi,\Phi,Z_0^*)=Z^*+\Psi(p,\xi,\Phi,Z_0^*)$$

we get

$$\begin{aligned}
&\|h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_1,a_1}\\
&\quad\le\|\lambda^2Q_{-\omega}(\tilde L_U[\Psi(p,\xi,\Phi,Z_0^*)]_0+\tilde L_U[\Psi(p,\xi,\Phi,Z_0^*)]_2)\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1}\\
&\qquad+\|\lambda^2Q_{-\omega}(\tilde L_U[Z^*]_0+\tilde L_U[Z^*]_2)\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1}\\
&\qquad+\|\lambda^2Q_{-\omega}\mathcal{K}_0[p,\xi]\|_{\nu_1,a_1}.
\end{aligned}$$

We claim that for $j=0$ and $j=2$:

$$\begin{aligned}
\|\lambda^2Q_{-\omega}\tilde L_U[\Psi(p,\xi,\Phi,Z_0^*)]_j\,\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1}\le CT^\sigma\lambda_*(0)^\Theta(&\|\Phi\|_E+\|\dot p\|_{L^\infty(-T,T)}\\
&+\|\dot\xi\|_{L^\infty(0,T)}+\|Z_0^*\|_*).
\end{aligned}\tag{11.4}$$

Indeed, let $\psi=\Psi(p,\xi,\Phi,Z_0^*)$. From (2.8) we get, for $j=0$ and $j=2$:

$$|\lambda^2Q_{-\omega}\tilde L_U[\psi]_j|\le C\frac{\lambda_*}{(1+|y|)^3}\|\nabla_x\psi\|_{L^\infty}.$$

We use $\nu_1<1$ (c.f. (11.1)) and $a_1<3$ to estimate for $|y|\le 2R$

$$\frac{\lambda_*}{(1+|y|)^3}\le\frac{\lambda_*^{\nu_1}}{(1+|y|)^{a_1}}\lambda_*(0)^{1-\nu_1}.$$

Then for $|y|\le 2R$ and $j=0,2$:

$$|\lambda^2Q_{-\omega}\tilde L_U[\psi]_j|\le C\frac{\lambda_*^{\nu_1}}{(1+|y|)^{a_1}}\lambda_*(0)^{1-\nu_1}\|\nabla_x\psi\|_{L^\infty}.$$

By the definition of the norm $\|\ \|_{\sharp,\Theta,\gamma}$ (c.f. (5.44)) and Proposition 5.6 we have

$$\begin{aligned}
\|\nabla_x\psi\|_{L^\infty}&\le C\lambda_*(0)^\Theta\|\Psi(p,\xi,\Phi,Z_0^*)\|_{\sharp,\Theta,\gamma}\\
&\le C\lambda_*(0)^\Theta T^\sigma(\|\Phi\|_E+\|\dot p\|_{L^\infty(-T,T)}+\|\dot\xi\|_{L^\infty(0,T)}+\|Z_0^*\|_*).
\end{aligned}$$

Hence for $j=0,2$

$$\begin{aligned}
|\lambda^2Q_{-\omega}\tilde L_U[\psi]_j|\le C\frac{\lambda_*^{\nu_1}}{(1+|y|)^{a_1}}T^\sigma\lambda_*(0)^\Theta(&\|\Phi\|_E+\|\dot p\|_{L^\infty(-T,T)}+\|\dot\xi\|_{L^\infty(0,T)}\\
&+\|Z_0^*\|_*),
\end{aligned}$$

and therefore we see that (11.4) is valid.

Next we claim that

$$\|\lambda^2 Q_{-\omega}\tilde{L}_U[Z^*]_j\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1} \le CT^\sigma\|Z_0\|_*, \tag{11.5}$$

for $j = 0, 2$ and some $\sigma > 0$,

Indeed, we use estimate (14.9) of Lemma 14.2 to obtain for $j = 0, 2$:

$$|\lambda^2 Q_{-\omega}\tilde{L}_U[Z^*]_j\,\chi_{\mathcal{D}_{2R}}| \le C\frac{\lambda_*}{(1+\rho)^3}|\log\varepsilon|\|Z_0\|_*,$$

where $\varepsilon > 0$ satisfies (**??**). Since $\nu_1 < 1$ (11.1), we get

$$\|\lambda^2 Q_{-\omega}\tilde{L}_U[Z^*]_j\,\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1} \le C\lambda_*(0)^{1-\nu_1}|\log\lambda_*(0)|\|Z_0\|_*.$$

This implies (11.5).

Next we estimate $\lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi]$. We claim that

$$\|\lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi]\|_{\nu_1,a_1} \le CT^\sigma\|\dot{p}\|_{L^\infty(-T,T)}. \tag{11.6}$$

Indeed, consider $\mathcal{K}_{01}$ given in (3.17). We have

$$|\lambda^2 Q_{-\omega}\mathcal{K}_{01}[p,\xi]| \le C\frac{\lambda_*}{(1+\rho)^3}\int_{-T}^t |\dot{p}(s)k(z,t-s)|\,ds.$$

But

$$\int_{-T}^t |\dot{p}(s)k(z,t-s)|\,ds \le \int_{-T}^{t-(r^2+\lambda_*(t)^2)} \dots ds + \int_{t-(r^2+\lambda_*(t)^2)}^t \dots ds$$

and

$$\begin{aligned}\int_{-T}^{t-(r^2+\lambda_*(t)^2)} |\dot{p}(s)k(z,t-s)|\,ds &\le C\int_{-T}^{t-(r^2+\lambda_*(t)^2)} \frac{|\dot{p}(s)|}{t-s}\,ds \\ &\le C\|\dot{p}\|_{L^\infty}(|\log(\lambda_*)| + |\log(1+\rho)|)\end{aligned}$$

$$\int_{t-(r^2+\lambda_*(t)^2)}^t |\dot{p}(s)k(z,t-s)|\,ds \le C\|\dot{p}\|_{L^\infty}.$$

This implies that

$$\begin{aligned}\|\lambda^2 Q_{-\omega}\tilde{L}_U[\mathcal{K}_{01}[p,\xi]]\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1} &\le C\lambda_*(0)^{1-\nu_1}\|\dot{p}\|_{L^\infty(-T,T)} \\ &\le CT^\sigma\|\dot{p}\|_{L^\infty(-T,T)},\end{aligned}$$

for some $\sigma > 0$ (by (11.1)). The estimate for $\mathcal{K}_{02}$ is similar, and we obtain (11.6).

Combining (11.4), (11.5), and (11.6) we obtain

$$\|h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_1,a_1} \le CT^\sigma(\|\Phi\|_E + \|\dot{p}\|_{L^\infty(-T,T)} + \|Z_0^*\|_*). \tag{11.7}$$

Then thanks to Proposition 5.1 we get (11.2).

The proof of (11.3) is very similar, using that

$$\begin{aligned}&\|h_1[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)] - h_1[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]\|_{\nu_1,a_1} \\ &\quad\le \|\lambda^2 Q_{-\omega}\tilde{L}_U[\Psi(p,\xi,\Phi_1,Z_0^*) - \Psi(p,\xi,\Phi_2,Z_0^*)]_0\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1} \\ &\qquad+ \|\lambda^2 Q_{-\omega}\tilde{L}_U[\Psi(p,\xi,\Phi_1,Z_0^*) - \Psi(p,\xi,\Phi_2,Z_0^*)]_2\chi_{\mathcal{D}_{2R}}\|_{\nu_1,a_1} \\ &\quad\le CT^\sigma\lambda_*(0)^\Theta\|\Psi(p,\xi,\Phi_1,Z_0^*) - \Psi(p,\xi,\Phi_2,Z_0^*)\|_{\sharp,\Theta,\gamma} \\ &\quad\le CT^\sigma\lambda_*(0)^\Theta\|\Phi_1-\Phi_2\|_E,\end{aligned} \tag{11.8}$$

by Corollary 5.1. This proves (11.3).

Estimate of $\mathcal{F}_2$. Assume that

$$\nu_2 < 1 - \beta(a_2 - 2). \tag{11.9}$$

If $\|\Phi\|_E \le 1$ then

$$\|\mathcal{F}_2(\Phi)\|_{\nu_2,a_2-2} \le CT^\sigma(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*), \tag{11.10}$$

for some $\sigma > 0$, and if $\|\Phi_1\|_E$, $\|\Phi_2\|_E \le 1$ then

$$\|\mathcal{F}_2(\Phi_1) - \mathcal{F}_2(\Phi_2)\|_{\nu_2,a_2-2} \le CT^\sigma \|\Phi_1 + \Phi_2\|_E. \tag{11.11}$$

By Proposition 5.2

$$\|\mathcal{F}_2(\Phi)\|_{\nu_2,a_2-2} \le C\|h_2[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_2,a_2}$$

We estimate from the definition of h_2 in (5.12) and recalling that $\Psi^*(p,\xi,\Phi,Z_0^*) = Z^* + \Psi(p,\xi,\Phi,Z_0^*)$

$$\begin{aligned}\|h_2[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_2,a_2} \le{}& \|\lambda^2 Q_{-\omega}\tilde L_U[\Psi(p,\xi,\Phi,Z_0^*)]_1^{(0)}\|_{\nu_2,a_2}\\ &+ \|\lambda^2 Q_{-\omega}\tilde L_U[Z^*]_1^{(0)}\chi_{\mathcal{D}_{2R}}\|_{\nu_2,a_2}\\ &+ \|\lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]]\|_{\nu_2,a_2}.\end{aligned}$$

We claim that

$$\begin{aligned}\|\lambda^2 Q_{-\omega}\tilde L_U[\Psi(p,\xi,\Phi,Z_0^*)]_1^{(0)}\chi_{\mathcal{D}_{2R}}\|_{\nu_2,a_2} \le{}& CT^\sigma\lambda_*(0)^\Theta(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)}\\ &+ \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*).\end{aligned} \tag{11.12}$$

Indeed, let $\psi = \Psi(p,\xi,\Phi,Z_0^*)$. From (5.10) we get

$$|\lambda^2 Q_{-\omega}\tilde L_U[\psi]_1^{(0)}| \le C\frac{\lambda_*}{(1+|y|)^2}\|\nabla_x\psi\|_{L^\infty}.$$

We use $1-\nu_2-\beta(a_2-2) > 0$ to estimate for $|y| \le 2R$

$$\frac{\lambda_*}{(1+|y|)^2} \le \frac{\lambda_*^{\nu_2}}{(1+|y|)^{a_2}}\lambda_*(0)^{1-\nu_2}R^{a_2-2}.$$

Then for $|y| \le 2R$

$$|\lambda^2 Q_{-\omega}\tilde L_U[\psi]_1^{(0)}| \le C\frac{\lambda_*^{\nu_2}}{(1+|y|)^{a_2}}\lambda_*(0)^{1-\nu_2-\beta(a_2-2)}\|\nabla_x\psi\|_{L^\infty}.$$

By the definition of the norm $\| \ \|_{\sharp,\Theta,\gamma}$ (c.f. (5.44)) and Proposition 5.6 we then obtain

$$\begin{aligned}|\lambda^2 Q_{-\omega}\tilde L_U[\psi]_1^{(0)}| \le{}& C\frac{\lambda_*^{\nu_2}}{(1+|y|)^{a_2}}\lambda_*(0)^{1-\nu_2-\beta(a_2-2)+\Theta}(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)}\\ &+ \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*),\end{aligned}$$

which implies (11.12).

A similar argument, using (14.9) of Lemma 14.2, gives

$$\|\lambda^2 Q_{-\omega}\tilde L_U[Z^*]_1^{(0)}\chi_{\mathcal{D}_{2R}}\|_{\nu_2,a_2} \le CT^\sigma\|Z_0\|_*, \tag{11.13}$$

for some $\sigma > 0$.

Next we estimate $\lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]$ (c.f. (3.19)). We have

$$|\lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]| \le C\frac{\lambda_*}{1+\rho^2}|\dot\xi(t)|$$

and then we see that

$$\|\lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]]\|_{\nu_2,a_2} \le CT^\sigma\|\dot\xi\|_{L^\infty(0,T)}. \tag{11.14}$$

Combining (11.12), (11.13), and (11.14) we obtain

$$\|h_2[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_1,a_1} \le CT^\sigma(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*).$$

Then thanks to Proposition 5.2 we get (11.10).

The proof of (11.11) is very similar, using that

$$\begin{aligned}
&\|h_2[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)] - h_2[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]\|_{\nu_2,a_2}\\
&\quad\le \|\lambda^2 Q_{-\omega}\tilde L_U[\Psi(p,\xi,\Phi_1,Z_0^*) - \Psi(p,\xi,\Phi_2,Z_0^*)]_1^{(0)}\chi_{\mathcal{D}_{2R}}\|_{\nu_2,a_2}\\
&\quad\le CT^\sigma\lambda_*(0)^\Theta\|\Psi(p,\xi,\Phi_1,Z_0^*) - \Psi(p,\xi,\Phi_2,Z_0^*)\|_{\sharp,\Theta,\gamma}\\
&\quad\le CT^\sigma\lambda_*(0)^\Theta\|\Phi_1-\Phi_2\|_E,
\end{aligned}$$

by Corollary 5.1. This proves (11.11).

Estimate of $\mathcal{F}_3$. Assume that

$$\nu_3 < \min\Big(1+\Theta+\sigma_1, 1+\Theta+2\gamma\beta, \nu_1+\frac12\delta\beta(a_1-2)\Big), \tag{11.15}$$

where $\sigma_1>0$ is the constant appearing in (5.36).

We claim that if $\|\Phi\|_E\le 1$ then

$$\|\mathcal{F}_3(\Phi)\|_{**,\nu_3}\le C\lambda_*(0)^\sigma, \tag{11.16}$$

for some $\sigma>0$, and if $\|\Phi_1\|_E, \|\Phi_2\|_E\le 1$ then

$$\|\mathcal{F}_3(\Phi_1)-\mathcal{F}_3(\Phi_3)\|_{**,\nu_3}\le C\lambda_*(0)^\sigma\|\Phi_1-\Phi_2\|_E. \tag{11.17}$$

Let $a\in(1,2)$. By Proposition 5.3

$$\|\mathcal{F}_3(\Phi)\|_{**,\nu_3}\le C\Big\|h_3[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)] + \sum_{j=1}^2 c_{0j}^*[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]w_\rho^2 Z_{0j}\Big\|_{\nu_3,a}.$$

We estimate first $\|h_3[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_3,a}$. and for this we recall (5.13):

$$h_3[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)] = \lambda^2 Q_{-\omega}(\tilde L_U[\Psi^*(p,\xi,\Phi,Z_0^*)]_1 - \tilde L_U[\Psi^*]_1^{(0)})\chi_{\mathcal{D}_{2R}}.$$

Let us write $\Psi=\Psi^*(p,\xi,\Phi,Z_0^*)$ and define

$$\widetilde\Psi(x,t) = D_x\Psi(\xi(t),t)(x-\xi(t)).$$

Then

$$\tilde L_U[\Psi]_1^{(0)} = \tilde L_U[\tilde\Psi]_1.$$

We can then estimate

$$\begin{aligned}
|\tilde L_U[\Psi]_1 - \tilde L_U[\widetilde\Psi]_1| &\le C\lambda_*^{-1}\frac{1}{(1+\rho)^2}|\nabla\Psi(\xi(t)+\lambda(t)y,t) - \nabla\Psi(\xi(t),t)|\\
&\le C\lambda_*^{-1}\frac{1}{(1+\rho)^2}(\lambda(t)|y|)^{2\gamma}\lambda_*^\Theta(\lambda_* R)^{-2\gamma}\|\Psi\|_{\sharp,\Theta,\gamma},
\end{aligned}$$

and this gives

$$|h_3[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]|\le C\lambda_*^{1+\Theta}R^{-2\gamma}\frac{1}{(1+\rho)^{2-2\gamma}}\|\Psi^*(p,\xi,\Phi,Z_0^*)\|_{\sharp,\Theta,\gamma}. \tag{11.18}$$

By choosing

$$\nu_3<1+\Theta+2\gamma\beta$$

and

$$a=2-2\gamma$$

(since $\gamma\in(0,\frac12)$ we have $a\in(1,2)$), we get

$$\|h_3[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]\|_{\nu_3,a}\le CT^\sigma\|\Psi^*(p,\xi,\Phi,Z_0^*)\|_{\sharp,\Theta,\gamma}. \tag{11.19}$$

Next we estimate $c_{0j}^*[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]$ (defined in (5.31)) and for this we use Proposition 5.5 with

$$a(t) = a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)].$$

We need to check that

$$A = a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)](T)$$

satisfies

$$\mathrm{Re}(A) < 0, \quad \frac{1}{C_1} \le |A| \le C_1, \tag{11.20}$$

and estimate $\|a(\cdot)-a(T)\|_{\Theta,l-1}$, $[a]_{\gamma,m,l-1}$.

To do the computations we need a formula, which is proved later in this section.

Lemma 11.1. *Let $\Psi(x) \in \mathbb{R}^3$ be a continuous function in Ω and write*

$$\Psi = \begin{pmatrix} \psi \\ \psi_3 \end{pmatrix}, \quad \psi \in \mathbb{C} = \mathbb{R}^2. \tag{11.21}$$

Then

$$a_0^{(0)}[p,\xi,\Psi](t) = -\frac{1}{4\pi}\int_{B_{2R}} \rho^2 w_\rho^3 \left[\mathrm{div}(\psi^*) + i\,\mathrm{curl}(\psi)\right] dy, \tag{11.22}$$

where $\mathrm{div}(\psi)$ *and* $\mathrm{curl}(\psi)$ *are evaluated at $\xi(t)+\lambda(t)y$.*

Note that $a_0^{(0)}[p,\xi,\Psi]$ is linear in Ψ and recall that $\Psi^*(p,\xi,\Phi,Z_0^*) = \Psi(p,\xi,\Phi,Z_0^*) + Z^{*0} + Z^{*1}$.

Then to see that (11.20) holds, we use (11.22) to get

$$a_0^{(0)}[p,\xi,\Psi](T) = \mathrm{div}(\psi)(\xi(T),T) + i\,\mathrm{curl}(\psi)(\xi(T),T), \tag{11.23}$$

using the same notation (11.21). By Corollary 5.6

$$\begin{aligned}|\nabla_x\Psi(p,\xi,\Phi,Z_0^*)| &\le CT^\sigma\lambda_*(0)^\Theta(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|\dot\xi\|_{L^\infty(0,T)} + \|Z_0^*\|_*)\\ &\le CT^\sigma\end{aligned}$$

and by the hypothesis (5.9) and Lemma 9.4 we have

$$\begin{aligned}|\nabla_x Z^{*0}| &\le \alpha_0\\ |\nabla_x Z^{*1}| &\le CT^\sigma|\log T|.\end{aligned}$$

It follows that

$$a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)](T) = \mathrm{div}(z_0^{*0})(q) + i\,\mathrm{curl}(z_0^{*0})(q) + O(T^{\sigma/2})$$

which thanks to (5.6) imply (11.20) for $T>0$ sufficiently small.

Next we estimate $\|a(\cdot)-a(T)\|_{\Theta,l-1}$, $[a]_{\gamma,m,l-1}$. For this we state here some auxiliary results that we prove later on.

Lemma 11.2. *Suppose that $\|\Psi\|_{\sharp,\Theta,\gamma}<\infty$ and p, ξ satisfy (5.1), (5.2). Then*

$$|a_0^{(0)}[p,\xi,\Psi](t) - a_0^{(0)}[p,\xi,\Psi](T)| \le C\lambda_*(t)^\Theta\|\Psi\|_{\sharp,\Theta,\gamma}, \tag{11.24}$$

and

$$|a_0^{(0)}[p,\xi,\Psi](t) - a_0^{(0)}[p,\xi,\Psi](s)| \le C\lambda_*(t)^\Theta\frac{(t-s)^\gamma}{\lambda_*(t)^{2\gamma}R(t)^{2\gamma}}\|\Psi\|_{\sharp,\Theta,\gamma}, \tag{11.25}$$

for $s\le t$ in $[0,T]$ with $t-s\le\frac{1}{10}(T-t)$.

We decompose

$$a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)] = a_0^{(0)}[p,\xi,\Psi(p,\xi,\Phi,Z_0^*)] + a_0^{(0)}[p,\xi,Z^{*0}] + a_0^{(0)}[p,\xi,Z^{*1}].$$

Using the choice of m (5.34) we see that if $l<1+2\Theta$ then

$$\begin{aligned}&\|a_0^{(0)}[p,\xi,\Psi(p,\xi,\Phi,Z_0^*)](\cdot) - a_0^{(0)}[p,\xi,\Psi(p,\xi,\Phi,Z_0^*)](T)\|_{\Theta,l-1}\\ &\le C|\log T|^{l-1-\Theta}\|\Psi(p,\xi,\Phi,Z_0^*)\|_{\sharp,\Theta,\gamma}.\end{aligned} \tag{11.26}$$

Similarly, if $l < 1 + 2m$ then

$$[a_0^{(0)}[p,\xi,\Psi(p,\xi,\Phi,Z_0^*)]]_{\gamma,m,l-1} \le C|\log T|^{l-1-m}\|\Psi(p,\xi,\Phi,Z_0^*)\|_{\sharp,\Theta,\gamma}. \tag{11.27}$$

On the other hand, by the assumption (5.6) we have

$$\|a_0^{(0)}[p,\xi,Z^{*0}](\cdot) - a_0^{(0)}[p,\xi,Z^{*0}](T)\|_{\Theta,l-1} \le CT^\sigma \tag{11.28}$$

and

$$[a_0^{(0)}[p,\xi,Z^{*0}]]_{\gamma,m,l-1} \le CT^\sigma. \tag{11.29}$$

for some $\sigma > 0$.

Next, using the hypothesis on Z_0^{*1} in (5.8) and the estimates in Lemmas 9.2, 9.3, and 9.4 we get

$$\|a_0^{(0)}[p,\xi,Z^{*1}](\cdot) - a_0^{(0)}[p,\xi,Z^{*1}](T)\|_{\Theta,l-1} \le CT^\sigma \tag{11.30}$$

and

$$[a_0^{(0)}[p,\xi,Z^{*1}]]_{\gamma,m,l-1} \le CT^\sigma. \tag{11.31}$$

for some $\sigma > 0$.

Combining (11.26), (11.27), (11.28), (11.29), (11.30), and (11.31), and using Proposition 5.6 we deduce that

$$\|a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)](\cdot) - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)](T)\|_{\Theta,l-1} \le CT^\sigma \tag{11.32}$$

and

$$[a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]]_{\gamma,m,l-1} \le CT^\sigma \tag{11.33}$$

for some $\sigma > 0$.

Then, applying Proposition 5.5 we get

$$\left|\frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2|Z_{01}|^2} e^{-i\omega}\mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*]\right](t)\right| \le C\lambda_*^{1+\Theta+\sigma_1} \tag{11.34}$$

where $\sigma_1 \in (0,\gamma(\alpha-1+2\beta))$ is the constant in (5.36) and the constant C above depends on the estimates (11.32) and (11.33).

Let us look at the remaining terms in c_{0j}^*. First we note that c_0^* can be rewritten as

$$\begin{aligned} c_0^*[p,\xi,\Psi^*] = \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2|Z_{01}|^2} e^{-i\omega}\Big(\mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*]\right](t) + a_0^{(2)}[p,\xi,\Psi^*](t)\Big) \\ - (c_0[h_1[p,\xi,\Psi^*]] - \tilde c_0[h_1[p,\xi,\Psi^*]]). \end{aligned} \tag{11.35}$$

Indeed, we have

$$h[p,\xi,\Psi^*] - h_1[p,\xi,\Psi^*] = \lambda^2 Q_{-\omega}\tilde L_U[\Psi^*]_1\chi_{\mathcal{D}_{2R}} + \lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi].$$

By definition of c_{0j} (5.14)

$$c_{0j}[\lambda^2 Q_{-\omega}\tilde L_U[\Psi^*]_1\chi_{\mathcal{D}_{2R}}] = \lambda^2\frac{1}{\int_{\mathbb{R}^2} w_\rho^2|Z_{lj}|^2}\int_{B_{2R}} Q_{-\omega}\tilde L_U[\Psi^*]_1\cdot Z_{0j}(y)\,dy.$$

On the other hand, from the definition of $a_0^{(l)}$ (5.30)

$$\begin{aligned} \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2|Z_{0j}|^2} e^{-i\omega}a_0^{(l)}[p,\xi,\Psi] &= -\frac{\lambda^2}{\int_{\mathbb{R}^2} w_\rho^2|Z_0j|^2}\int_{B_{2R}}\Big(Q_{-\omega}\tilde L_U[\Psi]_l\cdot Z_{01} + iQ_{-\omega}\tilde L_U[\Psi]_l\cdot Z_{02}\Big)\,dy \\ &= -c_0[\lambda^2 Q_{-\omega}\tilde L_U[\Psi]_l\chi_{\mathcal{D}_{2R}}]. \end{aligned}$$

Since

$$c_0[\lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]] = 0,$$

by orthogonality, we deduce that

$$c_0[h[p,\xi,\Psi^*]] - h_1[p,\xi,\Psi^*]] = -\frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} e^{-i\omega} a_0^{(1)}[p,\xi,\Psi].$$

From here we deduce formula (11.35).

Using Lemma 11.3 we have

$$\left| \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{01}|^2} e^{-i\omega} a_0^{(2)}[p,\xi,\Psi^*](t) \right| \le C\lambda_*^{1+\Theta+2\gamma\beta} \|\Psi^*\|_{\sharp,\Theta,\gamma} \tag{11.36}$$

Using estimate (5.18) of Proposition 5.1 we find that

$$|c_0[h_1[p,\xi,\Psi^*]] - \tilde c_0[h_1[p,\xi,\Psi^*]]| \le C\lambda_*^{\nu_1} R^{-\frac12\delta(a_1-2)} \|h_1\|_{\nu_1,a_1},$$

and using (11.7) we get

$$|c_0[h_1[p,\xi,\Psi^*]] - \tilde c_0[h_1[p,\xi,\Psi^*]]| \le CT^\sigma \lambda_*^{\nu_1} R^{-\frac12\delta(a_1-2)} (\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|Z_0^*\|_*). \tag{11.37}$$

Assuming (11.15) and using (11.34), (11.36), and (11.37) we deduce that

$$|c_0^*[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)](t)| \le CT^\sigma \lambda_*^{\nu_3},$$

This combined with (11.19) implies (11.16).

We prove now the Lipschitz estimate (11.11). Let us write

$$\Phi_i = (\phi_{i1},\phi_{i2},\phi_{i3},\phi_{i4}), \quad \phi_i = \phi_{i1}+\phi_{i2}+\phi_{i3}+\phi_{i4}$$

and recall that

$$\mathcal{F}_3(\Phi) = \mathcal{T}_{\lambda,3}\Big(h_3[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)] + \sum_{j=1}^2 c_{0j}^*[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)] w_\rho^2 Z_{0j} \Big)$$

Then by Proposition 5.3, and taking $a \in (1,2)$ we get

$$\begin{aligned} &\|\mathcal{F}_3(\Phi_1) - \mathcal{F}_3(\Phi_2)\|_{**,\nu_3} \\ &\le C\|h_3[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)] - h_3[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]\|_{\nu_3,a} \\ &\quad + \sum_{j=1,2} \|(c_{0j}^*[p,\xi,\Psi(p,\xi,\Phi_1,Z_0^*)+Z^*] - c_{0j}^*[p,\xi,\Psi(p,\xi,\Phi_2,Z_0^*)+Z^*]) w_\rho^2 Z_{0j}\|_{\nu_3,a}. \end{aligned} \tag{11.38}$$

We claim that

$$\|h_3[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)] - h_3[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]\|_{\nu_3,a} \le C\lambda_*(0)^\sigma \|\Phi_1-\Phi_2\|_E \tag{11.39}$$

and

$$\begin{aligned} &\|(c_{0j}^*[p,\xi,\Psi(p,\xi,\Phi_1,Z_0^*)+Z^*] - c_{0j}^*[p,\xi,\Psi(p,\xi,\Phi_2,Z_0^*)+Z^*]) w_\rho^2 Z_{0j}\|_{\nu_3,a} \\ &\quad \le C\lambda_*(0)^\sigma \|\Phi_1-\Phi_2\|_E \end{aligned} \tag{11.40}$$

for some $\sigma > 0$. In (11.39) $a = 2-2\gamma$.

Let us write

$$\Psi_i = \Psi^*(p,\xi,\Phi_i,Z_0^*)$$

and define

$$\widetilde\Psi_i(x,t) = D_x\Psi_i(\xi(t),t)(x-\xi(t)).$$

Then

$$\tilde L_U[\Psi_i]_1^{(0)} = \tilde L_U[\tilde\Psi_i]_1.$$

We estimate as in (11.18) noting that h_3 (c.f. (5.13)) is linear in Ψ:

$$
\begin{aligned}
&|h_3[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)] - h_3[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]| \\
&\le C\lambda_*^{1+\Theta} R^{-2\gamma} \frac{1}{(1+|y|)^{2-2\gamma}} \|\Psi^*(p,\xi,\Phi_1,Z_0^*)] - \Psi^*(p,\xi,\Phi_2,Z_0^*)]\|_{\sharp,\Theta,\gamma}.
\end{aligned}
$$

Then from Corollary 5.1 and using that $\nu_3 < 1+\Theta+2\gamma\beta$, $a = 2-2\gamma$ we get (11.39).

To prove (11.40) we obtain the Lipschitz property for each term in the expression of c_0^* given in (11.35). Using the Lipschitz property of $\mathcal{R}_0$ (12.2), we have

$$
\begin{aligned}
&\left|\mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)]\right](t) - \mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]\right](t)\right| \\
&\quad \le C\lambda_*^{\Theta+\sigma_1}\Big([a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*) - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]_{\gamma,m,l-1} \\
&\qquad + T^{\Theta-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)](\cdot) - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)](T) \\
&\qquad - (a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)](\cdot) - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)](T))\|_{\Theta,l-1} \\
&\qquad + C\frac{T^{\Theta-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\Theta,l-1}|a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)(T) - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)(T)|\Big).
\end{aligned}
\tag{11.41}
$$

As in (11.26) from (11.24), and using that $l < 1+2\Theta$, we get

$$
\begin{aligned}
&\|a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)](\cdot) - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)](T) \\
&\qquad - (a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)](\cdot) - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)](T))\|_{\Theta,l-1} \\
&\le C|\log T|^{l-1-\Theta}\|\Psi^*(p,\xi,\Phi_1,Z_0^*) - \Psi^*(p,\xi,\Phi_2,Z_0^*)\|_{\sharp,\Theta,\gamma}
\end{aligned}
$$

and by Corollary 5.1

$$
\begin{aligned}
&\|a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)](\cdot) - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)](T) \\
&\qquad - (a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)](\cdot) - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)](T))\|_{\Theta,l-1} \\
&\le C|\log T|^{l-1-\Theta}T^\sigma\|\Phi_1-\Phi_2\|_E.
\end{aligned}
\tag{11.42}
$$

Similarly, if $l < 1+2m$, from (11.25) we have

$$
\begin{aligned}
&[a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)] - a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]]_{\gamma,m,l-1} \\
&\le C|\log T|^{l-1-m}\|\Psi^*(p,\xi,\Phi_1,Z_0^*) - \Psi^*(p,\xi,\Phi_2,Z_0^*)\|_{\sharp,\Theta,\gamma} \\
&\le C|\log T|^{l-1-m}T^\sigma\|\Phi_1-\Phi_2\|_E.
\end{aligned}
\tag{11.43}
$$

Combining (11.41), (11.42) and (11.43) we obtain

$$
\begin{aligned}
&\left|\mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)]\right](t) - \mathcal{R}_0\left[a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]\right](t)\right| \\
&\le CT^\sigma\lambda_*^{\Theta+\sigma_1}\|\Phi_1-\Phi_2\|_E,
\end{aligned}
\tag{11.44}
$$

(for possibly a smaller $\sigma > 0$).

To control the term involving $a_0^{(2)}[p,\xi,\Psi^*]$ we use the following estimate, whose proof is given at the end of the section.

Lemma 11.3. *Suppose that $\|\Psi\|_{\sharp,\Theta,\gamma} < \infty$ and p, ξ satisfy (5.1), (5.2). Then for $l = 1,2$ we have*

$$
|a_0^{(l)}[p,\xi,\Psi](t)| \le C\lambda_*^{\Theta+2\gamma\beta}\|\Psi\|_{\sharp,\Theta,\gamma}.
\tag{11.45}
$$

Using (11.45) and Corollary 5.1 to estimate

$$\begin{aligned}|a_0^{(2)}&[p,\xi,\Psi(p,\xi,\Phi_1,Z_0^*)+Z^*](t)-a_0^{(2)}[p,\xi,\Psi(p,\xi,\Phi_2,Z_0^*)+Z^*](t)|\\&\leq C\lambda_*^{\Theta+2\gamma\beta}\|\Psi(p,\xi,\Phi_1,Z_0^*)-\Psi(p,\xi,\Phi_2,Z_0^*)\|_{\sharp,\Theta,\gamma}\\&\leq C\lambda_*^{\Theta+2\gamma\beta}\|\Phi_1-\Phi_2\|_E.\end{aligned}\tag{11.46}$$

To estimate $c_0[h_1[p,\xi,\Psi^*]]-\tilde c_0[h_1[p,\xi,\Psi^*]]$ we use estimate (5.18) and (11.8) to obtain

$$\begin{aligned}\Big|&c_0[h_1[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)]]-\tilde c_0[h_1[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)]]\\&\quad-(c_0[h_1[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]]-\tilde c_0[h_1[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)]])\Big|\\&\leq C\lambda_*^{\nu_1}R^{-\frac12\delta(a_1-2)}\|h_1[p,\xi,\Psi^*(p,\xi,\Phi_1,Z_0^*)]-h_1[p,\xi,\Psi^*(p,\xi,\Phi_2,Z_0^*)]\|_{\nu_1,a_1}\\&\leq CT^\sigma\lambda_*(0)^\Theta\lambda_*^{\nu_1}R^{-\frac12\delta(a_1-2)}\|\Phi_1-\Phi_2\|_E.\end{aligned}\tag{11.47}$$

From estimates (11.44), (11.46) and (11.47) and the condition (11.15) we deduce the validity of (11.40).

Estimate of $\mathcal F_4$. Assume

$$\nu_4<1.\tag{11.48}$$

We claim that if $\|\Phi\|_E\leq 1$ then

$$\|\mathcal F_4(\Phi)\|_{***,\nu_4}\leq C\lambda_*(0)^\sigma\tag{11.49}$$

and if $\|\Phi_1\|_E,\|\Phi_2\|_E\leq 1$ then

$$\|\mathcal F_4(\Phi_1)-\mathcal F_4(\Phi_2)\|_{***,\nu_4}\leq C\lambda_*(0)^\sigma\|\Phi_1-\Phi_2\|_E,\tag{11.50}$$

for some $\sigma>0$.

Indeed, by Proposition 5.4

$$\|\mathcal F_4(\Phi)\|_{***,\nu_4}\leq C\sum_{j=1,2}\|c_{-1,j}[h_1[p,\xi,\Psi^*(p,\xi,\Phi,Z_0^*)]]w_\rho^2Z_{-1,j}\|_{\nu_4,a},$$

where we have fixed any $a\in(2,3)$.

Let $\Psi^*=\Psi(p,\xi,\Phi,Z_0^*)+Z^*$. Recalling the definition of h_1 (5.11), and of $c_{-1,j}$ (5.14),

$$\begin{aligned}|c_{-1,j}[h_1[p,\xi,\Psi(p,\xi,\Phi,Z_0^*)+Z^*]]\,|&\leq C\lambda_*^2|c_{-1,j}[\tilde L_U[\Psi^*]_1\,\chi_{\mathcal D_{2R}}]|\\&\quad+C\lambda_*^2|c_{-1,j}[\tilde L_U[\Psi^*]_2\,\chi_{\mathcal D_{2R}}]|\\&\quad+C\lambda_*^2|c_{-1,j}[\mathcal K_0]|.\end{aligned}$$

But from (2.5)

$$|c_{-1,j}[\tilde L_U[\Psi^*]\chi_{\mathcal D_{2R}}]|\leq C\lambda_*\log(R)\|\nabla_x\Psi^*\|_{L^\infty(\Omega\times(0,T))}.\tag{11.51}$$

To estimate the term involving $\mathcal K_0$ (c.f. (3.17), (3.18)) let us rewrite it as

$$\mathcal K_0[p,\xi]=-\frac1\lambda\rho w_\rho^2I_{01}[p]Q_\omega E_1-\frac1\lambda w_\rho^2I_{02}[p]Q_\omega E_2$$

where

$$\begin{aligned}I_{01}[p](r,t)&=\int_{-T}^t\mathrm{Re}(\dot p(s)e^{-i\omega(t)})(2k+rk_z+\frac14\cos wzk_z-\frac14\cos wz^2k_{zz}w)\,ds\\&\quad-\dot\lambda(t)\\I_{02}[p](r,t)&=\int_{-T}^t\mathrm{Im}(\dot p(s)e^{-i\omega(t)})(2k+\frac14zk_z-\frac14z^2k_{zz})\,ds,\end{aligned}$$

k and its derivatives are evaluated at $(z,t-s)$, $z=\sqrt{r^2+\lambda(t)^2}$ and $k(z,t)=2\frac{1-e^{-\frac{z^2}{4t}}}{z^2}$, and $r=|x-\xi(t)|$.

We claim that if p satisfies (5.1), then

$$|I_{0j}[p](r,t)| \le C, \quad j=1,2. \tag{11.52}$$

Indeed, let us compute the first term, the rest being similar. We have

$$\begin{aligned}\int_{-T}^{t} |\dot p(s) k(z,t-s)|\,ds \le C \int_{-T}^{t-z^2} |\dot\lambda_*(s) k(z,t-s)|\,ds \\ + C\int_{t-z^2}^{t} |\dot\lambda_*(s) k(z,t-s)|\,ds.\end{aligned}$$

For the first term we have

$$\int_{-T}^{t-z^2} |\dot\lambda_*(s) k(z,t-s)|\,ds \le C. \tag{11.53}$$

Indeed, assuming $T-t\le z^2$ we have

$$\begin{aligned}\int_{-T}^{t-z^2} |\dot\lambda_*(s) k(z,t-s)|\,ds &\le C \int_{-T}^{t-z^2} \frac{|\dot\lambda_*(s)|}{t-s}\,ds \\ &= C\int_{-T}^{t-(T-t)} \ldots + C\int_{t-(T-t)}^{t-z^2} \ldots.\end{aligned}$$

Then

$$\begin{aligned}\int_{-T}^{t-(T-t)} \frac{|\dot\lambda_*(s)|}{t-s}\,ds &\le C|\log T| \int_{-T}^{t-(T-t)} \frac{1}{(T-s)|\log(T-s)|^2}\,ds \\ &\le C,\end{aligned}$$

and

$$\begin{aligned}\int_{t-(T-t)}^{t-z^2} \frac{|\dot\lambda_*(s)|}{t-s}\,ds &\le \frac{C|\log T|}{|\log(T-t)|^2}\int_{t-(T-t)}^{t-z^2} \frac{1}{t-s}\,ds \\ &\le C|\dot\lambda_*(t)||\log z|.\end{aligned}$$

Assuming that $T-t\ge z^2$ we have

$$\int_{-T}^{t-z^2} \frac{|\dot\lambda_*(s)|}{t-s}\,ds \le C.$$

Now observe that

$$\begin{aligned}|\dot\lambda_*(t)|\,|\log z|\,\chi_{\mathcal{D}_{2R}} &= |\dot\lambda_*(t)|\,|\log(r^2+\lambda_*(t)^2)|\,\chi_{\mathcal{D}_{2R}} \\ &\le C|\dot\lambda_*(t)|\,|\log\lambda_*(t)|\,\chi_{\mathcal{D}_{2R}} \le C,\end{aligned}$$

and this proves (11.53).

Finally

$$\begin{aligned}\int_{t-z^2}^{t} |\dot\lambda_*(s) k(z,t-s)|\,ds &\le \frac{C}{z^2}\int_{t-z^2}^{t} |\dot\lambda_*(s)|\,ds \\ &\le C.\end{aligned}$$

Combining this with (11.53) we obtain (11.52).

Using (11.52) we find that

$$\lambda_*^2 |c_{-1,j}[\mathcal{K}_0 \chi_{\mathcal{D}_{2R}}]| \le C\lambda_*. \tag{11.54}$$

From (11.51), (11.54) we obtain

$$|c_{-1,j}[h_1[p,\xi,\Psi(p,\xi,\Phi,Z_0^*)+Z^*]]\,| \le C\lambda_* \log(R)\|\nabla_x \Psi^*\|_{L^\infty(\Omega\times(0,T))} + C\lambda_*. \tag{11.55}$$

But

$$\|\nabla_x \Psi^*\|_{L^\infty(\Omega\times(0,T))} \le \lambda_*(0)^\Theta \|\Psi(p,\xi,\Phi,Z_0^*)\|_{\sharp,\Theta,\gamma} + \|\nabla_x Z^*\|_{L^\infty(\Omega\times(0,T))}$$

and by Proposition 5.6

$$\begin{aligned} \|\nabla_x \Psi^*\|_{L^\infty(\Omega\times(0,T))} &\le C\lambda_*(0)^\Theta T^\sigma(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|Z_0^*\|_*) \\ &\quad + \|\nabla_x Z^*\|_{L^\infty(\Omega\times(0,T))}. \end{aligned} \tag{11.56}$$

Since Z^* solves the heat equation in Ω with initial condition Z_0^* and Z_0^* satisfies (5.9), we have by Lemma 9.4

$$\|\nabla_x Z^*\|_{L^\infty(\Omega\times(0,T))} \le C\alpha_0 + C|\log T|\|Z_0^{*1}\|_*. \tag{11.57}$$

Combining (11.55), (11.56), (11.57)we find

$$\begin{aligned} &|c_{-1,j}[h_1[p,\xi,\Psi(p,\xi,\Phi,Z_0^*) + Z^*]]\,| \\ &\le C\lambda_* \log(R)\lambda_*(0)^\Theta T^\sigma(\|\Phi\|_E + \|\dot p\|_{L^\infty(-T,T)} + \|Z_0^*\|_*) \\ &\quad + C\lambda_* \log(R)\alpha_0 + C\lambda_* \log(R)|\log T|\,\|Z_0^{*1}\|_* + C\lambda_* \end{aligned}$$

Since $\nu_4 < 1$ we obtain (11.49).

The proof of (11.50) is similar.

With the previous estimates we can give now the proof of Proposition 5.7.

Proof of Proposition 5.7. From (11.2), (11.10), (11.16), (11.49) we see that for $T > 0$ small, $\mathcal{F}$ maps $\overline{\mathcal{B}}_1$ of E into itself. Estimates (11.3), (11.11), (11.17), (11.50) show that $\mathcal{F}$ is a contraction. □

Proof of Lemma 11.1. Let us use the notation (11.21). From (2.7) we see that

$$\begin{aligned} &e^{i\omega}\left[Q_{-\omega}\tilde L_U[\Psi]_0\cdot Z_{01} + iQ_{-\omega}\tilde L_U[\Psi]_0\cdot Z_{02}\right] \\ &= \lambda^{-1}\rho^2 w_\rho^3 e^{i\omega}\left[\operatorname{div}(e^{-i\omega}\psi) + i\operatorname{curl}(e^{-i\omega}\psi)\right] \\ &= \lambda^{-1}\rho^2 w_\rho^3\left[\operatorname{div}(\psi) + i\operatorname{curl}(\psi)\right]. \end{aligned}$$

Hence from the definition (5.30)

$$a_0^{(0)}[p,\xi,\Psi](t) = -\frac{1}{4\pi}\int_{B_{2R}} \rho^2 w_\rho^3\left[\operatorname{div}(\psi) + i\operatorname{curl}(\psi)\right]\,dy \tag{11.58}$$

where ψ is evaluated at (x,t) with $x = \xi(t) + \lambda(t)y$. □

Proof of Lemma 11.2. From formula (11.58) we have

$$\begin{aligned} &|a_0^{(0)}[p,\xi,\Psi](t) - a_0^{(0)}[p,\xi,\Psi](T)| \\ &\le C\int_{B_{2R}} \frac{1}{(1+|y|)^3}|\nabla_x\psi(\xi(T),T) - \nabla_x\psi(\xi(t)+\lambda(t)y,t)|\,dy. \end{aligned}$$

By definition of the norm $\|\ \|_{\sharp,\Theta,\gamma}$, (5.44), we have

$$\begin{aligned} &|\nabla_x\psi(\xi(T),T) - \nabla_x\psi(\xi(t)+\lambda(t)y,t)| \\ &\le |\nabla_x\psi(\xi(T),T) - \nabla_x\psi(\xi(T),t)| + |\nabla_x\psi(\xi(T),t) - \nabla_x\psi(\xi(t)+\lambda(t)y,t)| \\ &\le \|\psi\|_{\sharp,\Theta,\gamma}\big(\lambda_*^\Theta + (\lambda\rho + |\xi(T)-\xi(t)|)^{2\gamma}\lambda_*^\Theta(\lambda_* R)^{-2\gamma}\big) \\ &\le \|\psi\|_{\sharp,\Theta,\gamma}\big(\lambda_*^\Theta + (\rho+1)^{2\gamma}\lambda_*^\Theta R^{-2\gamma}\big). \end{aligned}$$

Therefore

$$\begin{aligned} |a_0^{(0)}[p,\xi,\Psi](t) - a_0^{(0)}[p,\xi,\Psi](T)| &\le \big(\lambda_*^\Theta + \lambda_*^\Theta R^{-2\gamma}\big)\|\Psi\|_{\sharp,\Theta,\gamma} \\ &\le C\lambda_*^\Theta\|\Psi\|_{\sharp,\Theta,\gamma}. \end{aligned}$$

From here we get (11.24).

Let us prove (11.25). Let $s \le t$ be in $[0,T]$ with $t-s \le \frac{1}{10}(T-t)$. Using (11.58) we see that

$$
\begin{aligned}
&|a_0^{(0)}[p,\xi,\Psi](t) - a_0^{(0)}[p,\xi,\Psi](s)| \\
&\le C \int_{B_{2R}} \frac{1}{(1+|y|)^3} |\nabla_x \psi(\xi(t)+\lambda(t)y,t) - \nabla_x \psi(\xi(s)+\lambda(s)y,s)|\, dy
\end{aligned}
$$

We estimate

$$
\begin{aligned}
&|\nabla_x \psi(\xi(t)+\lambda(t)y,t) - \nabla_x \psi(\xi(s)+\lambda(s)y,s)| \\
&\le |\nabla_x \psi(\xi(t)+\lambda(t)y,t) - \nabla_x \psi(\xi(s)+\lambda(s)y,t)| \\
&\quad + \nabla_x \psi(\xi(s)+\lambda(s)y,t) - \nabla_x \psi(\xi(s)+\lambda(s)y,s)| \\
&\le \left[(|\xi(t)-\xi(s)| + |\lambda(t)-\lambda(s)|\,|y|)^{2\gamma} + (t-s)^\gamma \right] \lambda_*(t)^\Theta (\lambda_*(t)R(t))^{-2\gamma} \|\Psi\|_{\sharp,\Theta,\gamma} \\
&\le \left[(\|\dot\xi\|_{L^\infty} + \|\dot\lambda\|_{L^\infty}|y|)^{2\gamma}(t-s)^{2\gamma} + (t-s)^\gamma \right] \lambda_*(t)^\Theta (\lambda_*(t)R(t))^{-2\gamma} \|\Psi\|_{\sharp,\Theta,\gamma}.
\end{aligned}
$$

Using (5.1), (5.2) and integrating we find (11.25).

□

Proof of Lemma 11.3. To estimate $a_0^{(l)}[p,\xi,\Psi](t)$ for $l=1,2$ we freeze the function $\nabla_x\Psi(x,t)$ at the point $(\xi(t),t)$ and then notice that if $\nabla_x\Psi$ were a constant in space, then

$$
\int_{B_{2R}} Q_{-\omega}\tilde L_U[\Psi]_l \cdot Z_{0j}\, dy = 0, \quad l=1,2,\ j=1,2,
$$

because the function $Q_{-\omega}\tilde L_U[\Psi]_l$ is in Fourier mode l. This allows us to write

$$
\begin{aligned}
a_0^{(l)}[p,\xi,\Psi](t) = -\frac{\lambda}{4\pi} e^{i\omega} \int_{B_{2R}} \Big(& Q_{-\omega}\big(\tilde L_U[\Psi]_l - \tilde L_U[\widetilde\Psi]_l\big)\cdot Z_{01} \\
& + i Q_{-\omega}\big(\tilde L_U[\Psi]_l - \tilde L_U[\widetilde\Psi]_l\big)\cdot Z_{02}\Big)\, dy
\end{aligned}
$$

where

$$
\widetilde\Psi(x,t) = D_x\Psi(\xi(t),t)(x-\xi(t)).
$$

But

$$
\begin{aligned}
|\tilde L_U[\Psi]_l - \tilde L_U[\widetilde\Psi]_l| &\le C\lambda_*^{-1}\frac{1}{(1+\rho)^2}|\nabla\Psi(\xi(t)+\lambda(t)y,t) - \nabla\Psi(\xi(t),t)| \\
&\le C\lambda_*^{-1}\frac{1}{(1+\rho)^2}(\lambda(t)|y|)^{2\gamma}\lambda_*^\Theta(\lambda_* R)^{-2\gamma}\|\Psi\|_{\sharp,\Theta,\gamma},
\end{aligned}
$$

which implies

$$
|a_0^{(l)}[p,\xi,\Psi](t)| \le C\lambda_*^{\Theta+2\gamma\beta}\|\Psi\|_{\sharp,\Theta,\gamma}
$$

□

12. Adjusting the parameters

In this section we prove that the last equations of the gluing system (5.37)–(5.43) can be solved, by adjusting the parameter functions $p=\lambda e^{i\omega}$ and ξ, as stated in Proposition 5.9, thus concluding the proof of Theorem 1.

We recall from Section 5 that (12.3) is equivalent to

$$
\mathcal B_0[p] = a_0^{(0)}[p,\xi,\Psi^*] + \mathcal R_0\left[a_0^{(0)}[p,\xi,\Psi^*]\right], \quad t\in[0,T] \tag{12.1}
$$

where $\Psi^* = \Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)$. We recall that $\mathcal B_0$ is the integral operator defined in (4.6) which has the approximate form

$$
\mathcal B_0[p] = \int_{-T}^{t-\lambda^2} \frac{\dot p(s)}{t-s}ds + O\big(\|\dot p\|_\infty\big).
$$

In Proposition 5.5 we constructed an approximate inverse $\mathcal{P}$ of the operator $\mathcal{B}_0$, so that given a satisfying (5.25), $p := \mathcal{P}[a]$, satisfies the equation

$$\mathcal{B}_0[p] = a + \mathcal{R}_0[a], \quad \text{in } [0,T],$$

for a small remainder $\mathcal{R}_0[a]$. The proof of that proposition gives the decomposition

$$\mathcal{P}[a] = p_{0,\kappa} + \mathcal{P}_1[a] + \mathcal{P}_2[a],$$

where $p_{0,\kappa}$ is defined in (13.2) and $\kappa = \kappa[a] \in \mathbb{C}$. The next result gives additional properties of the operators $\kappa[a]$, $\mathcal{P}_1[a]$, $\mathcal{P}_2[a]$.

Proposition 12.1. *Let us make the same assumptions as in Proposition 5.5. Let*

$$\kappa = \kappa[a], \quad p_1 = \mathcal{P}_1[a], \quad p_2 = \mathcal{P}_2[a].$$

Then

$$\begin{aligned}
\kappa[a] &= a(T)\Big(1 + O\Big(\frac{1}{|\log T|}\Big)\Big),\\
|\dot p_1(t) - \dot p_{0,\kappa}(t)| &\le C\frac{|\log T|^{1-\sigma}\log(|\log T|)^2}{|\log(T-t)|^{3-\sigma}},\\
|\ddot p_1(t)| &\le C\frac{|\log T|}{|\log(T-t)|^3(T-t)},
\end{aligned}$$

$$\begin{aligned}
\|\dot p_2\|_{\Theta,l} &\le C\big(T^{\frac12+\sigma-\Theta} + \|a(\cdot)-a(T)\|_{\Theta,l-1}\big),\\
[\dot p_2]_{\gamma,m,l} &\le C(|\log T|^{l-3}T^{\alpha_0-m-\gamma} + T^{\Theta}\frac{\log|\log T|}{|\log T|}\|a(\cdot)-a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}),
\end{aligned}$$

where $\alpha_0 > 0$ is some fixed some constant and $\sigma > 0$ is arbitrary (with C depending on σ), and

$$\begin{aligned}
&|\mathcal{R}_0[a_1](t) - \mathcal{R}_0[a_2]|\\
&\quad\le C\frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}\cdot\Big([a_1-a_2]_{\gamma,m,l-1}\\
&\qquad + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1(\cdot)-a_1(T)-(a_2(\cdot)-a_2(T))\|_{\mu,l-1}\\
&\qquad + C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}|a_1(T)-a_2(T)|\Big),
\end{aligned} \tag{12.2}$$

for a_1, a_2 satisfying the assumptions of Proposition 5.5.

Proof of Proposition 5.9. Let $\Psi(p,\xi,\Phi,Z_0^*)$ be the solution to equation (5.37) constructed in Proposition 5.6. Let $\Phi(p,\xi,Z_0^*)$ denote the solution of (5.46) constructed in Proposition 5.7. In (5.42)-(5.43) we replace Ψ^* by $\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)$. Then to find a solution of the full system (5.37)–(5.43) it is sufficient to find p, ξ such that

$$c_{0j}[h(p,\xi,\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*))](t) - c_{0j}^*[p,\xi,\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)](t) = 0 \tag{12.3}$$

$$c_{1j}[h(p,\xi,\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*))](t) = 0 \tag{12.4}$$

for all $t \in (0,T)$, $j = 1,2$.

Then it is natural to define the space $X_1 := \mathbb{C} \times \tilde X_1$ where

$$\tilde X_1 := \{p_1 \in C([-T,T;\mathbb{C}]) \cap C^1([-T,T;\mathbb{C}]) \mid p_1(T) = 0,\ \|p_1\|_{*,3-\sigma} < \infty\}.$$

Let us rewrite equation (12.4) as follows. By (5.14), (12.4) is equivalent to

$$\int_{\mathbb{R}^2} h[p,\xi,\Psi^*]\cdot Z_{1j}(y)\,dy = 0, \quad t \in (0,T),\ j = 1,2,$$

and recalling (4.1), this is equivalent to

$$\lambda \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*] \cdot Z_{1j} + \lambda \int_{B_{2R}} \mathcal{K}_1[p,\xi] \cdot Z_{1j} = 0,$$

which yields the following equation

$$\dot{\xi}_j = \frac{1}{4\pi}(1+(2R)^{-2}) \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*] \cdot Z_{1j}, \quad j=1,2. \tag{12.5}$$

We reformulate (12.1)-(12.5) as the fixed point problem

$$[p,\xi] = \mathcal{A}[p,\xi] \quad \text{in } \mathcal{B} \tag{12.6}$$

where the space $\mathcal{B}$ will be introduced below and the operator $\mathcal{A} = [\mathcal{A}_1, \mathcal{A}_2]$ is defined by

$$\mathcal{A}_1[p,\xi] = \mathcal{P}\left[a_0^{(0)}[p,\xi,\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)]\right]$$
$$\mathcal{A}_2[p,\xi] = q - \int_t^T b[p,\xi](s)\,ds$$

with

$$b_{1j}[p,\xi](t) = \frac{1}{4\pi}(1+(2R)^{-2}) \int_{B_{2R}} Q_{-\omega} \tilde{L}_U[\Psi^*(p,\xi,\Phi(p,\xi,Z_0^*),Z_0^*)] \cdot Z_{1j}.$$

To define $\mathcal{B}$ consider the closed ball

$$\mathcal{B}_1 = \overline{B}_{l_1}(\kappa_0) \times \overline{B}_{l_2}(0) \subset X_1,$$

where $\kappa_0 = \operatorname{div} z_0^{*0}(q) + i \operatorname{curl} z_0^{*0}(q)$ with z_0^{*0} so that

$$Z_0^{*0}(x) = \begin{bmatrix} z_0^{*0}(x) \\ z_{03}^{*0}(x) \end{bmatrix}, \quad z_0^{*0}(x) = z_{01}^{*0}(x) + i z_{02}^{*0}(x),$$

and $Z_0^* = Z_0^{*0} + Z_0^{*1}$ is the initial condition as described in (5.9). Here the numbers l_1, l_2 are given by

$$l_1 = T^\sigma, \quad l_2 = C_0 |\log T|^{1-\sigma} \log^2(|\log T|),$$

with $\sigma > 0$ small and and $C_0 > 0$ is a fixed large constant. We consider ξ in the space

$$X_2 = \{\xi \in C^1([0,T];\mathbb{R}^2) \ : \ \dot{\xi}(T) = 0\}$$

endowed with the norm

$$\|\xi\|_{X_2} = \|\xi\|_{L^\infty(0,T)} + \sup_{t\in(0,T)} \lambda_*(t)^{-\sigma} |\dot{\xi}(t)|$$

where $\sigma \in (0,1)$ is fixed. In X_2 we consider the closed ball $\mathcal{B}_2 := \overline{B}_1(\xi^*)$, where $\xi^* \equiv q \in \Omega$. We consider the Banach space $X := X_1 \times X_2$ and its closed ball $\mathcal{B} := \mathcal{B}_1 \times \mathcal{B}_2$. We formulate the fixed point problem (12.6) in $\mathcal{B}$. We claim that $\mathcal{A}(\mathcal{B}) \subset \mathcal{B}$ and that $\mathcal{A}$ is a contraction mapping on $\mathcal{B}$ for the norm $\| \ \|_X$. This is consequence of the various bounds and Lipschitz estimates derived in §13 for the operator $\mathcal{P}$ and in §5 for the operators Ψ^* and Φ.

□

13. The λ-ω system

In this section we prove Proposition 5.5, on approximate solvability of the equation

$$\mathcal{B}_0[p](t) = a(t), \quad t \in [0,T),$$

where $\mathcal{B}_0$ is the operator defined in (4.6) and $a : [0,T] \to \mathbb{C}$ is a given continuous function.

Consistently with the discussion in section 4, we assume that $\mathrm{Re}(a(T)) < 0$. We will construct an operator $\mathcal{P}$ that to a function a in a suitable class assigns $p = \mathcal{P}[a]$ such that

$$\mathcal{B}_0[p](t) = a(t) + \mathcal{R}_0[a](t), \quad \text{in } [0,T). \tag{13.1}$$

so that $\mathcal{R}_0[a](t)$ is a suitably small.

13.1. **Preliminaries.** We construct the function p in Proposition 5.5 by linearization, and the first approximation is a function p_κ that deals with the case of constant a.

First we introduce some notation. We work with $\kappa \in \mathbb{C}$ and let $p_{0,\kappa}$ be the function

$$p_{0,\kappa}(t) = \kappa|\log T| \int_t^T \frac{1}{|\log(T-s)|^2}\, ds, \quad t \leq T, \tag{13.2}$$

so that

$$\dot{p}_{0,\kappa}(t) = -\frac{\kappa|\log T|}{|\log(T-t)|^2}. \tag{13.3}$$

We will always consider

$$\frac{1}{C_1} \leq |\kappa| \leq C_1 \tag{13.4}$$

where $C_1 > 1$ is a large fixed constant and therefore we have

$$\frac{1}{\tilde{C}_1}\lambda_* \leq |p_{0,\kappa}| \leq \tilde{C}_1\lambda_*,$$

with $\tilde{C}_1 > 0$.

The first term in the function p constructed in Proposition 5.5 is a function close to $p_{0,\kappa}$ that actually more or less solves (13.1) in the case that a is constant.

Lemma 13.1. *Given $\kappa \in \mathbb{C}$ satisfying (13.4), there is a function $p_\kappa : [-T,T] \to \mathbb{C}$, a constant $c(\kappa) \in \mathbb{C}$, and $\mathcal{R}_1(\kappa)(t)$ such that*

$$\mathcal{B}_0[p_\kappa](t) = c(\kappa) + \mathcal{R}_1(\kappa)(t) \tag{13.5}$$

for $t \in [0,T]$, where $\mathcal{R}_1(\kappa)(t)$ satisfies

$$|\mathcal{R}_1(\kappa)(t)| \leq C\lambda_*^{\alpha_0} \tag{13.6}$$

for some $\alpha_0 > 0$.

We have additional estimates for p_κ and the remainder $\mathcal{R}_1(\kappa)$ constructed above. The function p_κ can be decomposed as

$$p_\kappa = p_{0,\kappa} + p_{1,\kappa}.$$

Here $p_{0,\kappa}$ is defined in (13.2). The function $p_{1,\kappa}$ satisfies: given $k \in (1,2)$ there is C such that

$$\|p_{1,\kappa}\|_{*,k+1} \leq C|\log T|^{k-1}\log^2(|\log T|) \tag{13.7}$$

and

$$\|p_{1,\kappa_1} - p_{1,\kappa_2}\|_{*,k+1} \leq C|\log T|^{k-1}\log^2(|\log T|)|\kappa_1 - \kappa_2| \tag{13.8}$$

for κ_1, κ_2 satisfying (13.4), where the norm $\| \ \|_{*,k}$ is defined for $g \in C([-T,T];\mathbb{C}) \cap C^1([-T,T);\mathbb{C})$ with

$$g(T) = 0$$

and $k > 0$ by

$$\|g\|_{*,k} = \sup_{t\in[-T,T]} |\log(T-t)|^k |\dot{g}(t)|, \tag{13.9}$$

(here $\dot{g} = \frac{d}{dt}g$).

The remainder, satisfies together with (13.6) the estimate for the derivative in t:

$$\left|\frac{d}{dt}\mathcal{R}_1(\kappa)(t)\right| \leq C\lambda_*^{\alpha_0-1} \tag{13.10}$$

and Lipschitz estimates

$$|\mathcal{R}_1(\kappa_1)(t) - \mathcal{R}_1(\kappa_2)(t)| \le C\lambda_*^{\alpha_0}|\kappa_1 - \kappa_2| \tag{13.11}$$

$$\left|\frac{d}{dt}\mathcal{R}_1(\kappa_1)(t) - \frac{d}{dt}\mathcal{R}_1(\kappa_2)(t)\right| \le C\lambda_*^{\alpha_0 - 1}|\kappa_1 - \kappa_2|, \tag{13.12}$$

for κ_1, κ_2 satisfying (13.4).

The proof of Lemma 13.1 and estimates (13.7), (13.8), (13.10), (13.11), and (13.12) are in section 13.4.

For the proof of Proposition 5.5 and Lemma 13.1 it will be useful to isolate the main part of the operator $\mathcal{B}_0$, defined in (4.6). Given the asymptotic expansion of Γ_l in (4.5) we write

$$\mathcal{B}_0[p] = \mathcal{I}[p] + \tilde{\mathcal{B}}[p],$$

where

$$\mathcal{I}[p] := \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{t-s}\, ds \tag{13.13}$$

$$\tilde{\mathcal{B}}[p] := \tilde{\mathcal{B}}_1[p] + \tilde{\mathcal{B}}_2[p] + \tilde{\mathcal{B}}_3[p] + \tilde{\mathcal{B}}_4[p] + \tilde{\mathcal{B}}_5[p], \tag{13.14}$$

and

$$\begin{cases} \tilde B_1[p](t) = e^{i\omega(t)} \displaystyle\int_{-T}^{t-\lambda_*(t)^2} \frac{\operatorname{Re}(\dot p(s)e^{-i\omega(t)})}{t-s}\Big(\Gamma_1\big(\frac{\lambda(t)^2}{t-s}\big) - 1\Big)\, ds \\ \tilde B_2[p](t) = ie^{i\omega(t)} \displaystyle\int_{-T}^{t-\lambda_*(t)^2} \frac{\operatorname{Im}(\dot p(s)e^{-i\omega(t)})}{t-s}\Big(\Gamma_2\big(\frac{\lambda(t)^2}{t-s}\big) - 1\Big)\, ds \\ \tilde B_3[p](t) = e^{i\omega(t)} \displaystyle\int_{t-\lambda_*(t)^2}^{t} \frac{\operatorname{Re}(\dot p(s)e^{-i\omega(t)})}{t-s}\Gamma_1\big(\frac{\lambda(t)^2}{t-s}\big)\, ds \\ \tilde B_4[p](t) = ie^{i\omega(t)} \displaystyle\int_{t-\lambda_*(t)^2}^{t} \frac{\operatorname{Im}(\dot p(s)e^{-i\omega(t)})}{t-s}\Gamma_2\big(\frac{\lambda(t)^2}{t-s}\big)\, ds \\ \tilde B_5[p](t) = -\dot\lambda(t)e^{i\omega(t)} = -\operatorname{Re}(\dot p(t)), \end{cases} \tag{13.15}$$

and we use the notation $p(t) = \lambda(t)e^{i\omega(t)}$.

To prove Proposition 5.5, we take p of the form

$$p = p_\kappa + p_2,$$

where p_κ is the function constructed in Lemma 13.1, for some $\kappa \in \mathbb{C}$ to be determined. The function $p_2(t)$ will have the property

$$p_2(t) = o(p_\kappa(t)),$$

as $t \to T$.

We would like that

$$\mathcal{I}[p_\kappa](t) + \mathcal{I}[p_2](t) + \tilde{\mathcal{B}}[p_\kappa + p_2](t) \approx a(t). \tag{13.16}$$

Given $\alpha > 0$, let us decompose

$$\mathcal{I}[p] = S_\alpha[\dot p] + R_\alpha[\dot p]$$

where S_α, R_α are defined as in (5.28), (5.29), that is

$$S_\alpha[g](t) = g(t)[(1+\alpha)\log(T-t) - 2\log(\lambda_*(t))] + \int_{-T}^{t-(T-t)^{1+\alpha}} \frac{g(s)}{t-s}\, ds$$

$$R_\alpha[g](t) = -\int_{t-(T-t)^{1+\alpha}}^{t-\lambda_*(t)^2} \frac{g(t)-g(s)}{t-s}\, ds.$$

The idea is to replace $\mathcal{I}[p_2]$ by $S_\alpha[\dot p_2]$ in (13.16) to make this equation more manageable, that is, we consider

$$S_\alpha[\dot p_2] + \tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa] + \mathcal{R}_1(\kappa) = a(t), \quad t \in [0,T],$$

where we have used (13.5).

We introduce one more modification, so as to have a more convenient problem to treat. Let us split

$$S_\alpha[g] = L_0[g] + L_1[g]$$

where

$$\begin{aligned} L_0[g] &= (1-\alpha)|\log(T-t)|g(t) \\ L_1[g] &= (4\log(|\log(T-t)|) - 2\log(\kappa) - 2\log(|\log(T)|))g(t) \\ &\quad + \int_{-T}^{t-(T-t)^{1+\alpha}} \frac{g(s)}{t-s}\,ds. \end{aligned} \tag{13.17}$$

Let η be a smooth cut-off function such that

$$\eta(s) = 1 \quad \text{for } s \geq 0, \quad \eta(s) = 0 \quad \text{for } s \leq -\frac{1}{4}. \tag{13.18}$$

We actually introduce one more modification to (13.16). For this, it is convenient that a is defined in $[-T,T]$. So, given a function $a : [0,T] \to \mathbb{C}$ satisfying the hypotheses of Proposition 5.5, we extend a continuously by constant for $t \leq 0$.

The equation that we are going to solve is the following one:

$$L_0[\dot p_2] + \eta(\frac{t}{T})L_1[\dot p_2] + \tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa] = a(t) - \mathcal{R}_1(\kappa) + c \quad \text{in } [-T,T] \tag{13.19}$$

for some constant c. Later on we shall show that it is possible to adjust κ so that $c = 0$.

13.2. **Construction of a solution to (13.20).** Since in (13.19) the terms $a(t)$ and $\mathcal{R}_1(\kappa)$ have similar behavior, we will consider just

$$L_0[\dot p_2] + \eta(\frac{t}{T})L_1[\dot p_2] + \tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa] = a(t) + c \quad \text{in } [-T,T] \tag{13.20}$$

Consider the norm $\|\ \|_{\mu,l}$ defined in (5.23).

Lemma 13.2. *Let $\mu, \alpha \in (0, \frac{1}{2})$ and $l \in \mathbb{R}$. Assume that $\frac{1}{C_1} \leq |a(T)| \leq C_1$ and*

$$T^\mu |\log T|^{1+\sigma-l} \|a(\cdot) - a(T)\|_{\mu,l-1} \leq C_1, \tag{13.21}$$

for some $\sigma > 0$ fixed. Then if $T > 0$ is small there is a solution p_2 to (13.20) for some $c \in \mathbb{C}$. Moreover this solution satisfies

$$\|\dot p_2\|_{\mu,l} \leq C\|a(\cdot) - a(T)\|_{\mu,l-1}. \tag{13.22}$$

For the proof of this lemma we consider the linear equation

$$L_0[g] + \eta(\frac{t}{T})L_1[g] = f + c \quad \text{in } [-T,T]. \tag{13.23}$$

We will assume that $f(T) = 0$, and hence $c = L_1[g](T)$ because all other terms in the equation vanish at T. Thanks to the cut-off function $\eta(\frac{t}{T})$, we need only to consider the values of $L_1[g](t)$ for $t \geq -\frac{T}{4}$. Then in the definition of $L_1[g]$, $t - (T-t)^{1+\alpha} \geq t - \frac{1}{2}(T-t) \geq -T$ of $T > 0$ is small.

For the right hand side of (13.23) we take the space $C([-T,T];\mathbb{C})$ with $f(T) = 0$ and the norm $\|f\|_{\mu,l-1}$.

The next lemma asserts the solvability of (13.23) in the weighted spaces introduced above.

Lemma 13.3. *Let $\alpha \in (0, \frac{1}{2})$ and $T > 0$ be sufficiently small. Assume $\|f\|_{\mu,l-1} < \infty$ where $\mu \in (0,1)$, $l \in \mathbb{R}$. Then for $T > 0$ small there is a solution $S[f]$ of (13.23) that defines a linear operator of f and such that*

$$\|S[f]\|_{\mu,l} \leq C\|f\|_{\mu,l-1}. \tag{13.24}$$

Proof. We consider (13.23) as a fixed point problem of the form

$$g = L_0^{-1}\left[f - \eta(\frac{t}{T})\left(L_1[g](t) - L_1[g](T)\right)\right], \tag{13.25}$$

where L_0^{-1} is defined the formula

$$L_0^{-1}[f](t) = \frac{1}{(1-\alpha)}\frac{f(t)}{|\log(T-t)|}. \tag{13.26}$$

It is clear that

$$\|L_0^{-1}[f]\|_{\mu,l} \le \frac{1}{1-\alpha}\|f\|_{\mu,l-1}. \tag{13.27}$$

We claim that

$$\|L_1[g](\cdot) - L_1[g](T)\|_{\mu,l-1} \le \Big(\alpha + \frac{C\log|\log T|}{|\log T|}\Big)\|g\|_{\mu,l}. \tag{13.28}$$

Indeed consider the term

$$\begin{aligned}
&|(4\log(|\log(T-t)|) - 2\log(\kappa) - 2\log(|\log(T)|))g(t)| \\
&\le C\log|\log(T-t)|\frac{(T-t)^\mu}{|\log(T-t)|^l}\|g\|_{\mu,l},
\end{aligned}$$

and this gives

$$\|(4\log(|\log(T-t)|) - 2\log(\kappa) - 2\log(|\log(T)|))g(t)\|_{\mu,l-1} \le \frac{C\log|\log T|}{|\log T|}\|g\|_{\mu,l}.$$

To estimate the integral term we decompose

$$\int_{-T}^{t-(T-t)^{1+\alpha}}\frac{g(s)}{t-s}\,ds - \int_{-T}^{T}\frac{g(s)}{T-s}\,ds = I_1 + I_2 + I_3$$

where

$$\begin{aligned}
I_1 &= \int_{t-(T-t)/2}^{t-(T-t)^{1+\alpha}}\frac{g(s)}{t-s}\,ds \\
I_2 &= \int_{-T}^{t-(T-t)/2} g(s)\left(\frac{1}{t-s} - \frac{1}{T-s}\right)ds \\
I_3 &= \int_{t-(T-t)/2}^{T}\frac{g(s)}{T-s}\,ds.
\end{aligned}$$

Then

$$\begin{aligned}
|I_1| &\le \|g\|_{\mu,l}\int_{t-(T-t)/2}^{t-(T-t)^{1+\alpha}}\frac{(T-s)^\mu}{|\log(T-s)|^l(t-s)}\,ds \\
&= \|g\|_{\mu,l}\int_{(T-t)^{1+\alpha}}^{(T-t)/2}\frac{(T-t+r)^\mu}{|\log(T-t+r)|^l\,r}\,dr \\
&= \|g\|_{\mu,l}\frac{(T-t)^\mu}{|\log(T-t)|^l}\int_{(T-t)^{1+\alpha}}^{(T-t)/2}\frac{1+O(\frac{r}{T-t})}{r}\,dr \\
&\le \|g\|_{\mu,l}\frac{(T-t)^\mu}{|\log(T-t)|^l}(\alpha|\log(T-t)| + C).
\end{aligned}$$

$$\begin{aligned}
|I_2| &\le \|g\|_{\mu,l}(T-t)\int_{-T}^{t-(T-t)/2}\frac{(T-s)^\mu}{|\log(T-s)|^l(T-s)(t-s)}\,ds \\
&\le C\|g\|_{\mu,l}\frac{(T-t)^\mu}{|\log(T-t)|^l}.
\end{aligned}$$

and

$$
\begin{aligned}
|I_3| &\le \|g\|_{\mu,l} \int_{t-(T-t)/2}^{T} \frac{(T-s)^{\mu-1}}{|\log(T_s)|^l}\, ds \\
&\le C\|g\|_{\mu,l} \frac{(T-t)^{\mu}}{|\log(T-t)|^l}.
\end{aligned}
$$

These estimates imply (13.28). Then this inequality combined with (13.27) shows that

$$
\left\| L_0^{-1}\left[\eta(\frac{t}{T})\,(L_1[g](t) - L_1[g](T)) \right] \right\|_{\mu,l} \le \frac{1}{1-\alpha}\Big(\alpha + \frac{C\log|\log T|}{|\log T|}\Big)\|g\|_{\mu,l}.
$$

Then for $\alpha \in (0, \frac{1}{2})$ and $T > 0$ sufficiently small this operator is a contraction and we obtain the conclusion of the lemma. □

Proof of Lemma 13.2. Let S denote the linear operator constructed in Lemma 13.3.

Then to find a solution to (13.20) it is sufficient to find a solution p_2 of the fixed point problem

$$
p_2 = \mathcal{A}[p_2] \tag{13.29}
$$

where $\tilde p = \mathcal{A}[p_2]$ is defined by $\tilde p(T) = 0$ and

$$
\frac{d\tilde p}{dt} = S\left[-\big(\tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa]\big) + a(t) - a(T) \right].
$$

Let $M_1 = C_0\|a(\cdot) - a(T)\|_{\mu,l-1}$, where C_0 is a sufficiently large fixed constant. We claim that if $T > 0$ is sufficiently small then $\mathcal{A}$ is a contraction in ball $\overline{B}_{M_1}$ of the space of complex valued functions $p_2 \in C^1([-T, T])$ with $p_2(T) = 0$ and with the norm $\|\dot p_2\|_{\mu,l}$. Note that with this norm we have

$$
|p_2(t)| \le C\|\dot p_2\|_{\mu,l} \frac{(T-t)^{\mu+1}}{|\log(T-t)|^l}. \tag{13.30}
$$

In particular, thanks to (13.21), if $\|\dot p_2\|_{\mu,l} \le M_1$, then

$$
\left|\frac{p_2}{\lambda_*}\right| + \left|\frac{\dot p_2}{\dot\lambda_*}\right| \ll 1
$$

for $T > 0$ small.

Let us verify that $\mathcal{A}$ maps $\overline{B}_{M_1}$ into itself. Let $p_2 \in \overline{B}_{M_1}$. By (13.24) we have

$$
\|\mathcal{A}[p_2]\|_{\mu,l} \le C\Big(\|\tilde{\mathcal{B}}[p_\kappa + p_2] - \tilde{\mathcal{B}}[p_\kappa]\|_{\mu,l-1} + \|a(\cdot) - a(T)\|_{\mu,l-1}\Big). \tag{13.31}
$$

We claim that for $p_1, p_2 \in \overline{B}_{M_1}$ we have

$$
\|\tilde{\mathcal{B}}[p_\kappa + p_1] - \tilde{\mathcal{B}}[p_\kappa + p_2]\|_{\mu,l-1} \le C\frac{1}{|\log T|}\|\dot p_1 - \dot p_2\|_{\mu,l}. \tag{13.32}
$$

Assuming for now this estimate let us continue with proving that $\mathcal{A}$ maps $\overline{B}_{M_1}$ into itself. Let $p_1 \in \overline{B}_{M_1}$. By (13.31) and (13.32)

$$
\|\mathcal{A}[p_2]\|_{\mu,l} \le C\frac{M_1}{|\log T|} + C\|a(\cdot) - a(T)\|_{\mu,l-1} \le M_1,
$$

if $T > 0$ is small. Also thanks to (13.24) and (13.32) we see that $\mathcal{A}$ is a contraction in $\overline{B}_{M_1}$. This finishes the proof of the lemma. □

Proof of (13.32). We will prove that

$$
\|\tilde{\mathcal{B}}_j[p_\kappa + p_1] - \tilde{\mathcal{B}}_j[p_\kappa + p_2]\|_{\mu,l-1} \le C\frac{1}{|\log T|}\|\dot p_1 - \dot p_2\|_{\mu,l}, \tag{13.33}
$$

holds for $j = 1, \ldots, 5$, with $\tilde{\mathcal{B}}_j$ defined in (13.15). The estimate for $\tilde B_5$ is direct from the definition.

Let us prove that (13.33) holds for $\tilde{\mathcal{B}}_1$.

For this write

$$
\tilde{\mathcal{B}}_1[p_\kappa + p_1] - \tilde{\mathcal{B}}_1[p_\kappa + p_2] = D_{1,a}
$$

where

$$D_{i,a} = \frac{(p_\kappa + p_1)(t)}{|(p_\kappa + p_1)(t)|} \operatorname{Re}\left(\frac{(\bar p_\kappa + \bar p_1)(t)}{|(p_\kappa + p_1)(t)|} \tilde{\mathcal{B}}_{i,a}[p_\kappa + p_1](t) \right) - \frac{(p_\kappa + p_2)(t)}{|(p_\kappa + p_2)(t)|} \operatorname{Re}\left(\frac{(\bar p_\kappa + \bar p_2)(t)}{|(p_\kappa + p_2)(t)|} \tilde{\mathcal{B}}_{i,a}[p_\kappa + p_2](t) \right), \tag{13.34}$$

and

$$\tilde{\mathcal{B}}_{i,a}[p](t) = \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{t-s} \left(\Gamma_i \left(\frac{|p(t)|^2}{t-s} \right) - 1 \right) ds. \tag{13.35}$$

We write

$$D_{i,a} = \int_0^1 \frac{d}{d\zeta} \left[\frac{(p_\kappa + p_\zeta)(t)}{|(p_\kappa + p_\zeta)(t)|} \operatorname{Re}\left(\frac{(\bar p_\kappa + \bar p_\zeta)(t)}{|(p_\kappa + p_\zeta)(t)|} \tilde{\mathcal{B}}_{i,a}[p_\kappa + p_\zeta](t) \right) \right] d\zeta$$

where $p_\zeta = \zeta p_1 + (1-\zeta) p_2$. Let us analyze the terms in this expression. For this we note that

$$\left| \frac{d}{d\zeta} \frac{(p_\kappa + p_\zeta)(t)}{|(p_\kappa + p_\zeta)(t)|} \right| \le 2 \frac{|p_1(t) - p_2(t)|}{|(p_\kappa + p_\zeta)(t)|}.$$

Using (13.30) and (13.108), which will be proved later on, we get

$$\begin{aligned} &\left| \operatorname{Re}\left(\frac{(\bar p_\kappa + \bar p_\zeta)(t)}{|(p_\kappa + p_\zeta)(t)|} \tilde{\mathcal{B}}_{i,a}[p_\kappa + p_\zeta](t) \right) \frac{d}{d\zeta} \frac{(p_\kappa + p_\zeta)(t)}{|(p_\kappa + p_\zeta)(t)|} \right| \\ &\le C \frac{|p_1(t) - p_2(t)|}{|(p_\kappa + p_\zeta)(t)|} \frac{|\log T|}{|\log(T-t)|^2} \\ &\le C \frac{(T-t)^\mu}{|\log(T-t)|^l} \|\dot p_1 - \dot p_2\|_{\mu,l}. \end{aligned}$$

Let us consider

$$\begin{aligned} &\frac{d}{d\zeta} \tilde{\mathcal{B}}_{i,a}[p_\kappa + p_\zeta](t) \\ &= \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p_1(s) - \dot p_2(s)}{t-s} \left(\Gamma_i \left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{t-s} \right) - 1 \right) ds \\ &\quad + 2(p_\kappa(t) + p_\zeta(t)) \cdot (p_1(t) - p_2(t)) \\ &\quad\ \cdot \int_{-T}^{t-\lambda_*(t)^2} \frac{(\dot p_\kappa + \dot p_\zeta)(s)}{(t-s)^2} \Gamma_i' \left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{t-s} \right) ds. \end{aligned} \tag{13.36}$$

We estimate the first term using (4.5) (here $\sigma \in (0,1)$):

$$\begin{aligned} &\left| \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p_1(s) - \dot p_2(s)}{t-s} \left(\Gamma_i \left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{t-s} \right) - 1 \right) ds \right| \\ &\le C \int_{-T}^{t-\lambda_*(t)^2} \frac{|\dot p_1(s) - \dot p_2(s)|}{t-s} \frac{|(p_\kappa + p_\zeta)(t)|^{2\sigma}}{(t-s)^\sigma} ds \\ &\le C \|\dot p_1 - \dot p_2\|_{\mu,l}\, \lambda_*(t)^{2\sigma} \int_{-T}^{t-\lambda_*(t)^2} \frac{(T-s)^\mu}{(t-s)^{1+\sigma} |\log(T-s)|^l} ds \end{aligned}$$

and by Lemma 13.11

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{(T-s)^\mu}{(t-s)^{1+\sigma} |\log(T-s)|} ds \le \frac{C (T-t)^\mu}{\lambda_*(t)^{2\sigma} |\log(T-t)|^l}.$$

Therefore

$$\left|\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p_1(s) - \dot p_2(s)}{t-s}\left(2\Gamma_i\left(\frac{|(p_\kappa+p_\zeta)(t)|^2}{t-s}\right)-1\right)ds\right|$$
$$\le C\frac{(T-t)^\mu}{|\log(T-t)|^l}\|\dot p_1-\dot p_2\|_{\mu,l}.$$

For the second term in (13.36) we take $\sigma\in(0,1)$ and compute

$$\left|(p_\kappa(t)+p_\zeta(t))\cdot(p_1(t)-p_2(t))\int_{-T}^{t-\lambda_*(t)^2}\frac{(\dot p_\kappa+\dot p_\zeta)(s)}{(t-s)^2}\Gamma_i'\left(\frac{|(p_\kappa+p_\zeta)(t)|^2}{t-s}\right)\right|$$
$$\le C\lambda_*(t)\|\dot p_1-\dot p_2\|_{\mu,l}\frac{(T-t)^{1+\mu}}{|\log(T-t)|^l}\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot\lambda_*(s)}{(t-s)^2}\left(\frac{|(p_\kappa+p_\zeta)(t)|^2}{t-s}\right)^{-\sigma}ds$$
$$\le C\lambda_*(t)^{1-2\sigma}\|\dot p_1-\dot p_2\|_{\mu,l}\frac{(T-t)^{1+\mu}}{|\log(T-t)|^l}\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot\lambda_*(s)}{(t-s)^{2-\sigma}}ds$$
$$\le C\frac{(T-t)^\mu}{|\log(T-t)|^l}\|\dot p_1-\dot p_2\|_{\mu,l}.$$

Thus we have proved that

$$|D_{i,a}|\le C\frac{(T-t)^\mu}{|\log(T-t)|}\|p_1\|_{*,\mu}. \tag{13.37}$$

Let us prove that (13.33) holds for $\tilde{\mathcal B}_3$. For this note that

$$\tilde{\mathcal B}_3[p_\kappa+p_1]-\tilde{\mathcal B}_3[p_\kappa+p_2]=D_{1,b}$$

where

$$D_{i,b}=\frac{(p_\kappa+p_1)(t)}{|(p_\kappa+p_1)(t)|}\operatorname{Re}\left(\frac{(\bar p_\kappa+\bar p_1)(t)}{|(p_\kappa+p_1)(t)|}\tilde{\mathcal B}_{i,b}[p_\kappa+p_1](t)\right)$$
$$-\frac{(p_\kappa+p_2)(t)}{|(p_\kappa+p_2)(t)|}\operatorname{Re}\left(\frac{(\bar p_\kappa+\bar p_2)(t)}{|(p_\kappa+p_2)(t)|}\tilde{\mathcal B}_{i,b}[p_\kappa+p_2](t)\right),$$

and

$$\tilde{\mathcal B}_{i,b}[p](t)=\int_{t-\lambda_*(t)^2}^t\frac{\dot p(s)}{t-s}\Gamma_i\left(\frac{|p(t)|^2}{t-s}\right)ds.$$

The estimate of $D_{i,b}$ is very similar compared to $D_{i,a}$, the only difference appearing in

$$\frac{d}{d\zeta}\tilde{\mathcal B}_{i,b}[p_\kappa+p_\zeta](t)$$
$$=\int_{t-\lambda_*(t)^2}^t\frac{\dot p_1(s)-\dot p_2(s)}{t-s}\Gamma_i\left(\frac{|(p_\kappa+p_\zeta)(t)|^2}{t-s}\right)ds$$
$$+2(p_\kappa(t)+p_\zeta(t))\cdot(p_1(t)-p_2(t))\int_{t-\lambda_*(t)^2}^t\frac{(\dot p_\kappa+\dot p_\zeta)(s)}{(t-s)^2}\Gamma_i'\left(\frac{|(p_\kappa+p_\zeta)(t)|^2}{t-s}\right)ds.$$

We estimate the first term above

$$\left|\int_{t-\lambda_*(t)^2}^t\frac{\dot p_1(s)-\dot p_2(s)}{t-s}\Gamma_i\left(\frac{|(p_\kappa+p_\zeta)(t)|^2}{t-s}\right)ds\right|$$
$$\le\frac{C}{|(p_\kappa+p_\zeta)(t)|^2}\int_{t-\lambda_*(t)^2}^t|\dot p_1(s)-\dot p_2(s)|\,ds$$
$$\le C\frac{(T-t)^\mu}{|\log(T-t)|^l}\|\dot p_1-\dot p_2\|_{\mu,l}.$$

The second term is estimated by

$$\begin{aligned}
&\left|(p_\kappa(t)+p_\zeta(t))\cdot(p_1(t)-p_2(t))\int_{t-\lambda_*(t)^2}^t \frac{(\dot p_\kappa+\dot p_\zeta)(s)}{(t-s)^2}\Gamma_i'\left(\frac{|(p_\kappa+p_\zeta)(t)|^2}{t-s}\right)ds\right| \\
&\le C\lambda_*(t)\frac{(T-t)^{1+\mu}}{|\log(T-t)|^l}\|\dot p_1-\dot p_2\|_{\mu,l}\frac{1}{\lambda_*(t)^4}\int_{t-\lambda_*(t)^2}^t \frac{|\log T|}{|\log(T-s)|^2}\,ds \\
&\le C\frac{(T-t)^\mu}{|\log(T-t)|^l}\|\dot p_1-\dot p_2\|_{\mu,l}.
\end{aligned}$$

We conclude that

$$|D_{i,b}|\le C\frac{(T-t)^\mu}{|\log(T-t)|^l}\|\dot p_1-\dot p_2\|_{\mu,l}. \tag{13.38}$$

From (13.37), (13.38) and similar estimates for $D_{i,a}$ and $D_{i,b}$ with the real part replaced by the imaginary part, we obtain (13.32). □

We have also a Lipschitz property of the solution constructed in Lemma 13.2.

Lemma 13.4. *Let $\mu,\alpha\in(0,\frac12)$ and $l\in\mathbb{R}$. Assume that for $j=1,2$, a_j satisfies $\frac{1}{C_1}\le|a_j(T)|\le C_1$ and (13.21), and let κ_1, κ_2 satisfy (13.4). Then for $T>0$ is small the solution $p_2[a,\kappa]$ to (13.20) constructed in Lemma 13.2 satisfies*

$$\|\dot p_2[a_1,\kappa_1]-\dot p_2[a_2,\kappa_1]\|_{\mu,l}\le C\|a_1(\cdot)-a_1(T)-(a_2(\cdot)-a_2(T))\|_{\mu,l-1} \tag{13.39}$$

$$\|\dot p_2[a_1,\kappa_1]-\dot p_2[a_1,\kappa_2]\|_{\mu,l}\le C\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}|\kappa_1-\kappa_2|. \tag{13.40}$$

Proof. To prove (13.39) we compute formally the directional derivative of p_2

$$p_2'(t)=\frac{d}{ds}p_2[a+sa_1,\kappa](t)\big|_{s=0}$$

where a, a_1 are functions satisfying (13.21) and $\frac{1}{C_1}\le|a(T)|\le C_1$.

From equation (13.20) we get

$$L_0[\dot p_2']+\eta(\frac{t}{T})L_1[\dot p_2']+D\tilde{\mathcal B}[p_\kappa+p_2](p_2')=a_1(t)+c'\quad\text{in }[-T,T]. \tag{13.41}$$

and where $D\tilde{\mathcal B}[p](v)(t)$ is defined as

$$\frac{d}{ds}\tilde{\mathcal B}[p+sv](t)\big|_{s=0},$$

and is given by

$$\begin{aligned}
D\tilde{\mathcal B}[p](v)=&\left(\frac{v}{|p|}-\frac{p(p\cdot v)}{|p|^3}\right)\operatorname{Re}\left(\frac{\bar p}{|p|}\tilde{\mathcal B}_1[p]\right)+\frac{p}{|p|}\operatorname{Re}\left(\left(\frac{\bar v}{|p|}-\frac{\bar p(p\cdot v)}{|p|^3}\right)\tilde{\mathcal B}_1[p]\right)\\
&+\frac{p}{|p|}\operatorname{Re}\left(\frac{\bar p}{|p|}D\tilde{\mathcal B}_1[p](v)\right)\\
&+i\left(\frac{v}{|p|}-\frac{p(p\cdot v)}{|p|^3}\right)\operatorname{Im}\left(\frac{\bar p}{|p|}\tilde{\mathcal B}_2[p]\right)+i\frac{p}{|p|}\operatorname{Im}\left(\left(\frac{\bar v}{|p|}-\frac{\bar p(p\cdot v)}{|p|^3}\right)\tilde{\mathcal B}_2[p]\right)\\
&+i\frac{p}{|p|}\operatorname{Im}\left(\frac{\bar p}{|p|}D\tilde{\mathcal B}_2[p](v)\right)\\
&-\operatorname{Re}(\dot v),
\end{aligned}$$

where

$$\tilde{\mathcal B}_j[p](t)=\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p(s)}{t-s}\left(\Gamma_j(\frac{|p(t)|^2}{t-s})-1\right)ds+\int_{t-\lambda_*(t)^2}^t\frac{\dot p(s)}{t-s}\Gamma_1(\frac{|p(t)|^2}{t-s})\,ds$$

and

$$
\begin{aligned}
D\tilde{\mathcal{B}}_j[p](v)(t) = &\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot v(s)}{t-s}\left(\Gamma_j\left(\frac{|p(t)|^2}{t-s}\right)-1\right) ds \\
&+ 2p(t)\cdot v(t)\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{(t-s)^2}\Gamma_j'\left(\frac{|p(t)|^2}{t-s}\right) ds \\
&+ \int_{t-\lambda_*(t)^2}^{t} \frac{\dot v(s)}{t-s}\Gamma_j\left(\frac{|p(t)|^2}{t-s}\right) ds \\
&+ 2p(t)\cdot v(t)\int_{t-\lambda_*(t)^2}^{t} \frac{\dot p(s)}{t-s}\Gamma_j'\left(\frac{|p(t)|^2}{t-s}\right) ds.
\end{aligned}
$$

The operator $D\tilde{\mathcal{B}}[p_\kappa+p_2]$ satisfies the estimate

$$
\|D\tilde{\mathcal{B}}[p_\kappa+p_2](v)\|_{\mu,l-1} \le \frac{C}{|\log T|}\|v\|_{\mu,l}, \tag{13.42}
$$

which follows from (13.32). Using the above estimate together with equation (13.41), and Lemma 13.3 we deduce (13.39).

For the proof of (13.40) we proceed similarly, computing the derivative of equation (13.20) with respect to κ

$$
L_0[\dot p_2'] + \eta(\frac{t}{T})L_1[\dot p_2'] + D\tilde{\mathcal{B}}[p_\kappa+p_2](p_2') = -D\tilde{\mathcal{B}}[p_\kappa+p_2](p_\kappa') + D\tilde{\mathcal{B}}[p_\kappa](p_\kappa') + c', \tag{13.43}
$$

in $[-T,T]$, where now $()' = \frac{d}{d\kappa}$. We claim that

$$
|D\tilde{\mathcal{B}}[p_\kappa+p_2](p_\kappa') - D\tilde{\mathcal{B}}[p_\kappa](p_\kappa')| \le C\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}\frac{(T-t)^\mu}{|\log(T-t)|^l}. \tag{13.44}
$$

We have to consider the two terms above together and see a cancellation to get the correct estimate. The first term of $D\tilde{\mathcal{B}}[p_\kappa+p_2](p_\kappa') - D\tilde{\mathcal{B}}[p_\kappa](p_\kappa')$ is

$$
\begin{aligned}
d_1 &= \left(\frac{p_\kappa'}{|p_\kappa+p_2|} - \frac{(p_\kappa+p_2)((p_\kappa+p_2)\cdot p_\kappa')}{|p_\kappa+p_2|^3}\right)\mathrm{Re}\left(\frac{\overline{p_\kappa+p_2}}{|p_\kappa+p_2|}\tilde{\mathcal{B}}_1[p_\kappa+p_2]\right) \\
&\quad - \left(\frac{p_\kappa'}{|p_\kappa|} - \frac{p_\kappa(p_\kappa\cdot p_\kappa')}{|p_\kappa|^3}\right)\mathrm{Re}\left(\frac{\bar p_\kappa}{|p_\kappa|}\tilde{\mathcal{B}}_1[p_\kappa]\right) \\
&= e_1+e_2+e_3+e_4
\end{aligned} \tag{13.45}
$$

where

$$
\begin{aligned}
e_1 &= \left(\frac{p_\kappa'}{|p_\kappa+p_2|} - \frac{p_\kappa'}{|p_\kappa|}\right)\mathrm{Re}\left(\frac{\overline{p_\kappa+p_2}}{|p_\kappa+p_2|}\tilde{\mathcal{B}}_1[p_\kappa+p_2]\right) \\
e_2 &= \left(-\frac{(p_\kappa+p_2)((p_\kappa+p_2)\cdot p_\kappa')}{|p_\kappa+p_2|^3} + \frac{p_\kappa(p_\kappa\cdot p_\kappa')}{|p_\kappa|^3}\right)\mathrm{Re}\left(\frac{\overline{p_\kappa+p_2}}{|p_\kappa+p_2|}\tilde{\mathcal{B}}_1[p_\kappa+p_2]\right) \\
e_3 &= \left(\frac{p_\kappa'}{|p_\kappa|} - \frac{p_\kappa(p_\kappa\cdot p_\kappa')}{|p_\kappa|^3}\right)\mathrm{Re}\left(\left(\frac{\overline{p_\kappa+p_2}}{|p_\kappa+p_2|} - \frac{\overline{p_\kappa}}{|p_\kappa|}\right)\tilde{\mathcal{B}}_1[p_\kappa+p_2]\right) \\
e_4 &= \left(\frac{p_\kappa'}{|p_\kappa|} - \frac{p_\kappa(p_\kappa\cdot p_\kappa')}{|p_\kappa|^3}\right)\mathrm{Re}\left(\frac{\bar p_\kappa}{|p_\kappa|}(\tilde{\mathcal{B}}_1[p_\kappa+p_2] - \tilde{\mathcal{B}}_1[p_\kappa])\right)
\end{aligned}
$$

(We recall that p_κ is the function constructed in Lemma 13.1.) Note that

$$\begin{aligned}\left|\frac{p_\kappa'}{|p_\kappa+p_2|}-\frac{p_\kappa'}{|p_\kappa|}\right| &\le C\frac{|p_\kappa'|}{|p_\kappa|^2}|p_2|\\ &\le C\frac{|\log(T-t)|^2}{|\log T|(T-t)}(T-t)^{1+\mu}|\log(T-t)|^{-l}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}\\ &= C(T-t)^\mu\frac{|\log(T-t)|^{-l+2}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}\end{aligned}$$

This together with (13.108) shows that

$$|e_j|\le C(T-t)^\mu|\log(T-t)|^{-l}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}, \tag{13.46}$$

for $j=1$. The proof of (13.46) for $j=2,3$ is similar. Finally, (13.46) for $j=4$ follows from the proof of (13.32). This shows that the expression (13.45) satisfies

$$|d_1|\le C(T-t)^\mu|\log(T-t)|^{-l}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}.$$

The other terms in $D\tilde{\mathcal{B}}[p_\kappa+p_2](p_\kappa'+p_2')-D\tilde{\mathcal{B}}[p_\kappa](p_\kappa')$ are handled similarly and we omit the details.

Using Lemma 13.3 and estimates (13.42), (13.44) we deduce (13.40). □

13.3. **Hölder estimate of the solution.** We will show in this section that the solution constructed in Lemma 13.2 has some Hölder regularity inherited from the one of a.

We then have the following result, where the Hölder semi norm $[\ ]_{\gamma,m,l}$ is defined in (5.24).

Lemma 13.5. *Let $\alpha\in(0,\frac12)$, $\mu,\gamma\in(0,1)$, $m\le\mu-\gamma$, $l\in\mathbb{R}$. Assume that $\mathrm{Re}(a(T))<0$ with $\frac{1}{C_1}\le\mathrm{Re}(a(T))\le C_1$ and*

$$T^\mu|\log T|^{1+\sigma-l}\|a(\cdot)-a(T)\|_{\mu,l-1}+[a]_{\gamma,m,l-1}\le C_1, \tag{13.47}$$

for some $\sigma>0$. Then the solution p_2 constructed in Lemma 13.2 satisfies

$$\begin{aligned}[\dot p_2]_{\gamma,m,l}\lesssim{}&\frac{T^\mu}{|\log T|}\left(T^{-\gamma-m}+\log|\log T|\right)\|a(\cdot)-a(T)\|_{\mu,l-1}\\ &+[a(\cdot)-a(T)]_{\gamma,m,l-1}.\end{aligned} \tag{13.48}$$

For the proof we need an estimate for the operator S constructed in Lemma 13.3.

Lemma 13.6. *Let S denote the linear operator constructed in Lemma 13.3. Assume $\mu,\gamma\in(0,1)$,*

$$m\le\mu-\gamma \tag{13.49}$$

and $l\in\mathbb{R}$. Then S satisfies

$$[S(f)]_{\gamma,m,l}\le C\Big([f]_{\gamma,m,l-1}+T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|f\|_{\mu,l-1}\Big). \tag{13.50}$$

Proof. The proof uses the fixed point characterization (13.25) of the operator S and the mapping properties of L_0 and L_1 with respect to the Hölder semi-norm $[\ ]_{\gamma,m,l}$.

The operator L_0^{-1} defined by (13.26), that is,

$$L_0^{-1}[f](t)=\frac{1}{(1-\alpha)}\frac{f(t)}{|\log(T-t)|},$$

satisfies: if $m\le\mu-\gamma$ then

$$[L_0^{-1}[f]\ ]_{\gamma,m,l}\le\frac{1}{1-\alpha}[f]_{\gamma,m,l-1}+CT^{\mu-m-\gamma}|\log T|^{-1}\|f\|_{\mu,l-1}. \tag{13.51}$$

Indeed, writing $g(t) = \frac{f(t)}{|\log(T-t)|}$ we have, for $-T \le s \le t \le T$ such that $t-s \le \frac{1}{10}(T-t)$:

$$\begin{aligned} |g(t)-g(s)| &\le \frac{|f(t)-f(s)|}{|\log(T-t)|} + |f(s)| \left| \frac{1}{|\log(T-t)|} - \frac{1}{|\log(T-s)|} \right| \\ &\le [f]_{\gamma,m,l-1}(t-s)^\gamma \frac{(T-t)^m}{|\log(T-t)|^l} + |f(s)| \left| \frac{1}{|\log(T-t)|} - \frac{1}{|\log(T-s)|} \right|. \end{aligned}$$

But

$$\begin{aligned} &|f(s)| \left| \frac{1}{|\log(T-t)|} - \frac{1}{|\log(T-s)|} \right| \\ &\le C\|f\|_{\mu,l-1} \frac{(T-t)^\mu}{|\log(T-t)|^{l-1}} \frac{t-s}{(T-t)|\log(T-t)|^2} \\ &\le C\|f\|_{\mu,l-1} \frac{(T-t)^{\mu-\gamma}(t-s)^\gamma}{|\log(T-t)|^{l+1}}. \end{aligned}$$

But by (13.49),

$$(T-t)^{\mu-\gamma} \le CT^{\mu-\gamma-m}(T-t)^m.$$

It follows that

$$\begin{aligned} |f(s)| \left| \frac{1}{|\log(T-t)|} - \frac{1}{|\log(T-s)|} \right| &\le C\|f\|_{\mu,l-1} \frac{T^{\mu-\gamma-m}(T-t)^m(t-s)^\gamma}{|\log(T-t)|^{l+1}} \\ &\le C\|f\|_{\mu,l-1} |\log T|^{-1} T^{\mu-\gamma-m} \frac{(T-t)^m(t-s)^\gamma}{|\log(T-t)|^l} \end{aligned}$$

This proves (13.51).

Let L_1 be defined defined in (13.17) and

$$\bar{L}_1[g](t) = \eta(\frac{t}{T})(L_1[g](t) - L_1[g](T)),$$

where η is defined in (13.18). Then we claim that if (13.49) holds then

$$[\bar{L}_1[g]]_{\gamma,m,l-1} \le \left(\alpha + C\frac{\log|\log T|}{|\log T|}\right)([g]_{\gamma,m,l} + T^{\mu-m-\gamma}\|g\|_{\mu,l}). \tag{13.52}$$

Let

$$-T \le t_1 < t_2 < T, \quad t_2 - t_1 \le \frac{T-t_2}{10} \tag{13.53}$$

and then note that

$$\bar{L}_1[g](t_2) - \bar{L}_1[g](t_1) = h_1 + h_2$$

with

$$\begin{aligned} h_1 &= \left(\eta(\frac{t_2}{T}) - \eta(\frac{t_1}{T})\right)(L_1[g](t_2) - L_1[g](T)) \\ h_2 &= \eta(\frac{t_1}{T})(L_1[g](t_2) - L_1[g](t_1)). \end{aligned}$$

Then

$$|h_1| \le C\frac{t_2 - t_1}{T}|L_1[g](t_2) - L_1[g](T)|$$

and by (13.28)

$$\begin{aligned}|h_1| &\le C\frac{t_2-t_1}{T}\frac{(T-t_2)^{\mu}}{|\log(T-t_2)|^{l-1}}\left(\alpha+\frac{C\log|\log T|}{|\log T|}\right)\|g\|_{\mu,l}\\ &\le C(t_2-t_1)^{\gamma}\frac{(T-t_2)^{\mu-\gamma}}{|\log(T-t_2)|^{l-1}}\left(\alpha+\frac{C\log|\log T|}{|\log T|}\right)\|g\|_{\mu,l}\\ &\le C(t_2-t_1)^{\gamma}\frac{T^{-\mu}(T-t_2)^{\mu-\gamma}}{|\log(T-t_2)|^{l-1}}\left(\alpha+\frac{C\log|\log T|}{|\log T|}\right)T^{\mu}\|g\|_{\mu,l}\\ &\le C(t_2-t_1)^{\gamma}\frac{(T-t_2)^{m}}{|\log(T-t_2)|^{l-1}}\left(\alpha+\frac{C\log|\log T|}{|\log T|}\right)T^{\mu-m-\gamma}\|g\|_{\mu,l}.\end{aligned}$$

To estimate h_2 we only need to consider $t_1\ge-\frac{T}{4}$ because of the cut-off function. It is convenient to split

$$L_1=L_{11}+L_{12}$$

where

$$\begin{aligned}L_{11}[g](t)&=(4\log(|\log(T-t)|)-2\log(\kappa)-2\log(|\log(T)|))g(t)\\ L_{12}[g](t)&=\int_{-T}^{t-(T-t)^{1+\alpha}}\frac{g(s)}{t-s}\,ds.\end{aligned}$$

Then

$$\begin{aligned}L_{12}[g](t_2)-L_{12}[g](t_1)&=\int_{(T-t_2)^{1+\alpha}}^{(T-t_1)^{1+\alpha}}\frac{g(t_2-r)}{r}\,ds\\ &\quad+\int_{(T-t_1)^{1+\alpha}}^{(T-t_2)/2}\frac{g(t_2-r)-g(t_1-r)}{r}\,dr\\ &\quad+\int_{(T-t_2)/2}^{(T-t_1)/2}\frac{g(t_1-r)}{r}\,dr.\end{aligned}\tag{13.54}$$

Note that assuming $T>0$ small and (13.53) we have that $(T-t_1)^{1+\alpha}\le(T-t_2)/2$. We estimate

$$\begin{aligned}&\left|\int_{(T-t_2)^{1+\alpha}}^{(T-t_1)^{1+\alpha}}\frac{g(t_2-r)}{r}\,dr\right|\\ &\le\|g\|_{\mu,l}\int_{(T-t_2)^{1+\alpha}}^{(T-t_1)^{1+\alpha}}\frac{(T-t_2+r)^{\mu}}{|\log(T-t_2+r)|^{l}}\frac{1}{r}\,dr.\end{aligned}$$

But

$$\begin{aligned}\int_{(T-t_2)^{1+\alpha}}^{(T-t_1)^{1+\alpha}}\frac{(T-t_2+r)^{\mu}}{|\log(T-t_2+r)|^{l}}\frac{1}{r}\,dr&\le C\frac{(T-t_2)^{\mu}}{|\log(T-t_2)|^{l}}\frac{t_2-t_1}{T-t_2}\\ &\le C(t_2-t_1)^{\gamma}\frac{(T-t_2)^{\mu-\gamma}}{|\log(T-t_2)|^{l}}\\ &\le\frac{C}{|\log T|}(t_2-t_1)^{\gamma}\frac{(T-t_2)^{\mu-\gamma}}{|\log(T-t_2)|^{l-1}}.\end{aligned}$$

Therefore

$$\left|\int_{(T-t_2)^{1+\alpha}}^{(T-t_1)^{1+\alpha}}\frac{g(t_2-r)}{r}\,dr\right|\le\frac{C}{|\log T|}(t_2-t_1)^{\gamma}\frac{(T-t_2)^{m}}{|\log(T-t_2)|^{l-1}}T^{\mu-m-\gamma}\|g\|_{u,l-1}.\tag{13.55}$$

We estimate the second term:

$$\begin{aligned}
&\left| \int_{(T-t_1)^{1+\alpha}}^{(T-t_2)/2} \frac{g(t_2-r)-g(t_1-r)}{r}\, dr \right| \\
&\leq [g]_{\gamma,m,l} \int_{(T-t_1)^{1+\alpha}}^{(T-t_2)/2} \frac{(T-t_2+r)^m}{|\log(T-t_2+r)|^l} \frac{(t_2-t_1)^\gamma}{r}\, dr \\
&= [g]_{\gamma,m,l}(t_2-t_1)^\gamma \frac{(T-t_2)^m}{|\log(T-t_2)|^l} \\
&\qquad \cdot \int_{(T-t_1)^{1+\alpha}}^{(T-t_2)/2} \left| 1 + \frac{\log(1+\frac{r}{T-t_2})}{\log(T-t_2)} \right|^{-l} (1+\frac{r}{T-t_2})^m \frac{1}{r}\, dr,
\end{aligned}$$

and we estimate

$$\begin{aligned}
&\int_{(T-t_1)^{1+\alpha}}^{(T-t_2)/2} \left| 1 + \frac{\log(1+\frac{r}{T-t_2})}{\log(T-t_2)} \right|^{-l} (1+\frac{r}{T-t_2})^m \frac{1}{r}\, dr \\
&= \int_{(T-t_1)^{1+\alpha}}^{(T-t_2)/2} \frac{1}{r}\left(1 + O(\frac{r}{T-t_2}) \right) dr \\
&\leq \alpha|\log(T-t_2)| + C.
\end{aligned}$$

With this we deduce

$$\begin{aligned}
&\left| \int_{(T-t_1)^{1+\alpha}}^{(T-t_2)/2} \frac{g(t_2-r)-g(t_1-r)}{r}\, dr \right| \\
&\leq [g]_{\gamma,m,l}(t_2-t_1)^\gamma \frac{(T-t_2)^m}{|\log(T-t_2)|^l} (\alpha|\log(T-t_2)| + C) \\
&\leq [g]_{\gamma,m,l}(t_2-t_1)^\gamma \frac{(T-t_2)^m}{|\log(T-t_2)|^{l-1}} \left(\alpha + \frac{C}{|\log T|} \right). \qquad (13.56)
\end{aligned}$$

For the third term in (13.54) we compute

$$\left| \int_{(T-t_2)/2}^{(T-t_1)/2} \frac{g(t_1-r)}{r}\, dr \right| \leq \|g\|_{\mu,l} \int_{(T-t_2)/2}^{(T-t_1)/2} \frac{(T-t_1+r)^\mu}{|\log(T-t_1+r)|^l} \frac{1}{r}\, dr$$

and we estimate the integral

$$\begin{aligned}
\int_{(T-t_2)/2}^{(T-t_1)/2} \frac{(T-t_1+r)^\mu}{|\log(T-t_1+r)|^l} \frac{1}{r}\, dr &\leq C \frac{(T-t_1)^\mu}{|\log(T-t_1)|^l} \frac{t_2-t_1}{T-t_2} \\
&\leq C(t_2-t_1)^\gamma \frac{(T-t_1)^{\mu-\gamma}}{|\log(T-t_1)|^l}.
\end{aligned}$$

Since $m \leq -\gamma$, we obtain

$$\left| \int_{(T-t_2)/2}^{(T-t_1)/2} \frac{g(t_1-r)}{r}\, dr \right| \leq \frac{C}{|\log T|}(t_2-t_1)^\gamma \frac{(T-t_1)^m}{|\log(T-t)|^{l-1}} T^{\mu-m-\gamma}\|g\|_{\mu,l}. \qquad (13.57)$$

From (13.55), (13.56) and (13.57) we obtain

$$[L_{12}[g]]_{\gamma,m,l-1} \leq \left(\alpha + C\frac{\log|\log T|}{|\log T|} \right) ([g]_{\gamma,m,l} + T^{\mu-m-\gamma}\|g\|_{\mu,l}). \qquad (13.58)$$

Next we analyze L_{11}. The largest term in $L_{11}[g](t_2) - L_{11}[g](t_1)$ is

$$\log(|\log(T-t_2)|)g(t_2) - \log(|\log(T-t_1)|)g(t_1) = l_1 + l_2$$

where

$$l_1 = [\log(|\log(T-t_2)|) - \log(|\log(T-t_1)|)]g(t_2)$$
$$l_2 = \log(|\log(T-t_2)|)(g(t_2) - g(t_1)).$$

Then

$$\begin{aligned}|l_1| &\le C\frac{t_2-t_1}{(T-t_2)|\log(T-t_2)|}\frac{(T-t_2)^\mu}{|\log(T-t_2)|^l}\|g\|_{\mu,l}\\ &\le C(t_2-t_1)^\gamma\frac{(T-t_2)^{\mu-\gamma}}{|\log(T-t_2)|^{l+1}}\|g\|_{\mu,l}\\ &\le C(t_2-t_1)^\gamma\frac{(T-t_2)^m}{|\log(T-t_2)|^{l-1}}T^{\mu-m-\gamma}|\log T|^{-2}\|g\|_{\mu,l}\end{aligned}$$

and

$$\begin{aligned}|l_2| &\le \log(|\log(T-t_1)|)(t_2-t_1)^\gamma\frac{(T-t_2)^m}{|\log(T-t_2)|^l}[g]_{\gamma,m,l}\\ &\le C\frac{\log|\log T|}{|\log T|}(t_2-t_1)^\gamma\frac{(T-t_2)^m}{|\log(T-t_2)|^{l-1}}[g]_{\gamma,m,l}.\end{aligned}$$

From this we obtain

$$[L_{11}[g]]_{\gamma,m,l-1} \le C\frac{\log|\log T|}{|\log T|}[g]_{\gamma,m,l}. \tag{13.59}$$

From (13.58) and (13.59) we obtain the validity of (13.52).

Then the conclusion of the lemma is obtained from the contraction mapping theorem. □

Proof of Lemma 13.5. In Lemma 13.2, p_2 is constructed as the solution of the fixed point problem (13.29). From this equation and (13.50) we get

$$\begin{aligned}[\dot p_2]_{\gamma,m,l} &\lesssim [\tilde{\mathcal{B}}[p_\kappa+p_2] - \tilde{\mathcal{B}}[p_\kappa]]_{\gamma,m,l-1} + [a(\cdot)-a(T)]_{\gamma,m,l-1}\\ &\quad + T^\mu\frac{\log|\log T|}{|\log T|}\|\tilde{\mathcal{B}}[p_\kappa+p_2] - \tilde{\mathcal{B}}[p_\kappa]\|_{\mu,l-1}\\ &\quad + T^\mu\frac{\log|\log T|}{|\log T|}\|a(\cdot)-a(T)\|_{\mu,l-1}.\end{aligned} \tag{13.60}$$

We have the following estimate

$$\begin{aligned}&\big[\tilde{\mathcal{B}}[p_\kappa+p_1] - \tilde{\mathcal{B}}[p_\kappa+p_2]\big]_{\gamma,m,l-1}\\ &\quad\lesssim \frac{1}{|\log T|}\Big([\dot p_1-\dot p_2]_{\gamma,m,l} + T^{\mu-\gamma-m}\|\dot p_1-\dot p_2\|_{\mu,l}\Big),\end{aligned} \tag{13.61}$$

for p_1, p_2 in $\overline{B}_{M_1}$ (with $M_1 = C_0\|a(\cdot)-a(T)\|_{\mu,l-1}$ defined in Lemma 13.2). We give a sketch of a proof below.

From (13.60), (13.61) and (13.32) we get

$$\begin{aligned}[\dot p_2]_{\gamma,m,l} &\lesssim \frac{1}{|\log T|}\Big([\dot p_2]_{\gamma,m,l} + T^{\mu-\gamma-m}\|\dot p_2\|_{\mu,l}\Big)\\ &\quad + [a(\cdot)-a(T)]_{\gamma,m,l-1} + T^\mu\frac{\log|\log T|}{|\log T|^2}\|\dot p_2\|_{\mu,l}\\ &\quad + T^\mu\frac{\log|\log T|}{|\log T|}\|a(\cdot)-a(T)\|_{\mu,l-1}.\end{aligned}$$

Therefore for $T > 0$ small, and using the estimate (13.22) we get

$$[\dot p_2]_{\gamma,m,l} \lesssim \frac{T^\mu}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|a(\cdot) - a(T)\|_{\mu,l-1} + [a(\cdot) - a(T)]_{\gamma,m,l-1}.$$

This is (13.48). □

Proof of (13.61). We do just some of the terms in the difference. Consider $D_{i,a}$ defined in (13.34) and let us analyze the following term in this difference

$$f(t) := \frac{(p_\kappa + p_2)(t)}{|(p_\kappa + p_2)(t)|}\operatorname{Re}\left(\frac{(\bar p_\kappa + \bar p_2)(t)}{|(p_\kappa + p_2)(t)|}(\tilde{\mathcal{B}}_{i,a}[p_\kappa + p_1](t) - \tilde{\mathcal{B}}_{i,a}[p_\kappa + p_2](t))\right).$$

We want to measure the Hölder seminorm of this expression, and for this let $t, s \in [-T, T]$ with $0 \le t - s \le \frac{1}{10}(T - t)$. In the expression $f(t) - f(s)$ there are several terms, and let us consider

$$(\tilde{\mathcal{B}}_{i,a}[p_\kappa + p_1](t) - \tilde{\mathcal{B}}_{i,a}[p_\kappa + p_2](t)) - (\tilde{\mathcal{B}}_{i,a}[p_\kappa + p_1](s) - \tilde{\mathcal{B}}_{i,a}[p_\kappa + p_2](s)).$$

Writing

$$\tilde{\mathcal{B}}_{i,a}[p_\kappa + p_1](t) - \tilde{\mathcal{B}}_{i,a}[p_\kappa + p_2](t) = \int_0^1 \frac{d}{d\zeta}\tilde{\mathcal{B}}_{i,a}[p_\kappa + p_\zeta](t)$$

where $p_\zeta = \zeta p_1 + (1-\zeta)p_2$ we see that it is sufficient to estimate $\tilde f(t) - \tilde f(s)$ where

$$\tilde f(t) = \tilde f_1(t) + \tilde f_2(t)$$

and

$$\begin{aligned}\tilde f_1(t) &= \int_{T+t}^{\lambda_*(t)^2} \frac{\dot p_1(t-z) - \dot p_2(t-z)}{z}\left(\Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{z}\right) - 1\right) dz\\ \tilde f_2(t) &= 2(p_\kappa(t) + p_\zeta(t)) \cdot (p_1(t) - p_2(t))\\ &\quad \cdot \int_{-T}^{t-\lambda_*(t)^2} \frac{(\dot p_\kappa + \dot p_\zeta)(s)}{(t-s)^2}\Gamma_i'\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{t-s}\right) ds.\end{aligned}$$

In fact we claim that

$$\begin{aligned}|\tilde f(t) - \tilde f(s)| &\le C[\dot p_1 - \dot p_2]_{\gamma,m,l}(t-s)^\gamma \frac{(T-t)^m}{|\log(T-t)|^l}\\ &\quad + C\|\dot p_1 - \dot p_2\|_{\mu,l}\frac{T^{\mu-\gamma-m}(T-t)^m(t-s)^\gamma}{|\log(T-t)|^l}.\end{aligned} \tag{13.62}$$

We estimate

$$|\tilde f_1(t) - \tilde f_1(s)| \le I + II$$

where

$$I = \int_{\lambda_*(s)^2}^{T+t} \left| \frac{\dot p_1(t-z) - \dot p_2(t-z)}{z} - \frac{\dot p_1(s-z) - \dot p_2(s-z)}{z} \right| \cdot \left| \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{z} \right) - 1 \right| dz$$

$$II = \int_{\lambda_*(s)^2}^{T+t} \left| \frac{\dot p_1(s-z) - \dot p_2(s-z)}{z} \right| \cdot \left| \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{z} \right) - \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(s)|^2}{z} \right) \right| dz$$

$$III = \int_{\lambda_*(s)^2}^{\lambda_*(t)^2} \left| \frac{\dot p_1(s-z) - \dot p_2(s-z)}{z} \left(\Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(s)|^2}{z} \right) - 1 \right) \right| dz$$

$$IV = \int_{T+s}^{T+t} \left| \frac{\dot p_1(t-z) - \dot p_2(t-z)}{z} \left(\Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{z} \right) - 1 \right) \right| dz$$

We have

$$I \le [\dot p_1 - \dot p_2]_{\gamma,m,l}(t-s)^\gamma \int_{\lambda_*(s)^2}^{T+t} \frac{(T-(t-z))^m}{|\log(T-(t-z))|^l} \frac{1}{z} \cdot \left| \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{z} \right) - 1 \right| dz,$$

but

$$\begin{aligned} &\int_{\lambda_*(s)^2}^{T+t} \frac{(T-(t-z))^m}{|\log(T-(t-z))|^l} \frac{1}{z} \left| \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{z} \right) - 1 \right| dz \\ &\quad = \int_{-T}^{t-\lambda_*(s)^2} \frac{(T-z)^m}{|\log(T-z)|^l (t-z)} \left| \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{t-z} \right) - 1 \right| dz \\ &\quad \le C\lambda_*(t)^{2\sigma} \int_{-T}^{t-\lambda_*(s)^2} \frac{(T-z)^m}{|\log(T-z)|^l (t-z)^{1+\sigma}} dz \\ &\quad \le C \frac{(T-t)^m}{|\log(T-t)|^l}, \end{aligned}$$

by Lemma 13.11, and we get

$$I \le C[\dot p_1 - \dot p_2]_{\gamma,m,l}(t-s)^\gamma \frac{(T-t)^m}{|\log(T-t)|^l}. \tag{13.63}$$

We next estimate

$$\begin{aligned} II &\le \int_{-T+s-t}^{s-\lambda_*(s)^2} \left| \frac{\dot p_1(z) - \dot p_2(z)}{s-z} \right| \left| \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{s-z} \right) - \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(s)|^2}{s-z} \right) \right| dz \\ &\le C\|\dot p_1 - \dot p_2\|_{\mu,l} \int_{-T+s-t}^{s-\lambda_*(s)^2} \frac{(T-z)^\mu}{|\log(T-z)|^l} \frac{1}{s-z} \cdot \left| \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(t)|^2}{s-z} \right) - \Gamma_i\left(\frac{|(p_\kappa + p_\zeta)(s)|^2}{s-z} \right) \right| dz \end{aligned}$$

We have

$$\begin{aligned}
&\left|\Gamma_i\left(\frac{|(p_\kappa+p_\zeta)(t)|^2}{s-z}\right)-\Gamma_i\left(\frac{|(p_\kappa+p_\zeta)(s)|^2}{s-z}\right)\right|\\
&\leq\left|\Gamma_i'\left(\frac{|\tilde p(t)|^2}{s-z}\right)\right|\left|\frac{|(p_\kappa+p_\zeta)(t)|^2}{s-z}-\frac{|(p_\kappa+p_\zeta)(s)|^2}{s-z}\right|\\
&\leq C\left(\frac{\lambda_*(t)^2}{s-z}\right)^{-\sigma}\lambda_*(t)\frac{|(p_\kappa+p_\zeta)(t)-(p_\kappa+p_\zeta)(s)|}{s-z},
\end{aligned}\tag{13.64}$$

where $\sigma\in(0,1)$. Since $p_\kappa=p_{0,\kappa}+p_1$ and p_1 has the estimate (13.7) for its derivative, we get

$$|p_\kappa(t)-p_\kappa(s)|\leq C\frac{|\log T|}{|\log(T-t)|^2}(t-s).$$

On the other hand

$$\begin{aligned}
|p_\zeta(t)-p_\zeta(s)|&\leq(\|\dot p_1\|_{\mu,l}+\|\dot p_2\|_{\mu,l})\frac{(T-t)^\mu}{|\log(T-t)|^l}(t-s)\\
&\leq(\|\dot p_1\|_{\mu,l}+\|\dot p_2\|_{\mu,l})\frac{T^\mu}{|\log T|^{l-1}}\frac{|\log T|}{|\log(T-t)|^2}(t-s)
\end{aligned}$$

so that

$$\begin{aligned}
&|(p_\kappa+p_\zeta)(t)-(p_\kappa+p_\zeta)(s)|\\
&\quad\leq C\frac{|\log T|}{|\log(T-t)|^2}\left(1+(\|\dot p_1\|_{\mu,l}+\|\dot p_2\|_{\mu,l})\frac{T^\mu}{|\log T|^{l-1}}\right)(t-s).
\end{aligned}$$

Since we are assuming (13.47) and we have $\|\dot p_j\|_{\mu,l}\leq C_0\|a(t)-a(T)\|_{\mu,l-1}$, we have

$$T^\mu|\log T|^{1+\sigma-l}\|p_j\|_{\mu,l}\leq C,$$

for simplicity we will use the estimate

$$|(p_\kappa+p_\zeta)(t)-(p_\kappa+p_\zeta)(s)|\leq C\frac{|\log T|}{|\log(T-t)|^2}(t-s).$$

Then

$$\begin{aligned}
II\leq C\|\dot p_1-\dot p_2\|_{\mu,l}\lambda_*(t)^{1-2\sigma}\frac{|\log T|}{|\log(T-t)|^2}(t-s)&\\
\cdot\int_{-T+s-t}^{s-\lambda_*(s)^2}\frac{(T-z)^\mu}{|\log(T-z)|^l}\frac{1}{(s-z)^{2-\sigma}}\,dz.&
\end{aligned}$$

But

$$\int_{-T+s-t}^{s-\lambda_*(s)^2}\frac{(T-z)^\mu}{|\log(T-z)|^l}\frac{1}{(s-z)^{2-\sigma}}\,dz\leq C\frac{(T-t)^\mu}{|\log(T-t)|^l}\frac{1}{\lambda_*(t)^{2(1-\sigma)}}$$

and therefore

$$\begin{aligned}
II&\leq C\|\dot p_1-\dot p_2\|_{\mu,l}\frac{|\log T|}{|\log(T-t)|^2}\frac{(T-t)^\mu(t-s)}{|\log(T-t)|^l}\frac{1}{\lambda_*(t)}\\
&\leq C\|\dot p_1-\dot p_2\|_{\mu,l}\frac{T^{\mu-\gamma-m}(T-t)^m(t-s)^\gamma}{|\log(T-t)|^l}.
\end{aligned}\tag{13.65}$$

Let us estimate $I\!I\!I$. We have for $\sigma \in (0,1)$:

$$
\begin{aligned}
I\!I\!I &= \int_{\lambda_*(s)^2}^{\lambda_*(t)^2} \left| \frac{\dot p_1(s-z) - \dot p_2(s-z)}{z} \left(\Gamma_i \left(\frac{|(p_\kappa + p_\zeta)(s)|^2}{z} \right) - 1 \right) \right| dz \\
&\le C \|\dot p_1 - \dot p_2\|_{\mu,l} \lambda_*(s)^{2\sigma} \int_{\lambda_*(s)^2}^{\lambda_*(t)^2} \frac{1}{z^{1+\sigma}} \frac{(T-(s-z))^\mu}{|\log(T-(s-z))|^l} \, dz \\
&\le C \|\dot p_1 - \dot p_2\|_{\mu,l} \frac{(T-t)^\mu}{|\log(T-t)|^l} \lambda_*(s)^{2\sigma} \left(\lambda_*(s)^{-2\sigma} - \lambda_*(t)^{-2\sigma} \right) \\
&\le C \|\dot p_1 - \dot p_2\|_{\mu,l} \frac{(T-t)^\mu}{|\log(T-t)|^l} \lambda_*(s)^{2\sigma} \lambda_*(s)^{-1-2\sigma} |\dot\lambda_*(s)| (t-s) \\
&\le C \|\dot p_1 - \dot p_2\|_{\mu,l} \frac{(T-t)^{\mu-1}(t-s)}{|\log(T-t)|^l} \\
&\le C \|\dot p_1 - \dot p_2\|_{\mu,l} \frac{(T-t)^\mu}{|\log(T-t)|^l} \frac{t-s}{T-t} \\
& C \|\dot p_1 - \dot p_2\|_{\mu,l} \frac{T^{\mu-\gamma-m}(T-t)^m (t-s)^\gamma}{|\log(T-t)|^l} .
\end{aligned} \tag{13.66}
$$

Next we handle $I\!V$:

$$
\begin{aligned}
I\!V &= \int_{T+s}^{T+t} \left| \frac{\dot p_1(s-z) - \dot p_2(s-z)}{z} \left(\Gamma_i \left(\frac{|(p_\kappa + p_\zeta)(s)|^2}{z} \right) - 1 \right) \right| dz \\
&\le C \|\dot p_1 - \dot p_2\|_{\mu,l} \lambda_*(s)^{2\sigma} \int_{T+s}^{T+t} \frac{1}{z^{1+\sigma}} \frac{(T-(s-z))^\mu}{|\log(T-(s-z))|^l} \, dz \\
&= C \|\dot p_1 - \dot p_2\|_{\mu,l} \lambda_*(s)^{2\sigma} \int_{-T}^{-T-t+s} \frac{1}{(s-z)^{1+\sigma}} \frac{(T-z)^\mu}{|\log(T-z)|^l} \, dz \\
&\le C \|\dot p_1 - \dot p_2\|_{\mu,l} \lambda_*(s)^{2\sigma} \frac{T^{1+\mu-\sigma}(t-s)}{|\log T|^l} \\
& C \|\dot p_1 - \dot p_2\|_{\mu,l} \frac{T^{\mu-\gamma-m}(T-t)^m (t-s)^\gamma}{|\log(T-t)|^l} .
\end{aligned} \tag{13.67}
$$

From (13.63), (13.65), (13.66) and (13.67) we deduce that

$$
\begin{aligned}
|\tilde f_1(t) - \tilde f_1(s)| &\le C [\dot p_1 - \dot p_2]_{\gamma,m,l} (t-s)^\gamma \frac{(T-t)^m}{|\log(T-t)|^l} \\
&\quad + C \|\dot p_1 - \dot p_2\|_{\mu,l} \frac{T^{\mu-\gamma-m}(T-t)^m (t-s)^\gamma}{|\log(T-t)|^l} .
\end{aligned}
$$

The estimate for $\tilde f_2$ is the same and we get (13.62).

□

We will also need a Lipschitz estimate of p_2 as a function of κ and $a(t)$ in the semi norm $[\ ]_{\gamma,m,l}$.

Lemma 13.7. *Let $\alpha \in (0, \frac{1}{2})$, $\mu, \gamma \in (0,1)$, $m \le \mu - \gamma$, $l \in \mathbb{R}$. Assume that for $j = 1,2$, we have $\mathrm{Re}(a_j(T)) < 0$ with $\frac{1}{C_1} \le \mathrm{Re}(a_j(T)) \le C_1$ and*

$$
T^\mu |\log T|^{1+\sigma-l} \|a_j(\cdot) - a_j(T)\|_{\mu,l-1} + [a_j]_{\gamma,m,l-1} \le C_1,
$$

for some $\sigma > 0$, and that κ_1, κ_2 satisfy (13.4). Then the solution $p_2 = p_2[a,\kappa]$ constructed in Lemma 13.2 satisfies

$$
\begin{aligned}
&[\dot p_2[a_1,\kappa_1] - \dot p_2[a_2,\kappa_1]]_{\gamma,m,l}\\
&\quad\lesssim [a_1-a_2]_{\gamma,m,l-1}\\
&\qquad + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1(\cdot)-a_1(T)-(a_2(\cdot)-a_2(T))\|_{\mu,l-1},
\end{aligned}
\tag{13.68}
$$

and

$$
[\dot p_2[a_1,\kappa_1] - \dot p_2[a_1,\kappa_2]]_{\gamma,m,l} \le C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}|\kappa_1-\kappa_2|. \tag{13.69}
$$

Proof. To prove this result we proceed as in the proof of Lemma 13.4, estimating the directional derivative

$$
p_2'(t) = \frac{d}{ds}p_2[a+sa_1,\kappa](t)\big|_{s=0}.
$$

This function satisfies (13.41).

The operator $D\tilde{\mathcal{B}}[p_\kappa+p_2]$ satisfies the estimate

$$
[D\tilde{\mathcal{B}}[p_\kappa+p_2](v)]_{\gamma,\mu,l-1} \le C\frac{1}{|\log T|}\Big([\dot v]_{\gamma,m,l} + T^{\mu-\gamma-m}\|\dot v\|_{\mu,l}\Big) \tag{13.70}
$$

which follows from (13.61). Using the above estimate together with equation (13.41), and Lemma 13.6 we deduce

$$
\begin{aligned}
[\dot p_2']_{\gamma,m,l} &\lesssim [D\tilde{\mathcal{B}}[p_\kappa+p_2](p_2')]_{\gamma,m,l-1} + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|D\tilde{\mathcal{B}}[p_\kappa+p_2](p_2')\|_{\mu,l-1}\\
&\quad + [a_1]_{\gamma,m,l-1} + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1\|_{\mu,l-1}\\
&\lesssim \frac{1}{|\log T|}\Big([\dot p_2']_{\gamma,m,l} + T^{\mu-\gamma-m}\|\dot p_2'\|_{\mu,l}\Big)\\
&\quad + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|D\tilde{\mathcal{B}}[p_\kappa+p_2](p_2')\|_{\mu,l-1}\\
&\quad + [a_1]_{\gamma,m,l-1} + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1\|_{\mu,l-1}.
\end{aligned}
$$

For $T > 0$ small and using (13.32) we get

$$
[\dot p_2']_{\gamma,m,l} \lesssim \frac{T^{\mu-\gamma-m}}{|\log T|}\|\dot p_2'\|_{\mu,l} + [a_1]_{\gamma,m,l-1} + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1\|_{\mu,l-1}.
$$

Estimate (13.39) translates into

$$
\|\dot p_2'\|_{\mu,l} \lesssim \|a_1(\cdot)-a_1(T)\|_{\mu,l-1} \tag{13.71}
$$

so that in the end we find that

$$
[\dot p_2']_{\gamma,m,l} \lesssim [a_1]_{\gamma,m,l-1} + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1\|_{\mu,l-1}.
$$

This is (13.68).

For the proof of (13.69) we proceed similarly. We have the derivative of (13.20) with respect to κ in (13.43), where $()' = \frac{d}{d\kappa}$. We claim that

$$
[D\tilde{\mathcal{B}}[p_\kappa+p_2](p_\kappa') - D\tilde{\mathcal{B}}[p_\kappa](p_\kappa')]_{\gamma,m,l-1} \le C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}, \tag{13.72}
$$

and give a proof of it later on.

From equation (13.43) and (13.70), (13.72) we deduce that

$$
\begin{aligned}
[\dot p_2']_{\gamma,m,l} &\le C[D\tilde{\mathcal B}[p_\kappa+p_2](p_2')]_{\gamma,m,l-1} + C[D\tilde{\mathcal B}[p_\kappa+p_2](p_\kappa') - D\tilde{\mathcal B}[p_\kappa](p_\kappa')]_{\gamma,m,l-1}\\
&\quad + CT^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|D\tilde{\mathcal B}[p_\kappa+p_2](p_2')\|_{\mu,l-1}\\
&\quad + CT^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|D\tilde{\mathcal B}[p_\kappa+p_2](p_\kappa') - D\tilde{\mathcal B}[p_\kappa](p_\kappa')\|_{\mu,l-1}\\
&\le \frac{C}{|\log T|}\Big([\dot p_2']_{\gamma,m,l} + T^{\mu-\gamma-m}\|\dot p_2'\|_{\mu,l}\Big) + C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}\\
&\quad + CT^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|D\tilde{\mathcal B}[p_\kappa+p_2](p_2')\|_{\mu,l-1}\\
&\quad + CT^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|D\tilde{\mathcal B}[p_\kappa+p_2](p_\kappa') - D\tilde{\mathcal B}[p_\kappa](p_\kappa')\|_{\mu,l-1}.
\end{aligned}
$$

Then from (13.71), (13.42), and (13.44) we get

$$
[\dot p_2']_{\gamma,m,l} \le C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}.
$$

Let us prove (13.72). Let us analyze the following term in the expression $D\tilde{\mathcal B}[p_\kappa+p_2](p_\kappa') - D\tilde{\mathcal B}[p_\kappa](p_\kappa')$

$$
\frac{p_\kappa}{|p_\kappa|}\operatorname{Re}\left(\frac{\bar p_\kappa}{|p_\kappa|}D\tilde{\mathcal B}_{1,a}[p_\kappa+p_2](p_\kappa') - D\tilde{\mathcal B}_{1,a}[p_\kappa](p_\kappa')\right)
$$

where $\mathcal B_{1,a}$ is defined in (13.35). We write

$$
D\tilde{\mathcal B}_{1,a}[p_\kappa+p_2](p_\kappa') - D\tilde{\mathcal B}_{1,a}[p_\kappa](p_\kappa') = f_1(t)+f_2(t)
$$

where

$$
\begin{aligned}
f_1(t) &= \int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p_\kappa'(s)}{t-s}\left(\Gamma_1\left(\frac{|(p_\kappa+p_2)(t)|^2}{t-s}\right) - \Gamma_1\left(\frac{|p_\kappa(t)|^2}{t-s}\right)\right)ds\\
f_2(t) &= 2(p_\kappa+p_2)(t)\cdot p_\kappa'(t)\int_{-T}^{t-\lambda_*(t)^2}\frac{(\dot p_\kappa+\dot p_2)(s)}{(t-s)^2}\Gamma_j'\left(\frac{|(p_\kappa+p_2)(t)|^2}{t-s}\right)ds\\
&\quad - 2p_\kappa(t)\cdot p_\kappa'(t)\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p_\kappa(s)}{(t-s)^2}\Gamma_j'\left(\frac{|p_\kappa(t)|^2}{t-s}\right)ds.
\end{aligned}
$$

Let us compute the Hölder semi norm $[\ ]_{\gamma,m,l}$ of f_1. Let then $t,s\in[-T,T]$, $0\le t-s\le \frac{1}{10}(T-t)$. We rewrite

$$
f_1(t) = \int_{\lambda_*(t)^2}^{T+t}\frac{\dot p_\kappa'(t-z)}{z}\left(\Gamma_1\left(\frac{|(p_\kappa+p_2)(t)|^2}{z}\right) - \Gamma_1\left(\frac{|p_\kappa(t)|^2}{z}\right)\right)dz
$$

and then

$$
\begin{aligned}
f_1(t)-f_1(s) &= \int_{\lambda_*(t)^2}^{T+t}\frac{\dot p_\kappa'(t-z)}{z}\left(\Gamma_1\left(\frac{|(p_\kappa+p_2)(t)|^2}{z}\right) - \Gamma_1\left(\frac{|p_\kappa(t)|^2}{z}\right)\right)dz\\
&\quad - \int_{\lambda_*(s)^2}^{T+s}\frac{\dot p_\kappa'(s-z)}{z}\left(\Gamma_1\left(\frac{|(p_\kappa+p_2)(s)|^2}{z}\right) - \Gamma_1\left(\frac{|p_\kappa(s)|^2}{z}\right)\right)dz\\
&= I + II + III + IV
\end{aligned}
$$

where

$$
\begin{aligned}
I &= -\int_{\lambda_*(s)^2}^{\lambda_*(t)^2} \frac{\dot p_\kappa'(s-z)}{z}\left(\Gamma_1\left(\frac{|(p_\kappa+p_2)(s)|^2}{z}\right)-\Gamma_1\left(\frac{|p_\kappa(s)|^2}{z}\right)\right) dz\\
II &= \int_{T+s}^{T+t} \frac{\dot p_\kappa'(t-z)}{z}\left(\Gamma_1\left(\frac{|(p_\kappa+p_2)(t)|^2}{z}\right)-\Gamma_1\left(\frac{|p_\kappa(t)|^2}{z}\right)\right) dz\\
III &= \int_{\lambda_*(t)^2}^{T+s} \left(\frac{\dot p_\kappa'(t-z)}{z}-\frac{\dot p_\kappa'(s-z)}{z}\right)\left(\Gamma_1\left(\frac{|(p_\kappa+p_2)(t)|^2}{z}\right)-\Gamma_1\left(\frac{|p_\kappa(t)|^2}{z}\right)\right) dz\\
IV &= \int_{\lambda_*(t)^2}^{T+s} \frac{\dot p_\kappa'(s-z)}{z}\left(\Gamma_1\left(\frac{|(p_\kappa+p_2)(t)|^2}{z}\right)-\Gamma_1\left(\frac{|p_\kappa(t)|^2}{z}\right)\right.\\
&\qquad\qquad \left. -\left[\Gamma_1\left(\frac{|(p_\kappa+p_2)(s)|^2}{z}\right)-\Gamma_1\left(\frac{|p_\kappa(s)|^2}{z}\right)\right]\right) dz
\end{aligned}
$$

We have, with a computation similar to (13.64), and using the bound for $\dot p_\kappa$ that we obtain from $p_\kappa = \kappa p_{0,\kappa} + p_1$, (13.7) and (13.22):

$$
\begin{aligned}
III &\le C\frac{|\log(T-t)|}{|\log(T-t)|^2}(t-s)\int_{\lambda_*(t)^2}^{T+s}\frac{1}{z}\left|\Gamma_1\left(\frac{|(p_\kappa+p_2)(t)|^2}{z}\right)-\Gamma_1\left(\frac{|p_\kappa(t)|^2}{z}\right)\right| dz\\
&\le C\frac{|\log(T-t)|}{|\log(T-t)|^2}(t-s)\lambda_*(t)^{1-2\sigma}\,||p_\kappa(t)+p_2(t)|-|p_\kappa(t)||\int_{\lambda_*(t)^2}^{T+s}\frac{1}{z^{2-\sigma}}\,dz\\
&\le C\frac{|\log(T-t)|}{|\log(T-t)|^2}(t-s)\lambda_*(t)^{-1}\|p_2\|_{\mu,l}\frac{(T-t)^\mu}{|\log(T-t)|^l}\\
&\le C\|p_2\|_{\mu,l}\frac{(T-t)^{\mu-1}}{|\log(T-t)|^l}(t-s)\\
&\le C\|a(\cdot)-a(T)\|_{\mu,l-1}\frac{T^{\mu-\gamma-m}(T-t)^m}{|\log(T-t)|^l}(t-s)^\gamma.
\end{aligned}
$$

Now we use (13.22) and find that

$$
III \le C\|a(\cdot)-a(T)\|_{\mu,l}\frac{T^{\mu-\gamma-m}(T-t)^m}{|\log(T-t)|^l}(t-s)^\gamma.
$$

For IV we claim that the same estimate holds

$$
IV \le C\|a(\cdot)-a(T)\|_{\mu,l-1}\frac{T^{\mu-\gamma-m}(T-t)^m}{|\log(T-t)|^l}. \tag{13.73}
$$

Let us estimate IV. For this we write

$$
\begin{aligned}
&\Gamma_1\left(\frac{|(p_\kappa+p_2)(t)|^2}{z}\right)-\Gamma_1\left(\frac{|p_\kappa(t)|^2}{z}\right)\\
&\qquad\qquad -\left[\Gamma_1\left(\frac{|(p_\kappa+p_2)(s)|^2}{z}\right)-\Gamma_1\left(\frac{|p_\kappa(s)|^2}{z}\right)\right]\\
&= \int_0^1 \frac{d}{d\zeta}\left[\Gamma_1\left(\frac{|(p_\kappa+\zeta p_2)(t)|^2}{z}\right)-\Gamma_1\left(\frac{|(p_\kappa+\zeta p_2)(s)|^2}{z}\right)\right] d\zeta\\
&= 2\int_0^1 \left[\Gamma_1'\left(\frac{|(p_\kappa+\zeta p_2)(t)|^2}{z}\right)\frac{(p_\kappa+\zeta p_2)(t)\cdot p_2(t)}{z}\right.\\
&\qquad\qquad \left. -\Gamma_1'\left(\frac{|(p_\kappa+\zeta p_2)(s)|^2}{z}\right)\frac{(p_\kappa+\zeta p_2)(s)\cdot p_2(s)}{z}\right] d\zeta\\
&= A_1+A_2+A_3
\end{aligned}
$$

$$A_1 = 4(t-s)\int_0^1\int_0^1 \Gamma_1''\left(\frac{|(p_\kappa+\zeta p_2)(t_v)|^2}{z}\right)\frac{(p_\kappa+\zeta p_2)(t_v)\cdot p_2(t_v)}{z} \cdot \frac{(p_\kappa+\zeta p_2)(t_v)\cdot(\dot p_\kappa+\zeta \dot p_2)(t_v)}{z}\,dvd\zeta$$

$$A_2 = 2(t-s)\int_0^1\int_0^1 \Gamma_1'\left(\frac{|(p_\kappa+\zeta p_2)(t_v)|^2}{z}\right)\frac{(\dot p_\kappa+\zeta \dot p_2)(t_v)\cdot p_2(t_v)}{z}\,dvd\zeta$$

$$A_3 = 2(t-s)\int_0^1\int_0^1 \Gamma_1'\left(\frac{|(p_\kappa+\zeta p_2)(t_v)|^2}{z}\right)\frac{(p_\kappa+\zeta p_2)(t_v)\cdot \dot p_2(t_v)}{z}\,dvd\zeta$$

where $t_v = vt+(1-v)s$.

Then we need to estimate

$$\int_{\lambda_*(t)^2}^{T+s}\frac{\dot p_\kappa'(s-z)}{z}A_j\,dz$$

for $j=1,2,3$. Using (13.22), we have (with $\sigma\in(0,1)$ fixed)

$$\begin{aligned}
&\left|\int_{\lambda_*(t)^2}^{T+s}\frac{\dot p_\kappa'(s-z)}{z}A_1\,dz\right| \\
&\leq C(t-s)|(p_\kappa+\zeta p_2)(t_v)|^{-2\sigma}|p_2(t_v)||(\dot p_\kappa+\zeta\dot p_2)(t_v)| \\
&\qquad\cdot\int_{\lambda_*(t)^2}^{T+s}\frac{1}{z^{2-\sigma}}|\dot\lambda_*(s-z)|\,dz \\
&\leq C(t-s)\lambda_*(t)^{-2\sigma}\|a(\cdot)-a(T)\|_{\mu,l-1}\frac{(T-t)^{\mu+1}}{|\log(T-t)|^l}\lambda_*(t)^{2\sigma-2}|\dot\lambda_*(t)|^2 \\
&\leq C\|a(\cdot)-a(T)\|_{\mu,l-1}(t-s)\frac{(T-t)^{\mu+1}}{|\log(T-t)|^l}\lambda_*(t)^{-2}|\dot\lambda_*(t)|^2 \\
&\leq C\|a(\cdot)-a(T)\|_{\mu,l-1}(t-s)\frac{(T-t)^{\mu-1}}{|\log(T-t)|^l} \\
&\leq C\|a(\cdot)-a(T)\|_{\mu,l-1}(t-s)^\gamma\frac{T^{\mu-\gamma-m}(T-t)^m}{|\log(T-t)|^l}.
\end{aligned}\tag{13.74}$$

Next, again using (13.22),

$$\begin{aligned}
&\left|\int_{\lambda_*(t)^2}^{T+s}\frac{\dot p_\kappa'(s-z)}{z}A_1\,dz\right| \\
&\leq C(t-s)|\dot\lambda_*(t)|^2\lambda_*(t)^{-2\sigma}\frac{(T-t)^{\mu+1}}{|\log(T-t)|^l}\|p_2\|_{\mu,l}\int_{\lambda_*(t)^2}^{T+s}\frac{1}{z^{2-\sigma}}\,dz \\
&\leq C(t-s)|\dot\lambda_*(t)|^2\lambda_*(t)^{-2\sigma}\frac{(T-t)^{\mu+1}}{|\log(T-t)|^l}\|p_2\|_{\mu,l}\lambda_*(t)^{2\sigma-2} \\
&\leq C(t-s)\frac{(T-t)^{\mu-1}}{|\log(T-t)|^l}\|p_2\|_{\mu,l} \\
&\leq C(t-s)^\gamma\frac{T^{\mu-\gamma-m}(T-t)^m}{|\log(T-t)|^l}\|p_2\|_{\mu,l} \\
&\leq C(t-s)^\gamma\frac{T^{\mu-\gamma-m}(T-t)^m}{|\log(T-t)|^l}\|a(\cdot)-a(T)\|_{\mu,l-1}.
\end{aligned}\tag{13.75}$$

In similar way we get

$$\left|\int_{\lambda_*(t)^2}^{T+s}\frac{\dot p_\kappa'(s-z)}{z}A_1\,dz\right| \leq C(t-s)^\gamma\frac{T^{\mu-\gamma-m}(T-t)^m}{|\log(T-t)|^l}\|a(\cdot)-a(T)\|_{\mu,l-1}.\tag{13.76}$$

Combining (13.74), (13.75) and (13.76) we get (13.73).

With similar estimate for I and II we get

$$[f_1]_{\gamma,m,l} \le C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a(T)\|_{\mu,l-1}.$$

The estimate of f_2 is similar and we omit the detalis.

□

Proof of Proposition 5.5. By Lemma 13.2 there is p_2 satisfying (13.19), where we have used this lemma with a replaced by $a-\mathcal{R}_1(\kappa)$, with $\mathcal{R}_1(\kappa)$ being the remainder appearing in (13.5).

Note that by (13.6) and using the assumption $\Theta<\alpha_0$, we have

$$\|\mathcal{R}_1(\kappa)\|_{\Theta,l-1} \le T^{\alpha_0-\Theta}|\log T|^{l-1}. \tag{13.77}$$

Therefore from (13.22) we find

$$\|\dot p_2\|_{\Theta,l} \le C\big(T^{\alpha_0-\Theta}|\log T|^{l-1} + \|a(\cdot)-a(T)\|_{\Theta,l-1}\big). \tag{13.78}$$

In equation (13.19) the constant c depends on κ and we claim that it is possible to choose κ satisfying (13.4) such that $c=0$. Evaluating (13.19) at $t=T$ we find

$$\int_{-T}^{T} \frac{\dot p_\kappa(s)+\dot p_2(s)}{T-s}\,ds = a(T)+c. \tag{13.79}$$

We consider then the equation $c=0$ with κ as an un known, that is, we look for κ satisfying

$$\int_{-T}^{T} \frac{\dot p_\kappa(s)+\dot p_2(s)}{T-s}\,ds = a(T). \tag{13.80}$$

(Note that p_2 also depends on κ.)

Let

$$f(\kappa,a) = \int_{-T}^{T} \frac{\dot p_\kappa(s)+\dot p_2(s)}{T-s}\,ds.$$

We claim that

$$f(\kappa,a) = \kappa + \tilde f(\kappa,a) \tag{13.81}$$

where $\tilde f$ satisfies

$$|\tilde f(\kappa_1,a)-\tilde f(\kappa_2,a)| \le \frac{C}{|\log T|}|\kappa_1-\kappa_2| \tag{13.82}$$

for κ_1, κ_2 satisfying (13.4). To prove this we write

$$f(\kappa,a) = f_0(\kappa)+f_1(\kappa)+f_2(\kappa,a)$$

where

$$\begin{aligned} f_0(\kappa) &= \int_{-T}^{T} \frac{\dot p_{0,\kappa}(s)}{T-s}\,ds = \kappa\int_{-T}^{T} \frac{\dot p_{0,1}(s)}{T-s}\,ds \\ f_1(\kappa) &= \int_{-T}^{T} \frac{\dot p_{1,\kappa}(s)}{T-s}\,ds \\ f_2(\kappa,a) &= \int_{-T}^{T} \frac{\dot p_2(s)}{T-s}\,ds. \end{aligned}$$

By the explicit formula (13.2)

$$f_0(\kappa) = 1+\tilde f_0(\kappa),$$

where $\tilde f_0$ satisfies (13.82). The functions $f_1(\kappa)$, $f_2(\kappa,a)$ also satisfy (13.82), which follows from the Lipschitz estimates (13.8) and (13.40). This proves (13.81).

Using (13.81) it follows that there exists a unique κ so that (13.80) holds. Moreover

$$\kappa[a] = a(T) + \kappa_1[a] \tag{13.83}$$

where the function κ_1 satisfies

$$|\kappa_1[a_1] - \kappa_1[a_2]| \le C\|a_1(\cdot) - a_1(T) - (a_2(\cdot) - a_2(T))\|_{\mu,l-1}\frac{T^\mu}{|\log T|^l}. \tag{13.84}$$

for a_1, a_2 satisfy $\mathrm{Re}(a_j(T)) < 0$ with $\frac{1}{C_1} \le \mathrm{Re}(a_j(T)) \le C_1$ and

$$T^\mu |\log T|^{1+\sigma-l}\|a_j(\cdot) - a_j(T)\|_{\mu,l-1} + [a_j]_{\gamma,m,l-1} \le C_1,$$

for some $\sigma > 0$. To prove this, we have to estimate

$$\begin{aligned}|f(\kappa, a_1) - f(\kappa, a_2)| &= |f_2(\kappa, a_1) - f_2(\kappa, a_2)|\\ &\le \int_{-T}^{T} \frac{|\dot p_{2,a_1}(s) - \dot p_{2,a_2}(s)|}{T-s}\, ds\end{aligned}$$

and using (13.39)

$$\begin{aligned}|f(\kappa, a_1) - f(\kappa, a_2)| &\le \|\dot p_2[a_1] - \dot p_2[a_2]\|_{\mu,l}\int_{-T}^{T} \frac{(T-s)^{\mu-1}}{|\log(T-s)|^l}\, ds\\ &\le C\|a_1(\cdot) - a_1(T) - (a_2(\cdot) - a_2(T))\|_{\mu,l-1}\frac{T^\mu}{|\log T|^l}.\end{aligned}$$

Now let us prove the estimate (5.27). For this we note that what we left out in (13.19) is $R_\alpha[\dot p_2]$ and hence the remainder $\mathcal{R}_0[a]$ is just $R_\alpha[\dot p_2]$. By Lemma 13.5 we have

$$\begin{aligned}[\dot p_2]_{\gamma,m,l} \le\ & C\frac{T^\Theta}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|a(\cdot) - a(T)\|_{\Theta,l-1}\\ &+ C\frac{T^\Theta}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|\mathcal{R}_1(\kappa)\|_{\Theta,l-1}\\ &+ C[a(\cdot) - a(T)]_{\gamma,m,l-1}\\ &+ C[\mathcal{R}_1(\kappa)]_{\gamma,m,l-1}.\end{aligned}$$

Using (13.10) we see that for $s \le t$ in $[0,T]$ such that $t - s \le \frac{1}{10}(T-t)$ we have

$$\frac{|\mathcal{R}_1(t) - \mathcal{R}_1(s)|}{(t-s)^\gamma} \le \lambda_*(t)^{\alpha_0-\gamma}$$

and since $m \le \Theta - \gamma$, $\Theta < \alpha_0$ by hypothesis, we get

$$[\mathcal{R}_1(\kappa)]_{\gamma,m,l-1} \le C\lambda_*(0)^\sigma$$

for some $\sigma > 0$.

Using this and (13.77) we obtain

$$[\dot p_2]_{\gamma,m,l} \lesssim T^\sigma + C\frac{T^\Theta}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1},$$

for some $\sigma > 0$. Then

$$\begin{aligned}|R_\alpha[\dot p_2]| &\le \int_{t-(T-t)^{1+\alpha}}^{t-\lambda_*(t)^2} \frac{|\dot p_2(t) - \dot p_2(s)|}{t-s}\, ds\\ &\le C[\dot p_2]_{\gamma,m,l}\int_{t-(T-t)^{1+\alpha}}^{t-\lambda_*(t)^2} (T-s)^m|\log(T-s)|^{-l}\frac{(t-s)^\gamma}{t-s}\, ds\\ &\le C[\dot p_2]_{\gamma,m,l}\frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}\end{aligned} \tag{13.85}$$

and therefore

$$|R_\alpha[\dot p_2]| \le C\Big(T^\sigma + C\frac{T^\Theta}{|\log T|}\left(T^{-\gamma-m} + \log|\log T|\right)\|a(\cdot)-a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}\Big) \cdot \frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}.$$

We now prove that $\mathcal{R}_0[a]$ is Lipschitz with respect to a. Let a_1, a_2 satisfy $\mathrm{Re}(a_j(T)) < 0$ with $\frac{1}{C_1} \le \mathrm{Re}(a_j(T)) \le C_1$ and

$$T^\mu |\log T|^{1+\sigma-l}\|a_j(\cdot)-a_j(T)\|_{\mu,l-1} + [a_j]_{\gamma,m,l-1} \le C_1,$$

for some $\sigma>0$. We let $\kappa_j = \kappa_j[a_j]$ as found in (13.83).

We have by the same computation as in (13.85)

$$|R_\alpha[\dot p_2[a_1,\kappa_1] - R_\alpha[\dot p_2[a_2,\kappa_2]| \le C[\dot p_2[a_1,\kappa_1] - \dot p_2[a_2,\kappa_2]]_{\gamma,m,l}\frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}.$$

Using (13.68) and (13.69) we then get

$$\begin{aligned}
&[\dot p_2[a_1,\kappa_1] - \dot p_2[a_2,\kappa_2]]_{\gamma,m,l}\\
&\le [a_1-a_2]_{\gamma,m,l-1}\\
&\quad + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1(\cdot)-a_1(T)-(a_2(\cdot)-a_2(T))\|_{\mu,l-1}\\
&\quad + C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}|\kappa_1-\kappa_2|,
\end{aligned}$$

and using (13.84)

$$\begin{aligned}
&[\dot p_2[a_1,\kappa_1] - \dot p_2[a_2,\kappa_2]]_{\gamma,m,l}\\
&\le [a_1-a_2]_{\gamma,m,l-1}\\
&\quad + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1(\cdot)-a_1(T)-(a_2(\cdot)-a_2(T))\|_{\mu,l-1}\\
&\quad + C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}|a_1(T)-a_2(T)|\\
&\quad + C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}\|a_1(\cdot)-a_1(T)-(a_2(\cdot)-a_2(T))\|_{\mu,l-1}\frac{T^\mu}{|\log T|^l}.
\end{aligned}$$

Using the assumption

$$T^\mu|\log T|^{-l}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1} \le C_1|\log T|^{-1-\sigma}$$

we may write

$$\begin{aligned}
&[\dot p_2[a_1,\kappa_1] - \dot p_2[a_2,\kappa_2]]_{\gamma,m,l}\\
&\le [a_1-a_2]_{\gamma,m,l-1}\\
&\quad + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1(\cdot)-a_1(T)-(a_2(\cdot)-a_2(T))\|_{\mu,l-1}\\
&\quad + C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}|a_1(T)-a_2(T)|.
\end{aligned}$$

Thus, finally we get

$$
\begin{aligned}
&|R_\alpha[\dot p_2[a_1,\kappa_1] - R_\alpha[\dot p_2[a_2,\kappa_2]| \\
&\quad \le C\frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}\cdot\Big([a_1-a_2]_{\gamma,m,l-1} \\
&\qquad + T^{\mu-m-\gamma}\frac{\log|\log T|}{|\log T|}\|a_1(\cdot)-a_1(T)-(a_2(\cdot)-a_2(T))\|_{\mu,l-1} \\
&\qquad + C\frac{T^{\mu-\gamma-m}}{|\log T|}\|a_1(\cdot)-a_1(T)\|_{\mu,l-1}|a_1(T)-a_2(T)|\Big).
\end{aligned}
$$

□

13.4. **Proof of Lemma 13.1.** To do this we look for p_κ of the form

$$p_\kappa = p_{0,\kappa} + p_1,$$

where $p_{0,\kappa}$ is defined in (13.2), and we would like

$$\mathcal{I}[p_{0,\kappa}] + \mathcal{I}[p_1] + \tilde{\mathcal{B}}[p_{0,\kappa}+p_1](t) - c(\kappa) = O((T-t)^{\alpha_0}) \quad \text{for } t\in[0,T]. \tag{13.86}$$

The idea is to replace in (13.86) the operator $\mathcal{I}[p_1]$ by $S_{\alpha_0}[\dot p_1]$ defined in (5.28) and try to solve the corresponding equation. We claim that if $\alpha_0>0$ is small, then we can find p_1 such that

$$\mathcal{I}[p_{0,\kappa}] + S_{\alpha_0}[\dot p_1] + \tilde{\mathcal{B}}[p_{0,\kappa}+p_1](t) - c(\kappa) = 0 \quad \text{in } [0,T], \tag{13.87}$$

for some $c(\kappa)$. This means that instead of (13.86) we have obtained

$$\mathcal{B}_0[p_{0,,\kappa}+p_1] - c(\kappa) = R_{\alpha_0}[\dot p_1] \quad \text{in } [0,T].$$

The second step is to prove that there is κ such that $c(\kappa)=A$. The final step is to show that

$$|R_{\alpha_0}[\dot p_1]| \le C(T-t)^{\alpha_0},$$

and this implies (13.86).

Construction of a solution to (13.87). To obtain a function p satisfying (13.87) we formulate a fixed point problem as follows.

We decompose

$$S_{\alpha_0}[g] = \tilde L_0[g] + \tilde L_1[g]$$

where

$$\tilde L_0[g](t) = (1-\alpha_0)|\log(T-t)|g(t) + \int_{-T}^t \frac{g(s)}{T-s}\,ds$$

and $\tilde L_1$ contains all other terms, that is,

$$
\begin{aligned}
\tilde L_1[g](t) = &\int_{t-(T-t)}^{t-(T-t)^{1+\alpha_0}} \frac{g(s)}{t-s}\,ds - \int_{t-(T-t)}^t \frac{g}{T-s}\,ds \\
&+ \int_{-T}^{t-(T-t)} g(s)\left(\frac{1}{t-s}-\frac{1}{T-s}\right)ds \\
&+ (4\log(|\log(T-t)|) - 2\log(|\log(T)|))g(t).
\end{aligned}
$$

Given a continuous function f in $[-T,T]$ with a certain modulus of continuity at T, we would like to find g such that

$$S_{\alpha_0}[\dot g] = f \quad \text{in } [-T,T].$$

We will not quite obtain this, but we will solve a modified version of this equation. Let η be a smooth cut-off function such that

$$\eta(s)=1 \quad \text{for } s\ge 0, \qquad \eta(s)=0 \quad \text{for } s\le -\frac14. \tag{13.88}$$

We will be able to find a function g such that

$$\tilde{L}_0[\dot{g}] + \eta(\frac{t}{T})\tilde{L}_1[\dot{g}] = f + c \quad \text{in } [-T, T]. \tag{13.89}$$

We use the norm $\| \|_{*,k}$ defined in (13.9) for the solution g of the above equation. For the right hand side of (13.89) we take the space $C([-T,T];\mathbb{C})$ with $f(T) = 0$ and the norm

$$\|f\|_{**,k} = \sup_{t\in[-T,T]} |\log(T-t)|^k |f(t)|. \tag{13.90}$$

Note that in (13.89) the expression $\eta(\frac{t}{T})\tilde{L}_1[\dot{g}](t)$ is well defined for g of class C^1 in $[-T, T)$. Indeed, because of the cut-off function, $\tilde{L}_1[\dot{g}](t)$ needs to be computed only for $t \geq -\frac{T}{4}$, and for $t \geq -\frac{T}{4}$ the integrals appearing in $L_1[\dot{g}]$ are well defined, since they start at either at $-T$ or $t - \frac{1}{2}(T-t) = \frac{3}{2}t - \frac{1}{2}T \geq -T$.

The next lemma gives the solvability of (13.89) in the weighted spaces introduced above. Let

$$\Upsilon = \frac{2-\alpha_0}{1-\alpha_0}$$

Lemma 13.8. *Let $C_2 > 1$ be fixed, κ satisfying (13.4), and assume that $k > \Upsilon - 1$. Then, there is $\bar{\alpha}_0 > 0$, so that for $0 < \alpha_0 \leq \bar{\alpha}_0$, and $T > 0$ small, there is a linear operator T_1 such that $g = T_1[f]$ satisfies (13.89) for some constant c and*

$$\|g\|_{*,k+1} + |c| \leq \frac{C}{k+1-\Upsilon}\|f\|_{**,k}. \tag{13.91}$$

The constant C is independent of T, α_0.

Let

$$E(t) := \mathcal{I}[p_{0,\kappa}](t), \tag{13.92}$$
$$\tilde{E}(t) = E(t) - E(T),$$

where $\mathcal{I}$ is given by (13.13), and consider the fixed point problem

$$p_1 = \mathcal{A}[p_1] \tag{13.93}$$

where

$$\mathcal{A}[p_1] = T_1\big[-\eta\tilde{E} - \tilde{\mathcal{B}}[p_{0,\kappa} + p_1]\big], \tag{13.94}$$

where η is the cut-off function defined in (13.88).

Note that if p_1 is a solution of (13.93) then p_1 satisfies

$$\tilde{L}_0[\dot{p}_1] + \eta(\frac{t}{T})\tilde{L}_1[\dot{p}_1] = \eta\tilde{E} - \tilde{\mathcal{B}}[p_{0,\kappa} + p_1](t) + c$$

in $[-T, T]$ for some constant c. This implies that p_1 satisfies

$$S_{\alpha_0}[\dot{p}_1] + \tilde{\mathcal{B}}[p_{0,\kappa} + p_1] - E = c$$

in $[0, T]$ for some possibly different constant c. This is precisely the equation (13.87).

Note that it is not necessary to subtract $\tilde{\mathcal{B}}[p_{0,\kappa} + p_1](T)$ in the argument of T_1 in (13.93), because for the class of functions p_1 that we consider we have $\tilde{\mathcal{B}}[p_{0,\kappa} + p_1](T) = 0$, see (13.108) later on.

Proposition 13.1. *Let $k > 0$, $k < 2$ close to 2 and $\alpha_0 > 0$ small. Then for $T > 0$ small there is a function p_1 satisfying (13.93) and moreover*

$$\|p_1\|_{*,k+1} \leq M \tag{13.95}$$

where

$$M = C_0|\log(T)|^{k-1}\log(|\log(T)|)^2, \tag{13.96}$$

with C_0 a fixed large constant.

Moreover, if we denote by $p_1(\kappa)$ the solution just constructed, we have, for κ_1, κ_2 satisfying (13.4)

$$\|p_1(\kappa_1) - p_1(\kappa_2)\|_{*,k+1} \leq C|\log T|^{k-1}\log(|\log T|)^2\,|\kappa_1 - \kappa_2|. \tag{13.97}$$

The rest of the subsection is devoted to the proof of Proposition 13.1.

We start with the construction of the linear operator T_1 in Lemma 13.8. We want to find an inverse for $\tilde{L}_0$, namely given f find g such that $\tilde{L}_0[\dot{g}] = f$. To do this, we differentiate this equation and we get

$$\ddot{g}(t) + \frac{2-\alpha_0}{1-\alpha_0}\frac{\dot{g}(t)}{(T-t)|\log(T-t)|} = \frac{1}{1-\alpha_0}\frac{\dot{f}(t)}{|\log(T-t)|}. \tag{13.98}$$

Then we can write a particular solution for $\dot{g}$ to (13.98) as

$$\dot{g}(t) = \frac{f(t)}{(1-\alpha_0)|\log(T-t)|} + \frac{\Upsilon - 1}{1-\alpha_0}|\log(T-t)|^{-\Upsilon}\int_t^T \frac{|\log(T-s)|^{\Upsilon-2}}{T-s}f(s)\,ds, \tag{13.99}$$

where $\Upsilon = \frac{2-\alpha_0}{1-\alpha_0}$ and where we have assumed that $\frac{|\log(T-s)|^{\Upsilon-2}}{T-s}f(s)$ is integrable near T (for example $f(s) = O(|\log(T-s)|^{-k})$ with $k > \Upsilon - 1$ suffices).

Define the operator

$$T_0[f] = g, \tag{13.100}$$

where g is such that $\dot{g}$ is given by (13.99) and $g(T) = 0$. Note that $g = T_0[f]$ solves (13.98) and therefore

$$\tilde{L}_0[\dot{g}] = f + c,$$

for some constant c.

Lemma 13.9. *Assume $k > \Upsilon - 1$. Then for $f \in C([-T,T];\mathbb{C})$ with $f(T) = 0$*

$$\|T_0[f]\|_{*,k+1} \leq \frac{C}{k+1-\Upsilon}\|f\|_{**,k}.$$

The constant is independent of Υ (if Υ is bounded), k, T.

Proof. This is direct from (13.99). □

Proof of Lemma 13.8. We construct g as a solution of the fixed point problem

$$g = T_0\left[f - \eta(\frac{t}{T})\tilde{L}_1[g]\right].$$

where T_0 is the operator constructed in (13.100) and η is the cut-off function (13.88).

By Lemma 13.9

$$\|T_0[\tilde{L}_1[g]]\|_{*,k+1} \leq \frac{C}{k+1-\Upsilon}\|\tilde{L}_1[g]\|_{**,k}.$$

Let us analyze the different terms in $\tilde{L}_1$, which we denote by

$$\tilde{L}_1 = \sum_{j=1}^4 \tilde{L}_{1j}$$

where

$$\tilde{L}_{11}[g](t) = \int_{t-(T-t)}^{t-(T-t)^{1+\alpha_0}} \frac{\dot{g}(s)}{t-s}\,ds \tag{13.101}$$

$$\tilde{L}_{12}[g](t) = \int_{t-(T-t)}^{t} \frac{\dot{g}(s)}{T-s}\,ds$$

$$\tilde{L}_{13}[g](t) = \int_{-T}^{t-(T-t)} \dot{g}(s)\left(\frac{1}{t-s} - \frac{1}{T-s}\right)ds$$

$$\tilde{L}_{14}[g](t) = (4\log(|\log(T-t)|) - 2\log(|\log(T)|))\dot{g}(t).$$

Then we have

$$
\begin{aligned}
|\tilde L_{11}[g](t)| &\leq \|g\|_{*,k+1} \int_{t-(T-t)}^{t-(T-t)^{1+\alpha_0}} \frac{1}{(t-s)|\log(T-s)|^{k+1}}\, ds \\
&\leq \|g\|_{*,k+1} \frac{1}{|\log(T-t)|^{k+1}} \int_{t-(T-t)}^{t-(T-t)^{1+\alpha_0}} \frac{1}{t-s}\, ds \\
&\leq \|g\|_{*,k+1} \frac{\alpha_0}{|\log(T-t)|^{k}}
\end{aligned}
$$

and therefore

$$
\|\tilde L_{11}[g]\|_{**,k} \leq \alpha_0 \|g\|_{*,k+1}.
$$

We also find that

$$
\begin{aligned}
|\tilde L_{12}[g](t)| &\leq \|g\|_{*,k+1} \int_{t-(T-t)}^{t} \frac{1}{(T-s)|\log(T-s)|^{k+1}}\, ds \\
&\leq \|g\|_{*,k+1} \frac{1}{|\log(T-t)|^{k+1}} \int_{t-(T-t)}^{t} \frac{1}{T-s}\, ds \\
&\leq \|g\|_{*,k+1} \frac{1}{|\log(T-t)|^{k+1}} \int_{t-(T-t)}^{t} \frac{1}{T-s}\, ds \\
&\leq \|g\|_{*,k+1} \frac{1}{|\log(T-t)|^{k+1}} \log(2),
\end{aligned}
$$

which implies

$$
\|\tilde L_{12}[g]\|_{**,k} \leq \frac{\log(2)}{|\log(T)|} \|g\|_{*,k+1}.
$$

Concerning $\tilde L_{13}$ we have

$$
\begin{aligned}
|\tilde L_{13}[g](t)| &\leq \|g\|_{*,k+1} \int_{-T}^{t-(T-t)} \frac{1}{|\log(T-s)|^{k+1}} \left(\frac{1}{t-s} - \frac{1}{T-s} \right) ds \\
&\leq C\|g\|_{*,k+1} (T-t) \int_{-T}^{t-(T-t)} \frac{1}{(T-s)^2|\log(T-s)|^{k+1}}\, ds \\
&\leq C\|g\|_{*,k+1} \frac{1}{|\log(T-t)|^{k+1}}
\end{aligned}
$$

and this gives

$$
\|\tilde L_{13}[g]\|_{**,k} \leq \frac{C}{|\log(T)|} \|g\|_{*,k+1}.
$$

Finally

$$
|\tilde L_{14}[g](t)| \leq C\|g\|_{*,k+1} \frac{\log(|\log(T-t)|) + \log(|\log(T)|)}{|\log(T-t)|^{k+1}}
$$

and hence using that $\kappa = O(1)$ we get

$$
\|(\tilde L_{14}[g]\|_{**,k} \leq \frac{C \log(|\log(T)|)}{|\log T|} \|g\|_{*,k+1}.
$$

Therefore

$$
\begin{aligned}
\|T_0[\tilde L_1[g]]\|_{*,k+1} &\leq \frac{C}{k+1-\Upsilon} \|\tilde L_1[g]\|_{**,k} \\
&\leq \frac{C}{k+1-\Upsilon} \left(\alpha_0 + \frac{1}{|\log T|} + \frac{\log|\log T|}{|\log T|} \right) \|g\|_{*,k+1}.
\end{aligned}
$$

we get a contraction if $\alpha_0 > 0$ is fixed small and then $T > 0$ is sufficiently small. □

Next we need an estimate for the error E defined in (13.92).

Lemma 13.10. *Let $p_{0,\kappa}$ be given by (13.2) and assume $\kappa \in \mathbb{C}$ satisfies (13.4). Then*

$$|E(t) - E(T)| \le C \frac{|\log T| \log|\log(T-t)|}{|\log(T-t)|^2}, \quad -\frac{T}{4} \le t \le T. \tag{13.102}$$

Proof. By definition we have

$$E(t) = \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p_{0,\kappa}(s)}{t-s}\, ds.$$

Let $t \in [-\frac{T}{4}, T]$ and let us write

$$\begin{aligned}
E(t) &= \int_{-T}^{t-(T-t)/5} \frac{\dot p_{0,\kappa}(s)}{t-s}\, ds + \int_{t-(T-t)/5}^{t-p_0(t)^2} \frac{\dot p_{0,\kappa}(s)}{t-s}\, ds \\
&= \int_{-T}^{t} \frac{\dot p_{0,\kappa}(s)}{T-s}\, ds - \int_{t-(T-t)/5}^{t} \frac{\dot p_{0,\kappa}(s)}{T-s}\, ds \\
&\quad + \int_{-T}^{t-(T-t)/5} \dot p_{0,\kappa}(s) \left(\frac{1}{t-s} - \frac{1}{T-s} \right) ds \\
&\quad + \int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{\dot p_{0,\kappa}(s)}{t-s}\, ds.
\end{aligned}$$

We estimate

$$\begin{aligned}
\left| \int_{t-(T-t)/5}^{t} \frac{\dot p_{0,\kappa}(s)}{T-s}\, ds \right| &\le C\kappa |\log T| \int_{t-(T-t)/5}^{t} \frac{1}{(T-s)|\log(T-s)|^2}\, ds \\
&\le \frac{C\kappa |\log T|}{(T-t)|\log(T-t)|^2} \int_{t-(T-t)/5}^{t} ds \\
&\le \frac{C\kappa|\log T|}{|\log(T-t)|^2},
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_{-T}^{t-(T-t)/5} \dot p_{0,\kappa}(s) \left(\frac{1}{t-s} - \frac{1}{T-s} \right) ds \right| \\
&\quad \le C\kappa |\log(T)| (T-t) \int_{-T}^{t-(T-t)/5} \frac{1}{|\log(T-s)|^2 (t-s)(T-s)}\, ds \\
&\quad \le C\kappa |\log(T)| (T-t) \int_{-T}^{t-(T-t)/5} \frac{1}{|\log(T-s)|^2 (T-s)^2}\, ds \\
&\quad \le \frac{C\kappa|\log(T)|}{|\log(T-t)|^2}.
\end{aligned}$$

With the fourth term in E we proceed as follows

$$\begin{aligned}
&\int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{\dot p_{0,\kappa}(s)}{t-s}\, ds \\
&= \dot p_{0,\kappa}(t) \int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{1}{t-s}\, ds - \int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{\dot p_{0,\kappa}(t) - \dot p_{0,\kappa}(s)}{t-s}\, ds \\
&= \dot p_{0,\kappa}(t) (\log(T-t) - 2\log(\lambda_*)) - \int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{\dot p_{0,\kappa}(t) - \dot p_{0,\kappa}(s)}{t-s}\, ds.
\end{aligned}$$

But

$$\left|\int_{t-(T-t)/5}^{t-\lambda_*(t)^2} \frac{\dot p_{0,\kappa}(t)-\dot p_{0,\kappa}(s)}{t-s}\,ds\right| \le \sup_{s\le t}|\ddot p_{0,\kappa}(s)|(T-t) \le \frac{C\kappa|\log(T)|}{|\log(T-t)|^3}.$$

Therefore we have obtained

$$E=\int_{-T}^{t}\frac{\dot p_{0,\kappa}(s)}{T-s}\,ds+\dot p_{0,\kappa}(t)(\log(T-t)-2\log(\lambda_*))+O(\frac{\kappa|\log(T)|}{|\log(T-t)|^2}).$$

We note that

$$\dot p_{0,\kappa}(t)|\log(T-t)|+\int_0^t\frac{\dot p_{0,\kappa}(s)}{T-s}\,ds=c$$

for some constant c. Indeed, by (13.3)

$$\begin{aligned}\frac{d}{dt}\left(\dot p_{0,\kappa}(t)|\log(T-t)|+\int_0^t\frac{\dot p_{0,\kappa}(s)}{T-s}\,ds\right)&=\ddot p_{0,\kappa}(t)|\log(T-t)|+2\frac{\dot p_{0,\kappa}(t)}{T-t}\\&=\frac{\frac{d}{dt}(\dot p_{0,\kappa}(t)|\log(T-t)|^2)}{|\log(T-t)|}\\&=0.\end{aligned}$$

This shows that

$$E(t)=E(T)+O(\frac{|\log T|[\log(|\log T|)+\log(|\log(T-t)|)]}{|\log(T-t)|^2}),$$

which implies the estimate (13.102). □

Proof of Proposition 13.1. Let T_1 be the operator constructed in Lemma 13.8 for $T>0$, $\alpha_0>0$ small and $\mathcal{A}$ defined in (13.94).

We will apply inequality (13.91) with $k<2$ close to 2. The constant in this inequality remains bounded as $\alpha_0\to 0^+$, because $\Upsilon=\frac{2-\alpha_0}{1-\alpha_0}\to 2$ as $\alpha_0\to 0^+$.

For the poof we use the norm (13.90) with $k<2$, k close to 2 so $k+1<3$ is close to 3. We work with p_1 in the space $X=C([-T,T];\mathbb{C})\cap C^1([-T,T);\mathbb{C})$ with the norm $\|\cdot\|_{*,k+1}$ defined in (13.9). By Lemma 13.8

$$\|\mathcal{A}[p_1]\|_{*,k+1}\le C\Big(\|\eta\tilde E\|_{**,k}+\|\tilde{\mathcal{B}}[p_{0,\kappa}+p_1](t)-\tilde{\mathcal{B}}[p_{0,\kappa}+p_1](T)\|_{**,k}\Big),\tag{13.103}$$

and by Lemma 13.10

$$\|\eta\tilde E\|_{**,k}\le C_E|\log T|^{k-1}\log(|\log T|),\tag{13.104}$$

for some $C_E>0$. We take in X the closed ball $\overline{B}_M(0)$ of center 0 and radius M given by (13.96) with $C_0>0$ suitably large. The proof of Proposition 13.1 consists in showing that $\mathcal{A}:\overline{B}_M(0)\to\overline{B}_M(0)$ is a contraction. The estimates required for this are the following: for $\|p_1\|_{*,k+1}\le M$ we have

$$\|\tilde{\mathcal{B}}[p_{0,\kappa}+p_1]\|_{**,k}\le C|\log(T)|^{k-1},\tag{13.105}$$

and for $\|p_i\|_{*,k+1}\le M$, $i=1,2$ we have

$$\|\tilde{\mathcal{B}}[p_{0,\kappa}+p_1]-\tilde{\mathcal{B}}[p_{0,\kappa}+p_2]\|_{**,k}\le\frac{C}{|\log T|}\|p_1-p_2\|_{*,k+1}.\tag{13.106}$$

These inequalities are proved in Lemmas 13.12 and 13.13 below.

Form these estimates we see that $\mathcal{A}$ is a contraction in the ball $\overline{B}_M$. Indeed, from (13.103), (13.104) and (13.105) we have

$$\begin{aligned}\|\mathcal{A}[p_1]\|_{*,k+1}&\le C\cdot C_E|\log T|^{k-1}\log(|\log T|)+C|\log(T)|^{k-1}\\&\le C_0|\log T|^{k-1}\log(|\log T|)^2\end{aligned}$$

by fixing C_0 large. Therefore $\mathcal{A} : \overline{B}_M(0) \to \overline{B}_M(0)$.

Next, for $\|p_i\|_{*,k+1} \le M$, $i = 1, 2$, by Lemma 13.8 and (13.106) we get

$$\begin{aligned}\|\mathcal{A}[p_1] - \mathcal{A}[p_2]\|_{*,k+1} &\le C\|\tilde{\mathcal{B}}[p_{0,\kappa} + p_1] - \tilde{\mathcal{B}}[p_{0,\kappa} + p_2]\|_{**,k} \\ &\le \frac{C}{|\log T|}\|p_1 - p_2\|_{*,k+1}.\end{aligned}$$

The proof of (13.97) will be given in Corollary 13.1 below. □

Lemma 13.11.

a) If $a > 1$, $b > 0$ then

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{1}{(t-s)^a|\log(T-s)|^b}\,ds \le C\frac{\lambda_*(t)^{2(1-a)}}{|\log(T-t)|^b}, \quad t \in [0,T].$$

b) If $\mu \in (0,1)$, $l \in \mathbb{R}$ then

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{(T-s)^\mu}{(t-s)^2|\log(T-s)|^l}\,ds \le C\frac{(T-t)^\mu}{\lambda_*(t)^2|\log(T-t)|^l}.$$

Proof. Let us start with property a). Consider first $t \in [0,T]$. Then we can write

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{1}{(t-s)^a|\log(T-s)|^b}\,ds = \int_{-T}^{t-(T-t)} \ldots + \int_{t-(T-t)}^{t-\lambda_*(t)^2}.$$

Then

$$\begin{aligned}\int_{-T}^{t-(T-t)} \frac{1}{(t-s)^a|\log(T-s)|^b}\,ds &\le C\int_{-T}^{t-(T-t)} \frac{1}{(T-s)^a|\log(T-s)|^b}\,ds \\ &\le C\frac{(T-t)^{1-a}}{|\log(T-t)|^b}\,ds \\ &\le C\frac{\lambda_*(t)^{2(1-a)}}{|\log(T-t)|^b}.\end{aligned}$$

The other integral is

$$\begin{aligned}\int_{t-(T-t)}^{t-\lambda_*(t)^2} \frac{1}{(t-s)^a|\log(T-s)|^b}\,ds &\le \frac{1}{|\log(T-t)|^b}\int_{t-(T-t)}^{t-\lambda_*(t)^2} \frac{1}{(t-s)^a}\,ds \\ &\le C\frac{\lambda_*(t)^{2(1-a)}}{|\log(T-t)|^b}.\end{aligned}$$

Now consider $t \in [-T, 0]$:

$$\begin{aligned}\int_{-T}^{t-\lambda_*(t)^2} \frac{1}{(t-s)^a|\log(T-s)|^b}\,ds &\le \frac{1}{|\log(T-t)|^b}\int_{-T}^{t-\lambda_*(t)^2} \frac{1}{(t-s)^a}\,ds \\ &\le C\frac{\lambda_*(t)^{2(1-a)}}{|\log(T-t)|^b}.\end{aligned}$$

As for property b). Again consider first $t \in [0,T]$. Then

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{(T-s)^\mu}{(t-s)^2|\log(T-s)|^l}\,ds = \int_{-T}^{t-(T-t)} \ldots + \int_{t-(T-t)}^{t-\lambda_*(t)^2}.$$

Then

$$\begin{aligned}\int_{-T}^{t-(T-t)} \frac{(T-s)^\mu}{(t-s)^2|\log(T-s)|^l}\,ds &\le C\int_{-T}^{t-(T-t)} \frac{(T-s)^{\mu-2}}{|\log(T-s)|^l}\,ds \\ &\le C\frac{(T-t)^{\mu-1}}{|\log(T-t)|^l},\end{aligned}$$

and

$$\int_{t-(T-t)}^{t-\lambda_*(t)^2} \frac{(T-s)^\mu}{(t-s)^2|\log(T-s)|^l}\,ds \le C\frac{(T-t)^\mu}{|\log(T-t)|^l}\int_{t-(T-t)}^{t-\lambda_*(t)^2}\frac{1}{(t-s)^2}\,ds$$
$$\le C\frac{(T-t)^\mu}{\lambda_*(t)^2|\log(T-t)|^l}.$$

The case $t\in[-T,0]$ is handled similarly as in part a). □

Lemma 13.12. *Let M be given by (13.96), that is, $M = C|\log T|^{k-1}\log(|\log T|)^2$. For $\|p_1\|_{*,k+1}\le M$ we have*

$$\|\tilde{\mathcal{B}}[p_{0,\kappa}+p_1](\cdot)-\tilde{\mathcal{B}}[p_{0,\kappa}+p_1](T)\|_{**,k}\le C|\log(T)|^{k-1}.$$

Proof. For these estimates it is useful to notice that with the choice of M, if $\|p_1\|_{*,k+1}\le M$ we have

$$\left|\frac{p_1}{p_{0,\kappa}}\right|\le C_0\frac{\log(|\log T|)^2}{|\log T|}\ll 1$$

for $T>0$ small.

Recall $\tilde B$ given by (13.14). The estimate

$$\|\tilde{\mathcal{B}}_5[p_{0,\kappa}+p_1](\cdot)-\tilde{\mathcal{B}}_5[p_{0,\kappa}+p_1](T)\|_{**,k}\le C|\log(T)|^{k-1}$$

is direct from the definition.

For the other terms let us write

$$\begin{cases}\tilde B_1[p](t)=\dfrac{p(t)}{|p(t)|}\,\mathrm{Re}\left(\dfrac{\bar p(t)}{|p(t)|}\tilde{\mathcal{B}}_{1,a}[p](t)\right)\\ \tilde B_2[p](t)=\dfrac{p(t)}{|p(t)|}\,\mathrm{Im}\left(\dfrac{\bar p(t)}{|p(t)|}\tilde{\mathcal{B}}_{2,a}[p](t)\right)\\ \tilde B_3[p](t)=\dfrac{p(t)}{|p(t)|}\,\mathrm{Re}\left(\dfrac{\bar p(t)}{|p(t)|}\tilde{\mathcal{B}}_{1,b}[p](t)\right)\\ \tilde B_4[p](t)=\dfrac{p(t)}{|p(t)|}\,\mathrm{Im}\left(\dfrac{\bar p(t)}{|p(t)|}\tilde{\mathcal{B}}_{2,b}[p](t)\right)\end{cases}\tag{13.107}$$

where

$$\tilde{\mathcal{B}}_{i,a}[p](t):=\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p(s)}{t-s}\left(\Gamma_i\Big(\frac{\lambda(t)^2}{t-s}\Big)-1\right)ds$$
$$\tilde{\mathcal{B}}_{i,b}[p](t):=\int_{t-\lambda_*(t)^2}^{t}\frac{\dot p(s)}{t-s}\Gamma_i\Big(\frac{\lambda(t)^2}{t-s}\Big)\,ds.$$

Then to prove the statement of the lemma it is sufficient to show that

$$|\tilde{\mathcal{B}}_{i,a}[p](t)|+|\tilde{\mathcal{B}}_{i,b}[p](t)|\le C\frac{|\log T|}{|\log(T-t)|^2}.\tag{13.108}$$

Using Lemma 13.11 and (4.5) we find for any $\sigma\in(0,1)$,

$$|\tilde{\mathcal{B}}_{i,a}[p](t)|\le C\lambda_*(t)^{2\sigma}\int_{-T}^{t-\lambda_*(t)^2}\frac{|\dot p(s)|}{(t-s)^{1+\sigma}}\,ds$$
$$\le C\lambda_*(t)^{2\sigma}|\log T|\int_{-T}^{t-\lambda_*(t)^2}\frac{1}{(t-s)^{1+\sigma}|\log(T-s)|^2}\,ds$$
$$\le C\frac{|\log T|}{|\log(T-t)|^2}.$$

Similarly

$$|\tilde{\mathcal{B}}_{i,b}[p](t)| \le C\frac{1}{\lambda_*(t)^2}\int_{t-\lambda_*(t)^2}^t |\dot p(s)|\,ds \le C\frac{|\log T|}{|\log(T-t)|^2}.$$

This proves (13.108). ☐

Lemma 13.13. *Let M be given by (13.96). For $\|p_i\|_{*,k+1} \le M$, $i=1,2$ we have*

$$\|\tilde{\mathcal{B}}[p_{0,\kappa}+p_1] - \tilde{\mathcal{B}}[p_{0,\kappa}+p_2]\|_{**,k} \le \frac{C}{|\log T|}\|p_1-p_2\|_{*,k+1}.$$

Proof. Again the estimate is direct for $\tilde{\mathcal{B}}_5$. To prove the corresponding inequality for $\tilde{\mathcal{B}}_1$, we use (13.107) and write

$$\begin{aligned}
D_{1,a} &:= \tilde{B}_1[p_\kappa+p_1](t) - \tilde{B}_1[p_\kappa+p_2](t)\\
D_{2,a} &:= \tilde{B}_2[p_\kappa+p_1](t) - \tilde{B}_2[p_\kappa+p_2](t)\\
D_{1,b} &:= \tilde{B}_3[p_\kappa+p_1](t) - \tilde{B}_3[p_\kappa+p_2](t)\\
D_{2,b} &:= \tilde{B}_4[p_\kappa+p_1](t) - \tilde{B}_4[p_\kappa+p_2](t).
\end{aligned}$$

We claim that

$$|D_{i,a}| \le \frac{C}{|\log(T-t)|^{k+1}}\|p_1-p_2\|_{*,k+1}, \tag{13.109}$$

$$|D_{i,b}| \le \frac{C}{|\log(T-t)|^{k+1}}\|p_1-p_2\|_{*,k+1}. \tag{13.110}$$

To prove this, let us consider $D_{1,a}$ and write

$$D_{1,a} = \int_0^1 \frac{d}{d\zeta}\left[\frac{(p_{0,\kappa}+p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\operatorname{Re}\left(\frac{(\bar p_{0,\kappa}+\bar p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa}+p_\zeta](t)\right)\right]d\zeta$$

where $p_\zeta = \zeta p_1 + (1-\zeta)p_2$, and note that

$$\begin{aligned}
&\frac{d}{d\zeta}\left[\frac{(p_{0,\kappa}+p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\operatorname{Re}\left(\frac{(\bar p_{0,\kappa}+\bar p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa}+p_\zeta](t)\right)\right]\\
&=\operatorname{Re}\left(\frac{(\bar p_{0,\kappa}+\bar p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa}+p_\zeta](t)\right)\frac{d}{d\zeta}\frac{(p_{0,\kappa}+p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\\
&\quad+\frac{(p_{0,\kappa}+p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\operatorname{Re}\left(\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa}+p_\zeta](t)\frac{d}{d\zeta}\frac{(\bar p_{0,\kappa}+\bar p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\right)\\
&\quad+\frac{(p_{0,\kappa}+p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\operatorname{Re}\left(\frac{(\bar p_{0,\kappa}+\bar p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\frac{d}{d\zeta}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa}+p_\zeta](t)\right).
\end{aligned}\tag{13.111}$$

But

$$\begin{aligned}
&\left|\frac{d}{d\zeta}\frac{(p_{0,\kappa}+p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\right|\\
&=\left|\frac{(p_1-p_2)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|} - \frac{(p_{0,\kappa}+p_\zeta)(t)[(p_{0,\kappa}+p_\zeta)(t)\cdot(p_1-p_2)(t)]}{|(p_{0,\kappa}+p_\zeta)(t)|^3}\right|\\
&\le 2\frac{|(p_1-p_2)(t)|}{|(p_{0,\kappa}+p_\zeta)(t)|}
\end{aligned}$$

Using (13.108)

$$\begin{aligned}&\left|\operatorname{Re}\left(\frac{(\bar p_{0,\kappa}+\bar p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa}+p_\zeta](t)\right)\frac{d}{d\zeta}\frac{(p_{0,\kappa}+p_\zeta)(t)}{|(p_{0,\kappa}+p_\zeta)(t)|}\right|\\&\le C\frac{|(p_1-p_2)(t)|}{|(p_{0,\kappa}+p_\zeta)(t)|}\frac{|\log T|}{|\log(T-t)|^2}\\&\le \frac{C}{|\log(T-t)|^{k+1}}\|p_1-p_2\|_{*,k+1}.\end{aligned}$$

The second term in (13.111) is estimated analogously.

Let us consider

$$\begin{aligned}&\frac{d}{d\zeta}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa}+p_\zeta](t)\\&=\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p_1(s)-\dot p_2(s)}{t-s}\left(\Gamma_i\left(\frac{|(p_{0,\kappa}+p_\zeta)(t)|^2}{t-s}\right)-1\right)ds\\&\quad+2(p_{0,\kappa}(t)+p_\zeta(t))\cdot(p_1(t)-p_2(t))\\&\qquad\cdot\int_{-T}^{t-\lambda_*(t)^2}\frac{(\dot p_{0,\kappa}+\dot p_\zeta)(s)}{(t-s)^2}\Gamma_i'\left(\frac{|(p_{0,\kappa}+p_\zeta)(t)|^2}{t-s}\right)ds.\end{aligned}\tag{13.112}$$

We estimate the first term above using (4.5). For $\sigma\in(0,1)$ we have

$$\begin{aligned}&\left|\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p_1(s)-\dot p_2(s)}{t-s}\left(\Gamma_i\left(\frac{|(p_{0,\kappa}+p_\zeta)(t)|^2}{t-s}\right)-1\right)ds\right|\\&\le C\int_{-T}^{t-\lambda_*(t)^2}\frac{|\dot p_1(s)-\dot p_2(s)|}{t-s}\left(\frac{|(p_{0,\kappa}+p_\zeta)(t)|^2}{t-s}\right)^\sigma ds\\&\le C\|p_1-p_2\|_{*,k+1}\lambda_*(t)^{2\sigma}\int_{-T}^{t-\lambda_*(t)^2}\frac{1}{(t-s)^{2\sigma}|\log(T-s)|^{k+1}}ds\end{aligned}$$

and by Lemma 13.11

$$\int_{-T}^{t-\lambda_*(t)^2}\frac{1}{(t-s)^{2\sigma}|\log(T-s)|^{k+1}}ds\le\frac{C}{\lambda_*(t)^{2\sigma}|\log(T-t)|^{k+1}}.$$

Therefore

$$\begin{aligned}&\left|\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p_1(s)-\dot p_2(s)}{t-s}\left(\Gamma_i\left(\frac{|(p_{0,\kappa}+p_\zeta)(t)|^2}{t-s}\right)-1\right)ds\right|\\&\le\frac{C}{|\log(T-t)|^{k+1}}\|p_1-p_2\|_{*,k+1}.\end{aligned}$$

For the second term in (13.112) we compute, for $\sigma \in (0,1)$,

$$\begin{aligned}
&\left| (p_{0,\kappa}(t) + p_\zeta(t)) \cdot (p_1(t) - p_2(t)) \int_{-T}^{t-\lambda_*(t)^2} \frac{(\dot p_{0,\kappa} + \dot p_\zeta)(s)}{(t-s)^2} \Gamma_i' \left(\frac{|(p_{0,\kappa} + p_\zeta)(t)|^2}{t-s} \right) \right| \\
&\leq C\lambda_*(t) \cdot |p_1(t) - p_2(t)| \int_{-T}^{t-\lambda_*(t)^2} \frac{|(\dot p_{0,\kappa} + \dot p_\zeta(s))|}{(t-s)^2} \left(\frac{|(p_{0,\kappa} + p_\zeta)(t)|^2}{t-s} \right)^{-\sigma} \\
&\leq C\lambda_*(t)^{1-\sigma} \|p_1 - p_2\|_{*,k+1} \frac{T-t}{|\log(T-t)|^{k+1}} \int_{-T}^{t-\lambda_*(t)^2} \frac{|\log T|}{(t-s)^{2-\sigma} |\log(T-s)|^2} \, ds \\
&\leq C\lambda_*(t)^{1-\sigma} \|p_1 - p_2\|_{*,k+1} \frac{T-t}{|\log(T-t)|^{k+1}} \frac{|\log T|}{\lambda_*(t)^{2-2\sigma} |\log(T-t)|^2} \\
&\leq C \frac{T-t}{|\log(T-t)|^{k+1}} \frac{|\log T|^{\sigma-1} (T-t)^{\sigma-1}}{|\log(T-t)|^{2\sigma-2}} \frac{|\log T|}{|\log(T-t)|^2} \|p_1 - p_2\|_{*,k+1} \\
&\leq \frac{C}{|\log(T-t)|^{k+1}} \|p_1 - p_2\|_{*,k+1}.
\end{aligned}$$

Thus we have obtained the estimate (13.109).

The estimate of $D_{1,b}$ is very similar, the only difference appears in

$$\begin{aligned}
&\frac{d}{d\zeta} \tilde{\mathcal{B}}_{1,b}[p_{0,\kappa} + p_\zeta](t) \\
&= 2 \int_{t-\lambda_*(t)^2}^{t} \frac{\dot p_1(s) - \dot p_2(s)}{t-s} \Gamma_i \left(\frac{|(p_{0,\kappa} + p_\zeta)(t)|^2}{t-s} \right) ds \\
&\quad + 4 (p_{0,\kappa}(t) + p_\zeta(t)) \cdot (p_1(t) - p_2(t)) \\
&\qquad \cdot \int_{t-\lambda_*(t)^2}^{t} \frac{(\dot p_{0,\kappa} + \dot p_\zeta)(s)}{(t-s)^2} \Gamma_i' \left(\frac{|(p_{0,\kappa} + p_\zeta)(t)|^2}{t-s} \right) ds.
\end{aligned}$$

We estimate the first term above

$$\begin{aligned}
&\left| \int_{t-\lambda_*(t)^2}^{t} \frac{\dot p_1(s) - \dot p_2(s)}{t-s} \Gamma_i \left(\frac{|(p_{0,\kappa} + p_\zeta)(t)|^2}{t-s} \right) ds \right| \\
&\leq \frac{C}{|(p_{0,\kappa} + p_\zeta)(t)|^2} \int_{t-\lambda_*(t)^2}^{t} |\dot p_1(s) - \dot p_2(s)| \, ds \\
&\leq \frac{C}{|\log(T-t)|^{k+1}} \|p_1 - p_2\|_{*,k+1}.
\end{aligned}$$

The second term is estimated by

$$\begin{aligned}
&\left| (p_{0,\kappa}(t) + p_\zeta(t)) \cdot (p_1(t) - p_2(t)) \int_{t-\lambda_*(t)^2}^{t} \frac{(\dot p_{0,\kappa} + \dot p_\zeta)(s)}{(t-s)^2} \Gamma_i' \left(\frac{|(p_{0,\kappa} + p_\zeta)(t)|^2}{t-s} \right) ds \right| \\
&\leq C\lambda_*(t) \frac{T-t}{|\log(T-t)|^{k+1}} \|p_1 - p_2\|_{*,k+1} \frac{1}{\lambda_*(t)^4} \int_{t-\lambda_*(t)^2}^{t} \frac{|\log T|}{|\log(T-s)|^2} \, ds \\
&\leq \frac{C}{|\log(T-t)|^{k+1}} \|p_1 - p_2\|_{*,k+1}.
\end{aligned}$$

We conclude the validity of (13.110). Estimates (13.109) and (13.110) give the result of the lemma. □

Estimate for the second derivative.

Lemma 13.14. *Let p_1 be the solution constructed in Proposition 13.1. Then*

$$|\ddot{p}_1(t)| \le C\frac{|\log T|}{|\log(T-t)|^3(T-t)} \tag{13.113}$$

$$\Big|\frac{d^3}{dt^3}p_1(t)\Big| \le C\frac{|\log T|}{|\log(T-t)|^3(T-t)^2}. \tag{13.114}$$

Proof. The fixed point problem (13.93) gives a solution to

$$\tilde{L}_0[\dot{p}_1] + \eta(\frac{t}{T})\tilde{L}_1[p_1] + \eta E + \tilde{\mathcal{B}}[p_{0,\kappa}+p_1](t) = c, \quad t\in[-T,T],$$

for some constant c. We differentiate this equation with respect to t and get (formally)

$$\begin{aligned}(1-\alpha_0)|\log(T-t)|\ddot{p}_1 &+ 2(1-\alpha_0)\frac{\dot{p}_1}{T-t} + \frac{1}{T}\eta'(\frac{t}{T})\tilde{L}_1[p_1] + \eta(\frac{t}{T})\frac{d}{dt}\tilde{L}_1[p_1]\\ &+\eta\frac{d}{dt}E + \frac{1}{T}\eta'(\frac{t}{T})E + \frac{d}{dt}\tilde{\mathcal{B}}[p_{0,\kappa}+p_1] = 0.\end{aligned} \tag{13.115}$$

We will use this equation and the norm $\|\ \|_{\mu,l}$ defined in (5.23) to prove that

$$\|\ddot{p}_1\|_{-1,3} \le C|\log T|, \tag{13.116}$$

which is the same as (13.113).

We rewrite equation (13.115) as

$$(1-\alpha_0)|\log(T-t)|\ddot{p}_1 + \eta(\frac{t}{T})\tilde{L}_1[\ddot{p}_1] + \mathcal{U}[\ddot{p}_1](t) = h, \tag{13.117}$$

where h is a function satisfying

$$|h(t)| \le C\frac{|\log T|}{|\log(T-t)|^2(T-t)}, \tag{13.118}$$

and $\mathcal{U}$ is the linear operator

$$\begin{aligned}\mathcal{U}[\ddot{p}_1](t) = &\frac{p(t)}{|p(t)|}\operatorname{Re}\left(\frac{\bar{p}(t)}{|p(t)|}\mathcal{U}_1[\ddot{p}_1]\right) + i\frac{p(t)}{|p(t)|}\operatorname{Im}\left(\frac{\bar{p}(t)}{|p(t)|}\mathcal{U}_2[\ddot{p}_1]\right)\\ &-\operatorname{Re}(\ddot{p}_1(t))\end{aligned} \tag{13.119}$$

where

$$\begin{aligned}\mathcal{U}_j[\ddot{p}_1](t) = &\int_{-T}^{t-\lambda_*(t)^2}\frac{\ddot{p}_1(s)}{t-s}\left(\Gamma_j\left(\frac{|p(t)|^2}{t-s}\right)-1\right)ds\\ &+\int_{t-\lambda_*(t)^2}^{t}\frac{\ddot{p}_1(s)}{t-s}\Gamma_j\left(\frac{|p(t)|^2}{t-s}\right)ds.\end{aligned} \tag{13.120}$$

In (13.119) and (13.120) p is given by $p_{0,\kappa}+p_1$.

We will verify this for a few terms (the other are analogous). One of the terms in h is $\frac{d}{dt}\tilde{E}(t)$. With a calculation similar to Lemma 13.10 we get

$$\left|\frac{d}{dt}\tilde{E}(t)\right| \le C\frac{|\log T|}{|\log(T-t)|^2(T-t)}.$$

Next we compute $\frac{d}{dt}\tilde L_{11}[\dot p_1]$ with $\tilde L_{11}$ defined in (13.101):

$$\begin{aligned}\frac{d}{dt}\tilde L_{11}[\dot p_1] &= \frac{d}{dt}\int_{(T-t)^{1+\alpha_0}}^{T-t}\frac{\dot p_1(t-r)}{r}\,dr\\ &= \int_{(T-t)^{1+\alpha_0}}^{T-t}\frac{\ddot p_1(t-r)}{r}\,dr - \frac{\dot p_1(t-(T-t))}{T-t}\\ &\quad + (1+\alpha_0)\frac{\dot p_1(t-(T-t)^{1+\alpha_0})}{T-t}\\ &= \tilde L_{11}[\ddot p_1] - \frac{\dot p_1(t-(T-t))}{T-t} + (1+\alpha_0)\frac{\dot p_1(t-(T-t)^{1+\alpha_0})}{T-t}. \end{aligned} \tag{13.121}$$

Of this expression $\tilde L_{11}[\ddot p_1]$ appears in the left hand side of (13.117), while the other terms are part of h, and they are estimated by

$$\begin{aligned}\left|\frac{\dot p_1(t-(T-t))}{T-t}\right| &\le C\frac{|\log(T)|^{k-1}\log(|\log(T)|)^2}{|\log(T-t)|^{k+1}(T-t)} \le C\frac{|\log T|}{|\log(T-t)|^2(T-t)}\\ \left|\frac{\dot p_1(t-(T-t)^{1+\alpha_0})}{T-t}\right| &\le C\frac{|\log(T)|^{k-1}\log(|\log(T)|)^2}{|\log(T-t)|^{k+1}(T-t)} \le C\frac{|\log T|}{|\log(T-t)|^2(T-t)}.\end{aligned}$$

Similar computations for $\tilde L_{1j}$, $j=2,3,4$ give that $\frac{d}{dt}\tilde L_1[\dot p_1]$ can be decomposed as $\tilde L_1[\ddot p_1]$ plus terms that belong to h and have the estimate (13.118).

Next we analyze $\frac{d}{dt}\tilde{\mathcal B}[p_{0,\kappa}+p_1]$ where $\tilde B$ is defined in (13.14). The first term in $\frac{d}{dt}\tilde{\mathcal B}[p_{0,\kappa}+p_1]$ is

$$\begin{aligned}\frac{d}{dt}\tilde{\mathcal B}_1[p_{0,\kappa}+p_1] &= \frac{d}{dt}\left[\frac{p(t)}{|p(t)|}\operatorname{Re}\left(\frac{\bar p(t)}{|p(t)|}\tilde{\mathcal B}_{1,a}[p](t)\right)\right]\\ &= \left(\frac{\dot p}{|p|}-\frac{p(p\cdot\dot p)}{|p|^3}\right)\operatorname{Re}\left(\frac{\bar p(t)}{|p(t)|}\tilde{\mathcal B}_{1,a}[p](t)\right)\\ &\quad + \frac{p(t)}{|p(t)|}\operatorname{Re}\left(\frac{\bar p(t)}{|p(t)|}\frac{d}{dt}\tilde{\mathcal B}_{1,a}[p](t)\right)\\ &\quad + \frac{p(t)}{|p(t)|}\operatorname{Re}\left(\left(\frac{\dot p}{|p|}-\frac{p(p\cdot\dot p)}{|p|^3}\right)\tilde{\mathcal B}_{1,a}[p](t)\right),\end{aligned} \tag{13.122}$$

where $\tilde{\mathcal B}_{i,a}[p]$ is given in (13.35).

The first and third term in (13.122) are part of h, and they are bounded, using (13.108), by

$$\begin{aligned}\left|\left(\frac{\dot p}{|p|}-\frac{p(p\cdot\dot p)}{|p|^3}\right)\operatorname{Re}\left(\frac{\bar p(t)}{|p(t)|}\tilde{\mathcal B}_{1,a}[p](t)\right)\right| &\le C\frac{|\dot\lambda_*|}{\lambda_*}\left|\tilde{\mathcal B}_{1,a}[p]\right|\\ &\le \frac{C}{T-t}\frac{|\log T|}{|\log(T-t)|^2}.\end{aligned}$$

We then compute the second term in (13.122), changing variables $t-s=r$:

$$\begin{aligned}
\frac{d}{dt}\tilde{\mathcal{B}}_{1,a}[p](t) &= \frac{d}{dt}\int_{-T}^{t-\lambda_*(t)^2} \frac{\dot p(s)}{t-s}\Big(\Gamma_i(\frac{\lambda(t)^2}{t-s})-1\Big)\,ds \qquad (13.123)\\
&= \frac{d}{dt}\int_{T+t}^{\lambda_*(t)^2} \frac{\dot p(t-r)}{r}\Big(\Gamma_i(\frac{\lambda(t)^2}{r})-1\Big)\,dr\\
&= \frac{\dot p(t-\lambda_*(t)^2)}{\lambda_*(t)^2}\Big(\Gamma_i(\frac{\lambda(t)^2}{\lambda_*(t)^2})-1\Big)2\lambda_*(t)\dot\lambda_*(t)\\
&\quad - \frac{\dot p(-T)}{T+t}\Big(\Gamma_i(\frac{\lambda(t)^2}{T+t})-1\Big)\\
&\quad + \int_{T+t}^{\lambda_*(t)^2} \frac{\ddot p(t-r)}{r}\Big(\Gamma_i(\frac{\lambda(t)^2}{r})-1\Big)\,dr\\
&\quad + 2\lambda(t)\dot\lambda(t)\int_{T+t}^{\lambda_*(t)^2} \frac{\dot p(t-r)}{r^2}\Gamma_i'(\frac{\lambda(t)^2}{r})\,dr.
\end{aligned}$$

The first, second and fourth terms are part of h, and they have the correct estimate. Indeed, we have

$$\begin{aligned}
\left|\frac{\dot p(t-\lambda_*(t)^2)}{\lambda_*(t)^2}\Big(\Gamma_i(\frac{\lambda(t)^2}{\lambda_*(t)^2})-1\Big)2\lambda_*(t)\dot\lambda_*(t)\right| &\le C\frac{\dot\lambda_*(t)^2}{\lambda_*(t)}\\
&\le C\frac{|\log T|^2}{|\log(T-t)|^4}\frac{|\log(T-t)|^2}{|\log T|(T-t)}\\
&= C\frac{|\log T|}{|\log(T-t)|^2(T-t)},
\end{aligned}$$

$$\left|\frac{\dot p(-T)}{T+t}\Big(\Gamma_i(\frac{\lambda(t)^2}{T+t})-1\Big)\right| \le \frac{C}{|\log T|T} \le C\frac{|\log T|}{|\log(T-t)|^2(T-t)},$$

and

$$\begin{aligned}
&\left|2\lambda(t)\dot\lambda(t)\int_{T+t}^{\lambda_*(t)^2}\frac{\dot p(t-r)}{r^2}\Gamma_i'(\frac{\lambda(t)^2}{r})\,dr\right|\\
&= \left|2\lambda(t)\dot\lambda(t)\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p(s)}{(t-s)^2}\Gamma_i'(\frac{\lambda(t)^2}{t-s})\,ds\right|\\
&\le C\lambda_*(t)|\dot\lambda_*(t)|\int_{-T}^{t-\lambda_*(t)^2}\frac{|\dot p(s)|}{(t-s)^2}\left|\Gamma_i'(\frac{\lambda(t)^2}{t-s})\right|\,ds\\
&\le C\lambda_*(t)|\dot\lambda_*(t)|\int_{-T}^{t-\lambda_*(t)^2}\frac{|\dot\lambda_*(s)|}{(t-s)^2}\Big(\frac{t-s}{\lambda(t)^2}\Big)^\sigma\,ds\\
&\le C\lambda_*(t)^{1-2\sigma}|\dot\lambda_*(t)||\log T|\int_{-T}^{t-\lambda_*(t)^2}\frac{1}{|\log(T-s)|^2}\frac{1}{(t-s)^{2-\sigma}}\,ds
\end{aligned}$$

But

$$\begin{aligned}
\int_{-T}^{t-(T-t)}\frac{1}{|\log(T-s)|^2}\frac{1}{(t-s)^{2-\sigma}}\,ds &\le C\int_{-T}^{t-(T-t)}\frac{1}{|\log(T-s)|^2}\frac{1}{(T-s)^{2-\sigma}}\,ds\\
&\le C\frac{(T-t)^{\sigma-1}}{|\log(T-t)|^2}
\end{aligned}$$

and

$$\int_{t-(T-t)}^{t-\lambda_*(t)^2} \frac{1}{|\log(T-s)|^2} \frac{1}{(t-s)^{2-\sigma}}\, ds \le C \frac{1}{|\log(T-t)|^2} \int_{t-(T-t)}^{t-\lambda_*(t)^2} \frac{1}{(t-s)^{2-\sigma}}\, ds$$
$$\le C \frac{1}{|\log(T-t)|^2} \lambda_*(t)^{2\sigma-2}$$

In summary

$$\left| 2\lambda(t)\dot\lambda(t) \int_{T+t}^{\lambda_*(t)^2} \frac{\dot p(t-r)}{r^2} \Gamma_i'(\frac{\lambda(t)^2}{r})\, dr \right|$$
$$\le C\lambda_*(t)^{1-2\sigma} |\dot\lambda_*(t)| |\log T| \frac{1}{|\log(T-t)|^2} \lambda_*(t)^{2\sigma-2}$$
$$= C \frac{|\log(T-t)|^2}{|\log T|(T-t)} \frac{|\log T|^2}{|\log(T-t)|^4}$$
$$= C \frac{|\log T|}{(T-t)|\log(T-t)|^2}.$$

Finally, the third term in (13.123) is

$$\int_{T+t}^{\lambda_*(t)^2} \frac{\ddot p(t-r)}{r} \Big(\Gamma_i(\frac{\lambda(t)^2}{r}) - 1 \Big)\, dr = \int_{-T}^{t-\lambda_*(t)^2} \frac{\ddot p(s)}{t-s} \Big(\Gamma_i(\frac{\lambda(t)^2}{t-s}) - 1 \Big)\, ds$$

which is part of $\mathcal{U}[\ddot p_1]$.

To deduce estimate (13.113) we use (13.117) to get

$$\|\ddot p_1\|_{-1,3} \le C\|\eta(\frac{t}{T}\tilde L_1[\dot p_1]\|_{-1,2} + C\|\mathcal{U}[\dot p_1]\|_{-1,2} + C\|h\|_{-1,2}.$$

We note that (13.118) gives

$$\|h\|_{-1,2} \le C|\log T|, \tag{13.124}$$

and we claim that

$$\|\tilde L_1[\dot p]\|_{j,m} \le \left(\alpha_0 + \frac{C}{|\log T|} \right) \|\dot p\|_{j,m+1}, \tag{13.125}$$

and that

$$\|\mathcal{U}[g]\|_{j,m} \le \frac{C}{|\log T|} \|g\|_{j,m+1}. \tag{13.126}$$

Indeed,

$$|\tilde L_1[\dot p](t)| \le \|\dot p\|_{j,m+1} \int_{t-(T-t)/2}^{t-(T-t)^{1+\alpha_0}} \frac{(T-s)^j}{|\log(T-s)|^{m+1}(t-s)}\, ds$$
$$\le \frac{\|\dot p\|_{j,m+1}}{|\log(T-t)|^{m+1}} \int_{t-(T-t)/2}^{t-(T-t)^{1+\alpha_0}} \frac{(T-s)^j}{(t-s)}\, ds,$$

but

$$\int_{t-(T-t)/2}^{t-(T-t)^{1+\alpha_0}} \frac{(T-s)^j}{(t-s)}\, ds = \int_{(T-t)^{1+\alpha_0}}^{(T-t)/2} \frac{(T-t+r)^j}{r}\, dr$$
$$= (T-t)^j \int_{(T-t)^{1+\alpha_0}}^{(T-t)/2} \frac{1+O(\frac{r}{T-t})}{r}\, dr$$
$$\le (T-t)^j \left(\alpha_0 |\log(T-t)| + C \right).$$

Thus

$$|\tilde{L}_1[g](t)| \leq \left(\alpha_0 + \frac{C}{|\log(T-t)|}\right) \frac{(T-t)^j}{|\log(T-t)|^m} \|\dot{p}\|_{j,m+1},$$

and this proves (13.125).

To obtain (13.126) we compute

$$\begin{aligned}
&\int_{-T}^{t-\lambda_*(t)^2} \frac{|g(s)|}{t-s} \left|\Gamma_j\left(\frac{|p(t)|^2}{t-s}\right) - 1\right| ds \\
&\leq \|\ddot{p}_1\|_{j,m+1} \int_{-T}^{t-\lambda_*(t)^2} \frac{(T-s)^j}{|\log(T-s)|^{m+1}} \frac{1}{t-s} \left|\Gamma_j\left(\frac{|p(t)|^2}{t-s}\right) - 1\right| ds.
\end{aligned}$$

We then split the integral into $\int_{-T}^{t-(T-t)} \dots$ and $\int_{t-(T-t)}^{t-\lambda_*(t)^2} \dots$ and estimate (with $\sigma \in (0,1)$ and using (4.5)):

$$\begin{aligned}
&\int_{-T}^{t-(T-t)} \frac{(T-s)^j}{|\log(T-s)|^{m+1}} \frac{1}{t-s} \left|\Gamma_j\left(\frac{|p(t)|^2}{t-s}\right) - 1\right| ds \\
&\leq C\lambda_*(t)^{2\sigma} \int_{-T}^{t-(T-t)} \frac{(T-s)^{j-1-\sigma}}{|\log(T-s)|^{m+1}} ds \\
&\leq C\lambda_*(t)^{2\sigma} \frac{(T-t)^{j-\sigma}}{|\log(T-t)|^{m+1}} \\
&\leq \frac{C}{|\log T|} \frac{(T-t)^j}{|\log(T-t)|^m},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t-(T-t)}^{t-\lambda_*(t)^2} \frac{(T-s)^j}{|\log(T-s)|^{m+1}} \frac{1}{t-s} \left|\Gamma_j\left(\frac{|p(t)|^2}{t-s}\right) - 1\right| ds \\
&\leq C\lambda_*(t)^{2\sigma} \frac{(T-s)^j}{|\log(T-t)|^{m+1}} \int_{t-(T_t)}^{t-\lambda_*(t)^2} (t-s)^{-1-\sigma} ds \\
&\leq C \frac{(T-s)^j}{|\log(T-t)|^{m+1}} \\
&\leq \frac{C}{|\log T|} \frac{(T-s)^j}{|\log(T-t)|^m}.
\end{aligned}$$

Therefore

$$\int_{-T}^{t-\lambda_*(t)^2} \frac{|g(s)|}{t-s} \left|\Gamma_j\left(\frac{|p(t)|^2}{t-s}\right) - 1\right| ds \leq \frac{C}{|\log T|} \frac{(T-s)^j}{|\log(T-t)|^m} \|\ddot{p}_1\|_{j,m+1}.$$

This proves (13.126).

Then using (13.124), (13.125), and (13.126) we obtain (13.116).

The proof of (13.114) is analogous, differentiating (13.117) with respect to t once more. We omit the details. □

Lipschitz estimates. Proposition 13.1 defines a function that to κ satisfying (13.4) associates $p_1(\kappa)$, which is the unique fixed point of $\mathcal{A}$ in the ball $\{\|p_1\|_{*,k+1} \leq M\}$, $M = C_0|\log(T)|^{k-1}\log(|\log(T)|)^2$.

The next result gives several Lipschitz estimates of this map.

Corollary 13.1. *Let $k \in (0,2)$. For κ_1, κ_2 satisfying (13.4) we have*

$$\|p_1(\kappa_1) - p_1(\kappa_2)\|_{*,k+1} \leq C|\log T|^{k-1}\log(|\log T|)^2\,|\kappa_1 - \kappa_2|. \tag{13.127}$$

Proof. The Lipschitz property for p_1 results from a standard argument once we show that the dependence of $\mathcal{A}$ (see (13.94)) is Lipschitz with respect to κ. Indeed, let us make the dependence of $\mathcal{A}$ on κ explicit by writing $\mathcal{A}[p_1,\kappa] = T_1[-\eta\tilde{E}(\kappa) - \tilde{B}[p_{0,\kappa}+p_1]]$. We claim that for $\|p_1\|_{*,k+1} \le M$ (defined in (13.96)) we have

$$\|\mathcal{A}[p_1,\kappa_1] - \mathcal{A}[p_1,\kappa_2]\|_{*,k+1} \le C|\log T|^{k-1}\log(|\log T|)^2\,|\kappa_1-\kappa_2|.$$

Since $\tilde{E}(\kappa)$ is linear in κ, using Lemma 13.8, we need only to prove that

$$\|\tilde{B}[p_{0,\kappa_1}+p_1] - \tilde{B}[p_{0,\kappa_2}+p_1]\|_{**,k} \le C|\log T|^{k-1}\,|\kappa_1-\kappa_2|. \tag{13.128}$$

The proof is similar to the one of Lemma 13.13. We use the notation in that lemma and we give details for one of the terms. Let us consider

$$\begin{aligned}\widetilde{D}_{i,a} &= \frac{(p_{0,\kappa_1}+p_1)(t)}{|(p_{0,\kappa_1}+p_1)(t)|}\operatorname{Re}\left(\frac{(\bar{p}_{0,\kappa_1}+\bar{p}_1)(t)}{|(p_{0,\kappa_1}+p_1)(t)|}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa_1}+p_1](t)\right)\\ &\quad - \frac{(p_{0,\kappa_2}+p_1)(t)}{|(p_{0,\kappa_2}+p_1)(t)|}\operatorname{Re}\left(\frac{(\bar{p}_{0,\kappa_2}+\bar{p}_1)(t)}{|(p_{0,\kappa_2}+p_1)(t)|}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa_2}+p_1](t)\right).\end{aligned}$$

We claim that

$$\left|\widetilde{D}_{i,a}\right| \le C|\kappa_1-\kappa_2|\frac{|\log T|}{|\log(T-t)|^2}. \tag{13.129}$$

To prove this, we write

$$\widetilde{D}_{i,a} = \int_0^1 \frac{d}{d\zeta}\left[\frac{(p_{0,\kappa_\zeta}+p_1)(t)}{|(p_{0,\kappa_\zeta}+p_1)(t)|}\operatorname{Re}\left(\frac{(\bar{p}_{0,\kappa_\zeta}+\bar{p}_1)(t)}{|(p_{0,\kappa_\zeta}+p_1)(t)|}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa_\zeta}+p_1](t)\right)\right]d\zeta,$$

where

$$p_{0,\kappa_\zeta} = \zeta p_{0,\kappa_1} + (1-\zeta)\zeta p_{0,\kappa_2}.$$

Let us study

$$\begin{aligned}&\left|\frac{d}{d\zeta}\frac{(p_{0,\kappa_\zeta}+p_1)(t)}{|(p_{0,\kappa_\zeta}+p_1)(t)|}\right|\\ &= \left|\frac{(\kappa_1-\kappa_2)p_{0,1}(t)}{|(p_{0,\kappa_\zeta}+p_1)(t)|} - \frac{(p_{0,\kappa_\zeta}+p_1)(t)[(p_{0,\kappa_\zeta}+p_1)(t)\cdot(\kappa_1-\kappa_2)p_{0,1}(t)]}{|(p_{0,\kappa_\zeta}+p_1)(t)|^2}\right|\\ &\le 2\frac{|(\kappa_1-\kappa_2)p_{0,1}(t)|}{|(p_{0,\kappa_\zeta}+p_1)(t)|}\\ &\le C|\kappa_1-\kappa_2|.\end{aligned}$$

Using (13.108)

$$\begin{aligned}&\left|\operatorname{Re}\left(\frac{(\bar{p}_{0,\kappa_\zeta}+\bar{p}_1)(t)}{|(p_{0,\kappa_\zeta}+p_1)(t)|}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa_\zeta}+p_1](t)\right)\frac{d}{d\zeta}\frac{(p_{0,\kappa_\zeta}+p_1)(t)}{|(p_{0,\kappa_\zeta}+p_1)(t)|}\right|\\ &\le C|\kappa_1-\kappa_2|\frac{|\log T|}{|\log(T-t)|^2}.\end{aligned}$$

Let us analyze

$$\begin{aligned}&\frac{d}{d\zeta}\tilde{\mathcal{B}}_{i,a}[p_{0,\kappa_\zeta}+p_1](t)\\ &= (\kappa_1-\kappa_2)\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot{p}_{0,1}(s)}{t-s}\left(2\Gamma_i\left(\frac{|(p_{0,\kappa_\zeta}+p_1)(t)|^2}{t-s}\right)-1\right)ds\\ &\quad + 4(\kappa_1-\kappa_2)(p_{0,\kappa_\zeta}(t)+p_1(t))\cdot p_{0,1}\\ &\qquad\cdot\int_{-T}^{t-\lambda_*(t)^2}\frac{(\dot{p}_{0,\kappa_\zeta}+\dot{p}_1)(s)}{(t-s)^2}\Gamma_i'\left(\frac{|(p_{0,\kappa_\zeta}+p_1)(t)|^2}{t-s}\right)ds.\end{aligned}$$

We estimate the first term above

$$\begin{aligned}
&\left|(\kappa_1-\kappa_2)\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p_{0,1}(s)}{t-s}\left(2\Gamma_i\left(\frac{|(p_{0,\kappa_\zeta}+p_1)(t)|^2}{t-s}\right)-1\right)ds\right|\\
&\le C|\kappa_1-\kappa_2|\int_{-T}^{t-\lambda_*(t)^2}\frac{|\dot\lambda_*(s)|}{t-s}\left(\frac{|(p_{0,\kappa_\zeta}+p_1)(t)|^2}{t-s}\right)^{\sigma}ds\\
&\le C|\kappa_1-\kappa_2|\lambda_*(t)^{2\sigma}\int_{-T}^{t-\lambda_*(t)^2}\frac{|\log T|}{(t-s)^{1+\sigma}|\log(T-s)|^2}\,ds
\end{aligned}$$

and by Lemma 13.11

$$\int_{-T}^{t-\lambda_*(t)^2}\frac{1}{(t-s)^{1+\sigma}|\log(T-s)|^2}\,ds\le\frac{C}{\lambda_*(t)^{2\sigma}|\log(T-t)|^2}.$$

Therefore

$$\begin{aligned}
&\left|(\kappa_1-\kappa_2)\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p_{0,1}(s)}{t-s}\left(2\Gamma_i\left(\frac{|(p_{0,\kappa_\zeta}+p_1)(t)|^2}{t-s}\right)-1\right)ds\right|\\
&\le C|\kappa_1-\kappa_2|\frac{|\log T|}{|\log(T-t)|^2}
\end{aligned}$$

For the second term in (13.112) we compute

$$\begin{aligned}
&\Bigg|4(\kappa_1-\kappa_2)(p_{0,\kappa_\zeta}(t)+p_1(t))\cdot p_{0,1}\\
&\qquad\cdot\int_{-T}^{t-\lambda_*(t)^2}\frac{(\dot p_{0,\kappa_\zeta}+\dot p_1)(s)}{(t-s)^2}\Gamma_i'\left(\frac{|(p_{0,\kappa_\zeta}+p_1)(t)|^2}{t-s}\right)ds\Bigg|\\
&\le C|\kappa_1-\kappa_2|\lambda_*(t)^2\int_{-T}^{t-\lambda_*(t)^2}\frac{|\dot\lambda_*(s)|}{(t-s)^2}\left(\frac{|(p_{0,\kappa_\zeta}+p_1)(t)|^2}{t-s}\right)^{-\sigma}ds\\
&\le C|\kappa_1-\kappa_2|\lambda_*(t)^{2-2\sigma}\int_{-T}^{t-\lambda_*(t)^2}\frac{|\dot\lambda_*(s)|}{(t-s)^{2-\sigma}}\,ds\\
&\le C|\kappa_1-\kappa_2|\lambda_*(t)^{2-2\sigma}\int_{-T}^{t-\lambda_*(t)^2}\frac{|\log T|}{(t-s)^{2-\sigma}|\log(T-s)|^2}\,ds\\
&\le C|\kappa_1-\kappa_2|\frac{|\log T|}{|\log(T-t)|^2}.
\end{aligned}$$

The last term in $\widetilde D_{i,a}$ is estimated similarly and we obtain (13.129).
From (13.112) we obtain

$$\|\widetilde D_{i,a}\|_{**,k}\le C|\kappa_1-\kappa_2||\log T|^{k-1}.$$

The other terms in the expression $\|\tilde B[p_{0,\kappa_1}+p_1]-\tilde B[p_{0,\kappa_2}+p_1]\|_{**,k}$ are estimated similarly and we find (13.128). □

We will also need a Lipschitz estimate for $\ddot p_1$ in the norm $\|\ \|_{-1,3}$ and $\frac{d^3}{dt^3}p_1$ in the norm $\|\ \|_{-2,3}$.

Lemma 13.15. *For κ_1,κ_2 satisfying (13.4) we have*

$$\|\ddot p_1(\kappa_1)-\ddot p_1(\kappa_2)\|_{-1,3}\le C|\log T|\,|\kappa_1-\kappa_2| \tag{13.130}$$

$$\left\|\frac{d^3}{dt^3}p_1(\kappa_1)-\frac{d^3}{dt^3}p_1(\kappa_2)\right\|_{-2,3}\le C|\log T|\,|\kappa_1-\kappa_2| \tag{13.131}$$

Proof. For the proof we proceed formally estimating $p_1':=\frac{d}{d\kappa}p_1(\kappa)$ in the norm $\|\ \|_{-1,3}$ (defined in (5.23)).

We start from equation (13.117) and differentiate with respect to κ to get

$$(1-\alpha_0)|\log(T-t)|\ddot{p}_1' + \eta(\frac{t}{T})\tilde{L}_1[\ddot{p}_1'] + \mathcal{U}[\ddot{p}_1'](t) = \tilde{h}, \tag{13.132}$$

where $p_1' = \frac{dp_1}{d\kappa}$ and $\dot{()} = \frac{d}{dt}$.

We claim that

$$\|\tilde{h}\|_{-1,2} \le C|\log T|. \tag{13.133}$$

We show here a few steps of the computation.

One term in $\tilde{h}$ is $\frac{\partial}{\partial\kappa}\frac{\partial}{\partial t}\tilde{E}$. But we have

$$\left|\frac{d}{dt}\tilde{E}(t)\right| \le C\frac{|\log T|}{|\log(T-t)|^2(T-t)}.$$

and $\tilde{E}$ is linear in κ, so that

$$\left\|\frac{\partial}{\partial\kappa}\frac{\partial}{\partial t}\tilde{E}\right\|_{-1,2} \le C|\log T|.$$

Other terms in $\tilde{h}$ come from the computation of $\frac{d}{dt}\tilde{L}_1$. In particular from (13.121) we get

$$\frac{\dot{p}_1'(t-(T-t))}{T-t}, \quad \frac{\dot{p}_1'(t-(T-t)^{1+\alpha_0})}{T-t}.$$

Using (13.127) we have

$$\|p_1'\|_{*,k+1} \le C|\log T|^{k-1}\log(|\log T|)^2. \tag{13.134}$$

This implies

$$\left|\frac{\dot{p}_1'(t-(T-t))}{T-t}\right| \le C\frac{|\log T|^{k-1}\log(|\log T|)^2}{|\log(T-t)|^{k+1}(T-t)} \le C\frac{|\log T|}{|\log(T-t)|^2(T-t)}$$
$$\left|\frac{\dot{p}_1'(t-(T-t)^{1+\alpha_0})}{T-t}\right| \le C\frac{|\log T|^{k-1}\log(|\log T|)^2}{|\log(T-t)|^{k+1}(T-t)} \le C\frac{|\log T|}{|\log(T-t)|^2(T-t)}.$$

The other terms in h come from the computation of $\frac{d}{d\kappa}\frac{d}{dt}\tilde{\mathcal{B}}[p_{0,\kappa}+p_1]$. Let us consider $\frac{d}{dt}\tilde{\mathcal{B}}_1$ as in (13.122) and compute

$$\begin{aligned}\frac{d}{d\kappa}\frac{d}{dt}\tilde{\mathcal{B}}_1[p_{0,\kappa}+p_1] &= \frac{d}{d\kappa}\left[\left(\frac{\dot{p}}{|p|} - \frac{p(p\cdot\dot{p})}{|p|^3}\right)\operatorname{Re}\left(\frac{\bar{p}(t)}{|p(t)|}\tilde{\mathcal{B}}_{1,a}[p](t)\right)\right] \\ &\quad + \frac{d}{d\kappa}\left[\frac{p(t)}{|p(t)|}\operatorname{Re}\left(\frac{\bar{p}(t)}{|p(t)|}\frac{d}{dt}\tilde{\mathcal{B}}_{1,a}[p](t)\right)\right] \\ &\quad + \frac{d}{d\kappa}\left[\frac{p(t)}{|p(t)|}\operatorname{Re}\left(\left(\frac{\dot{p}}{|p|} - \frac{p(p\cdot\dot{p})}{|p|^3}\right)\tilde{\mathcal{B}}_{1,a}[p](t)\right)\right],\end{aligned} \tag{13.135}$$

where $p = p_{0,\kappa} + p_1$. The first term above is

$$\begin{aligned}&\frac{d}{d\kappa}\left[\left(\frac{\dot{p}}{|p|} - \frac{p(p\cdot\dot{p})}{|p|^3}\right)\operatorname{Re}\left(\frac{\bar{p}(t)}{|p(t)|}\tilde{\mathcal{B}}_{1,a}[p](t)\right)\right] \\ &= \left[\frac{d}{d\kappa}\left(\frac{\dot{p}}{|p|} - \frac{p(p\cdot\dot{p})}{|p|^3}\right)\right]\operatorname{Re}\left(\frac{\bar{p}(t)}{|p(t)|}\tilde{\mathcal{B}}_{1,a}[p](t)\right) \\ &\quad + \left(\frac{\dot{p}}{|p|} - \frac{p(p\cdot\dot{p})}{|p|^3}\right)\operatorname{Re}\left(\left[\frac{d}{d\kappa}\frac{\bar{p}(t)}{|p(t)|}\right]\tilde{\mathcal{B}}_{1,a}[p](t)\right) \\ &\quad + \left(\frac{\dot{p}}{|p|} - \frac{p(p\cdot\dot{p})}{|p|^3}\right)\operatorname{Re}\left(\frac{\bar{p}(t)}{|p(t)|}\frac{d}{d\kappa}\tilde{\mathcal{B}}_{1,a}[p](t)\right).\end{aligned}$$

Using (13.134) and (13.108) we obtain

$$\left|\left[\frac{d}{d\kappa}\left(\frac{\dot{p}}{|p|} - \frac{p(p\cdot\dot{p})}{|p|^3}\right)\right]\operatorname{Re}\left(\frac{\bar{p}(t)}{|p(t)|}\tilde{\mathcal{B}}_{1,a}[p](t)\right)\right| \le C\frac{|\log T|}{(T-t)|\log(T-t)|^2},$$

and

$$\left|\left(\frac{\dot p}{|p|}-\frac{p(p\cdot\dot p)}{|p|^3}\right)\mathrm{Re}\left(\left[\frac{d}{d\kappa}\frac{\bar p(t)}{|p(t)|}\right]\tilde{\mathcal{B}}_{1,a}[p](t)\right)\right|\le C\frac{|\log T|}{(T-t)|\log(T-t)|^2}.$$

Next we analyze

$$\begin{aligned}\frac{d}{d\kappa}\tilde{\mathcal{B}}_{1,a}[p]&=\frac{d}{d\kappa}\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p(s)}{t-s}\left(\Gamma_i\left(\frac{|p(t)|^2}{t-s}\right)-1\right)ds\\&=\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p'(s)}{t-s}\left(\Gamma_i\left(\frac{|p(t)|^2}{t-s}\right)-1\right)ds\\&\quad+2p(t)\cdot p'(t)\int_{-T}^{t-\lambda_*(t)^2}\frac{\dot p(s)}{(t-s)^2}\Gamma_i'\left(\frac{|p(t)|^2}{t-s}\right)ds.\end{aligned}$$

These terms are also part of $\tilde h$ and using (13.134) we get that

$$\left|\frac{d}{d\kappa}\tilde{\mathcal{B}}_{1,a}[p]\right|\le\frac{C|\log T|}{|\log(T-t)|^2}.$$

Combining this with the estimate

$$\frac{|\dot p|}{|p|}\le\frac{C}{T-t}$$

we get

$$\left|\left(\frac{\dot p}{|p|}-\frac{p(p\cdot\dot p)}{|p|^3}\right)\mathrm{Re}\left(\frac{\bar p(t)}{|p(t)|}\frac{d}{d\kappa}\tilde{\mathcal{B}}_{1,a}[p](t)\right)\right|\le C\frac{|\log T|}{(T-t)|\log(T-t)|^2},$$

which is the desired estimate for this part of $\tilde h$.

Let us consider now the second term in (13.135):

$$\begin{aligned}\frac{d}{d\kappa}\left[\frac{p(t)}{|p(t)|}\mathrm{Re}\left(\frac{\bar p(t)}{|p(t)|}\frac{d}{dt}\tilde{\mathcal{B}}_{1,a}[p](t)\right)\right]&=\left[\frac{d}{d\kappa}\frac{p(t)}{|p(t)|}\right]\mathrm{Re}\left(\frac{\bar p(t)}{|p(t)|}\frac{d}{dt}\tilde{\mathcal{B}}_{1,a}[p](t)\right)\\&\quad+\frac{p(t)}{|p(t)|}\mathrm{Re}\left(\left[\frac{d}{d\kappa}\frac{\bar p(t)}{|p(t)|}\right]\frac{d}{dt}\tilde{\mathcal{B}}_{1,a}[p](t)\right)\\&\quad+\frac{p(t)}{|p(t)|}\mathrm{Re}\left(\frac{\bar p(t)}{|p(t)|}\frac{d}{d\kappa}\left[\frac{d}{dt}\tilde{\mathcal{B}}_{1,a}[p](t)\right]\right).\end{aligned}$$

Of these terms, let us analyze the last one. We compute, using the expression (13.123)

$$\begin{aligned}\frac{d}{d\kappa}\frac{d}{dt}\tilde{\mathcal{B}}_{1,a}[p](t)&=\frac{d}{d\kappa}\left[\frac{\dot p(t-\lambda_*(t)^2)}{\lambda_*(t)^2}\Big(\Gamma_i(\frac{\lambda(t)^2}{\lambda_*(t)^2})-1\Big)2\lambda_*(t)\dot\lambda_*(t)\right]\\&\quad-\frac{d}{d\kappa}\left[\frac{\dot p(-T)}{T+t}\Big(\Gamma_i(\frac{\lambda(t)^2}{T+t})-1\Big)\right]\\&\quad+\frac{d}{d\kappa}\left[\int_{T+t}^{\lambda_*(t)^2}\frac{\ddot p(t-r)}{r}\Big(\Gamma_i(\frac{\lambda(t)^2}{r})-1\Big)dr\right]\\&\quad+\frac{d}{d\kappa}\left[2\lambda(t)\dot\lambda(t)\int_{T+t}^{\lambda_*(t)^2}\frac{\dot p(t-r)}{r^2}\Gamma_i'(\frac{\lambda(t)^2}{r})\,dr\right].\end{aligned}$$

Let us examine the third term above (after changing variables)

$$
\begin{aligned}
&\frac{d}{d\kappa}\int_{-T}^{t-\lambda_*(t)^2} \frac{\ddot p(s)}{t-s}\Big(\Gamma_i(\frac{|p(t)|^2}{t-s})-1\Big)\,ds\\
&=\int_{-T}^{t-\lambda_*(t)^2} \frac{\ddot p_{0,\kappa}'(s)}{t-s}\Big(\Gamma_i(\frac{|p(t)|^2}{t-s})-1\Big)\,ds\\
&\quad+\int_{-T}^{t-\lambda_*(t)^2} \frac{\ddot p_1'(s)}{t-s}\Big(\Gamma_i(\frac{|p(t)|^2}{t-s})-1\Big)\,ds\\
&\quad+2p(t)\cdot p'(t)\int_{-T}^{t-\lambda_*(t)^2} \frac{\ddot p(s)}{(t-s)^2}\Gamma_i'(\frac{|p(t)|^2}{t-s})\,ds.
\end{aligned}
\tag{13.136}
$$

The same computations as before show that

$$
\left|\int_{-T}^{t-\lambda_*(t)^2} \frac{\ddot p_{0,\kappa}'(s)}{t-s}\Big(\Gamma_i(\frac{|p(t)|^2}{t-s})-1\Big)\,ds\right|\le C\frac{|\log T|}{(T-t)|\log(T-t)|^2}
$$

$$
\left|2p(t)\cdot p'(t)\int_{-T}^{t-\lambda_*(t)^2} \frac{\ddot p(s)}{(t-s)^2}\Gamma_i'(\frac{|p(t)|^2}{t-s})\,ds\right|\le C\frac{|\log T|}{(T-t)|\log(T-t)|^2}
$$

and these terms are part of $\tilde h$. The second term in (13.136) is part of $\mathcal{U}[\ddot p_1']$.

Using the equation (13.132) and the estimates (13.133) for $\tilde h$ and (13.125), (13.126) we deduce that

$$
\Big\|\frac{d}{d\kappa}p_1\Big\|_{-1,3}\le C|\log T|,
$$

which gives (13.130).

The proof of (13.131) is similar, and we omit the details. □

Estimate of the remainder. Next we use the previous results on p_1 to obtain an estimate of $R_{\alpha_0}[\dot p_1]$.

Lemma 13.16. *Let p_1 be the solution constructed in Proposition 13.1. Then*

$$
|R_{\alpha_0}[\dot p_1](t)|\le C\frac{|\log T|}{|\log(T-t)|^3}(T-t)^{\alpha_0},
\tag{13.137}
$$

and for κ_1, κ_2 satisfying (13.4) we have

$$
|R_{\alpha_0}[\dot p_1(\kappa_1)]-R_{\alpha_0}[\dot p_1(\kappa_2)]|\le C\frac{|\log T|}{|\log(T-t)|^3}(T-t)^{\alpha_0}|\kappa_1-\kappa_2|.
\tag{13.138}
$$

Proof. We have, thanks to (13.113)

$$
\begin{aligned}
|R_{\alpha_0}[\dot p_1](t)|&\le\int_{t-(T-t)^{1+\alpha_0}}^{t-\lambda_*(t)^2}\frac{|\dot p_1(t)-\dot p_1(s)|}{t-s}\,ds\\
&\le\sup_{r\in(t-(T-t)^{1+\alpha_0},t-\lambda_*(t)^2)}|\ddot p_1(r)|(T-t)^{1+\alpha_0}\\
&\le C\frac{|\log T|}{|\log(T-t)|^3}(T-t)^{\alpha_0}.
\end{aligned}
$$

This proves (13.137).

The proof of (13.138) is similar, using (13.130). □

Lemma 13.17. *Let p_1 be the solution constructed in Proposition 13.1. Then*

$$
\Big|\frac{d}{dt}R_{\alpha_0}[\dot p_1](t)\Big|\le C\frac{|\log T|}{|\log(T-t)|^3}(T-t)^{\alpha_0-1},
\tag{13.139}
$$

$$
\Big|\frac{d}{dt}R_{\alpha_0}[\dot p_1(\kappa_1)](t)-\frac{d}{dt}R_{\alpha_0}[\dot p_1(\kappa_2)](t)\Big|\le C\frac{|\log T|}{|\log(T-t)|^3}(T-t)^{\alpha_0-1}|\kappa_1-\kappa_2|.
\tag{13.140}
$$

Proof. We compute

$$\begin{aligned}\frac{d}{dt}R_{\alpha_0}[\dot p_1](t) &= -\frac{d}{dt}\int_{t-(T-t)^{1+\alpha_0}}^{t-\lambda_*(t)^2}\frac{\dot p_1(t)-\dot p_1(s)}{t-s}\,ds\\ &= -\frac{d}{dt}\int_{\lambda_*(t)^2}^{(T-t)^{1+\alpha_0}}\frac{\dot p_1(t)-\dot p_1(t-r)}{r}\,dr\\ &= -(1+\alpha_0)\frac{\dot p_1(t-(T-t)^{1+\alpha_0})}{T-t} - 2(1+\alpha_0)\frac{\dot p_1(t-\lambda_*(t)^2)}{\lambda_*(t)}\dot\lambda_*(t)\\ &\quad -\int_{\lambda_*(t)^2}^{(T-t)^{1+\alpha_0}}\frac{\ddot p_1(t)-\ddot p_1(t-r)}{r}\,dr.\end{aligned}$$

By (13.113) and (13.114) we get (13.139).

The proof of (13.140) is similar, using (13.130) and (13.131).

□

14. The equation for Z_1^*

In this section we discuss the stability of the blow-up phenomenon predicted in Theorem 1 and prove Theorem 2. We consider the class of initial conditions that lead to blow-up at a given point as described in §5.1. The solution has the form

$$u(x,t) = U_{\lambda(t),\omega(t),\xi(t)} + \varphi + a(|\varphi|^2)U_{\lambda(t),\omega(t),\xi(t)}$$

where $a(s) = \sqrt{1-s}-1$ and

$$\varphi(x,t) = \Pi_{U^\perp_{\lambda(t),\omega(t),\xi(t)}}\left[\tilde Z^*(x,t) + \Phi(\lambda,\omega,\xi)(x,t) + \psi(x,t) + \eta\phi(x,t)\right],$$

where the point $\xi(T)\in\Omega$ is prescribed. Changing slightly the proof we can achieve that the value $\xi(0)=q$ be prescribed. Let us denote $\varepsilon = \lambda(0)$. A simple application of implicit function theorem to the system of equations determining (λ,ω,ξ) leads to the fact that the blow-up time T and the final point $\xi(T)$ can be regarded as functions of arbitrary small values $\varepsilon>0$ and points $q\in\Omega$.

The functions (λ,ω,ξ) as well as ψ and ϕ have Lipschitz dependence in $p:=(\varepsilon,q)$ and Z^* in suitable topologies. We relabel

$$\omega(p) := \omega(0),\quad U_p := U_{\varepsilon,\omega(p),q},\quad \tilde\Phi(p)(x) = \Phi(\lambda,\omega,\xi)(x,0)+\psi(x,0)$$

so that the initial condition of the solution above becomes

$$u_0(p) = U_p + \Pi_{U_p^\perp}[Z^*+\tilde\Phi(p)] + a(|\Pi_{U_p^\perp}[Z^*+\tilde\Phi(p)]|^2)U_p.$$

A generic initial condition close to

$$U_{p_0} + \Pi_{U_{p_0}^\perp}[Z_0^*+\tilde\Phi(p_0)] + a(|\Pi_{U_{p_0}^\perp}[Z_0^*+\tilde\Phi(p_0)]|^2)U_{p_0}$$

with values in S^2 can be written in the form

$$v(x;\varphi_1) := U_{p_0} + \Pi_{U_{p_0}^\perp}[Z_0^*+\tilde\Phi(p_0)+\varphi_1] + a(|\Pi_{U_{p_0}^\perp}[Z_0^*+\tilde\Phi(p_0)+\varphi_1]|^2)U_{p_0}$$

where φ_1 is a small function, otherwise arbitrary. We shall show that if φ_1 is sufficiently small in C^2-topology and it lies on a certain codimension-1 manifold, then problem (3.1) with initial condition $u_0(x) = v(x;\varphi_1)$ has blow-up as predicted. Thus what we need is that for suitable

$$\zeta = (\varepsilon,q,Z^*) = \zeta_0+\zeta_1,\quad \zeta_1 = (\varepsilon_1,q_1,Z_1^*)$$

we have that

$$v(\cdot;\varphi_1) = u_0(p). \tag{14.1}$$

It is convenient to measure the size of ζ_1 with respect to the norm (see (5.7)),

$$\|p_1\| := |q_1| + |\varepsilon_1| + \|Z_1^*\|_*.$$

We expand $u_0(p)$ around $p = p_0$ and get

$$u_0(\zeta) = U_{\zeta_0} + \varphi(\zeta) + a(|\varphi(\zeta)|^2))\, U_{\zeta_0},$$

where

$$\begin{aligned}\varphi(\zeta) =& \Pi_{U_{\zeta_0}^{\perp}}[Z^* + \tilde\Phi(\zeta) + (U_\zeta - U_{\zeta_0})(1-\gamma(\zeta) + a(p))],\\ \gamma(\zeta) =& U_p \cdot (Z^* + \tilde\Phi(\zeta))\\ a(p) =& a(|\Pi_{U_{\zeta}^{\perp}}[Z^* + \tilde\Phi(\zeta)]|^2).\end{aligned}$$

Therefore, equation (14.1) becomes

$$\Pi_{U_{\zeta_0}^{\perp}}[Z_0^* + \tilde\Phi(\zeta_0) + \varphi_1] = \Pi_{U_{\zeta_0}^{\perp}}[Z^* + \tilde\Phi(\zeta) + (U_\zeta - U_{p_0})(1-\gamma + a)]$$

or, equivalently

$$\Pi_{U_{\zeta_0}^{\perp}}[Z_1^* + \tilde\Phi(\zeta) - \tilde\Phi(\zeta_0) + (U_\zeta - U_{\zeta_0})(1-\gamma(\zeta) + a(\zeta)) - \varphi_1] \;=\; 0.$$

We will get a solution to this equation if we find a constant c_0 such that

$$Z_1^* + \tilde\Phi(\zeta_0 + \zeta_1) - \tilde\Phi(\zeta_0) + (U_\zeta - U_{\zeta_0})(1-\gamma(\zeta) + a(\zeta)) = \varphi_1 + c_0 U_{\zeta_0}$$

Let us consider the functions $Z_{lj}(y)$ defined in (2.2), $l = 0, 1$, $j = 1, 2$, with $y = \frac{x-q}{\varepsilon}$. We introduce the following intermediate problem: we want to find a function Z_1^* and five constants c_0, c_{lj} such that

$$Z_1^* + \tilde\Phi(\zeta_0 + p_1) - \tilde\Phi(\zeta_0) + (U_\zeta - U_{\zeta_0})(1-\gamma(\zeta) + a(\zeta)) = \varphi_1 + c_0 U_{\zeta_0} + c_{lj} Z_{lj} \tag{14.2}$$

and the following five real constraints hold for the function $Z_1^*(x)$:

$$\operatorname{div} z_1^*(\zeta_0) = 0, \quad \operatorname{curl} Z_1^*(q_0) = 0, \quad Z_1^*(q_0) = 0. \tag{14.3}$$

Summation convention is used in (14.2).

To make the argument more transparent, we consider a simplified linearized version of (14.2)-(14.3), in which lower order terms are neglected, and only the constants associated to mode 0 (associated to dilations and rotations) are considered. Thus we consider the model equation for Z_1^*,

$$\begin{cases} Z_1^* + \Phi_0[Z_1^*] = \varphi_1 + \displaystyle\sum_{j=1}^{2} c_{0j} Z_{0j}, \\ \operatorname{div} z_1^*(q_0, 0) = 0, \quad \operatorname{curl} z_1^*(q_0, 0) = 0. \end{cases} \tag{14.4}$$

where

$$\Phi_0[Z_1^*](r) = \begin{pmatrix} \phi_0[Z_1^*](r,t) \\ 0 \end{pmatrix}$$

with

$$\phi_0[Z_1^*](r) = r e^{i\theta} \int_{-T}^{0} \dot p(s) k(r^2 + \varepsilon^2, -s)\, ds, \quad k(\zeta, t) = 2\frac{1 - e^{-\frac{\zeta}{4t}}}{\zeta}, \tag{14.5}$$

where $p(t) = \lambda(t) e^{i\omega(t)}$, $r = |x - q_0|$, $\varepsilon = \lambda(0)$, and $p = p[Z_1^*]$ is such that the following equation is satisfied

$$\begin{cases} \dot p(t) |\log(T-t)| + \displaystyle\int_{-T}^{t} \frac{\dot p(s)}{T-s}\, ds = \operatorname{div} \tilde z_1(q,t) + i \operatorname{curl} \tilde z_1(q,t), \quad t \in [0,T]. \\ \qquad p(T) = 0, \end{cases} \tag{14.6}$$

where

$$\begin{cases} \partial_t \tilde Z_1(x,t) = \Delta \tilde Z_1(x,t) \quad \text{in } \Omega \times (0,T) \\ \tilde Z_1(x,0) = Z_1^*(x) \quad x \in \Omega \\ \tilde Z_1(x,t) = 0 \quad (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{14.7}$$

and we use the notation

$$\tilde{Z}_1 = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_{1,3} \end{pmatrix}, \quad Z_1^* = \begin{pmatrix} z_1^* \\ z_{1,3}^* \end{pmatrix}.$$

The main result here is the solvability of (14.4).

Proposition 14.1. *Assume $\|\varphi_1\|_*$ is finite. Then for $T>0$ sufficiently small equation (14.4) has a unique solution Z_1^*, c_{01}, $c_{0,2}$ and moreover*

$$\|Z_1^*\|_* + |c_{01}| + |c_{02}| \le C\|\varphi_1\|_*.$$

We can obtain a similar result if all constraints and constants are considered, with essentially the same proof as that below. On the other hand, to derive the corresponding result to the full problem (14.2)-(14.3), we need to use the linearized version and contraction mapping principle. For that we need to use the precise Lipschitz estimates of the solution of the inner-outer gluing system on the parameters involved as done in §5 and §13. The C^1 character of the manifold predicted in Theorem 2 follows from the fixed point characterization and the implicit function theorem.

We devote the rest of this section to the proof of the proposition, whose main step is the following estimate.

Lemma 14.1. *Assume that*

$$\operatorname{div} z_1^*(q_0) = 0, \quad \operatorname{curl} z_1^*(q_0) = 0. \tag{14.8}$$

Then

$$\|\Phi_0[Z_1^*]\|_* \le \frac{C}{|\log T|}\|Z_1^*\|_*.$$

To prove this we need a corollary of Lemma 9.1 adapted to the norm $\|\ \|_*$ defined in (5.7) is the following.

Lemma 14.2. *Suppose $Z_1^* \in C^2(\overline{\Omega})$ satisfies*

$$|\nabla_x Z_1^*(x)| \le |\log\varepsilon|, \quad x\in\Omega$$

$$|D_x^2 Z_1^*(x)| \le \frac{|\log\varepsilon|^{\frac12}}{|x-q_0|+\varepsilon} \quad x\in\Omega.$$

Then the solution $\tilde{Z}_1$ of (14.7) satisfies

$$|\nabla_x \tilde{Z}_1(x,t)| \le |\log\varepsilon|, \quad t\ge 0, \tag{14.9}$$

and

$$|\nabla_x \tilde{Z}_1(x,t) - \nabla_x \tilde{Z}_1(x,T)| \le C\begin{cases} |\log\varepsilon| & \text{if } 0\le t\le \varepsilon^2 \\ |\log\varepsilon|^{\frac12}\frac{T-t}{T}(1+\log(\frac{T}{t})) & \text{if } \varepsilon^2 \le t\le T. \end{cases}$$

Proof. As in Lemma 9.1 we consider the function given by Duhamel's formula in $\mathbb{R}^2$ and then decompose the solution as a sum of the one in $\mathbb{R}^2$ and a smooth one in Ω with zero initial condition.

From (9.3) and $|\nabla_x Z_1^*(x)| \le |\log\varepsilon|$ we get (14.9).

For $0\le t\le\varepsilon^2$ we get

$$|\nabla_x \tilde{Z}_1(x,t) - \nabla_x \tilde{Z}_1(x,T)| \le C|\log\varepsilon|$$

from (14.9). For $\varepsilon^2\le t\le T$ from Lemma 9.1 we obtain

$$|\nabla_x \tilde{Z}_1(x,t) - \nabla_x \tilde{Z}_1(x,T)| \le C|\log\varepsilon|^{1/2}\frac{\sqrt{T}-\sqrt{t}}{\sqrt{T}}\left(1+\log\Big(\frac{T}{t}\Big)\right).$$

□

Proof of Lemma 14.1. Let $f(t) = \operatorname{div} \tilde z_1(q,t) + i \operatorname{curl} \tilde z_1(q,t)$. Differentiating (14.6) we find

$$\frac{d}{dt}\left(\dot p(t)|\log(T-t)|^2\right) = |\log(T-t)|\dot f(t).$$

This can be integrated explicitly and we get

$$\dot p(t) = -\frac{1}{|\log(T-t)|^2}\int_t^T |\log(T-s)|\dot f(s)\,ds + \frac{c|\log T|}{|\log(T-t)|^2}$$

for some constant c to be determined. Integrating by parts we find that

$$\dot p(t) = \frac{f(t)-f(T)}{|\log(T-t)|} + \frac{1}{|\log(T-t)|^2}\int_t^T \frac{f(s)-f(T)}{T-s}\,ds + \frac{c|\log T|}{|\log(T-t)|^2}.$$

This function is defined for $t \in [0,T]$ and we need to extend it to $[-T,T]$ to make sense of (14.6). A possible extension is $\dot p(t) = \dot p(0)$ for $t \in [-T,0]$ but this makes this lemma too simple and not useful to adapt to the real situation. For this reason we make the analysis with the following extension. Define

$$\dot p_1(t) = \frac{f(t)-f(T)}{|\log(T-t)|} + \frac{1}{|\log(T-t)|^2}\int_t^T \frac{f(s)-f(T)}{T-s}\,ds \tag{14.10}$$

so that

$$\dot p(t) = \dot p_1(t) + \frac{c|\log T|}{|\log(T-t)|^2} \quad \text{for } t\in[0,T]$$

Then define

$$\dot p(t) = \dot p_1(0) + \frac{c|\log T|}{|\log(T-t)|^2}, \quad t\in[-T,0]. \tag{14.11}$$

We want to estimate

$$\phi_0[Z_1^*](r) = re^{i\theta}\int_{-T}^0 \dot p(s)k(r^2+\varepsilon^2,-s)\,ds, \quad k(\zeta,t) = 2\frac{1-e^{-\frac{\zeta}{4t}}}{\zeta},$$

which, thanks to (14.11) depends only on $\dot p_1(0)$ and c. Therefore we need to estimate these quantities. We claim that

$$\dot p_1(0) = \frac{f(0)-f(T)+O(\frac{\log(|\log T|)}{|\log T|^{1/2}})\|Z_1^*\|_*}{|\log T|}(1+O(\frac{1}{|\log T|})) \tag{14.12}$$

$$c = f(T)(1+O(\frac{1}{|\log T|})) + O(\frac{1}{|\log T|})f(0) + O(\frac{\log(|\log T|)}{|\log T|^{1/2}})\|Z_1^*\|_*. \tag{14.13}$$

To obtain these estimates we note that evaluating equation (14.6) at $t=0$ we get

$$\dot p_1(0)(|\log T|+\log 2) + c(1+O(\frac{1}{|\log T|})) = f(0) \tag{14.14}$$

and evaluating equation (14.6) at $t=T$ we get

$$\int_{-T}^T \frac{\dot p_1(s)}{T-s}\,ds + c(1+O(\frac{1}{|\log T|})) = f(T). \tag{14.15}$$

Thus we need to estimate $\int_{-T}^T \frac{\dot p_1(s)}{T-s}\,ds$ where p_1 is given (14.10). We have

$$\begin{aligned}\int_{-T}^T \frac{\dot p_1(s)}{T-s}\,ds &= \int_{-T}^0 \frac{\dot p_1(s)}{T-s}\,ds + \int_0^T \frac{\dot p_1(s)}{T-s}\,ds \\ &= \dot p_1(0)\log 2 + \int_0^T \frac{\dot p_1(s)}{T-s}\,ds.\end{aligned}$$

To estimate $\int_0^T \frac{\dot p_1(s)}{T-s}\,ds$ we write

$$\dot p_1 = \dot p_{1a} + \dot p_{1b}$$

with

$$\dot p_{1a}(t) = \frac{f(t)-f(T)}{|\log(T-t)|}$$

$$\dot p_{1b}(t) = \frac{1}{|\log(T-t)|^2}\int_t^T \frac{f(s)-f(T)}{T-s}\,ds.$$

We compute

$$\int_0^T \frac{\dot p_{1a}(s)}{T-s}\,ds = \int_0^{\frac{T}{|\log T|}} ... + \int_{\frac{T}{|\log T|}}^T ....$$

By Lemma 14.2 we have that

$$|f(t)-f(T)| \le C\|Z_1^*\|_* \begin{cases} \log(|\log T|)|\log T|^{1/2}\dfrac{T-t}{T}, & \dfrac{T}{|\log T|} \le t \le T \\ |\log T|, & 0 \le t \le \dfrac{T}{|\log T|}. \end{cases} \tag{14.16}$$

which in particular implies

$$|\dot p_{1a}(t)| \le C\frac{\|Z_1^*\|_*}{|\log(T-t)|} \begin{cases} \log(|\log T|)|\log T|^{1/2}\dfrac{T-t}{T}, & \dfrac{T}{|\log T|} \le t \le T \\ |\log T|, & 0 \le t \le \dfrac{T}{|\log T|}. \end{cases}$$

Therefore

$$\int_0^{\frac{T}{|\log T|}} \frac{|\dot p_{1a}(s)|}{T-s}\,ds \le \frac{C}{|\log T|}\|Z_1^*\|_*,$$

and

$$\begin{aligned} &\int_{\frac{T}{|\log T|}}^T \frac{|\dot p_{1a}(s)|}{T-s}\,ds \\ &\le C\log(|\log T||\log T|^{1/2})\|Z_1^*\|_* \int_{\frac{T}{|\log T|}}^T \frac{1}{T-s}\frac{T-s}{T}\frac{1}{|\log(T-s)|}\,ds \\ &\le C\frac{\log(|\log T|)|\log T|^{1/2}}{|\log T|}\|Z_1^*\|_*. \end{aligned}$$

It follows that

$$\int_0^T \frac{|\dot p_{1a}(s)|}{T-s}\,ds \le C\frac{\log(|\log T|)}{|\log T|^{1/2}}\|Z_1^*\|_*. \tag{14.17}$$

By (14.16), we find that

$$|\dot p_{1b}(t)| \le C\|Z_1^*\|_* \begin{cases} \dfrac{\log(|\log T|)|\log T|^{1/2}}{|\log(T-t)|^2}\dfrac{T-t}{T}, & \dfrac{T}{|\log T|} \le t \le T \\ \dfrac{\log(|\log T|)}{|\log T|^{3/2}}, & 0 \le t \le \dfrac{T}{|\log T|}. \end{cases}$$

This implies that

$$\int_0^T \frac{|\dot p_{1b}(s)|}{T-s}\,ds \le \frac{\log(|\log T|)}{|\log T|^{3/2}}\|Z_1^*\|_*. \tag{14.18}$$

From (14.17) and (14.18) we find that

$$\left|\int_0^T \frac{\dot p_1(s)}{T-s}\,ds\right| \le \frac{\log(|\log T|)}{|\log T|^{1/2}}\|Z_1^*\|_*.$$

Therefore (14.15) gives

$$\dot p_1(0)\log 2 + O(\frac{\log(|\log T|)}{|\log T|^{1/2}})\|Z_1^*\|_* + c(1+O(\frac{1}{|\log T|})) = f(T). \tag{14.19}$$

Equations (14.14) and (14.19) form a system

$$\begin{bmatrix} |\log T| + \log 2 & 1+O(\frac{1}{|\log T|}) \\ \log 2 & 1+O(\frac{1}{|\log T|}) \end{bmatrix} \begin{bmatrix} \dot p_1(0) \\ c \end{bmatrix} = \begin{bmatrix} f(0) \\ f(T) + O(\frac{\log(|\log T|)}{|\log T|^{1/2}})\|Z_1^*\|_* \end{bmatrix}$$

for $\dot p_1(0)$ and c, and solving we get (14.12), (14.13).

We use (14.12), (14.13) to estimate ϕ_0 given by (14.5):

$$\phi_0[Z_1^*](r) = re^{i\theta}\int_{-T}^0 \dot p(s) k(r^2+\varepsilon^2, -s)\, ds, \quad k(\zeta,t) = 2\frac{1-e^{-\frac{\zeta}{4t}}}{\zeta}.$$

We start with the L^∞ part of the norm. Let us consider first the case $r^2+\varepsilon^2 \le T$. We note that

$$\begin{aligned}\int_{-T}^0 \dot p(s) k(r^2+\varepsilon^2,-s)\, ds = &\int_{-T}^{-(r^2+\varepsilon^2)} \dot p(s) k(r^2+\varepsilon^2,-s)\, ds \\ &+ \int_{-(r^2+\varepsilon^2)}^0 \dot p(s) k(r^2+\varepsilon^2,-s)\, ds. \end{aligned}\tag{14.20}$$

Then

$$\begin{aligned}\int_{-T}^{-(r^2+\varepsilon^2)} \dot p(s) k(r^2+\varepsilon^2,-s)\, ds = \dot p(0) &\int_{-T}^{-(r^2+\varepsilon^2)} k(r^2+\varepsilon^2,-s)\, ds \\ &+ \int_{-T}^{-(r^2+\varepsilon^2)} (\dot p(s) - \dot p(0)) k(r^2+\varepsilon^2,-s)\, ds. \end{aligned}\tag{14.21}$$

But $\dot p(0) = \dot p_1(0) + \frac{c}{|\log T|}$. From (14.12) and (14.13) we find

$$\dot p_1(0) + \frac{c}{|\log T|} = \frac{f(0)}{|\log T|} + (|f(0)|+|f(T)|) O(\frac{1}{|\log T|^2}) + O(\frac{\log(|\log T|)}{|\log T|^{3/2}})\|Z_1^*\|_*. \tag{14.22}$$

The hypothesis (14.8) means $f(0) =$ and hence

$$\dot p_1(0) + \frac{c}{|\log T|} = (|f(0)|+|f(T)|) O(\frac{1}{|\log T|^2}) + O(\frac{\log(|\log T|)}{|\log T|^{3/2}})\|Z_1^*\|_*.$$

Since $f(t) = O(|\log T| \|Z_1^*\|_*)$ we find

$$\dot p(0) = \dot p_1(0) + \frac{c}{|\log T|} = O(\frac{1}{|\log T|})\|Z_1^*\|_*.$$

Therefore

$$\begin{aligned}\left| \dot p(0) \int_{-T}^{-(r^2+\varepsilon^2)} k(r^2+\varepsilon^2,-s)\, ds \right| &\le \frac{C}{|\log T|}\|Z_1^*\|_* \int_{-T}^{-(r^2+\varepsilon^2)} \frac{1}{-s}\, ds \\ &\le \frac{C}{|\log T|}\|Z_1^*\|_* \left|\log(\frac{r^2+\varepsilon^2}{T})\right|. \end{aligned}$$

For the second integral in (14.21), note that

$$\begin{aligned}
&\left|\int_{-T}^{-(r^2+\varepsilon^2)} (\dot p(s)-\dot p(0))k(r^2+\varepsilon^2,-s)\,ds\right| \\
&\le |c||\log T|\int_{-T}^{-(r^2+\varepsilon^2)}\left[\frac{1}{|\log(T-s)|^2}-\frac{1}{|\log T|^2}\right]\frac{1}{-s}\,ds \\
&\le C\frac{|c|}{|\log T|^2}\left|\log(\frac{r^2+\varepsilon^2}{T})\right|.
\end{aligned}$$

Since $|c|\le C|\log T|\|Z_1^*\|_*$ we get

$$\left|\int_{-T}^{-(r^2+\varepsilon^2)} (\dot p(s)-\dot p(0))k(r^2+\varepsilon^2,-s)\,ds\right| \le C\frac{\|Z_1^*\|_*}{|\log T|}\left|\log(\frac{r^2+\varepsilon^2}{T})\right|.$$

For the second integral in (14.20) we have

$$\begin{aligned}
\left|\int_{-(r^2+\varepsilon^2)}^{0} \dot p(s)k(r^2+\varepsilon^2,-s)\,ds\right| &\le \|\dot p\|_{L^\infty}\int_{-(r^2+\varepsilon^2)}^{0}|k(r^2+\varepsilon^2,-s)|\,ds \\
&\le \frac{C}{|\log T|}\|Z_1^*\|_*.
\end{aligned}$$

In summary, when $r^2+\varepsilon^2\le T$ we obtain

$$\left|\int_{-T}^{0} \dot p(s)k(r^2+\varepsilon^2,-s)\,ds\right| \le C\frac{\|Z_1^*\|_*}{|\log T|}\left[1+\left|\log(\frac{r^2+\varepsilon^2}{T})\right|\right]. \tag{14.23}$$

When $r^2+\varepsilon^2\ge T$ we obtain directly

$$\left|\int_{-T}^{0} \dot p(s)k(r^2+\varepsilon^2,-s)\,ds\right| \le C\frac{\|Z_1^*\|_*}{|\log T|}. \tag{14.24}$$

Both estimates show that

$$\|\phi_0[Z_1^*](r)\|_{L^\infty}\le C\frac{\|Z_1^*\|_*}{|\log T|}.$$

For first derivatives we have

$$\partial_r\phi_0 = e^{i\theta}\int_{-T}^{0}\dot p(s)k(r^2+\varepsilon^2,-s)\,ds + 2r^2e^{i\theta}\int_{-T}^{0}\dot p(s)k_\zeta(r^2+\varepsilon^2,-s)\,ds.$$

Using (14.23), (14.24)

$$\frac{1}{|\log\varepsilon|}\sup_{\Omega}\left|\int_{-T}^{-(r^2+\varepsilon^2)}\frac{\dot p(s)}{-s}\,ds\right| \le \frac{C}{|\log T|}\|Z_1^*\|_*.$$

($r=|x-q_0|$). We get at worst the same estimate for the other terms of $\nabla_x\phi_0$.

For the second derivatives we proceed similarly. We have, for instance,

$$\partial_{rr}\phi_0 = 6re^{i\theta}\int_{-T}^{0}\dot p(s)k_\zeta(r^2+\varepsilon^2,-s)\,ds + 2r^3e^{i\theta}\int_{-T}^{0}\dot p(s)k_{\zeta\zeta}(r^2+\varepsilon^2,-s)\,ds.$$

Let us analyze the term $re^{i\theta}\int_{-T}^0\dot p(s)k_\zeta(r^2+\varepsilon^2,-s)\,ds$. Assume $r^2+\varepsilon^2\le T$ and split

$$re^{i\theta}\int_{-T}^{0}\dot p(s)k_\zeta(r^2+\varepsilon^2,-s)\,ds = re^{i\theta}\int_{-T}^{-(r^2+\varepsilon^2)}\dots + re^{i\theta}\int_{-(r^2+\varepsilon^2)}^{0}\dots$$

For the first term we estimate in a similar way to (14.21), since for $r^2+\varepsilon^2\le|s|$, $s\in[-T,-r^2-\varepsilon^2]$ we have $|k_\zeta(r^2+\varepsilon^2,-s)\le\frac{C}{|s|}$. and we get

$$\left|re^{i\theta}\int_{-T}^{-(r^2+\varepsilon^2)}\dot p(s)k_\zeta(r^2+\varepsilon^2,-s)\,ds\right| \le C\frac{\|Z_1^*\|_*}{|\log T|}r\left[1+\left|\log(\frac{r^2+\varepsilon^2}{T})\right|\right].$$

For the second term we have

$$re^{i\theta}\int_{-(r^2+\varepsilon^2)}^0 \dot p(s)k_\zeta(r^2+\varepsilon^2,-s)\,ds = re^{i\theta}\dot p(0)\int_{-(r^2+\varepsilon^2)}^0 k_\zeta(r^2+\varepsilon^2,-s)\,ds + re^{i\theta}\int_{-(r^2+\varepsilon^2)}^0 (\dot p(s)-\dot p(0))k_\zeta(r^2+\varepsilon^2,-s)\,ds.$$

But using (14.13) and $|f(t)|\le C|\log T|\,\|Z_1^*\|_*$ we find

$$\left|re^{i\theta}\int_{-(r^2+\varepsilon^2)}^0 (\dot p(s)-\dot p(0))k_\zeta(r^2+\varepsilon^2,-s)\,ds\right| \le \frac{C}{r+\varepsilon}|c|\frac{1}{|\log T|^2} \le \frac{C}{|\log T|}\frac{1}{r+\varepsilon}\|Z_1^*\|_*.$$

Since $\dot p(0) = \dot p_1(0)+\frac{c}{|\log T|}$, using (14.22) and (14.8) we obtain

$$|\dot p(0)|\le \frac{C}{|\log T|}\|Z_1^*\|_*$$

and hence

$$\frac{1}{|\log T|^{1/2}}\sup_\Omega (r+\varepsilon)\left|re^{i\theta}\int_{-(r^2+\varepsilon^2)}^0 \dot p(s)k_\zeta(r^2+\varepsilon^2,-s)\,ds\right| \le \frac{C}{|\log T|^{3/2}}\|Z_1^*\|_*.$$

Similar estimates for the other terms show the validity of

$$\|\Phi_0[Z_1^*]\|_*\le \frac{C}{|\log T|}\|Z_1^*\|_*,$$

which is the desired conclusion. □

Proof of Proposition 14.1. We look for a solution of (14.4) in the space of functions

$$\mathcal{Z} = \{Z_1^*\in C^2(\overline\Omega) : \|Z_1^*\|_*<\infty,\ \operatorname{div} z_1^*(q_0)=0,\ \operatorname{curl} z_1^*(q_0)=0\}.$$

To determinte c_{0j} we apply divergence and curl (14.4) at q_0 to obtain

$$\begin{aligned} c_{01} &= \varepsilon\left(\operatorname{div}\phi_0[Z_1^*](q_0,0) - \operatorname{div}\varphi_1(q_0)\right)\\ c_{02} &= \varepsilon\left(\operatorname{curl}\phi_0[Z_1^*](q_0,0) - \operatorname{curl}\varphi_1(q_0)\right). \end{aligned}$$

With this equation (14.4) becomes the fixed point problem

$$Z_1^* = \mathcal{F}[Z_1^*] + \varphi_1 + \operatorname{div}\varphi_1(q_0)\varepsilon Z_{01} + \operatorname{curl}\varphi_1(q_0)\varepsilon Z_{02}. \tag{14.25}$$

where

$$\mathcal{F}[Z_1^*] = -\Phi_0[Z_1^*] - \operatorname{div}\phi_0[Z_1^*](q_0,0)\varepsilon Z_{01} - \operatorname{curl}\phi_0[Z_1^*](q_0,0)\varepsilon Z_{02}$$

By Lemma 14.1 we get

$$\begin{aligned} |\operatorname{div}\phi_0[Z_1^*](q_0,0) + i\operatorname{curl}\phi_0[Z_1^*](q_0,0)| &\le C|\log\varepsilon|\|\Phi_0[Z_1^*]\|_*\\ &\le C\|Z_1^*\|_*. \end{aligned}$$

But

$$\|\varepsilon Z_{0j}\|_*\le \frac{C}{|\log T|^{1/2}}.$$

This and Lemma 14.1 shows that

$$\|\mathcal{F}[Z_1^*]\|_*\le \frac{C}{|\log T|^{1/2}}\|Z_1^*\|_*.$$

By the contraction mapping principle, equation (14.25) has a unique fixed point in $\mathcal{Z}$. □

15. Estimates for the solution of the heat equation

15.1. **Proof of Lemma 8.1.** The proof of the estimates is done by analyzing the solution ψ of

$$\begin{cases} \partial_t\psi_0 = \Delta\psi_0 + f \quad \text{in } \mathbb{R}^2\times(0,T), \\ \psi_0(x,0) = 0 \quad x\in\mathbb{R}^2, \end{cases} \tag{15.1}$$

defined by Duhamel's formula

$$\psi_0(x,t) = \int_0^t \int_{\mathbb{R}^2} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{4\pi(t-s)} f(y,s)\,dyds.$$

assuming

$$|f(x,t)| \le \chi_{\{|y|\le 2\lambda_*(s)R(s)\}}\lambda_*(t)^{\nu-2}R(t)^{-a}. \tag{15.2}$$

The solution to (8.5) is then given by $\psi = \psi_0 + \psi_1$ where ψ_1 solves the homogeneous heat equation in $\Omega\times(0,T)$ with boundary condition given by $-\psi_0$. In the sequel we prove that the estimates (8.6)–(8.7) are valid for ψ_0. Then the conclusion for ψ_1 follows from standard parabolic estimates. In what follows we denote by ψ the solution to (15.1) given by Duhamel's formula.

Proof of (8.6). We have, using the heat kernel,

$$\begin{aligned}\psi(x,t) &= C\int_0^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{t-s}\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t-s)}}\,dyds \\ &= C\int_0^t \lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x - z|^2}\,dzds\end{aligned}$$

where $\tilde x = x(t-s)^{-1/2}$. First we estimate

$$\begin{aligned}&\int_0^{t-(T-t)} \lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x-z|^2}\,dzds \\ &\le C\int_0^{t-(T-t)} \frac{\lambda_*(s)^{\nu}R(s)^{2-a}}{t-s}ds \\ &\le C\int_0^{t-(T-t)} \frac{\lambda_*(s)^{\nu}R(s)^{2-a}}{T-s}ds \\ &\le C\lambda_*(0)^{\nu}R(0)^{2-a}.\end{aligned} \tag{15.3}$$

Consider the integrals $\int_{t-(T-t)}^{t-\lambda_*(t)^2}$ and $\int_{t-\lambda_*(t)^2}^{t}$. We have

$$\begin{aligned}&\int_{t-(T-t)}^{t-\lambda_*(t)^2} \lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x-z|^2/4}\,dzds \\ &\le C\int_{t-(T-t)}^{t-\lambda_*(t)^2} \frac{\lambda_*(s)^{\nu}R(s)^{2-a}}{t-s}\,ds \\ &\le C\lambda_*(t)^{\nu}R(t)^{2-a}|\log(T-t)|.\end{aligned} \tag{15.4}$$

For the second part we have

$$\begin{aligned}&\int_{t-\lambda_*(t)^2}^{t} \lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x-z|^2/4}\,dzds \\ &\le C\int_{t-\lambda_*(t)^2}^{t} \lambda_*(s)^{\nu-2}R(s)^{-a}\,ds \\ &\le C\lambda_*(t)^{\nu}R(t)^{-a}.\end{aligned} \tag{15.5}$$

From (15.3), (15.4), (15.5),we deduce

$$|\psi(x,t)| \le C\lambda_*(0)^\nu R(0)|\log T|.$$

which is the desired estimate (8.6). □

Proof of (8.7). Using the heat kernel we have

$$|\psi(x,t) - \psi(x,T)| \le I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_1 &= \int_0^{t-(T-t)} \int_{\mathbb{R}^2} |G(x-y,t-s) - G(x-y,T-s)||f(y,s)|\, dyds \\
I_2 &= \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} |G(x-y,t-s) - G(x-y,T-s)||f(y,s)|\, dyds \\
I_3 &= \int_t^T \int_{\mathbb{R}^2} |G(x-y,t-s) - G(x-y,T-s)||f(y,s)|\, dyds.
\end{aligned}$$

We estimate the first integral

$$I_1 \le (T-t)\int_0^1 \int_0^{t-(T-t)} \lambda_*(s)^{\nu-2}R(s)^{-a} \int_{|y|\le 2\lambda_*(s)R(s)} |\partial_t G(x-y,t_v-s)|\, dydsdv,$$

where $t_v = vT + (1-v)(T-t)$. We have

$$\begin{aligned}
&\int_{|y|\le 2\lambda_*(s)R(s)} |\partial_t G(x-y,t_v-s)|\, dy \\
&\le \frac{C}{(t_v-s)^2}\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)}}\left(1+\frac{|x-y|^2}{t_v-s}\right) dy \\
&\le C\frac{1}{(t_v-s)}\int_{|z|\le 2\lambda_*(s)R(s)(t_v-s)^{-1/2}} e^{-|\tilde x - z|^2}\left(1+|\tilde x - z|^2\right) dz.
\end{aligned}$$

We then get

$$\begin{aligned}
&\int_0^{t-(T-t)} \lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|y|\le 2\lambda_*(s)R(s)} |\partial_t G(x-y,t_v-s)|\, dyds \\
&\le C\int_0^{t-(T-t)} \frac{\lambda_*(s)^\nu R(s)^{2-a}}{(t_v-s)^2}\, ds \\
&\le C\int_0^{t-(T-t)} \frac{\lambda_*(s)^\nu R(s)^{2-a}}{(T-s)^2}\, ds \\
&\le C\frac{|\log T|^{\nu-\beta(2-a)}(T-t)^{\nu-\beta(2-a)}}{|\log(T-t)|^{2\nu-2\beta(2-a)}},
\end{aligned}$$

where we have used $R(t) = \lambda_*(t)^{-\beta}$. Therefore

$$I_1 \le C\lambda_*(t)^\nu R(t)^{2-a}.$$

Next we estimate I_2:

$$\begin{aligned}
I_2 \le{}& \int_{t-(T-t)}^{t}\int_{|y|\le 2\lambda_*(s)R(s)} |G(x-y,t-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\, dyds \\
&+ \int_{t-(T-t)}^{t}\int_{|y|\le 2\lambda_*(s)R(s)} |G(x-y,T-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\, dyds.
\end{aligned}$$

These two integrals are very similar. Let us compute the first one

$$
\begin{aligned}
&\int_{t-(T-t)}^{t}\int_{|y|\le 2\lambda_*(s)R(s)} |G(x-y,t-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\,dyds\\
&\le C\int_{t-(T-t)}^{t}\frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{t-s}\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t-s)}}\,dyds\\
&= C\int_{t-(T-t)}^{t}\lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4}\,dzds.
\end{aligned}
$$

We split this integral in $\int_{t-(T-t)}^{t-\lambda_*(t)^2}\ldots+\int_{t-\lambda_*(t)^2}^{t}$ and estimate

$$
\begin{aligned}
&\int_{t-(T-t)}^{t-\lambda_*(t)^2}\lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4}\,dzds\\
&\le C\int_{t-(T-t)}^{t-\lambda_*(t)^2}\frac{\lambda_*(s)^{\nu}R(s)^{2-a}}{t-s}\,ds\\
&\le C\lambda_*(t)^{\nu}R(t)^{2-a}|\log(T-t)|.
\end{aligned}
$$

For the second part we have

$$
\begin{aligned}
&\int_{t-\lambda_*(t)^2}^{t}\lambda_*(s)^{\nu-2}R(s)^{-a}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4}\,dzds\\
&\le C\lambda_*(t)^{\nu}R(t)^{-a},
\end{aligned}
$$

and therefore, summarizing,

$$
\begin{aligned}
&\int_{t-(T-t)}^{t}\int_{|y|\le 2\lambda_*(s)R(s)} |G(x-y,t-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\,dyds\\
&\le C\lambda_*(t)^{\nu}R(t)^{2-a}|\log(T-t)|.
\end{aligned}
$$

Similar computations show that

$$
\begin{aligned}
&\int_{t-(T-t)}^{t}\int_{|y|\le 2\lambda_*(s)R(s)} |G(x-y,T-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\,dyds\\
&\le C\lambda_*(t)^{\nu}R(t)^{2-a}|\log(T-t)|
\end{aligned}
$$

and we obtain

$$
I_2\le C\lambda_*(t)^{\nu}R(t)^{2-a}|\log(T-t)|.
$$

Finally

$$
\begin{aligned}
I_3 &\le C\int_t^T\int_{|y|\le 2\lambda_*(s)R(s)}\frac{e^{-\frac{|x-y|^2}{4(T-s)}}}{T-s}\lambda_*(s)^{\nu-2}R(s)^{-a}\,dyds\\
&\le C\int_t^T\frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{T-s}\int_{|z|\le 2\lambda_*(s)R(s)(T-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4}\,dzds\\
&\le C\int_t^T\frac{\lambda_*(s)^{\nu}R(s)^{2-a}}{(T-s)}ds\\
&\le C\lambda_*(t)^{\nu}R(t)^{2-a}.
\end{aligned}
$$

This finishes the proof of (8.7). □

Proof of (8.8). Using the heat kernel we have

$$
\begin{aligned}
|\nabla\psi(x,t)| &\le C\int_0^t \lambda_*(s)^{\nu-2}R(s)^{-a}\frac{1}{(t-s)^2}\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t-s)}}|x-y|\,dyds\\
&= C\int_0^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x-z|^2/4}|\tilde x-z|\,dzds\\
&\le C\int_0^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}(1+|z|)\,dzds
\end{aligned}
$$

where $\tilde x=(t-s)^{-1/2}x$. Then

$$
\begin{aligned}
&\int_0^{t-\frac{1}{10}(T-t)} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}(1+|z|)\,dzds\\
&\le C\int_0^{t-\frac{1}{10}(T-t)}\frac{\lambda_*(s)^{\nu}R(s)^{2-a}}{(t-s)^{3/2}}\,ds\\
&\le C\int_0^{t-\frac{1}{10}(T-t)}\frac{\lambda_*(s)^{\nu}R(s)^{2-a}}{(T-s)^{3/2}}\,ds\\
&= C|\log T|^{\nu-\beta(2-a)}\int_0^{t-\frac{1}{10}(T-t)}\frac{(T-s)^{\nu-\beta(2-a)-3/2}}{|\log(T-s)|^{2(\nu-\beta(2-a))}}\,ds\\
&= C|\log T|^{\nu-\beta(2-a)}\frac{T^{\nu-\beta(2-a)-1/2}}{|\log T|^{2(\nu-\beta(2-a))}},
\end{aligned}
$$

and here we need $\nu-1/2-\beta(2-a)>0$. Since $\beta<\frac{1}{2}$, we have

$$
\begin{aligned}
|\log T|^{\nu-\beta(2-a)}\frac{T^{\nu-\beta(2-a)-1/2}}{|\log T|^{2(\nu-\beta(2-a))}} &\le C\lambda_*(0)^{\nu}R(0)^{2-a}T^{-1/2}\\
&\le C\lambda_*(0)^{\nu-1}R(0)^{1-a}.
\end{aligned}
$$

To estimate the integral

$$
\int_{t-\frac{1}{10}(T-t)}^t \lambda_*(s)^{\nu-2}R(s)^{-a}\frac{1}{(t-s)^{1/2}}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}(1+|z|)\,dzds
$$

let us define

$$
g(v)=\int_0^{2v} e^{-\rho^2/4}(1+\rho)\rho\,d\rho
$$

so that

$$
\begin{aligned}
&\int_{t-\frac{1}{10}(T-t)}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}}\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}(1+|z|)\,dzds\\
&=\int_{t-\frac{1}{10}(T-t)}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}}\,g\left(\frac{\lambda_*(s)R(s)}{(t-s)^{1/2}}\right)ds
\end{aligned}
$$

We change variables

$$
\frac{t-s}{\lambda_*(s)^2R(s)^2}=u
$$

and note that

$$
\frac{du}{ds}=-\frac{1}{\lambda_*(s)^2R(s)^2}\left(1+\frac{2\dot\lambda_*(s)}{\lambda_*(s)}(t-s)+\frac{2\dot R(s)}{R(s)}(t-s)\right).
$$

Then

$$
\begin{aligned}
&\int_{t-\frac{1}{10}(T-t)}^{t} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}} \int_{|z|\leq 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}(1+|z|)\,dzds \\
&= \int_0^{\frac{T-t}{10\lambda_*(t-\frac{1}{10}(T-t))^2 R(t-\frac{1}{10}(T-t))^2}} \lambda_*(s(u))^{\nu-1}R(s(u))^{1-a}u^{-1/2}g(u^{-1/2}) \\
&\qquad \cdot \left| 1 + \frac{2\dot\lambda_*(s)}{\lambda_*(s)}(t-s) + \frac{2\dot R(s)}{R(s)}(t-s) \right| du \\
&\leq C\lambda_*(t)^{\nu-1}R(t)^{1-a}.
\end{aligned}
$$

This establishes (8.8). ∎

Proof of (8.9). Using the heat kernel we have

$$
\begin{aligned}
&\partial_{x_i}\psi(x,t) - \partial_{x_i}\psi(x,T) \\
&= \int_0^t \int_{\mathbb{R}^2} \left(\partial_{x_i}G(x-y,t-s) - \partial_{x_i}G(x-y,T-s)\right) f(y,s)\,dyds \\
&\quad - \int_0^T \int_{\mathbb{R}^2} \partial_{x_i}G(x-y,T-s) f(y,s)\,dyds,
\end{aligned}
$$

and so

$$
|\partial_{x_i}\psi(x,t) - \partial_{x_i}\psi(x,T)| \leq I_1 + I_2 + I_3,
$$

where

$$
\begin{aligned}
I_1 &= \int_0^{t-(T-t)} \int_{\mathbb{R}^2} |\partial_{x_i}G(x-y,t-s) - \partial_{x_i}G(x-y,T-s)| f(y,s)\,dyds \\
I_2 &= \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} |\partial_{x_i}G(x-y,t-s) - \partial_{x_i}G(x-y,T-s)| f(y,s)\,dyds \\
I_3 &= \int_t^T \int_{\mathbb{R}^2} |\partial_{x_i}G(x-y,T-s)| f(y,s)\,dyds.
\end{aligned}
$$

For the first integral, we have

$$
\begin{aligned}
I_1 \leq C(T-t)\int_0^1 \int_0^{t-(T-t)} &\frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_v-s)^{5/2}} \Bigg\{ \\
&\int_{|y|\leq 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) dy \Bigg\} dsdv
\end{aligned}
$$

where $t_v = vT + (1-v)(T-t)$. Changing variables

$$
\begin{aligned}
&\int_{|y|\leq 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) dy \\
&= (t_v-s)\int_{|z|\leq 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x_v - z|^2/4}(|\tilde x_v - z| + |\tilde x_v - z|^3)\,dz
\end{aligned}
$$

where $\tilde x_v = x(t_v-s)^{-1/2}$. We then need to estimate

$$
\begin{aligned}
&\int_0^{t-(T-t)} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_v-s)^{3/2}} \int_{|z|\leq 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x_v - z|^2/4}(|\tilde x_v - z| + |\tilde x_v - z|^3)\,dzds \\
&\leq C\int_0^{t-(T-t)} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(T-s)^{3/2}} \int_0^{2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-\rho^2/4}(1+\rho^3)\rho\,d\rho ds.
\end{aligned}
$$

We note that in the range $T/2 \le t \le T$ we have $\lambda_*(s)R(s)(t-s)^{-1/2} \le 1$ for all $0 \le s \le t-(T-t)$. Therefore

$$
\begin{aligned}
&\int_0^{t-(T-t)} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(T-s)^{3/2}} \int_0^{2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-\rho^2/4}(1+\rho^3)\rho\, d\rho ds \\
&\le C\int_0^{t-(T-t)} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(T-s)^{3/2}} \lambda_*(s)R_0(s)(t-s)^{-1/2} ds \\
&\le C\int_0^{t-(T-t)} \frac{\lambda_*(s)^{\nu-1}R(s)^{1-a}}{(T-s)^{2}} ds \\
&\le C(T-t)^{-1}\lambda_*(t)^{\nu-1}R(t)^{1-a}.
\end{aligned}
$$

Therefore

$$
I_1 \le C\lambda_*(t)^{\nu-1}R(t)^{1-a}.
$$

To estimate I_2 it is sufficient to bound the terms

$$
\begin{aligned}
&\int_{t-(T-t)}^t \int_{|y|\le 2\lambda_*(s)R(s)} |\nabla_x G(x-y,t-s)|\lambda_*(s)^{\nu-2}R^{-a}\, dyds, \\
&\int_{t-(T-t)}^t \int_{|y|\le 2\lambda_*(s)R(s)} |\nabla_x G(x-y,T-s)|\lambda_*(s)^{\nu-2}R^{-a}\, dyds.
\end{aligned}
$$

Let us start with:

$$
\begin{aligned}
&\int_{t-(T-t)}^t \int_{|y|\le 2\lambda_*(s)R(s)} |\nabla_x G(x-y,t-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\, dyds \\
&\le C\int_{t-(T-t)}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}} \int_{|z|(t-s)^{1/2}\le 2\lambda_*(s)R(s)} e^{-|\tilde x - z|^2}|\tilde x - z|\, dzds, \\
&\le C\int_{t-(T-t)}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}} \int_0^{2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-\rho^2}(1+\rho)\rho\, d\rho ds,
\end{aligned}
$$

where $\tilde x = (t-s)^{-1/2}x$. We note that for $s \in [t-(T-t), t]$ the inequality

$$
\frac{\lambda_*(s)R(s)}{(t-s)^{1/2}} \le 1
$$

is equivalent to $s \le s^*$ for some $s^* \in (t-(T-t), t)$, and that for $s \le s^*$

$$
\int_0^{2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-\rho^2}(1+\rho)\rho\, d\rho \le C\lambda_*(s)^2R(s)^2(t-s)^{-1}.
$$

Then

$$
\begin{aligned}
&\int_{t-(T-t)}^{s^*} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}} \int_0^{2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-\rho^2}(1+\rho)\rho\, d\rho ds \\
&\le C\lambda_*(t)^{\nu}R(t)^{2-a}\int_{t-(T-t)}^{s^*} \frac{1}{(t-s)^{3/2}}\, ds \\
&\le C\lambda_*(t)^{\nu-1}R(t)^{1-a}.
\end{aligned}
$$

The integral on $[s^*, t]$ is estimated

$$
\begin{aligned}
&\int_{s^*}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}} \int_0^{2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-\rho^2}(1+\rho)\rho\, d\rho ds \\
&\le C\lambda_*(t)^{\nu-2}R(t)^{-a}\int_{s^*}^t \frac{1}{(t-s)^{1/2}} ds \\
&\le C\lambda_*(t)^{\nu-1}R(t)^{1-a}.
\end{aligned}
$$

In the same way we get

$$\int_{t-(T-t)}^{t}\int_{|y|\le 2\lambda_*(s)R(s)} |\nabla_x G(x-y,T-s)|\lambda_*(s)^{-1}R^{-a}\,dyds \le C\lambda_*(t)^{\nu-1}R(t)^{1-a},$$

and therefore

$$I_2 \le C\lambda_*(t)^{\nu-1}R(t)^{1-a}.$$

We deal now with I_3:

$$\begin{aligned}
I_3 &\le C\int_t^T \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(T-s)^2}\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(T-s)}}|x-y|\,dyds\\
&\le C\int_t^T \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(T-s)^{1/2}}\int_{|z|(T-s)^{1/2}\le 2\lambda_*(s)R(s)} e^{-|\tilde x-z|^2/4}|\tilde x-z|\,dzds\\
&\le C\int_t^T \frac{\lambda_*(s)^{\nu}R(s)^{2-a}}{(T-s)^{3/2}}\,ds\\
&\le C\int_t^T \frac{\lambda_*(s)^{\nu-\beta(2-a)}}{(T-s)^{3/2}}\,ds\\
&= C|\log T|^{\nu-\beta(2-a)}\int_t^T \frac{(T-s)^{\nu-\beta(2-a)-3/2}}{|\log(T-s)|^{2\nu-2\beta(2-a)}}\,ds\\
&\le |\log T|^{\nu-\beta(2-a)}\frac{(T-t)^{\nu-1/2-\beta(2-a)}}{|\log(T-t)|^{2-2\beta(2-a)}}\\
&\le C\lambda_*(t)^{\nu-1}R(t)^{1-a}.
\end{aligned}$$

This proves (8.9). ☐

Proof of (8.10). Let $0<t_1<t_2<T$. We assume that $t_2<2t_1$. In the other case a similar proof gives the result. Let f be given by (15.2). Using the heat kernel we have

$$|\partial_{x_i}\psi(x,t)-\partial_{x_i}\psi(x,T)| \le I_1+I_2+I_3,$$

where

$$\begin{aligned}
I_1 &= \int_0^{t_1-(t_2-t_1)}\int_{\mathbb{R}^2} |\partial_{x_i}G(x-y,t_1-s)-\partial_{x_i}G(x-y,t_2-s)|f(y,s)\,dyds\\
I_2 &= \int_{t_1-(t_2-t_1)}^{t_1}\int_{\mathbb{R}^2} |\partial_{x_i}G(x-y,t_1-s)|f(y,s)\,dyds\\
I_3 &= \int_{t_1-(t_2-t_1)}^{t_2}\int_{\mathbb{R}^2} |\partial_{x_i}G(x-y,t_2-s)|f(y,s)\,dyds.
\end{aligned}$$

For the first integral, we have

$$\begin{aligned}
I_1 &\le (t_2-t_1)\int_0^1\int_0^{t_1-(t_2-t_1)}\int_{\mathbb{R}^2}|\partial_t\partial_{x_i}G(x-y,t_v-s)|f(y,s)\,dydsdv\\
&\le (t_2-t_1)\int_0^1\int_0^{t_1-(t_2-t_1)}\frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_v-s)^{5/2}}\Bigg\{\\
&\qquad\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)}}\left(\frac{|x-y|}{(t_v-s)^{1/2}}+\frac{|x-y|^3}{(t_v-s)^{3/2}}\right)\,dy\Bigg\}dsdv
\end{aligned}$$

where $t_v = vt_2 + (1-v)t_1$. Changing variables

$$\begin{aligned}
&\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) dy \\
&= (t_v-s) \int_{|z|\le 2\lambda_*(s)R(s)(t_v-s)^{-1/2}} e^{-|\tilde{x}_v - z|^2/4} (|\tilde{x}_v - z| + |\tilde{x}_v - z|^3)\, dz
\end{aligned}$$

where $\tilde{x}_v = x(t_v-s)^{-1/2}$. But

$$\begin{aligned}
&\int_{|z|\le 2\lambda_*(s)R(s)(t_v-s)^{-1/2}} e^{-|\tilde{x}_v - z|^2/4} (|\tilde{x}_v - z| + |\tilde{x}_v - z|^3)\, dz \\
&\le \int_{|z|\le 2\lambda_*(s)R(s)(t_v-s)^{-1/2}} e^{-|z|^2/4} (|z| + |z|^3)\, dz \\
&\le C(\lambda_*(s)R(s)(t_v-s)^{-1/2})^{\mu}
\end{aligned}$$

for any $0<\mu<1$

Therefore

$$\begin{aligned}
I_1 &\le C(t_2-t_1) \int_0^{t_1-(t_2-t_1)} \frac{\lambda_*(s)^{\nu-2} R(s)^{-a}}{(t_v-s)^{3/2}} \int_0^{2\lambda_*(s)R(s)(t_v-s)^{-1/2}} e^{-\rho^2/4}(1+\rho^3)\rho \, d\rho ds \\
&\le C(t_2-t_1) \int_0^{t_1-(t_2-t_1)} \frac{\lambda_*(s)^{\mu+\nu-2} R(s)^{\mu-a}}{(t_2-s)^{3/2+\mu/2}} ds.
\end{aligned}$$

Recall that $R(t) = \lambda_*(t)^{-\beta}$. If $\mu+\nu-2-\beta(\mu-a)\le 0$ we have

$$\begin{aligned}
&\int_0^{t_1-(t_2-t_1)} \frac{\lambda_*(s)^{\mu+\nu-2} R(s)^{\mu-a}}{(t_2-s)^{3/2+\mu/2}} ds \\
&\le \lambda_*(t_1)^{\mu+\nu-2} R(t_1)^{\mu-a} \int_0^{t_1-(t_2-t_1)} \frac{1}{(t_2-s)^{3/2+\mu/2}} ds \\
&\le C\lambda_*(t_1)^{\mu+\nu-2} R(t_1)^{\mu-a} (t_2-t_1)^{-1/2-\mu/2}.
\end{aligned}$$

If $b := \mu+\nu-2-\beta(\mu-a) > 0$

$$\int_0^{t_1-(t_2-t_1)} \frac{\lambda_*(s)^b}{(t_2-s)^{3/2+\mu/2}}\, ds = \int_0^{t_1-(T-t_1)} \dots ds + \int_{t_1-(T-t_1)}^{t_1-(t_2-t_1)} \dots ds,$$

and, assuming $b - 1/2 - \mu/2 < 0$ (which we have),

$$\begin{aligned}
\int_0^{t_1-(T-t_1)} \frac{\lambda_*(s)^b}{(t_2-s)^{3/2+\mu/2}}\, ds &\le C \int_0^{t_1-(T-t_1)} \frac{\lambda_*(s)^b}{(T-s)^{3/2+\mu/2}}\, ds \\
&= \int_0^{t_1-(T-t_1)} \frac{|\log T|^b (T-s)^{b-3/2-\mu/2}}{|\log(T-s)|^b}\, ds \\
&\le C \frac{|\log T|^b (T-t_1)^{b-1/2-\mu/2}}{|\log(T-t_1)|^b} \\
&\le C\lambda_*(t_2)^b (t_2-t_1)^{-1/2-\mu/2},
\end{aligned}$$

while

$$\int_{t_1-(T-t_1)}^{t_1-(t_2-t_1)} \frac{\lambda_*(s)^b}{(t_2-s)^{3/2+\mu/2}}\, ds \le C\lambda_*(t_2)^b (t_2-t_1)^{-1/2-\mu/2}.$$

In any case we obtain

$$I_1 \le C\lambda_*(t_2)^{\mu+\nu-2} R(t_2)^{\mu-a} (t_2-t_1)^{1/2-\mu/2}.$$

To estimate I_2 we have

$$\begin{aligned} I_2 &= \int_{t_1-(t_2-t_1)}^{t_1} \int_{|y|\le 2\lambda_*(s)R(s)} |\nabla_x G(x-y,t_1-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\,dyds \\ &\le C\int_{t_1-(t_2-t_1)}^{t_1} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_1-s)^{1/2}} \int_{|z|(t-s)^{1/2}\le 2\lambda_*(s)R(s)} e^{-|\tilde{x}-z|^2}|\tilde{x}-z|\,dzds, \\ &\le C\int_{t_1-(t_2-t_1)}^{t_1} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_1-s)^{1/2}} \int_0^{2\lambda_*(s)R(s)(t_1-s)^{-1/2}} e^{-\rho^2}(1+\rho)\rho\,d\rho ds, \end{aligned}$$

where $\tilde{x} = (t-s)^{-1/2}x$. But then, for $0<\mu<1$:

$$\begin{aligned} &\int_{t_1-(t_2-t_1)}^{t_1} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_1-s)^{1/2}} \int_0^{2\lambda_*(s)R(s)(t_1-s)^{-1/2}} e^{-\rho^2}(1+\rho)\rho\,d\rho ds \\ &\le C\int_{t_1-(t_2-t_1)}^{t_1} \frac{\lambda_*(s)^{\nu-2+\mu}R(s)^{-a+\mu}}{(t_1-s)^{1/2+\mu/2}}\,ds \\ &\le C\lambda_*(t_2)^{\nu-2+\mu}R(t_2)^{-a+\mu}(t_2-t_1)^{1/2-\mu/2}. \end{aligned}$$

We deal now with I_3:

$$\begin{aligned} I_3 &\le C\int_{t_1-(t_2-t_1)}^{t_2} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_2-s)^2} \int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_2-s)}}|x-y|\,dyds \\ &\le C\int_{t_1-(t_2-t_1)}^{t_2} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_2-s)^{1/2}} \int_{|z|\le 2\lambda_*(s)R(s)(t_2-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4}|\tilde{x}-z|\,dzds \\ &\le C\int_{t_1-(t_2-t_1)}^{t_2} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_2-s)^{1/2}} \int_{|z|\le 2\lambda_*(s)R(s)(t_2-s)^{-1/2}} e^{-|z|^2/4}(1+|z|)\,dzds. \end{aligned}$$

For $\mu\in(0,1)$ we have the inequality

$$\int_{|z|\le A} e^{-|z|^2/4}(1+|z|)\,dz \le CA^\mu.$$

Therefore

$$\begin{aligned} &\int_{t_1-(t_2-t_1)}^{t_2} \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t_2-s)^{1/2}} \int_{|z|\le 2\lambda_*(s)R(s)(t_2-s)^{-1/2}} e^{-|z|^2/4}(1+|z|)\,dzds \\ &\le C\int_{t_1-(t_2-t_1)}^{t_2} \frac{\lambda_*(s)^{\mu+\nu-2}R(s)^{\mu-a}}{(t_2-s)^{1/2+\mu/2}}\,ds \\ &\le C\lambda_*(t_2)^{\mu+\nu-2}R(t_2)^{\mu-a}(t_2-t_1)^{1/2-\mu/2}. \end{aligned}$$

This proves (8.10). □

Proof of (8.11). We have

$$\begin{aligned} &|\nabla\psi(x_1,t)-\nabla\psi(x_2,t)| \\ &\le \int_0^t \int_{|y|\le 2\lambda_*(s)R(s)} |\nabla_x G(x_1-y,t-s)-\nabla_x G(x_2-y,t-s)|\lambda_*(s)^{\nu-2}R(s)^{-a} \end{aligned}$$

Let $L=|x_1-x_2|$. We decompose the integral:

$$\int_0^t ...\,ds = \int_0^{t-L^2} ...\,ds + \int_{t-L^2}^t ...\,ds,$$

and in the first one we estimate:

$$\begin{aligned}
&\int_0^{t-L^2}\int_{|y|\le 2\lambda_*(s)R(s)} |\nabla_x G(x_1-y,t-s)-\nabla_x G(x_2-y,t-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\,dyds\\
&\le L\int_0^1\int_0^{t-L^2}\int_{|y|\le 2\lambda_*(s)R(s)} |D_x^2 G(x_v-y,t-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\,dydsdv
\end{aligned}$$

where $x_v = vx_2+(1-v)x_1$. We note that

$$\partial_{x_ix_j}G(x,t) = -\frac{\delta_{ij}}{8\pi t^2}e^{-\frac{|x|^2}{t}} + \frac{x_ix_j}{16\pi t^3}e^{-\frac{|x|^2}{t}},$$

and hence

$$\begin{aligned}
&\int_0^{t-L^2}\int_{|y|\le 2\lambda_*(s)R(s)} |\nabla_x G(x_1-y,t-s)-\nabla_x G(x_2-y,t-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\,dyds\\
&\le L\int_0^1\int_0^{t-L^2}\frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^2}\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x_v-y|^2}{4(t-s)}}\left(1+\frac{|x_v-y|^2}{t-s}\right)dydsdv
\end{aligned}$$

We change variables and estimate

$$\begin{aligned}
&\int_{|y|\le 2\lambda_*(s)R(s)} e^{-\frac{|x_v-y|^2}{4(t-s)}}\left(1+\frac{|x_v-y|^2}{t-s}\right)dy\\
&\qquad = (t-s)\int_{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x_v-z|^2/4}\left(1+|\tilde x_v-z|^2\right)dz
\end{aligned}$$

where $\tilde x_v = x_v(t-s)^{-1/2}$. Then we note that

$$\int_{|z|\le A} e^{-|\tilde x_v-z|^2/4}\left(1+|\tilde x_v-z|^2\right)dz \le \int_{|z|\le A} e^{-|z|^2/4}\left(1+|z|^2\right)dz \le CA^{\mu}$$

for any $\mu\in(0,1)$. Hence

$$\begin{aligned}
&\int_0^{t-L^2}\int_{|y|\le 2\lambda_*(s)R(s)} |\nabla_x G(x_1-y,t-s)-\nabla_x G(x_2-y,t-s)|\lambda_*(s)^{\nu-2}R(s)^{-a}\,dyds\\
&\le CL\int_0^{t-L^2}\frac{\lambda_*(s)^{\nu-2+\mu}R(s)^{-a+\mu}}{(t-s)^{1+\mu/2}}\,ds\\
&\le CL^{1-\mu}\,\lambda_*(t)^{\nu-2+\mu}R(t)^{\mu-a}\\
&= CL^{\gamma}\,\lambda_*(t)^{\nu-1-\gamma}R(t)^{1-a-\gamma}
\end{aligned}$$

where $\gamma = 1-\mu$.

Next deal with the integral $\int_{t-L^2}^t\int_{\mathbb{R}^2}\dots dyds$. We split

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla_x G(x_1-y,t-s)-\nabla_x G(x_2-y,t-s)|\chi_{\{|y|\le 2\lambda_*(s)R(s)\}}\lambda_*(s)^{\nu-2}R(s)^{-a}\,dy\\
&= \int_{A_1}\dots dy + \int_{A_2}\dots dy
\end{aligned}$$

where

$$A_1 = \{x\in\mathbb{R}^2 \,:\, |x-\bar x|\le 3L\}, \qquad A_2 = \{x\in\mathbb{R}^2 \,:\, |x-\bar x|\ge 3L\},$$

and $\bar x = \frac12(x_1+x_2)$.

We estimate

$$\begin{aligned}
&\int_{t-L^2}^t \int_{A_2} |\nabla_x G(x_1-y,t-s) - \nabla_x G(x_2-y,t-s)| \chi_{\{|y|\le 2\lambda_*(s)R(s)\}} \lambda_*(s)^{\nu-2} R(s)^{-a}\, dyds \\
&\le L \int_0^1 \int_{t-L^2}^t \int_{A_2} |D_x^2 G(x_v - y, t-s)| \chi_{\{|y|\le 2\lambda_*(s)R(s)\}} \lambda_*(s)^{\nu-2} R(s)^{-a}\, dydsdv \\
&\le L \int_0^1 \int_{t-L^2}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^2} \int_{A_2} e^{-\frac{|x_v-y|^2}{4(t-s)}} \left(1 + \frac{|x_v-y|^2}{t-s}\right) \chi_{\{|y|\le 2\lambda_*(s)R(s)\}}\, dydsdv \\
&\le L \int_0^1 \int_{t-L^2}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)} \int_{|z-\tilde x|\ge 3L(t-s)^{-1/2}} e^{-\frac{|\tilde x_v - z|^2}{4}} \left(1 + |\tilde x_v - z|^2\right) dzdsdv \\
&\le L \int_0^1 \int_{t-L^2}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)} \frac{(t-s)^{1/2}}{L^{1/2}}\, dsdv \\
&\le L \lambda_*(t)^{\nu-2} R(t)^{-a}
\end{aligned}$$

and since $L \le 2\lambda_*(t)R(t)$

$$\begin{aligned}
&\int_{t-L^2}^t \int_{A_2} |\nabla_x G(x_1-y,t-s) - \nabla_x G(x_2-y,t-s)| \frac{\lambda_*(s)^\sigma}{|y|+\lambda_*(s)} \\
&\le C L^\gamma \lambda_*(t)^{\nu-1-\gamma} R(t)^{1-a-\gamma}.
\end{aligned}$$

Now we consider the integral over A_1. By symmetry it is only necessary to estimate

$$\begin{aligned}
&\int_{t-L^2}^t \int_{|y-x_1|\le 4L} |\nabla_x G(x_1-y,t-s)| \chi_{\{|y|\le 2\lambda_*(s)R(s)\}} \lambda_*(s)^{\nu-2} R(s)^{-a}\, dyds \\
&\le \int_{t-L^2}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^2} \int_{|y-x_1|\le 4L} e^{-\frac{|x_1-y|^2}{4(t-s)}} |x_1 - y| \chi_{\{|y|\le 2\lambda_*(s)R(s)\}}\, dyds \\
&\le \int_{t-L^2}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}} \int_{|z-\tilde x_1|\le 4L(t-s)^{-1/2}} e^{-\frac{|\tilde x_1 - z|^2}{4}} |\tilde x_1 - y| \\
&\quad \cdot \chi_{\{|z|\le 2\lambda_*(s)R(s)(t-s)^{-1/2}\}}\, dyds \\
&\le C \int_{t-L^2}^t \frac{\lambda_*(s)^{\nu-2}R(s)^{-a}}{(t-s)^{1/2}}\, ds \\
&\le C L \lambda_*(t)^{\nu-2} R(t)^{-a},
\end{aligned}$$

and since $L \le 2\lambda_*(t)R(t)$

$$\begin{aligned}
&\int_{t-L^2}^t \int_{|y-x_1|\le 4L} |\nabla_x G(x_1-y,t-s)| \chi_{\{|y|\le 2\lambda_*(s)R(s)\}} \lambda_*(s)^{\nu-2} R(s)^{-a}\, dyds \\
&\le C L^\gamma \lambda_*(t)^{\nu-1-\gamma} R(t)^{1-a-\gamma}.
\end{aligned}$$

□

15.2. **Proof of Lemma 8.2.** .

Proof of (8.12). We have, using the heat kernel,

$$\begin{aligned}
\psi(x,t) &= C \int_0^t \frac{\lambda_*(s)^m}{t-s} \int_{|y|\ge \lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{1}{|y|^2}\, dyds \\
&= C \int_0^t \frac{\lambda_*(s)^m}{t-s} \int_{|z|\ge \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x - z|^2/4} \frac{1}{|z|^2}\, dzds
\end{aligned}$$

where $\tilde x = x(t-s)^{-1/2}$.

We note that for $v > 0$

$$\int_{|z|\geq v} e^{-|\tilde{x}-z|^2/4}\frac{1}{|z|^2}\,dz \leq C\begin{cases}|\log v| & \text{if } v\leq \frac{1}{2}\\ 1 & \text{if } v\geq\frac{1}{2}.\end{cases} \tag{15.6}$$

Indeed, if $v \geq \frac{1}{2}$

$$\begin{aligned}\int_{|z|\geq v} e^{-|\tilde{x}-z|^2/4}\frac{1}{|z|^2}\,dz &\leq 4\int_{|z|\geq\frac{1}{2}} e^{-|\tilde{x}-z|^2/4}\,dz\\ &\leq 4\int_{\mathbb{R}^2} e^{-|z|^2/4}\,dz \leq C.\end{aligned}$$

If $v \leq \frac{1}{2}$ then

$$\begin{aligned}\int_{|z|\geq v} e^{-|\tilde{x}-z|^2/4}\frac{1}{|z|^2}\,dz &= \int_{v\leq|z|\leq 1}\frac{1}{|z|^2}\,dz + \int_{|z|\geq 1} e^{-|\tilde{x}-z|^2/4}\,dz\\ &\leq C|\log v|.\end{aligned}$$

Using (15.6)

$$\begin{aligned}&\int_0^{t-\lambda_*(t)^2R(t)^2}\frac{\lambda_*(s)^m}{t-s}\int_{|z|\geq\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4}\frac{1}{|z|^2}\,dzds\\ &\leq C\int_0^{t-\lambda_*(t)^2R(t)^2}\frac{\lambda_*(s)^m}{t-s}|\log(t-s)|ds\\ &\leq CT^m|\log T|^{2-m}.\end{aligned}$$

Next consider the integral $\int_{t-\lambda_*(t)^2R(t)^2}^t\ldots ds$. In this range $s \geq t-\lambda_*(t)^2R(t)^2 \geq t-\lambda_*(s)^2R(s)^2$ and so $\lambda_*(s)R(s)(t-s)^{-1/2}\geq 1$. Therefore, using (15.6), we get

$$\begin{aligned}&\int_{t-\lambda_*(t)^2R(t)^2}^t\frac{\lambda_*(s)^m}{t-s}\int_{|z|\geq\lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4}\frac{1}{|z|^2}\,dzds\\ &\leq \int_{t-\lambda_*(t)^2R(t)^2}^t\frac{\lambda_*(s)^m}{t-s}\frac{t-s}{\lambda_*(s)^2R(s)^2}\,ds\\ &\leq C\lambda_*(t)^m \leq CT^m|\log T|^{2-m},\end{aligned}$$

and we deduce the desired estimate (8.12). □

Proof of (8.13). Assume $t\in[\frac{3}{4}T,T]$. Using the heat kernel we have

$$|\psi(x,t)-\psi(x,T)| \leq I_1+I_2+I_3,$$

where

$$\begin{aligned}I_1 &= \int_0^{t-(T-t)}\int_{\mathbb{R}^2}|G(x-y,t-s)-G(x-y,T-s)||f(y,s)|\,dyds,\\ I_2 &= \int_{t-(T-t)}^t\int_{\mathbb{R}^2}G(x-y,t-s)|f(y,s)|\,dyds,\\ I_3 &= \int_{t-(T-t)}^T\int_{\mathbb{R}^2}G(x-y,T-s)|f(y,s)|\,dyds.\end{aligned}$$

We estimate the first integral

$$I_1 \leq (T-t)\int_0^1\int_0^{t-(T-t)}\int_{|y|\geq\lambda_*(s)R(s)}|\partial_t G(x-y,t_v-s)|\frac{\lambda_*(s)^m}{|y|^2}\,dydsdv,$$

where $t_v = vT + (1-v)t$. We have

$$\begin{aligned}
&\int_{|y|\geq \lambda_*(s)R(s)} |\partial_t G(x-y, t_v - s)| \frac{\lambda_*(s)^m}{|y|^2}\, dy \\
&\leq C \frac{\lambda_*(s)^m}{(t_v-s)^2} \int_{|y|\geq \lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(1 + \frac{|x-y|^2}{t_v - s}\right) \frac{1}{|y|^2}\, dy \\
&\leq C \frac{\lambda_*(s)^m}{(t_v-s)^2} \int_{|z|\geq \lambda_*(s)R(s)(t_v-s)^{-1/2}} e^{-\frac{|\tilde{x}-z|^2}{4}} \left(1 + |\tilde{x}-z|^2\right) \frac{1}{|z|^2}\, dz.
\end{aligned}$$

Note that $t_v \leq T$ so that

$$\frac{\lambda_*(s)R(s)}{(t_v-s)^{1/2}} \geq \frac{\lambda_*(s)R(s)}{(T-s)^{1/2}} \geq \frac{\lambda_*(t)R(t)}{(T-t)^{1/2}}.$$

Similarly to (15.6) we have

$$\int_{|z|\geq \lambda_*(s)R(s)(t_v-s)^{-1/2}} e^{-\frac{|\tilde{x}-z|^2}{4}} \left(1 + |\tilde{x}-z|^2\right) \frac{1}{|z|^2}\, dz \leq C|\log(T-t)|.$$

Therefore

$$\begin{aligned}
&\int_0^{t-(T-t)} \int_{|y|\geq \lambda_*(s)R(s)} |\partial_t G(x-y, t_v-s)| \frac{\lambda_*(s)^m}{|y|^2}\, dyds \\
&\leq C|\log(T-t)| \int_0^{t-(T-t)} \frac{\lambda_*(s)^m}{(t_v-s)^2}\, ds \\
&= C|\log(T-t)||\log T|^m \int_0^{t-(T-t)} \frac{(T-s)^{m-2}}{|\log(T-s)|^{2m}}\, ds \\
&\leq C|\log T|^m \frac{(T-t)^{m-1}}{|\log(T-t)|^{2m-1}},
\end{aligned}$$

and this shows that

$$I_1 \leq C|\log T|^m \frac{(T-t)^m}{|\log(T-t)|^{2m-1}}.$$

Next we estimate I_2:

$$\begin{aligned}
I_2 &\leq C \int_{t-(T-t)}^{t} \frac{\lambda_*(s)^m}{t-s} \int_{|y|\geq \lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{1}{|y|^2}\, dyds \\
&= C \int_{t-(T-t)}^{t} \frac{\lambda_*(s)^m}{t-s} \int_{|z|\geq \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde{x}-z|^2/4} \frac{1}{|z|^2}\, dzds.
\end{aligned}$$

We note that for $s \in [t-(T-t), t]$ the inequality

$$\frac{\lambda_*(s)R(s)}{(t-s)^{1/2}} \leq 1$$

is equivalent to $s \leq s^*$ for some $s^* \in (t-(T-t), t)$. For $s \leq s^*$ we use (15.6) and obtain

$$\begin{aligned}
I_2 &\leq C \int_{t-(T-t)}^{s^*} \frac{\lambda_*(s)^m}{t-s} |\log(t-s)|\, ds \leq C\lambda_*(t)^m |\log \lambda_*(t)|^2 \\
&\leq C|\log T|^m \frac{(T-t)^m}{|\log(T-t)|^{2m-2}}
\end{aligned}$$

Finally, for I_3, using (15.6), we get

$$
\begin{aligned}
I_3 &\le C\int_t^T \frac{\lambda_*(s)^m}{T-s}\int_{|y|\ge \lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(T-s)}}\frac{1}{|y|^2}\,dyds\\
&\le C\int_t^T \frac{\lambda_*(s)^m}{T-s}\int_{|z|\ge \lambda_*(s)R(s)(T-s)^{-1/2}} e^{-|\tilde x-z|^2/4}\frac{1}{|z|^2}\,dyds\\
&\le C\int_t^T \frac{\lambda_*(s)^m}{T-s}|\log(T-s)|\,ds\\
&\le C|\log T|^m\frac{(T-t)^m}{|\log(T-t)|^{2m-1}}.
\end{aligned}
$$

This finishes the proof of (8.13) when $t\in[\frac34 T,T]$. For $t\in[0,\frac34 T]$ estimate (8.13) follows from (8.12). □

Proof of (8.14). Using the heat kernel we have

$$
\begin{aligned}
|\nabla\psi(x,t)| &\le C\int_0^t \frac{\lambda_*(s)^m}{(t-s)^2}\int_{|y|\ge \lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t-s)}}\frac{|x-y|}{|y|^2}\,dyds\\
&= C\int_0^t \frac{\lambda_*(s)^m}{(t-s)^{3/2}}\int_{|z|\ge \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|\tilde x-z|^2/4}\frac{|\tilde x-z|}{|z|^2}\,dzds\\
&\le C\int_0^t \frac{\lambda_*(s)^m}{(t-s)^{3/2}}\int_{|z|\ge \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}\frac{1+|z|}{|z|^2}\,dzds
\end{aligned}
$$

where $\tilde x=(t-s)^{-1/2}x$. We find that

$$
\begin{aligned}
&\int_0^{t-(T-t)} \frac{\lambda_*(s)^m}{(t-s)^{3/2}}\int_{|z|\ge \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}\frac{1+|z|}{|z|^2}\,dzds\\
&\le C\int_0^{t-(T-t)} \frac{\lambda_*(s)^m}{(t-s)^{3/2}}|\log(t-s)|ds\\
&\le CT^{m-\frac12}|\log T|^{2-2m}.
\end{aligned}
$$

Let us estimate

$$
\int_{t-(T-t)}^t \frac{\lambda_*(s)^m}{(t-s)^{3/2}}\int_{|z|\ge \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}\frac{1+|z|}{|z|^2}\,dzds.
$$

We note that the for $s\in[t-(T-t),t]$ the inequality $\frac{\lambda_*(s)R(s)}{(t-s)^{1/2}}\le 1$ is equivalent to $s\le s^*$ for some $s^*\in(t-(T-t),t)$.

Then for $s\le s^*$ we use the estimate

$$
\int_{|z|\ge \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}\frac{1+|z|}{|z|^2}\,dz\le C|\log(t-s)|,
$$

and hence

$$
\begin{aligned}
&\int_{t-(T-t)}^{s^*} \frac{\lambda_*(s)^m}{(t-s)^{3/2}}\int_{|z|\ge \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4}\frac{1+|z|}{|z|^2}\,dzds\\
&\le C\int_{t-(T-t)}^{s^*} \frac{\lambda_*(s)^m}{(t-s)^{3/2}}|\log(t-s)|\,ds\\
&\le C\frac{\lambda_*(t)^{m-1}|\log(T-t)|}{R(t)}\le C\frac{T^{m-1}|\log T|^{2-m}}{R(T)}.
\end{aligned}
$$

For the remaining integral

$$
\begin{aligned}
&\int_{s^*}^t \frac{\lambda_*(s)^m}{(t-s)^{3/2}} \int_{|z|\geq \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2/4} \frac{1+|z|}{|z|^2}\, dzds \\
&\leq C \int_{s^*}^t \frac{\lambda_*(s)^m}{(t-s)^{3/2}} \frac{(t-s)^{3/2}}{\lambda_*(s)^3 R(s)^3}\, ds \\
&\leq C\frac{\lambda_*(t)^m}{R(t)} \leq C\frac{T^{m-1}|\log T|^{1-m}}{R(T)}.
\end{aligned}
$$

Gathering the previous estimates we deduce (8.14). □

Proof of (8.15). Using the heat kernel we have

$$
\begin{aligned}
&\partial_{x_i}\psi(x,t) - \partial_{x_i}\psi(x,T) \\
&= \int_0^t \int_{\mathbb{R}^2} \left(\partial_{x_i} G(x-y,t-s) - \partial_{x_i} G(x-y,T-s)\right) f(y,s)\, dyds \\
&\quad - \int_0^T \int_{\mathbb{R}^2} \partial_{x_i} G(x-y,T-s) f(y,s)\, dyds
\end{aligned}
$$

and so

$$
|\partial_{x_i}\psi(x,t) - \partial_{x_i}\psi(x,T)| \leq I_1 + I_2 + I_3,
$$

where

$$
\begin{aligned}
I_1 &= \int_0^{t-(T-t)} \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y,t-s) - \partial_{x_i} G(x-y,T-s)|\, |f(y,s)|\, dyds, \\
I_2 &= \int_{t-(T-t)}^t \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y,t-s)|\, |f(y,s)|\, dyds, \\
I_3 &= \int_{t-(T-t)}^T \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y,T-s)|\, |f(y,s)|\, dyds.
\end{aligned}
$$

Let us estimate I_1:

$$
\begin{aligned}
I_1 &\leq (T-t) \int_0^1 \int_0^{t-(T-t)} \lambda_*(s)^m \int_{|y|\geq \lambda_*(s)R(s)} |\partial_t \partial_{x_i} G(x-y,t_v-s)| \frac{1}{|y|^2}\, dydsdv \\
&\leq C(T-t) \int_0^1 \int_0^{t-(T-t)} \frac{\lambda_*(s)^m}{(t_v-s)^{5/2}} \\
&\qquad \cdot \int_{|y|\geq \lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) \frac{1}{|y|^2}\, dy\, dsdv
\end{aligned}
$$

where $t_v = vT + (1-v)t$. Changing variables

$$
\begin{aligned}
&\int_{|y|\geq \lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) \frac{1}{|y|^2}\, dy \\
&= \int_{|z|\geq \lambda_*(s)R(s)(t_v-s)^{-1/2}} e^{-|\tilde{x}_v - z|^2/4} (|\tilde{x}_v - z| + |\tilde{x}_v - z|^3) \frac{1}{|z|^2}\, dz \\
&\leq C|\log(t_v-s)|.
\end{aligned}
$$

where $\tilde{x}_v = x(t_v - s)^{-1/2}$. Then

$$\begin{aligned}
&\int_0^{t-(T-t)} \frac{\lambda_*(s)^m}{(t_v-s)^{5/2}} \Big\{ \int_{|y|\geq \lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) \frac{1}{|y|^2}\, dy \Big\}\, ds \\
&\leq C \int_0^{t-(T-t)} \frac{\lambda_*(s)^m}{(T-s)^{5/2}} |\log(t_v - s)|\, ds \\
&\leq C \frac{\lambda_*(t)^m |\log(T-t)|}{(T-t)^{3/2}}
\end{aligned}$$

Thus we have found that

$$\begin{aligned}
I_1 &\leq C \frac{\lambda_*(t)^m |\log(T-t)|}{(T-t)^{1/2}} = C|\log T|^m (T-t)^{m-\frac{1}{2}} |\log(T-t)|^{1-2m} \\
&\leq C \frac{\lambda_*(t)^{m-1} |\log(T-t)|}{R(t)}
\end{aligned}$$

since $R = \lambda_*^{-\beta}$ and $\beta \in (0, \frac{1}{2})$.

We estimate I_2

$$\begin{aligned}
I_2 &\leq C \int_{t-(T-t)}^t \frac{\lambda_*(s)^m}{(t-s)^2} \int_{|y|\geq \lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{|x-y|}{|y|^2}\, dyds \\
&= C \int_{t-(T-t)}^t \frac{\lambda_*(s)^m}{(t-s)^{3/2}} \int_{|z|\geq \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-\frac{|\tilde{x}-z|^2}{4(t-s)}} \frac{|\tilde{x}-z|}{|z|^2}\, dzds \\
&\leq C \int_{t-(T-t)}^t \frac{\lambda_*(s)^m}{(t-s)^{3/2}} \int_{|z|\geq \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2} \frac{1+|z|}{|z|^2}\, dzds,
\end{aligned}$$

where $\tilde{x} = (t-s)^{-1/2} x$. We note that the for $s \in [t-(T-t), t]$ the inequality $\frac{\lambda_*(s)R(s)}{(t-s)^{1/2}} \leq 1$ is equivalent to $s \leq s^*$ for some $s^* \in (t-(T-t), t)$. Then for $s \leq s^*$ we use the estimate

$$\int_{|z|\geq \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2} \frac{1+|z|}{|z|^2}\, dz \leq C|\log(t-s)|,$$

and hence

$$\begin{aligned}
&\int_{t-(T-t)}^t \frac{\lambda_*(s)^m}{(t-s)^{3/2}} \int_{|z|\geq \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2} \frac{1+|z|}{|z|^2}\, dzds \\
&\leq C \int_{t-(T-t)}^{s^*} \frac{\lambda_*(s)^m}{(t-s)^{3/2}} |\log(t-s)|\, ds \\
&\leq C \frac{\lambda_*(t)^{m-1} |\log(T-t)|}{R(t)}.
\end{aligned}$$

The integral on $[s^*, t]$ is estimated by

$$\begin{aligned}
&\int_{s^*}^t \frac{\lambda_*(s)^m}{(t-s)^{3/2}} \int_{|z|\geq \lambda_*(s)R(s)(t-s)^{-1/2}} e^{-|z|^2} \frac{1+|z|}{|z|^2}\, dzds \\
&\leq C \frac{\lambda_*(t)^{m-2}}{R(t)^2} \int_{s^*}^t \frac{1}{(t-s)^{1/2}} ds \\
&\leq C \frac{\lambda_*(t)^{m-1}}{R(t)}
\end{aligned}$$

In summary we get

$$I_2 \leq C \frac{\lambda_*(t)^{m-1} |\log(T-t)|}{R(t)}.$$

Finally, we estimate I_3:

$$
\begin{aligned}
I_3 &\le C\int_t^T \frac{\lambda_*(s)^m}{(T-s)^2}\int_{|y|\ge\lambda_*(s)R(s)} e^{-\frac{|x-y|^2}{4(T-s)}}\frac{|x-y|}{|y|^2}\,dyds\\
&\le C\int_t^T \frac{\lambda_*(s)^m}{(T-s)^{3/2}}\int_{|z|\ge\lambda_*(s)R(s)(T-s)^{-1/2}} e^{-|\tilde x-z|^2/4}\frac{|\tilde x-z|}{|z|^2}\,dzds\\
&\le C\int_t^T \frac{\lambda_*(s)^m}{(T-s)^{3/2}}\frac{(T-s)^{1/2}}{\lambda_*(s)R(s)}\,ds\\
&\le C\frac{\lambda_*(t)^{m-1}}{R(t)}.
\end{aligned}
$$

Combining the estimates on I_1, I_2 and I_3 we deduce (8.15). □

15.3. **Proof of Lemma 8.3.**

Proof of (8.16). Let us show (8.16). We have, using the heat kernel,

$$
\begin{aligned}
|\psi(x,t)| &\le C\int_0^t \frac{1}{t-s}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t-s)}}\,dyds\\
&\le Ct.
\end{aligned}
$$

□

Proof of (8.17). Using the heat kernel we have

$$
|\psi(x,t)-\psi(x,T)| \le I_1+I_2+I_3,
$$

where

$$
\begin{aligned}
I_1 &= \int_0^{t-(T-t)}\int_{\mathbb{R}^2} |G(x-y,t-s)-G(x-y,T-s)|\,dyds\\
I_2 &= \int_{t-(T-t)}^t\int_{\mathbb{R}^2} |G(x-y,t-s)-G(x-y,T-s)|\,dyds\\
I_3 &= \int_t^T\int_{\mathbb{R}^2} |G(x-y,t-s)-G(x-y,T-s)|\,dyds.
\end{aligned}
$$

We estimate the first integral

$$
I_1 \le (T-t)\int_0^1\int_0^{t-(T-t)}\int_{\mathbb{R}^2} |\partial_t G(x-y,t_v-s)|\,dydsdv,
$$

where $t_v = vT+(1-v)(T-t)$. We have

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |\partial_t G(x-y,t_v-s)|\,dy\\
&\le \frac{C}{(t_v-s)^2}\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t_v-s)}}\left(1+\frac{|x-y|^2}{t_v-s}\right)dy\\
&\le C\frac{1}{(t_v-s)}\int_{\mathbb{R}^2} e^{-|\tilde x-z|^2}\left(1+|\tilde x-z|^2\right)dz.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_0^{t-(T-t)}\int_{\mathbb{R}^2} |\partial_t G(x-y,t_v-s)|\,dyds\\
&\le C\int_0^{t-(T-t)}\frac{1}{(T-s)}\,ds\\
&\le C|\log(T-t)|
\end{aligned}
$$

and hence

$$I_1 \le C(T-t)|\log(T-t)|.$$

Next we estimate $I_2 \le I_{2,1} + I_{2,2}$, where

$$I_{2,1} = \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} |G(x-y,t-s)|\, dyds$$
$$I_{2,2} = \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} |G(x-y,T-s)|\, dyds.$$

Let us compute the first one

$$\begin{aligned} I_{2,1} &\le C \int_{t-(T-t)}^{t} \frac{1}{t-s} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t-s)}}\, dyds \\ &= C \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} e^{-|\tilde{x}-z|^2/4}\, dzds \\ &\le C \int_{t-(T-t)}^{t} ds \\ &\le C(T-t) \end{aligned}$$

A similar computation yields the same bound for the second integral and we obtain

$$I_2 \le C(T-t).$$

We estimate

$$\begin{aligned} I_3 &\le C \int_t^T \frac{1}{T-s} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(T-s)}}\, dyds \\ &= C \int_t^T \int_{\mathbb{R}^2} e^{-|\tilde{x}-z|^2}\, dyds \\ &\le C(T-t). \end{aligned}$$

This finishes the proof of (8.17). □

Proof of (8.20). Let $0 < t_1 < t_2 < T$. Using the heat kernel we have

$$|\partial_{x_i}\psi(x,t_2) - \partial_{x_i}\psi(x,t_1)| \le I_1 + I_2 + I_3,$$

where

$$I_1 = \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y,t_1-s) - \partial_{x_i} G(x-y,t_2-s)|\, dyds$$
$$I_2 = \int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y,t_1-s) - \partial_{x_i} G(x-y,t_2-s)|\, dyds$$
$$I_3 = \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y,t_2-s)|\, dyds.$$

For the first integral, we have

$$\begin{aligned} I_1 &\le (t_2-t_1) \int_0^1 \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^2} |\partial_t \partial_{x_i} G(x-y,t_v-s)|\, dydsdv \\ &\le (t_2-t_1) \int_0^1 \int_0^{t_1-(t_2-t_1)} \frac{1}{(t_v-s)^{5/2}} \Big\{ \\ &\qquad \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) dy \Big\} dsdv \end{aligned}$$

where $t_v = vt_2 + (1-v)(t_2-t_1)$.

Changing variables

$$\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) dy$$
$$= (t_v-s) \int_{\mathbb{R}^2} e^{-|\tilde{x}_v - z|^2/4} (|\tilde{x}_v - z| + |\tilde{x}_v - z|^3)\, dz$$

where $\tilde{x}_v = x(t_v-s)^{-1/2}$. We then need to estimate

$$\int_0^{t_1-(t_2-t_1)} \frac{1}{(t_v-s)^{3/2}} \int_{\mathbb{R}^2} e^{-|\tilde{x}_v-z|^2/4} (|\tilde{x}_v - z| + |\tilde{x}_v - z|^3)\, dzds$$
$$\leq C \int_0^{t_1-(t_2-t_1)} \frac{1}{(t_v-s)^{3/2}} ds$$
$$\leq C(t_2-t_1)^{-1/2}.$$

This yields

$$I_1 \leq C(t_2-t_1)^{1/2}.$$

To estimate I_2 it suffices to bound separately the terms:

$$\int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y, t_1-s)|\, dyds$$
$$\int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y, t_2-s)|\, dyds.$$

We have

$$\int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y, t_1-s)|\, dyds$$
$$\leq C \int_{t_1-(t_2-t_1)}^{t_1} \frac{1}{(t_1-s)^2} \int_{\mathbb{R}^2} e^{-\frac{|x-y|}{4(t_1-s)}} |x-y|\, dyds$$
$$\leq C \int_{t_1-(t_2-t_1)}^{t_1} \frac{1}{(t_1-s)^{1/2}}\, ds$$
$$\leq C(t_2-t_1)^{1/2}.$$

In a similar way we find that

$$\int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y, t_2-s)| \leq C(t_2-t_1)^{1/2},$$

and we obtain

$$I_2 \leq C(t_2-t_1)^{1/2}.$$

Finally

$$I_3 \leq C \int_{t_1}^{t_2} \frac{1}{(t_2-s)^2} \int_{\mathbb{R}^2} e^{-\frac{|x-y|}{4(t_2-s)}} |x-y|\, dyds$$
$$\leq C \int_{t_1}^{t_2} \frac{1}{(t_2-s)^{1/2}} \int_{\mathbb{R}^2} e^{-|\tilde{x}-z|} |\tilde{x}-z|\, dyds$$
$$\leq C(t_2-t_1)^{1/2}.$$

This proves (8.20).

□

Proof of (8.19). Using the heat kernel we have

$$|\partial_{x_i}\psi(x,t) - \partial_{x_i}\psi(x,T)| \le I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_0^{t-(T-t)} \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y,t-s) - \partial_{x_i} G(x-y,T-s)|\,|f(y,s)|\,dyds \\ I_2 &= \int_{t-(T-t)}^t \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y,t-s) - \partial_{x_i} G(x-y,T-s)|\,|f(y,s)|\,dyds \\ I_3 &= \int_t^T \int_{\mathbb{R}^2} |\partial_{x_i} G(x-y,T-s)|\,|f(y,s)|\,dyds. \end{aligned}$$

Let us estimate I_1:

$$\begin{aligned} I_1 &\le (T-t) \int_0^1 \int_0^{t-(T-t)} \int_{\mathbb{R}^2} |\partial_t \partial_{x_i} G(x-y,t_v-s)|\,dydsdv \\ &\le C(T-t) \int_0^1 \int_0^{t-(T-t)} \frac{1}{(t_v-s)^{5/2}} \Big\{ \\ &\qquad \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) dy \Big\} dsdv \end{aligned}$$

where $t_v = vT + (1-v)(T-t)$. Changing variables

$$\begin{aligned} &\int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t_v-s)}} \left(\frac{|x-y|}{(t_v-s)^{1/2}} + \frac{|x-y|^3}{(t_v-s)^{3/2}} \right) \frac{1}{|y|+\lambda_*(s)}\,dy \\ &= (t_v-s) \int_{\mathbb{R}^2} e^{-|\tilde x_v - z|^2/4} (|\tilde x_v - z| + |\tilde x_v - z|^3)\,dz \end{aligned}$$

where $\tilde x_v = x(t_v-s)^{-1/2}$. We then need to estimate

$$\begin{aligned} &\int_0^{t-(T-t)} \frac{1}{(t_v-s)^{3/2}} \int_{\mathbb{R}^2} e^{-|\tilde x_v - z|^2/4} (|\tilde x_v - z| + |\tilde x_v - z|^3)\,dzds \\ &\le C \int_0^{t-(T-t)} \frac{1}{(T-s)^{3/2}} ds \\ &\le C(T-t)^{-1/2} \end{aligned}$$

Therefore

$$I_1 \le C(T-t)^{1/2}$$

To estimate I_2 it is sufficient to bound the terms

$$\begin{aligned} I_{2,1} &= \int_{t-(T-t)}^t \int_{\mathbb{R}^2} |\nabla_x G(x-y,t-s)|\,dyds, \\ I_{2,2} &= \int_{t-(T-t)}^t \int_{\mathbb{R}^2} |\nabla_x G(x-y,T-s)|\,dyds. \end{aligned}$$

Let us start with:

$$\begin{aligned} I_{2,1} &\le C \int_{t-(T-t)}^t \frac{1}{(t-s)^2} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(t-s)}} |x-y|\,dyds \\ &= C \int_{t-(T-t)}^t \frac{1}{(t-s)^{1/2}} \int_{\mathbb{R}^2} e^{-\frac{|\tilde x - z|^2}{4(t-s)}} |\tilde x - z|\,dzds \\ &\le C(T-t)^{1/2}. \end{aligned}$$

We obtain similarly $I_{2,2} \le C(T-t)^{1/2}$ and therefore

$$I_2 \le C(T-t)^{1/2}.$$

Finally, let us handle I_3:

$$\begin{aligned} I_3 &\le C \int_t^T \frac{1}{(T-s)^2} \int_{\mathbb{R}^2} e^{-\frac{|x-y|^2}{4(T-s)}} |x-y| \, dyds \\ &\le C \int_t^T \frac{1}{(T-s)^{1/2}} \int_{\mathbb{R}^2} e^{-|\tilde{x}-z|^2/4} |\tilde{x}-z| \, dzds \\ &\le C(T-t)^{1/2}. \end{aligned}$$

This concludes the proof of (8.19). □

16. Derivative of the exterior problem

Let $f(y,t)$ be a function satisfying

$$|f(y,t)| \le \lambda_*(t)^\nu R(t)^{-a} \chi_{B_R(t)},$$

and let $\psi[\lambda,\xi]$ be the solution of

$$\begin{cases} \psi_t = \Delta_x \psi + \dfrac{1}{\lambda(t)^2} f(\dfrac{x-\xi}{\lambda}, t) \quad \text{in } \mathbb{R}^2 \times (0,T) \\ \psi(x,0) = 0 \quad x \in \mathbb{R}^2, \end{cases}$$

given by Duhamel's formula.

Let

$$\tilde{\psi}[\lambda,\xi](y,t) = \psi[\lambda,\xi](\xi(t)+\lambda(t)y,t).$$

We consider the directional derivative with respect to λ of $\tilde{\psi}$ in the direction of λ_1, defined by

$$D_\lambda \tilde{\psi}[\lambda,\xi][\lambda_1] = \lim_{s\to 0} \frac{1}{s} \left(\tilde{\psi}[\lambda+s\lambda_1,\xi] - \tilde{\psi}[\lambda,\xi] \right)$$

and also the directional derivative with respect to ξ of $\tilde{\psi}$ in the direction of ξ_1, defined by

$$D_\xi \tilde{\psi}[\lambda,\xi][\xi_1] = \lim_{s\to 0} \frac{1}{s} \left(\tilde{\psi}[\lambda,\xi+s\xi_1] - \tilde{\psi}[\lambda,\xi] \right).$$

In the rest of the section we always assume the following conditions:

$$\begin{gathered} |\dot{\xi}(t)| \le C, \quad \text{in } (0,T) \\ |\dot{\lambda}(t)| \le C, \quad \text{in } (0,T) \\ C_1\lambda_*(t) \le \lambda(t) \le C_2\lambda_*(t), \quad \text{in } (0,T) \\ R(t) = \lambda_*(t)^{-\beta}, \quad \beta < \frac{1}{2}, \end{gathered}$$

where $C, C_1, C_2 > 0$.

16.1. **Derivative with respect to λ.** The proofs of the estimates below are based on Duhamel's formula for the solution:

$$\psi(x,t) = \int_0^t \int_{\mathbb{R}^2} \frac{\exp(-\frac{|x-x'|^2}{4(t-s)})}{t-s} \frac{1}{\lambda(s)^2} f(\frac{x'-\xi(s)}{\lambda(s)}, s) \, dx'ds.$$

We change variables writing $x = \xi(t)+\lambda(t)y$ and $x' = \xi(s)+\lambda(s)y'$. Then

$$\tilde{\psi}(y,t) = \int_0^t \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{t-s} f(y',s) \, dy'ds,$$

and we obtain the following formula for the directional derivative:

$$\begin{aligned} &D_\lambda\tilde\psi[\lambda_1](y,t)\\ &= -\frac12\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}\\ &\qquad\cdot(\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y')\cdot(\lambda_1(t)y-\lambda_1(s)y')f(y',s)\,dy'ds.\end{aligned}$$

Lemma 16.1. *We have*

$$|D_\lambda\tilde\psi[\lambda,\xi](\lambda_1)(y,t)|\le C\Big(\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}+\|\dot\lambda_1\|_{L^\infty}\Big)\lambda_*(0)^\nu R(0)^{2-a},$$

and

$$\begin{aligned}&|D_\lambda\tilde\psi[\lambda,\xi](\lambda_1)(y,t)-D_\lambda\tilde\psi[\lambda,\xi](\lambda_1)(y,T)|\\ &\le C\Big(\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}+\|\dot\lambda_1\|_{L^\infty}\Big)\lambda_*(t)^\nu R(t)^{2-a},\end{aligned}$$

for $|y|\le R(t)$, $t\in(0,T)$.

Proof. One of the terms is

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}(\xi(t)-\xi(s))\cdot\lambda_1(t)yf(y',s)\,dy'ds\Big|\\ &\le C\lambda_*(t)R(t)\Big\|\frac{\lambda_1}{\lambda_*}\Big\|_{L^\infty}\|\dot\xi\|_{L^\infty}\int_0^t(t-s)^{-1}\lambda_*(s)^\nu R(s)^{-a}\\ &\qquad\int_{|y'|\le R(s)}\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})\,dy'ds\\ &\le C\lambda_*(t)R(t)\Big\|\frac{\lambda_1}{\lambda_*}\Big\|_{L^\infty}\|\dot\xi\|_{L^\infty}\int_0^t(t-s)^{-1}\lambda_*(s)^\nu R(s)^{-a}\\ &\qquad\int_{|y'|\le R(s)}\Big(\frac{t-s}{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}\Big)^\mu\,dy'ds\qquad(\mu\in(0,1))\\ &\le C\lambda_*(t)R(t)\Big\|\frac{\lambda_1}{\lambda_*}\Big\|_{L^\infty}\|\dot\xi\|_{L^\infty}\int_0^t(t-s)^{-1+\mu}\lambda_*(s)^{\nu-2\mu}R(s)^{-a}\int_{|y'|\le R(s)}|y'|^{-2\mu}\,dy'ds\\ &\le C\lambda_*(t)R(t)\Big\|\frac{\lambda_1}{\lambda_*}\Big\|_{L^\infty}\|\dot\xi\|_{L^\infty}\int_0^t(t-s)^{-1+\mu}\lambda_*(s)^{\nu-2\mu}R(s)^{-a+2-2\mu}\,ds\\ &\le C\lambda_*(t)R(t)\Big\|\frac{\lambda_1}{\lambda_*}\Big\|_{L^\infty}\|\dot\xi\|_{L^\infty}\lambda_*(t)^{\nu-\mu}R(t)^{-a+2-2\mu}\end{aligned}$$

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}\\ &\qquad\cdot(\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y')\cdot(\lambda_1(t)y-\lambda_1(s)y')f(y',s)\,dy'ds.\end{aligned}$$

□

Lemma 16.2. *For any $\sigma>0$ there is C such that*

$$\begin{aligned}&|\nabla_x D_\lambda\tilde\psi[\lambda,\xi](\lambda_1)(y,t)-\nabla_x D_\lambda\tilde\psi[\lambda,\xi](\lambda_1)(y,T)|\\ &\le C\Big(\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}+\|\dot\lambda_1\|_{L^\infty}\Big)\lambda_*(t)^{\nu-1-\sigma}R(t)^{1-a},\end{aligned}\tag{16.1}$$

for $|y|\le R(t)$, $t\in(0,T)$.

Proof. We compute

$$\partial_{y_i} D_\lambda \tilde{\psi}[\lambda_1](y,t) = D_1 + D_2 + D_3 \tag{16.2}$$

where

$$\begin{aligned}
D_1(y,t) &= -\frac{1}{4}\lambda(t)\int_0^t\int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^3}(\xi_i(t)-\xi_i(s)+\lambda(t)y_i-\lambda(s)y_i') \\
&\quad\cdot(\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y')\cdot(\lambda_1(t)y-\lambda_1(s)y')f(y',s)\,dy'ds \\
D_2(y,t) &= -\frac{1}{2}\lambda(t)\int_0^t\int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}(\lambda_1(t)y_i-\lambda_1(s)y_i')f(y',s)\,dy'ds \\
D_3(y,t) &= -\frac{1}{2}\lambda_1(t)\int_0^t\int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{(t-s)})}{4(t-s)^2} \\
&\quad\cdot(\xi_i(t)-\xi_i(s)+\lambda(t)y_i-\lambda(s)y_i')f(y',s)\,dy'ds.
\end{aligned}$$

Since

$$\partial_{x_i} D_\lambda\tilde{\psi}[\lambda_1](\frac{x-\xi}{\lambda},t) = \frac{1}{\lambda}\partial_{y_i}D_\lambda\tilde{\psi}[\lambda_1](\frac{x-\xi}{\lambda},t),$$

to obtain (16.1) it is sufficient to prove that the functions

$$\begin{cases} g_1(y,t) = -4\dfrac{1}{\lambda(t)}D_1(y,t) \\ g_2(y,t) = -2\dfrac{1}{\lambda(t)}D_2(y,t) \\ g_3(y,t) = -2\dfrac{1}{\lambda(t)}D_3(y,t) \end{cases} \tag{16.3}$$

satisfy the estimate

$$|g_j(y,t)-g_j(y,T)| \le C\left(\left\|\frac{\lambda_1}{\lambda}\right\|_{L^\infty} + \|\dot\lambda_1\|_{L^\infty}\right)\lambda_*(t)^{\nu-1-\sigma}R(t)^{1-a}, \tag{16.4}$$

for $|y|\le R(t)$ and $j=1,2,3$. We will do the computation in detail for

$$g_2(y,t) = \int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}(\lambda_1(t)y_i-\lambda_1(s)y_i')f(y',s)\,dy'ds.$$

The corresponding inequalities for g_1 and g_3 are similar and we omit their proof.

To prove (16.4) for $j=2$, let us write

$$\begin{aligned}
&g_2(y,t)-g_2(y,T) \\
&= \int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}(\lambda_1(t)y_i-\lambda_1(s)y_i')f(y',s)\,dy'ds \\
&\quad -\int_0^T\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(T)-\xi(s)+\lambda(T)y-\lambda(s)y'|^2}{4(T-s)})}{(T-s)^2}(\lambda_1(T)y_i-\lambda_1(s)y_i')f(y',s)\,dy'ds \\
&= A_0+A_1+A_2+A_3,
\end{aligned}$$

where

$$A_0 = \lambda_1(t) y_i \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} f(y',s)\, dy' ds$$

$$A_1 = -\int_0^{t-(T-t)} \int_{\mathbb{R}^2} \Big[\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} - \frac{\exp(-\frac{|\xi(T)-\xi(s)+\lambda(T)y-\lambda(s)y'|^2}{4(T-s)})}{(T-s)^2} \Big] \cdot \lambda_1(s) y_i' f(y',s)\, dy' ds$$

$$A_2 = \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} (\lambda_1(t) y_i - \lambda_1(s) y_i') f(y',s)\, dy' ds$$

$$A_3 = -\int_{t-(T-t)}^{T} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(T)-\xi(s)+\lambda(T)y-\lambda(s)y'|^2}{4(T-s)})}{(T-s)^2} (\lambda_1(T) y_i - \lambda_1(s) y_i') f(y',s)\, dy' ds$$

Estimate of A_0. We claim that

$$|A_0| \le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a}. \tag{16.5}$$

Indeed, we have

$$\begin{aligned} &\int_0^{t-(T-t)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} |f(y',s)|\, dy' ds \\ &\le C \int_0^{t-(T-t)} \frac{1}{(T-s)^2} \lambda_*(s)^\nu R(s)^{2-a}\, ds \\ &\le C \frac{1}{(T-t)} \lambda_*(t)^\nu R(t)^{2-a} \end{aligned} \tag{16.6}$$

and therefore, for $|y| \le R(t)$,

$$\begin{aligned} |A_0| &\le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a} \lambda_*(t) R(t)^2 \\ &\le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a}. \end{aligned}$$

Estimate of A_1. We claim that

$$\begin{aligned} |A_1| \le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R^{1-a} \Big[(\|\dot\xi\|_{L^\infty} + \|\dot\lambda\|_{L^\infty} |y|) \frac{\lambda_*(t)^2 R(t)^2}{(T-t)^{1/2}} \\ + \frac{\lambda_*(t)^2 R(t)^2}{T-t} \Big]. \end{aligned} \tag{16.7}$$

We express

$$\begin{aligned}
&\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}-\frac{\exp(-\frac{|\xi(T)-\xi(s)+\lambda(T)y-\lambda(s)y'|^2}{4(T-s)})}{(T-s)^2}\\
&=\int_0^1\frac{d}{dv}\frac{\exp(-\frac{|\xi(t_v)-\xi(s)+\lambda(t_v)y-\lambda(s)y'|^2}{4(t_v-s)})}{(t_v-s)^2}\,dv,\qquad (t_v=vt+(1-v)T)\\
&=-(T-t)\int_0^1\frac{d}{dv}\exp(-\frac{|\xi(t_v)-\xi(s)+\lambda(t_v)y-\lambda(s)y'|^2}{4(t_v-s)})\\
&\cdot\Bigg[-2\frac{(\xi(t_v)-\xi(s)+\lambda(t_v)y-\lambda(s)y')\cdot(\dot\xi(t_v)+\dot\lambda(t_v)y)}{4(t_v-s)^3}\\
&\quad+2\frac{|\xi(t_v)-\xi(s)+\lambda(t_v)y-\lambda(s)y'|^2}{4(t_v-s)^4}-\frac{2}{(t_v-s)^3}\Bigg]\,dv,
\end{aligned}\tag{16.8}$$

So

$$A_1=A_{11}+A_{12}+A_{13}$$

where

$$\begin{aligned}
A_{11}&=2(T-t)\int_0^{t-(T-t)}\int_{\mathbb{R}^2}\int_0^1\exp(-\frac{|\xi(t_v)-\xi(s)+\lambda(t_v)y-\lambda(s)y'|^2}{4(t_v-s)})\\
&\qquad\frac{(\xi(t_v)-\xi(s)+\lambda(t_v)y-\lambda(s)y')\cdot(\dot\xi(t_v)+\dot\lambda(t_v)y)}{4(t_v-s)^3}\lambda_1(s)y_i'f(y',s)\,dvdy'ds\\
A_{12}&=-2(T-t)\int_0^{t-(T-t)}\int_{\mathbb{R}^2}\int_0^1\exp(-\frac{|\xi(t_v)-\xi(s)+\lambda(t_v)y-\lambda(s)y'|^2}{4(t_v-s)})\\
&\qquad\frac{|\xi(t_v)-\xi(s)+\lambda(t_v)y-\lambda(s)y'|^2}{4(t_v-s)^4}\lambda_1(s)y_i'f(y',s)\,dvdy'ds\\
A_{13}&=2(T-t)\int_0^{t-(T-t)}\int_{\mathbb{R}^2}\int_0^1\exp(-\frac{|\xi(t_v)-\xi(s)+\lambda(t_v)y-\lambda(s)y'|^2}{4(t_v-s)})\\
&\qquad\frac{1}{(t_v-s)^3}\lambda_1(s)y_i'f(y',s)\,dvdy'ds.
\end{aligned}$$

To estimate A_{11} we use $e^{-|z|^2}z\le C$ and get

$$\begin{aligned}
|A_{11}|&\le C(T-t)\int_0^{t-(T-t)}\int_{\mathbb{R}^2}\int_0^1\frac{|\dot\xi(t_v)+\dot\lambda(t_v)y|}{(t_v-s)^{5/2}}|\lambda_1(s)||y_i'f(y',s)|\,dvdy'ds\\
&\le C(T-t)(\|\dot\xi\|_{L^\infty}+\|\dot\lambda\|_{L^\infty}|y|)\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\\
&\quad\cdot\int_0^1\int_0^{t-(T-t)}\frac{\lambda(s)}{(t_v-s)^{5/2}}\int_{\mathbb{R}^2}|y_i'f(y',s)|\,dy'dsdv
\end{aligned}$$

But

$$\begin{aligned}
\int_{\mathbb{R}^2}|y_i'f(y',s)|\,dy'&\le\lambda(s)^\nu R(s)^{-a}\int_{|y'|\le R(s)}|y'|\,dy'\\
&=C\lambda(s)^\nu R(s)^{3-a}.
\end{aligned}\tag{16.9}$$

Therefore

$$\begin{aligned}
\int_0^{t-(T-t)}\frac{\lambda(s)}{(t_v-s)^{5/2}}\int_{\mathbb{R}^2}|y_i'f(y',s)|\,dy'ds&\le\int_0^{t-(T-t)}\frac{\lambda(s)^{\nu+1}R(s)^{3-a}}{(t_v-s)^{5/2}}\\
&\le C\frac{\lambda(t)^{\nu+1}R(t)^{3-a}}{(T-t)^{3/2}}
\end{aligned}$$

and then

$$|A_{11}| \le C(\|\dot\xi\|_{L^\infty} + \|\dot\lambda\|_{L^\infty}|y|)\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\frac{\lambda(t)^{\nu+1}R(t)^{3-a}}{(T-t)^{1/2}}. \tag{16.10}$$

For A_{12} we have, using (16.9),

$$\begin{aligned}|A_{12}| &\le (T-t)\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\int_0^1\int_0^{t-(T-t)}\frac{1}{(t_v-s)^3}\lambda(s)\int_{\mathbb{R}^2}|y_i'f(y',s)|\,dy'dsdv\\ &\le (T-t)\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\int_0^1\int_0^{t-(T-t)}\frac{\lambda(s)^{\nu+1}R(s)^{3-a}}{(t_v-s)^3}\,dsdv\\ &\le C\,\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\frac{\lambda(t)^{\nu+1}R(t)^{3-a}}{T-t}.\end{aligned} \tag{16.11}$$

Let us estimate A_{13}:

$$\begin{aligned}|A_{13}| &\le 2(T-t)\int_0^{t-(T-t)}\int_{\mathbb{R}^2}\int_0^1\frac{1}{(t_v-s)^3}\lambda_1(s)|y_i'f(y',s)|\,dvdy'ds\\ &\le (T-t)\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\int_0^1\int_0^{t-(T-t)}\frac{\lambda(s)^{\nu+1}R(s)^{3-a}}{(t_v-s)^3}\,dsdv\\ &\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\frac{\lambda(t)^{\nu+1}R(t)^{3-a}}{T-t}.\end{aligned} \tag{16.12}$$

Combining (16.10), (16.11), and (16.12) we obtain (16.7).

Estimate of A_2. We claim that

$$|A_2| \le C\left(\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty} + \|\dot\lambda_1\|_{L^\infty}\right)\lambda_*(t)^{\nu-1-\sigma}R(t)^{1-a}, \quad \text{for } |y|\le R(t). \tag{16.13}$$

Indeed, let us make the dependence on the variables more explicit by writing

$$A_2(y,t) = \int_{t-(T-t)}^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}(\lambda_1(t)y_i-\lambda_1(s)y_i')\\ f(y',s)\,dy'ds.$$

Let us define

$$\tilde A_2(y,t) = \int_{t-(T-t)}^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\lambda(s)(y-y')|^2}{4(t-s)})}{(t-s)^2}\lambda_1(s)(y_i-y_i')f(y',s)\,dy'ds.$$

We claim that

$$|\tilde A_2(y,t)| \le C\lambda_*(t)^{-\sigma}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a} \tag{16.14}$$

and

$$|A_2(y,t)-\tilde A_2(y,t)| \le C\left(\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}+\|\dot\lambda_1\|_{L^\infty}\right)\lambda_*(t)^{\nu-1}R(t)^{1-a}. \tag{16.15}$$

Let us prove (16.14). Let $\mu \in (1, \frac{3}{2})$ be fixed. Then

$$
\begin{aligned}
|\tilde{A}_2(y,t)| \le & \int_{t-(T-t)}^{t} \frac{1}{(t-s)^2} \lambda_*(s)^{\nu} R(s)^{-a} \\
& \quad \int_{|y'|\le R(s)} \exp\Big(-\frac{|\lambda(s)(y-y')|^2}{4(t-s)}\Big)|\lambda_1(s)(y_i-y_i')|\, dy' ds \\
& \le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \int_{t-(T-t)}^{t} \frac{1}{(t-s)^2} \lambda_*(s)^{\nu+1} R(s)^{-a} \\
& \quad \int_{|y'|\le R(s)} \left(\frac{t-s}{|\lambda(s)(y-y')|^2}\right)^{\mu} |y_i - y_i'|\, dy' ds \\
& \le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \int_{t-(T-t)}^{t} (t-s)^{\mu-2} \lambda_*(s)^{\nu+1-2\mu} R(s)^{-a} \\
& \quad \int_{|y'|\le R(s)} |y-y'|^{1-2\mu}\, dy' ds \\
& \le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \int_{t-(T-t)}^{t} (t-s)^{\mu-2} \lambda_*(s)^{\nu+1-2\mu} R(s)^{3-2\mu-a} ds \\
& \le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} (T-t)^{\mu-1} \lambda_*(t)^{\nu+1-2\mu} R(t)^{3-2\mu-a} \\
& \le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a} \lambda_*(t)^{1-\mu} R(t)^{2-2\mu} \\
& \le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a} \lambda_*(t)^{-\sigma},
\end{aligned}
$$

where $\sigma = (1-\mu)(1-2\beta)$. This proves (16.14).

To prove (16.15) we define

$$
\widehat{A}_2(y,t) = \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\lambda(s)(y-y')|^2}{4(t-s)})}{(t-s)^2} (\lambda_1(t) y_i - \lambda_1(s) y_i') \, f(y',s)\, dy' ds,
$$

and show that

$$
|A_2(y,t) - \widehat{A}_2(y,t)| \le C \Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a} \tag{16.16}
$$

and

$$
|\tilde{A}_2(y,t) - \widehat{A}_2(y,t)| \le C \|\dot{\lambda}_1\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a}. \tag{16.17}
$$

To prove (16.16) we write

$$
\begin{aligned}
& A_2(y,t) - \widehat{A}_2(y,t) \\
& = \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)}) - \exp(-\frac{|\lambda(s)(y-y')|^2}{4(t-s)})}{(t-s)^2} \\
& \qquad \cdot (\lambda_1(t) y_i - \lambda_1(s) y_i') f(y',s)\, dy' ds.
\end{aligned}
$$

Note that

$$\begin{aligned}
&\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})-\exp(-\frac{|\lambda(s)(y-y')|^2}{4(t-s)})\\
&=\int_0^1\frac{d}{d\tau}\exp(-\frac{|\tau(\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y)+\lambda(s)(y-y')|^2}{4(t-s)})\,d\tau\\
&=-\frac{1}{2(t-s)}\int_0^1\exp(-\frac{|\tau(\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y)+\lambda(s)(y-y')|^2}{4(t-s)})\\
&\quad\cdot[\tau(\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y)+\lambda(s)(y-y')]\cdot[\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y]\,d\tau.
\end{aligned}\tag{16.18}$$

Fix any $\mu\in(1,\frac{3}{2})$. Then

$$\begin{aligned}
&|A_2(y,t)-\widehat{A}_2(y,t)|\\
&=\frac{1}{2}\Big|\int_{t-(T-t)}^t\frac{1}{(t-s)^3}\int_{\mathbb{R}^2}\int_0^1\exp(-\frac{|\tau(\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y)+\lambda(s)(y-y')|^2}{4(t-s)})\\
&\quad\cdot[\tau(\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y)+\lambda(s)(y-y')]\cdot[\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y]\,d\tau\\
&\qquad\cdot(\lambda_1(t)y_i-\lambda_1(s)y_i')f(y',s)\,dy'ds\Big|\\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\int_{t-(T-t)}^t\int_0^1(t-s)^{\mu-2}\lambda_*(s)^\nu R(s)^{-a}\\
&\qquad\int_{|y'|\le R(s)}|\tau(\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y)+\lambda(s)(y-y')|^{1-2\mu}\\
&\qquad(\|\dot\xi\|_{L^\infty}+\|\dot\lambda\|_{L^\infty}|y|)(\lambda(t)|y|+\lambda(s)|y'|)\,dy'd\tau ds\\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}(\|\dot\xi\|_{L^\infty}+\|\dot\lambda\|_{L^\infty}R(t))\int_{t-(T-t)}^t(t-s)^{\mu-2}\lambda_*(s)^\nu R(s)^{-a}\lambda_*(s)^{1-2\mu}R(s)^{3-2\mu}\\
&\qquad(\lambda(t)R(t)+\lambda(s)R(s))\,ds\\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}(\|\dot\xi\|_{L^\infty}+\|\dot\lambda\|_{L^\infty}R(t))(T-t)^{\mu-1}\lambda_*(t)^{\nu+2-2\mu}R(t)^{4-2\mu-a}\\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu+1-\mu}R(t)^{5-2\mu-a}\\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a}\lambda_*(t)^{2-\mu}R(t)^{4-2\mu}\\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a},
\end{aligned}$$

and we obtain (16.16).

Finally we show the validity of (16.17). We have

$$\begin{aligned}
&|\tilde{A}_2(y,t) - \widehat{A}_2(y,t)| \\
&= \left| \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\lambda(s)(y-y')|^2}{4(t-s)})}{(t-s)^2} (\lambda_1(s) - \lambda_1(t)) y_i f(y',s)\, dy' ds \right| \\
&\le C \|\dot{\lambda}_1\|_{L^\infty} R(t) \int_{t-(T-t)}^{t} (t-s)^{\mu-1} \int_{|y'|\le R(s)} |\lambda(s)(y-y')|^{-2\mu} f(y',s)\, dy' ds \\
&\le C \|\dot{\lambda}_1\|_{L^\infty} R(t) \int_{t-(T-t)}^{t} (t-s)^{\mu-1} \lambda_*(s)^{\nu} R(s)^{-a} \lambda(s)^{-2\mu} R(s)^{2-2\mu}\, ds \\
&\le C \|\dot{\lambda}_1\|_{L^\infty} R(t) (T-t)^{\mu} \lambda(t)^{\nu-2\mu} R(t)^{-a+2-2\mu} \\
&\le C \|\dot{\lambda}_1\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a} \lambda_*(t)^{1-\mu} R(t)^{2-2\mu} \\
&\le C \|\dot{\lambda}_1\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a},
\end{aligned}$$

where we have chosen $\mu \in (0,1)$. This is (16.17).

Estimate of A_3. Finally we have

$$|A_3| \le C \left(\left\| \frac{\lambda_1}{\lambda} \right\|_{L^\infty} + \|\dot{\lambda}_1\|_{L^\infty} \right) \lambda_*(t)^{\nu-1} R(t)^{1-a}, \quad \text{for } |y| \le R(t). \tag{16.19}$$

The proof is similar to that of (16.13) and we omit it.

Combining (16.5), (16.7), (16.13) and (16.19) we obtain the validity of (16.4) for $j = 2$. □

Lemma 16.3. *For any $\sigma > 0$ and $\gamma \in (0,1)$ there is C such that*

$$\begin{aligned}
&|\nabla_x D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y_1,t) - \nabla_x D_\lambda \tilde{\psi}[\lambda,\xi](\lambda_1)(y_2,t)| \\
&\le C \Big(\frac{|y_1 - y_2|}{R(t)} \Big)^{\gamma} \Big(\Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} + \|\dot{\lambda}_1\|_{L^\infty} \Big) \lambda_*(t)^{\nu-1-\sigma} R(t)^{1-a},
\end{aligned}$$

for $|y_1|$, $|y_2| \le R(t)$, $t \in (0,T)$.

Proof. Using formula (16.2), it is sufficient to show that the functions g_1, g_2 and g_3 defined in (16.3) satisfy

$$|g_j(y_1,t) - g_j(y_2,t)| \le C \Big(\frac{|y_1 - y_2|}{R(t)} \Big)^{\gamma} \Big(\Big\| \frac{\lambda_1}{\lambda} \Big\|_{L^\infty} + \|\dot{\lambda}_1\|_{L^\infty} \Big) \lambda_*(t)^{\nu-1-\sigma} R(t)^{1-a},$$

for $|y_1|$, $|y_2| \le R(t)$, $t \in (0,T)$.

We do the computations in detail for g_2. The functions g_1, g_3 are treated similarly.

Let $t \in (0,T)$, $|y_1|$, $|y_2| \le R(t)$ and define

$$L = 3|y_1 - y_0|, \quad \bar{y} = \frac{y_0 + y_1}{2}.$$

Let us write

$$g_2(y,t) = \int_0^t \int_{\mathbb{R}^2} G_2(y,y',t,s)\, dy' ds,$$

where

$$G_2(y,y',t,s) = \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} (\lambda_1(t) y_i - \lambda_1(s) y_i') f(y',s). \tag{16.20}$$

We the compute

$$g_2(y_1,t) - g_2(y_2,t) = B_1 + B_2 + B_3$$

where

$$\begin{aligned} B_1 &= \int_0^{t-(T-t)} \int_{\mathbb{R}^2} (G_2(y_1,y',t,s) - G_2(y_0,y',t,s))\, dy'ds \\ B_2 &= \int_{t-(T-t)}^t \int_{|y'-\bar y|\le L} G_2(y_1,y',t,s)\, dy'ds \\ &\quad - \int_{t-(T-t)}^t \int_{|y'-\bar y|\le L} G_2(y_0,y',t,s)\, dy'ds \\ B_3 &= \int_{t-(T-t)}^t \int_{|y'-\bar y|\ge L} (G_2(y_1,y',t,s) - G_2(y_0,y',t,s))\, dy'ds. \end{aligned}$$

Estimate of B_1. We claim that for $t \in (0,T)$ and $|y_1|$, $|y_0| \le R(t)$ we have

$$|B_1| \le C \left\| \frac{\lambda_1}{\lambda} \right\|_{L^\infty} \frac{|y_1 - y_0|}{R(t)} \lambda_*(t)^{\nu-1} R(t)^{1-a}. \tag{16.21}$$

To prove this, let us set

$$\tilde G_2(y,z,y',t,s) = \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} (\lambda_1(t) z_i - \lambda_1(s) y_i') f(y',s), \tag{16.22}$$

so that

$$\begin{aligned} B_1 &= \int_0^{t-(T-t)} \int_{\mathbb{R}^2} (\tilde G_2(y_1,y_1,y',t,s) - \tilde G_2(y_0,y_0,y',t,s))\, dy'ds \\ &= B_{1,1} + B_{1,2}, \end{aligned}$$

where

$$\begin{aligned} B_{1,1} &= \int_0^{t-(T-t)} \int_{\mathbb{R}^2} (\tilde G_2(y_1,y_1,y',t,s) - \tilde G_2(y_1,y_0,y',t,s))\, dy'ds \\ B_{1,2} &= \int_0^{t-(T-t)} \int_{\mathbb{R}^2} (\tilde G_2(y_1,y_0,y',t,s) - \tilde G_2(y_0,y_0,y',t,s))\, dy'ds. \end{aligned}$$

We claim that

$$|B_{1,1}| \le C \left\| \frac{\lambda_1}{\lambda} \right\|_{L^\infty} \frac{|y_1 - y_0|}{R(t)} \lambda_*(t)^{\nu-1} R(t)^{1-a}, \tag{16.23}$$

and that

$$|B_{1,2}| \le C \left\| \frac{\lambda_1}{\lambda} \right\|_{L^\infty} \frac{|y_1 - y_0|}{R(t)} \lambda_*(t)^{\nu-1} R(t)^{1-a}. \tag{16.24}$$

From these two estimates we get (16.21).

To prove (16.23) we note that

$$B_{1,1} = \lambda_1(t)(y_1 - y_0) \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_1-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} f(y',s)\, dy'ds.$$

We have already have obtained in (16.6) the estimate

$$\begin{aligned} &\int_0^{t-(T-t)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} |f(y',s)|\, dy'ds \\ &\le C \frac{1}{(T-t)} \lambda_*(t)^\nu R(t)^{2-a} \\ &\le C\, \lambda_*(t)^{\nu-1} R(t)^{2-a} \end{aligned}$$

and (16.23) follows.

To prove (16.24) we write

$$B_{1,2} = B_{1,2,1} + B_{1,2,2}$$

where

$$B_{1,2,1} = \lambda_1(t) y_0^i \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_1-\lambda(s)y'|^2}{4(t-s)}) - \exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_0-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} f(y',s)\,dy'ds$$

$$B_{1,2,2} = \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_1-\lambda(s)y'|^2}{4(t-s)}) - \exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_2-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} \lambda_1(s) y_i' f(y',s)\,dy'ds.$$

We have

$$\begin{aligned}
&\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_1-\lambda(s)y'|^2}{4(t-s)}) - \exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_0-\lambda(s)y'|^2}{4(t-s)}) \\
&= \int_0^1 \frac{d}{d\tau} \exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_\tau-\lambda(s)y'|^2}{4(t-s)})\,d\tau, \quad y_\tau = \tau y_1 + (1-\tau)y_0 \\
&= -\frac{1}{2}\lambda(t) \int_0^1 \exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_\tau-\lambda(s)y'|^2}{4(t-s)}) \\
&\qquad \frac{(\xi(t)-\xi(s)+\lambda(t)y_\tau-\lambda(s)y')\cdot(y_1-y_0)}{t-s}\,d\tau
\end{aligned}$$

Then we get

$$\begin{aligned}
&\Big|\int_0^{t-(T-t)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_1-\lambda(s)y'|^2}{4(t-s)}) - \exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_0-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} \\
&\qquad f(y',s)\,dy'ds\Big| \\
&\le C|y_1-y_0| \int_0^1 \int_0^{t-(T-t)} (T-s)^{-5/2} \lambda_*(s)^{\nu+1} R(s)^{2-a}\,ds d\tau \\
&= C|y_1-y_0|(T-t)^{-3/2}\lambda_*(t)^{\nu+1}R(t)^{2-a}.
\end{aligned}$$

Therefore

$$\begin{aligned}
|B_{1,2,1}| &\le C|y_1-y_0||\lambda_1(t)||y_0|(T-t)^{-3/2}\lambda_*(t)^{\nu+1}R(t)^{2-a} \\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}|y_1-y_0||\lambda_1(t)||y_0|(T-t)^{-3/2}\lambda_*(t)^{\nu+2}R(t)^{3-a}.
\end{aligned}$$

Similarly

$$\begin{aligned}
|B_{1,2,2}| &\le C|y_1-y_0|\,\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty} \int_0^{t-(T-t)} \frac{1}{(T-s)^{5/2}}\lambda_*(s)^{\nu+2}R(s)^{3-a}\,ds \\
&\le C|y_1-y_0|\,\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty} \frac{1}{(T-t)^{3/2}}\lambda_*(t)^{\nu+2}R(t)^{3-a}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|B_{1,2}| &\le C\,\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty} \frac{|y_1-y_0|}{R(t)}\lambda_*(t)^{\nu-1}R(t)^{1-a}\lambda_*(t)^{3/2}R(t)^3 \\
&\le C\,\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty} \frac{|y_1-y_0|}{R(t)}\lambda_*(t)^{\nu-1}R(t)^{1-a}.
\end{aligned}$$

This proves (16.24).

Estimate of B_2. We claim that for any $\sigma > 0$ and $\gamma \in (0,1)$ there is C such that for $t \in (0,T)$ and $|y_1|$, $|y_0| \le R(t)$ we have

$$|B_2| \le C\Big(\frac{|y_1-y_0|}{R}\Big)^\gamma \Big(\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty} + \|\dot\lambda_1\|_{L^\infty}\Big)\lambda_*(t)^{-\sigma}\lambda_*(t)^{\nu-1}R(t)^{1-a} \tag{16.25}$$

Let us recall that $L = 3|y_1-y_0|$. Let us define

$$I(y,z) = \int_{t-(T-t)}^t \int_{|y'-\bar y|\le L} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}(\lambda_1(t)z_i - \lambda_1(s)y_i') \\ f(y',s)\,dy'ds,$$

so that

$$B_2 = I(y_1,y_1) - I(y_0,y_0).$$

We claim that for any $\sigma>0$ and $\gamma\in(0,1)$ there is C such that for $t\in(0,T)$, $|y|$, $|z|\le R(t)$ we have

$$|I(y,z)| \le C\Big(\frac{L}{R}\Big)^\gamma \Big(\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty} + \|\dot\lambda_1\|_{L^\infty}\Big)\lambda_*(t)^{-\sigma}\lambda_*(t)^{\nu-1}R(t)^{1-a}. \tag{16.26}$$

To prove (16.26) we define two quantities:

$$\tilde I(y) = \int_{t-(T-t)}^t \int_{|y'-\bar y|\le L} \frac{e^{-\frac{|\lambda(s)(y-y')|^2}{4(t-s)}}}{(t-s)^2}\lambda_1(s)(y_i-y_i')f(y',s)\,dy'ds$$

$$\hat I(y) = \int_{t-(T-t)}^t \int_{|y'-\bar y|\le L} \frac{e^{-\frac{|\lambda(s)(y-y')|^2}{4(t-s)}}}{(t-s)^2}(\lambda_1(t)y_i-\lambda_1(s)y_i')f(y',s)\,dy'ds.$$

We claim that for any $\sigma>0$ and $\gamma\in(0,1)$ there is C such that for $t\in(0,T)$, $|y|\le R(t)$ we have

$$|\tilde I(y)| \le C\Big(\frac{L}{R}\Big)^\gamma \Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{-\sigma}\lambda_*(t)^{\nu-1}R(t)^{1-a}, \tag{16.27}$$

and that

$$|\hat I(y) - \tilde I(y)| \le C\|\dot\lambda_1\|_{L^\infty}\frac{L}{R}\lambda_*(t)^{\nu-1}R(t)^{1-a} \tag{16.28}$$

$$|I(y,z) - \hat I(y)| \le C\frac{L}{R}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a}, \tag{16.29}$$

for $|z|\le R(t)$.

To prove (16.27) let $\mu\in(1,\frac32)$ be fixed. Then

$$\begin{aligned}
|\tilde I(y)| &\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\int_{t-(T-t)}^t \lambda_*(s)^{\nu+1}R(s)^{-a}(t-s)^{-2} \\
&\qquad \int_{|y'-\bar y|\le L,\ |y'|\le R(s)}\Big(\frac{t-s}{|\lambda(s)(y-y')|^2}\Big)^\mu |y-y'|\,dy'ds \\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\int_{t-(T-t)}^t \lambda_*(s)^{\nu+1-2\mu}R(s)^{-a}(t-s)^{\mu-2}\int_{|y'-\bar y|\le L,\ |y'|\le R(s)}|y-y'|^{1-2\mu}\,dy'ds \\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}L^{3-2\mu}\int_{t-(T-t)}^t \lambda_*(s)^{\nu+1-2\mu}R(s)^{-a}(t-s)^{\mu-2}\,ds \\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}L^{3-2\mu}\lambda_*(t)^{\nu+1-2\mu}R(t)^{-a}(T-t)^{\mu-1} \\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}L^{3-2\mu}\lambda_*(t)^{\nu-1}R(t)^{1-a}\lambda_*(t)^{1-\mu}R(t)^{-1} \\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\Big(\frac{L}{R}\Big)^{3-2\mu}\lambda_*(t)^{\nu-1}R(t)^{1-a}\lambda_*(t)^{1-\mu}R(t)^{2-2\mu}.
\end{aligned}$$

This proves (16.27) with $\gamma = 3-2\mu\in(0,1)$ and $\sigma = (\mu-1)(1-2\beta)$.

Next we prove (16.28). For this, let $\mu \in (0,1)$ be fixed. Then

$$|\hat{I}(y) - \tilde{I}(y)| = \left| y_i \int_{t-(T-t)}^{t} \int_{|y'-\bar{y}|\le L} \frac{\exp(-\frac{|\lambda(s)(y-y')|^2}{4(t-s)})}{(t-s)^2} (\lambda_1(t) - \lambda_1(s)) f(y',s)\, dy' ds \right|$$
$$\le |y| \|\dot{\lambda}_1\|_{L^\infty} \int_{t-(T-t)}^{t} (t-s)^{-1} \lambda_*(s)^{\nu} R(s)^{-a}$$
$$\int_{|y'-\bar{y}|\le L,\ |y'|\le R(s)} \left(\frac{t-s}{\lambda(s)(y-y')|^2} \right)^{\mu} dy' ds.$$

Thus we get

$$\begin{aligned}
|\hat{I}(y) - \tilde{I}(y)| &\le |y| \|\dot{\lambda}_1\|_{L^\infty} \int_{t-(T-t)}^{t} (t-s)^{\mu-1} \lambda_*(s)^{\nu-2\mu} R(s)^{-a} L^{2-2\mu}\, ds \\
&\le C|y| \|\dot{\lambda}_1\|_{L^\infty} (T-t)^{\mu} \lambda_*(t)^{\nu-2\mu} R(t)^{-a} L^{2-2\mu} \\
&\le C|y| \|\dot{\lambda}_1\|_{L^\infty} \frac{L^{2-2\mu}}{R} \lambda_*(t)^{\nu-1} R(t)^{1-a} \lambda_*(t)^{1-\mu} \\
&\le C \|\dot{\lambda}_1\|_{L^\infty} \frac{L}{R} \lambda_*(t)^{\nu-1} R(t)^{1-a} \lambda_*(t)^{1-\mu} R^{2-2\mu} \\
&\le C \|\dot{\lambda}_1\|_{L^\infty} \frac{L}{R} \lambda_*(t)^{\nu-1} R(t)^{1-a},
\end{aligned}$$

and we obtain (16.28).

We prove next (16.29). Let $\mu \in (0,1)$ be fixed and estimate:

$$\begin{aligned}
&|I(y,z) - \hat{I}(y)| \\
&= \Big| \int_{t-(T-t)}^{t} \int_{|y'-\bar{y}|\le L} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)}) - \exp(-\frac{|\lambda(s)(y-y')|^2}{4(t-s)})}{(t-s)^2} \\
&\qquad (\lambda_1(t) z_i - \lambda_1(s) y_i') f(y',s)\, dy' ds \Big| \\
&\le C \int_{t-(T-t)}^{t} (t-s)^{-3} \int_{|y'-\bar{y}|\le L} \int_0^1 \exp(-\frac{|\tau[\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y] + \lambda(s)(y-y')|^2}{4(t-s)}) \\
&\qquad |\tau[\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y] + \lambda(s)(y-y')|\, |\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y| \\
&\qquad |(\lambda_1(t) z_i - \lambda_1(s) y_i') f(y',s)|\, d\tau dy' ds
\end{aligned}$$

Hence

$$
\begin{aligned}
&|I(y,z)-\hat I(y)|\\
&\le C\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}(\|\dot\xi\|_{L^\infty}+\|\dot\lambda\|_{L^\infty}R(t))\int_0^1\int_{t-(T-t)}^t (t-s)^{\mu-2}\lambda_*(s)^\nu R(s)^{-a}\\
&\qquad\int_{|y'-\bar y|\le L}|\tau[\xi(t)-\xi(s)+(\lambda(t)-\lambda(s))y]+\lambda(s)(y-y')|^{1-2\mu}\,dy'\\
&\qquad(\lambda_*(t)R(t)+\lambda_*(s)R(s)\,dsd\tau\\
&\le CL^{3-2\mu}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}(\|\dot\xi\|_{L^\infty}+\|\dot\lambda\|_{L^\infty}R(t))\lambda_*(t)R(t)\int_{t-(T-t)}^t (t-s)^{\mu-2}\lambda_*(s)^{\nu+1-2\mu}R(s)^{-a}\,ds\\
&\le CL^{3-2\mu}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}(\|\dot\xi\|_{L^\infty}+\|\dot\lambda\|_{L^\infty}R(t))\lambda_*(t)R(t)(T-t)^{\mu-1}\lambda_*(t)^{\nu+1-2\mu}R(t)^{-a}\\
&\le C\frac{L}{R}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-\mu+1}R(t)^{5-a-2\mu}\\
&\le C\frac{L}{R}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a}\lambda_*(t)^{2-\mu}R(t)^{4-2\mu}\\
&\le C\frac{L}{R}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a}.
\end{aligned}
$$

This proves (16.29).

Combining (16.27), (16.28) and (16.29) we deduce (16.26).

Estimate of B_3. Let us recall that $L=|y_1-y_0|$. We claim that for any $\sigma>0$ and $\gamma\in(0,1)$ there is C such that for $t\in(0,T)$ and $|y_1|$, $|y_0|\le R(t)$ we have

$$
|B_3|\le C\Big(\frac{L}{R}\Big)^\gamma\Big(\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}+\|\dot\lambda_1\|_{L^\infty}\Big)\lambda_*(t)^{-\sigma}\lambda_*(t)^{\nu-1}R(t)^{1-a}. \tag{16.30}
$$

We decompose

$$
B_3=B_{3,1}+B_{3,2}
$$

where

$$
\begin{aligned}
B_{3,1}&=\int_{t-(T-t)}^t\int_{|y'-\bar y|\ge L}(\tilde G_2(y_1,y_1,y',t,s)-\tilde G_2(y_0,y_1,y',t,s))\,dy'ds\\
B_{3,2}&=\int_{t-(T-t)}^t\int_{|y'-\bar y|\ge L}(\tilde G_2(y_0,y_1,y',t,s)-\tilde G_2(y_0,y_0,y',t,s))\,dy'ds
\end{aligned}
$$

and $\tilde G_2$ is defined in (16.22).

We claim that

$$
|B_{3,1}|\le C\frac{L}{R}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a} \tag{16.31}
$$

$$
|B_{3,2}|\le C\Big(\frac{L}{R}\Big)^\gamma\Big(\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}+\|\dot\lambda_1\|_{L^\infty}\Big)\lambda_*(t)^{-\sigma}\lambda_*(t)^{\nu-1}R(t)^{1-a}. \tag{16.32}
$$

From the two above estimates we obtain (16.30).

To prove (16.31), we note that

$$
\begin{aligned}
B_{3,1}=\int_{t-(T-t)}^t\int_{|y'-\bar y|\ge L}&\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_1-\lambda(s)y'|^2}{4(t-s)})-\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_0-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}\\
&(\lambda_1(t)z_i-\lambda_1(s)y_i')f(y',s)\,dy'ds.
\end{aligned}
$$

Let $\mu \in (1, \frac{3}{2})$ and estimate

$$
\begin{aligned}
|B_{3,1}| \le \Big| & \int_{t-(T-t)}^{t} \int_{|y'-\bar y|\ge L} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_1-\lambda(s)y'|^2}{4(t-s)}) - \exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_0-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} \\
& (\lambda_1(t)z_i - \lambda_1(s)y_i')f(y',s)\,dy'ds \Big| \\
\le & \int_{t-(T-t)}^{t} (t-s)^{-2}\lambda_*(s)^{\nu}R(s)^{-a}\int_0^1 \\
& \int_{|y'-\bar y|\ge L, |y'|\le R(s)} \lambda(t)\frac{(t-s)^{\mu-1}}{|\xi(t)-\xi(s)+\lambda(t)y_\tau-\lambda(s)y'|^{2\mu-1}} \\
& (|\lambda_1(t)z_i| + |\lambda_1(s)y_i'|)\,d\tau dy'ds \\
\le & C|y_1-y_0|\int_0^1\int_{t-(T-t)}^{t}(t-s)^{\mu-3}\lambda_*(s)^{\nu+1}R(s)^{-a} \\
& \int_{|y'-\bar y|\ge L} |\xi(t)-\xi(s)+\lambda(t)y_\tau-\lambda(s)y'|^{1-2\mu} \\
& (|\lambda_1(t)z_i| + |\lambda_1(s)y_i'|)\,d\tau dy'ds \\
\le & C|y_1-y_0|L^{3-2\mu}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)R(t)\int_{t-(T-t)}^{t}(t-s)^{\mu-3}\lambda_*(s)^{\nu+2-2\mu}R(s)^{-a}\,ds \\
\le & C|y_1-y_0|L^{3-2\mu}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu+3-2\mu}R(t)^{1-a}(T-t)^{\mu-2} \\
= & C\frac{|y_1-y_0|^{4-2\mu}}{R^{4-2\mu}}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a}\lambda_*(t)^{2-\mu}R(t)^{4-2\mu} \\
= & C\frac{|y_1-y_0|}{R}\Big\|\frac{\lambda_1}{\lambda}\Big\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a}.
\end{aligned}
$$

This proves (16.31).

We prove the estimate (16.32) for $B_{3,2}$. For a fixed $\mu > 1$ we have

$$
\begin{aligned}
& \int_{t-(T-t)}^{t}\int_{|y'-\bar y|\ge L}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}|f(y',s)|\,dy'ds \\
& \le C\int_{t-(T-t)}^{t}(t-s)^{-2}\lambda_*(s)^{\nu}R(s)^{-a} \\
& \qquad \int_{|y'-\bar y|\ge L}\left(\frac{t-s}{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}\right)^{\mu} dy'ds \\
& \le CL^{2-2\mu}\int_{t-(T-t)}^{t}(t-s)^{\mu-2}\lambda_*(s)^{\nu-2\mu}R(s)^{-a}\,ds \\
& \le CL^{2-2\mu}(T-t)^{\mu-1}\lambda_*(t)^{\nu-2\mu}R(t)^{-a}
\end{aligned}
$$

so that

$$
\begin{aligned}
|B_{3,2}| &= \left| \lambda_1(t)(y_1^i - y_0^i) \int_{t-(T-t)}^{t} \int_{|y'-\bar y|\geq L} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y_0-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} f(y',s)\,dy'ds \right| \\
&\leq C|y_1-y_0|^{3-2\mu}\lambda_*(t)^{\nu-\mu}R(t)^{-a} \\
&= C\frac{|y_1-y_0|^{3-2\mu}}{R^{3-2\mu}}\lambda_*(t)^{\nu-1}R(t)^{1-a}\lambda_*(t)^{1-\mu}R(t)^{2-2\mu} \\
&= C\frac{|y_1-y_0|^{3-2\mu}}{R^{3-2\mu}}\lambda_*(t)^{\nu-1-\sigma}R(t)^{1-a}
\end{aligned}
$$

□

Lemma 16.4. *For any $\sigma > 0$ and $\gamma \in (0,1)$ there is C such that*

$$
\begin{aligned}
&|\nabla_x D_\lambda \tilde\psi[\lambda,\xi](\lambda_1)(y,t_2) - \nabla_x D_\lambda \tilde\psi[\lambda,\xi](\lambda_1)(y,t_1)| \\
&\leq C...
\end{aligned}
$$

for $|y_1|$, $|y_2| \leq R(t)$, $t \in (0,T)$.

Proof. Using formula (16.2), it is sufficient to show that the functions g_1, g_2 and g_3 defined in (16.3) satisfy

$$
|g_j(y,t_2) - g_j(y,t_1)| \leq C...
$$

for $|y_1|$, $|y_2| \leq R(t)$, $t \in (0,T)$.

We do the computations in detail for g_2. The functions g_1, g_3 are treated similarly.

Let t_1, $t_2 \in (0,T)$, $|y| \leq R(t)$ with $t_1 \leq t_2$ and $t_2 \leq t_1 - \frac{1}{10}(T-t_1)$. Let us write

$$
g_2(y,t) = \int_0^t \int_{\mathbb{R}^2} G_2(y,y',t,s)\,dy'ds,
$$

where G_2 is defined in (16.20).

We the compute

$$
g_2(y,t_2) - g_2(y,t_1) = B_1 + B_2 + B_3
$$

where

$$
\begin{aligned}
B_1 &= \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^2} (G_2(y,y',t_2,s) - G_2(y,y',t_1,s))\,dy'ds \\
B_2 &= \int_{t_1-(t_2-t_1)}^{t_2} \int_{\mathbb{R}^2} G_2(y,y',t_2,s)\,dy'ds \\
B_3 &= -\int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^2} G_2(y,y',t_1,s)\,dy'ds
\end{aligned}
$$

Estimate of B_1. We claim that for $t \in (0,T)$ and $|y_1|$, $|y_0| \leq R(t)$ we have

$$
|B_1| \leq C\left(\|\dot\lambda_1\|_{L^\infty} + \left\|\frac{\lambda_1}{\lambda}\right\|_{L^\infty}\right)\lambda_*(t)^{\nu-1}R(t)^{1-a}\frac{(t_2-t_1)^\gamma}{(\lambda_*(t)R(t))^{2\gamma}} \tag{16.33}
$$

To prove this, let us write

$$
B_1 = B_{1,1} + B_{1,2} + B_{1,3},
$$

where

$$B_{1,1} = \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t_2)-\xi(s)+\lambda(t_2)y-\lambda(s)y'|^2}{4(t_2-s)}) - \exp(-\frac{|\xi(t_1)-\xi(s)+\lambda(t_1)y-\lambda(s)y'|^2}{4(t_2-s)})}{(t_2-s)^2}$$
$$(\lambda_1(t_2)y_i - \lambda_1(s)y_i')f(y',s)\,dy'ds$$

$$B_{1,2} = \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^2} \Big(\frac{\exp(-\frac{|\xi(t_1)-\xi(s)+\lambda(t_1)y-\lambda(s)y'|^2}{4(t_2-s)})}{(t_2-s)^2} - \frac{\exp(-\frac{|\xi(t_1)-\xi(s)+\lambda(t_1)y-\lambda(s)y'|^2}{4(t_1-s)})}{(t_1-s)^2} \Big)$$
$$(\lambda_1(t_2)y_i - \lambda_1(s)y_i')f(y',s)\,dy'ds$$

$$B_{1,3} = (\lambda_1(t_2) - \lambda_1(t_1))y_i \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t_1)y-\lambda(s)y'|^2}{4(t_1-s)})}{(t_1-s)^2} f(y',s)\,dy'ds.$$

We estimate $B_{1,1}$:

$$\begin{aligned}
|B_{1,1}| &\le C \int_0^{t_1-(t_2-t_1)} \frac{1}{(t_2-s)^3} \int_{\mathbb{R}^2} \big| |\xi(t_2)-\xi(s)+\lambda(t_2)y-\lambda(s)y'|^2 - |\xi(t_1)-\xi(s)+\lambda(t_1)y-\lambda(s)y'|^2 \big| \\
&\quad (|\lambda_1(t_2)y| + |\lambda_1(s)y'|)\,|f(y',s)|\,dy'ds \\
&\le C \int_0^{t_1-(t_2-t_1)} \frac{1}{(t_2-s)^3} \int_{\mathbb{R}^2} \big(|\xi(t_2)-\xi(s)+\lambda(t_2)y-\lambda(s)y'|^2 + |\xi(t_1)-\xi(s)+\lambda(t_1)y-\lambda(s)y'|^2 \big) \\
&\quad \cdot \big(|\xi(t_2)-\xi(t_1)| + |\lambda(t_2)-\lambda(t_1)|\,|y| \big) \\
&\quad \cdot (|\lambda_1(t_2)y| + |\lambda_1(s)y'|)\,|f(y',s)|\,dy'ds \\
&\le C(t_2-t_1)(\|\dot\xi\|_{L^\infty} + \|\dot\lambda\|_{L^\infty} R(t)) \int_0^{t_1-(t_2-t_1)} \frac{1}{(t_2-s)^3} \lambda(s)^\nu R(s)^{-a} \\
&\quad \cdot \int_{|y'|\le R(s)} \big(|\xi(t_2)-\xi(s)+\lambda(t_2)y-\lambda(s)y'| + |\xi(t_1)-\xi(s)+\lambda(t_1)y-\lambda(s)y'| \big) \\
&\quad \cdot (|\lambda_1(t_2)y| + |\lambda_1(s)y'|)\,dy'ds
\end{aligned}$$

Let us estimate $B_{1,3}$. Using (16.6) we get

$$\begin{aligned}
|B_{1,3}| &\le C\lambda_*(t)^{\nu-1}R(t)^{3-a}|\lambda_1(t_2) - \lambda_1(t_1)| \\
&\le C\|\dot\lambda_1\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a}R(t)^2(t_2-t_1) \\
&\le C\|\dot\lambda_1\|_{L^\infty}\lambda_*(t)^{\nu-1}R(t)^{1-a}\frac{(t_2-t_1)^\gamma}{(\lambda_*(t)R(t))^{2\gamma}}
\end{aligned}$$

□

16.2. **Derivative with respect to ξ.**

Lemma 16.5. *Assume that*

$$\begin{gathered}
|\dot\xi(t)| \le C, \quad \text{in } (0,T) \\
|\dot\lambda(t)| \le C, \quad \text{in } (0,T) \\
C_1\lambda_*(t) \le \lambda(t) \le C_2\lambda_*(t), \quad \text{in } (0,T) \\
R(t) = \lambda_*(t)^{-\beta}, \quad \beta < \frac{1}{2},
\end{gathered}$$

where $C, C_1, C_2 > 0$. Then there is C such that

$$\begin{aligned}
&|\nabla_x D_\xi \tilde\psi[\lambda,\xi](\xi_1)(y,t) - \nabla_x D_\xi \tilde\psi[\lambda,\xi](\xi_1)(y,T)| \\
&\le C\Big(\Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R} \Big\|_{L^\infty} + \Big\| \frac{\dot\xi_1}{R} \Big\|_{L^\infty} \Big) \lambda_*(t)^{\nu-1}R(t)^{1-a},
\end{aligned} \tag{16.34}$$

for $|y| \le R(t)$, $t \in (0,T)$.

Proof. By Duhamel's formula

$$\psi(x,t)=\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|x-x'|^2}{4(t-s)})}{t-s}\frac{1}{\lambda(s)^2}f(\frac{x'-\xi(s)}{\lambda(s)},s)\,dx'ds.$$

We change variables writing $x=\xi(t)+\lambda(t)y$ and $x'=\xi(s)+\lambda(s)y'$. Then

$$\tilde\psi(y,t)=\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{t-s}f(y',s)\,dy'ds,$$

and

$$\begin{aligned}&D_\xi\tilde\psi[\xi_1](y,t)\\&=-\frac12\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}\\&\qquad\cdot(\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y')\cdot(\xi_1(t)-\xi_1(s))f(y',s)\,dy'ds.\end{aligned}$$

We compute

$$\partial_{y_i}D_\xi\tilde\psi[\xi_1](y,t)=D_1+D_2 \tag{16.35}$$

where

$$\begin{aligned}D_1(y,t)&=-\frac14\lambda(t)\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^3}(\xi_i(t)-\xi_i(s)+\lambda(t)y_i-\lambda(s)y_i')\\&\qquad\cdot(\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y')\cdot(\lambda_1(t)y-\lambda_1(s)y')f(y',s)\,dy'ds\\D_2(y,t)&=-\frac12\lambda(t)\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}(\xi_1^i(t)-\xi_1^i(s))f(y',s)\,dy'ds.\end{aligned}$$

To obtain (16.34) it is sufficient to prove that the functions

$$\begin{cases}g_1(y,t)=-4\dfrac{1}{\lambda(t)}D_1(y,t)\\[2mm] g_2(y,t)=-2\dfrac{1}{\lambda(t)}D_2(y,t)\end{cases}\tag{16.36}$$

satisfy the estimate

$$|g_j(y,t)-g_j(y,T)|\le C\Big(\Big\|\frac{\xi_1(\cdot)-\xi_1(T)}{\lambda R}\Big\|_{L^\infty}+\Big\|\frac{\dot\xi_1}{R}\Big\|_{L^\infty}\Big)\lambda_*(t)^{\nu-1}R(t)^{1-a},\tag{16.37}$$

for $|y|\le R(t)$, $t\in(0,T)$ and $j=1,2,3$. We will do the computation in detail for

$$g_2(y,t)=\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}(\xi_1^i(t)-\xi_1^i(s))f(y',s)\,dy'ds.$$

To prove (16.37) for $j=2$, let us write

$$\begin{aligned}&g_2(y,t)-g_2(y,T)\\&=\int_0^t\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2}(\xi_1^i(t)-\xi_1^i(s))f(y',s)\,dy'ds\\&\quad-\int_0^T\int_{\mathbb{R}^2}\frac{\exp(-\frac{|\xi(T)-\xi(s)+\lambda(T)y-\lambda(s)y'|^2}{4(T-s)})}{(T-s)^2}(\xi_1^i(T)-\xi_1^i(s))f(y',s)\,dy'ds\\&=A_0+A_1+A_2+A_3,\end{aligned}$$

where

$$
\begin{aligned}
A_0 &= (\xi_1^i(t) - \xi_1^i(T)) \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} f(y',s)\, dy'ds \\
A_1 &= \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \Big[\frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} - \frac{\exp(-\frac{|\xi(T)-\xi(s)+\lambda(T)y-\lambda(s)y'|^2}{4(T-s)})}{(T-s)^2} \Big] \\
&\qquad \cdot (\xi_1^i(T) - \xi_1^i(s)) f(y',s)\, dy'ds \\
A_2 &= \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} (\xi_1^i(t) - \xi_1^i(s)) f(y',s)\, dy'ds \\
A_3 &= -\int_{t-(T-t)}^{T} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(T)-\xi(s)+\lambda(T)y-\lambda(s)y'|^2}{4(T-s)})}{(T-s)^2} (\xi_1^i(T) - \xi_1^i(s)) f(y',s)\, dy'ds
\end{aligned}
$$

Estimate of A_0. We claim that

$$
|A_0| \le C \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a}. \tag{16.38}
$$

Indeed, thanks to (16.6) we get

$$
\begin{aligned}
|A_0| &\le C \frac{|\xi_1(t) - \xi_1(T)|}{\lambda(t)R(t)} \lambda_*(t)^{\nu-1} R(t)^{1-a} (\lambda(t) R(t)^2) \\
&\le C \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R(t)^{1-a}.
\end{aligned}
$$

Estimate of A_1. We claim that

$$
\begin{aligned}
|A_1| \le C \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R} \Big\|_{L^\infty} \lambda_*(t)^{\nu-1} R^{1-a} \Big[(\|\dot\xi\|_{L^\infty} + \|\dot\lambda\|_{L^\infty} |y|) \frac{\lambda_*(t)^2 R(t)^2}{(T-t)^{1/2}} \\
+ \frac{\lambda_*(t)^2 R(t)^2}{T-t} \Big].
\end{aligned} \tag{16.39}
$$

Indeed, using (16.8) we find that

$$
A_1 = A_{11} + A_{12} + A_{13}
$$

where

$$
\begin{aligned}
A_{11} &= 2(T-t) \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \int_0^1 \exp(-\frac{|\xi(t_v) - \xi(s) + \lambda(t_v) y - \lambda(s) y'|^2}{4(t_v - s)}) \\
&\qquad \frac{(\xi(t_v) - \xi(s) + \lambda(t_v) y - \lambda(s) y') \cdot (\dot\xi(t_v) + \dot\lambda(t_v) y)}{4(t_v - s)^3} (\xi_1^i(T) - \xi_1^i(s)) f(y',s)\, dv dy' ds \\
A_{12} &= -2(T-t) \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \int_0^1 \exp(-\frac{|\xi(t_v) - \xi(s) + \lambda(t_v) y - \lambda(s) y'|^2}{4(t_v - s)}) \\
&\qquad \frac{|\xi(t_v) - \xi(s) + \lambda(t_v) y - \lambda(s) y'|^2}{4(t_v - s)^4} (\xi_1^i(T) - \xi_1^i(s)) f(y',s)\, dv dy' ds \\
A_{13} &= 2(T-t) \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \int_0^1 \exp(-\frac{|\xi(t_v) - \xi(s) + \lambda(t_v) y - \lambda(s) y'|^2}{4(t_v - s)}) \\
&\qquad \frac{1}{(t_v - s)^3} (\xi_1^i(T) - \xi_1^i(s)) f(y',s)\, dv dy' ds.
\end{aligned}
$$

To estimate A_{11} we use $e^{-|z|^2} z \le C$ and get

$$
\begin{aligned}
|A_{11}| &\le C(T-t) \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \int_0^1 \frac{|\dot\xi(t_v) + \dot\lambda(t_v) y|}{(t_v - s)^{5/2}} |(\xi_1^i(T) - \xi_1^i(s)) f(y',s)| \, dv dy' ds \\
&\le C(T-t)(\|\dot\xi\|_{L^\infty} + \|\dot\lambda\|_{L^\infty}|y|) \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R} \Big\|_{L^\infty} \\
&\quad \cdot \int_0^1 \int_0^{t-(T-t)} \frac{\lambda(s) R(s)}{(t_v - s)^{5/2}} \int_{\mathbb{R}^2} |f(y',s)| \, dy' ds dv
\end{aligned}
$$

But

$$
\int_{\mathbb{R}^2} |f(y',s)| \, dy' \le C \lambda(s)^\nu R(s)^{2-a}. \tag{16.40}
$$

Therefore

$$
\begin{aligned}
\int_0^{t-(T-t)} \frac{\lambda(s) R(s)}{(t_v - s)^{5/2}} \int_{\mathbb{R}^2} |f(y',s)| \, dy' ds &\le \int_0^{t-(T-t)} \frac{\lambda(s)^{\nu+1} R(s)^{3-a}}{(t_v - s)^{5/2}} \\
&\le C \frac{\lambda(t)^{\nu+1} R(t)^{3-a}}{(T-t)^{3/2}}
\end{aligned}
$$

and then

$$
|A_{11}| \le C(\|\dot\xi\|_{L^\infty} + \|\dot\lambda\|_{L^\infty}|y|) \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R} \Big\|_{L^\infty} \frac{\lambda(t)^{\nu+1} R(t)^{3-a}}{(T-t)^{1/2}}. \tag{16.41}
$$

For A_{12} we have, using (16.40),

$$
\begin{aligned}
|A_{12}| &\le (T-t) \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R} \Big\|_{L^\infty} \int_0^1 \int_0^{t-(T-t)} \frac{1}{(t_v - s)^3} \lambda(s) R(s) \int_{\mathbb{R}^2} |f(y',s)| \, dy' ds dv \\
&\le (T-t) \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R} \Big\|_{L^\infty} \int_0^1 \int_0^{t-(T-t)} \frac{\lambda(s)^{\nu+1} R(s)^{3-a}}{(t_v - s)^3} \, ds dv \\
&\le C \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{R} \Big\|_{L^\infty} \frac{\lambda(t)^{\nu+1} R(t)^{3-a}}{T-t}.
\end{aligned} \tag{16.42}
$$

Let us estimate A_{13}:

$$
\begin{aligned}
|A_{13}| &\le 2(T-t) \int_0^{t-(T-t)} \int_{\mathbb{R}^2} \int_0^1 \frac{1}{(t_v - s)^3} |\xi_1(s) - \xi_1(T)| \, |f(y',s)| \, dv dy' ds \\
&\le (T-t) \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{R} \Big\|_{L^\infty} \int_0^1 \int_0^{t-(T-t)} \frac{\lambda(s)^{\nu+1} R(s)^{3-a}}{(t_v - s)^3} \, ds dv \\
&\le C \Big\| \frac{\xi_1(\cdot) - \xi_1(T)}{\lambda R} \Big\|_{L^\infty} \frac{\lambda(t)^{\nu+1} R(t)^{3-a}}{T-t}.
\end{aligned} \tag{16.43}
$$

Combining (16.41), (16.42), and (16.43) we obtain (16.39).

Estimate of A_2. We claim that

$$
|A_2| \le C \Big\| \frac{\dot\xi_1}{R} \Big\|_{L^\infty} \lambda(t)^{\nu-1} R(t)^{1-a}. \tag{16.44}
$$

Indeed, taking $\mu \in (0,1)$, we find that

$$
\begin{aligned}
|A_2| &\le \int_{t-(T-t)}^{t} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)^2} |\xi_1^i(t)-\xi_1^i(s)|\,|f(y',s)|\,dy'ds \\
&\le \Big\|\frac{\dot\xi_1}{R}\Big\|_{L^\infty} \int_{t-(T-t)}^{t} \lambda(s)^\nu R(s)^{1-a} \int_{\mathbb{R}^2} \frac{\exp(-\frac{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}{4(t-s)})}{(t-s)}\,dy'ds \\
&\le C\Big\|\frac{\dot\xi_1}{R}\Big\|_{L^\infty} \int_{t-(T-t)}^{t} \lambda(s)^\nu R(s)^{1-a}(t-s)^{-1} \int_{|y'|\le R(s)} \Big(\frac{t-s}{|\xi(t)-\xi(s)+\lambda(t)y-\lambda(s)y'|^2}\Big)^\mu dy'ds \\
&\le C\Big\|\frac{\dot\xi_1}{R}\Big\|_{L^\infty} \int_{t-(T-t)}^{t} \lambda(s)^{\nu-2\mu} R(s)^{1-a}(t-s)^{-1+\mu} R(s)^{2-2\mu}\,ds \\
&\le C\Big\|\frac{\dot\xi_1}{R}\Big\|_{L^\infty} (T-t)^\mu \lambda(t)^{\nu-2\mu} R(t)^{3-2\mu-a} \\
&\le C\Big\|\frac{\dot\xi_1}{R}\Big\|_{L^\infty} \lambda(t)^{\nu-1} R(t)^{1-a} \lambda(t)^{1-\mu} R(t)^{2-2\mu} \\
&\le C\Big\|\frac{\dot\xi_1}{R}\Big\|_{L^\infty} \lambda(t)^{\nu-1} R(t)^{1-a}
\end{aligned}
$$

Estimate of A_3. Finally we have

$$
|A_3| \le C\Big\|\frac{\dot\xi_1}{R}\Big\|_{L^\infty} \lambda(t)^{\nu-1} R(t)^{1-a}. \tag{16.45}
$$

The proof is similar to that of (16.44) and we omit it.

Combining (16.38), (16.39), (16.44) and (16.45) we obtain the validity of (16.37) for $j=2$.

□

17. The formula for Φ_0

We look for an approximate solution of

$$
\partial_t \Phi_0 = \Delta \Phi_0 - \frac{2\dot p(t)}{r}\begin{bmatrix} e^{i\theta} \\ 0 \end{bmatrix}.
$$

We take Φ_0 of the form

$$
\Phi_0 = \begin{bmatrix} \varphi_0(r,t)e^{i\theta} \\ 0 \end{bmatrix}
$$

and then find that

$$
\partial_t \varphi_0 = \Delta\varphi_0 - \frac{1}{r^2}\varphi_0 - \frac{2\dot p(t)}{r}. \tag{17.1}
$$

First we construct a solution φ to

$$
\partial_t \varphi = \Delta\varphi - \frac{1}{r^2}\varphi + \frac{1}{r}. \tag{17.2}
$$

We look for φ in self-similar form

$$
\varphi(r,t) = t^\alpha q(\frac{r}{\sqrt t}).
$$

Then (17.2) is equivalent to

$$
t^{\alpha-1}\left(\alpha q - \frac12 \xi q'(\xi)\right) = t^{\alpha-1}\left(q'' + \frac1\xi q' - \frac{1}{\xi^2}q\right) + \frac{\sqrt t}{\xi}, \quad \xi = \frac{r}{\sqrt t}.
$$

Then we take $\alpha = \frac12$ and look for q solving

$$
0 = q'' + \frac1\xi q' - \frac{1}{\xi^2} q - \frac12\left(q - \xi q'(\xi)\right) + \frac1\xi. \tag{17.3}
$$

Note that $q_1(\xi) = \xi$ is in the kernel of the homogeneous operator. The other element in the kernel is given by

$$q_2(\xi) = q_1(\xi) \int q_1(x)^{-2} e^{-\int a}\, dx$$

where

$$a(\xi) = \frac{1}{\xi} + \frac{1}{2}\xi,$$

so that

$$e^{\int a} = \xi e^{\frac{1}{4}\xi^2}.$$

Then we can choose

$$q_2(\xi) = \xi \int_\xi^\infty \frac{1}{x^3} e^{-\frac{1}{4}x^2}\, dx.$$

This function decays as $\xi \to \infty$ and $q_2(\xi) \sim \frac{1}{\xi}$ as $\xi \to 0$.

A solution of equation (17.3) is given by

$$q(\xi) = -q_1(\xi) \int q_1(x)^{-2} e^{-\int a} \int \frac{1}{y} e^{\int a} q_1(y)\, dy dx$$

Then

$$q(\xi) = -\xi \int \frac{1}{x^3} e^{-\frac{1}{4}x^2} \int y e^{\frac{1}{4}y^2}\, dy dx$$

We want a solution bounded as $\xi \to 0$ and $q(\xi) \to 0$ as $\xi \to \infty$. We choose

$$\begin{aligned} q(\xi) &= \xi \int_\xi^\infty \frac{1}{x^3} e^{-\frac{1}{4}x^2} \int_0^x y e^{\frac{1}{4}y^2}\, dy dx \\ &= 2\xi \int_\xi^\infty \frac{1}{x^3} e^{-\frac{1}{4}x^2} (e^{\frac{1}{4}x^2} - 1)\, dx \\ &= 2\xi \int_\xi^\infty \frac{1 - e^{-\frac{1}{4}x^2}}{x^3}\, dx. \end{aligned} \tag{17.4}$$

Note that

$$q(\xi) \sim \begin{cases} -\frac{1}{2}\xi \log \xi & \xi \to 0^+, \\ \frac{1}{\xi} & \xi \to \infty. \end{cases}$$

We have then

$$\varphi(r,t) = \sqrt{t} q(\frac{r}{\sqrt{t}}).$$

A solution to (17.1) is given by

$$\varphi_0(r,t) = \int_0^t \dot{g}(s) \varphi(r, t-s)\, ds + g(0)\varphi(r,t), \tag{17.5}$$

where $g(t) = -2\dot{p}(t)$. Indeed,

$$\begin{aligned} (\partial_t - \partial_{rr} - \frac{1}{r}\partial_r - \frac{1}{r^2})\varphi_0 &= \int_0^t \dot{g}(s)(\partial_t - \partial_{rr} - \frac{1}{r}\partial_r - \frac{1}{r^2})\varphi(r, t-s)\, ds \\ &= \frac{1}{r}(g(t) - g(0)) + \frac{1}{r} g(0) \\ &= \frac{g(t)}{r}. \end{aligned}$$

Let us integrate (17.5) by parts

$$\varphi_0(r,t) = g(s)\varphi(r,t-s)\big|_{s=0}^{t} + \int_0^t g(s)\partial_t\varphi(r,t-s)\,ds + g(0)\varphi(r,t)$$
$$= \int_0^t g(s)\partial_t\varphi(r,t-s)\,ds.$$

We now compute, using (17.4)

$$\partial_t\varphi(r,t) = \frac{1}{2}\frac{1}{\sqrt{t}}\left(q(\xi)-\xi q'(\xi)\right)$$
$$= \frac{1}{\sqrt{t}}\frac{1-e^{-\frac{1}{4}\xi^2}}{\xi}$$
$$= \frac{1-e^{-\frac{r^2}{4t}}}{r}.$$

Therefore

$$\varphi_0(r,t) = -2\int_0^t p(s)\frac{1-e^{\frac{r^2}{4(t-s)}}}{r}\,ds$$
$$= -2r\int_0^t \frac{\dot p(s)}{t-s}\frac{1-e^{\frac{r^2}{4(t-s)}}}{\frac{r^2}{t-s}}\,ds$$
$$= -r\int_0^t \frac{\dot p(s)}{t-s}k\Big(\frac{r^2}{t-s}\Big)\,ds$$

where

$$k(\zeta) = 2\frac{1-e^{-\frac{\zeta}{4}}}{\zeta}$$

$$\Phi_0(r,t) = -\int_0^t \frac{\dot p(s)}{t-s}k\Big(\frac{r^2}{t-s}\Big)\,ds\begin{bmatrix} re^{i\theta}\\ 0\end{bmatrix}.$$

We regularize this function by modifying the definition of Φ_0 as follows:

$$\Phi_0(r,t) = -\int_0^t \frac{\dot p(s)}{t-s}k\Big(\frac{r^2+\lambda(t)^2}{t-s}\Big)\,ds\begin{bmatrix} re^{i\theta}\\ 0\end{bmatrix}.$$

Acknowledgements:

The research of J. Wei is partially supported by NSERC of Canada. J. Dávila and M. del Pino have been supported by grants Fondecyt 1130360, 1150066, Fondo Basal CMM and Millenium Nucleus CAPDE NC130017.

References

[1] S.B. Angenent, J. Hulshof and H. Matano, *The radius of vanishing bubbles in equivariant harmonic map flow from D^2 to S^2*. SIAM J. Math. Anal. 41 (2009), 1121–1137.

[2] M. Bertsch, R. Dal Passo, and R. van der Hout, *Nonuniqueness for the heat flow of harmonic maps on the disk.* Arch. Rat. Mech. Anal. 161 (2002), no. 2, 93–112.

[3] J.B. van den Berg, J. Hulshof and J.R. King, *Formal asymptotics of bubbling in the harmonic map heat flow.* SIAM J. Appl. Math. 63 (2003), 1682–1717.

[4] J. B. van den Berg, J. F. Williams, *(In-)stability of singular equivariant solutions to the Landau-Lifshitz-Gilbert equation.* European J. Appl. Math. 24 (2013), no. 6, 921–948.

[5] K.-C. Chang, *Heat flow and boundary value problem for harmonic maps.* Ann. Inst. H. Poincare Anal. Non Lineaire 6 (1989), no. 5, 363–395.

[6] K.-C. Chang, W.Y. Ding and R. Ye, *Finite-time blow-up of the heat flow of harmonic maps from surfaces.* J. Differential Geom. 36 (1992), no. 2, 507–515.

[7] J. Chen and G. Tian, *Compactification of moduli space of harmonic mappings.* Comment. Math. Helv. 74 (1999), no. 2, 201–237.

[8] Y.M. Chen, M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps.* Math. Z. 201 (1989), no. 1, 83–103.
[9] C. Cortázar, M. del Pino, M. Musso, *Green's function and infinite-time bubbling in the critical nonlinear heat equation.* Preprint arXiv:1604.07117.
[10] P. Daskalopoulos, M. del Pino, N. Sesum, *Type II ancient compact solutions to the Yamabe flow.* J. Reine Angew Math., to appear.
[11] M. del Pino, M. Kowalczyk, J. Wei *Concentration on curves for nonlinear Schrödinger equations.* Comm. Pure Appl. Math. 60 (2007), no. 1, 113–146.
[12] M. del Pino, M. Kowalczyk, J. Wei *On De Giorgi's conjecture in dimension* $N = 9$. Ann. of Math. (2) 174 (2011), no. 3, 1485–1569.
[13] W. Ding, G. Tian, *Energy identity for a class of approximate harmonic maps from surfaces.* Comm. Anal. Geom. 3 (1995), no. 3-4, 543–554.
[14] J. Eells Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. 86 (1964), 109–160.
[15] A. Freire, *Uniqueness for the harmonic map flow in two dimensions.* Calc. Var. Partial Differential Equations 3 (1995), no. 1, 95–105.
[16] M. Guan, S. Gustafson and T. P. Tsai, *Global existence and blow-up for harmonic map heat flow.* J. Differential Equations 246 (2009), no. 1, 1–20.
[17] F.H. Lin, *Gradient estimates and blow-up analysis for stationary harmonic maps.* Annal Math. 149 (1999), 785–829.
[18] F.H. Lin, C.Y. Wang, *Energy identity of harmonic map flows from surfaces at finite singular time.* Calc. Var. Partial Differential Equations 6 (1998), 369–380.
[19] F.H. Lin, C.Y. Wang, *The analysis of harmonic maps and their heat flows.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.
[20] G.M. Lieberman, *Second order parabolic differential equations.* World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[21] F. Merle, H. Zaag,*Stability of the blow-up profile for equations of the type* $u_t = \Delta u + |u|^{p-1}u$. Duke Math. J. 86 (1997), no. 1, 143–195.
[22] F. Merle, P. Raphaël, I. Rodnianski, *Blowup dynamics for smooth data equivariant solutions to the critical Schrodinger map problem.* Invent. Math. 193 (2013), no. 2, 249–365.
[23] J. Qing,*On singularities of the heat flow for harmonic maps from surfaces into spheres.* Comm. Anal. Geom. 3 (1995), no. 1-2, 297–315.
[24] J. Qing, *A remark on the finite time singularity of the heat flow for harmonic maps.* Calc. Var. Partial Differential Equations 17 (2003), no. 4, 393–403.
[25] Qing, J., Tian, G. *Bubbling of the heat flows for harmonic maps from surfaces.* Comm. Pure Appl. Math. 50 (1997), no. 4, 295–310.
[26] P. Raphaël, *Existence and stability of a solution blowing up on a sphere for an* L^2*-supercritical nonlinear Schrödinger equation.* Duke Math. J. 134 (2006), no. 2, 199–258.
[27] P. Raphaël, I. Rodnianksi, *Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems.* Publ. Math. Inst. Hautes Études Sci. 115 (2012), 1–122.
[28] P. Raphaël, R. Schweyer, *Stable blowup dynamics for the 1-corotational energy critical harmonic heat flow.* Comm. Pure Appl. Math. 66 (2013), no. 3, 414–480.
[29] P. Raphaël, Schweyer, *Quantized slow blow-up dynamics for the corotational energy-critical harmonic heat flow.* Anal. PDE 7 (2014), no. 8, 1713–1805.
[30] I. Rodnianski, J. Sterbenz, *On the formation of singularities in the critical* $O(3)$*-model.* Ann. of Math. (2) 172 (2010), no. 1, 187–242.
[31] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces.* Comment. Math. Helv. 60 (1985), no. 4, 558–581.
[32] P. M. Topping, *Winding behaviour of finite-time singularities of the harmonic map heat flow.* Math. Z. 247 (2004), 279–302.
[33] P. M. Topping, *Repulsion and quantization in almost-harmonic maps, and asymptotics of the harmonic map flow.* Annals of Math. 159 (2004), 465–534.
[34] P. M. Topping, *Reverse bubbling and nonuniqueness in the harmonic map flow.* Int. Math. Res. Not. 2002, no. 10, 505–520.
[35] R. van der Hout, *On the nonexistence of finite time bubble trees in symmetric harmonic map heat flows from the disk to the 2-sphere.* J. Differential Equations 192 (2003), no. 1, 188–201.
[36] C.Y. Wang, *Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets* Houston J. Math. 22(3) (1996), 559–590.

Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.
Email address: jdavila@dim.uchile.cl

Departamento de Ingeniería Matemática and Centro de Modelamiento Matemático (UMI 2807 CNRS), Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.
Email address: delpino@dim.uchile.cl

Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2
Email address: jcwei@math.ubc.ca