# INFINITELY MANY POSITIVE SOLUTIONS FOR NONLINEAR SCHRÖDINGER SYSTEM WITH NONSYMMETRIC FIRST ORDER

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ABSTRACT. We consider the following nonlinear Schrödinger system:

$$\begin{cases} -\Delta u + \lambda_1 u = u^3 + \beta u v^2 & \text{in } \Omega \\ -\Delta v + \lambda_2 v = v^3 + \beta v u^2 & \text{in } \Omega \\ u > 0, v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $\lambda_1, \lambda_2 > 0$ and $\beta \in \mathbb{R}$ is a coupling number. We prove that for $\beta \leq -1$ this problem has infinitely many positive solutions.

## 1. Introduction

We consider the following nonlinear Schrödinger system:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 & \text{in } \Omega \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta v u^2 & \text{in } \Omega \\ u > 0, v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain ($N \leq 3$), $\lambda_1, \lambda_2, \mu_1, \mu_2 > 0$ are parameters and $\beta \in \mathbb{R}$ is a coupling number.

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This type of system arises in the study of solitary wave solutions of the following two component Schrödinger system (also called Gross-Pitaevskii equations):
(1.2)

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1 &= \frac{\bar{h}^2}{2m}\Delta\Phi_1 - V_1(x)\Phi_1 + \mu_1|\Phi_1|^2\Phi_1 + \beta|\Phi_2|^2\Phi_1, \text{ for } x\in\Omega, t>0 \\ -i\frac{\partial}{\partial t}\Phi_2 &= \frac{\bar{h}^2}{2m}\Delta\Phi_2 - V_2(x)\Phi_2 + \mu_2|\Phi_2|^2\Phi_2 + \beta|\Phi_1|^2\Phi_2, \text{ for } x\in\Omega, t>0 \\ \Phi_i &= \Phi_i(x,t)\in\mathbb{C}, i=1,2 \\ \Phi_i(x,t) &= 0, \text{ for } x\in\partial\Omega, t>0, i=1,2. \end{cases}$$

The above system models a binary mixture of Bose-Einstein condensate in two different hyperfine states $|1>$ and $|2>$ (see [13]). Physically, $\Phi_1$ and $\Phi_2$ are the corresponding condensate amplitudes, $\Omega\subset\mathbb{R}^N$ is the domain for condensate dwelling, $\bar{h}$ is the Planck constant divided by $2\pi$, $m$ is atom mass and $V_j$ is the trapping potential for the $j-th$ hyperfine state. $\mu_j$ and $\beta$ are the interaction of the states. The sign of $\beta$ determines the interaction of the states $|1>$ with $|2>$. When $\beta<0$, it means that the interaction is repulsive and for $\beta>0$ the interaction is attractive.

System (1.1) has attracted much attraction in recent years. Many results concerning the existence and multiplicity of solutions are obtained, under the suitable assumptions on the parameters $\beta$ and $\lambda's$. See for example [1, 2, 7, 8, 9, 10, 12, 14, 17, 18, 19, 20, 21, 22, 23, 31, 32, 33, 27, 28] and references therein.

In this paper, we assume that $\mu_1,\mu_2>0$, which implies that the self interaction of the single states $|1>$ and $|2>$ are attractive. We also consider the case of $\beta<0$, which implies that the interaction is repulsive.

When $N\le 3$ and $\beta<0$, the existence of at least one least energy solution can be proved easily ([18]). For the full symmetric case, that is, $\mu_1=\mu_2,\lambda_1=\lambda_2$, the existence of infinitely many positive solutions are obtained by Wei-Weth [32] (in the radial case) and Dancer-Wei-Weth [12] (in the general domain case). The basic idea of [12] and [32] is to find suitable Ljusternik-Schnirlman category theory for obtaining positive solutions for elliptic systems. To this end, they observed that when $\lambda_1=\lambda_2$ and $\mu_1=\mu_2$ system (1.1) is invariant under the reflection $(u,v)\to\sigma(u,v)=(v,u)$. (This is called $Z_2$ symmetry.) This invariance is essential for the multiplicity results of [12] and [32]. (Tian and Wang [29] generalized to $Z_N$ symmetry and obtained infinitely many positive solutions for symmetric $N$ elliptic systems.) On the other hand, when $\lambda_1=\lambda_2$, and $\beta\in(-\sqrt{\mu_1\mu_2},0)$ system (1.1) admits a special solution of the type $(c_1u,c_2u)$, where $c_1>0,c_2>0$ and $u$ is a solution of the single scalar equation. Using this fact and bifurcation theory, Bartsch-Dancer-Wang [7] proved the existence of infinitely many positive radial solutions to (1.1) when $\beta\le-\sqrt{\mu_1\mu_2}$ and $\Omega$ is a ball. As far as we know, the question of the existence of infinitely many positve solution to system (1.1) when $\lambda_1 \neq \lambda_2$ is largely open. In this paper we partially solve the question by assuming that

$$\mu_1 = \mu_2, \lambda_1 \neq \lambda_2. \tag{1.3}$$

In other words, we asssume symmetry in the leading order term but allow non-symmetry in the lower order term. Thus by rescaling we consider the following problem

$$\begin{cases} -\Delta u + \lambda_1 u = u^3 + \beta u v^2 & \text{in } \Omega \\ -\Delta v + \lambda_2 v = v^3 + \beta v u^2 & \text{in } \Omega \\ u > 0, v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Regarding (1.4) we prove the following.

**Theorem 1.1** Let $N \leq 2$ and (1.3) hold.

*(a) If $\beta \leq -1$, then system (1.4) admits a sequence $(u_k, v_k)_k$ of solution with $||u_k||_{L^\infty(\Omega)} + ||v_k||_{L^\infty(\Omega)} \to \infty$;*

*(b) For any positive integer $k$, there exist number $\beta_k > -1$ and such that for $\beta < \beta_k$, system (1.4) has at least $k$ pairs $(u, v), (v, u)$ of solutions.*

The idea of our proof is by applying the perturbation arguments of single scalar equations, which was originally due to Bahri-Lions [4] (see also Rabinowitz [24], Bahri-Berestycki [5, 6]).

Let us recall that for single scalar equation

$$-\Delta u = f(x, u) \text{ in } \Omega, \;\; u = 0 \text{ on } \partial\Omega \tag{1.5}$$

it is known that when $f(x, u) = -f(x, -u)$, problem (1.5) has infinitely many solutions. A long-standing open question is whether or not there are infinitely many solutions for general $f(x, u)$. Bahri-Lions [4] showed that it is possible to obtain infinitely many solutions if $f(x, u) = g(x, u) + h(x, u)$ where $g(x, u)$ is odd and $h(x, u)$ is lower order terms.

We follow the idea of [4]. To this end we need to make use of some of properties associated with the symmetric functional corresponding to the full *symmetric* system. (Here and throughout the paper, we say that the system is full symmetric if the system is invariant under the reflection $(u, v) \to \sigma(u, v) = (v, u)$. We also call the energy functional corresponding to the system a *symmetric* functional.) Following the idea of [4] and [5], our proof relies on appropriate estimates on the growth of some critical values of the unperturbed problem (full *symmetric* problem in our situation). These estimates are obtained through a general abstract result which is obtained by the connection of Morse index and critical values. The restriction $N = 2$ is needed for these estimates. We will show that the critical values $C_k$ of the *symmetric* functional are *stable* in some sense. It means that after the perturbation, the level sets of the two functionals (*symmetric* functional and the perturbed functional) are not too far from each other. As a result, if $C_k$ grows fast enough, one may hope to get a critical value of the perturbed functional, and therefore to get the solution of the original problem.

The organization of the paper is as follows: In Section 2, we will derive the estimates concerning the critical values of *symmetric* functional on an Nehari manifold $M$ (see Section 2 for the definition of $M$). The proof of Theorem 1.1 will be delayed in Section 3.

## 2. Estimates via Morse index

Based on the idea of Bahari and Lions [4], in this section, we first study the growth of critical value for the *symmetric* system. Without loss of generality, we may assume $\lambda_1 = \lambda_2 = 1, \mu_1 = \mu_2 = 1$. We consider the following system:

$$\begin{cases} -\Delta u + u = u^3 + \beta u v^2 & \text{in } \Omega \\ -\Delta v + v = v^3 + \beta v u^2 & \text{in } \Omega \\ u > 0, v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Let $H := H_0^1(\Omega) \times H_0^1(\Omega)$ be equipped with the norm $||(u,v)|| = ||u||_{H_0^1(\Omega)} + ||v||_{H_0^1(\Omega)}$. We consider the *symmetric* functional $E \in C^2(H, \mathbb{R})$ defined by:

$$E(u,v) = \frac{1}{2}(||u||^2_{H_0^1(\Omega)} + ||v||^2_{H_0^1(\Omega)}) - \frac{1}{4}\int_\Omega (|u|^4 + |v|^4)dx - \frac{\beta}{2}\int_\Omega u^2 v^2 dx, \tag{2.2}$$

where and in the following of the paper, $||u||^2_{H_0^1(\Omega)} = \int_\Omega (|\nabla u|^2 + |u^2|)dx, u \in H_0^1(\Omega)$. Moreover, for a function $u \in L^s(\Omega)$, we denote by $|u|_s$ the usual $L^s$ norm of $u$.

We first define an Nehari type manifold:

$$\begin{aligned} M: &= \{(u,v) \in H, u, v \neq 0 | E_u(u,v)u = 0, E_v(u,v)v = 0\} \\ &= \{(u,v) \in H, u, v \neq 0 | ||u||^2_{H_0^1(\Omega)} - \beta\int_\Omega u^2v^2dx = \int_\Omega |u|^4 dx, \\ &\quad ||v||^2_{H_0^1(\Omega)} - \beta\int_\Omega u^2v^2dx = \int_\Omega |v|^4dx\}. \end{aligned}$$

Then it is easy to see that all nontrivial critical points $(u,v)$ of $E$ are contained in $M$. Moreover by Lemma 4.2 in [12], we deduce that

(i) $M$ is a $C^2$ submanifold of $H$ with codimension two.

(ii) If $(u,v)$ is a critical point of the restriction of the functional $E$ to $M$, then $(u,v)$ is a nontrivial critical point of $E$ in $H$.

(iii) $E(u,v) = \frac{1}{4}(||u||^2_{H^1_0(\Omega)} + ||v||^2_{H^1_0(\Omega)})$, for $(u,v) \in M$.

(iv) $E_M := E|_M : M \to \mathbb{R}$ satisfies the Palais-Smale condition ((P.S) condition in short).

We denote by $E^*(u,v) := E|_M, K_c := \{(u,v) \in M | E^*(u,v) = c, (E^*)'(u,v) = 0\}$, and $E^*_c := \{(u,v) \in M | E^*(u,v) \leq c\}$. Note that for every $c \in \mathbb{R}$ the functional $E^*$ and the sets $M, K_c, E^*_c$ are invariant with respect to the reflection $\sigma : H \to H, \sigma(u,v) = (v,u)$.

Obviously (2.1) is invariant under the reflection $(u,v) \to \sigma(u,v) = (v,u)$. This invariance is essential for the multiplicity result of [12] which relies on an invariant of Lyusternik-Schirelman theory on the Nehari manifold $M$. The importance of this manifold is that it contains all solutions of (2.1). Moreover, it is invariant under the reflection $\sigma$, and $\sigma$ has no fixed points in $M$ if $\beta \leq -1$.

**Definition 2.1** *For any closed $\sigma-$ invariant subset $A \subset M$, we define the genus $\gamma(A) = \inf\{k \in \mathbb{N}|$ there exists a continuous map $h : A \to \mathbb{R}^k \backslash \{0\}$ with $h(\sigma(u,v)) = -h(u,v)$ for all $(u,v) \in A\}$. If no such map $h$ exists, we set $\gamma(A) = \infty$; In particular, if $A$ contains a fixed point of $\sigma, \gamma(A) = \infty$, and clearly, we have $\gamma(\emptyset) = 0$.*

For reader's convenience, we list some properties of the genus $\gamma$ (see [12]).

**Lemma 2.2** *Let $A, B \subset M$ be closed and $\sigma-$invariant. It holds that:*

*(i) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*

*(ii) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*

*(iii) If $h : A \to M$ is continuous and $\sigma$ equivariant, then $\gamma(A) \leq \gamma(\overline{h(A)})$.*

*(iv) If $A$ is compact and does not contain the fixed point of $\sigma$. Then $\gamma(A) < \infty$, and there exists a relative open $\sigma-$invariant neighborhood $N$ of $A$ in $M$ such that $\gamma(A) = \gamma(\bar{N})$.*

*(v) If $S$ is the boundary of a bounded symmetric neighborhood of zero in $k-$dimensional normal vector space and $\psi : S \to M$ is a continuous map satisfying $\psi(-u) = \sigma(\psi(u))$, then $\gamma(\psi(S)) \geq k$.*

Now let us define the nondecreasing sequence of Lyustermik-Schnirelmann type critical levels associated to the genus $\gamma$ by:

$$C_k := \inf\{c \in \mathbb{R} | \gamma(E^*_c) \geq k, k \in \mathbb{N}\}. \tag{2.3}$$

Set

$$c(\beta) := \inf\{E(u,v) | (u,v) \in M \text{ is a fixed point of } \sigma\}.$$

Then it follows from [12] that :

(1) $c(\beta)=\infty$ for $\beta\leq -1$;
(2)For every $k\in\mathbb{N}$, $C_k<\infty$ is bounded independent of $\beta<0$;
(3)$C_k\to\bar{c}$ as $k\to\infty$ with $c(\beta)\leq\bar{c}\leq\infty$;
(4) If $C_k<\infty$, then $K_{C_k}\neq\emptyset$ and $E_{C_k}$ contains at least $k$ pairs $(u,v),(v,u)$ of critical points of $E$.

In order to establish a connection between Morse indices and min-max critical value, we introduce the following:

**Definition 2.3** *Let $(u_0,v_0)$ be a solution of the full symmetric system (2.1). Set*

$$\begin{aligned}\partial_{\mathbf{\Delta}} &:= \begin{pmatrix} -\Delta, & 0 \\ 0, & -\Delta \end{pmatrix} + \begin{pmatrix} 1+3u_0^2+\beta v_0^2, & 2\beta u_0 v_0 \\ 2\beta u_0 v_0, & 1+3v_0^2+\beta u_0^2, \end{pmatrix}\\ &:= \mathbf{\Delta}+\mathbf{V}\end{aligned}$$

We define the general Morse index of the functional $E$ at $(u_0,v_0)$ to be the dimension of the negative and null eigenspace of the operator $\partial_{\mathbf{\Delta}}$ acting on the space $L^2(\Omega)\times L^2(\Omega)$. We denote this finite integer by $\mu(u_0,v_0)$.

**Remark 2.4** *Note that the Hession $D^2E(u,v)$ of the functional $E$ with $(u,v)$ being a critical point of $E$ is Fredholm type $(Id+K)$ with $K$ is a compact operator. Therefore, the general Morse index $\mu(u,v)$ at $(u,v)$ is finite and this is remains true after small perturbation of the functional.*

**Definition 2.5** *We say $(u,v)$ is a non-degenerate critical point of $E$ if the operator $\partial_{\mathbf{\Delta}}$ is invertible.*

As we mentioned in the introduction, in order to use the perturbation arguments, one of the key points is the appropriate estimate on the growth of some of the critical values for the full *symmetric* problem (or unperturbed problem). The following proposition provides a lower bound on the Morse index $\mu(u,v)$ of the critical point $(u,v)$.

**Proposition 2.6** *For any $k\in\mathbb{N}$, there exists a critical point $(u_k,v_k)$ of $E^*$ such that $E^*(u_k,v_k)=C_k$ and $\mu(u_k,v_k)\geq k$, where $C_k$ is defined as in* (2.3).

To prove Proposition 2.6, we need the following invariant deformation lemma due to [12].

**Lemma A** *Let $c\in\mathbb{R}$, and let $N\subset M$ be a relative open $\sigma-$ invariant neighborhood of the critical set $K_c$. Then there exists $\epsilon>0$ and a $C^1$ deformation $\eta:[0,1]\times M^{c+\epsilon}\setminus N\to M^{c+\epsilon}$ such that for all $(u,v)\in M^{c+\epsilon}\setminus N$ and $s\in[0,1]$,*

*it holds:*
(2.4)
$$\eta(0,(u,v))=(u,v), \eta(1,(u,v))\in M^{c-\epsilon} \text{ and } \sigma[\eta(s,(u,v))]=\eta(s,\sigma(u,v))$$

**Proof of Proposition 2.6.** We may first assume that all the critical points of $E^*$ at the energy level $C_k$ are non-degenerate (this can be completed through an application of Sard-Smale's theorem to $E^*$ to be considered as a functional on $M/\mathbb{Z}_2$, the quotient space of $M$ under the map $\sigma(u,v)=(v,u)$ (see [3] and [5]).

Assume $(u_0,v_0)$ is a non-degenerate critical point of $E^*$ at the energy level $C_k$. For the sake of simplicity, we may assume that $(u_0,v_0)$ is the only critical point at the energy level $E^*(u_0,v_0)$ (In the case when there are other critical points please see Remark 2.7 below). By definition 2.5, the Hession of $E^*$ at $(u_0,v_0)$ is invertiable, i. e. the operator $\partial_{\mathbf{\Delta}}$ defines an isomorphism between $T_{(u_0,v_0)}M$, the tangent space to $M$ at $(u_0,v_0)$ and its dual. At the same time, by Remark 2.4, we may assume that $(u_0,v_0)$ has finite Morse index $\mu:=\mu(u_0,v_0)$. Thus, by Morse theory there exists a coordinate chart $\varphi$ around $(u_0,v_0)$ descried by $(\mathbf{U},\mathbf{V})$ with $U=((u_1,v_1),...,(u_\mu,v_\mu))$ such that the functional
$$E^*(u,v)=E^*(u_0,v_0)+||\mathbf{V}||^2-||\mathbf{U}||^2,$$
where $\varphi((u,v))=(\mathbf{U},\mathbf{V}), \varphi((u_0,v_0))=(\mathbf{O},\mathbf{O}), ||\mathbf{U}||^2+||\mathbf{V}||^2$ is the norm in the Hilbert space $H$. For simplicity, we may assume $E^*((u_0,v_0))=0$. Let $H_\mu$ be the subspace of $H$ such that $H_\mu=\mathrm{span}\{E_1,...,E_\mu\}$, and $H_\mu^\perp$ is its orthogonal completes , $(E_1,...,E_\mu)$ is the first $\mu$ orthogonal basis of the space $H$ with coordinates $(\mathbf{U},\mathbf{U})$.

Let $D_\mu^\epsilon:=\{(u,v)\in H_\mu|||(u,v)||^2<\epsilon\}$ and $\partial D_\mu^\epsilon:=\{(u,v)\in H_\mu, ||(u,v)||^2=\epsilon\}$. Then by using the invariant Morse lemma A, we have a local deformation of the set $E^*_{E^*((u_0,v_0))+\epsilon}$ onto the set $E^*_{E^*((u_0,v_0))-\epsilon}\cup D_\mu^\epsilon\cup\partial D_\mu^\epsilon$. Moreover, we have
$$E^*_{E^*((u_0,v_0))-\epsilon}\cap D_\mu^\epsilon=\partial D_\mu^\epsilon$$
and
$$E^*(u,v)\leq E^*(u_0,v_0) \text{ for any } (u,v)\in D_\mu^\epsilon.$$
Note that in our case the critical points appear in pairs $((u_0,v_0),\sigma(u_0,v_0))$, the retraction by deformation can be take to be $\sigma-$invariant. More precisely, for $\epsilon>0$ small enough, we have a continuous deformation retractor $\eta:[0,1]\times E^*_{C_k+\epsilon}\to E^*_{C_k+\epsilon}$ such that

(i) $\eta(0,x)=x$;
(ii) $\sigma(\eta(s,(u,v)))=\eta(s,\sigma(u,v))$;
(iii) $\eta(s,(u,v))=(u,v)$, for all $(u,v)\in E^*_{C_k-\epsilon}\cup D_\mu^\epsilon$;

(iv) $\eta(1,(u,v)) \in E^*_{C_k-\epsilon} \cup D^\epsilon_\mu$.

Recall that $C_k := \inf\{c \in \mathbb{R} | \gamma(E^*_c) \geq k, k \in \mathbb{N}\}$, we may choose $c_0$ with $\gamma(E^*_{c_0}) \geq k$ such that

$$E^*(u,v) \leq C_k + \epsilon.$$

Let $B := \eta(1, E^*_{c_0})$, then by (iv), we have $B \subset E^*_{C_k-\epsilon} \cup D^\epsilon_\mu$ and since $c_0 \leq C_k$, $\gamma(B) \geq k$. Set $\tilde{B} := \overline{B \setminus D^\epsilon_\mu} \cup \partial D^\epsilon_\mu$, it follows from the definition of $C_k$, we have $\gamma(\tilde{B}) < k$. As a result, there is a continuous map $h : \tilde{B} \to \mathbb{R}^{k-1} \setminus \{0\}$ with $h(\sigma(u,v)) = -h(u,v)$.

In the following we proceed using a contradiction argument. Assume that the Morse index $\mu < k$. Note that the $\mathbb{R}^{k-1} \setminus \{0\} \cong S^{k-1}$, the spheres of $\mathbb{R}^{k-1}$, and the homotopy groups of $S^{k-1}$ of order strictly less than $k-1$ are zero. Hence the map $h$ can be extended from $\partial D^\epsilon_\mu$ to $D^\epsilon_\mu$. Thus we obtain a continuous map $\tilde{h} : \tilde{B} \cup D^\epsilon_\mu \cup \partial D^\epsilon_\mu \to S^{k-1}$ (with $h(\sigma(u,v)) = -h(u,v)$). So we have a map $\tilde{h}$ from $B$ to $S^{k-1}$ with $h(\sigma(u,v)) = -h(u,v)$, whis is a contradiction with $\gamma(B) \geq k$. Therefore we have $\mu \geq k$. □

**Remark 2.7** *In the case when there are other critical points, which are all non-degenerate, hence of finite number since the functional $E^*$ satisfies the (P.S.) condition, without loss of generality, we may assume that there are $m$ critical points $(u_1,v_1),...,(u_m,v_m)$ with Morse index $\mu_1,...,\mu_m$. We then have a retraction by deformation of $E^*_{C_k+\epsilon}$ onto $E^*_{C_k-\epsilon} \cup D^\epsilon_{\mu_1} \cup ... \cup D^\epsilon_{\mu_m}$ satisfying*

$$E^*_{C_k-\epsilon} \cap D^\epsilon_{\mu_i} = \partial D^\epsilon_{\mu_i}, i = 1,2,...,m$$

*and*

$$E^*(u,v) \leq C_k, \text{ for all } (u,v) \in D^\epsilon_{\mu_i}, i = 1,2,...,m,$$

*where the definition of the sets $D^\epsilon_{\mu_i}$ is the same as $D^\epsilon_\mu$. Then by repeating same arguments as we did in the proof of Proposition 2.6 , we obtain $\sup \mu_i \geq k, i = 1,2,...,k$, which again implies that the desired result.* □

**Proposition 2.8** *Let $(u,v) \in H$ be the critical point of the functional $E^*$, and $\tilde{\mu}(u,v)$ be the dimension of the negative and null eigenspace of the operator $\partial_{\mathbf{\Delta}}$ with domain $H^2(\Omega) \cap H^1_0(\Omega) \times H^2(\Omega) \cap H^1_0(\Omega)$. Then we have*

$$\tilde{\mu}(u,v) \leq C\Big(\int_\Omega (|u|^q + |v|^q)dx\Big)^{\frac{1}{\gamma}}, \text{ for any } q \in (2,\infty), \gamma > 1. \tag{2.5}$$

*where $C$ is a constant independent of $(u,v)$.*

**Proof.** Suppose that $(u_0,v_0)$ is a critical point of $E^*$. Recall that the Morse index of the $(u_0,v_0)$ is defined as the dimension of the negative and null eigenspace of the operator $\partial_{\boldsymbol{\Delta}} = -\partial_\Delta + P(u_0, v_0)$ acting on the space $\tilde{H} := [H^2(\Omega) \cap H_0^1(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)]$, where

$$-\partial_\Delta = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}$$

$$P(u_0, v_0) = \begin{pmatrix} 1 - 3u_0^2 - \beta v_0^2 & 2\beta u_0 v_0 \\ 2\beta u_0 v_0 & 1 - 3v_0^2 - \beta u_0^2 \end{pmatrix}.$$

Since we are working in the space $H := H_0^1(\Omega) \times H_0^1(\Omega)$, and $\Omega$ is a bounded domain, the operator $\partial_{\boldsymbol{\Delta}}$ is of Fredholm type, and the operator $P(u_0, v_0)$ is compact. Therefore, the desired result in Proposition 2.8 is an easy consequence of the celebrated semi-classical inequality due to Simon [26] for the case of $N = 2$ (see also Cwickel [11], Lieb [16] and Rosenbljum [25] for $N \geq 3$). It states that for $N = 2$, if $V$ has bounded support satisfying $V \in L^\gamma(\mathbb{R}^N)$ for any $\gamma > 1$, then the number of negative or zero eigenvalues (counted with their multiplicity) of the Schrödinger operator $(-\Delta + V)$ on $L^2(\mathbb{R}^N)$ is bounded by $C_0(\int_{\mathbb{R}^N} |V^-|^\gamma dx)^{\frac{1}{\gamma}}$ for some constant $C_0$ dependent on $\gamma$ and on the measure of the support of $V$. Applying the above result, we obtain that there exist a constant $C$ such that

$$\tilde{\mu}(u_0, v_0) \leq C \int_\Omega (|u_0|^q + |v_0|^q) dx, \text{ for any } \ q = 2\gamma, \forall \gamma \in (1, \infty).$$

□

By using Proposition 2.6 and Proposition 2.8, we are ready to prove the following:

**Proposition 2.9** *There exists a positive constant $C_1$ such that, for all $k \in \mathbb{N}$, it holds that*

$$C_k \geq C_1 k^2.$$

**Proof.** By Proposition 2.6, we may assume $(u_k, v_k)$ is the critical point of $E^*$ at the energy level $C_k$ such that the Morse index $\mu(u_k, v_k)$ of the critical point $(u_k, v_k)$ satisfying $\mu(u_k, v_k) \geq k$. On the other hand, by Proposition 2.8, we have

$$\mu(u_k, v_k) \leq C(\int_\Omega (|u_k|^{2\gamma} + |v_k|^{2\gamma}) dx)^{\frac{1}{\gamma}}, \text{ for any } \gamma \in (1, \infty),$$

and hence

$$k \leq C(\int_\Omega (|u_k|^{2\gamma} + |v_k|^{2\gamma}) dx)^{\frac{1}{\gamma}}, \text{ for any } \gamma \in (1, \infty).$$

It follows from Hölder inequality,

$$k \le C\left(\left(\int_\Omega (|u_k|^4+|v_k|^4)dx\right)^{\frac{2\gamma}{4}}\right)^{\frac{1}{\gamma}}, \text{ for any } \gamma \in (1,\infty). \tag{2.6}$$

On the other hand, we observe that $C_k = E^*(u_k, v_k) = \frac{1}{4}(||u_k||^2 + ||v_k||^2)$. Hence, we deduces from (2.6) that there exists a constant $C_1$ such that

$$C_k \ge C_1 k^2.$$

□

## 3. Perturbation Functional and Proof of Theorem 1.1

We consider the following perturbed system

$$\begin{cases} -\Delta u + u = u^3 + \beta v^2 u + \lambda u & \text{in } \Omega \\ -\Delta v + v = v^3 + \beta u^2 v - \lambda v & \text{in } \Omega \\ u > 0, v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\lambda > 0$ is considered as a perturbation parameter. We use the same notation as in the previous section and define the perturbed functional on $H := H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$E^\lambda = E + \frac{1}{2}\lambda^2 \int_\Omega (u^2 - v^2)dx, \tag{3.2}$$

where $E$ is the *symmetric* functional corresponding to the full *symmetric* system.

**Remark 3.1** *It is easy to check that the functional $E^\lambda$ is of class $C^2$. Moreover, by using the similar arguments as in* [12]*, we can see that the restriction of the functional $E^\lambda$ to $M$ satisfies the $(P.S)$ condition.*

Motivated by the idea of [6], in order to find the critical points of the functional $E^\lambda$, we need to establish the criteria of non-contractibility properties of the level sets of the functionals $E^\lambda$ and the *symmetric* one $E$. For the *symmetric* functional $E$, such a criterion is provided by using Kransnosel'skii's theory (see [15], Cha.VI). For the perturbed functional $E^\lambda$, by using the $Palais-Smale$ condition and the fact that $M$ is a $C^{1,1}$ manifold, we have the following invariant criteria. Since the proof is just an easy modification of Theorem 2.5 in [6], we omit it.

**Lemma 3.2** *Let $E, E^* \in C^0(M, \mathbb{R})$ be two functions such that $E \in C^1(\tilde{E}_a, \mathbb{R})$, $E^* \in C^1(\tilde{E}^*_c, \mathbb{R})$, $E, E^*$ satisfy $(P.S)_a$ and $(P.S)_c$ conditions, respectively. Assume further that $E^*$ is bounded from below on $M$ and is symmetric in the sense* *that the functional is $\sigma$ invariant, i.e. $E^*(\sigma(u,v)) = E^*(u,v)$. Let $C_k$ be defined in* (2.3). *If there exist $k \in \mathbb{N}, \epsilon > 0$ and $a \in \mathbb{R}$ such that $c \leq C_k$ and*

$$E^*_{C_k+\epsilon} \subset E_a \subset E_{a+\epsilon} \subset E^*_{C_{k+1}-\epsilon},$$

*then $E$ has at least one critical value in $[a, +\infty)$. Here $\tilde{E}_a := \{(u,v) \in M | E(u,v) \geq a\}$, $\tilde{E}^*_c := \{(u,v) \in M | E^*(u,v) \geq c\}$.*

According to Lemma 3.2, to prove Theorem 1.1, it will be sufficient to show that there are infinitely many distinct values of $k \in \mathbb{N}$ and $\epsilon_k > 0, a_k \in \mathbb{R}$ such that $E^*_{C_k+\epsilon_k} \subset E^\lambda_{a_k} \subset E^*_{C_{k+1}-\epsilon_k}$. We first have:

**Lemma 3.3** *There exist $C_0 \geq 1$, $\delta > 0$ such that for any $C \geq C_0$, it holds that*

$$E^*_C \subset E^\lambda_{C_1} \subset E^*_{C_2},$$

*where $C_1 = C + \delta C^{\frac{1}{2}}, C_2 = C_1 + \delta C_1^{\frac{1}{2}}$.*

**Proof.** By the definition of the functionals $E^*$ and $E^\lambda$, we observe that for any sequence $(u_n, v_n) \in M, E^*(u_n, v_n) \to +\infty \Leftrightarrow E^\lambda(u_n, v_n) \to +\infty$. Hence, there exists a constant $C_0 \geq 1$ such that for any $(u,v) \in M$ if $E^*(u,v) \geq C_0$ then $E^\lambda(u,v) \geq 1$. Similarly we have if $E^\lambda(u,v) \geq C_0$ for some $C_0 \geq 1$, then $E^*(u,v) \geq 1$. On the other hand, by the definition of $E^\lambda$ and $E^*$, clearly, for any $(u,v) \in M$ with $E^*(u,v) \geq 1$ and $E^\lambda(u,v) \geq 1$, it holds that

$$|E^\lambda(u,v) - E^*(u,v)| = \frac{1}{2}\lambda| \int_\Omega (u^2 - v^2)dx| \leq \delta_1[E^*(u,v)]^{\frac{1}{2}}, \text{ for some } \delta_1 > 0.$$

On the other hand, for $E^* \geq C_0$ for some $C_0 \geq 1$, we can deduce from the above inequality that there exists some constant $\delta_2$ (related to $\lambda$) such that

$$|E^*(u,v) - E^\lambda(u,v)| \leq \delta_2[E^\lambda(u)]^{\frac{1}{2}}$$

Thus for any $C \geq C_0$, one has the desired inclusions with $\delta := max(\delta_1, \delta_2)$. $\square$

**Proof of Theorem 1.1** $(a)$ Let $(u_k, v_k)_k$ be a sequence of critical points with critical value $b_k$ satisfying $C_k + \epsilon \leq b_k \leq C_{k+1} - \epsilon$ and such that $E^\lambda(u_k, v_k) \to +\infty$. By the proof of Lemma 3.3, we see that $E^\lambda(u_k, v_k) \to +\infty \Leftrightarrow E^*(u_k, v_k) \to +\infty$. On the other hand, it is easy to check that $E^*(u_k, v_k) = ||u_k||^2 + ||v_k||^2 \to +\infty$, which yields that

$$|\Omega|^4(|u_k|^4_\infty + |v_k|^4_\infty) \geq |u_k|^4_4 + |v_k|^4_4 \geq ||u_k||^2 + ||v_k||^2$$

which implies that $||u_k||_\infty + ||v_k||_\infty \to \infty$ as $k \to \infty$.

$(b)$ By Lemma 3.3, we have for any $C \geq 1$,

$$E^*_C \subset E^\lambda_{C_1} \subset E^*_{C_2},$$

where $C_1 = C + \delta C^{\frac{1}{2}}, C_2 = C_1 + \delta C_1^{\frac{1}{2}}, \delta > 0$ is a constant. Since $C \geq 1$, we have $C \geq C^{\frac{1}{2}}$ and it turns out that

$$C_2 \leq C + \tilde{\delta} C^{\frac{1}{2}} \text{ with } \tilde{\delta} := \delta + \delta(1+\delta)^{\frac{1}{2}}.$$

In view of the results of Lemma 3.2, in order to show the existence of infinitely many solutions of (1.4), it will suffice to prove that for infinitely many distinct values of $k$, we have $C_{k+1} > C_k + \tilde{\delta} C_k^{\frac{1}{4}}$. Indeed, once $C_{k+1} > C_k + \tilde{\delta} C_k^{\frac{1}{2}} := \nu(C_k)$, we can choose $\epsilon_k > 0$ small enough so that $C_{k+1} - \epsilon_k > \nu(C_k + \epsilon_k)$ and we write $a_k := \nu(C_k + \epsilon_k)$ to obtain that

$$E^*_{C_k+\epsilon_k} \subset E^\lambda_{a_k} \subset E^*_{a_k} \subset E^*_{C_{k+1}-\epsilon_k}.$$

Thus, for any such $k$, there exists a critical value of $E^\lambda$ in $(a_k, +\infty)$. Note that $a_k \geq C_k$ and $\lim_{k\to\infty} C_k = \infty$. The existence of infinitely many such $k \in N$ implies the existence of infinitely many distinct critical values of $E^\lambda$ which converges to $\infty$.

Now we return to the proof of the existence of infinitely many distinct values of $k$ such that $C_{k+1} > C_k + \tilde{\delta} C_k^{\frac{1}{2}}$. We prove this by contradiction arguments. Suppose that there exists $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ one has $C_{k+1} \leq \nu(C_k)$, that is

$$C_{k+1} \leq C_k + \tilde{\delta} C_k^{\frac{1}{2}}, \forall k \geq k_0.$$

In the following we will show that

$$C_{k+1} \leq Ck^2, \forall k \geq 1.$$

Indeed, this follows from the results in [5]. For reader's convience, we give the sketch of the proof. Let $\delta_k := k^{-2} C_k$. Note that for $t > 0$, $(1+t)^2 \geq 1 + 2t$, it follows from $0 \leq C_{k+1} - C_k \leq \delta C_k^{\frac{1}{2}}$ that

$$2\delta_{k+1} k^{-1} + \delta_{k+1} - \delta_k \leq \delta \delta_k^{\frac{1}{2}} k^{-1}.$$

There are two cases to be considered
(i) $\delta_{k+1} \leq \delta_k$;
(ii) $\delta_k \leq \delta_{k+1} \leq \frac{1}{2}\delta\delta_k^{\frac{1}{2}}$.
Each of these two case implies that $\delta_k \leq [\frac{1}{2}\delta]^2$. Thus we have

$$\delta_{k+1} \leq \max\{\delta_k, \frac{1}{4}\delta^2\}, \forall k \geq k_0.$$

As a result we obtain that the sequence $\delta_k$ is bounded which implies that

$$C_{k+1} \leq Ck^2, \forall k \geq 1. \tag{3.3}$$

But by Proposition 2.9, we know that

$$C_k \geq Ck^2$$

which contraction with (3.3). This completes the proof of Theorem 1.1.

$\square$

## References


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