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ASYMPTOTIC BEHAVIOR OF TOUCH-DOWN SOLUTIONS AND GLOBAL BIFURCATIONS FOR AN ELLIPTIC PROBLEM WITH A SINGULAR NONLINEARITY

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Abstract We consider the following problem

$$-\Delta u = \frac{\lambda}{(1-u)^2} \ \text{ in } \Omega, \ \ u = 0 \ \text{ on } \partial\Omega, \ \ 0 < u < 1 \ \text{ in } \Omega$$

where Ω is a rather symmetric domain in $\mathbb{R}^2$. We prove that there exists a $\lambda_* > 0$ such that for $\lambda \in (0, \lambda_*)$ the minimal solution is unique. Then we analyze the asymptotic behavior of touch-down solutions, i.e., solutions with $\max_\Omega u_i(0) \to 1$. We show that after a rescaling, the solution will be asymptotically symmetric. As a consequence, we show that the branch of positive solutions must undergo infinitely many bifurcations as the maximums of the solutions on the branch go to 1 (possibly only changes of direction). This gives a positive answer to some open problems in [12]. Our result is new even in the radially symmetric case. Central to our analysis is the monotonicity formula, one-dimensional Sobloev inequality, and classification of solutions to a supercritical problem

$$\Delta U = \frac{1}{U^2} \ \text{ in } \mathbb{R}^2, \ U(0) = 1, U(z) \geq 1.$$

1. **Introduction.** We consider the structure of positive solutions to the following problem

$$(S_\lambda) \qquad -\Delta u = \frac{\lambda}{(1-u)^2} \ \text{ in } \Omega, \ \ 0 < u < 1 \ \text{ in } \Omega, \ \ u = 0 \ \text{ on } \partial\Omega$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^2$ is very well behaved (see [7]). More precisely, we consider domain Ω in $\mathbb{R}^2$ which is Lipschitz, $0 \in \Omega$, Ω is invariant under the 2 reflections in the coordinate planes and such that if $0 < t < s < \tilde{t}_i$, $(I - P_i)D_{i,s} \subseteq (I - P_i)D_{i,t}$. Here P_i is the orthogonal projection onto span e_i, $D_{i,s} = \{x \in \Omega : \ x_i = s\}$, $\tilde{t}_i = \sup\{x_i : \ x \in \Omega\}$ and $\{e_i\}$ is the usual basis for $\mathbb{R}^2$. Examples of the domains include balls, ellipses, rectangles, etc.

S_λ models a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at +1. When a voltage-represented here by λ-is applied, the membrane deflects towards the ceiling plate and a snap-through may occur when it exceeds a certain critical value λ^* (pull-in voltage). This creates a so-called "pull-in instability" which greatly affects the design of many devices (see [11] and [26] for a detailed discussion on MEMS devices). Note that two-dimensional domains are of real physical relevance.

In recent papers [12]-[14] and [10], the authors studied the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \frac{\lambda g(x)}{(1-u)^2} \ \text{ in } \Omega \\ 0 < u < 1 \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega \end{cases}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $g \in C(\overline{\Omega})$ is a nonnegative function. They gave a detailed study on the minimal solutions of the problem (P_λ) with different forms of $g(x)$. The following theorem was obtained.

Theorem 1.1. *(Theorem 1.1-1.3 in* [12]*): Suppose $g \in C(\overline{\Omega})$ is a nonnegative function on Ω. Then, there exists a finite $\lambda^* > 0$ such that*

1. If $0 \le \lambda < \lambda^*$, there exists a unique minimal solution $\underline{u}_\lambda$ of (P_λ) such that $\mu_{1,\lambda}(\underline{u}_\lambda) > 0$. (See (1.1) below.) Moreover, $\underline{u}_\lambda \to 0$ as $\lambda \to 0$.

2. If $\lambda > \lambda^*$, there is no solution for (P_λ).

3. If $1 \le N \le 7$, then-by means of energy estimates-one has

$$\sup_{\lambda \in (0,\lambda^*)} \|\underline{u}_\lambda\|_\infty < 1$$

and consequently $u^* = \lim_{\lambda \uparrow \lambda^*} \underline{u}_\lambda$ is a solution of (P_{λ^*}) such that

$$\mu_{1,\lambda^*}(u^*) = 0.$$

4. If $g(x) = |x|^\alpha$ and Ω is the unit ball, then $u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}$ and $\lambda^* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$, provided $N \ge 8$ and $0 \le \alpha \le \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$.

Issues such as uniqueness, multiplicity and other qualitative properties of the solutions for (P_λ) are still far from being well understood, even in the radially symmetric case. In their paper [12], Ghoussoub and Guo present some numerical evidence for various conjectures relating the case $g(x) = |x|^\alpha$. They conjectured that for $2 \le N \le 7$ and $\alpha \ge 0$, there exists an infinite number of branches of solutions to (P_λ). In a more recent paper [10], Esposito, Ghoussoub and Guo found a second solution of (P_λ) for $\lambda \in (\lambda^* - \delta, \lambda^*)$ and showed that this second solution is a mountain pass solution. To establish these results, the authors need some special forms of $g(x)$ to guarantee that the solutions u_λ of (P_λ) possess the property $\|u_\lambda\|_\infty < 1$.

There already exist in the literature many interesting results concerning the properties of the branch of solutions for Dirichlet boundary value problems of the form $-\Delta u = \lambda h(u)$ where h is a regular nonlinearity (for example of the form e^u or $(1+u)^p$ for $p > 1$). See, for example, [4], [5], [21], [22] and the references therein. The singular situation was first considered in a very general context in [24].

A solution $\underline{u}_\lambda$ of the equation (S_λ) is called *minimal* if $\underline{u}_\lambda \in C^2(\Omega)$ satisfies $\underline{u}_\lambda \le u$ in Ω for any solution u of (S_λ). Throughout this paper, unless otherwise specified, solutions for (S_λ) are considered to be in the classical sense. Now for any solution u of (S_λ), one can introduce the linearized operator at u defined by:

$$L_{u,\lambda} = -\Delta - \frac{2\lambda}{(1-u)^3},$$

and its corresponding eigenvalues $\{\mu_{k,\lambda} : \ k = 1, 2, \ldots\}$. Note that the first eigenvalue is simple and is given by:

$$\mu_{1,\lambda}(u) = \inf\Big\{\Big\langle L_{u,\lambda}\phi, \phi\Big\rangle_{H_0^1(\Omega)} : \ \phi \in C_0^\infty(\Omega), \ \int_\Omega |\phi(x)|^2 dx = 1\Big\} \tag{1.1}$$

with the infimum being attained at a first eigenfunction ϕ_1, while the second eigenvalue is given by the formula:

$$\mu_{2,\lambda}(u) = \inf\Big\{\Big\langle L_{u,\lambda}\phi, \phi\Big\rangle_{H_0^1(\Omega)} : \ \phi \in C_0^\infty(\Omega), \ \int_\Omega |\phi(x)|^2 dx = 1, \ \int_\Omega \phi(x)\phi_1(x) dx = 0\Big\}.$$

This construction can then be iterated to obtain the k-th eigenvalue $\mu_{k,\lambda}(u)$ with the convention that eigenvalues are repeated according to their multiplicities.

In this paper, we shall give a positive answer to the conjecture B in [12] in two dimensional case. We show that for our well-behaved domain $\Omega \subset \mathbb{R}^2$, there exists $\lambda_* > 0$ such that for $\lambda < \lambda_*$, the minimal solution is the only solution to (S_λ). Furthermore, (S_λ) has an infinite number of branches of solutions. Note that in our case here, the maximum of the solution of (S_λ) will be close to 1 and this makes the problem more difficult to deal with. From now on, we assume Ω to be the symmetric domain as defined at the beginning of introduction.

Let D denote the component of $\{(u,\lambda) \in C(\overline{\Omega}) \times \mathbf{R}^+ : \ -\Delta u = \frac{\lambda}{(1-u)^2}, \ 0 < u < 1 \text{ in } \Omega, \ u = 0$ on $\partial\Omega\}$ containing $(0,0)$ in its closure. Note that we can talk about the component since it is a simple curve near the end point. We will show that $\inf\{\lambda > 0 : \ (u,\lambda) \in D \text{ for some } u\} > 0$ so that the only solution for each small positive λ is the unique minimal solution near 0 in $C(\overline{\Omega})$. It is convenient to add $(0,0)$ to D.

We first have the following theorem which gives exact asymptotic behavior for the touch-down solutions.

Theorem 1.2. *Let (λ_i, u_i) be a sequence of solutions of (S_λ) such that*

$$\lambda_i \to \lambda_0 \neq 0, \max_{x\in\Omega} u_i(x) = u_i(0) \to 1. \tag{1.2}$$

Then the rescaled function

$$U_i(y) = \frac{1 - u_i(\epsilon_i^{3/2}\lambda_i^{-1/2}y)}{\epsilon_i}, \quad y \in \Omega_i := \{y : \epsilon_i^{3/2}\lambda_i^{-1/2}y \in \Omega\}, \tag{1.3}$$

where $\epsilon_i = 1 - u_i(0)$, approaches to U in $C^2_{loc}(\mathbb{R}^2)$ as $i \to \infty$, where $U = U(z)$ is the unique radially symmetric solution to

$$\Delta U = \frac{1}{U^2} \text{ in } \mathbb{R}^2, \quad U(0) = 1, U(z) > 1. \tag{1.4}$$

Using Theorem 1.2, we derive the following theorem on bifurcation of solutions

Theorem 1.3. *(1) $\lambda_* = \inf\{\lambda > 0 : (u,\lambda) \in D$ for some $u\} > 0$.*
(2) D has infinitely many bifurcation points.

Remark 1.4. We actually prove somewhat more. We prove that D contains a piecewise analytic continuous curve T such that the implicit function theorem applies (to solve for u as a function of λ) except at isolated points of T and there exists a sequence $(u_i, \lambda_i) \in T$ such that $||u_i||_\infty \to 1$ as $i \to \infty$ and each (u_i, λ_i) is either a point where T changes direction (i.e. the branch T locally "bends back" or (u_i, λ_i) is a point of secondary bifurcation. This holds for any choice of T. For generic symmetric Ω, we see that the former holds (that is the points (u_i, λ_i) are points where the branch changes direction.)

Remark 1.5. Theorem 1.2 and 1.3 can be extended to the following more general elliptic equation of the form

$$(S_{\lambda,p}) \qquad -\Delta u = \frac{\lambda}{(1-u)^p} \text{ in } \Omega, \ \ 0 < u < 1 \text{ in } \Omega, \ \ u = 0 \text{ on } \partial\Omega$$

where $1 \leq p < 3$. The critical observation is that we need to use the Sobolev inequality (2.2). We leave the details to interested readers. Note that when $p = 1$, $S_{\lambda,p}$ arises in the study of singular minimal hypersurfaces with symmetry. See [23], [27] and the references therein. For general $p > 0$, $S_{\lambda,p}$ also arises in relation to chemical catalyst kinetics (see [2] and [8]).

The main difficulty in proving Theorem 1.2 is the classification of solutions to the following problem (after blowing up):

$$\Delta U = \frac{1}{U^2} \text{ in } \mathbb{R}^2, \quad U(0) = 1, U(z) > 1. \tag{1.5}$$

Problem (1.5) can be considered as a **supercritical** problem in $\mathbb{R}^2$. To show radial symmetry of all solutions to (1.5), we make critical use of the one-dimensional Sobolev inequality (Lemma 2.1) and the monotonicity formula (Lemma 2.2).

The organization of the paper is as follows: in Section 2, we collect several important estimates, including Sobolev inequality, monotonicity formula, classification theorem and regularity theorems. In Section 3, we prove the critical theorem on the radial symmetry of solutions to (1.5) and therefore prove Theorem 1.2. In Section 4, we prove (1) of Theorem 1.3. Here again we need to use monotonicity formula and Sobolev inequality. Finally, we adopt Dancer's idea in [7] to prove (2) of Theorem 1.3 in Section 5.

After the paper was finished, we were informed by the referee a paper of P. Esposito [9], where he proved that the Morse index of the touch-down solutions must approach $+\infty$ by energy method. His results hold true for general convex domains and for higher dimensions ($2 \leq N \leq 7$). However the exact asymptotic behavior of the solutions is not analyzed in [9] and in particular our Theorems 1.2 and 1.3 seem to be new.

2. **Some Preliminaries.** We collect several important preliminaries in this section.

First, we have the following one-dimensional Sobolev inequality which plays a key role in our estimate. The proof of it can be found in [1]:

Lemma 2.1. *(a) (Proposition 1.3 of* [1]*.) It holds*

$$\Big(\int_0^\pi u^{-2}\Big)\Big(\int_0^\pi (u^2-u_\theta^2)\Big) \le \pi^2, \tag{2.1}$$

for all u such that $u>0, u(\theta+\pi)=u(\theta)$.

(b) (Proposition 1.4 of [1]*.) For any $\beta\in(\frac{9}{16},1)$ and for u satisfying $u>0, u(\theta+\pi)=u(\pi), \int_0^\pi \frac{\cos(2\theta)}{u^2}=\int_0^\pi\frac{\sin(2\theta)}{u^2}=0$, there exists $\lambda_\beta\ge 1$ such that*

$$\Big(\int_0^\pi u^{-2}\Big)\Big(\int_0^\pi (u^2-\beta u_\theta^2)\Big) \le \lambda_\beta. \tag{2.2}$$

Next, we recall the following monotonicity formula from [16].

Lemma 2.2. *((2.14) of* [16]*.) If $z\in H^1(\Omega)$ with $\int_\Omega z^{-1}dx<\infty$ is a nonnegative stationary solution of the equation*

$$\Delta z = \frac{\lambda}{z^2} \quad in\ \Omega, \tag{2.3}$$

then the function

$$\mathcal{E}_z(r) := -\frac{3}{2}r^{-4/3}\int_{B(0,r)}\frac{\lambda}{z}dx + \frac{1}{4}\frac{d}{dr}\Big[r^{-4/3}\int_{\partial B(0,r)} z^2 dS\Big] - \frac{1}{4}r^{-7/3}\int_{\partial B(0,r)} z^2 dS$$

is a nondecreasing function of r, where $B(0,r)\subset\Omega$ is the ball with center at 0 and radius r.

We are also in need of the following classification theorem.

Theorem 2.3. *(Theorem 1.2 of* [17]*.) The solutions to the following problem*

$$\Delta U = \frac{1}{U^2} \quad in\ \mathbb{R}^2, \quad U(0)=1\,, U(z)>1, \tag{2.4}$$

are radially symmetric if

$$\lim_{|z|\to+\infty} |z|^{-\frac{2}{3}}U(z) = (\frac{4}{9})^{-1/3} \tag{2.5}$$

Finally, we state the following useful lemma.

Lemma 2.4. *(Lemma 2.2 of* [20]*.) Assume φ is a nonnegative smooth function on $\bar{B_r}\subset\mathbb{R}^2$ such that $\Delta\varphi+\varphi^2\ge 0$. Then there exists a universal constant $\eta_0>0$ such that $\int_{B_r}\varphi dx\le\eta_0$ implies $\varphi(x)\le\frac{c}{r^2}\int_{B_r}\varphi$ for $x\in B_{r/2}$. Here c is an absolute constant.*

3. **A radial symmetry result in $\mathbb{R}^2$.** In this section, we shall provide a crucial radial symmetry result in $\mathbb{R}^2$. Let $(u_i,\lambda_i)\in D$ with $||u_i||_\infty\to 1$ and $\lambda_i\to\alpha>0$ as $i\to\infty$. If we define $\epsilon_i = 1-||u_i||_\infty$ and

$$U_i(y) = \frac{1-u_i(\epsilon_i^{3/2}\lambda_i^{-1/2}y)}{\epsilon_i}, \quad y\in\Omega_i := \{y:\ \epsilon_i^{3/2}\lambda_i^{-1/2}y\in\Omega\}, \tag{3.1}$$

then U_i satisfies the equation

$$\Delta U_i = \frac{1}{U_i^2} \quad \text{in } \Omega_i \tag{3.2}$$

and $U_i\to U$ in $C^2_{loc}(\mathbb{R}^2)$ as $i\to\infty$, where $U=U(z)$ satisfies

$$\Delta U = \frac{1}{U^2} \quad \text{in } \mathbb{R}^2, \quad U(0)=1, \ \ U(z)\ge 1. \tag{3.3}$$

The main result of this section is the following classification theorem, which proves Theorem 1.2.

Theorem 3.1. *If U is defined as above, then U is radially symmetric in $\mathbb{R}^2$.*

To prove Theorem 3.1, we use Theorem 2.3. It is enough to verify (2.5). The rest of the section is to prove (2.5). We will achieve this by a series of lemmas.

To this end, we first need to analyze the behavior of u_i near the boundary of Ω. Let $z_i = 1-u_i$. We see that z_i satisfies the problem

$$\Delta z_i = \frac{\lambda_i}{z_i^2} \text{ in } \Omega, \quad 0 < z_i < 1 \text{ in } \Omega, \quad z_i = 1 \text{ on } \partial\Omega. \tag{3.4}$$

Moreover, $\min_\Omega z_i \to 0$ as $i \to \infty$. It is clear that $\epsilon_i = \min_\Omega z_i$ and

$$U_i(y) = \frac{z_i(\epsilon_i^{3/2}\lambda_i^{-1/2}y)}{\epsilon_i} \quad \text{for } y \in \Omega_i. \tag{3.5}$$

In the following, we consider the problem (3.4). By our symmetric assumptions on Ω and a standard Method of Moving Planes as in [15], we see that z_i satisfies

$$x_j(\partial z_i/\partial x_j) \geq 0, j = 1, 2, \quad \text{on } \Omega. \tag{3.6}$$

This implies that z_i is even in x_j for $j = 1, 2$. Therefore, the minimum of z_i attains at 0.

Choose a large ball $B(0, r_0) \subset\subset \Omega$ with center at 0 and radius $r_0 > 0$ and $\text{dist}(\partial B(0, r_0), \partial\Omega) > 0$. The following lemma analyzes the behavior of z_i far from 0.

Lemma 3.2. *For any $0 < \beta < 1$, there exists $0 < \kappa := \kappa(\beta) < 1$ independent of i such that for all i sufficiently large,*

$$\kappa \leq z_i(x) \leq 1 \quad \textit{for } x \in \Omega \backslash B(0, \beta r_0). \tag{3.7}$$

Proof. Let σ_1 and $\varphi_1 > 0$ in Ω be the first eigenvalue and eigenfunction of the problem

$$-\Delta\varphi = \sigma\varphi \text{ in } \Omega, \quad \varphi = 0 \text{ on } \partial\Omega.$$

Then multiplying φ_1 on both the sides of the equation (3.4) and integrating it on Ω, we see that

$$\sigma_1 \int_\Omega (1 - z_i)\varphi_1 \, dx = \lambda_i \int_\Omega \frac{\varphi_1}{z_i^2} dx. \tag{3.8}$$

Since $0 < z_i < 1$, we see from (3.8) that

$$\int_\Omega \frac{\varphi_1}{z_i^2} dx \leq C.$$

Here we have used the fact that $\lambda_i \to \alpha > 0$ as $i \to \infty$. This implies that for any small $0 < \delta < \text{dist}(\partial B(0, r_0), \partial\Omega)/10$,

$$\int_{\Omega_\delta} \frac{dx}{z_i^2} \leq C,$$

where $\Omega_\delta = \{x \in \Omega : \text{ dist}(x, \partial\Omega) > \delta\}$ (note that $B(0, r_0) \subset\subset \Omega_\delta$). The Hölder inequality implies

$$\int_{\Omega_\delta} \frac{dx}{z_i} \leq \Big(\int_{\Omega_\delta} \frac{dx}{z_i^2}\Big)^{1/2} |\Omega|^{1/2} \leq C. \tag{3.9}$$

By the standard moving plane argument, we see that for any $x \in \Omega_\delta \backslash B(0, \beta r_0)$, there exists a piece of cone Γ_x with vertex at x with (i) $\text{meas}(\Gamma_x) \geq \gamma > 0$, where γ is independent of x, (ii) $\Gamma_x \subset \Omega_\delta$, (iii) $\frac{1}{z_i(y)} \geq \frac{1}{z_i(x)}$ for any $y \in \Gamma_x$. Thus, it follows from (i)-(iii) and (3.9) that

$$\frac{1}{z_i(x)} \leq \frac{1}{\text{meas}(\Gamma_x)} \int_{\Gamma_x} \frac{dy}{z_i(y)} \leq \gamma^{-1} \int_{\Omega_\delta} \frac{dy}{z_i(y)} \leq C. \tag{3.10}$$

This implies that

$$z_i(x) \geq C \quad \text{for } x \in \Omega_\delta \backslash B(0, \beta r_0). \tag{3.11}$$

By (3.6), $z_i(x) \geq C$ for $x \in \Omega \backslash \Omega_\delta$. Thus, the κ exists and the proof is completed. □

Next, we consider the Emden-Fowler transformation of (3.4). Let $R_i := \lambda_i^{1/2}\epsilon_i^{-3/2} r_0$. We see that

$$U_i(R_i) = R_i^{2/3} r_0^{-2/3} \lambda_i^{-1/3} z_i(r_0).$$

Therefore, if we define

$$v_i(s, \theta) := |y|^{-2/3} U_i(y), \quad |y| = e^s, \quad R_i = e^{T_i} \tag{3.12}$$

we easily see that for $(s, \theta) \in [0, T_i] \times S^1$, $v_i := v_i(s, \theta)$ satisfies the equation:

$$v_{ss} + \frac{4}{3} v_s + v_{\theta\theta} + \frac{4}{9} v = \frac{1}{v^2}. \tag{3.13}$$

Since $z_i(x) := z_i(x_1, x_2)$ is even of x_1 and x_2 respectively, we see that for any fixed s, $v_i(s,\cdot)$ is a π-periodic function. Moreover, using the fact that $U_i \to U$ as $i \to \infty$ and the definition of $U_i(R_i)$ we obtain that for i sufficiently large,

$$v_i(0,\theta) = O(1), \ \ v_i(T_i,\theta) = O(1) \ \text{ for } \theta \in S^1. \tag{3.14}$$

Moreover, for any $0 < \tilde{\beta} < 1$ independent of i

$$v_i(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [0,\tilde{\beta}] \times S^1 \tag{3.15}$$

and

$$v_i(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [\tilde{\beta}T_i, T_i] \times S^1. \tag{3.16}$$

(3.15) and (3.16) can be obtained from the fact $U_i \to U$ as $i \to \infty$ and Lemma 3.2. Furthermore, easy calculations imply that

$$(v_i)_s(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [0,\tilde{\beta}] \times S^1, \tag{3.17}$$

$$(v_i)_\theta(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [0,\tilde{\beta}] \times S^1, \tag{3.18}$$

$$(v_i)_s(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [\tilde{\beta}T_i, T_i] \times S^1 \tag{3.19}$$

and

$$(v_i)_\theta(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [\tilde{\beta}T_i, T_i] \times S^1 \tag{3.20}$$

We easily see that (3.17) and (3.18) hold by using the fact that $U_i \to U$ in $C^1_{loc}(\mathbb{R}^2)$ as $i \to \infty$. To obtain (3.19) and (3.20), we notice that, since $z_i(x) \geq \kappa$ for $x \in \Omega\backslash B(0,\beta r_0)$, the equation of z_i and the regularity of Δ imply that there exists $C > 0$ independent of i such that

$$|\nabla z_i(x)| \leq C \ \text{ for } x \in \Omega\backslash B(0,\beta r_0).$$

This implies (3.19) and (3.20).

Denoting $w_i(s) = \int_{S^1} v_i^2(s,\theta)d\theta$, $\overline{v}_i(s) = \int_{S^1} v_i(s,\theta)d\theta$ and for convenience, omitting the subscript i from w_i, $\overline{v}_i$ and v_i in the following, we see that

$$w_{ss} + \frac{4}{3}w_s + \frac{8}{9}w = 2\int_{S^1}\frac{ds}{v} + 2\int_{S^1}(v_\theta^2 + v_s^2)ds \tag{3.21}$$

$$\overline{v}_{ss} + \frac{4}{3}\overline{v}_s + \frac{4}{9}\overline{v} = \int_{S^1}\frac{d\theta}{v^2}. \tag{3.22}$$

Moreover, it follows from (3.6) that

$$v_s(s,\theta) + \frac{2}{3}v(s,\theta) \geq 0 \ \text{ for } (s,\theta) \in (-\infty, T_i] \times S^1. \tag{3.23}$$

The next lemma gives L^2-control of v_ξ.

Lemma 3.3.

$$\int_0^s \int_{S^1} v_\xi^2(\xi,\theta)d\theta d\xi \leq C \ \textit{ for } s \in [0,T_i],$$

where $C > 0$ *is independent of* i.

Proof. Multiplying $v_s(s,\theta)$ on both the sides of (3.13) and integrating it on $[0,T_i] \times S^1$, using the facts in (3.14), (3.17), (3.18), (3.19), (3.20), we see that

$$\int_0^{T_i}\int_{S^1} v_s^2(s,\theta)dsd\theta \leq C$$

where C is independent of i. Thus,

$$\int_0^s \int_{S^1} v_s^2(\xi,\theta)d\theta d\xi \leq C \ \text{ for } s \in [0,T_i]. \tag{3.24}$$

This completes the proof. □

The following lemma is the most important estimate we need.

Lemma 3.4. *There exists* $C > 0$ *independent of* i *such that*

$$\max_{s\in[0,T_i]} \overline{v}(s) \leq C \tag{3.25}$$

and

$$\max_{s\in[0,T_i]} \int_{S^1}\frac{d\theta}{v^2(s,\theta)} \leq C. \tag{3.26}$$

Proof. We prove (3.25) and (3.26) by the following steps.

Step 1: We show that
$$\int_{S^1} \frac{d\theta}{v^2(s,\theta)} \le C \int_{S^1} v(s,\theta) d\theta \quad \text{for } s \in (-\infty, T_i]. \tag{3.27}$$
In fact, from (3.22) and (3.23) we see that
$$\begin{aligned} \overline{v}_{sss} + \frac{4}{3}\overline{v}_{ss} + \frac{4}{9}\overline{v}_s &= -2\int_{S^1} \frac{v_s}{v^3} \\ &\le \frac{4}{3}\int_{S^1} \frac{d\theta}{v^2} \\ &= \frac{4}{3}\Big[\overline{v}_{ss} + \frac{4}{3}\overline{v}_s + \frac{4}{9}\overline{v}\Big]. \end{aligned}$$
Setting $q(s) = \overline{v}_s - \frac{4}{3}\overline{v}$, we see that
$$q_{ss} + \frac{4}{3}q_s + \frac{4}{9}q \le 0, \tag{3.28}$$
i.e.
$$(e^{\frac{2}{3}s}q(s))_{ss} \le 0 \quad \text{for } s \in (-\infty, T_i]. \tag{3.29}$$
Note that
$$e^{\frac{2}{3}s}(\overline{v}_s(s) + \frac{2}{3}\overline{v}(s)) \to 0, \quad e^{\frac{2}{3}s}\overline{v}(s) \to 2\pi \quad \text{as } s \to -\infty.$$
We see that
$$e^{\frac{2}{3}s}q(s) \to -4\pi < 0 \quad \text{as } s \to -\infty. \tag{3.30}$$
This implies that
$$(e^{\frac{2}{3}s}q(s))_s \to 0 \quad \text{as } s \to -\infty. \tag{3.31}$$
It follows from (3.29) and (3.31) that
$$(e^{\frac{2}{3}s}q(s))_s \le 0 \quad \text{for } s \in (-\infty, T_i] \tag{3.32}$$
and hence
$$e^{\frac{2}{3}s}q(s) \le -4\pi \quad \text{for } s \in (-\infty, T_i].$$
Thus,
$$\overline{v}_s \le \frac{4}{3}\overline{v} \quad \text{in } (-\infty, T_i]. \tag{3.33}$$
(3.33) together with the fact $\overline{v}_s + \frac{2}{3}\overline{v} \ge 0$ implies
$$|\overline{v}_s(s)| \le \frac{4}{3}\overline{v}(s) \quad \text{for } s \in (-\infty, T_i]. \tag{3.34}$$
On the other hand, we see from (3.32) that
$$q_s(s) + \frac{2}{3}q(s) \le 0 \quad \text{for } s \in (-\infty, T_i] \tag{3.35}$$
and
$$q_s(s) + \frac{2}{3}q(s) = \overline{v}_{ss}(s) - \frac{2}{3}\overline{v}_s(s) - \frac{8}{9}\overline{v}(s) = \int_{S^1} v^{-2}d\theta - 2\overline{v}_s(s) - \frac{4}{3}\overline{v}(s). \tag{3.36}$$
Then, (3.35), (3.36) and (3.34) imply that
$$\int_{S^1} \frac{d\theta}{v^2(s,\theta)} \le 2\overline{v}_s(s) + \frac{4}{3}\overline{v}(s) \le C\overline{v}(s) \quad \text{for } s \in (-\infty, T_i]. \tag{3.37}$$
This finishes the proof of Step 1.

Step 2: We prove that
$$\int_{S^1} \frac{d\theta}{v^2(s,\theta)} \le C\Big(1 + \int_{S^1} v_s^2\Big). \tag{3.38}$$
To prove (3.38), we consider the function
$$J(s) = \int_{S^1} \Big[\frac{1}{2}v_\theta^2(s,\theta) - \frac{1}{2}v_s^2(s,\theta) - \frac{2}{9}v^2(s,\theta) - \frac{1}{v(s,\theta)}\Big] d\theta. \tag{3.39}$$
Multiplying (3.13) by v_s and integrating on S^1, we obtain that
$$J'(s) = \frac{4}{3}\int_{S^1} v_s^2(s,\theta)d\theta > 0. \tag{3.40}$$

Then

$$J(0) \le J(s) \le J(T_i) \text{ for } s \in [0, T_i].$$

It follows from (3.15)-(3.20) that

$$|J(s)| \le C \text{ for } s \in [0, T_i]$$

and hence

$$\Big| \int_{S^1} v_\theta^2(s,\theta)d\theta - \frac{4}{9}\int_{S^1} v^2(s,\theta)d\theta \Big| \le C\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta + \int_{S^1} \frac{d\theta}{v(s,\theta)}\Big). \tag{3.41}$$

On the other hand, it follows from the Young's inequality and (3.37) that

$$\int_{S^1} \frac{d\theta}{v(s,\theta)} \le C\Big(\int_{S^1} \frac{d\theta}{v^2(s,\theta)} + 1\Big) \le C(\overline{v}(s) + 1).$$

Then we see from (3.41) and the Young's inequality that for any $0 < \tilde{\epsilon} < 1/10$,

$$\begin{aligned} &\int_{S^1} v_\theta^2(s,\theta)d\theta - \frac{4}{9}\int_{S^1} v^2(s,\theta)d\theta \\ &\le C\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta + \overline{v}(s)\Big) \\ &\le C\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta\Big) + C\tilde{\epsilon}\int_{S^1} v^2(s,\theta)d\theta + C(\tilde{\epsilon}). \end{aligned}$$

Choosing $\tilde{\epsilon}$ such that $\frac{4}{9} + C\tilde{\epsilon} < 1$ and using the Sobolev's inequality (2.1), we obtain that

$$\int_{S^1} v^2(s,\theta)d\theta \le C\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta + \frac{1}{\int_{S^1} \frac{1}{v^2(s,\theta)}d\theta}\Big). \tag{3.42}$$

(Note that we can use the Sobolev's inequality (2.1) here since for each s, $v(s,\cdot)$ is a π-periodic function.) Since the convexity of the function $p(s) = s^2$ and (3.37) imply that

$$\int_{S^1} v^2(s,\theta)d\theta \ge C\overline{v}^2 \ge C\Big(\int_{S^1} \frac{d\theta}{v^2(s,\theta)}\Big)^2 \tag{3.43}$$

we see from (3.42) that

$$\Big(\int_{S^1} \frac{d\theta}{v^2(s,\theta)}\Big)^3 \le C\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta\Big)\Big(\int_{S^1} \frac{d\theta}{v^2(s,\theta)}\Big) + C. \tag{3.44}$$

Therefore,

$$\int_{S^1} \frac{d\theta}{v^2(s,\theta)} \le C\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta\Big). \tag{3.45}$$

This finishes the proof of Step 2.

Step 3: We prove (3.25) and (3.26).

Going back to (3.22), we see that

$$\overline{v}_{ss} + \frac{4}{3}\overline{v}_s + \frac{4}{9}\overline{v} = \int_{S^1} \frac{d\theta}{v^2(s,\theta)} \le C\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta\Big).$$

This implies

$$\Big[e^{\frac{2}{3}s}\Big(\overline{v}_s(s) + \frac{2}{3}\overline{v}(s)\Big)\Big]' \le Ce^{\frac{2}{3}s}\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta\Big)$$

and (integrating from 0 to s)

$$\overline{v}_s(s) + \frac{2}{3}\overline{v}(s) \le C\Big(1 + \int_0^s \int_{S^1} e^{\frac{2}{3}(\xi - s)} v_\xi^2(\xi,\theta)d\theta d\xi\Big) \le C \text{ for } s \in [0, T_i] \tag{3.46}$$

where we use Lemma 3.3 and (3.14) and (3.17). Let s_i be the point satisfies $\overline{v}(s_i) = \max_{s\in[0,T_i]} \overline{v}(s)$. We have three cases here: (i) $s_i = 0$, (ii) $s_i = T_i$, (iii) $s_i \in (0, T_i)$. For the first two cases, we see from (3.14) that $\overline{v}(s_i) \le C$ and C is independent of i. For the third case, we see that $\overline{v}'(s_i) = 0$ and (3.46) implies that

$$\max_{s\in[0,T_i]} \overline{v}(s) = \overline{v}(s_i) \le C.$$

Moreover, it can be easily seen from (3.37) that

$$\max_{s\in[0,T_i]} \int_{S^1} \frac{d\theta}{v^2(s,\theta)} \le \max_{s\in[0,T_i]} \overline{v}(s) \le C.$$

This completes the proof. □

Lemma 3.5. *There exists $C > 0$ independent of i such that*

$$v(s,\theta) \geq C > 0 \quad \text{for } (s,\theta) \in [0,T_i] \times S^1. \tag{3.47}$$

Proof. The main idea is to consider $\frac{1}{v^3}$ and then use Lemma 2.4 and Lemma 3.4.

Let (s^i, θ^i) be the point where $v(s,\theta)$ attains its minimum $v_{min} := \min_{(s,\theta)\in[0,T_i]\times S^1} v(s,\theta)$. Then we have three cases here again: (i) $s^i = 0$, (ii) $s^i = T_i$, (iii) $s^i \in (0,T_i)$. For the first two cases, we see from (3.14) that $v_{min} \geq C > 0$. We only need to consider the last case. Note that we can also assume that $s^i \geq \tilde{\beta}$ and $s^i \leq \tilde{\beta} T_i$ for some $0 < \tilde{\beta} < 1/2$ and i sufficiently large. Otherwise, we see from (3.15) and (3.16) that $v_{min} \geq C > 0$.

Let $m(s,\theta) = \frac{1}{v^3(s,\theta)}$. Then m satisfies the equation

$$\begin{aligned} \Delta m &= \frac{12}{v^5}|\nabla v|^2 + 4\frac{v_s}{v^4} + \frac{4}{3}\frac{1}{v^3} - \frac{3}{v^6} \\ &\geq -\frac{4}{3}m_s - 3m^2 + \frac{4}{3}m. \end{aligned}$$

Therefore, m satisfies

$$\Delta m + \frac{4}{3}m_s + 3m^2 \geq 0 \quad \text{for } (s,\theta) \in [0,T_i] \times S^1$$

and

$$\max_{(s,\theta)\in[0,T_i]\times S^1} m(s,\theta) := m_{max} = m(s^i,\theta^i).$$

Define $\hat{m}(s,\theta) = e^{\frac{2}{3}(s-s^i)} m(s,\theta)$. An easy calculation implies that $\hat{m}$ satisfies

$$\Delta \hat{m} + 3e^{-\frac{2}{3}(s-s^i)}\hat{m} \geq 0.$$

We can choose $\delta > 0$ and $C = e^{\frac{2}{3}\delta}$ independent of i such that

$$\Delta \hat{m} + C\hat{m}^2 \geq 0 \quad \text{for } (s,\theta) \in [s^i - \delta, s^i + \delta] \times S^1. \tag{3.48}$$

We need to show that such δ exists, that is, $[s^i - \delta, s^i + \delta] \subset [0, T_i]$. Since $s^i \geq \tilde{\beta}$ and $s^i \leq \tilde{\beta} T_i$ for $0 < \tilde{\beta} < 1/2$, we can choose $0 < \delta < \tilde{\beta}/10$. We see that $s^i + \delta \leq \tilde{\beta} T_i + \delta \leq T_i$ for i sufficiently large (note that $T_i \to \infty$ as $i \to \infty$). Moreover, $s^i - \delta \geq s^i - \tilde{\beta} \geq 0$. Setting $\tilde{m}(s,\theta) = C\hat{m}(s,\theta)$, we see

$$\Delta \tilde{m} + \tilde{m}^2 \geq 0 \quad \text{for } (s,\theta) \in [s^i - \delta, s^i + \delta] \times S^1.$$

By Lemma 2.4, we see that there exists $\eta_0 > 0$ independent of i such that for any $r > 0$ if $\int_{B_r} \tilde{m}dx \leq \eta_0$, then

$$\tilde{m}(x) \leq \frac{C}{r^2}\int_{B_r} \tilde{m}(x)dx \quad \text{for } x \in B_{r/2}$$

where $B_r = \{x = (s,\theta) : \ |x - x^i| < r, \ x^i = (s^i,\theta^i)\} \subset [s^i - \delta, s^i + \delta] \times S^1$. Now we choose $0 < r = \tilde{C}^{-1}\eta_0 v_{min}$, where $\tilde{C} = 2Ce^{\frac{2}{3}\delta}\hat{C}$, $\hat{C} > 0$ is the constant such that $\int_{S^1} v^{-2}(s,\theta)d\theta \leq \hat{C}$. We can assume $4r < \delta$ such that $B_r \subset (s^i - \delta, s^i + \delta) \times S^1$. Otherwise, $r \geq \delta/4$, $v_{min} \geq \frac{\delta}{4}\tilde{C}\eta_0 > 0$ and this is our conclusion. We obtain that

$$\begin{aligned} \int_{B_r} \tilde{m}(x)dx &= \int_{B_r} \frac{Ce^{\frac{2}{3}(s-s^i)}}{v^3}dx \\ &\leq \int_{s_i - r}^{s_i + r}\int_{S^1} \frac{Ce^{\frac{2}{3}(s-s^i)}}{v^3}d\theta ds \\ &\leq \int_{s_i - r}^{s_i + r}\Big(\frac{Ce^{\frac{2}{3}(s-s^i)}}{v_{min}}\Big)\int_{S^1}\frac{d\theta ds}{v^2(s,\theta)} \\ &\leq \tilde{C}v_{min}^{-1} r = \eta_0. \end{aligned}$$

Thus,

$$Cv_{min}^{-3} = \tilde{m}(x^i) \leq \frac{C}{r^2}\int_{B_r}\tilde{m}(x)dx = Cv_{min}^{-2}.$$

This implies that

$$v_{min} \geq C > 0.$$

This completes the proof. □

Lemma 3.6. *There exists $C > 0$ independent of i such that*

$$v(s,\theta) \le C \quad \text{for } (s,\theta) \in [0,T_i] \times S^1 \tag{3.49}$$

and

$$|v_s(s,\theta)| \le C \quad \text{for } (s,\theta) \in [0,T_i] \times S^1. \tag{3.50}$$

Proof. We first show that

$$\max_{s\in[0,T_i]} \int_{S^1} v^2(s,\theta)d\theta \le C. \tag{3.51}$$

Then we apply regularity theory.

It follows from (3.21) that

$$\int_{S^1} v_\theta^2(s,\theta)d\theta - \frac{4}{9}\int_{S^1} v^2(s,\theta)d\theta = \frac{1}{2}\Big[w_{ss}(s) + \frac{4}{3}w_s(s)\Big] - \int_{S^1} v_s^2(s,\theta)d\theta - \int_{S^1} \frac{d\theta}{v(s,\theta)}.$$

Thus,

$$\int_{S^1} v_\theta^2(s,\theta)d\theta - \frac{4}{9}\int_{S^1} (v - \frac{1}{2\pi}\overline{v})^2 d\theta = \frac{1}{2}\Big[w_{ss}(s) + \frac{4}{3}w_s(s)\Big] - \int_{S^1} v_s^2(s,\theta)d\theta - \int_{S^1} \frac{d\theta}{v(s,\theta)} + \frac{4}{9\pi}\int_{S^1} v\overline{v}d\theta - \frac{2}{9\pi}\overline{v}^2(s). \tag{3.52}$$

Using the inequality

$$\int_{S^1} (v - \frac{1}{2\pi}\overline{v})^2 d\theta \le \int_{S^1} v_\theta^2(s,\theta)d\theta$$

we see from (3.52) that

$$\int_{S^1} v_\theta^2(s,\theta)d\theta \le C\Big(1 + \overline{v}^2(s) + \int_{S^1}\frac{d\theta}{v(s,\theta)} + \int_{S^1} v_s^2(s,\theta)d\theta\Big). \tag{3.53}$$

By the embedding theorem, we see that

$$v^2(s,\theta) \le C\int_{S^1} v_\theta^2 d\theta \le C\Big(1 + \overline{v}^2(s) + \int_{S^1}\frac{d\theta}{v(s,\theta)} + \int_{S^1} v_s^2(s,\theta)d\theta\Big). \tag{3.54}$$

On the other hand, multiplying $v_s(s,\theta)$ on both the sides of (3.13) and integrating it on $(0,s)\times S^1$, we obtain

$$\int_{S^1} v_\theta^2(s,\theta)d\theta = \int_{S^1} v_s^2(s,\theta)d\theta + \frac{4}{9}\int_{S^1} v^2(s,\theta)d\theta + 2\int_{S^1}\frac{d\theta}{v(s,\theta)} + f(s) \tag{3.55}$$

where

$$\begin{aligned} f(s) \quad = \quad & -2\int_{S^1}\frac{d\theta}{v(0,\theta)} + \int_{S^1} v_\theta^2(0,\theta)d\theta - \frac{4}{9}\int_{S^1} v^2(0,\theta)d\theta \\ & - \int_{S^1} v_s^2(0,\theta)d\theta + \frac{8}{3}\int_0^s\int_{S^1} v_\xi^2(\xi,\theta)d\theta d\xi. \end{aligned}$$

We easily see from (3.14), (3.17), (3.18) and Lemma 3.3 that $|f(s)| \le C$ for $s \in [0,T_i]$. Thus, combining (3.21) and (3.55), we see that

$$w_{ss} + \frac{4}{3}w_s = 4\int_{S^1} v_s^2(s,\theta)d\theta + 6\int_{S^1}\frac{d\theta}{v(s,\theta)} + 2f(s). \tag{3.56}$$

Integrating (3.54) on S^1 and using Lemmas 3.4 and 3.5, we see that

$$w(s) \le C\Big(1 + \int_{S^1} v_s^2(s,\theta)d\theta\Big). \tag{3.57}$$

(3.57), (3.56) and the facts that $|f(s)| \le C$, $v(s,\theta) \ge C$ imply that

$$w(s) \le C\Big(1 + w_{ss}(s) + \frac{4}{3}w_s(s)\Big). \tag{3.58}$$

If $w(s)$ attains its maximum $\max_{s\in[0,T_i]} w(s)$ at s_i^*, then there are three cases for s_i^*: (i) $s_i^* = 0$, (ii) $s_i^* = T_i$, (iii) $s_i^* \in (0,T_i)$. For the first two cases, we see from (3.14) that

$$w(s_i^*) \le C.$$

For the last case, we see that $w_{ss}(s_i^*) \le 0$ and $w_s(s_i^*) = 0$. Thus, it follows from (3.58) that

$$w(s_i^*) \le C.$$

Thus,

$$\max_{s\in[0,T_i]} \int_{S^1} v^2(s,\theta)d\theta \le C. \tag{3.59}$$

Now we obtain (3.49). We see from (3.13) that

$$\Delta v + \frac{4}{3}v_s + \Big(\frac{4}{9} - \frac{1}{v^3}\Big)v = 0. \tag{3.60}$$

It is known from Lemma 3.5 that the function $1/v^3$ is bounded for $(s,\theta) \in [0,T_i] \times S^1$. Then if $\max_{(s,\theta)\in[0,T_i]\times S^1} v(s,\theta)$ attains at the point (s_*^i, θ_*^i), we also have three cases: (i) $s_*^i = 0$, (ii) $s_*^i = T_i$, (iii) $s_*^i \in (0, T_i)$. For the first two cases, we see from (3.14) that

$$v(s_*^i, \theta_*^i) \leq C.$$

We only need to consider the third case. Note that we can assume $s_*^i \geq \tilde{\beta}$ and $s_*^i \leq \tilde{\beta} T_i$ for some $0 < \tilde{\beta} < 1/2$. On the contrary, we directly see from (3.15) and (3.16) that $v(s_*^i, \theta_*^i) \leq C$. This is our conclusion. Thus, we can find $0 < \delta_* < \tilde{\beta}/10$, $0 < r_* < 4\delta_*$ independent of i such that

$$B_{r_*}(s_*^i, \theta_*^i) \subset [s_*^i - \delta_*, s_*^i + \delta_*] \times S^1 \subset [0, T_i] \times S^1,$$

where $B_{r_*}(s_*^i, \theta_*^i)$ is the ball with center at (s_*^i, θ_*^i) and radius r_*. Then Theorem 8.17 of [18] implies that

$$\sup_{(s,\theta)\in B_{r_*}(s_*^i,\theta_*^i)} v(s,\theta) \leq C\Big(\int_{B_{r_*}(s_*^i,\theta_*^i)} v^2 dx + 1\Big) \leq C \tag{3.61}$$

here we have used (3.59). Therefore,

$$\max_{(s,\theta)\in[0,T_i]\times S^1} v(s,\theta) = v(s_*^i, \theta_*^i) \leq C. \tag{3.62}$$

By arguments similar to those in the proof of (3.62), we can also obtain that

$$\max_{(s,\theta)\in[0,T_i]\times S^1} |v_s(s,\theta)| \leq C.$$

Indeed, we see that v_s satisfies the equation

$$\Delta v_s + \frac{4}{3}v_{ss} + \Big[\frac{4}{9} + 2v^{-3}\Big]v_s = 0. \tag{3.63}$$

If $\max_{(s,\theta)\in[0,T_i]\times S^1} |v_s(s,\theta)| = \max_{(s,\theta)\in[0,T_i]\times S^1} v_s(s,\theta)$ and attains at the point $(\hat{s}^i, \hat{\theta}^i)$, we also have three cases: (i) $\hat{s}^i = 0$, (ii) $\hat{s}^i = T_i$, (iii) $\hat{s}^i \in (0, T_i)$. For the first two cases, we see from (3.17), (3.19) that

$$v_s(\hat{s}^i, \hat{\theta}^i) \leq C.$$

We only need to consider the third case. Similarly, we can also assume that $\hat{s}^i \geq \tilde{\beta}$ and $\hat{s}^* \leq \tilde{\beta} T_i$ for some $0 < \tilde{\beta} < 1/2$. Since the function v^{-3} is bounded, Theorem 8.17 of [GT] and the arguments exactly same as those in the proof of (3.62) imply that

$$\max_{(s,\theta)\in[0,T_i]\times S^1} v_s(s,\theta) = v_s(\hat{s}^i, \hat{\theta}^i) \leq C.$$

If $\max_{(s,\theta)\in[0,T_i]\times S^1} |v_s(s,\theta)| = \max_{(s,\theta)\in[0,T_i]\times S^1}(-v_s(s,\theta))$, noticing that $-v_s$ satisfies the equation in (3.63), we can use the same arguments to obtain

$$\max_{(s,\theta)\in[0,T_i]\times S^1} (-v_s(s,\theta)) \leq C.$$

This completes the proof. □

Finally, we can complete the proof of Theorem 3.1.

Proof of Theorem 3.1.

As we remarked earlier, we just need to verify (2.5) in Theorem 2.3.

It follows from Lemmas 3.5 and 3.6 that for all i sufficiently large,

$$v_i(s,\theta) \leq C \ \text{ for } (s,\theta) \in [0,T_i] \times S^1, \tag{3.64}$$

$$|(v_i)_s(s,\theta)| \leq C \ \text{ for } (s,\theta) \in [0,T_i] \times S^1, \tag{3.65}$$

$$v_i(s,\theta) \geq C > 0 \ \text{ for } (s,\theta) \in [0,T_i] \times S^1. \tag{3.66}$$

Therefore, it follows from the regularity of Δ that

$$v_i \to V \ \text{ in } C^1_{loc}([0,\infty) \times S^1) \text{ as } i \to \infty$$

and V satisfies the equation

$$V_{ss} + \frac{4}{3}V_s + V_{\theta\theta} + \frac{4}{9}V = \frac{1}{V^2} \ \text{ for } (s,\theta) \in [0,\infty) \times S^1.$$

Meanwhile, we see from (3.66) that $V(s,\theta) \geq C$. It follows from Theorem 1.3 of [17] that

$$\lim_{s\to\infty} V(s,\theta) = \tilde{\kappa}, \quad \tilde{\kappa} = \Big(\frac{4}{9}\Big)^{-1/3}. \tag{3.67}$$

This proves (2.5) and completes the proof. □

4. **Uniqueness of solutions for small voltage.** In the following we focus on the uniqueness of solutions of (S_λ) when λ is small enough. It is known from Theorem 1.1 that there exists a unique minimal solution $\underline{u}_\lambda$ of (S_λ) for $0 \leq \lambda < \lambda^*$. In Theorem 5.5 of [12], the authors showed that for every $M > 0$ there exists $0 < \lambda_1^*(M) < \lambda^*$ such that for $\lambda \in (0, \lambda_1^*(M))$, (S_λ) has a unique solution u_λ satisfying $||\frac{1}{(1-u_\lambda)^3}||_{L^{1+\epsilon}(\Omega)} \leq M$, where $0 < \epsilon < 1$ is a small number.

In this section, we shall show that there exists $0 < \lambda_* < \lambda^*$ such that for $\lambda \in (0, \lambda_*)$, (S_λ) has a unique solution, i.e., the minimal solution $\underline{u}_\lambda$.

Theorem 4.1. *There exists $0 < \lambda_* < \lambda^*$ such that for $\lambda \in (0, \lambda_*)$, (S_λ) has a unique solution, i.e. the minimal solution $\underline{u}_\lambda$.*

Note that Theorem 4.1 is not proven even in the ball case. Thus our result is new even in radially symmetric case.

We prove this theorem by a contradiction argument. Our main idea is to use arguments in Lemma 3.2-Lemma 3.4 and the monotonicity formula.

Suppose on the contrary. We see that there are sequences $\{\lambda_i\}$ and $\{u_i\} \equiv \{u_{\lambda_i}\}$ with $\lambda_i \to 0$ as $i \to \infty$ such that u_i is a non-minimal solution of (S_{λ_i}). A solution u is said to be a non-minimal solution of (S_λ), if $0 \leq u < 1$ in Ω and there exists another solution $0 \leq v < 1$ of (S_λ) and a point $x \in \Omega$ such that $u(x) > v(x)$.

We consider two cases here:

(i) there is a $0 < \rho < 1$ such that $||u_i||_\infty \leq 1 - \rho$ for all i (we can choose subsequences if necessary).

(ii) $||u_i||_\infty \to 1$ as $i \to \infty$ (we can choose subsequences if necessary).

We easily see from Theorem 5.5 of [12] that the first case does not occur since for this case (S_{λ_i}) has a unique minimal solution, but u_i is a non-minimal solution by our assumption. Note that there exists $M > 0$ such that

$$||\frac{1}{(1-u_i)^3}||_{L^{1+\epsilon}(\Omega)} \leq M$$

We only need to consider the second case. Defining $z_i = 1 - u_i$, we see that $\min_\Omega z_i \to 0$ as $i \to \infty$. We shall use the same notation as in Section 3. Note that z_i satisfies the problem

$$\Delta z_i = \frac{\lambda_i}{z_i^2} \ \text{ in } \Omega, \quad z_i = 1 \ \text{ on } \partial\Omega. \tag{4.1}$$

We define $\epsilon_i = \min_\Omega z_i$ and

$$\hat{U}_i(y) = \frac{z_i(\epsilon_i^{3/2} y)}{\epsilon_i}, \ y \in \Omega_i := \{y : \ \epsilon_i^{3/2} y \in \Omega\}. \tag{4.2}$$

Then

$$\Delta \hat{U}_i = \lambda_i \hat{U}_i^{-2} \ \text{ in } \Omega_i, \quad \hat{U}_i(0) = 1, \quad \hat{U}_i(z) \geq 1. \tag{4.3}$$

We shall show that such $\hat{U}_i$ does not exist.

First we use Pohozaev's identity to show that Lemma 3.2 still holds in this case.

Lemma 4.2. *For any $\beta \in (0,1)$, there is $\kappa := \kappa(\beta) \in (0,1)$ independent of i such that for i sufficiently large,*

$$z_i(x) \geq \kappa \ \textit{for } x \in \Omega \backslash B(0, \beta r_0), \tag{4.4}$$

where r_0 is given in Lemma 3.2.

Proof. By the well-known Pohozaev identity, we have that

$$\int_{\partial\Omega} < x, n > \Big(\frac{\partial z_i}{\partial n}\Big)^2 dS = 2\lambda_i \int_\Omega \frac{dx}{z_i} - \lambda_i \int_{\partial\Omega} \frac{< x, n >}{z_i} dS \tag{4.5}$$

where n is the unit outward normal vector of $\partial\Omega$. By the assumptions on Ω, we see that $< x, n > \geq \text{dist}(0, \partial\Omega) \geq C > 0$ for $x \in \partial\Omega$. Therefore, it follows from (4.5) that

$$\int_{\partial\Omega} \Big(\frac{\partial z_i}{\partial n}\Big)^2 dS \leq C\lambda_i \int_\Omega \frac{dx}{z_i}. \tag{4.6}$$

On the other hand, it follows from the equation of z_i that

$$\int_{\partial\Omega} \frac{\partial z_i}{\partial n} dS = \lambda_i \int_\Omega \frac{dx}{z_i^2}.$$

Thus,

$$\Big(\lambda_i \int_\Omega (z_i)^{-2} dx\Big)^2 \le C \int_{\partial\Omega} \Big(\frac{\partial z_i}{\partial n}\Big)^2 dS \le C\lambda_i \int_\Omega (z_i)^{-1} dx \le C\lambda_i \Big(\int_\Omega (z_i)^{-2} dx\Big)^{1/2}.$$

This implies

$$\int_\Omega (z_i)^{-2} dx \le C\lambda_i^{-2/3}. \tag{4.7}$$

By the standard moving plane argument as that in the proof of Lemma 3.2, we see that for any $x \in \Omega\backslash B(0,\beta r_0)$, there exists a piece of cone with vertex at x: Γ_x with (i) meas$(\Gamma_x) \ge \gamma > 0$, where γ is independent of x, (ii) $\Gamma_x \subset \Omega$, (iii) $z_i^{-2}(y) \ge z_i^{-2}(x)$ for any $y \in \Gamma_x$. Thus, it follows from (i)-(iii) and (4.7) that

$$z_i^{-2}(x) \le \frac{1}{\text{meas}(\Gamma_x)} \int_{\Gamma_x} z_i^{-2}(y) dy \le \gamma^{-1} C\lambda_i^{-2/3}. \tag{4.8}$$

This implies that

$$\lambda_i z_i^{-2}(x) \le C\lambda_i^{1/3} \ \text{ for } x \in \Omega\backslash B(0,\beta r_0). \tag{4.9}$$

Let k_i be the solution of the problem

$$\begin{cases} \Delta k_i = C\lambda_i^{1/3} & \text{in } \Omega\backslash B(0,\beta r_0), \\ k_i = 1 & \text{on } \partial\Omega \\ k_i = 0 & \text{on } \partial B(0,\beta r_0). \end{cases}$$

Then the maximum principle implies that

$$z_i \ge k_i \ \text{ in } \Omega\backslash B(0,\beta r_0).$$

Since $k_i = k_0 + C\lambda_i^{1/3} k_1$, where

$$\begin{cases} \Delta k_0 = 0 & \text{in } \Omega\backslash B(0,\beta r_0), \\ k_0 = 1 & \text{on } \partial\Omega \\ k_i = 0 & \text{on } \partial B(0,\beta r_0), \end{cases}$$

and

$$\begin{cases} \Delta k_1 = 1 & \text{in } \Omega\backslash B(0,\beta r_0), \\ k_0 = 0 & \text{on } \partial\Omega \\ k_i = 0 & \text{on } \partial B(0,\beta r_0), \end{cases}$$

we see that

$$z_i \ge k_0 + C\lambda_i^{1/3} k_1.$$

Note that the maximum principle implies $k_0 > 0$ in $\Omega\backslash B(0,\beta r_0)$ and $|k_1(x)| \le C$ for $x \in \Omega\backslash B(0,\beta r_0)$. We see that

$$z_i(x) \ge C \ \text{ for } x \in \Omega\backslash B(0,\tau\beta r_0) \text{ and } i \text{ sufficiently large} \tag{4.10}$$

where $\tau > 1$ is close to 1. The arbitrary of β implies our conclusion. This completes the proof. □

Proof of Theorem 4.1

Let $\hat{U}_i$ be defined at (4.2) and $R_i = \epsilon_i^{-3/2} r_0$. We see that

$$\hat{U}_i(R_i) = R_i^{2/3} r_0^{-2/3} z_i(r_0).$$

Therefore, if we define

$$v_i(s,\theta) = |y|^{-2/3}\hat{U}_i(y), \quad |y| = e^s, \tag{4.11}$$

we easily see that for $s \in [0, T_i]$ (where $e^{T_i} = R_i$), v_i satisfies the equation:

$$v_{ss} + \frac{4}{3} v_s + v_{\theta\theta} + \frac{4}{9} v = \lambda_i v^{-2}. \tag{4.12}$$

Moreover, by the definition of $\hat{U}_i(R_i)$ and the fact that $\hat{U}_i \to 1$ in $C^1_{loc}(\mathbb{R}^2)$ as $i \to \infty$, we obtain that for i sufficiently large,

$$v_i(0,\theta) = O(1), \quad v_i(T_i,\theta) = O(1) \ \text{ for } \theta \in S^1. \tag{4.13}$$

We also see that for any $0 < \tilde{\beta} < 1$ independent of i such that

$$v_i(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [0,\tilde{\beta}] \times S^1 \tag{4.14}$$

and

$$v_i(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [\tilde\beta T_i, T_i] \times S^1. \tag{4.15}$$

(4.14) and (4.15) are obtained from the fact that $\hat U_i \to 1$ in $C^1_{loc}(\mathbb{R}^2)$ as $i \to \infty$ and Lemma 4.2. Moreover, easy calculations show that

$$(v_i)_s(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [0, \tilde\beta] \times S^1 \tag{4.16}$$

and

$$(v_i)_s(s,\theta) = O(1) \ \text{ for } (s,\theta) \in [\tilde\beta T_i, T_i] \times S^1. \tag{4.17}$$

Exactly the same arguments as those in the proof of Lemma 3.4 imply

$$\max_{s\in[0,T_i]} \int_{S^1} v_i(s,\theta) d\theta \le C \tag{4.18}$$

and

$$\lambda_i \max_{s\in[0,T_i]} \int_{S^1} \frac{d\theta}{v_i^2(s,\theta)} \le C. \tag{4.19}$$

Thus,

$$\lambda_i \int_{S^1} \frac{d\theta}{v_i(s,\theta)} \le C\lambda_i \Big(\int_{S^1} \frac{d\theta}{v_i^2(s,\theta)} \Big)^{1/2} \ \text{ for } s \in [0, T_i]$$

and hence

$$\lambda_i \int_{S^1} \frac{d\theta}{v_i(s,\theta)} \le C\lambda_i^{1/2} \ \text{ for } s \in [0, T_i]. \tag{4.20}$$

Now we use the monotonicity formula in Lemma 2.2: since $\hat U_i \in C^2(\Omega_i)$, we easily see that $\hat U_i$ is stationary. By Lemma 2.2, the function

$$\mathcal{E}_{\hat U_i}(r) := -\frac{3}{2} r^{-4/3} \int_{B(0,r)} \frac{\lambda_i}{\hat U_i} dy + \frac{1}{4}\frac{d}{dr}\Big[r^{-4/3} \int_{\partial B(0,r)} \hat U_i^2 dS \Big] - \frac{1}{4} r^{-7/3} \int_{\partial B(0,r)} \hat U_i^2 dS$$

is a nondecreasing function of r. Moreover, a simple calculation implies that under the changes:

$$v_i(s,\theta) = |y|^{-2/3} \hat U_i, \quad |y| = e^s$$

the function $\mathcal{E}_{\hat U_i}(r)$ is just a positive multiple of $\mathcal{E}_{v_i}(s) = (w_i)'(s) - 6h_i(s)$. Hence $\mathcal{E}_{v_i}(s)$ is a nondecreasing function of s for $s \in [0, T_i]$, where

$$w_i(s) = \int_{S^1} v_i^2(s,\theta) d\theta, \quad h_i(s) = \lambda_i \int_{-\infty}^s e^{\frac{4}{3}(\xi - s)} \int_{S^1} \frac{1}{v_i}. \tag{4.21}$$

We see from (4.20) that

$$h_i(s) \le C\lambda_i^{1/2} \ \text{ for } s \in [0, T_i]. \tag{4.22}$$

Now we claim that

$$w_i'(T_i) = o(1) - \frac{4}{3} w_i(T_i) \tag{4.23}$$

where $w_i(s) = \int_{S^1} v_i^2(s,\theta) d\theta$. Indeed, it follows from the equation of $\overline{v}_i$ that

$$\begin{aligned}
\overline{v}_i'(T_i) + \frac{2}{3}\overline{v}_i(T_i) &= \int_{-\infty}^{T_i} e^{\frac{2}{3}(s - T_i)} \int_{S^1} \frac{\lambda_i}{v_i^2} d\theta \\
&\le C\lambda_i \int_{\tilde\beta T_i}^{T_i} e^{\frac{2}{3}(s-T_i)} ds + \int_{-\infty}^{\tilde\beta T_i} e^{\frac{2}{3}(s-T_i)} \int_{S^1} \frac{\lambda_i}{v_i^2} d\theta \\
&\le C\lambda_i (1 - e^{\frac{2}{3}(\tilde\beta - 1)T_i}) + Ce^{\frac{2}{3}(\tilde\beta - 1)T_i} \\
&= o(1) \ \text{ for } i \text{ sufficiently large}
\end{aligned}$$

Here we have used (4.15) and (4.19). Thus, we see that

$$\begin{aligned}
w_i'(T_i) &= 2\int_{S^1} v_i(T_i,\theta)(v_i)_s(T_i,\theta) d\theta \\
&= 2\int_{S^1} v_i(T_i,\theta)[(v_i)_s(T_i,\theta) + \frac{2}{3} v_i(T_i,\theta)] d\theta - \frac{4}{3} w_i(T_i) \\
&= O(1)[\overline{v}_i'(T_i) + \frac{2}{3}\overline{v}_i(T_i)] - \frac{4}{3} w_i(T_i) \\
&= o(1) - \frac{4}{3} w_i(T_i).
\end{aligned}$$

This is our claim. Thus, it follows from (4.23) that

$$w_i'(T_i) \le -C$$

and

$$\mathcal{E}_{v_i}(T_i) \le -C.$$

The monotonicity of $\mathcal{E}_{v_i}(s)$ of s implies

$$w_i'(s) - 6h_i(s) \le -C \quad \text{for } s \in [0, T_i]. \tag{4.24}$$

Then

$$h_i(s) \ge C(1 + w_i'(s)) \quad \text{for } s \in [0, T_i]. \tag{4.25}$$

Integrating (4.25) from 0 to T_i and using (4.13), (4.22), we see that

$$\lambda_i^{1/2} T_i \ge C T_i + C$$

and

$$\lambda_i^{1/2} \ge C > 0 \tag{4.26}$$

This is a contradiction. This completes the proof of Theorem 4.1. □

5. **Proof of Theorem 1.3.** In this section we complete the proof of Theorem 1.3. (1) of Theorem 1.3 follows from Theorem 4.1. So we just need to prove (2). By using Theorem 3.1, the main ideas of the proof of (2) are similar to those in [7].

We first note that, by the implicit theorem, for $\lambda \in (0, \lambda^*)$, the operator $I - \lambda A'(\underline{u}_\lambda)$ is invertible. Here $A(u) = (-\Delta)^{-1} \frac{1}{(1-u)^2}$. Thus, $(\lambda, \underline{u}_\lambda)$ for $0 < \lambda < \lambda^*$ is a simple curve of D. We can argue as in section 2.1 of Buffoni, Dancer and Toland [3] and [4]-[5] in the space $C^1(\overline{\Omega}) \times \mathbb{R}$, to find a analytic curve $\lambda = \tilde{\lambda}(t)$, $u = \tilde{u}(t)$ for $t \ge 0$ such that $\|\tilde{u}(t)\|_\infty \to 1$ as $t \to \infty$, $(\tilde{u}(t), \tilde{\lambda}(t)) \in D$ for $t \ge 0$, $(\tilde{u}(0), \tilde{\lambda}(0)) = (0, 0)$ and $I - \tilde{\lambda}(t) A'(\tilde{u}(t))$ is invertible except at isolated points. Note that we allow the curve $(\tilde{u}(t), \tilde{\lambda}(t))$ to have isolated intersections and $\|\cdot\|_{1,\infty}$ is the usual norm on $C^1(\overline{\Omega})$. If we now use the usual trick of finding a minimal continuum in $\{(\tilde{u}(t), \tilde{\lambda}(t)) : t \ge 0\}$ joining $(0, 0)$ to "infinity", we obtain a curve with no self intersections but is only piecewise analytic and continuous and $I - \lambda A'(u)$ is invertible except at isolated points. To obtain a minimal irreducible continuum as in Whyburn [28], we can use arguments as in [7]. Let us denote this curve by T and parameterize it by $(\hat{u}(t), \hat{\lambda}(t))$ for $t \ge 0$. Let $\mu_{i,\hat{\lambda}(t)}(\hat{u}(t))$ be the ith eigenvalue counting multiplicity of

$$-\Delta - \frac{2\hat{\lambda}(t)}{(1 - \hat{u}(t))^3} I \tag{5.1}$$

on Ω with Dirichlet boundary condition. (The definition of $\mu_{i,\hat{\lambda}(t)}(\hat{u}(t))$ is given in Section 1.) By our comments above, $\mu_{i,\hat{\lambda}(t)}(\hat{u}(t))$ are continuous, piecewise analytic and have only isolated zeros. We will show blow that $\mu_{i,\hat{\lambda}(t)}(\hat{u}(t)) < 0$ for large t. This means that for any $\zeta > 0$, (5.1) has at least ζ negative eigenvalues for t large. Hence we see that there is a sequence $\{t_i\}$ with $t_i \to \infty$ as $i \to \infty$ such that the number of negative eigenvalues of (5.1) (counting multiplicity) changes at t_i. (Recall that $\mu_{i,\hat{\lambda}(0)}(\hat{u}(0)) = \mu_i(-\Delta) \to +\infty$ as $i \to \infty$). Each $(\hat{u}(t_i), \hat{\lambda}(t_i))$ must be a bifurcation point. Otherwise the solutions near $(\hat{u}(t_i), \hat{\lambda}(t_i))$ are a curve parameterized by λ, the critical groups of these solutions must be locally independent of λ by homotopy invariance of the critical groups (where critical groups are defined in Chang [6]). By the formula for the critical groups at a non-degenerate point (see [6], p.33), this implies that the number of negative eigenvalues of the linearization counting multiplicity must be constant in a deleted neighborhood of $(\hat{u}(t_i), \hat{\lambda}(t_i))$ which contradicts our choice of t_i. (There is a minor technical point here. We need to work in the space $H_0^1(\Omega)$. We choose $\|\hat{u}(t_i)\|_\infty < \tau < 1$ and then smoothly truncate the function $\frac{1}{(1-s)^2}$ such that it equals $\frac{1}{(1-\tau)^2}$ for $1 > s > \tau$ so the equation makes sense on $H_0^1(\Omega)$. Note that the truncation will not affect the solutions close to $(\hat{u}(t_i), \hat{\lambda}(t_i))$ in $H_0^1(\Omega) \times \mathbb{R}$.)

To prove our claim on $\mu_{i,\hat{\lambda}(t)}(\hat{u}(t))$ for large t, we need to consider positive solutions (u_i, λ_i) of (S_λ) such that $\lambda_i \to \kappa \in (0, \infty)$ as $i \to \infty$ and $\|u_i\|_\infty \to 1$ as $i \to \infty$. (Note that Theorem 4.1 implies that $\lambda_i \not\to 0$ as $i \to \infty$). Thus, we see that there is t_i with $t_i \to \infty$ such that $\hat{\lambda}(t_i) = \lambda_i$ and $\hat{u}(t_i) = u_i$. We use a blowing up argument as in Section 3. If we define $\epsilon_i = 1 - \|u_i\|_\infty$ and

$$U_i(y) = \frac{1 - u_i(\epsilon_i^{3/2} \lambda_i^{-1/2} y)}{\epsilon_i}, \quad y \in \Omega_i := \{y : \epsilon_i^{3/2} \lambda_i^{-1/2} y \in \Omega\},$$

then U_i is defined on a "large" domain Ω_i, $U_i(0) = \min_{\Omega_i} U_i = 1$. A rather standard limiting argument shows that a subsequence of the U_i converges uniformly on compact set to a positive solution U of $\Delta U = \frac{1}{U^2}$ on $\mathbb{R}^2$ such that $U(0) = 1$, $U(z) \geq 1$. It follows easily from the equation that $U(z) > 0$ for all z. By the Gidas-Ni-Nirenberg theorem [15] and our assumptions, $x_j(\partial u_i/\partial x_j) \leq 0$ on Ω (where no summation is intended). After the rescaling and the limit argument, we find that $x_j(\partial U/\partial x_j) \geq 0$ on $\mathbb{R}^2$ and U is even in x_j. By Theorem 3.1, we conclude that U is radially symmetric and hence unique.

We now claim that the solution q of

$$-h''(r) - \frac{1}{r}h'(r) = \frac{2}{U^3}h(r), \quad h(0) = 1 \tag{5.2}$$

has infinitely many positive zeros. To prove this claim, we first notice that it follows from [17] that $r^{-2/3}U(r) \to (4/9)^{-1/3}$ as $r \to \infty$. Hence $\frac{2}{U^3} \sim \frac{8}{9}r^{-2}$ as $r \to \infty$. On the other hand, by explicitly solving the equation (it is an Euler equation), one finds that any non-trivial solution of

$$-k'' - \frac{1}{r}k' - (\mu/r^2)k = 0 \tag{5.3}$$

has infinitely many (and unbounded) positive zeros if $\mu > 0$. (Note that under the changes: $r = e^s$ and $\tilde{k}(s) = k(r)$, we see that $\tilde{k}(s)$ satisfies the equation

$$\tilde{k}''(s) + \mu\tilde{k}(s) = 0.$$

It is easily seen that $\tilde{k}(s)$ has infinitely many positive zeroes for any $\mu > 0$.) Thus, we can easily deduce that q has infinitely many positive zeros. Our claim holds.

We now in the position to complete the proof of Theorem 1.3. If $N > 0$ and σ is small and negative, we see by continuous dependence that the solution $\tilde{q}$ of

$$-h''(r) - \frac{1}{r}h'(r) = \frac{2}{U^3}h(r) + \sigma h, \quad h(0) = 1 \tag{5.4}$$

has at least N positive zeros. Note that the solution of (5.2) is unique. Let h_i be the function defined to be $\tilde{q}(|x|)$ for $|x|$ between the ith and $(i+1)$th the zeros of $\tilde{q}$ and to be zero otherwise. Then $h_i \in H^1(\mathbb{R}^2)$, h_i are orthogonal (in $L^2(\mathbb{R}^2)$ or $H^1(\mathbb{R}^2)$) and by multiplying (5.4) by h_i and integrating between these zeros we see that

$$Q(h) = \int_{\mathbb{R}^2} \left[\frac{1}{2}|\nabla h|^2 - \frac{1}{U^3}h^2\right]$$

is strictly negative at each h_i. Hence the span of h_i is an $(N-1)$-dimensional subspace of $C_0^\infty(\mathbb{R}^2)$ such that $Q(h) < \tilde{\mu} < 0$ if h is in the unit sphere of T. Since h_i has compact support it follows easily that there is an $(N-1)$-dimensional subspace of $H_0^1(\Omega_i)$ such that

$$\int_{\Omega_i} |\nabla h(y)|^2 - \frac{2(1 - \|\hat{u}(t_i)\|_\infty)^3}{(1 - \hat{u}(t_i)(\tau_i y))^3}h^2(y) < 0$$

where $\tau_i = (1 - \|\hat{u}(t_i)\|_\infty)^{3/2}[\hat{\lambda}(t_i)]^{-1/2}$ for large t_i if h is in the unit sphere in T. (Note that Ω_i, which is Ω rescaled has the property that each function in T is supported in Ω_i for large i.)

Hence returning to the original scaling we see that there is an $(N-1)$-dimensional subspace T_i of $H_0^1(\Omega)$ such that

$$\int_{\Omega} |\nabla h|^2 - \frac{2\hat{\lambda}(t)}{(1 - (\hat{u}(t))^3}h^2 < 0$$

for h is in the unit sphere of T_i and t large. By the variational characterization of eigenvalues, this implies that $\mu_{i,\hat{\lambda}(t)}(\hat{u}(t)) < 0$ for $1 \leq i \leq N-1$ if t is large. Since N is arbitrary, this proves our claim and completes the proof of Theorem 1.3.

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