EXISTENCE, STABILITY AND METASTABILITY OF POINT CONDENSATION PATTERNS GENERATED BY GRAY-SCOTT SYSTEM

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ABSTRACT. Of concern are point condensation patterns in the generalized Gray-Scott model in higher dimensional case. We first establish the existence of boundary condensation or interior condensation under some assumptions on the geometry of the domain. Then we study the stability of such patterns. It turns out the stability of point condensations depends on three factors: the domain geometry, the boundary conditions and the exponents in the reaction terms. We also classify the instability and metastability of point condensations.

1. INTRODUCTION

Recently there is a great interest in the study of self-replicating patterns observed in the irreversible Gray-Scott model which governs the chemical reactions $\mathcal{U} + 2\mathcal{V} \to 3\mathcal{V}$ and $\mathcal{V} \to \mathcal{P}$ in a gel reactor, where $\mathcal{U}$ and $\mathcal{V}$ are two chemical species, $\mathcal{V}$ catalyzes its own reaction with $\mathcal{U}$ and $\mathcal{P}$ is an inert product. See [8], [11], [12], [31], [30], [32], [33] for more details. Letting $U = U(x,t)$ and $V = V(x,t)$ denote the concentrations of $\mathcal{U}$ and $\mathcal{V}$, respectively, the pair of coupled reaction-diffusion equations governing these reactions is:

$$\begin{cases} V_t = D_V \Delta V - BV + UV^2, & x \in R^N \\ U_t = D_U \Delta U - UV^2 + A(1-U), & x \in R^N \end{cases} \tag{1.1}$$

where A denotes the rate at which $\mathcal{U}$ is fed from the reservoir into the reactor, the concentration of $\mathcal{V}$ in the reservoir is assumed to be zero, and $B = A + k$, where k is the rate at which $\mathcal{V}$ is converted to an inert product. Here the diffusivities, D_U and D_V, of $\mathcal{U}$ and $\mathcal{V}$, respectively, can be any chemically relevant positive numbers.

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A pulse may loosely be defined as a region of high V and low U (if the region shrinks to a point, then it is called point-condensation). The numerical simulations suggest that during a peak in V, it is observed that V is large while U becomes small. It was observed numerically that single pulse may be stable. When $N = 1$, pulse spitting was observed for $D_U = 1$ and $D_V = 0.01$. By contrast, in the two-dimensional numerical simulations, the spot replication was observed with $D_U = 2D_V = 2 \times 10^{-5}$. See Pearson [31] for more numerical results.

In [8], by using Mel'nikov method, Doelman, Kaper and Zegeling constructed single and multiple pulse solutions for (1.1) in the case $N = 1, D_U = 1, D_V = \delta^2 << 1$. In their paper [8], an important assumption is that $A \sim \delta^2, B \sim \delta^{2\alpha/3}$, where $\alpha \in [0, \frac{3}{2})$. In this case, they showed that $U = O(\delta^\alpha), V = O(\delta^{-\frac{\alpha}{3}})$. On the other hand, Reynonds, Ponce-Dawson and Pearson [32] gave a formal construction of single and multiple pulses in the case $N = 1, D_U = 1, D_V = \delta^2$.

In this paper, we study the following generalized Gray-Scott model (first discussed in [pp. 193, Section IV, [32]])

$$\begin{cases} V_t = D_V \Delta V - bV + U^q V^p, \quad x \in \Omega \\ U_t = D_U \Delta U - U^s V^r + a(1-U), \;\; x \in \Omega \end{cases} \tag{1.2}$$

where $p > 1, q > 0, s \geq 0, r > 0, \epsilon > 1, s(p-1) \leq qr, b > a > 0$ and $\Omega \subset R^N$ is a smooth bounded domain. (The condition that $s(p-1) \leq qr$ was introduced in [32] and will be needed in the existence of point condensation solutions. See (1.9) below.) In this paper, we take $a = O(1)$ and $b = O(1)$. We shall study the existence, stability, instability, and metastability of the stationary interior or boundary single-pulse solutions to the system (1.2) in the case when D_V is very small and D_U is very large.

More precisely, we consider the following system of elliptic equations

$$\begin{cases} \epsilon^2 \Delta V - bV + U^q V^p = 0, \quad x \in \Omega \\ D\Delta U - U^s V^r + a(1-U) = 0, \;\; x \in \Omega \end{cases} \tag{1.3}$$

where $\Omega \subset R^N$ $(N \leq 3)$ is a smooth and bounded domain, $a > 0, b > 0$ are fixed constants; $\epsilon > 0, D > 0$; $\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in R^N.

The exponents p, q, r, s are assumed to satisfy the condition

$$(A) \qquad 1 < p < (\frac{N+2}{N-2})_+, q > 0, r > 0, s > 0, \text{ and } 0 < s(p-1) < qr$$

where $(\frac{N+2}{N-2})_+ = \frac{N+2}{N-2}$ if $N \geq 3$;$= +\infty$ if $N = 1, 2$.

We assume that there is no flux of U through the boundary, i.e., we impose the Neumann boundary condition for U:

$$\frac{\partial U}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.4}$$

For V, we can impose either Neumann boundary condition

$$(N) \qquad \frac{\partial V}{\partial \nu} = 0 \text{ on } \partial\Omega$$

or Dirichlet boundary condition

$$(D) \qquad V = 0 \text{ on } \partial\Omega.$$

Problem (1.3) with (1.4) and (N) (or (D)) is denoted by $(1.3)_N$ (or $(1.3)_D$).

Note that the system (1.3) doesn't have a variational structure. Generally speaking the existence of solutions to (1.3) is very difficult. (The results of [8] give the existence in R^1. However it is hard to generalize to higher dimension.) In this paper, we follow the idea of Turing [34]. Namely we assume that $D_V = \epsilon^2$ is small and $D_U = D$ is large. This implies that for the quotient of the two diffusion coefficients

$$\frac{D_U}{D_V} \to \infty.$$

This is the situation described in the pioneering work of Turing [34]. His idea is that if, in the absence of diffusion, both U and V tend to a linearly stable uniform steady state, then, under certain conditions, spatially homogeneous patterns can evolve by diffusion-driven instability if the diffusion coefficients are different. Since diffusion is usually describing a stabilizing process this is a novel concept. This idea has been used in other systems, see [29].

Now we let $D \to +\infty$ first and assume that $U \to U_\infty$. Then U_∞ satisfies

$$\begin{cases} \Delta U_\infty = 0 \text{ in } \Omega \\ \frac{\partial U_\infty}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.5}$$

Hence U_∞ is a constant, say $U_\infty \equiv U_\epsilon$. To find out the constant U_ϵ, we integrate the equation for U, we obtain the following system

$$\begin{cases} \epsilon^2 \Delta V - bV + V^p U_\epsilon^q = 0, V > 0 & \text{in } \Omega, \\ a(1-U_\epsilon) - U_\epsilon^s \frac{1}{|\Omega|}\int_\Omega V^r = 0, \\ \frac{\partial V}{\partial \nu}|_{\partial\Omega} = 0 \ \text{ or } V|_{\partial\Omega} = 0. \end{cases} \tag{1.6}$$

(System (1.6) is called the *shadow system* of (1.3).)

Thus if we let

$$V_\epsilon(x) = U_\epsilon^{-q/(p-1)} v_\epsilon, \tag{1.7}$$

where $v_\epsilon(x)$ is a solution of the following equation

$$\begin{cases} \epsilon^2 \Delta v - bv + v^p = 0, v > 0 & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu}|_{\partial\Omega} = 0 \text{ or } v|_{\partial\Omega} = 0, \end{cases} \tag{1.8}$$

and U_ϵ satisfies

$$a(1-U_\epsilon) - U_\epsilon^{s - \frac{qr}{(p-1)}} \frac{1}{|\Omega|}\int_\Omega v_\epsilon^r = 0, \tag{1.9}$$

then (V_ϵ, U_ϵ) is a solution of (1.6).

We note that if $\int_\Omega v_\epsilon^p = o(1)$, then equation (1.9) has a unique solution of the form

$$U_\epsilon = (1+o(1))(\frac{1}{a|\Omega|}\int_\Omega v_\epsilon^r)^{-\frac{p-1}{qr-s(p-1)}} = o(1).$$

From now on, we always refer U_ϵ as the unique small solution to (1.9). (Note that (1.9) may contain other solutions which are large. See [44] in the case $p=2, q=1, r=2, s=1$.)

Equation (1.8) has been studied by many authors. It is known that equation (1.8) has both boundary spike solutions (in the Neumann boundary condition case) and interior spike solutions (both in the Neumann and Dirichlet case). For the boundary spike solutions, please see [4], [5], [13], [15], [17], [22], [23], [24], [35], [37], [38], [39], and the references therein. (When $p = \frac{N+2}{N-2}, N \geq 3$, boundary spike solutions of (1.8) have been studied in [1], [2], [3], [9], [10], [21], etc.) For the single interior spike solutions, please see [27], [36], [40], [41].

In particular, in [37], the author proved the following theorem on the existence of boundary spike solutions. (Let $H(P)$ be the mean curvature function on $\partial\Omega$. Set $G_B(P) := (\nabla^2_{\tau_P} H(P))$ where ∇_{τ_P} are the tangential derivatives of $P \in \partial\Omega$.)

Theorem A. *Let $P_0 \in \partial\Omega$ be a nondegenerate critical point of the mean curvature function $H(P)$. Then for ϵ sufficiently small, problem (1.8) with Neumann boundary condition has a solution $v_\epsilon^B(x)$ such that $v_\epsilon^B(x)$ has only one local maximum point P_ϵ and $P_\epsilon \in \partial\Omega$. Moreover $P_\epsilon \to P_0$ as $\epsilon \to 0$ and $v_\epsilon^B(y) := v_\epsilon^B(\epsilon y + P_\epsilon) \to w(y)$ as $\epsilon \to 0$ uniformly for $y \in \Omega_{\epsilon,P_\epsilon} := \{y | \epsilon y + P_\epsilon \in \bar{\Omega}\}$, where w is the unique solution of the following problem*

$$\begin{cases} \Delta w - bw + w^p = 0, w > 0 \quad in \ R^N, \\ w(0) = \max_{y \in R^N} w(y), w(y) \to 0 \ \ as \ |y| \to \infty. \end{cases} \tag{1.10}$$

Remark: It is known that the solution w to (1.10) is radial, unique and decays exponentially. (See [14], [18].)

In [40], the author constructed single interior spike solution to (1.8). We first introduced the following definition: For each $P_0 \in \Omega$, let

$$d\mu_{P_0}(z) = \lim_{\varepsilon \to 0} \frac{e^{-\frac{2|z-P_0|}{\varepsilon}} dz}{\int_{\partial\Omega} e^{-\frac{2|z-P_0|}{\varepsilon}} dz}. \tag{1.11}$$

$P_0 \in \Omega$ is called a *"nondegenerate peak point"* if the following holds: there exists $a \in R^N$ such that

$$\int_{\partial\Omega} e^{<z-P_0,a>} (z - P_0) d\mu_{P_0}(z) = 0 \tag{H1}$$

and

$$G_I(P_0) := \Big(\int_{\partial\Omega} e^{<z-P_0,a>} (z-P_0)_i (z-P_0)_j d\mu_{P_0}(z) \Big) \text{ is nonsingular.} \tag{H2}$$

(Here $(z - P_0)_i$ is the $i-$th component of $(z - P_0)$. The vector a in (H1) and (H2) is unique. For a proof of the above facts, see [41].)

In [40], the author proved the following two theorems.

Theorem B. *Let $P_0 \in \Omega$ be a nondegenerate peak point. Then for ϵ sufficiently small, problem (1.8) with Neumann boundary condition has a solution $v^I_{\epsilon,N}(x)$ such that $v^I_{\epsilon,N}(x)$ has only one local maximum point P_ϵ and $P_\epsilon \to P_0$ as $\epsilon \to 0$. Furthermore, $v^I_{\epsilon,N}(y) := v^I_{\epsilon,N}(\epsilon y + P_\epsilon) \to w(y)$ as $\epsilon \to 0$ uniformly for $y \in \Omega_{\epsilon,P_\epsilon}$, where w is the unique solution of (1.10).*

Theorem C. *Let $P_0 \in \Omega$ be a nondegenerate peak point. Then for ϵ sufficiently small, problem (1.8) with Dirichlet boundary condition has a solution $v^I_{\epsilon,D}(x)$ such that $v^I_{\epsilon,D}(x)$ has only one local maximum point P_ϵ and $P_\epsilon \to P_0$ as $\epsilon \to 0$. Furthermore, $v^I_{\epsilon,D}(y) := v^I_{\epsilon,D}(\epsilon y + P_\epsilon) \to w(y)$ as $\epsilon \to 0$ uniformly for $y \in \Omega_{\epsilon,P_\epsilon}$, where w is the unique solution of (1.10).*

We now begin to study system (1.3). Set

$$\tau := \frac{1}{D}.$$

Let $v^B_\epsilon, v^I_{\epsilon,N}, v^I_{\epsilon,D}$ be the solutions constructed in Theorems A-C. We denote the corresponding solutions to the shadow system (1.6) by $(V^B_\epsilon, U^B_\epsilon), (V^I_{\epsilon,N}, U^I_{\epsilon,N})$, $(V^I_{\epsilon,D}, U^I_{\epsilon,D})$ (defined by (1.7) and (1.9)). We shall construct the corresponding solutions to (1.3) in the case $\tau \to 0$.

We first consider the existence of boundary condensations for the system $(1.3)_N$.

Theorem 1.1. *Assume that (p, q, r, s) satisfies assumption (A). Suppose $P_0 \in \partial\Omega$ is a nondegenerate critical point of $H(P)$. Then for $\epsilon << 1, \tau << 1$, problem $(1.3)_N$ has a solution $(V^B_{\epsilon,\tau}(x), U^B_{\epsilon,\tau}(x))$ with the following properties:*

(a) $V^B_{\epsilon,\tau} = (U^B_\epsilon)^{-q/(p-1)}(v^B_\epsilon(x) + o(1))$,

(b) $U^B_{\epsilon,\tau} = U^B_\epsilon(1 + o(1))$,

where $v^B_\epsilon(x)$ is the solution constructed in Theorem A and U^B_ϵ is given by (1.9).

Our second theorem concerns the existence of interior spike solutions for $(1.3)_N$.

Theorem 1.2. *Assume that (p, q, r, s) satisfies assumption (A). Suppose $P_0 \in \Omega$ is a nondegenerate peak point. Then for $\epsilon << 1, \tau << 1$, problem $(1.3)_N$ has a solution $(V^I_{\epsilon,\tau,N}(x), U^I_{\epsilon,\tau,N}(x))$ with the following properties:*

(a) $V^I_{\epsilon,\tau,N} = (U^I_{\epsilon,N})^{-q/(p-1)}(v^I_{\epsilon,N}(x) + o(1))$,

(b) $U^I_{\epsilon,\tau,N} = U^I_{\epsilon,N}(1 + o(1))$,

where $v^I_{\epsilon,N}(x)$ is the solution constructed in Theorem B and $U^I_{\epsilon,N}$ is given by (1.9).

Similarly we can construct interior spike solutions for $(1.3)_D$.

Theorem 1.3. *Assume that* (p,q,r,s) *satisfies assumption (A). Suppose* $P_0 \in \Omega$ *is a nondegenerate peak point. Then for* $\epsilon << 1, \tau << 1$, *problem* $(1.3)_D$ *has a solution* $(V^I_{\epsilon,\tau,D}(x), U^I_{\epsilon,\tau,D}(x))$ *with the following properties:*

(a) $V^I_{\epsilon,\tau,D} = (U^I_{\epsilon,D})^{-q/(p-1)}(v^I_{\epsilon,D}(x)+o(1))$,

(b) $U^I_{\epsilon,\tau,D} = U^I_{\epsilon,D}(1+o(1))$,

where $v^I_{\epsilon,D}(x)$ *is the solution constructed in Theorem C and* $U^I_{\epsilon,D}$ *is given by (1.9).*

Next we study the stability and instability properties of the solutions constructed in Theorems 1.1-1.3. (Here we say a solution is linearly stable if all the eigenvalues of the associated linearized operator have negative real part, it is linearly unstable if the associated linearized operator has one eigenvalue with positive real part, and it is metastable if the eigenvalues of associated linearized operator either approach 0 or have strictly negative real parts.)

A remarkable fact is that although it is well-known that $v^B_\epsilon, v^I_{\epsilon,N}, v^I_{\epsilon,D}$ are all linearly unstable with respect to the equation (1.8) (see e.g., Theorem 2.1 of [19]), the corresponding solutions to the system (1.3) $(V^B_{\epsilon,\tau}, U^B_{\epsilon,\tau})$, $(V^I_{\epsilon,\tau,N}, U^I_{\epsilon,\tau,N})$, $(V^I_{\epsilon,\tau,D}, U^I_{\epsilon,\tau,D})$ may be stable with respect to (1.3). The shifting of positive eigenvalue in the single equation to negative eigenvalue in the system is an interesting problem which is of independent interest and will be studied in this paper.

It turns out that the stability and instability of single point condensation solutions depend on three factors: the exponents p,q,r,s, the geometry of the boundary and the domain, and the boundary conditions.

We say (p,q,r,s) satisfies assumption (B) if

$$(B) \qquad \text{either } r=2, 1<p\leq 1+\frac{4}{N}, \text{ or } \ r=p+1, \ \ 1<p<(\frac{N+2}{N-2})_+.$$

We say (p,q,r,s) satisfies assumption (C) if

$$(C) \qquad r=2, 1+\frac{4}{N}<p<(\frac{N+2}{N-2})_+, \ \frac{qr}{s}-(p-1)<c_0$$

where c_0 is a small number to be determined later.

We first consider the boundary spike solution $(V^B_{\epsilon,\tau}, U^B_{\epsilon,\tau})$.

Theorem 1.4. *For $\epsilon << 1, \tau << 1$, $(V^B_{\epsilon,\tau}, U^B_{\epsilon,\tau})$ is linearly unstable if either $P_0 \in \partial\Omega$ is a nondegenerate critical point of $H(P)$ such that $G_B(P_0)$ contains one positive eigenvalue or (p,q,r,s) satisfies assumption (C).*

$(V^B_{\epsilon,\tau}, U^B_{\epsilon,\tau})$ is linearly stable if $P_0 \in \partial\Omega$ is a nondegenerate critical point of $H(P)$ such that $G_B(P_0)$ contains no positive eigenvalue and (p,q,r,s) satisfies assumption (B).

$(V^B_{\epsilon,\tau}, U^B_{\epsilon,\tau})$ is metastable if $P_0 \in \partial\Omega$ is a nondegenerate critical point of $H(P)$ such that $G_B(P_0)$ contains at least one positive eigenvalue and (p,q,r,s) satisfies assumption (B).

We next consider $(V^I_{\epsilon,\tau,N}, U^I_{\epsilon,\tau,N})$.

Theorem 1.5. *Let P_0 be a nondegenerate peak point. Then for $\epsilon << 1, \tau << 1$, $(V^I_{\epsilon,\tau,N}, U^I_{\epsilon,\tau,N})$ is always linearly unstable.*

$(V^I_{\epsilon,\tau,N}, U^I_{\epsilon,\tau,N})$ is metastable if (p,q,r,s) satisfies assumption (B).

Finally we consider $(V^I_{\epsilon,\tau,D}, U^I_{\epsilon,\tau,D})$.

Theorem 1.6. *Let P_0 be a nondegenerate peak point. Then for $\epsilon << 1, \tau << 1$, $(V^I_{\epsilon,\tau,D}, U^I_{\epsilon,\tau,D})$ is linearly unstable if (p,q,r,s) satisfies assumption (C).*

$(V^I_{\epsilon,\tau,D}, U^I_{\epsilon,\tau,D})$ is linearly stable if (p,q,r,s) satisfies assumption (B).

Remarks: (1) From Theorems 1.3-1.6, we see that in the case when $r=2$ or $r=p+1$, we have a fairly good picture of the stability and instability of single spike solutions in the case when (p,q,r,s) satisfies assumption (A) and $a=O(1), b=O(1), D_U >> 1, D_V << 1$. Theorems 1.4-1.6 are still true if r is close to 2 or $p+1$. In other cases, it is unclear.

(2) We note that related stability result for the Gierer-Meinhardt system were obtained in [25] and [28] in the one dimensional case. In [43], the author studied the instability of interior spikes for the shadow system of Gierer-Meinhardt system in higher-dimensional case.

(3) The results in this paper can certainly be generalized to a large class of reaction-diffusion systems.

In conclusion, by taking $D_V = \epsilon^2 << 1$ and $D = D_U >> 1$, we obtain the shadow system which can be reduced to a single equation. By studying the shadow system, we obtain the existence and stability analysis of the original system (1.3). Here we assume that $a = O(1), b = O(1)$. The advantage of this approach is that we can deal with higher dimensions and boundary value problems while the disadvantage is that we can not deal with the strong coupling case (i.e., $D = O(1)$). On the other hand, in [8], they studied the strong coupling case in one dimension by dynamical system methods. But it seems hard to apply their methods to higher dimensions.

After the paper was submitted, we were kindly informed by the referees that some further results on the existence and stability of solutions to the Gray-Scott model are obtained in [6], [7], [16], [20] and [26]. In these papers system (1.1) has been studied on the unbounded domain $\Omega = R^1$ (or, in the numerical simulations, the interval Ω was taken so large that D_U could not be considered to be large). This implies that one cannot approximate $U(x)$ by the constant U_ϵ. $U(x)$ varies on a long spatial scale. As a consequence, both the existence problem and the stability problem have a significantly different character (they cannot be studied by the shadow system approach of this paper). Moreover, it has been found that on unbounded one-dimensional domains the relative magnitudes of a and b with respect to ϵ play a decisive role in the analysis.

In the rest of this section, we outline the proofs of Theorems 1.1-1.6. We shall concentrate on the proofs of Theorems 1.2 and 1.5 for $(V^I_{\epsilon,\tau,N}, U^I_{\epsilon,\tau,N})$. The proofs of the other cases are similar and will be left to the last section of this paper.

To avoid clumsy notations, we omit the indexes D, N, B, I for the moment. Furthermore, by suitable scaling, without loss of generality, we may assume that $b = 1$.

The proofs of Theorems 1.2 and 1.5 are based on perturbation arguments from $\tau = 0$ to small $\tau' s$. Note that when $\tau = 0$, we obtain the shadow system (1.6). For the purpose of perturbation, it is enough to obtain the eigenvalues of the system

$$\mathcal{L}_\infty \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \eta \end{pmatrix}, \lambda \in \mathcal{C}, \tag{1.12}$$

where $\mathcal{L}_\infty$ is the linearized operator of the system (1.6) at (V_ϵ, U_ϵ):

$$\mathcal{L}_\infty := \begin{pmatrix} \epsilon^2\Delta - 1 + pV_\epsilon^{p-1}U_\epsilon^q & qU_\epsilon^{q-1}V_\epsilon^p \\ -U_\epsilon^s \frac{1}{|\Omega|}\int_\Omega rV_\epsilon^{r-1}\cdot & -a - sU_\epsilon^{s-1}\frac{1}{|\Omega|}\int_\Omega V_\epsilon^r \end{pmatrix} \tag{1.13}$$

$\mathcal{L}_\infty$ is defined in the space $Y := H_\nu^2 \times R$ where $H_\nu^2 := \{u \in H^2(\Omega) | \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \partial\Omega\}$.

Since

$$V_\epsilon = U_\epsilon^{-q/(p-1)} v_\epsilon, U_\epsilon = (1 + o(1))(\frac{1}{a|\Omega|}\int_\Omega v_\epsilon^r)^{\frac{(p-1)}{qr - s(p-1)}},$$

by some simple computations (see Section 4) it is easy to see that the eigenvalues of problem (1.12) in Y are the same as the eigenvalues of the following eigenvalue problem

$$\epsilon^2\Delta\phi - \phi + pv_\epsilon^{p-1}\phi - \frac{rq}{s}\frac{\int_\Omega v_\epsilon^{r-1}\phi}{\int_\Omega v_\epsilon^r + a^{-1}s^{-1}(1 - U_\epsilon)^{-1}U_\epsilon\int_\Omega v_\epsilon^r(a+\lambda)} v_\epsilon^p = \lambda\phi, \phi \in H_\nu^2. \tag{1.14}$$

Let α_ϵ be an eigenvalue of (1.14). The following lemma will be proved in Section 4.

Lemma A. *(1) If $\alpha_\epsilon \to 0$ as $\epsilon \to 0$, then $\alpha_\epsilon = (1 + o(1))\tau^\epsilon$ where τ^ϵ is a small eigenvalue of L_ϵ in H_ν^2 where $L_\epsilon := \Delta - 1 + pv_\epsilon^{p-1}$.*

(2) If $\alpha_\epsilon \to \alpha_0 \neq 0$. Then α_0 is an eigenvalue of the following eigenvalue problem

$$\Delta\phi - \phi + pw^{p-1}\phi - \frac{rq}{s}\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p = \alpha_0\phi, \phi \in H^2(R^N), \tag{1.15}$$

where w is the unique solution of (1.10).

By Lemma A, we just need to know the eigenvalues of problem (1.15) with nonnegative real part and the small eigenvalues of L_ϵ. The small eigenvalues of L_ϵ are studied in [42] and [43]. It remains to study problem (1.15). A major contribution in this paper is to completely understand the eigenvalues of problem (1.15) in the following two cases

$$r = 2, \text{ or } r = p + 1.$$

In fact we shall prove (see Theorems 2.1 and 2.2)

Key Theorem: *(1) If (p,q,r,s) satisfies assumptions (A) and (B), then $Re(\alpha_0) < -c_1 < 0$ for some $c_1 > 0$, where $\alpha_0 \neq 0$ is an eigenvalue of problem (1.15).*

(2) If (p,q,r,s) satisfies assumptions (A) and (C), then problem (1.15) has an eigenvalue $\alpha_1 > 0$.

The structure of the paper is the following:

In Section 2, we study the eigenvalues of problem (1.15) and prove the Key Theorem.

In Section 3, we recall the small eigenvalues of L_ϵ obtained in [42] and [43].

In Section 4, we analyze the eigenvalues of $\mathcal{L}_\infty$.

In Section 5, we use perturbation arguments to prove Theorems 1.2 and 1.5.

In Section 6, we prove the other theorems.

Finally in appendix, we prove the reduction Lemma A.

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2. Eigenvalues of Problem (1.15)

In this section, we study the eigenvalues of problem (1.15).

Set

$$\gamma := \frac{qr}{s(p-1)}.$$

(By the assumption (A), $\gamma > 1$.)

Let w be the unique solution of (1.10) and set

$$L\phi := \Delta\phi - \phi + pw^{p-1}\phi - \gamma(p-1)\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w^p, \phi \in H^2(R^N).$$

Note that L is not selfadjoint if $r \neq p+1$.

Let

$$L_0 = \Delta - 1 + pw^{p-1},$$

$$X_0 := kernel(L_0) = span\{\frac{\partial w}{\partial y_j} | j = 1, ..., N\}.$$

It is well-known that L_0 admits the following set of eigenvalues

$$\mu_1 > 0, \mu_2 = ... = \mu_{N+1} = 0, \mu_{N+2} < 0, ... \tag{2.1}$$

where the eigenfunction corresponding to μ_1 is of constant sign. See, e.g. Theorem 2.1 in [19] and Theorem 2.1 in [43].

We observe also that if $L\phi = 0$, then $\phi - \gamma \frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w \in X_0$. So,

$$0 = \int_{R^N} w^{r-1}(\phi - \gamma \frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r} w) = (1-\gamma) \int_{R^N} w^{r-1}\phi$$

which implies that $\int_{R^N} w^{r-1}\phi = 0$ by assumption (A). Hence $\phi \in X_0$.

Note that

$$L_0 w = (p-1)w^p, L_0(\frac{1}{p-1} w + \frac{1}{2} x \nabla w) = w \tag{2.2}$$

and

$$\int_{R^N} (L_0^{-1} w) w = \int_{R^N} w(\frac{1}{p-1} w + \frac{1}{2} x \nabla w) = (\frac{1}{p-1} - \frac{N}{4}) \int_{R^N} w^2, \tag{2.3}$$

$$\int_{R^N} (L_0^{-1} w) w^p = \int_{R^N} w^p (\frac{1}{p-1} w + \frac{1}{2} x \nabla w)$$

$$= \int_{R^N} (L_0^{-1} w) \frac{1}{p-1} L_0 w = \frac{1}{p-1} \int_{R^N} w^2. \tag{2.4}$$

In this section, we shall prove the following two theorems.

Theorem 2.1. *Assume that* $r = 2$.

(1) Suppose that $1 < p \leq 1 + \frac{4}{N}$. *Let* $\alpha_0 \neq 0$ *be an eigenvalue of* L. *Then we have* $Re(\alpha_0) \leq -c_1$ *for some* $c_1 > 0$.

(2) If $1 + \frac{4}{N} < p < (\frac{N+2}{N-2})_+$ *and* $\frac{qr}{s} - (p-1) < c_0$ *for some* $c_0 > 0$, *then* L *has a positive eigenvalue* α_1.

Theorem 2.2. *Assume that*

$$r = p + 1, 1 < p < (\frac{N+2}{N-2})_+.$$

Let $\alpha_0 \neq 0$ *be an eigenvalue of* L. *Then we have* $Re(\alpha_0) \leq -c_2$ *for some* $c_2 > 0$.

Proof of Theorem 2.1:

We divide the proof into three cases.

Case 1: $r = 2, 1 < p < 1 + \frac{4}{N}$.

Since L is not self-adjoint, we introduce a new operator as follows:

$$L_1\phi := L_0\phi - (p-1)\frac{\int_{R^N} w\phi}{\int_{R^N} w^2} w^p - (p-1)\frac{\int_{R^N} w^p\phi}{\int_{R^N} w^2} w + (p-1)\frac{\int_{R^N} w^{p+1}\int_{R^N} w\phi}{(\int_{R^N} w^2)^2} w, \quad (2.5)$$

We have the following important lemma.

Lemma 2.3:

(1) L_1 is selfadjoint.

(2) The kernel of L_1 consists of $w, \frac{\partial w}{\partial y_j}, j = 1, ..., N$.

(3) There exists a positive constant $a_1 > 0$ such that

$$L_1(\phi,\phi)$$
$$:= \int_{R^N} (|\nabla\phi|^2 + \phi^2 - pw^{p-1}\phi^2) + \frac{2(p-1)\int_{R^N} w\phi \int_{R^N} w^p\phi}{\int_{R^N} w^2} - (p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}(\int_{R^N} w\phi)^2$$
$$\geq a_1 d^2_{L^2(R^N)}(\phi, X_1)$$

for all $\phi \in H^1(R^N)$, where $X_1 := span\{w, \frac{\partial w}{\partial y_j} | j = 1, ..., N\}$ and $d_{L^2(R^N)}$ means the distance in L^2-norm.

Proof: The first statement follows easily by direct verification.

For (2), it is easy to see that $X_1 \subset kernel(L_1)$. On the other hand, if $\phi \in Kernel(L_1)$, then we have

$$L_0\phi = c_1(\phi)w + c_2(\phi)w^p$$
$$= c_1(\phi)L_0(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) + c_2(\phi)L_0(\frac{w}{p-1})$$

where

$$c_1(\phi) = (p-1)\frac{\int_{R^N} w^p\phi}{\int_{R^N} w^2} - (p-1)\frac{\int_{R^N} w^{p+1}\int_{R^N} w\phi}{(\int_{R^N} w^2)^2}, c_2(\phi) = (p-1)\frac{\int_{R^N} w\phi}{\int_{R^N} w^2}.$$

Hence

$$\phi - c_1(\phi)(\frac{1}{p-1}w + \frac{1}{2}x\nabla w) - c_2(\phi)\frac{1}{p-1}w \in kernel(L_0).$$

Note that

$$c_1(\phi) = (p-1)c_1(\phi)\frac{\int_{R^N} w^p(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{\int_{R^N} w^2} - (p-1)c_1(\phi)\frac{\int_{R^N} w^{p+1}\int_{R^N} w(\frac{1}{p-1}w + \frac{1}{2}x\nabla w)}{(\int_{R^N} w^2)^2}$$

$$= c_1(\phi) - c_1(\phi)(p-1)(\frac{1}{p-1} - \frac{N}{4})\frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}$$

by (2.3) and (2.4).

This implies that $c_1(\phi) = 0, \phi \in X_1$.

It remains to prove (3). Suppose (3) is not true, then by (1) and (2) there exists (α, ϕ) such that

(i) α is real and positive,

(ii) $\phi \perp w$, $\phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \dots, N$, and

(iii) $L_1 \phi = \alpha \phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0 - \alpha)\phi = (p-1)\frac{\int_{R^N} w^p \phi}{\int_{R^N} w^2} w. \tag{2.6}$$

We first show that

$$\int_{R^N} w^p \phi \neq 0.$$

Suppose this is not the case. Then $\alpha > 0$ is an eigenvalue of L_0. By (2.1), ϕ has constant sign. This contradicts with the fact that $\phi \perp w$.

Therefore $\alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^{\perp}$. So (2.6) implies

$$\phi = (p-1)\frac{\int_{R^N} w^p \phi}{\int_{R^N} w^2}(L_0 - \alpha)^{-1} w.$$

Thus

$$\int_{R^N} w^p \phi = (p-1)\frac{\int_{R^N} w^p \phi}{\int_{R^N} w^2}\int_{R^N} ((L_0 - \alpha)^{-1} w) w^p,$$

$$\int_{R^N} w^2 = (p-1)\int_{R^N} ((L_0 - \alpha)^{-1} w) w^p,$$

$$\int_{R^N} w^2 = \int_{R^N} ((L_0 - \alpha)^{-1} w)((L_0 - \alpha) w + \alpha w),$$

$$0 = \alpha \int_{R^N} ((L_0 - \alpha)^{-1} w) w,$$

$$0 = \int_{R^N} ((L_0 - \alpha)^{-1} w) w. \tag{2.7}$$

Let $h_1(\alpha) = \int_{R^N} ((L_0 - \alpha)^{-1} w) w$, then $h_1(0) = \int_{R^N} (L_0^{-1} w) w = \int_{R^N} (\frac{1}{p-1} w + \frac{1}{2} x \cdot \nabla w) w = (\frac{1}{p-1} - \frac{N}{4}) \int_{R^N} w^2 > 0$ since $1 < p < 1 + \frac{4}{N}$. Moreover $h_1'(\alpha) = \int_{R^N} ((L_0 - \alpha)^{-2} w) w = \int_{R^N} ((L_0 - \alpha)^{-1} w)^2 > 0$. This implies $h_1(\alpha) > 0$ for all

$\alpha \in (0, \mu_1)$. Clearly, also $h_1(\alpha) < 0$ for $\alpha \in (\mu_1, \infty)$ (since $\lim_{\alpha \to +\infty} h_1(\alpha) = 0$). A contradiction to (2.7)!

□

We now finish the proof of Theorem 2.1 in Case 1.

Suppose that $L\phi = \alpha_0 \phi$ and $\alpha_0 \neq 0$. Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp kernel(L_0)$. Then we obtain two equations

$$L_0 \phi_R - (p-1)\gamma \frac{\int_{R^N} w\phi_R}{\int_{R^N} w^2} w^p = \alpha_R \phi_R - \alpha_I \phi_I, \tag{2.8}$$

$$L_0 \phi_I - (p-1)\gamma \frac{\int_{R^N} w\phi_I}{\int_{R^N} w^2} w^p = \alpha_R \phi_I + \alpha_I \phi_R. \tag{2.9}$$

Multiplying (2.8) by ϕ_R and (2.9) by ϕ_I and adding them together, we obtain

$$\begin{aligned} &-\alpha_R \int_{R^N} (\phi_R^2 + \phi_I^2) \\ &= L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I) \\ &+(p-1)(\gamma-2) \frac{\int_{R^N} w\phi_R \int_{R^N} w^p \phi_R + \int_{R^N} w\phi_I \int_{R^N} w^p \phi_I}{\int_{R^N} w^2} \\ &+(p-1) \frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2} [(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2] \end{aligned}$$

Multiplying (2.8) by w and (2.9) by w we obtain

$$(p-1)\int_{R^N} w^p \phi_R - \gamma(p-1) \frac{\int_{R^N} w\phi_R}{\int_{R^N} w^2} \int_{R^N} w^{p+1} = \alpha_R \int_{R^N} w\phi_R - \alpha_I \int_{R^N} w\phi_I$$

$$(p-1)\int_{R^N} w^p \phi_I - \gamma(p-1) \frac{\int_{R^N} w\phi_I}{\int_{R^N} w^2} \int_{R^N} w^{p+1} = \alpha_R \int_{R^N} w\phi_I + \alpha_I \int_{R^N} w\phi_R$$

Hence we have

$$\begin{aligned} &(p-1)\int_{R^N} w\phi_R \int_{R^N} w^2 \phi_R + (p-1)\int_{R^N} w\phi_I \int_{R^N} w^2 \phi_I \\ &= (\alpha_R + \gamma(p-1) \frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2})((\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2). \end{aligned}$$

Therefore we have

$$\begin{aligned} &-\alpha_R \int_{R^N} (\phi_R^2 + \phi_I^2) \\ &= L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I) \\ &+(p-1)(\gamma-2)(\frac{1}{p-1}\alpha_R + \gamma \frac{\int_{R^N} w^{p+1}}{\int_{R^N} w^2}) \frac{(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2}{\int_{R^N} w^2} \end{aligned}$$

$$+(p-1)\frac{\int_{R^N} w^{p+1}}{(\int_{R^N} w^2)^2}[(\int_{R^N} w\phi_R)^2 + (\int_{R^N} w\phi_I)^2].$$

Set

$$\phi_R = c_R w + \phi_R^\perp, \phi_R^\perp \perp X_1,$$

$$\phi_I = c_I w + \phi_I^\perp, \phi_I^\perp \perp X_1.$$

Then

$$\int_{R^N} w\phi_R = c_R \int_{R^N} w^2, \int_{R^N} w\phi_I = c_I \int_{R^N} w^2,$$

$$d^2_{L^2(R^N)}(\phi_R, X_1) = \|\phi_R^\perp\|^2_{L^2}, d^2_{L^2(R^N)}(\phi_I, X_1) = \|\phi_I^\perp\|^2_{L^2}.$$

By some simple computations we have

$$L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I)$$

$$+(\gamma-1)\alpha_R(c_R^2+c_I^2)\int_{R^N} w^2+(p-1)(\gamma-1)^2(c_R^2+c_I^2)\int_{R^N} w^{p+1}+\alpha_R(\|\phi_R^\perp\|^2_{L^2}+\|\phi_I^\perp\|^2_{L^2}) = 0$$

By Lemma 2.3 (3)

$$(\gamma-1)\alpha_R(c_R^2+c_I^2)\int_{R^N} w^2$$

$$+(p-1)(\gamma-1)^2(c_R^2+c_I^2)\int_{R^N} w^{p+1} + (\alpha_R + a_1)(\|\phi_R^\perp\|^2_{L^2} + \|\phi_I^\perp\|^2_{L^2}) \le 0$$

Since $\gamma > 1$, we must have

$$\alpha_R < 0.$$

This proves (1) of Theorem 2.1 in Case 1.

Case 2: $r = 2, 1 + \frac{4}{N} < p < (\frac{N+2}{N-2})_+, \frac{qr}{s} - (p-1) < c_0$.

In this case, we consider the following function

$$h_2(\alpha) := \alpha \int_{R^N} ((L_0 - \alpha)^{-1} w^p) w - \frac{s}{qr} \int_{R^N} w^2.$$

Note that for α sufficiently small, we have

$$\int_{R^N} w(L_0-\alpha)^{-1}w^p = \int_{R^N} wL_0^{-1}w^p + \alpha\int_{R^N} wL_0^{-2}w^p + O(\alpha^2)$$

$$= \frac{1}{p-1}\int_{R^N} w^2 + \frac{\alpha}{p-1}(\frac{1}{p-1} - \frac{N}{4})\int_{R^N} w^2 + O(\alpha^2).$$

Hence

$$h_2(\alpha) = (\frac{1}{p-1} - \frac{s}{qr})\int_{R^N} w^2 + \frac{\alpha}{p-1}(\frac{1}{p-1} - \frac{N}{4})\int_{R^N} w^2 + O(\alpha^2).$$

Since $(p-1)^{-1}-4N^{-1}<0$ and $(p-1)^{-1}-s(qr)^{-1}>0$, it is easy to show that there is an $\alpha_1>0$ sufficiently small such that $h_2(\alpha_1)=0$, provided that

$$\frac{\frac{1}{p-1}-\frac{s}{qr}}{\frac{1}{p-1}(\frac{1}{p-1}-\frac{N}{4})}=\frac{1-\gamma^{-1}}{\frac{1}{p-1}-\frac{N}{4}}$$

is sufficiently small. Now we put

$$\phi_1=(L_0-\alpha)^{-1}w^p.$$

Since $\int_{R^N}\phi_1 w=s(qr)^{-1}\int_{R^N}w^2$, it is easy to check that

$$L\phi_1=\alpha_1\phi_1.$$

Hence L has an positive eigenvalue α_1. Moreover the eigenfunction corresponding to α_1 can be chosen to be radial.

Case 3: $r=2, p=1+\frac{4}{N}$.

In this case we have

$$\int_{R^N}(L_0^{-1}w)w=\int_{R^N}w(\frac{1}{p-1}w+\frac{1}{2}x\nabla w)=0 \tag{2.10}$$

Set

$$\phi_0=\frac{1}{p-1}w+\frac{1}{2}x\nabla w. \tag{2.11}$$

We will follow the proof in Case 1. We first have the following lemma which is similar to Lemma 2.3.

Lemma 2.4:

(1) L_1 is selfadjoint;

(2) The kernel of L_1 consists of $w,\phi_0,\frac{\partial w}{\partial y_j}, j=1,...,N$.

(3) There exists a positive constant $a_2>0$ such that

$$L_1(\phi,\phi)$$
$$=\int_{R^N}(|\nabla\phi|^2+\phi^2-pw^{p-1}\phi^2)$$
$$+\frac{2(p-1)\int_{R^N}w\phi\int_{R^N}w^p\phi}{\int_{R^N}w^2}-(p-1)\frac{\int_{R^N}w^{p+1}}{(\int_{R^N}w^2)^2}(\int_{R^N}w\phi)^2$$
$$\geq a_2 d^2_{L^2(R^N)}(\phi,X_2)$$

for all $\phi\in H^1(R^N)$, where $X_2:=span\{w,\phi_0,\frac{\partial w}{\partial y_j}|j=1,...,N\}$.

Proof: The proof is similar to that of Lemma 2.3. □

Now we can finish the proof of Theorem 2.1 in Case 3. The proof is similar to that of Case 1, here we need to deal with an extra kernel ϕ_0.

Suppose that $L\phi = \alpha_0\phi$ and $\alpha_0 \neq 0$. Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp kernel(L_0)$. Then we obtain two equations

$$L_0\phi_R - (p-1)\gamma\frac{\int_{R^N} w\phi_R}{\int_{R^N} w^2}w^p = \alpha_R\phi_R - \alpha_I\phi_I, \tag{2.12}$$

$$L_0\phi_I - (p-1)\gamma\frac{\int_{R^N} w\phi_I}{\int_{R^N} w^2}w^p = \alpha_R\phi_I + \alpha_I\phi_R. \tag{2.13}$$

Set

$$\phi_R = c_R w + b_R\phi_0 + \phi_R^\perp, \phi_R^\perp \perp X_2,$$
$$\phi_I = c_I w + b_I\phi_0 + \phi_I^\perp, \phi_I^\perp \perp X_2.$$

Then similar to Case 1, we have

$$L_1(\phi_R, \phi_R) + L_1(\phi_I, \phi_I)$$
$$+(\gamma-1)\alpha_R(c_R^2 + c_I^2)\int_{R^N} w^2 + (p-1)(\gamma-1)^2(c_R^2 + c_I^2)\int_{R^N} w^{p+1}$$
$$+\alpha_R(b_R^2\int_{R^N}\phi_0^2 + b_I^2\int_{R^N}\phi_0^2 + \|\phi_R^\perp\|_{L^2}^2 + \|\phi_I^\perp\|_{L^2}^2) = 0$$

By Lemma 2.4 (3)

$$(\gamma-1)\alpha_R(c_R^2 + c_I^2)\int_{R^N} w^2$$
$$+(p-1)(\gamma-1)^2(c_R^2 + c_I^2)\int_{R^N} w^{p+1}$$
$$+\alpha_R(b_R^2\int_{R^N}\phi_0^2 + b_I^2\int_{R^N}\phi_0^2) + (\alpha_R + a_2)(\|\phi_R^\perp\|_{L^2}^2 + \|\phi_I^\perp\|_{L^2}^2) \le 0$$

If $\alpha_R \geq 0$, then necessarily we have

$$c_R = c_I = 0, \phi_R^\perp = 0, \phi_I^\perp = 0.$$

Hence

$$\phi_R = b_R\phi_0, \phi_I = b_I\phi_0.$$

This implies that

$$b_R L_0\phi_0 = (\alpha_R b_R - \alpha_I b_I)\phi_0, b_I L_0\phi_0 = (\alpha_R b_R + \alpha_I b_I)\phi_0,$$

which is impossible unless $b_R = b_I = 0$ or $\alpha_R = \alpha_I = 0$. A contradiction !

Combining Case 1, Case 2, and Case 3, we obtain Theorem 2.1. □

Proof of Theorem 2.2:

Let $r = p + 1$. L becomes

$$L = L_0 - \frac{qr}{s} \frac{\int_{R^N} w^p \cdot}{\int_{R^N} w^{p+1}} w^p.$$

We will follow the proof of Theorem 2.1. We first define a new operator.

$$L_2\phi = L_0\phi - (p-1)\frac{\int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} w^p. \tag{2.14}$$

We have the following lemma.

Lemma 2.5:

(1) L_2 is selfadjoint.

(2) The kernel of L_2 consists of $w, \frac{\partial w}{\partial y_j}, j = 1, ..., N$.

(3) There exists a positive constant $a_2 > 0$ such that

$$L_2(\phi, \phi)$$
$$:= \int_{R^N} (|\nabla \phi|^2 + \phi^2 - p w^{p-1}\phi^2) + \frac{(p-1)(\int_{R^N} w^p \phi)^2}{\int_{R^N} w^{p+1}}$$
$$\geq a_2 d^2_{L^2(R^N)}(\phi, X_1)$$

for all $\phi \in H^1(R^N)$.

Proof: The proof of (1) and (2) is similar to that of Lemma 2.3. We omit the details. It remains to prove (3).

Suppose (3) is not true, then by (1) and (2) there exists (α, ϕ) such that

(i) α is real and positive,

(ii) $\phi \perp w$, $\phi \perp \frac{\partial w}{\partial y_j}$, $j = 1, \ldots, N$, and

(iii) $L_2\phi = \alpha\phi$.

We show that this is impossible. From (ii) and (iii), we have

$$(L_0 - \alpha)\phi = \frac{(p-1)\int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} w^p. \tag{2.15}$$

We first show that

$$\int_{R^N} w^p \phi \neq 0.$$

Suppose this is not the case. Then α is an eigenvalue of L_0. By (2.1), ϕ has constant sign. This contradict the fact that $\phi \perp w$.

Therefore $\alpha \neq \mu_1, 0$, and hence $L_0 - \alpha$ is invertible in $X_0^\perp$. So (2.15) implies

$$\phi = \frac{(p-1)\int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} (L_0 - \alpha)^{-1} w^p.$$

Thus

$$\int_{R^N} w^p \phi = (p-1) \frac{\int_{R^N} w^p \phi}{\int_{R^N} w^{p+1}} \int_{R^N} ((L_0 - \alpha)^{-1} w^p) w^p,$$

$$\int_{R^N} w^{p+1} = (p-1) \int_{R^N} ((L_0 - \alpha)^{-1} w^p) w^p. \tag{2.16}$$

Let $h_3(\alpha) = (p-1)\int_{R^N}((L_0-\alpha)^{-1}w^p)w^p - \int_{R^N} w^{p+1}$, then $h_3(0) = (p-1)\int_{R^N}(L_0^{-1}w^p)w^p - \int_{R^N} w^{p+1} = 0$. Moreover $h_3'(\alpha) = (p-1)\int_{R^N}((L_0 - \alpha)^{-2}w^p)w^p = (p-1)\int_{R^N}((L_0-\alpha)^{-1}w^p)^2 > 0$. This implies $h_3(\alpha) > 0$ for all $\alpha \in (0, \mu_1)$. Clearly, also $h_3(\alpha) < 0$ for $\alpha \in (\mu_1, \infty)$. A contradiction to (2.16)!

□

We now finish the proof of Theorem 2.2. We follow the proof of Case 1 of Theorem 2.1.

Suppose that $L\phi = \alpha_0 \phi$ and $\alpha_0 \neq 0$. Let $\alpha_0 = \alpha_R + i\alpha_I$ and $\phi = \phi_R + i\phi_I$. Since $\alpha_0 \neq 0$, we can choose $\phi \perp kernel(L_0)$. Then we obtain two equations

$$L_0 \phi_R - (p-1)\gamma \frac{\int_{R^N} w^p \phi_R}{\int_{R^N} w^{p+1}} w^p = \alpha_R \phi_R - \alpha_I \phi_I, \tag{2.17}$$

$$L_0 \phi_I - (p-1)\gamma \frac{\int_{R^N} w^p \phi_I}{\int_{R^N} w^{p+1}} w^p = \alpha_R \phi_I + \alpha_I \phi_R. \tag{2.18}$$

Multiplying (2.17) by ϕ_R and (2.18) by ϕ_I and adding them together, we obtain

$$\begin{aligned} -\alpha_R \int_{R^N} (\phi_R^2 + \phi_I^2) \\ = L_2(\phi_R, \phi_R) + L_2(\phi_I, \phi_I) \\ +(p-1)(\gamma - 1) \frac{(\int_{R^N} w^p \phi_R)^2 + (\int_{R^N} w^p \phi_I)^2}{\int_{R^N} w^{p+1}} \end{aligned}$$

By Lemma 2.5 (3)

$$\alpha_R \int_{R^N} (\phi_R^2 + \phi_I^2) + a_2 d_{L^2}^2(\phi, X_1) + (p-1)(\gamma-1) \frac{(\int_{R^N} w^p \phi_R)^2 + (\int_{R^N} w^p \phi_I)^2}{\int_{R^N} w^{p+1}} \leq 0$$

Since $\gamma > 1$, this implies

$$\alpha_R < 0.$$

Theorem 2.2 is thus proved.

□

3. Eigenvalues of L_ϵ

In this section, we recall the small eigenvalues obtained in [42] and [43].

For each $u, v \in H^1(\Omega)$, we define

$$\langle u, v\rangle_\epsilon = \epsilon^{-N} \int_\Omega (\epsilon^2 \nabla u \cdot \nabla v + uv).$$

We denote $\langle u, u\rangle_\epsilon$ by $\|u\|_\epsilon^2$.

Let w be the unique solution of (1.10). For $P \in \Omega$, we set $w_{\epsilon,P}^N$ to be the unique solution in $H^1(\Omega)$ of

$$\begin{cases} \epsilon^2 \Delta u - u + w^p(\frac{x-P}{\epsilon}) = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

For $P \in \Omega$, we let

$$\varphi_{\epsilon,P}^N(x) = w((x-P)/\epsilon) - w_{\epsilon,P}^N(x).$$

Remark: The functions $w_{\epsilon,P}^N(x), \varphi_{\epsilon,P}^N(x)$ were introduced in [40].

Then we have

Theorem 3.1. *(See Theorem 1.2 in* [43]*.) The eigenvalue problem*

$$\begin{cases} \epsilon^2 \Delta \phi - \phi + p(v_{\epsilon,N}^I)^{p-1}\phi = \tau^\epsilon \phi \quad in\ \Omega \\ \frac{\partial \phi}{\partial \nu} = 0 \quad on\ \partial\Omega \end{cases} \tag{3.2}$$

admits the following set of eigenvalues

$$\tau_1^\epsilon = \mu_1 + o(1), \tau_2^\epsilon = o(1), ..., \tau_{N+1}^\epsilon = o(1), \tau_j^\epsilon = \mu_j + o(1),\ \ j \geq N+2.$$

Moreover, we have

$$\frac{\tau_j^\epsilon}{\varphi_{\epsilon,P_0}^N(P_0)} \to b_0 \lambda_{j-1}, j = 2, ..., N+1, \tag{3.3}$$

where $\lambda_j, j = 1, ..., N$ are the eigenvalues of $G_I(P_0)$ and

$$b_0 = 2d^{-2}(P_0, \partial\Omega) \frac{\int_{R^N} p w^{p-1} w' u_*'(r)}{\int_{R^N} (\frac{\partial w}{\partial y_1})^2 dy} < 0,$$

where $u_*(r)$ is the unique radial solution of the following problem

$$\Delta u - u = 0, u(0) = 1, u = u(r) \quad in\ R^N. \tag{3.4}$$

Furthermore, the eigenfunction corresponding to $\tau_{j+1}^\epsilon, j = 1, ..., N$ is given by the following:

$$\phi_j^\epsilon = \sum_{l=1}^{N} (g_{j,l} + o(1))\epsilon \frac{\partial w_{\epsilon,P}^N}{\partial P_l}|_{P=P_\epsilon},$$

where $\vec{g}_j = (g_{j,1}, ..., g_{j,N})^t$ is the eigenvector corresponding to λ_j, namely

$$G_I(P_0)\vec{g}_j = \lambda_j \vec{g}_j, j = 1, ..., N.$$

Remark: It was proved in [40] that

$$-\epsilon \log[-\varphi_{\epsilon,P_0}^N(P_0)] \to 2d(P_0, \partial\Omega) \quad \text{as } \epsilon \to 0$$

Therefore we have

$$\tau_j^\epsilon > 0, j = 2, ..., N+1. \tag{3.5}$$

4. Eigenvalues of $\mathcal{L}_\infty$

In this section, we study the shadow system at (V_ϵ, U_ϵ), where

$$V_\epsilon = (U_\epsilon)^{-\frac{q}{p-1}} v_\epsilon(x),$$

$$a(1 - U_\epsilon) - U_\epsilon^{s-qr/(p-1)} \frac{1}{|\Omega|} \int_\Omega v_\epsilon^r = 0$$

and $v_\epsilon(x)$ is the solution constructed by Theorem B. We show that (V_ϵ, U_ϵ) is nondegenerate. Moreover, we obtain exact eigenvalue estimates.

Let $\mathcal{L}_\infty$ be defined by (1.13). The domain of $\mathcal{L}_\infty$ is $Y = H_\nu^2 \oplus R$.

Let $(\alpha_\epsilon, \phi_\epsilon, \eta_\epsilon)$ be a solution of (1.12). Then we have

$$(a + \alpha_\epsilon + sU_\epsilon^{s-1} \frac{1}{|\Omega|} \int_\Omega V_\epsilon^r)\eta_\epsilon = -rU_\epsilon^s \frac{1}{|\Omega|} \int_\Omega V_\epsilon^{r-1} \phi_\epsilon \tag{4.1}$$

and

$$\epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + pV_\epsilon^{p-1} U_\epsilon^q \phi_\epsilon + \eta_\epsilon q U_\epsilon^{q-1} V_\epsilon^p = \alpha_\epsilon \phi_\epsilon, \phi_\epsilon \in H_\nu^2. \tag{4.2}$$

Substituting (4.1) into (4.2) and using (1.7) and (1.9), we obtain that

$$L_\infty \phi_\epsilon = \alpha_\epsilon \phi_\epsilon, \phi_\epsilon \in H_\nu^2 \tag{4.3}$$

where

$$L_\infty := \epsilon^2\Delta - 1 + pv_\epsilon^{p-1} - \frac{qr}{s}\frac{\int_\Omega v_\epsilon^{r-1}\cdot}{\int_\Omega v_\epsilon^r + a^{-1}s^{-1}(1-U_\epsilon)^{-1}U_\epsilon\int_\Omega v_\epsilon^r(a+\alpha_\epsilon)}v_\epsilon^p \tag{4.4}$$

in H^2_ν.

We have

Lemma A. *(1) If $\alpha_\epsilon \to 0$ as $\epsilon \to 0$, then $\alpha_\epsilon = (1+o(1))\tau^\epsilon$ where τ^ϵ is a small eigenvalue of L_ϵ in H^2_ν where $L_\epsilon := \Delta - 1 + pv_\epsilon^{p-1}$.*

(2) If $\alpha_\epsilon \to \alpha_0 \neq 0$. Then α_0 is an eigenvalue of the following eigenvalue problem

$$\Delta\phi - \phi + pw^{p-1}\phi - \frac{rq}{s}\frac{\int_{R^N} w^{r-1}\phi}{\int_{R^N} w^r}w^p = \alpha_0\phi, \phi \in H^2(R^N). \tag{4.5}$$

The proof of Lemma A is technical and is left to appendix.

Combining Theorems 2.1, 2.2 and Lemma A, we have

Theorem 4.1. *Let $\alpha_\epsilon^j, j = 1,2,...,$ be eigenvalues of $\mathcal{L}_\infty$.*

(1) If $r = 2, 1 < p \leq 1 + \frac{4}{N}$ or $r = p+1, 1 < p < (\frac{N+2}{N-2})_+$, then we have

$$\alpha_\epsilon^j = (1+o(1))\tau_{j+1}^\epsilon, j = 1,...,N, Re(\alpha_\epsilon^j) < -c_0 < 0, j > N.$$

(2) If $r = 2, 1 + \frac{4}{N} < p < (\frac{N+2}{N-2})_+$ and $\frac{qr}{s} - (p-1) < c_0$ for some $c_0 > 0$, then we have

$$Re(\alpha_\epsilon^1) > c_1 > 0.$$

Proof:

(1) If $r = 2, 1 < p \leq 1 + \frac{4}{N}$ or $r = p+1, 1 < p < (\frac{N+2}{N-2})_+$, then by Lemma A and Theorems 2.1 and 2.2, the first N eigenvalues of $\mathcal{L}_\infty$ are $(1+o(1))\tau_{j+1}^\epsilon, j = 1,...,N$. The $(N+1)-$st eigenvalue has a strictly negative real part.

(2) If $r = 2, 1+\frac{4}{N} < p < (\frac{N+2}{N-2})_+$, then by Theorem 2.1, L has an eigenvalue α_1 with positive real part. Moreover the eigenfunction corresponding to α_1 is radial. Since $\alpha_1 \neq 0$, a simple perturbation argument shows that $\mathcal{L}_\infty$ has an eigenvalue $(1+o(1))\alpha_1$.

□

5. Proofs of Theorems 1.2 and 1.5

In this section, we use perturbation arguments (similar to those in [24]) to construct solutions to (1.3) for $\tau = \frac{1}{D}$ small and thus finish the proofs of Theorems 1.2 and 1.5. Our main tool is the implicit function theorem.

We first decompose $L^2(\Omega)$ into

$$L^2(\Omega) = \mathcal{R} \oplus X_1,$$

where

$$X_1 := \{u \in L^2(\Omega) | \int_\Omega u(x)dx = 0\}$$

and the space of constant functions is identified with $\mathcal{R}$. Let $P : L^2(\Omega) \to X_1$ be the projection associated with this decomposition:

$$Pu = u - \frac{1}{|\Omega|}\int_\Omega u(x)dx.$$

Let

$$Z := H^2_\nu \cap X_1.$$

In accordance with the decomposition above, we set

$$U(x) = \xi + \psi(x), \int_\Omega \psi(x)dx = 0.$$

Define three operators

$$\mathcal{F}_1(\tau, V, \xi, \psi) := \epsilon^2 \Delta V - V + V^p(\xi+\psi)^q,$$
$$\mathcal{F}_2(\tau, V, \xi, \psi) := -\int_\Omega V^r(\xi+\psi)^s + a|\Omega| - a\xi|\Omega|,$$
$$\mathcal{F}_3(\tau, V, \xi, \psi) := \Delta\psi + \tau[-a\psi - P(V^r(\xi+\psi)^s)],$$

respectively.

Then for $N \leq 3$, since H^2_ν is compactly embedded into $L^\infty(\Omega)$,

$$\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$$

is an analytic mapping from an open set $\mathcal{R} \times \{V \in H^2_\nu | V(x) > 0 \text{ on } \Omega\} \times (\delta, +\infty) \times \{\psi \in Z | \|\psi\|_{L^\infty(\Omega)} < \delta\}$ into $L^2(\Omega) \times \mathcal{R} \times X_1$, where δ is a positive number.

Let (V_ϵ, U_ϵ) be a solution to the shadow system (1.6). Then we have

$$\mathcal{F}(0, V_\epsilon, U_\epsilon, 0) = 0. \tag{5.1}$$

If the matrix of partial derivatives with respect to (V, ξ, ψ) at $(\tau, V, \xi, \psi) = (0, V_\epsilon, U_\epsilon, 0)$, i.e.

$$D_{(V,\xi,\psi)}\mathcal{F}|_{(0,V_\epsilon,U_\epsilon,0)} \tag{5.2}$$

$$:= \begin{pmatrix} \epsilon^2\Delta - +pV_\epsilon^{p-1}U_\epsilon^q & qU_\epsilon^{q-1}V_\epsilon^p & qU_\epsilon^{q-1}V_\epsilon^p \\ -U_\epsilon^s r\int_\Omega V_\epsilon^{r-1}\cdot & -sU_\epsilon^{s-1}\int_\Omega V_\infty^r - a|\Omega| & -sU_\epsilon^{s-1}\int_\Omega V_\epsilon^r \\ 0 & 0 & \Delta \end{pmatrix}$$

is boundedly invertible, then by the implicit function theorem, we have a one parameter family of solutions $(V_{\epsilon,\tau}, \xi_{\epsilon,\tau}, \psi_{\epsilon,\tau}) \in Y \times \mathcal{R}_+ \times Z$ such that

$$\mathcal{F}(\tau, V_{\epsilon,\tau}, \xi_{\epsilon,\tau}, \psi_{\epsilon,\tau}) = 0 \tag{5.3}$$

for $|\tau|$ sufficiently small and

$$(V_{\epsilon,0}, \xi_{\epsilon,0}, \psi_{\epsilon,0}) = (V_\epsilon, U_\epsilon, 0). \tag{5.4}$$

Note that Δ under the homogeneous Neumann boundary condition is an isomorphism from Z onto X_1. Thus, $D_{(V,\xi,\psi)}\mathcal{F}|_{(0,V_\epsilon,U_\epsilon,0)}$ is boundedly invertible if and only if the linearized shadow operator

$$\mathcal{L}_\infty = \begin{pmatrix} \epsilon^2\Delta - 1 + pV_\epsilon^{p-1}U_\epsilon^q & qU_\epsilon^{q-1}V_\epsilon^p \\ -U_\epsilon^s \int_\Omega rV_\epsilon^{r-1}\cdot & -sU_\epsilon^{s-1}\int_\Omega V_\epsilon^r - a|\Omega| \end{pmatrix} \tag{5.5}$$

has a bounded inverse. By Lemma A and Theorem 3.1, we have that $\mathcal{L}_\infty$ has a trivial kernel if restricted to $L^2 \times \mathcal{R}$, so $\mathcal{L}_\infty$ has a bounded inverse (the norm of the inverse operator is of the order $|\varphi_{\epsilon,P_0}^N(P_0)|^{-1}$).

Theorem 1.2 is thus proved.

Proof of Theorem 1.5: By Lemma A and Theorem 3.1, we have that $\mathcal{L}_\infty$ has N eigenvalues

$$(1 + o(1))\tau_j^\epsilon, j = 2, ..., N + 1$$

where $\tau_j^\epsilon = O(|\varphi_{\epsilon,P_0}^N(P_0)|) = O(e^{-2d(P_0,\partial\Omega)/\epsilon})$ and $Re(\tau_j^\epsilon) > 0$. Hence for $\tau << 1$, $(V_{\epsilon,\tau,N}^I, U_{\epsilon,\tau,N}^I)$ is always linearly unstable.

The rest of the proof follows from Theorem 4.1.

□

6. Proofs of Theorems 1.1, 1.3, 1.4 and 1.6

In this section, we finish the proofs of other Theorems.

We first deal with the boundary spike case.

For $P \in \partial\Omega$, we set $w_{\epsilon,P}$ to be the unique solution in $H^1(\Omega)$ of

$$\begin{cases} \epsilon^2 \Delta u - u + w^p(\frac{x-P}{\epsilon}) = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.1}$$

Remark: The properties of $w_{\epsilon,P}$ was studied in [37].

Then we have

Theorem 6.1. *(See Theorem 1.3 of* [42].*) The following eigenvalue problem*

$$\begin{cases} \epsilon^2 \Delta \phi - \phi + p(v_\epsilon^B)^{p-1}\phi = \tau^\epsilon \phi \quad in\ \Omega, \\ \frac{\partial \phi}{\partial \nu} = 0 \quad on\ \partial\Omega, \end{cases} \tag{6.2}$$

admits the following set of eigenvalues

$$\tau_1^\epsilon = \mu_1 + o(1), \tau_2^\epsilon = o(1), ..., \tau_N^\epsilon = o(1), \tau_j^\epsilon = \mu_{j+1} + o(1), \ j \geq N+1.$$

Moreover,

$$\frac{\tau_{j+1}^\epsilon}{\epsilon^2} \to d_0 \Lambda_j, j = 1, ..., N-1, \tag{6.3}$$

where Λ_j are the eigenvalues of $G_B(P_0) := (\nabla^2_{\tau_{P_0}} H(P_0))$ and

$$d_0 = \frac{N-1}{N+1} \frac{\int_{R_+^N} (w')^2 y_N dy}{\int_{R_+^N} (\frac{\partial w}{\partial y_1})^2 dy} > 0.$$

Furthermore the eigenfunction corresponding to $\tau_{j+1}^\epsilon, j = 1, ..., N-1$ is given by the following:

$$\phi_j^\epsilon = \sum_{l=1}^{N-1} (A_{j,l} + o(1)) \epsilon \frac{\partial w_{\epsilon, P_\epsilon}}{\partial \tau_l(P_\epsilon)}$$

where P_ϵ is the unique local maximum of $v_\epsilon^B(x)$, $\vec{A}_j = (A_{j,1}, ..., A_{j,N-1})^t$ is the eigenvector of $G_B(P_0)$ corresponding to Λ_j, namely

$$G_B(P_0)\vec{A}_j = \Lambda_j \vec{A}_j, j = 1, ..., N-1.$$

We can now prove Theorems 1.1 and 1.4.

Proofs of Theorems 1.1 and 1.4:

Let P_0 be a nondegenerate critical point of $H(P)$.

By using Theorem 6.1 and Lemma A, the existence follows from the same proof of Section 4.

Now if

$$r = 2, 1 + \frac{4}{N} < p < (\frac{N+2}{N-2})_+, \frac{qr}{s} - (p-1) < c_0,$$

by Theorem 2.2, L has an eigenvalue $\alpha_1 > 0$. Moreover, the corresponding eigenfunction is radial. By a simple perturbation argument, problem (1.14) has an eigenvalue close to α_1 if τ is sufficiently small. This proves Theorem 1.4 in this case.

The other cases follows from Lemma A, Theorem 6.1 and Theorems 2.1-2.2.

□

We next consider the single interior spike case with the Dirichlet boundary conditions.

For $P \in \Omega$, we set $w_{\epsilon,P}^D$ to be the unique solution in $H^1(\Omega)$ of

$$\begin{cases} \epsilon^2 \Delta u - u + w^p(\frac{x-P}{\epsilon}) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{6.4}$$

Remark: The functions $w_{\epsilon,P}^D$ was introduced and studied in [27].

Let

$$\varphi_{\epsilon,P}^D(x) = w(\frac{x-P}{\epsilon}) - w_{\epsilon,P}^D.$$

Then we have

Theorem 6.2. *(See Theorem 1.2 in [43].) The eigenvalue problem*

$$\begin{cases} \epsilon^2 \Delta \phi - \phi + p(v_{\epsilon,D}^I)^{p-1}\phi = \tau^\epsilon \phi \quad in\ \Omega \\ \phi = 0 \quad on\ \partial\Omega \end{cases} \tag{6.5}$$

admits the following set of eigenvalues

$$\tau_1^\epsilon = \mu_1 + o(1), \tau_2^\epsilon = o(1), ..., \tau_{N+1}^\epsilon = o(1), \tau_j^\epsilon = \mu_j + o(1),\ j \geq N+2.$$

Moreover, we have

$$\frac{\tau_j^\epsilon}{\varphi_{\epsilon,P_0}^D(P_0)} \to b_0 \lambda_{j-1}, j = 2, ..., N+1, \tag{6.6}$$

where $\lambda_j, j = 1, ..., N$ are the eigenvalues of $G_I(P_0)$ and b_0 is defined in Theorem 3.1. Furthermore, the eigenfunction corresponding to $\tau_{j+1}^\epsilon, j =$

$1,...,N$ is given by the following:

$$\phi_j^\epsilon = \sum_{l=1}^N (g_{j,l} + o(1))\epsilon \frac{\partial w_{\epsilon,P}^D}{\partial P_l}|_{P=P_\epsilon},$$

where $\vec{g}_j = (g_{j,1}, ..., g_{j,N})^t$ is the eigenvector corresponding to λ_j, namely

$$G_I(P_0)\vec{g}_j = \lambda_j \vec{g}_j, j = 1, ..., N.$$

We can now finish the proofs of Theorem 1.3 and 1.6.

Proofs of Theorems 1.3 and 1.6: By using Theorem 6.2, the proof is almost identical to that of Theorems 1.2 and 1.5. Note that by Lemma 4.4 in [27]

$$-\epsilon \log(\varphi_{\epsilon,P_0}^D(P_0)) \to 2d(P_0, \partial\Omega)$$

as $\epsilon \to 0$.

□

Appendix: Proof of Lemma A

In this appendix, we prove Lemma A.

We first note that (2) of Lemma A follows easily by taking limit as $\epsilon \to 0$. We just need to prove Lemma A (1).

Let $(\alpha_\epsilon, \phi_\epsilon)$ satisfy (4.3) where $\alpha_\epsilon \to 0$ and $\|\phi_\epsilon\|_\epsilon = 1$.

Then we have

$$L_\epsilon(\phi_\epsilon) - \eta(\phi_\epsilon)v_\epsilon^p = \alpha_\epsilon \phi_\epsilon,$$

where

$$\eta(\phi_\epsilon) = \frac{qr}{s + a^{-1}(1-U_\epsilon)^{-1}U_\epsilon(a+\alpha_\epsilon)} \int_\Omega v_\epsilon^{r-1}\phi_\epsilon / (\int_\Omega v_\epsilon^r).$$

Set

$$\bar{\phi}_\epsilon = \phi_\epsilon - \frac{1}{p-1}\eta(\phi_\epsilon)v_\epsilon.$$

Then by a simple computation we have

$$L_\epsilon(\bar{\phi}_\epsilon) = \alpha_\epsilon(\bar{\phi}_\epsilon + c_\epsilon v_\epsilon \int_\Omega v_\epsilon^{r-1}\bar{\phi}_\epsilon), \tag{6.7}$$

where

$$c_\epsilon = \frac{qr}{(p-1)[s + a^{-1}(1-U_\epsilon)^{-1}U_\epsilon(a+\alpha_\epsilon)] - qr} \frac{1}{\int_\Omega v_\epsilon^r}.$$

Let ϕ_j^ϵ be the eigenfunction of τ_{j+1}^ϵ in Theorem 3.1. Set

$$\mathcal{C}_\epsilon := span\{\phi_j^\epsilon, j=1,...,N\} \subset L^2(\Omega),$$

$$\mathcal{K}_\epsilon := span\{\phi_j^\epsilon, j=1,...,N\} \subset H_\nu^2.$$

We decompose $\bar{\phi}_\epsilon$ as

$$\bar{\phi}_\epsilon = \sum_{j=1}^N b_j^\epsilon \phi_j^\epsilon + \bar{\phi}_\epsilon^\perp,$$

where $b_j^\epsilon \to b_j^0$, and $\bar{\phi}_\epsilon^\perp \perp \mathcal{K}_\epsilon$.

We first obtain that $\bar{\phi}_\epsilon^\perp$ satisfies

$$L_\epsilon(\bar{\phi}_\epsilon^\perp) = \sum_{j=1}^N (\alpha_\epsilon - \tau_{j+1}^\epsilon) b_j^\epsilon \phi_j^\epsilon + \alpha_\epsilon \bar{\phi}_\epsilon^\perp + \alpha_\epsilon v_\epsilon c_\epsilon \int_\Omega v_\epsilon \bar{\phi}_\epsilon^\perp + \alpha_\epsilon c_\epsilon (\sum_{j=1}^N b_j^\epsilon \int_\Omega v_\epsilon \phi_j^\epsilon) v_\epsilon. \tag{6.8}$$

Note that by Lemma 7.1 of [40],

$$\int_\Omega v_\epsilon \phi_j^\epsilon = O(\epsilon^N \varphi_{\epsilon,P_\epsilon}^N(P_\epsilon)), \qquad c_\epsilon \int_\Omega v_\epsilon \phi_j^\epsilon = O(\varphi_{\epsilon,P_\epsilon}^N(P_\epsilon)).$$

Then by a similar argument as in the proof of Propositions 6.1 and 6.2 in [40], we obtain that

$$L_\epsilon : \mathcal{K}_\epsilon^\perp \to \mathcal{C}_\epsilon^\perp$$

is an invertible map if $\epsilon > 0$ is small enough. Note that

$$c_\epsilon = O(\epsilon^{-N}), c_\epsilon \int_\Omega v_\epsilon \bar{\phi}_\epsilon^\perp = O(\|\bar{\phi}_\epsilon^\perp\|_\epsilon),$$

$$\alpha_\epsilon c_\epsilon v_\epsilon \int_\Omega v_\epsilon \bar{\phi}_\epsilon^\perp = O(\alpha_\epsilon \|\bar{\phi}_\epsilon^\perp\|_\epsilon^2).$$

Hence we have

$$\|\bar{\phi}_\epsilon^\perp\|_\epsilon = O(\varphi_{\epsilon,P_\epsilon}^N(P_\epsilon)|\alpha_\epsilon| \sum_{j=1}^N |b_j^\epsilon|).$$

Next we multiply both sides of (6.8) by ϕ_k^ϵ and integrate over Ω, we obtain

$$\sum_{j=1}^N (\alpha_\epsilon - \tau_{j+1}^\epsilon) b_j^\epsilon \int_\Omega \phi_k^\epsilon \phi_j^\epsilon = O(\varphi_{\epsilon,P_\epsilon}^N(P_\epsilon)|\alpha_\epsilon| \sum_{j=1}^N |b_j^\epsilon|).$$

Hence we have

$$(\alpha_\epsilon - \tau_{j+1}^\epsilon) b_j^\epsilon = O(\varphi_{\epsilon,P_\epsilon}^N(P_\epsilon)(|\alpha_\epsilon| + \sum_{j=1}^N |\tau_{j+1}^\epsilon|) \sum_{j=1}^N |b_j^\epsilon|),\ j = 1, \ldots, N.$$

Observe that $(b_1^\epsilon, ..., b_N^\epsilon) \neq (0, ..., 0)$ and moreover $b_j^\epsilon \to b_j^0 \neq 0$ for some j (along a sequence $\epsilon_n \to 0$). (Otherwise, $\|\bar{\phi}_\epsilon^\perp\|_\epsilon = o(1), \|\phi_\epsilon\|_\epsilon = o(1)$.) This shows that

$$\alpha_\epsilon - \tau_{j+1}^\epsilon = o(1)|\alpha_\epsilon|$$

for some $j = 1, ..., N$. By Theorem 3.1, this proves (1) of Lemma A.

□

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