Entire nonradial solutions for non-cooperative coupled elliptic system with critical exponents

Yuxia Guo
Department of Mathematics, Tsinghua University, Beijing, 100084, P.R.China
Bo Li
Department of Mathematics, Tsinghua University, Beijing, 100084, P.R.China
Juncheng Wei
Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

Abstract: We consider the following coupled elliptic system :

$$\begin{cases} -\Delta u & = \mu_1 u^{\frac{N+2}{N-2}} + \beta u^{\frac{2}{N-2}} v^{\frac{N}{N-2}} \quad \text{in } \mathbb{R}^N \\ -\Delta v & = \mu_2 v^{\frac{N+2}{N-2}} + \beta v^{\frac{2}{N-2}} u^{\frac{N}{N-2}} \quad \text{in } \mathbb{R}^N \\ u, v > 0, & u, v \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \qquad (S)$$

where $N = 3, 4, \mu_1, \mu_2$ are two positive constants and $\beta < 0$ is the coupling constant. We prove the existence of infinitely many positive nonradial solutions.

Keywords: Non-cooperative coupled systems, Critical exponents, Infinitely many non-radial solutions.

AMS Subject Classification:35B45, 25J25

1 Introduction

We consider the following coupled elliptic system

$$\begin{cases} \Delta u + \mu_1 u^{p-1} + \beta u^{\frac{p}{2}-1} v^{\frac{p}{2}} = 0 & \text{in } \mathbb{R}^N \\ \Delta v + \mu_2 v^{p-1} + \beta v^{\frac{p}{2}-1} u^{\frac{p}{2}} = 0 & \text{in } \mathbb{R}^N \\ u > 0, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \qquad (P)$$

where $p = 2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent, μ_1, μ_2 are two positive constants, and $\beta \neq 0$ is the coupling number. Physically, if $\beta > 0$, it means the attractive interaction of the states u and v. On the other hand, if $\beta < 0$, it means the repulsive interaction. More precisely, system (P) is closely related to the solitary wave solutions of the time-dependent m coupled nonlinear Schrödinger equations:

$$\begin{cases} -\sqrt{-1}\frac{\partial}{\partial t}\Phi_j = \Delta \Phi_j + \sum_{i\neq j} \beta_{ij} |\Phi_i|^{\frac{p}{2}-1} \Phi_j^{\frac{p}{2}}, & y \in \mathbb{R}^N, t > 0, \\ \Phi_j = \Phi_j(y,t) \in \mathcal{C}, & j = 1, ..., m, \\ \Phi_j(y,t) \to 0, & \text{as } |y| \to \infty, t > 0, \end{cases} \qquad (1.1)$$

where $\mu_j = \beta_{jj} > 0's$ are positive constants, $\beta'_{ij}s$ are coupling constants and the exponent $p > 2$. The above system has applications in many physical problems, especially in nonlinear optics (when $p = 4$). Physically, Φ_j denotes the j^{th} component of the beam in Kerr-like photo refractive media; The positive constant μ_j is for self-focusing in the j^{th} component of the beam; The coupling constant β_{ij} is the interaction between the j^{th} and i^{th} component of the beam. As $\beta_{ij} > 0$, the interaction is attractive, and the interaction is repulsive if $\beta_{ij} < 0$ (see [1] and references therein). In particular, when the spatial dimension is one, the system (1.1) is integrable and there are many analytical and numerical results on solitary wave solutions of the general m coupled nonlinear Schrödinger equations (see [6, 9, 10, 12]).

To obtain the solitary wave solutions of system (1.1), one set $\Phi_j(y,t) = e^{\sqrt{-1}\lambda_j t}u_j(y)$ and transform the system (1.1) to a steady state m coupled nonlinear Schrödinger equations:

$$\begin{cases} \Delta u_j - \lambda_j u_j + \mu_j u_j^{p-1} + \sum_{i \neq j} \beta_{ij} u_i^{\frac{p}{2}} u_j^{\frac{p}{2}-1} = 0, & y \in \mathbb{R}^N, \\ u_j(y) > 0, & y \in \mathbb{R}^N, j = 1, ..., m, \\ u_j(y) \to 0, & \text{as } |y| \to \infty. \end{cases} \tag{1.2}$$

Note that system (1.2) has a gradient structure with respect to the energy functional

$$\mathcal{E}[u_1, ..., u_m] = \frac{1}{2} \int_{\mathbb{R}^N} \sum_{i=1}^m [|\nabla u_i|^2 + \lambda_i u_i^2] - \frac{1}{p} \int_{\mathbb{R}^N} \sum_{i,j} \beta_{ij} |u_i|^{\frac{p}{2}} |u_j|^{\frac{p}{2}}. \tag{1.3}$$

In the case of subcritical, i.e. $p < \frac{2N}{N-2}$, the existence of ground state solutions for (1.2) may depend on coupling constant $\beta'_{ij}s$. More precisely, when all $\beta'_{ij}s$ are positive and the matrix $\sum = (|\beta_{ij}|)$ with $\beta_{jj} = \mu_j$ is positively definite, there exists a ground state solution which is radially symmetry. However if all $\beta'_{ij}s$ are negative, or one of $\beta'_{ij}s$ is negative and the matrix $\sum = (|\beta_{ij}|)$ is positively definite, then there is no ground state solutions (see [13] and [2]). For more related results for coupled nonlinear Schrödinger equations, we refer the reader to [3, 4, 5, 14, 21] and references therein.

To study the a priori estimates of solutions to (1.2), we have to study the existence and non-existence of the limiting elliptic system (P). In the case of $p < \frac{2N}{N-2}$ and $\beta_{12} > -\sqrt{\mu_1\mu_2}$, it has been proved that problem (P) has no classical solutions ([18]). On the other hand, if $\beta_{12} < -\sqrt{\mu_1\mu_2}$, non trivial solutions exist ([22]).

The purpose of the present paper is to study the critical case. So from now on we assume that $N \geq 3$ and $p = \frac{2N}{N-2}$. Observe that when $\beta = 0$, problem (P) decouples to the following (up to multiplication) well-known Yamabe problem

$$-\Delta u = u^{\frac{N+2}{N-2}}, u > 0 \text{ in } \mathbb{R}^N. \tag{1.4}$$

It is well known that all solutions to Yamabe problem (1.4) can be classified

$$U_{\varepsilon,x_0}(y) = (N(N-2))^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |y - x_0|^2} \right)^{\frac{N-2}{2}}.$$

On the other hand, it is known that when the coupling constant β is positive (the *cooperative* case), the only positive solutions to the system (P) are radially symmetric with the form $(u,v) = (c_1U, c_2U)$, where $U(y) = (N(N-2))^{\frac{N-2}{4}}(\frac{1}{1+|y|^2})^{\frac{N-2}{2}}$ is the solution of the equation (1.4) and c_1, c_2 are some positive constants (see [8]). In this paper we consider the case of non-cooperative, i.e. $\beta < 0$. We establish the following result, which seems to exhibit a new phenomena: $\forall$ fixed $\mu_1, \mu_2 > 0, \beta < 0$, problem (P) admits infinitely many positive nonradial finite energy solutions, whose energy can be arbitrarily large.

To explain the main ideas of the proof, we have to go back to equation (1.4). By remarks before, positive solutions to (1.4) are well classified. It is natural to ask wether or not there are finite energy non-radial sign changing solutions to (1.4). This was answered first by Ding [7]. His proof is variational: consider the functions of the form

$$u(x) = u(|x_1|, |x_2|), x = (x_1, x_2) \in S^N \subset \mathbb{R}^{N+1} = \mathbb{R}^k \times \mathbb{R}^{N-k}, k \geq 2. \tag{1.5}$$

The critical Sobolev embedding becomes compact and hence infinitely many sign changing solutions exist, thanks to the Ljusternik-Schnirelmann theory. See also [11]. Recently, del Pino, Musso, Pacard and Pistoia [15]-[16] gave another proof of countably many sign changing non-radial solutions. Their proof is more constructive: they built a sequence of solutions with one negative bump at the origin and large number of positive bumps in a polygon. This gives more precise information on such sign changing solutions.

It seems very difficult to apply variational method to obtain non-radial positive solutions to (P). So we turn to perturbative method as in [15]-[16]. First we observe that problem (P) is invariant under *rotation*, *reflection* and *Kelvin's transformation.* As in [15]-[16], we build a sequence of positive solutions with one positive bubble for u at the origin and large number of positive bubbles for v around a polygon. Since our system is coupled each other, in order to obtain a better control of the error terms, it is difficult to carry the reduction procedure by using the same norm in [15]-[16] (see also [23], [17], [19], [20]). We have to modify the norms. Moreover because of the coupling, the estimates in the reduction procedure is much more complicated than in [15]-[16]. We hope the method that we have delivered in this paper can be applied to general dimensions larger than 4. However, the difficulty of the proof of existence increases as the dimension N is getting larger. If $N \geq 5$, the powers of the nonlinear terms $u^{\frac{p}{2}-1}v^{\frac{p}{2}}$ are *sublinear* in u, and the operator becomes singular when we consider the linearized operator (however we think that this is only technical). Some new methods are needed. We will come back to this question in a forthcoming paper. In this paper, we mainly focus on the problem of dimension $N = 3, 4$. Indeed, from the view point of physics, the case $N = 3$ is more significant. Technically, the case of $N = 3$ is indeed more tricker than that of the case of $N = 4$. For reader's convenience, we first solve the problem of dimension 3, and leave the case of dimension 4 in the last section 5.

Our main result can be presented as follows:

Theorem 1.1 *Let $N = 3$ or 4. There exists some sufficiently large $k_0 \in \mathbb{N}$, such that for any $k \geq k_0$, the system (P) has a finite energy solution (u_k, v_k) of the following form:*

$$\begin{cases} u_k(y) &= u_*(y) \quad +\psi_k(y), \qquad y \in \mathbb{R}^N, \\ v_k(y) &= v_*(y) \quad +\varphi_k(y), \qquad y \in \mathbb{R}^N, \end{cases} \tag{UV}$$

where $\varepsilon_k \sim k^{-4} \ln^{-2} k$ for $N = 3$, $\varepsilon_k \sim k^{-3}$ for $N = 4$; $(u_*, v_*) := \left(\mu_1^{-\frac{N-2}{4}} U_{1,0}, \mu_2^{-\frac{N-2}{4}} \sum_{j=1}^{k} U_{\epsilon_k, x_j}\right)$,

$x_j = \left(\sqrt{1-\varepsilon_k^2}\cos(\frac{2(j-1)\pi}{k}), \sqrt{1-\varepsilon_k^2}\sin(\frac{2(j-1)\pi}{k}), \boldsymbol{0}\right) \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$, *for $j = 1, 2, \cdots, k$, and $\|\psi_k\|_* \to 0, \|\varphi_k\|_* \to 0$ as $k \to \infty$, where*

$$\|\varphi\|_* := \sup_{y \in \mathbb{R}^N} \left| v_*^{-1}(y) \cdot \varphi(y) \right|.$$

Acknowledgment. Y. Guo was supported by NSFC(11171171) and J. Wei was supported by a GRF grant from RGC of Hong Kong.

2 Approximation and Linearization

In this section and in the following sections 3 and 4, we deal with the case of $N = 3$. Note that in our proofs the values of the constants μ and β are not essential. Only the sign of them matters. So without loss of generality, we may assume $\mu_1 = \mu_2 = 1$ and $\beta = -1$. Namely we consider the following elliptic system

$$\begin{cases} -\Delta u = u^5 - u^2 v^3 \text{ in } \mathbb{R}^3, \\ -\Delta v = v^5 - v^2 u^3 \text{ in } \mathbb{R}^3, \\ u, v > 0 \text{ in } \mathbb{R}^3, u, v \in \mathcal{D}^{1,2}(\mathbb{R}^3). \end{cases} \tag{2.1}$$

An important observation is the following invariance: Let T_i be one of the following three maps, $i = 1, 2, 3$. Then if (u, v) is a solution of (2.1), $(T_i(u), T_i(v))$ is also a solution to (2.1). Here maps T_i are given by

(Rotation Invariance Map): for $\eta : \mathbb{R}^3 \to \mathbb{R}$, $(\bar{y}, y') \in \mathbb{R}^2 \times \mathbb{R}$,

$$T_1(\eta)(\bar{y}, y') = \eta\left(e^{\frac{2j\pi}{k}\sqrt{-1}}\bar{y}, y'\right), \quad j = 1, 2, 3, \cdots, k-1. \tag{C1}$$

(Reflection Invariance Map): for $\eta : \mathbb{R}^3 \to \mathbb{R}$, $(y_1, y_2, y_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$,

$$T_{2,1}(\eta)(y_1, y_2, y_3) = \eta(y_1, -y_2, y_3), \tag{C2.1}$$

$$T_{2,1}(\eta)(y_1, y_2, y_3) = \eta(y_1, y_2, -y_3). \tag{C2.2}$$

(Kelvin Invariance Map): for $\eta : \mathbb{R}^3 \to \mathbb{R}$, $y \in \mathbb{R}^3$,

$$T_3(\eta)(y) = |y|^{-5}\eta\left(\frac{y}{|y|^2}\right). \tag{C3}$$

Because of the three invariances, we can define a symmetry class as follows

$$H_s := \left\{(g_1, g_2) \in \left[\mathcal{D}^{1,2}(\mathbb{R}^3)\right]^2 \middle| \forall (\bar{x}, x') \in \mathbb{R}^2 \times \mathbb{R}, T_j(g_i)(\bar{x}, x') = g_i(\bar{x}, x'), j = 1, 2, 3, i = 1, 2\right\}$$

As in [15]-[16], the following approximation solution

$$(u_*, v_*) = \left(U_{1,0}, \sum_{j=1}^{k} U_{\varepsilon_k, x_j}\right)$$

where

$$x_j = \left(\sqrt{1-\varepsilon_k^2}\cos(\frac{2(j-1)\pi}{k}), \sqrt{1-\varepsilon_k^2}\sin(\frac{2(j-1)\pi}{k}), \mathbf{0}\right) \in \mathbb{R}^2 \times \mathbb{R}^{N-2},$$

for $j = 1, 2, \cdots, k$, belong to the symmetry class H_s. To simplify the notations, in the following of the paper, we will use ϵ instead of ϵ_k.

Let

$$\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) = \{(u, v) \in L^6(\mathbb{R}^3) \times L^6(\mathbb{R}^3) \Big| \nabla u \in L^2(\mathbb{R}^3), \nabla v \in L^2(\mathbb{R}^3)\},$$

with norm

$$||(\xi, \eta)||_{\mathcal{D}} := ||\xi||_{\mathcal{D}} + ||\eta||_{\mathcal{D}} = \langle \xi, \xi \rangle_{\mathcal{D}}^{\frac{1}{2}} + \langle \eta, \eta \rangle_{\mathcal{D}}^{\frac{1}{2}},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ is defined by:

$$\langle \xi_1, \xi_2 \rangle_{\mathcal{D}} := \int_{\mathbb{R}^3} \nabla \xi_1(y) \cdot \nabla \xi_2(y) dy, \quad \forall \xi_1, \xi_2 \in \mathcal{D}^{1,2}(\mathbb{R}^3),$$

Linearizing the equations (2.1) around (u_*, v_*), we obtain the following two linear operators L_0, L_{00}, namely:

$$\begin{cases} L_0(\psi, \varphi): & = \Delta\psi + 5u_*^4\psi - 2u_*v_*^3\psi - 3u_*^2v_*^2\varphi, \quad \forall (\psi, \varphi) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3), \\ L_{00}(\psi, \varphi): & = \Delta\varphi + 5v_*^4\varphi - 2u_*^3v_*\varphi - 3u_*^2v_*^2\psi \quad , \forall (\psi, \varphi) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3). \end{cases} \tag{L}$$

We rewrite the system (2.1) in terms of this linear operator $L = (L_0, L_{00})$ as:

$$\begin{cases} L_0(\psi, \varphi) & = -\left[N_{5,0}(\psi, \varphi) + N_{2,3}(\psi, \varphi)\right] & \text{in } \mathbb{R}^3 \\ L_{00}(\psi, \varphi) & = -\left[E + N_{0,5}(\psi, \varphi) + N_{3,2}(\psi, \varphi)\right] & \text{in } \mathbb{R}^3, \end{cases} \tag{LS}$$

where

$$E = v_*^5 - \sum_{i=1}^{k} U_{\varepsilon, x_i}^5;$$

$$N_{0,5}(\psi, \varphi) = (v_* + \varphi)^5 - v_*^5 - 5v_*^4\varphi;$$

$$N_{3,2}(\psi, \varphi) = -(u_* + \psi)^3(v_* + \varphi)^2 + 2u_*^3v_*\varphi + 3u_*^2v_*^2\psi;$$

$$N_{5,0}(\psi, \varphi) = (u_* + \psi)^5 - u_*^5 - 5u_*^4\psi;$$

$$N_{2,3}(\psi, \varphi) = -(u_* + \psi)^2(v_* + \varphi)^3 + 3u_*^2v_*^2\varphi + 2u_*v_*^3\psi.$$

In the following, we will focus on the solution to the following problem:

$$L(\psi,\varphi)=h, \tag{LN}$$

where $(\psi,\varphi)\in H_s$.

We first introduce the following weighted L^∞ norm and the weighted L^q norm:

$$\|\phi\|_*:=\sup_{y\in\mathbb{R}^N}\left|v_*^{-1}(y)\cdot\phi(y)\right|,$$

$$\|h\|_{**}:=\|(1+|y|)^{5-\frac{2N}{q}}h(y)\|_{L^q(\mathbb{R}^N)},\quad N=3,4.$$

It is well known that (see [15]-[16] and [23]) the set of bounded solutions of the decoupled homogeneous system

$$\Delta\psi+5u_*^4\psi=0,\ \Delta\varphi+5v_*^4\varphi=0 \tag{2.2}$$

is spanned by $8k$ functions (Z_l,Z_{js}), where $l=1,2,3,4$,$j=1,2,\cdots,k$;$s=1,2$ and

$$Z_l=\partial_{y_l}U_{1,0}(y),l=1,2,3,y=(y_1,y_2,y_3)\in\mathbb{R}^3,Z_4(y)=y\cdot\nabla U_{1,0}(y)+\frac{1}{2}U_{1,0}(y),y\in\mathbb{R}^3;$$

$$Z_{j,1}(y)=\partial_r U_{\varepsilon,x_j},Z_{j,2}(y)=\partial_\varepsilon U_{\varepsilon,x_j},$$

with $r=|x_j|=\sqrt{1-\varepsilon^2}$, $j=1,2,\cdots,k$.

The following proposition solves (LN) with general orthogonal conditions.

Proposition 2.1 *Let $h=(h_1,h_2)$ be a vector function such that $\|h\|_{**}=\|h_1\|_{**}+\|h_2\|_{**}<\infty$, and satisfy the following orthogonal condition (C_0) :*

$$\begin{cases}\displaystyle\int_{\mathbb{R}^3}Z_l(y)h_1(y)dy &=0,\quad l=1,2,3,4\\ \displaystyle\int_{\mathbb{R}^3}Z_{js}(y)h_2(y)dy &=0,\quad j=1,2,\cdots,k;\ s=1,2,\end{cases} \tag{C_0}$$

then the linear problem (LN) has a unique solution $(\psi,\varphi)=T(h_1,h_2)$ such that $\|(\psi,\varphi)\|_*:=\|\psi\|_*+\|\varphi\|_*<\infty$, and

$$\begin{cases}\displaystyle\int_{\mathbb{R}^3}U_{1,0}^4(y)Z_l(y)\psi(y)dy &=0,\quad l=1,2,3,4\\ \displaystyle\int_{\mathbb{R}^3}U_{\varepsilon,x_r}^4(y)Z_{js}(y)\varphi(y)dy &=0,\quad j=1,2,\cdots,k;\ s=1,2.\end{cases} \tag{PHSI}$$

Proof. For fixed k, let us consider the subspace

$$H=\{(\psi,\varphi)\Big|(\psi,\varphi)\text{satisfies }\ (PHSI)\}.$$

Then H is a Hilbert space under the induced inner product

$$\langle(\psi_1,\varphi_1),(\psi_2,\varphi_2)\rangle_H:=\int_{\mathbb{R}^3}\nabla\psi_1(y)\cdot\nabla\psi_2(y)dy+\int_{\mathbb{R}^3}\nabla\varphi_1(y)\cdot\nabla\varphi_2(y)dy.$$

The norm $\|\cdot\|_H$ on this space is defined by

$$\|(\psi,\varphi)\|_H := \langle(\psi,\varphi),(\psi,\varphi)\rangle_H^{\frac{1}{2}}.$$

By Hölder's inequality, we have

$$\begin{aligned}\|h_i\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} &\le C\|(1+|y|)^{-5+\frac{6}{q}}\|_{L^r(\mathbb{R}^3)}\|(1+|y|)^{5-\frac{6}{q}}h_i(y)\|_{L^q(\mathbb{R}^3)}\\ &\le C\left[\int_{\mathbb{R}^3}(1+|y|)^{-6}dy\right]^{\frac{1}{r}}\|h_i\|_{**},\end{aligned}$$

where $\frac{1}{q}+\frac{1}{r}=\frac{5}{6}, i=1,2$.

The following discussion is focused on the existence of the solution $(\psi,\varphi)\in H$, namely the solution satisfying the weak form of the system (P) in the space H, i.e., for any testing pair $(\xi_1,\xi_2)\in H$, it holds that:

$$\begin{cases}\int_{\mathbb{R}^3}\nabla\psi\cdot\nabla\xi_1 - \int_{\mathbb{R}^3}(5u_*^4\psi-2u_*v_*^3\psi-3u_*^2v_*^2\varphi)\cdot\xi_1 + \int_{\mathbb{R}^3}h_1\cdot\xi_1=0\\ \int_{\mathbb{R}^3}\nabla\varphi\cdot\nabla\xi_2 - \int_{\mathbb{R}^3}(5v_*^4\varphi-2u_*^3v_*\varphi-3u_*^2v_*^2\psi)\cdot\xi_2 + \int_{\mathbb{R}^3}h_2\cdot\xi_2=0.\end{cases} \tag{2.3}$$

Note that for any $(h_1,h_2)\in L^{\frac{6}{5}}(\mathbb{R}^3)\times L^{\frac{6}{5}}(\mathbb{R}^3)$, by the uniqueness of Riesz's theorem, we can define an injective, linear and bounded operator $A=(A_1,A_2): L^{\frac{6}{5}}(\mathbb{R}^3)\times L^{\frac{6}{5}}(\mathbb{R}^3)\to H$, such that $(\psi,\varphi)=A(h_1,h_2)=(A_1(h_1),A_2(h_2))$ and

$$\langle A_1(h_1),\xi_1\rangle_{\mathcal{D}} = -(h_1,\xi_1)_{L^2}; \qquad \langle A_2(h_2),\xi_2\rangle_{\mathcal{D}} = -(h_2,\xi_2)_{L^2}.$$

For convenience, we also define an operator $\tau=(\tau_1,\tau_2): H\to L^{\frac{6}{5}}(\mathbb{R}^3)\times L^{\frac{6}{5}}(\mathbb{R}^3)$ such that,

$$\tau_1(\psi,\varphi) = -5u_*^4\psi+2u_*v_*^3\psi+3u_*^2v_*^2\varphi;$$

$$\tau_2(\psi,\varphi) = -5v_*^4\varphi+2u_*^3v_*\varphi+3u_*^2v_*^2\psi.$$

Then the operator τ is compact due to the fact that $u_*^{\nu_1}(y)\cdot v_*^{\nu_2}(y)\sim\frac{1}{(1+|y|)^4}$, for $0\le\nu_1,\nu_2\le 4, \nu_1+\nu_2=4$, and $|y|\ge 2$. Furthermore, by Hölder's inequality, we have

$$\begin{aligned}\|u_*^{\nu_1}\cdot v_*^{\nu_2}\cdot\varphi\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} &\le C\|u_*^{\nu_1}v_*^{\nu_2}\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\cdot\|\varphi\|_{L^{2^*}(\mathbb{R}^3)}\\ &\le C(k^{-1}+k^{-\nu_2})\ln^{-\nu_2}k\cdot\|\varphi\|_{\mathcal{D}}\le C\|\varphi\|_{\mathcal{D}},\end{aligned}$$

and

$$\begin{aligned}\|u_*^{\nu_1}\cdot v_*^{\nu_2}\cdot\psi\|_{L^{\frac{6}{5}}(\mathbb{R}^3)} &\le C\|u_*^{\nu_1}v_*^{\nu_2}\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}\cdot\|\psi\|_{L^{2^*}(\mathbb{R}^3)}\\ &\le C(k^{-1}+k^{-\nu_2})\ln^{-\nu_2}k\cdot\|\psi\|_{\mathcal{D}}\le C\|\psi\|_{\mathcal{D}}.\end{aligned}$$

For the details of the estimate of $\|u_*^{\nu_1}v_*^{\nu_2}\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}$, we have used the result of $(v-ext)$ and $(v-int)$, which will be explained in the proof of the following Proposition 2.2.

Now we define an operator $B=A\circ\tau: H\to H$, then B is also a compact operator and the system (2.3) is simplified to the following form:

$$(I-B)(\psi,\varphi)=A(h_1,h_2).$$

A direct computation shows that

$$
\begin{aligned}
\Big\langle(\psi,\varphi),\ \ B(\widetilde{\psi},\widetilde{\varphi})\Big\rangle_H &= \Big\langle(\psi,\varphi),\ \ \Big(A_1\circ\tau_1(\widetilde{\psi},\widetilde{\varphi}), A_2\circ\tau_2(\widetilde{\psi},\widetilde{\varphi})\Big)\Big\rangle_H \\
&= \Big\langle\psi, A_1\circ\tau_1(\widetilde{\psi},\widetilde{\varphi})\Big\rangle_{\mathcal{D}} + \Big\langle\varphi, A_2\circ\tau_2(\widetilde{\psi},\widetilde{\varphi})\Big\rangle_{\mathcal{D}} \\
&= -\Big(\psi,\tau_1(\widetilde{\psi},\widetilde{\varphi})\Big)_{L^2} - \Big(\varphi,\tau_2(\widetilde{\psi},\widetilde{\varphi})\Big)_{L^2} \\
&= \int_{\mathbb{R}^3}\psi\cdot\Big(5u_*^4\cdot\widetilde{\psi} - 2u_*v_*^3\widetilde{\psi} - 3u_*^2v_*^2\widetilde{\varphi}\Big) + \int_{\mathbb{R}^3}\varphi\cdot\Big(5v_*^4\widetilde{\varphi} - 2u_*^3v_*\widetilde{\varphi} - 3u_*^2v_*^2\widetilde{\psi}\Big) \\
&= \int_{\mathbb{R}^3}\widetilde{\psi}\cdot(5u_*^4\psi - 2u_*v_*^3\psi - 3u_*^2v_*^2\varphi) + \int_{\mathbb{R}^3}\widetilde{\varphi}\cdot(5v_*^4\varphi - 2u_*^3v_*\varphi - 3u_*^2v_*^2\psi) \\
&= -\Big(\tau_1(\psi,\varphi),\widetilde{\psi}\Big)_{L^2} - (\tau_2(\psi,\varphi),\widetilde{\varphi})_{L^2} \\
&= \Big\langle A_1\circ\tau_1(\psi,\varphi),\ \ \widetilde{\psi}\Big\rangle_{\mathcal{D}} + \langle A_2\circ\tau_2(\psi,\varphi),\widetilde{\varphi}\rangle_{\mathcal{D}} \\
&= \Big\langle(A_1\circ\tau_1(\psi,\varphi),\ \ A_2\circ\tau_2(\psi,\varphi)),\ \ \Big(\widetilde{\psi},\widetilde{\varphi}\Big)\Big\rangle_H \\
&= \Big\langle B(\psi,\varphi),\Big(\widetilde{\psi},\widetilde{\varphi}\Big)\Big\rangle_H,
\end{aligned}
$$

where $(\cdot,\cdot)$ denote the usual inner product in $L^2(\mathbb{R}^3)$, which shows that B is also self-adjoint. Since the linear problem (LN) is equivalent to the equation of the operator:

$$(I-B)(\psi,\varphi) = A(h_1,h_2).$$

By the injectivity of the operator A, for $\forall(v_1,v_2)\in\ker(I-B)$,

$$(I-B)(v_1,v_2) = (0,0) = A(0,0),$$

which means that (v_1,v_2) is actually the solution to the homogeneous linear problem (LN). Therefore, there exist some constants $a_p, b_{js}, p=1,2,3,4; j=1,2,\cdots,k; s=1,2$, such that

$$
\begin{cases}
v_1(y) &= \sum\limits_{p=1}^{4} a_p\cdot Z_p(y), \qquad\qquad y\in\mathbb{R}^3, \\
v_2(y) &= \sum\limits_{s=1}^{2}\sum\limits_{j=1}^{k} b_{js}\cdot Z_{js}(y), \qquad y\in\mathbb{R}^3.
\end{cases}
$$

Putting this formula into the restriction system $(PHSI)$, we obtain that $a_p=0, b_{js}=0$, and $(v_1,v_2)=(0,0)$, hence $\ker(I-B)=\{0\}$, so that

$$R(I-B) = (\ker(I-B^*))^{\perp} = (\ker(I-B))^{\perp} = H.$$

This yields the uniqueness and existence of the solution for the system (LN). □

Proposition 2.2 *Under the same assumption of Proposition 2.1, there exists a large $k_0\in\mathbb{N}$, and a constant C independent of k, such that for any $k\geq k_0$, the solution (ψ_k,φ_k) to the linear problem (LN) is equivalent to the equation $(\psi_k,\varphi_k)=T((h_{1k},h_{2k}))$, then*

$$\|T((h_{1k},h_{2k}))\|_* = \|(\psi_k,\varphi_k)\|_* = \|\psi_k\|_* + \|\varphi_k\|_* \leq C(\|h_{1k}\|_{**} + \|h_{2k}\|_{**}) = C\|h_k\|_{**},$$

which shows that T is a bounded linear operator.

Proof. We prove the results by contradiction. Suppose that the conclusion does not hold true, then there exists a series of (ψ_k,φ_k) and (h_{1k},h_{2k}) satisfying

$$\|(\psi_k,\varphi_k)\|_* = \|\psi_k\|_* + \|\varphi_k\|_* \equiv 1; \quad \|h_k\|_{**} = \|h_{1k}\|_{**} + \|h_{2k}\|_{**} \to 0, \text{as} \quad k\to\infty.$$

We rewrite the linear problem (LN) into the following form:

$$\begin{cases} \Delta\psi_k + & \left(5u_*^4 - 2u_*\cdot v_*^3\right)\cdot \ \psi_k = & h_{1k} + 3u_*^2\cdot v_*^2\cdot\varphi_k; \\ \Delta\varphi_k + & \left(5v_*^4 - 2u_*^3\cdot v_*\right)\cdot\varphi_k = & h_{2k} + 3u_*^2\cdot v_*^2\cdot\psi_k. \end{cases}$$

Then the standard elliptic theory yields that

$$\|\psi_k\|_{L^\infty(B_0(\frac14))} \le C\|h_{1k} + 3u_*^2v_*^2\varphi_k\|_{L^q(B_0(\frac12))} \le C\left[\|h_{1k}\|_{**} + \|u_*^2v_*^2\varphi_k\|_{L^q(B_0(\frac12))}\right].$$

Note that $y\in B_0(\frac12)$, $|y|<\frac12$ and $|x_j| = \sqrt{1-\varepsilon^2}\sim 1$, we have $|y-x_j|\ge C(1+|y|)$ and

$$v_*(y) \le \sum_{j=1}^k\left(\frac{\varepsilon}{\varepsilon^2+|y-x_j|^2}\right)^{\frac12} \le C\frac{k\varepsilon^{\frac12}}{1+|y|} = Ck\varepsilon^{\frac12}u_*.$$

Thus

$$\|u_*^2\cdot v_*^2\cdot\varphi_k\|_{L^q(B_0(\frac12))} \le Ck^{-3}\ln^{-3}k\|\varphi_k\|_*\|u_*^5\|_{L^q(B_0(\frac12))} \le Ck^{-3}\ln^{-3}k\|\varphi_k\|_* \le Ck^{-3}\ln^{-3}k.$$

Hence, we obtain

$$\|\psi_k\|_{L^\infty(B_0(\frac14))} \le C\left(\|h_{1k}\|_{**} + k^{-3}\ln^{-3}k\right).$$

Let $\widehat{s}$ denote the Kelvin transform of s, that is $\widehat{s}(y) = |y|^{-1}s\left(\frac{y}{|y|^2}\right)$, then one can check that u_* and v_* are invariant under this Kelvin transform, hence

$$\Delta\widehat{\psi}_k + (5u_*^4 - 2u_*v_*^3)\widehat{\psi}_k = \widetilde{h}_{1k} + 3u_*^2v_*^2\widehat{\varphi}_k,$$

where $\widetilde{h}_{1k}(y) = |y|^{-5}h_{1k}\left(\frac{y}{|y|^2}\right)$, and

$$\|\widetilde{h}_{1k}\|_{L^q(B_0(4))} = \||y|^{5-\frac6q}h_{1k}(y)\|_{L^q(B_0^c(\frac14))} \le C\|h_{1k}\|_{**}.$$

For the sake of further estimate, we split the whole space $\mathbb{R}^3$ into two parts: namely, the interior region **INT** and the respective exterior region **EXT**.as

$$\mathbf{INT} := \left\{x\in\mathbb{R}^3 \middle| \exists j\in\{1,2,\cdots,k\}, s.t., |x-x_j| < \frac{\eta}{k}\right\}.$$

The exterior region **EXT** is defined as the complementary for **INT**, that is $\mathbf{EXT} = \mathbf{INT^c}$.
For $y\in EXT$, we observe that $|y-x_j|\ge\frac{\eta}{k}, \forall j = 1,2,\cdots,k$.

(1) If y lies in the inner part $EXT\cap B_0(2)$, we have two choices:

(i) $\exists$ some $i_0\in\{1,2,\cdots,k\}$ such that y is closest to this point i_0, but relatively far from all the other $x_j's(j\ne i_0)$, namely, $|y-x_j|\ge\frac12|x_j-x_{i_0}|\sim\frac{|j-i_0|}{k}$, for all $j\ne i_0$, then,

$$v_*(y)\le C\left(\sum_{j\ne i_0}\frac{k\varepsilon^{\frac12}}{|j-i_0|} + \frac{k\varepsilon^{\frac12}}{\eta}\right)\le Ck\ln k\cdot\varepsilon^{\frac12}\le Ck^{-1};$$

(ii) y is far from all $x_i's, i=1,2,\cdots,k$, such that $\exists$ some fixed constant $c_0>0$, $|y-x_i|\geq c_0, 1\leq i\leq k$, then

$$v_*(y)\leq Ck\varepsilon^{\frac{1}{2}}\leq Ck^{-1}\ln^{-1}k<Ck^{-1}.$$

(2) If $y\in EXT\cap B_0^c(2)$, then $|y|>2$ and $|y-x_i|>\eta/k$, we obtain that $|y-x_i|\sim 1+|y|$, since $|x_i|\sim 1, i=1,2,\cdots,k$, and $\varepsilon\sim k^{-4}\ln^{-2}k$, we have

$$v_*(y)\leq C\sum_{i=1}^{k}\frac{\varepsilon^{\frac{1}{2}}}{\varepsilon+|y-x_i|}\leq\frac{Ck\varepsilon^{\frac{1}{2}}}{1+|y|}. \qquad (v-ext)$$

For $y\in INT$, then $\exists j\in\{1,2,\cdots,k\}$ such that $y\in B_{x_j}(\eta/k)$. In this way, we can bring them all into the same concentration region around the origin, that is $w\in B_0\left(\frac{\eta}{k\varepsilon}\right)$. On the other hand, observe the fact that $\left|\frac{x_j-x_i}{\varepsilon}\right|\sim\frac{|j-i|}{k\varepsilon}(j\neq i)$, which dominates $|w|<\frac{\eta}{k\varepsilon}$ and $1<1+|\varepsilon w+x_j|\leq 1+\frac{\eta}{k}+|x_j|\leq 3$, hence

$$|v_*(x_j+\varepsilon w)|\leq C\frac{k\varepsilon^{-\frac{1}{2}}}{1+|w|}. \qquad (v-int)$$

And

$$\begin{aligned}
&\|-3u_*^2v_*^2\widehat{\varphi}_k\|_{L^q(B_0(4))}\\
\leq& C\|\varphi_k\|_*\cdot\left[\int_{B_0^c(\frac{1}{4})}v_*^{3q}(z)\cdot|z|^{3q-6}dz\right]^{\frac{1}{q}}\\
\leq& C\|\varphi_k\|_*\cdot\left[\left(\int_{EXT\cap B_0(2)}\right)+\left(\int_{EXT\setminus B_0(2)}\right)+\left(\sum_{j=1}^{k}\int_{B_{x_j}(\eta/k)}\right)v_*^{3q}(z)\cdot|z|^{3q-6}dz\right]^{\frac{1}{q}}\\
\leq& C\|\varphi_k\|_*\cdot\left[k^{-3q}\int_{B_0(2)}|z|^{3q-6}dz+k^{-3q}\ln^{-3q}k\int_{EXT\setminus B_0(2)}\frac{|z|^{3q-6}}{(1+|z|)^{3q}}dz\right.\\
&\left.+\sum_{j=1}^{k}\int_{B_{x_j}(\eta/k)}(1+|z|)^{3q-6}\left(v_*(z)\right)^{3q}dz\right]^{\frac{1}{q}}\\
\leq& Ck^{-3}+C\varepsilon^{\frac{3}{q}}\left[\sum_{j=1}^{k}\int_{B_0(\frac{\eta}{k\varepsilon})}(1+|\varepsilon w+x_j|)^{3q-6}\left(v_*(x_j+\varepsilon w)\right)^{3q}dw\right]^{\frac{1}{q}}\\
\leq& Ck^{-3}+C\varepsilon^{\frac{3}{q}}\left[\sum_{j=1}^{k}\int_{B_0(\frac{\eta}{k\varepsilon})}v_*^{3q}(x_j+\varepsilon w)dw\right]^{\frac{1}{q}}\\
\leq& Ck^{-3}+Ck^4\varepsilon^{\frac{3}{q}-\frac{3}{2}}\left[\int_0^{\frac{\eta}{k\varepsilon}}\frac{r^2dr}{(1+r)^{3q}}\right]^{\frac{1}{q}}\\
\leq& Ck^{-3}+Ck^{7-\frac{3}{q}}\varepsilon^{\frac{3}{2}}\\
\leq& Ck^{1-\frac{3}{q}}\ln^{-3}k.
\end{aligned}$$

Since $\frac{2}{3}<q<3$, it yields that $1-\frac{3}{q}<0$ and the norm of the remaining term vanishes as the parameter k increases. Moreover, we get the estimate in the outer region $B_0^c(\frac{1}{4})$ as the following:

$$\begin{aligned}\|\psi_k\|_{L^\infty(B_0^c(\frac{1}{4}))} \leq & \||y|\psi_k(y)\|_{L^\infty(B_0^c(\frac{1}{4}))} = \|\widehat{\psi}_k\|_{L^\infty(B_0(4))} \\ \leq & C\|\widetilde{h}_{1k}\|_{L^q(B_0(4))} + Ck^{1-\frac{3}{q}}\ln^{-3}k \\ \leq & C\left(\|h_{1k}\|_{**} + k^{1-\frac{3}{q}}\ln^{-3}k\right).\end{aligned}$$

Combining this result with that in the inner part $B_0(\frac{1}{4})$, we obtain,

$$\begin{aligned}&\|\psi_k\|_{L^\infty(\mathbb{R}^3)} \\ \leq & \|\psi_k\|_{L^\infty(B_0^c(\frac{1}{4}))} + \|\psi_k\|_{L^\infty(B_0(\frac{1}{4}))} \\ \leq & C\left(\|h_{1k}\|_{**} + k^{1-\frac{3}{q}}\ln^{-3}k + k^{-3}\ln^{-3}k\right) \\ \leq & C\left(\|h_{1k}\|_{**} + k^{1-\frac{3}{q}}\ln^{-3}k\right).\end{aligned}$$

Similarly

$$\|\varphi_k\|_{L^\infty(\mathbb{R}^3)} \leq C\left(\|h_{2k}\|_{**} + k^{1-\frac{3}{q}}\ln^{-3}k\right).$$

Therefore $\|\psi_k\|_{L^\infty(\mathbb{R}^3)} + \|\varphi_k\|_{L^\infty(\mathbb{R}^3)} \to 0$, as $k \to \infty$.

However, noting that $\|\psi_k\|_{**} + \|\varphi_k\|_{**} \equiv 1$, we can find some fixed constants $\theta_0 > 0, R > 0$, such that,

$$\|\psi_k\|_{L^\infty(B_0(R))} + \|\varphi_k\|_{L^\infty(B_0(R))} > \theta_0 > 0,$$

which is a contradiction with the fact that $\|\psi_k\|_{L^\infty(\mathbb{R}^3)} + \|\varphi_k\|_{L^\infty(\mathbb{R}^3)} \to 0$, as $k \to \infty$. □

Proposition 2.3 *Let h_{1k}, h_{2k} be such that*

$$T_j(h_{1k}) = h_{1k}, T_j(h_{2k}) = h_{2k}, \quad j = 1, 2, 3,$$

where T_js are the three invariance maps defined at the beginning of this section, then there exists a bounded linear operator T as that in Proposition 2.2, such that for any $k \geq k_0$, the problem (LN) admits a unique weak solution $(\psi_k, \varphi_k) = T(h_{1k}, h_{2k})$ such that $\|(\psi_k, \varphi_k)\|_* := \|\psi_k\|_* + \|\varphi_k\|_* < \infty$, and

$$\begin{cases} \int_{\mathbb{R}^3} U_{1,0}^4(y)Z_p(y)\psi_k dy & = 0, \quad p = 1, 2, 3, 4; \\ \int_{\mathbb{R}^3} u_{\varepsilon,x_j}^4(y)Z_{js}(y)\varphi_k(y)dy & = 0, \quad j = 1, 2, \cdots, k; s = 1, 2. \end{cases} \tag{PHSI}$$

Proof. In the proof of Proposition 2.1, it is sufficient to check the condition (C_0)

$$\begin{cases} \int_{\mathbb{R}^3} Z_p(y)h_{1k}(y)dy & = 0, \quad p = 1, 2, 3, 4; \\ \int_{\mathbb{R}^3} Z_{js}(y)h_{2k}(y)dy & = 0, \quad j = 1, 2, \cdots, k; \quad s = 1, 2. \end{cases} \tag{C_0}$$

By the oddness of Z_3 and the condition $(C2.1)-(C2.2)$, it is easy to verify that

$$\int_{\mathbb{R}^3} Z_3(y)h_{1k}(y)dy = 0.$$

For Z_1, Z_2, we consider the vector integral

$$I = \int_{\mathbb{R}^3} h_{1k}(y)\begin{bmatrix} Z_1(y) \\ Z_2(y) \end{bmatrix} dy = -\int_{\mathbb{R}^3} \frac{h_{1k}(y)}{(1+|y|^2)^{\frac{3}{2}}}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} dy,$$

then by the condition $(C1)$, we calculate that,

$$\begin{aligned} e^{\frac{2\pi}{k}\sqrt{-1}}I &= -\int_{\mathbb{R}^3} \frac{h_{1k}(y)}{(1+|y|^2)^{\frac{3}{2}}} e^{\frac{2\pi}{k}\sqrt{-1}}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} dy \\ &= -\int_{\mathbb{R}^3} \frac{h_1\left(e^{-\frac{2\pi}{k}\sqrt{-1}}(z_1,z_2),z_3\right)}{\left(1+\left|\left(e^{-\frac{2\pi}{k}\sqrt{-1}}(z_1,z_2),z_3\right)\right|^2\right)^{\frac{3}{2}}}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} dz \\ &= I, \end{aligned}$$

which yields that $I = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence we get that

$$\int_{\mathbb{R}^3} Z_1(y)h_{1k}(y)dy = \int_{\mathbb{R}^3} Z_2(y)h_{1k}(y)dy = 0.$$

For Z_4, observe that $Z_4(y) = \partial_\lambda\Big|_{\lambda=1}\left[\lambda^{\frac{1}{2}}U_{1,0}(\lambda y)\right]$, we define the function $I(\lambda)$ by

$$I(\lambda) = \lambda^{\frac{1}{2}}\int_{\mathbb{R}^3} U_{1,0}(\lambda y)h_{1k}(y)dy.$$

By changing variables $y \mapsto \frac{y}{|y|^2}$ and the condition $(C3)$, we have that $I(\lambda) = I\left(\frac{1}{\lambda}\right)$. Thus

$$\partial_\lambda I(\lambda)\Big|_{\lambda=1} = -\frac{1}{\lambda^2}\cdot\partial_s I(s)\Big|_{s=\frac{1}{\lambda},\lambda=1} = -\partial_\lambda I(\lambda)\Big|_{\lambda=1},$$

and

$$\partial_\lambda I(\lambda)\Big|_{\lambda=1} = \int_{\mathbb{R}^3} Z_4(y)h_{1k}(y)dy = 0.$$

Hence

$$\int_{\mathbb{R}^3} Z_i(y)\cdot h_{1k}(y)dy = 0, \quad i = 1,2,3,4.$$

Similarly

$$\int_{\mathbb{R}^3} Z_i(y)\cdot h_{2k}(y)dy = 0, \quad i = 1,2,3,4.$$

For the functions Z_{js}, we define the unit vectors as $e_j = \left(\cos\left(\frac{2(j-1)\pi}{k}\right), \sin\left(\frac{2(j-1)\pi}{k}\right), 0\right) \in \mathbb{R}^3, \ \ j = 1,2,\cdots,k.$. Then a direct computation shows that

$$\begin{aligned} \int_{\mathbb{R}^3} Z_{j,1}h_{2k}(y)dy &= \int_{\mathbb{R}^3} \frac{\partial}{\partial r}\left(\frac{\varepsilon}{\varepsilon^2+|y-x_j|^2}\right)^{\frac{1}{2}}\cdot h_{2k}(y)dy \\ &= \varepsilon^{\frac{1}{2}}\cdot\int_{\mathbb{R}^3} \frac{y\cdot e_j - r}{(\varepsilon^2+|y-x_j|^2)^{\frac{3}{2}}}\cdot h_{2k}(y)dy, \end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^3} Z_{j,2}(y)h_{2k}(y)dy\\
=&\int_{\mathbb{R}^3}\frac{\partial}{\partial\varepsilon}\left(\frac{\varepsilon}{\varepsilon^2+|y-x_j|^2}\right)^{\frac{1}{2}}\cdot h_{2k}(y)dy\\
=&\frac{1}{2}\cdot\varepsilon^{-\frac{1}{2}}\cdot\int_{\mathbb{R}^3}\frac{|y-x_j|^2-\varepsilon^2}{(\varepsilon^2+|y-x_j|^2)^{\frac{3}{2}}}\cdot h_{2k}(y)dy\\
=&\frac{1}{2}\cdot\varepsilon^{-\frac{1}{2}}\cdot\int_{\mathbb{R}^3}\frac{(|y|^2-1)h_{2k}(y)}{(\varepsilon^2+|y-x_j|^2)^{\frac{3}{2}}}dy-\frac{r}{\varepsilon}\int_{\mathbb{R}^3} Z_{j,1}(y)\cdot h_{2k}(y)dy.
\end{aligned}$$

It follows from the condition $(C3)$ that

$$\begin{aligned}
&\frac{1}{2}\cdot\varepsilon^{-\frac{1}{2}}\cdot\int_{\mathbb{R}^3}\frac{(|y|^2-1)h_{2k}(y)}{(\varepsilon^2+|y-x_j|^2)^{\frac{3}{2}}}dy\\
=&\frac{1}{2}\cdot\varepsilon^{-\frac{1}{2}}\cdot\int_{\mathbb{R}^3}\frac{(\left|\frac{y}{|y|^2}\right|^2-1)h_{2k}\left(\frac{y}{|y|^2}\right)}{\left(\varepsilon^2+\left|\frac{y}{|y|^2}-x_j\right|^2\right)^{\frac{3}{2}}}d\left(\frac{y}{|y|^2}\right)=\frac{1}{2}\cdot\varepsilon^{-\frac{1}{2}}\cdot\int_{\mathbb{R}^3}\frac{(1-|y|^2)h_{2k}(y)}{(\varepsilon^2+|y-x_j|^2)^{\frac{3}{2}}}dy\\
=&-\frac{1}{2}\cdot\varepsilon^{-\frac{1}{2}}\cdot\int_{\mathbb{R}^3}\frac{(|y|^2-1)h_{2k}(y)}{(\varepsilon^2+|y-x_j|^2)^{\frac{3}{2}}}dy=0.
\end{aligned}$$

Therefore

$$\int_{\mathbb{R}^3} Z_{j,2}(y)h_{2k}(y)dy=-\frac{r}{\varepsilon}\int_{\mathbb{R}^3} Z_{j,1}(y)\cdot h_{2k}(y)dy.$$

Now it is sufficient to prove that

$$\int_{\mathbb{R}^3} Z_{j,1}(y)\cdot h_{2k}(y)dy=0,\quad j=1,2,\cdots,k.$$

Let $I_j(t)=\int_{\mathbb{R}^3} w_\varepsilon(y-t\cdot x_j)\cdot h_{2k}(y)dy$ with $w_\varepsilon(y)=\varepsilon^{-\frac{1}{2}}U_{1,0}(\varepsilon^{-1}y)$, then

$$\partial_t\Big|_{t=1}I_j(t)=\int_{\mathbb{R}^3}\partial_t\Big|_{t=1}w_\varepsilon(y-t\cdot x_j)\cdot h_{2k}(y)dy=r\cdot\int_{\mathbb{R}^3} Z_{j,1}(y)\cdot h_{2k}(y)dy. \tag{2.4}$$

By changing the variables $y\mapsto\frac{y}{|y|^2}$ and the condition $(C2.1)-(C2.2)$, we obtain

$$\begin{aligned}
I_j(t)=&\int_{\mathbb{R}^3} w_\varepsilon\left(\frac{y}{|y|^2}-t\cdot x_j\right)\cdot h_{2k}\left(\frac{y}{|y|^2}\right)d\left(\frac{y}{|y|^2}\right)\\
=&\int_{\mathbb{R}^3} w_\varepsilon\left(\frac{y}{|y|^2}-t\cdot x_j\right)\cdot|y|^{-1}\cdot h_{2k}(y)dy\\
=&\int_{\mathbb{R}^3}\left(\frac{\varepsilon}{\varepsilon^2+t^2r^2}\right)^{\frac{1}{2}}\cdot\left(\frac{1}{\left|y-\frac{t\cdot x_j}{\varepsilon^2+t^2r^2}\right|^2+\frac{\varepsilon^2}{(\varepsilon^2+t^2r^2)^2}}\right)^{\frac{1}{2}}\cdot h_{2k}(y)dy\\
=&\int_{\mathbb{R}^3} w_{\varepsilon(t)}\left(y-r(t)\cdot x_j\right)\cdot h_{2k}(y)dy,
\end{aligned}$$

where $\varepsilon(t)=\frac{\varepsilon}{\varepsilon^2+t^2r^2}, r(t)=\frac{t}{\varepsilon^2+r^2t^2}$. Noticing that $\varepsilon(1)=\varepsilon;\quad r(1)=1$, we have,

$$
\begin{aligned}
&\partial_t\Big|_{t=1} I_j(t)\\
=&\left[\int_{\mathbb{R}^3}\partial_{\varepsilon(t)}w_{\varepsilon(t)}\left(y-r(t)x_j\right)h_{2k}(y)dy\cdot\varepsilon'(t)-\int_{\mathbb{R}^3}\partial_{r(t)}w_{\varepsilon(t)}\left(y-r(t)x_j\right)h_{2k}(y)dy\cdot r'(t)\right]\Bigg|_{t=1}\\
=&(-2\varepsilon r^2)\cdot\int_{\mathbb{R}^3}Z_{j,2}(y)h_{2k}(y)dy-(1-2r^2)\cdot\int_{\mathbb{R}^3}Z_{j,1}(y)h_{2k}(y)dy\\
=&(2r^3+2r^2-1)\cdot\int_{\mathbb{R}^3}Z_{j,1}(y)h_{2k}(y)dy.
\end{aligned}
\tag{2.5}
$$

Comparing the identities (2.4) and (2.5), we obtain

$$
r\cdot\int_{\mathbb{R}^3}Z_{j,1}(y)h_{2k}(y)dy=(2r^3+2r^2-1)\cdot\int_{\mathbb{R}^3}Z_{j,1}(y)h_{2k}(y)dy,\forall\ \ 0<r<1,
$$

hence

$$
\int_{\mathbb{R}^3}Z_{j,2}(y)h_{2k}(y)dy=-\frac{r}{\varepsilon}\int_{\mathbb{R}^3}Z_{j,1}(y)\cdot h_{2k}(y)dy=0,
$$

and the condition (C_0) holds as desired. □

3 Estimates of the error terms

In this section, we go back to the system (LS). Note that the problem (LS) is closely related to the problem (LN). However, these two problems are essentially different because the nonlinear data term h_k in (LS) depends on the solution (ψ_k,φ_k) itself, while the one in (LN) is independently given. We will present the precise asymptotic estimate for the nonlinear term h_k. Without loss of generality, we may assume that $\|\psi_k\|_*,\|\varphi_k\|_*<<1$. Recall the problem (LS) reads as:

$$
\begin{cases}
L_0(\psi_k,\varphi_k) &=-\left[N_{5,0}(\psi_k,\varphi_k)+N_{2,3}(\psi_k,\varphi_k)\right]:=h_{1k}(\psi_k,\varphi_k) \quad \text{in } \mathbb{R}^3\\
L_{00}(\psi_k,\varphi_k) &=-\left[E+N_{0,5}(\psi_k,\varphi_k)+N_{3,2}(\psi_k,\varphi_k)\right]:=h_{2k}(\psi_k,\varphi_k) \quad \text{in } \mathbb{R}^3,
\end{cases}
\tag{LS}
$$

where the nonlinear data term $h_k(\psi_k,\varphi_k)=(h_{1k}(\psi_k,\varphi_k),\quad h_{2k}(\psi_k,\varphi_k))$ is decomposed into five nonlinear error terms $E, N_{5,0}(\psi_k,\varphi_k), N_{2,3}(\psi_k,\varphi_k), N_{3,2}(\psi_k,\varphi_k), N_{0,5}(\psi_k,\varphi_k)$.

We will give estimates of each part by the following lemmas.

Lemma 3.1 $\|E\|_{**}\leq Ck^{1-\frac{3}{q}}\ln^{-5}k$.

Proof. Since the term E can be written into a form of a polynomial as the following:

$$
E=\left(\sum_{i=1}^{k}U_{\varepsilon,x_i}\right)^5-\sum_{i=1}^{k}U^5_{\varepsilon,x_i}=\sum_{\substack{i_1+i_2+\cdots+i_k=5\\ i_1,i_2,\cdots,i_k\in\mathbb{N}\\ i_1,i_2,\cdots,i_k\neq5}}\prod_{l=1}^{k}U^{i_l}_{\varepsilon,x_l},
$$

which is a sum of (k^5-k) terms, without loss of generality, we only consider the term $u_{\varepsilon,x_1}\cdot u^4_{\varepsilon,x_2}$. Recall that $\varepsilon\sim k^{-4}\ln^{-2}k$, $\frac{3}{2}<q<3$.

For $y \in EXT$, we have that $|y - x_1| \geq \frac{\eta}{k}$, and hence

$$\begin{aligned}
\left|U_{\varepsilon,x_1}(y) \cdot U^4_{\varepsilon,x_2}(y)\right| \leq & C\left(\frac{\varepsilon}{\varepsilon^2 + |y-x_1|^2}\right)^{\frac{1}{2}} \cdot \left(\frac{\varepsilon}{\varepsilon^2 + |y-x_2|^2}\right)^2 \\
\leq & C\left(\frac{\varepsilon}{\varepsilon^2 + \frac{\eta^2}{k^2}}\right)^{\frac{1}{2}} \cdot \frac{\varepsilon^{-2}}{1 + \left|\frac{y}{\varepsilon} - \frac{x_2}{\varepsilon}\right|^4} \\
\leq & C(k^2\varepsilon)^{\frac{1}{2}} \cdot \frac{\varepsilon^{-2}}{1 + \left|\frac{y}{\varepsilon} - \frac{x_2}{\varepsilon}\right|^4} \\
\leq & Ck^7 \ln^3 k \cdot \frac{1}{1 + \left|\frac{y}{\varepsilon} - \frac{x_2}{\varepsilon}\right|^4},
\end{aligned}$$

Let $w = \frac{y}{\varepsilon} - \frac{x_2}{\varepsilon}$, we have

$$\begin{aligned}
& \|U_{\varepsilon,x_1} \cdot U^4_{\varepsilon,x_2}\|_{**(EXT)} \\
\leq & C\|(1+|y|)^{5-\frac{6}{q}} u_{\varepsilon,x_1}(y) \cdot u^4_{\varepsilon,x_2}(y)\|_{L^q(EXT)} \\
\leq & Ck^7 \ln^3 k \cdot \varepsilon^{\frac{3}{q}} \cdot \left(\int_{\mathbb{R}^3} \frac{(1+|\varepsilon w + x_2|)^{5q-6} dw}{1+|w|^{4q}}\right)^{\frac{1}{q}} \\
\leq & Ck^7 \ln^3 k \cdot \varepsilon^{\frac{3}{q}} \left[\int_0^{\frac{1}{\varepsilon}} \frac{r^2 dr}{1+r^{4q}} + \int_{\frac{1}{\varepsilon}}^{\infty} \frac{\varepsilon^{5q-6} r^{5q-4} dr}{1+r^{4q}}\right]^{\frac{1}{q}} \\
\leq & Ck^7 \ln^3 k \cdot \varepsilon^{\frac{3}{q}} \varepsilon^{4-\frac{3}{q}} \\
\leq & Ck^{-9} \ln^{-5} k.
\end{aligned}$$

The desired estimation for E in the exterior region is the sum of $k^5 - k$ such terms, hence

$$\|E\|_{**(EXT)} \leq Ck^{-4} \ln^{-5} k.$$

For $y \in INT$, $\exists j \in \{1, 2, \cdots, k\}$ such that $|y - x_j| < \eta/k$. Similarly as what we have done in the previous section in **INT**, let $w = \frac{y - x_1}{\varepsilon}$, since $x_1 - x_2 \sim \frac{2\pi}{k}$, and for $|w| \leq \frac{\eta}{k\varepsilon}$ with $0 < \eta << 1$, the term $\left|\frac{x_1-x_2}{\varepsilon}\right|$ dominates $|w|$, hence the following estimates yields:

$$\begin{aligned}
&\|u_{\varepsilon,x_1}\cdot u^4_{\varepsilon,x_2}\|_{**(B_{x_1}(\eta/k))}\\
\leq &C\left\{\int_{B_{x_1}(\eta/k)}(1+|y|)^{5q-6}\cdot u^q_{\varepsilon,x_1}(y)\cdot u^{4q}_{\varepsilon,x_2}(y)dy\right\}^{\frac{1}{q}}\\
\leq &C\varepsilon^{\frac{3}{q}}\left[\int_{B_0(\frac{\eta}{k\varepsilon})}\left|u_{\varepsilon,x_1}(x_1+\varepsilon w)\cdot u^4_{\varepsilon,x_2}(x_1+\varepsilon w)\right|^q dw\right]^{\frac{1}{q}}\\
\leq &C\varepsilon^{\frac{3}{q}-\frac{5}{2}}\left[\int_{B_0(\frac{\eta}{k\varepsilon})}\left|\frac{1}{1+|w|}\cdot\frac{1}{1+\left|\frac{x_1-x_2}{\varepsilon}+w\right|^4}\right|^q dw\right]^{\frac{1}{q}}\\
\leq &C\varepsilon^{\frac{3}{q}-\frac{5}{2}}\left[\int_{B_0(\frac{\eta}{k\varepsilon})}\left|\frac{k^4\varepsilon^4}{1+|w|}\right|^q dw\right]^{\frac{1}{q}}\\
\leq &Ck^4\varepsilon^{\frac{3}{q}+\frac{3}{2}}\cdot\left[\int_{B_0(\frac{\eta}{k\varepsilon})}\frac{dw}{1+|w|^q}\right]^{\frac{1}{q}}\\
\leq &Ck^{-5-\frac{3}{q}}\cdot\ln^{-5}k.
\end{aligned}$$

At last, we sum all k concentration balls together and obtain that

$$\|u_{\varepsilon,x_1}\cdot u^4_{\varepsilon,x_2}\|_{**(INT)}\leq Ck^{-4-\frac{3}{q}}\ln^{-5}k,$$

and the estimation for E in the region **INT** is the sum of (k^5-k) such terms such that

$$\|E\|_{**(INT)}\leq Ck^{1-\frac{3}{q}}\ln^{-5}k.$$

Combining the results of exterior region and interior one together, we obtain the final estimate of the first error term E as

$$\|E\|_{**}\leq\|E\|_{**(EXT)}+\|E\|_{**(INT)}\leq Ck^{1-\frac{3}{q}}\ln^{-5}k.$$

□

The following lemma 3.2 is due to (Lemma 3.5 of [24]).

Lemma 3.2 *We have for any $t\geq -1$ and $q>1$*

$$|(1+t)^q-1-qt|\leq\begin{cases}C\min\{t^q,t^2\} & \text{if } 1<q\leq 2;\\ C(t^2+t^q) & \text{if } q>2.\end{cases}$$

Lemma 3.3

$$\|N_{5,0}(\psi_k,\varphi_k)\|_{**}\leq C\left[k^{1-\frac{3}{q}}\ln^{-2}k\|\psi_k\|^2_*+k^{1-\frac{3}{q}}\ln^{-5}k\|\psi_k\|^5_*\right]\leq Ck^{1-\frac{3}{q}}\ln^{-2}k\|\psi_k\|^2_*.$$

Proof. By Lemma 3.2, we have

$$\begin{aligned}|N_{5,0}(\psi_k,\varphi_k)| =&\left|(u_*+\psi_k)^5-u_*^5-5u_*^4\psi_k\right|\\ =&u_*^5\cdot\left|\left(1+\frac{\psi_k}{u_*}\right)^5-1-5\frac{\psi_k}{u_*}\right|\\ \le&Cu_*^5\cdot\left[\left|\frac{\psi_k}{u_*}\right|^2+\left|\frac{\psi_k}{u_*}\right|^5\right]\\ \le&C\left[|\psi_k|^2\cdot u_*^3+|\psi_k|^5\right]\\ \le&C\left[u_*^3\cdot v_*^2\|\psi\|_*^2+v_*^5\|\psi\|_*^5\right].\end{aligned}$$

Hence

$$\begin{aligned}\|N_{5,0}(\psi_k,\varphi_k)\|_{**}\le&\|N_{5,0}(\psi_k,\varphi_k)\|_{**(EXT)}+\sum_{j=1}^k\|N_{5,0}(\psi_k,\varphi_k)\|_{**\left(B_{x_j}(\eta/k)\right)}\\ \le&C\left[\|u_*^3\cdot v_*^2\|_{**(EXT)}+\sum_{j=1}^k\|u_*^3\cdot v_*^2\|_{**\left(B_{x_j}(\eta/k)\right)}\right]\cdot\|\psi_k\|_*^2\\ &+C\left[\|v_*^5\|_{**(EXT)}+\sum_{j=1}^k\|v_*^5\|_{**\left(B_{x_j}(\eta/k)\right)}\right]\cdot\|\psi_k\|_*^5.\end{aligned}$$

Recall the interior estimate $(v-int)$ and exterior estimate $(v-ext)$ of v_* in Proposition 2.2, we have

$$\begin{aligned}&\|u_*^3\cdot v_*^2\|_{**(EXT)}+\sum_{j=1}^k\|u_*^3\cdot v_*^2\|_{**\left(B_{x_j}(\eta/k)\right)}\\ \le&C\left\{\int_{EXT\setminus B_0(2)}\left[\left(\frac{1}{1+|y|}\right)^3\cdot\left(\frac{k\varepsilon^{\frac12}}{1+|y|}\right)^2\right]^q\cdot(1+|y|)^{5q-6}dy\right\}^{\frac1q}\\ &+Ck^{-2}\left\{\int_{EXT\cap B_0(2)}(1+|y|)^{2q-6}dy\right\}^{\frac1q}\cdot\\ &+C\sum_{j=1}^k\varepsilon^{\frac3q}\left\{\int_{B_0(\frac{\eta}{k\varepsilon})}v_*^{2q}(x_j+\varepsilon w)dw\right\}^{\frac1q}\\ \le&C\left(k^{-2}\ln^{-2}k+k^{-2}+k^{1-\frac3q}\ln^{-2}k\right)\\ \le&Ck^{1-\frac3q}\ln^{-2}k,\end{aligned}$$

and

$$
\begin{aligned}
&\|v_*^5\|_{**(EXT)} + \sum_{j=1}^{k} \|v_*^5\|_{**\left(B_{x_j}(\eta/k)\right)} \\
\leq & C\left\{\int_{EXT\setminus B_0(2)} \left[\left(\frac{k\varepsilon^{\frac{1}{2}}}{1+|y|}\right)^5\right]^q \cdot (1+|y|)^{5q-6} dy\right\}^{\frac{1}{q}} \\
&+ Ck^{-5}\left\{\int_{EXT\cap B_0(2)} (1+|y|)^{5q-6} dy\right\}^{\frac{1}{q}} \\
&+ Ck\varepsilon^{\frac{3}{q}}\left\{\int_{B_0(\frac{\eta}{k\varepsilon})} v_*^{5q}(x_j+\varepsilon w) dw\right\}^{\frac{1}{q}} \\
\leq & C\left(k^{-5}\ln^{-5} k + k^{-5} + k^{1-\frac{3}{q}}\ln^{-5} k\right) \\
\leq & Ck^{1-\frac{3}{q}}\ln^{-5} k.
\end{aligned}
$$

□

Lemma 3.4 *(1)*$\|N_{2,3}(\psi_k,\varphi_k)\|_{**} \leq Ck^{1-\frac{3}{q}} \cdot \ln^{-3} k$;
(2)$\|N_{3,2}(\psi_k,\varphi_k)\|_{**} \leq Ck^{1-\frac{3}{q}} \cdot \ln^{-2} k$.

Proof. We we split the term $N_{2,3}(\psi_k,\phi_k)$ into four parts so that Lemma 3.2 can be used, more precisely,

$$
\begin{aligned}
N_{2,3}(\psi_k,\varphi_k) =& -(u_*+\psi_k)^2\cdot(v_*+\varphi_k)^3 + 3u_*^2\cdot v_*^2\cdot\varphi_k + 2u_*\cdot v_*^3\cdot\psi_k \\
=& \quad \left[-(u_*+\psi_k)^2\cdot(v_*+\varphi_k)^3 + 3(u_*+\psi_k)^2\cdot(v_*+\varphi_k)^2\cdot\varphi_k + (u_*+\psi_k)^2\cdot v_*^3\right] \\
&+ \left[-3(u_*+\psi_k)^2\cdot(v_*+\varphi_k)^2\cdot\varphi_k + 3u_*^2\cdot u_*^2\cdot\varphi_k\right] \\
&+ \left[-(u_*+\psi_k)^2\cdot v_*^3 + 2(u_*+\psi_k)\cdot\psi_k\cdot v_*^3 + u_*^2\cdot v_*^3\right] \\
&+ \left[-2\psi_k^2\cdot v_*^3 - u_*^2\cdot v_*^3\right] \\
=:& I + II + III + IV.
\end{aligned}
$$

Firstly, by Lemma 3.2, we have

$$
\begin{aligned}
|I| \leq & C\,|u_*+\psi_k|^2\cdot|v_*+\varphi_k|^3\cdot\left|\left(\frac{v_*}{v_*+\varphi_k}\right)^3 + 3\frac{\varphi_k}{v_*+\varphi_k} - 1\right| \\
\leq & C\,|u_*+\psi_k|^2\cdot|v_*+\varphi_k|^3\cdot\left|\left(1+\frac{-\varphi_k}{v_*+\varphi_k}\right)^3 - 3\frac{-\varphi_k}{v_*+\varphi_k} - 1\right| \\
\leq & C\,|u_*+\psi_k|^2\cdot\left[|v_*+\varphi_k|\cdot|\varphi_k|^2 + |\varphi_k|^3\right] \\
\leq & Cv_*^3\cdot(u_*+\|\psi_k\|_*\cdot v_*)^2\cdot(1+2\|\varphi_k\|_*)\cdot\|\varphi_k\|_*^2.
\end{aligned}
$$

Hence, the estimates $(v - int)$ and $(v - ext)$ give that

$$
\begin{aligned}
\|I\|_{**} \leq & \|I\|_{**(EXT)} + \sum_{j=1}^{k} \|I\|_{**B_{x_j}(\eta/k)} \\
\leq & C\left[\int_{EXT\setminus B_0(2)} (1+|y|)^{5q-6} \cdot \left(\frac{k\varepsilon^{\frac{1}{2}}}{1+|y|}\right)^{3q} \cdot \left(\frac{1}{1+|y|} + \|\psi_k\|_* \cdot \frac{k\varepsilon^{\frac{1}{2}}}{1+|y|}\right)^{2q} dy\right]^{\frac{1}{q}} \cdot \|\varphi_k\|_*^2 \\
& + Ck^{-3}\left[\int_{EXT\cap B_0(2)} (1+|y|)^{5q-6} dy\right]^{\frac{1}{q}} \cdot \|\varphi_k\|_*^2 \\
& + Ck\varepsilon^{\frac{3}{q}}\left[\int_{B_0(\frac{\eta}{k\varepsilon})} \left(\frac{k\varepsilon^{-\frac{1}{2}}}{1+|w|}\right)^{3q} \cdot \left(\frac{1}{1+|\varepsilon w + x_j|} + \|\psi_k\|_* \cdot \frac{k\varepsilon^{-\frac{1}{2}}}{1+|w|}\right)^{2q} dw\right]^{\frac{1}{q}} \cdot \|\varphi_k\|_*^2 \\
\leq & C\left[k^{-3}\ln^{-3} k + k^{-3} + k^{1-\frac{3}{q}}\ln^{-3} k\right] \cdot \|\varphi_k\|_*^2 \\
\leq & Ck^{1-\frac{3}{q}}\ln^{-3} k\|\varphi_k\|_*^2.
\end{aligned}
$$

Similarly, as in the computation of I, we have

$$
\begin{aligned}
|II| \leq & \left|-3(u_* + \psi_k)^2 \cdot (v_* + \varphi_k)^2 \cdot \varphi_k + 3u_*^2 \cdot v_*^2 \cdot \varphi_k\right| \\
\leq & |\varphi_k| \cdot \Big[6u_* \cdot v_*^2 \cdot |\psi_k| + 3v_*^2 \cdot |\psi_k|^2 + 6u_*^2 \cdot v_* \cdot |\varphi_k| + 12u_* \cdot v_* \cdot |\varphi_k| \cdot |\psi_k| + 6v_* \cdot |\varphi_k| \cdot \psi_k^2 \\
& + 3u_*^2 \cdot \varphi_k^2 + 6u_* \cdot \varphi_k^2 \cdot |\psi_k| + 3\psi_k^2 \cdot \varphi_k^2\Big] \\
\leq & C\|\varphi_k\|_* \cdot v_* \cdot \Big[u_* \cdot v_*^3 \cdot \|\psi_k\|_* + v_*^4 \cdot \|\psi_k\|_*^2 + u_*^2 \cdot v_*^2 \cdot \|\varphi_k\|_* + u_* \cdot v_*^3 \cdot \|\varphi_k\|_* \cdot \|\psi_k\|_* \\
& + v_*^4 \cdot \|\varphi_k\|_* \cdot \|\psi_k\|_*^2 + u_*^2 \cdot v_*^2 \cdot \ |\varphi_k\|_*^2 + u_* \cdot v_*^3 \cdot \|\varphi_k\|_*^2 \cdot \|\psi_k\|_* + v_*^2\|\psi_k\|_*^2 \cdot \|\varphi_k\|_*^2\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
& \|II\|_{**} \\
\leq & \|II\|_{**(EXT)} + \sum_{j=1}^{k} \|II\|_{**B_{x_j}(\eta/k)} \\
\leq & C\left[\left(k^{-4}\ln^{-4} k + k^{-4} + k^{1-\frac{3}{q}}\ln^{-4} k\right) \cdot \|\varphi_k\|_* \cdot \|\psi_k\|_* + \left(k^{-3}\ln^{-3} k + k^{-3} + k^{1-\frac{3}{q}}\ln^{-3} k\right) \cdot \|\varphi_k\|_*^2\right] \\
\leq & Ck^{1-\frac{3}{q}}\ln^{-3} k \cdot \|\varphi_k\|_* \cdot \left(\ln^{-1} k \cdot \|\psi_k\|_* + \|\varphi_k\|_*\right).
\end{aligned}
$$

$$
\begin{aligned}
|III| = & \left|-(u_* + \psi_k)^2 \cdot v_*^3 + 2(u_* + \psi_k) \cdot \psi_k \cdot v_*^3 + u_*^2 \cdot v_*^3\right| \\
\leq & C|v_*|^3 \cdot (u_* + \psi_k)^2 \cdot \left|\left(1 + \frac{-\psi_k}{u_* + \psi_k}\right)^2 - 2\frac{-\psi_k}{u_* + \psi_k} - 1\right| \\
\leq & Cv_*^3 \cdot \psi_k^2 \\
\leq & Cv_*^5 \cdot \|\psi_k\|_*^2,
\end{aligned}
$$

and

$$\begin{aligned}\|III\|_{**} \leq & \|III\|_{**(EXT)} + \sum_{j=1}^{k} \|III\|_{**_{B_{x_j}(\eta/k)}} \\ \leq & C\left(k^{-5}\ln^{-5}k + k^{-5} + k^{1-\frac{3}{q}}\ln^{-5}k\right)\cdot \|\psi_k\|_*^2 \\ \leq & Ck^{1-\frac{3}{q}}\ln^{-5}k\cdot \|\psi_k\|_*^2.\end{aligned}$$

For the last term IV, a direct calculation shows that

$$|IV| \leq C\left(v_*^5\cdot\|\psi_k\|_*^2 + u_*^2 v_*^3\right),$$

hence

$$\begin{aligned}\|IV\|_{**} \leq & \|IV\|_{**(EXT)} + \sum_{j=1}^{k} \|IV\|_{**_{B_{x_j}(\eta/k)}} \\ \leq & C\left(k^{-5}\ln^{-5}k + k^{-5} + k^{1-\frac{3}{q}}\ln^{-5}k\right)\cdot\|\psi_k\|_*^2 + C\left(k^{-3}\ln^{-3}k + k^{-3} + k^{1-\frac{3}{q}}\ln^{-3}k\right) \\ \leq & Ck^{1-\frac{3}{q}}\left(\ln^{-5}k\cdot\|\psi_k\|_*^2 + \ln^{-3}k\right).\end{aligned}$$

Therefore

$$\begin{aligned}& \|N_{2,3}(\psi_k,\varphi_k)\|_{**} \\ \leq & \|I\|_{**} + \|II\|_{**} + \|III\|_{**} + \|IV\|_{**} \\ \leq & C\left[k^{1-\frac{3}{q}}\ln^{-3}k\|\varphi_k\|_*^2 + k^{1-\frac{3}{q}}\ln^{-4}k\|\varphi_k\|_*\|\psi_k\|_* + k^{1-\frac{3}{q}}\ln^{-3}k\left(\|\varphi_k\|_*^2+1\right) + k^{1-\frac{3}{q}}\ln^{-5}k\|\psi_k\|_*^2\right] \\ \leq & Ck^{1-\frac{3}{q}}\ln^{-3}k.\end{aligned}$$

By using the similar arguments, we obtain

$$\|N_{3,2}(\psi_k,\varphi_k)\|_{**} \leq Ck^{1-\frac{3}{q}}\cdot\ln^{-2}k$$

□

By Lemma 3.2 again, we have

$$\begin{aligned}& \|N_{0,5}(\psi_k,\varphi_k)\| \\ = & \left|(v_*+\varphi_k)^5 - v_*^5 - 5v_*^4\cdot\varphi_k\right| \\ = & v_*^5\cdot\left|\left(1+\frac{\varphi_k}{v_*}\right)^5 - 5\frac{\varphi_k}{v_*} - 1\right| \\ \leq & C\left(v_*^3\cdot|\varphi_k|^2 + |\varphi_k|^5\right) \\ \leq & Cv_*^5\cdot\|\varphi_k\|_*^2.\end{aligned}$$

Therefore, it is direct to obtain the following lemma without proof.

Lemma 3.5

$$\|N_{0,5}(\psi_k,\varphi_k)\|_{**} \leq Ck^{1-\frac{3}{q}}\cdot\ln^{-5}k\cdot\|\varphi_k\|_*^2.$$

4 Proof of the Theorem 1.1

We reduce the problem (LS) into a fixed point form, namely

$$(\psi_k,\varphi_k)=T\left(h_{1k}(\psi_k,\varphi_k),h_{2k}(\psi_k,\varphi_k)\right):=\mathcal{M}(\psi_k,\varphi_k),$$

where T is a bounded linear operator defined in Proposition 2.1, X is a Banach space defined as:

$$X:=\{(\psi,\varphi)\in C(\mathbb{R}^3)\times C(\mathbb{R}^3)\Big|\ \ \|(\psi,\varphi)\|_*:=||(\varphi,\psi)||_*\|\psi\|_*+\|\varphi\|_*<\rho\},$$

where ρ is a small positive number.

Since T is bounded, by the results of nonlinear data terms in the previous section, and the assumption of $\frac{3}{2}<q<3$, for k large enough, we have that

$$\begin{aligned}\|\mathcal{M}(\psi_k,\varphi_k)\|_* \leq& C\left[\|h_{1k}(\psi_k,\varphi_k)\|_{**}+\|h_{2k}(\psi_k,\varphi_k)\|_{**}\right]\\ \leq& C\left[\|N_{5,0}(\psi_k,\varphi_k)\|_{**}+\|N_{2,3}(\psi_k,\varphi_k)\|_{**}+\|N_{3,2}(\psi_k,\varphi_k)\|_{**}+\|N_{0,5}(\psi_k,\varphi_k)\|_{**}\right]\\ \leq& Ck^{1-\frac{3}{q}}\ln^{-2}k<\rho(\text{ for } k \text{ large}).\end{aligned}$$

This implies that the operator $\mathcal{M}$ maps X to X itself. In the following, we will show that $\mathcal{M}$ is a contraction mapping in the $\|\cdot\|_*$ norm. Choose any two elements (ψ_1,φ_1) and (ψ_2,φ_2) in X, we have

$$\begin{aligned}&\mathcal{M}(\psi_1,\varphi_1)-\mathcal{M}(\psi_2,\varphi_2)\\ =&T\Bigg\{\left[-(u_*+\psi_1)^5+(u_*+\psi_2)^5\right]\\ &+\left[(5u_*^4-2u_*v_*^3)\cdot(\psi_1-\psi_2)-3u_*^2v_*^2(\varphi_1-\varphi_2)\right]\\ &+\left[(u_*+\psi_1)^2\cdot((v_*+\varphi_1)^3-(v_*+\varphi_2)^3)+(v_*+\varphi_2)^3\cdot((u_*+\psi_1)^2-(u_*+\psi_2)^2)\right],\\ &\left[-(v_*+\varphi_1)^5+(v_*+\varphi_2)^5\right]\\ &+\left[-3u_*^2v_*^2(\psi_1-\psi_2)+(5v_*^4-2u_*^3v_*)(\varphi_1-\varphi_2)\right]\\ &+\left[(u_*+\psi_1)^3\cdot((v_*+\varphi_1)^2-(v_*+\varphi_2)^2)+(v_*+\varphi_2)^2\cdot((u_*+\psi_1)^3-(u_*+\psi_2)^3)\right]\Bigg\}\\ :=&T\Bigg\{N_1\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)+N_2\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)+N_3\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right),\\ &N_4\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)+N_5\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)+N_6\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\Bigg\}.\end{aligned}$$

For the first term $N_1\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)$, by the mean value theorem, $\exists\theta\in(0,1)$ such that

$$\begin{aligned}\left|N_1\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\right| =&5\left|u_*+\theta\cdot\psi_1+(1-\theta)\cdot\psi_2\right|^4\cdot|\psi_1-\psi_2|\\ \leq& C\left|u_*+(\|\psi_1\|_*+\|\psi_2\|_*)\cdot v_*\right|^4\cdot v_*\cdot\|\psi_1-\psi_2\|_*\\ \leq& C\left|u_*+\rho v_*\right|^4\cdot v_*\cdot\|\psi_1-\psi_2\|_*.\end{aligned}$$

By using the similar computation as in the Section 3, we have,

$$
\begin{aligned}
&\|N_1\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}\\
\leq&\|N_1\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**(EXT)}+\sum_{j=1}^{k}\|N_1\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**(B_{x_j}(\eta/k))}\\
\leq&C\left(k^{-1}\ln^{-1}k+k^{-1}+k^{1-\frac{3}{q}}\cdot\ln^{-1}k\right)\|\psi_1-\psi_2\|_*\\
\leq&Ck^{1-\frac{3}{q}}\cdot\ln^{-1}k\|\psi_1-\psi_2\|_*.
\end{aligned}
$$

In order to avoid the unnecessary repetitions, we briefly give the estimates of the rest of the nonlinear variances.

$$
\begin{aligned}
&|N_6\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)|\\
=&\left|(u_*+\psi_1)^3\cdot\left((v_*+\varphi_1)^2-(v_*+\varphi_2)^2\right)+(v_*+\varphi_2)^2\cdot\left((u_*+\psi_1)^3-(u_*+\psi_2)^3\right)\right|\\
\leq&C\left[|u_*+\psi_1|^3\cdot\left|(v_*+\varphi_1)^2-(v_*+\varphi_2)^2\right|+(v_*+\varphi_2)^2\cdot\left|(u_*+\psi_1)^3-(u_*+\psi_2)^3\right|\right]\\
\leq&C\left[(u_*+\rho\cdot v_*)^3\cdot v_*^2\cdot\|\varphi_1-\varphi_2\|_*+(u_*+\rho\cdot v_*)^2\cdot v_*^3\cdot\|\psi_1-\psi_2\|_*\right],
\end{aligned}
$$

it yields that

$$
\|N_6\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}\leq Ck^{1-\frac{3}{q}}\ln^{-2}k\cdot\left(\|\varphi_1-\varphi_2\|_*+\ln^{-1}k\cdot\|\psi_1-\psi_2\|_*\right).
$$

For the term $N_4\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)$, we use the mean value theorem again,

$$
\begin{aligned}
&|N_4\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)|\\
\leq&|v_*+(\|\varphi_1\|_*+\|\varphi_2\|_*)v_*|^4\cdot v_*\|\varphi_1-\varphi_2\|_*\\
\leq&Cv_*^5\cdot\|\varphi_1-\varphi_2\|_*,
\end{aligned}
$$

and hence

$$
\|N_4\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}\leq Ck^{1-\frac{3}{q}}\ln^{-5}k\cdot\|\varphi_1-\varphi_2\|_*.
$$

The estimate of $N_3\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)$ is similar to that of $N_6\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)$, we have

$$
\begin{aligned}
&|N_3\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)|\\
\leq&\left[(u_*+\psi_1)^2\cdot\left|(v_*+\varphi_1)^3-(v_*+\varphi_2)^3\right|+|v_*+\varphi_2|^3\cdot\left|(u_*+\psi_1)^2-(u_*+\psi_2)^2\right|\right]\\
\leq&C\left[(u_*+\rho\cdot v_*)^2\cdot v_*^3\cdot\|\varphi_1-\varphi_2\|_*+(u_*+\rho\cdot v_*)\cdot v_*^4\cdot\|\psi_1-\psi_2\|_*\right]
\end{aligned}
$$

and

$$
\|N_3\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}\leq Ck^{1-\frac{3}{q}}\ln^{-3}k\left(\|\varphi_1-\varphi_2\|_*+\ln^{-1}k\|\psi_1-\psi_2\|_*\right).
$$

For the last two terms $N_2\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)$ and $N_5\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)$, the calculus is a little bit easier, we have

$$
\begin{aligned}
&|N_2\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)|\\
\leq&C\left[u_*^4v_*\cdot\|\psi_1-\psi_2\|_*+u_*^2v_*^3\cdot\|\varphi_1-\varphi_2\|_*+u_*v_*^4\cdot\|\psi_1-\psi_2\|_*\right]
\end{aligned}
$$

and

$$\|N_2\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**} \le Ck^{1-\frac{3}{q}}\ln^{-1}k(\|\psi_1-\psi_2\|_*+\ln^{-2}k\|\varphi_1-\varphi_2\|_*);$$

$$\begin{aligned}&|N_5\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)|\\ \le& C\left[v_*^5\|\varphi_1-\varphi_2\|_*+u_*^3v_*^2\|\varphi_1-\varphi_2\|_*+u_*^2v_*^3\|\psi_1-\psi_2\|_*\right],\end{aligned}$$

and

$$\|N_5\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**} \le Ck^{1-\frac{3}{q}}\ln^{-2}k(\ln^{-1}k\|\psi_1-\psi_2\|_*+\|\varphi_1-\varphi_2\|_*).$$

Combining all the six estimates together, we have

$$\begin{aligned}&\|\mathcal{M}(\psi_1,\varphi_1)-\mathcal{M}(\psi_2,\varphi_2)\|_*\\ =&\|T\left\{(N_1+N_2+N_3,\quad N_4+N_5+N_6)\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\right\}\|_*\\ \le& C\Bigg[\|N_1\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}+\|N_2\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}+\|N_3\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}\\ &\quad+\|N_4\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}+\|N_5\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}+\|N_6\left((\psi_1,\varphi_1),(\psi_2,\varphi_2)\right)\|_{**}\Bigg]\\ \le& Ck^{1-\frac{3}{q}}\ln^{-1}k\cdot\left[\|\psi_1-\psi_2\|_*+\ln^{-1}k\cdot\|\varphi_1-\varphi_2\|_*\right]\\ \le& Ck^{1-\frac{3}{q}}\ln^{-1}k\cdot\|(\psi_1-\psi_2,\varphi_1-\varphi_2)\|_*.\end{aligned}$$

Therefore, we can find $k_0\in\mathbb{N}$ large enough such that, for any $k\ge k_0$, $Ck^{1-\frac{3}{q}}\ln^{-1}k<1$, which verifies that the operator $\mathcal{M}$ is indeed a contraction map, hence the unique existence has been proved by the Banach fixed point theorem. □

5 The case of $N=4$

In this section, we consider the case of $N=4$. The idea of the proof in this case is similar to the case of $N=3$. Hence, it is sufficient to give the sketch of the proof that mainly shows the difference between them. Similar as in the case of $N=3$, we reduce the problem to finding a solution of the linear operator $L'=(L'_0,L'_{00}):\mathcal{D}^{1,2}(\mathbb{R}^4)\times\mathcal{D}^{1,2}(\mathbb{R}^4)\to\mathcal{D}^{1,2}(\mathbb{R}^4)\times\mathcal{D}^{1,2}(\mathbb{R}^4)$. Then the system (S) gets rewritten in terms of this linear operator $L'=(L'_0,L'_{00})$ as:

$$\begin{cases} L'_0(\psi_k,\varphi_k) &= -\left[N'_{3,0}(\psi_k,\varphi_k)+N'_{1,2}(\psi_k,\varphi_k)\right] \quad \text{in } \mathbb{R}^4\\ L'_{00}(\psi_k,\varphi_k) &= -\left[E'+N'_{0,3}(\psi_k,\varphi_k)+N'_{2,1}(\psi_k,\varphi_k)\right] \quad \text{in } \mathbb{R}^4,\end{cases} \qquad (LS')$$

where L'_0,L'_{00} are defined as for any $(\psi,\varphi)\in\mathcal{D}^{1,2}(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$:

$$\begin{cases} L'_0(\psi,\varphi) &= \Delta\psi+3u_*^2\psi-2u_*v_*\varphi-v_*^2\psi \quad \text{in } \mathbb{R}^4\\ L'_{00}(\psi,\varphi) &= \Delta\varphi+3v_*^2\varphi-2u_*v_*\psi-u_*^2\varphi \quad \text{in } \mathbb{R}^4,\end{cases} \qquad (L')$$

where $E', N'_{0,3}, N'_{2,1}, N'_{3,0}, N'_{1,2}$ are defined as:

$$E' = v_*^3 - \sum_{i=1}^{k} u_{\varepsilon,x_i}^3;$$

$$N'_{0,3}(\psi,\varphi) = (v_* + \varphi)^3 - v_*^3 - 3v_*^2\varphi;$$

$$N'_{2,1}(\psi,\varphi) = -(u_* + \psi)^2(v_* + \varphi) + 2u_*v_*\psi + u_*^2\varphi;$$

$$N'_{3,0}(\psi,\varphi) = (u_* + \psi)^3 - 3u_*^2\psi - u_*^3;$$

$$N'_{1,2}(\psi,\varphi) = -(u_* + \psi)(v_* + \varphi)^2 + 2u_*v_*\varphi + v_*^2\psi.$$

Let T'_i be one of the following three invariance maps, $i = 1,2,3$:
(Rotation Invariance Map): for $\eta : \mathbb{R}^4 \to \mathbb{R},\ (\bar{y}, y') \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$T'_1(\eta)(\bar{y}, y') = \eta\left(e^{\frac{2j\pi}{k}\sqrt{-1}}\bar{y}, y'\right), \quad j = 1,2,3,\cdots,k-1. \qquad (C'1)$$

(Reflection Invariance Map): for $\eta : \mathbb{R}^4 \to \mathbb{R},\ (y_1,y_2,y_3,y_4) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$,

$$T'_2(\eta)(y_1,y_2,y_3,y_4) = \eta(y_1,-y_2,y_3,y_4) = \eta(y_1,y_2,-y_3,y_4) = \eta(y_1,y_2,y_3,-y_4). \qquad (C'2)$$

(Kelvin Invariance Map): for $\eta : \mathbb{R}^4 \to \mathbb{R},\ y \in \mathbb{R}^4$,

$$T'_3(\eta)(y) = |y|^{-6}\eta\left(\frac{y}{|y|^2}\right). \qquad (C'3).$$

Define the symmetric space

$$H'_s := \left\{(g_1,g_2) \in \left[\mathcal{D}^{1,2}(\mathbb{R}^4)\right]^2 \middle| \forall(\bar{x},x') \in \mathbb{R}^2\times\mathbb{R}^2, T_j(g_i)(\bar{x},x') = g_i(\bar{x},x'), i=1,2;\ \ j=1,2,3\right\}.$$

In the following it is sufficient to prove the invertibility problem of the linear operator $L' = (L'_0, L'_{00})$, namely, to find the solution to the problem:

$$L'(\psi,\varphi) = h, \qquad (LN'),$$

for $h = (h_1,h_2) \in H'_s$.

In this case, we note that the set of bounded solutions of the homogeneous system $L'(\psi,\varphi) = 0$ which is spanned by $10k$ functions $(Z'_p, Z'_{j,s})$, where $p = 1,2,\cdots,5$, $j = 1,2,\cdots,k$; $s = 1,2$. More precisely, we have: $Z'_l = \partial_{y_l}U_{1,0}(y), l = 1,2,3,4, y = (y_1,y_2,y_3,y_4) \in \mathbb{R}^4$, $Z'_5(y) = y\cdot\nabla U_{1,0}(y) + U_{1,0}(y), y \in \mathbb{R}^4$; $Z'_{j,1}(y) = \partial_r U_{\varepsilon_k,x_j}$, $Z'_{j,2}(y) = \partial_\varepsilon U_{\varepsilon_k,x_j}$, where $r = |x_j| = \sqrt{1-\varepsilon_k^2}$, $j = 1,2,\cdots,k$,$\varepsilon_k \sim k^{-3}$. To simplify the notations, In the following, we will use ε to denote ε_k.

Proposition 5.1 *Let $h = (h_1,h_2)$ be a vector function such that $\|h\|_{**} = \|h_1\|_{**} + \|h_2\|_{**} < \infty$, and h satisfy the orthogonal condition (H) :*

$$\begin{cases} \int_{\mathbb{R}^3} Z'_p(y)h_1(y)dy &= 0, \quad p = 1,2,3,4,5 \\ \int_{\mathbb{R}^3} Z'_{js}(y)h_2(y)dy &= 0, \quad j = 1,2,\cdots,k; s = 1,2, \end{cases} \qquad (H')$$

then the linear problem (LN') *has the unique solution* $(\psi, \varphi) = T'(h_1, h_2)$ *with* $\|(\psi, \varphi)\|_* := \|\psi\|_* + \|\varphi\|_* < \infty$, *moreover it holds that*

$$\begin{cases} \int_{\mathbb{R}^4} U_{1,0}^4(y) Z_p'(y) \psi dy &= 0, \quad p = 1, 2, 3, 4, 5 \\ \int_{\mathbb{R}^4} u_{\varepsilon, x_r}^4(y) Z_{js}'(y) \varphi(y) dy &= 0, \quad j = 1, 2, \cdots, k; s = 1, 2. \end{cases} \qquad (PHSI')$$

Remark 5.2 *By Proposition 5.1, it is sufficient to show the construction of the compact map* $\tau' = (\tau_1', \tau_2') : H \to L^{\frac{4}{3}}(\mathbb{R}^4) \times L^{\frac{4}{3}}(\mathbb{R}^4)$, *which is defined by:*

$$\tau_1'(\psi, \varphi) = -3u_*^2\psi + v_*^2\psi + 2u_*v_*\varphi, \quad \tau_2'(\psi, \varphi) = -3v_*^2\varphi + u_*^2\varphi + 2u_*v_*\psi.$$

Proof: Indeed, the operator τ' is compact due to the fact that $u_*^{\nu_1}(y) \cdot v_*^{\nu_2}(y) \sim \frac{1}{(1+|y|)^6}$, for $0 \le \nu_1, \nu_2 \le 3, \nu_1 + \nu_2 = 3$, and $|y| \ge 2$. Furthermore, by the Hölder inequality, we have

$$\|u_*^{\nu_1} \cdot v_*^{\nu_2} \cdot \varphi\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} \le C\|u_*^{\nu_1} v_*^{\nu_2}\|_{L^2(\mathbb{R}^4)} \cdot \|\varphi\|_{L^{2^*}(\mathbb{R}^4)} \le C\|\varphi\|_{\mathcal{D}},$$

and

$$\|u_*^{\nu_1} \cdot v_*^{\nu_2} \cdot \psi\|_{L^{\frac{4}{3}}(\mathbb{R}^4)} \le C\|u_*^{\nu_1} v_*^{\nu_2}\|_{L^2(\mathbb{R}^4)} \cdot \|\psi\|_{L^{2^*}(\mathbb{R}^4)} \le C\|\psi\|_{\mathcal{D}}.$$

Indeed, for fixed k, let us consider the subspace

$$H' = \{(\psi, \varphi) \in H_s' | (\psi, \varphi) \text{ satisfies } (PHSI')\},$$

similar to the case of H, the subspace H' can be given induced inner product which makes it Hilbert space.

For $y \in EXT \cap B_0^c(2)$, then $|y| > 2$ and $|y - x_i| > \eta/k$, we can obtain that $|y - x_i| \sim 1 + |y|$, since $|x_i| \sim 1, i = 1, 2, \cdots, k$, and $\varepsilon \sim k^{-3}$, we can get direct result as $|v_*(y)| \le C\frac{k\varepsilon}{1+|y|^2} \le \frac{Ck^{-2}}{1+|y|^2}$.

For $y \in EXT \cap B_0(2)$, we have two cases:

(1) $\exists$ some $i_0 \in \{1, 2, \cdots, k\}$, such that y is closest to this point i_0, but relatively far from all the other $x_j's(j \ne i_0)$, namely, $|y - x_j| \ge \frac{1}{2}|x_j - x_{i_0}| \sim \frac{|j-i_0|}{k}$, for all $j \ne i_0$, then

$$v_*(y) \le C\left(\sum_{j \ne i_0} \frac{k^2\varepsilon}{|j - i_0|^2} + \frac{k^2\varepsilon}{\eta^2}\right) \le C\varepsilon k^2 \le Ck^{-1};$$

(2) y is far from all $x_i s, i = 1, 2, \cdots, k$, such that $\exists$ some fixed constant $c_0 > 0$, $|y - x_i| \ge c_0, 1 \le i \le k$, and then $v_*(y) \le Ck\varepsilon \le Ck^{-2}$.

In conclusion, for $y \in EXT \cap B_0(2)$, $v_*(y)$ is uniformly estimated by k^{-1}.

Now we turn to the interior region $INT = \bigcup_{j=1}^k B_{x_j}(\frac{\eta}{k})$, which requires a scaling and transition transform of variables $y \mapsto w = \frac{y - x_j}{\varepsilon}, j = 1, 2, \cdots, k$. If y lies in one branch ball $B_{x_j}(\frac{\eta}{k})$, observe the fact that $\left|\frac{x_j - x_i}{\varepsilon}\right| \sim \frac{|j-i|}{k\varepsilon}(j \ne i)$, we have $|w| < \frac{\eta}{k\varepsilon}$. Hence

$$v_*(y) = v_*(\varepsilon w + x_j) = \frac{\varepsilon^{-1}}{1 + |w|^2} + \sum_{i \ne j} \frac{\varepsilon^{-1}}{1 + |w + \frac{x_j - x_i}{\varepsilon}|^2} \le \frac{Ck\varepsilon^{-1}}{1 + |w|^2}.$$

On the other hand, the L^2 estimate of the term $u_*^{\nu_1} v_*^{\nu_2}(y)$ can be given as follows:

$$\begin{aligned}
&\|u_*^{\nu_1} v_*^{\nu_2}(y)\|_{L^2} \\
\leq &\|u_*^{\nu_1} v_*^{\nu_2}(y)\|_{L^2(EXT)} + \sum_{j=1}^{k} \|u_*^{\nu_1} v_*^{\nu_2}(y)\|_{L^2(B_{x_j}(\eta/k))} \\
\leq &C\left[\int_{EXT\cap B_0(2)} + \int_{EXT\setminus B_0(2)} \frac{1}{(1+|y|^2)^{2\nu_1}} \cdot (v_*(y))^{2\nu_2}\, dy\right]^{\frac{1}{2}} \\
&+ C\sum_{j=1}^{k}\left[\int_{B_{x_j}(\frac{\eta}{k})} \frac{1}{(1+|y|^2)^{2\nu_1}} \cdot (v_*(y))^{2\nu_2}\, dy\right]^{\frac{1}{2}} \\
\leq &C\left[\int_{B_0(2)} \frac{1}{(1+|y|^2)^{2\nu_1}} dy \cdot k^{-2\nu_2} + \int_{B_0^c(2)} \frac{1}{(1+|y|^2)^{2\nu_1}} \cdot \left(\frac{k\varepsilon}{1+|y|^2}\right)^{2\nu_2} dy\right]^{\frac{1}{2}} \\
&+ C\sum_{j=1}^{k}\left[\int_{B_0(\frac{\eta}{k\varepsilon})} (v_*(\varepsilon w + x_j))^{2\nu_2} dw \cdot \varepsilon^4\right]^{\frac{1}{2}} \\
\leq &C\left(k^{-\nu_2} + (k\varepsilon)^{\nu_2} + k^{3\nu_2 - 1}\varepsilon^{\nu_2}\right) \leq C(k^{-1} + k^{-\nu_2}).
\end{aligned}$$

Therefore, the bound of the operator τ' is given by

$$\|\tau'(\psi, \varphi)\|_{L^{\frac{4}{3}}(\mathbb{R}^4)\times L^{\frac{4}{3}}(\mathbb{R}^4)} \leq C\left(k^{-1} + k^{-\nu_2}\right)\|(\psi,\varphi)\|_{H'}.$$

□

Proposition 5.3 *Under the same assumption of Proposition 2.1, we can find a large $k_0 \in \mathbb{N}$, and a constant C independent of k, such that for any $k \geq k_0$, the solution (ψ_k, φ_k) to the linear problem (Lin') is equivalent to the equation $(\psi_k, \varphi_k) = T'(h_k)$ with the data $h_k = (h_{1k}, h_{2k})$, we have the estimate*

$$\|T'(h_k)\|_* = \|(\psi_k, \varphi_k)\|_* = \|\psi_k\|_* + \|\varphi_k\|_* \leq C\left(\|h_{1k}\|_{**} + \|h_{2k}\|_{**}\right) = C\|h_k\|_{**},$$

which shows that T' is a bounded linear operator.

Remark 5.4 *It is worthy to remind that the Kelvin transform here is of 4 dimensional, so the explicit meanings of the Kelvin-type notations $\wedge$, $\sim$ of solutions and data terms are different from the 3 dimensional version in Proposition 2.2.*

We define the Kelvin transform of the solution as

$$\widehat{\psi}(y) = |y|^{-2}\psi(\frac{y}{|y|^2}), \quad \widehat{\varphi}(y) = |y|^{-2}\varphi(\frac{y}{|y|^2});$$

respectively, and the Kelvin transform of the error term reads

$$\widetilde{h_i}(y) = |y|^{-6} h_i(\frac{y}{|y|^2}), \quad i = 1, 2.$$

In the following, we need only to check the orthogonality restriction $(PHSI')$ so that the problem admits weak solutions. Under the following conditions, we can get the existence result of our problem which is essentially similar to Proposition 2.3.

Proposition 5.5 *Assume that the data term h_{1k}, h_{2k} satisfies all the conditions $(C'1), (C'2), (C'3)$, then for any $k \geq k_0$, the problem (Lin') admits a unique weak solution $(\psi_k, \varphi_k) = T'(h_{1k}, h_{2k})$ with $\|(\psi_k, \varphi_k)\|_* := \|\psi_k\|_* + \|\varphi_k\|_* < \infty$, and satisfies :*

$$\begin{cases} \int_{\mathbb{R}^4} U_{1,0}^4(y) Z_p'(y) \psi_k(y) dy &= 0, \quad p = 1, 2, 3, 4, 5; \\ \int_{\mathbb{R}^4} u_{\varepsilon,x_j}^4(y) Z_{js}'(y) \varphi_k(y) dy &= 0, \quad j = 1, 2, \cdots, k; s = 1, 2. \end{cases} \qquad (PHSI')$$

We point out that, all the propositions above stem from the linearized version of (LS') and the condition that the data terms are given in advance, such that they are independent of (ψ_k, φ_k). If we look back to (LS'), we can not avoid the appearance of (ψ, φ) in nonlinear data terms $E', N'_{3,0}, N'_{2,1}, N'_{1,2}, N'_{0,3}$. So we need to estimate them in terms of the norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ to induce the contraction map, which is what we should do in the rest of this paper.

Proposition 5.6

$$\|E'\|_{**} \leq Ck^{1-\frac{4}{q}}; \quad \|N'_{3,0}\|_{**} \leq Ck^{1-\frac{4}{q}} \|\psi_k\|_*^2;$$
$$\|N'_{2,1}\|_{**} \leq Ck^{1-\frac{4}{q}}; \quad \|N'_{1,2}\|_{**} \leq Ck^{1-\frac{4}{q}}; \quad \|N'_{0,3}\|_{**} \leq Ck^{1-\frac{4}{q}} \|\phi_k\|_*^2.$$

Proof. Since we can decompose E' into the sum of $(k^3 - k)$ terms as the following,

$$E' = \left(\sum_{i=1}^k u_{\varepsilon,x_i} \right)^3 - \sum_{i=1}^k u_{\varepsilon,x_i}^3 = \sum_{\substack{i_1+i_2+\cdots+i_k=3 \\ i_1,i_2,\cdots,i_k \in \mathbb{N} \\ i_1,i_2,\cdots,i_k \neq 3}} \prod_{l=1}^k u_{\varepsilon,x_l}^{i_l}.$$

Without loss of generality, it is sufficient to estimate only one term among them. We choose the term $u_{\varepsilon,x_1}(y) \cdot u_{\varepsilon,x_2}^2(y)$ as an example.

For $y \in EXT$, we have that

$$\begin{aligned} |u_{\varepsilon,x_1}(y) \cdot u_{\varepsilon,x_2}^2(y)| \leq & \frac{C\varepsilon}{\varepsilon^2 + |y - x_1|^2} \cdot \left(\frac{\varepsilon}{\varepsilon^2 + |y - x_2|^2} \right)^2 \\ \leq & \frac{C\varepsilon}{\varepsilon^2 + \left(\frac{\eta}{k}\right)^2} \cdot \frac{\varepsilon^{-2}}{1 + \left|\frac{y}{\varepsilon} - \frac{x_2}{\varepsilon}\right|^4} \leq \frac{Ck^2\varepsilon^{-1}}{1 + \left|\frac{y}{\varepsilon} - \frac{x_2}{\varepsilon}\right|^4}. \end{aligned}$$

Then, with the scaling transform $y \mapsto w = \frac{y - x_2}{\varepsilon}$, the exterior estimate of $u_{\varepsilon,x_1}(y) \cdot u_{\varepsilon,x_2}^2(y)$ writes:

$$\begin{aligned} & \|u_{\varepsilon,x_1}(y) \cdot u_{\varepsilon,x_2}^2(y)\|_{**(EXT)} \\ \leq & Ck^2\varepsilon^{-1} \cdot \left[\int_{\mathbb{R}^4} \frac{(1 + |\varepsilon w + x_2|)^{5q-8}}{1 + |\omega|^{4q}} d(\varepsilon w + x_2) \right]^{\frac{1}{q}} \\ \leq & Ck^2\varepsilon^{\frac{4}{q}-1} \cdot \left[\int_0^{\frac{1}{\varepsilon}} \frac{r^3 dr}{1 + r^{4q}} + \int_{\frac{1}{\varepsilon}}^{+\infty} \frac{\varepsilon^{5q-8} r^{5q-5} dr}{1 + r^{4q}} \right]^{\frac{1}{q}} \\ \leq & Ck^2\varepsilon^{\frac{4}{q}-1} \cdot \varepsilon^{4-\frac{4}{q}} \leq Ck^{-7}. \end{aligned}$$

For $y \in INT$, $\exists j \in \{1, 2, \cdots, k\}$ such that $|y - x_j| < \eta/k$, just as what we have done in Section 3, change the variables by $w = \frac{y-x_1}{\varepsilon}$, since $x_1 - x_2 \sim \frac{2\pi}{k}$, and for $|w| \le \frac{\eta}{k\varepsilon}$, where $0 < \eta << 1$, then the term $\left|\frac{x_1-x_2}{\varepsilon}\right|$ dominates $|w|$, hence

$$
\begin{aligned}
&\|u_{\varepsilon,x_1} \cdot u_{\varepsilon,x_2}^2\|_{**(B_{x_1}(\eta/k))} \\
\le& C\left\{\int_{B_{x_1}(\eta/k)} (1+|y|)^{5q-8}\, u_{\varepsilon,x_1}^q(y) \cdot u_{\varepsilon,x_2}^{2q}(y)dy\right\}^{\frac{1}{q}} \\
\le& C\varepsilon^{\frac{4}{q}}\left[\int_{B_0(\frac{\eta}{k\varepsilon})} \left|u_{\varepsilon,x_1}(x_1+\varepsilon w) \cdot u_{\varepsilon,x_2}^2(x_1+\varepsilon w)\right|^q dw\right]^{\frac{1}{q}} \\
\le& C\varepsilon^{\frac{4}{q}-3}\left[\int_{B_0(\frac{\eta}{k\varepsilon})} \left|\frac{1}{1+|w|^2} \cdot \frac{1}{1+\left|\frac{x_1-x_2}{\varepsilon}+w\right|^4}\right|^q dw\right]^{\frac{1}{q}} \\
\le& C\varepsilon^{\frac{4}{q}-3}\left[\int_{B_0(\frac{\eta}{k\varepsilon})} \left|\frac{k^4\varepsilon^4}{1+|w|^2}\right|^q dw\right]^{\frac{1}{q}} \\
\le& Ck^4\varepsilon^{\frac{4}{q}+1} \cdot \left[\int_{B_0(\frac{\eta}{k\varepsilon})} \frac{dw}{1+|w|^{2q}}\right]^{\frac{1}{q}} \\
\le& Ck^{6-\frac{4}{q}} \cdot \varepsilon^3 \le Ck^{-3-\frac{4}{q}}.
\end{aligned}
$$

For the estimate on the whole region INT, we sum all k concentration balls together and obtain that, $\|u_{\varepsilon,x_1} \cdot u_{\varepsilon,x_2}^4\|_{**(INT)} \le Ck^{-2-\frac{4}{q}}$,

Therefore, the energy term E' can be estimated by the following:

$$
\begin{aligned}
\|E'\|_{**} \le& \|E'\|_{**(EXT)} + \sum_{j=1}^{k} \|E'\|_{**B_{x_j}(\eta/k)} \\
\le& Ck^3 \cdot \left(\|u_{\varepsilon,x_1}(y) \cdot u_{\varepsilon,x_2}^2(y)\|_{**(EXT)} + \sum_{j=1}^{k} \|u_{\varepsilon,x_1}(y) \cdot u_{\varepsilon,x_2}^2(y)\|_{**B_{x_j}(\eta/k)}\right) \\
\le& Ck^3\left(k^{-7} + k^{-2-\frac{4}{q}}\right) \le Ck^{1-\frac{4}{q}}.
\end{aligned}
$$

Since the calculus of nonlinear data terms $N'_{3,0}, N'_{2,1}, N'_{1,2}, N'_{0,3}$ are so similar that it is reasonable to give the detail of $N'(3,0)$ explicitly and show the other results briefly.

For $N'(3,0)$, recall the important Lemma 3.2, by the assumption $\|\psi_k\|_*, \|\varphi_k\|_* << 1$, we can give the coarse estimate as follows:

$$
\begin{aligned}
&|N'_{3,0}(\psi_k, \varphi_k)| = \left|(u_* + \psi_k)^3 - 3u_*^2\psi_k - u_*^3\right| \\
=& u_*^3 \cdot |(1 + \frac{\psi_k}{u_*})^3 - 3\frac{\psi_k}{u_*} - 1| \le C(u_*|\psi_k|^2 + |\psi_k|^3) \le C(u_* v_*^2 + v_*^3)\|\psi_k\|_*^2
\end{aligned}
$$

and the norm of $N'_{3,0}$ can be calculated as:

$$\begin{aligned}
&\|N'_{3,0}(\psi_k,\varphi_k)\|_{**}\\
\leq& C\left[\int_{EXT\cap B_0(2)}+\int_{EXT\setminus B_0(2)}(1+|y|)^{5q-8}(u_*^q(y)\cdot v_*^{2q}(y)+v_*^{3q}(y))dy\right]^{\frac{1}{q}}\cdot\|\psi_k\|_*^2\\
&+C\sum_{j=1}^{k}\varepsilon^{\frac{4}{q}}\left[\int_{B_0(\eta/k\varepsilon)}(1+|\varepsilon w+x_j|)^{3q-8}v_*^{2q}(\varepsilon w+x_j)+(1+|\varepsilon w+x_j|)^{5q-8}v_*^{3q}(\varepsilon w+x_j)dw\right]^{\frac{1}{q}}\cdot\|\psi_k\|_*^2\\
\leq& C\left[\int_{B_0(2)}(1+|y|)^{5q-8}(k^{-2q}+k^{-3q})dy\right]^{\frac{1}{q}}\cdot\|\psi_k\|_*^2\\
&+C\left[\int_{\mathbb{R}^4}(1+|y|)^{5q-8}(\frac{k^{-4q}}{(1+|y|^2)^{3q}}+\frac{k^{-6q}}{(1+|y|^2)^{3q}})dy\right]^{\frac{1}{q}}\cdot\|\psi_k\|_*^2\\
&+Ck\varepsilon^{\frac{4}{q}}\left[\int_{B_0(\frac{\eta}{k\varepsilon})}\left(\frac{k\varepsilon^{-1}}{1+|w|^2}\right)^{2q}+\left(\frac{k\varepsilon^{-1}}{1+|w|^2}\right)^{3q}dw\right]^{\frac{1}{q}}\cdot\|\psi_k\|_*^2\\
\leq& Ck^{1-\frac{4}{q}}\|\psi_k\|_*^2
\end{aligned}$$

We finish our proof by the following remark.

Remark 5.7 *Now problem (LS') can be reduced into a fixed point form if we define another new operator $\mathcal{M}'$, namely, the fixed point result we desire is the following:*

$$(\psi_k,\varphi_k)=T'\left(h_{1k}(\psi_k,\varphi_k),h_{2k}(\psi_k,\varphi_k)\right):=\mathcal{M}'(\psi_k,\varphi_k),$$

where the operator $\mathcal{M}':X'\to X'$, the Banach space X' is defined as a small ball in a product space as following:

$$X':=\{(\psi,\varphi)\in C(\mathbb{R}^4)\times C(\mathbb{R}^4)\Big|\ \ \|\psi\|_*+\|\varphi\|_*\leq\rho\},$$

where ρ is a small positive number, and T' is the bounded linear operator defined in Proposition 5.3.

From the result of Proposition 5.6, and the bound of T' for $(\psi_k,\varphi_k)\in X'$, that $\mathcal{M}'$ maps X' to itself is direct. The proof of the contraction mapping $\mathcal{M}'$ is standard, and the difference from the proof of $\mathcal{M}$ is only technical, but not essential.

References

[1] N. Akhmedier, A. Ankiewicz, Partially coherent solutions on a finite background, *Phys. Rev. Lett.* 82, 1661(1999).

[2] A. Ambrosetti, E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, *C. R. Acad. Sci. Paris, Ser.* 1342 (2006), 453-458.

[3] A. Ambrosetti, E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, *J. London Math. Soc.* 75(2007), 67-82.

[4] A. Ambrosetti, A. Malchiodi and S. Secchi, Multiplicity results for some nonlinear Schrödinger equations with poentials, *Arch. Ration. Mech. Anal.* 159(2001), 253-271.

[5] T. Bartsch, Z.Q. Wang and J. Wei, Bound states for a coupled Schrödinger system, *J. Fixed Point Theory Appl.* 2(2007), 67-82.

[6] K. Chow, Periodic solutions for a system of four coupled nonlinear Schrödinger equations, *Phys. Rev. Lett.* A 285(2001), 319-326.

[7] W. Ding, On a conformally invariant elliptic equation on $\mathbb{R}^n$, *Communications on Mathematical Physics* 107(1986), 331-335.

[8] Y. Guo, J. Liu, Liouville type theorem for positive solutions of elliptic systems in $\mathbb{R}^N$, *Comm. Partial Differential Equations* 33(2008), 263-284.

[9] F. Hioe, Solitary waves for N coupled nonlinear Schrödinger equations, *Phys. Rev. Lett.* 82(1999), 1152-1155.

[10] F. Hioe and T. Salter, Special set and solution of coupled nonlinear Schrödinger equations, *J. Phys. A: Math. Gen.* 35(2002), 8913-8928.

[11] E. Hebey and M. Vaugon, Existence and multiplicity of nodal solutions for nonlinear elliptic equations with critical Sobolev growth, *J. Funct. Anal.* 119(1994), no.2, 298-318.

[12] T. Kanna and M. Lakshmannan, Exact soliton solutions, shape changing collisions and partially coherent solitions in coupled nonlinear Schrödinger equations, *Phys. Rev. Lett.* 86(2002), 5243.

[13] T. Lin, J. Wei, Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^N, N \leq 3$, *Comm. Math. Phys.* 255(2005), 629-753.

[14] Z. Liu, Z.Q. Wang , Multiple ground states of nonlinear Schrödinger system, *Comm. Math. Phys.* 282(2008), 721-731.

[15] M. Pino, M. Musso, F. Pacard and A. Pistoia, Large energy entire solutions for the Yamabe equation, *J. Diff. Eqns*, to appear.

[16] M. Pino, M. Musso, F. Pacard and A. Pistoia, Torus action on S^n and sign changing solutions for conformally invariant equations, *Annali della Scuola Normale Superiore*, to appear.

[17] M. Del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super critical Bahri-Coron's problem, *Calc. Var. Partial Differential Equations* 16 (2003) 113-145.

[18] P. Quittner and F. Souplet, Optimal Liouville-type theorems for noncooperative elliptic Schrodinger systems and applications, *Comm. Math. Phys.* to appear.

[19] O. Rey and J. Wei, Blowing up solutions for an elliptic Neumann problem with sub-or supercritical nonlinearity, I., $N = 3$, *J. Funct. Anal.* 212(2004), 472-499.

[20] O. Rey and J. Wei, Blowing up solutions for an elliptic Neumann problem with sub-or supercritical nonlinearity, II., $N \geq 4$, *Ann. Inst. H. Poincaré Anal. Non. Linéaire* 22(2005), 459-484.

[21] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger system, *Comm. Math. Phys.* 271 (2007), 119-221.

[22] H. Tavares, S. Terracini, G. Verzini and T. Weth, Existence and nonexistence of entire solutions for non-cooperative cubic elliptic system, *Comm. PDE* to appear.

[23] J. Wei, S. Yan, Infinitely many solutions for the prescribed scalar curvature problem on S^N, *Journal of Functional Analysis* 258 (2010), 3048-2081.

[24] F.Gladiali and M. Grossi, Supercritical elliptic problem with nonautonomous nonlinearities, to appear.

[25] W.Chen and C.Li, Classification of solutions of some nonlinear elliptic solutions, *Duke Math.J.*63(3) (1991),615-622.