Large energy entire solutions for the Yamabe type problem of polyharmonic operator

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Abstract: In this paper, we consider the following Yamabe type problem of polyharmonic operator :

$$\begin{cases} D_m u &= |u|^{\frac{4m}{N-2m}} u \quad \text{on } \mathbb{S}^N \\ u &\in H^m(\mathbb{S}^N), \end{cases} \tag{P}$$

where $N \geq 2m+1, m \in \mathbb{N}_+$, $\mathbb{S}^N$ is the unit sphere with the induced Riemannian metric $g = g_{\mathbb{S}^N}$, and D_m is the elliptic differential operator of $2m$ order given by

$$D_m = \prod_{k=1}^{m} (-\Delta_g + \frac{1}{4}(N-2k)(N+2k-2))$$

where Δ_g is the Laplace-Beltrami operator on $\mathbb{S}^N$. We will show that the problem (P) has infinitely many non-radial sign changing solutions.

Keywords: Polyharmonic equations, Critical exponents, Infinitely many non-radial solutions.

AMS Subject Classification: 35B45, 35J25.

1 Introduction

We consider the following Yamabe type problem for polyharmonic operator:

$$\begin{cases} D_m u &= |u|^{m^*-2} u \quad \text{on } \mathbb{S}^N \\ u &\in H^m(\mathbb{S}^N), \end{cases} \tag{P}$$

where $m^* = \frac{2N}{N-2m}, N \geq 2m+1, m \in \mathbb{N}_+$, $\mathbb{S}^N$ is the unit sphere with the induced Riemannian metric $g = g_{\mathbb{S}^N}$, and D_m is the elliptic differential operator of $2m$ order given by

$$D_m = \prod_{k=1}^{m} (-\Delta_g + \frac{1}{4}(N-2k)(N+2k-2))$$

where Δ_g is the Laplace-Beltrami operator on $\mathbb{S}^N$(see [7]).

The well known Yamabe problem, which stems from the conformal geometry, is the problem of finding some scalar curvature K in a compact Riemann manifold (M, g_0) of dimension $n \geq 2$. More precisely, for a given smooth function K defined on this manifold, we want to find a new metric g which is conformal to the original metric g_0 such that K is actually the scalar curvature under this new g. In the case of $m = 1$, the Yamabe problem read as:

$$\begin{cases} -\Delta_{\mathbb{S}^N} u + \frac{N(N-2)}{2} u - u^{\frac{N+2}{N-2}} = 0 \quad \text{on } \mathbb{S}^N \\ u > 0. \end{cases} \tag{1.1}$$

see ([3], [4], [31], [10]).

By using the stereo-graphic projection, the problem (1.1) can be reduced to:

$$\begin{cases} -\Delta u &= u^{\frac{N+2}{N-2}} & \text{in } \mathbb{R}^N \\ u &> 0 & \text{in } \mathbb{R}^N \\ u &\in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\int_{\mathbb{R}^N} |\nabla u|^2$. It is known that the only finite energy positive solution to (1.2) are given by the family of the functions (see [21]):

$$\mu^{-\frac{N-2}{2}} U(\mu^{-1}(x-\xi)), U(x) = (\frac{2}{1+|x|^2})^{\frac{N-2}{2}}, \xi \in \mathbb{R}^N, \mu > 0. \tag{1.3}$$

Moreover these functions are corresponding to the extremals for the critical Sobolev embedding (see [29]). And these functions are indeed all positive solutions of (1.2) even without the finite energy requirement (see [9]). It is natural to ask weather there are finite energy non-radial sign changing solutions to (1.2). This was first answered by Ding [10]. His proof is variational: consider the functions of the form

$$u(x) = u(|x_1|, |x_2|), x = (x_1, x_2) \in S^N \subset \mathbb{R}^{N+1} = \mathbb{R}^k \times \mathbb{R}^{N-k}, k \geq 2. \tag{1.4}$$

The critical Sobolev embedding becomes compact and hence infinitely many sign changing solutions exist, thanks to the Ljusternik-Schnirelmann theory. See also [17]. Recently, del Pino, Musso, Pacard and Pistoia [24]-[25] gave another proof of countablely many sign changing non-radial solutions. Their proof is more constructive: they built a sequence of solutions with one negative bump at the origin and large number of positive bumps in a polygon. This gives more precise information on such sign changing solutions.

On the other hand, the polyharmonic operator, in particular the biharmonic operator has found considerable interest due to its geometry roots in recent years. For instance, when $m = 2$, the problem (P) is related to the Paneitz operator, which was introduced by Paneitz [23] for smooth 4 dimensional Riemannian manifolds and was generalized by [8] to smooth N dimensional Riemannian manifolds. We refer the reader to the papers [2], [5], [6], [11], [12], [14], [15],

[16], [26], [27], [29], and the references therein, for various existence and multiplicity results on the polyharmonic operator and related problems. It is evident from these papers that the polyharmonic operator presents new and challenging features compared with the Laplace operator. However, few results are known for the Yamabe problem of polyharmonic operator. The purpose of the present paper is concerned on this topic.

Similar to in the case of $m = 1$, by using the stereo-graphic projection, the problem (P) can be reduced to the following problem in $\mathbb{R}^N$, namely

$$\begin{cases} (-\Delta)^m u & = |u|^{\frac{4m}{N-2m}} u \qquad \text{in } \mathbb{R}^N \\ \qquad u & \in D^{m,2}(\mathbb{R}^N), \end{cases} \tag{1.5}$$

where $N \geq 2m+1, m \in \mathbb{N}_+$, and $\mathcal{D}^{m,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm induced by the scalar product

$$(u,v) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Delta^{\frac{m}{2}} u \cdot \Delta^{\frac{m}{2}} v, & \text{if m is even,} \\ \displaystyle\int_{\mathbb{R}^N} \nabla\Delta^{\frac{m-1}{2}} \nabla\Delta^{\frac{m-1}{2}} v, & \text{if m is odd.} \end{cases} \tag{1.6}$$

Moreover it is known that the only finite energy positive solution to equation (1.5) are given by the family of the functions (see [21],[6]):

$$\mu^{-\frac{N-2m}{2}} U(\mu^{-1}(x-\xi)), U(x) = P_{m,N}^{\frac{N-2m}{4m}} (1+|x|^2)^{-\frac{N-2m}{2}},$$

where $P_{m,N} = \displaystyle\prod_{h=-m}^{m-1} (N+2h)$.

Generalizing the idea of Ding and using variational method, Bartsch and Weth [5] established an unbounded sequence of sign changing finite energy solutions to (1.5).

In this paper, following the idea in [24]-[25], we will construct a sequence of non-radial sign changing solutions for problem (1.5). Our result cover the case of Yamabe equations and the biharmonic equations.

Our main results are:

Theorem 1.1 *Let $m \geq 1$, $N \geq 2m+1$, and write $\mathbb{R}^N = \mathbb{C} \times \mathbb{R}^{N-2}$. Then for each k large enough, the problem (1.5) admits a finite energy solution of the form*

$$u_k(x) = U(x) - \sum_{j=1}^{k} \mu_k^{-\frac{N-2m}{2}} U(\mu_k^{-1}(x-\xi_j)) + o(1),$$

where $\xi_j = \sqrt{1-\mu_k^2}(e^{\frac{2\pi(j-1)}{k}\sqrt{-1}}, 0)$, $j = 1, 2, \cdots, k$, $U(x) = P_{m,N}^{\frac{N-2m}{4m}} (1+|x|^2)^{-\frac{N-2m}{2}}$, $\mu_k = \frac{\delta_k^{\frac{2}{N-2m}}}{k^2}$ for $N \geq 2m+2$, and $\mu_k = \frac{\delta_k^2}{k^3 \log^2 k}$ for $N = 2m+1$, and $o(1) \to 0$ uniformly as $k \to \infty$. δ_k is a positive number which depends on k only.

As a consequence, we have

Theorem 1.2 *Suppose that $N \geq 2m+1$, then problem (P) has infinitely many non-radial sign changing solutions.*

Remark 1.3 *The geometry picture of the sign-changing solution u is that it is positive near the center while negative in the region of the bubbles scattered around the Obata type solution in the middle.*

Remark 1.4 *We believe that similar result should also hold for the following Lane-Emden system*

$$\begin{cases} (-\Delta)^m u = |v|^{\alpha-1}v; \\ (-\Delta)^m v = |u|^{\beta-1}u. \end{cases} in \quad \mathbb{R}^N, \quad m \geq 1. \tag{1.7}$$

It is known that (see [18]), for $N > 2m$, $\alpha, \beta \geq 1$ but not equal to 1 such that

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} > \frac{N-2m}{N},$$

(1.7) *has no any positive solutions. On the other hand, for $N > 2m$, $\alpha, \beta \geq 1$ such that*

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} \leq \frac{N-2m}{N}.$$

(1.7) *admits infinitely positive solutions (see [19]), We conjecture that the following is true:*

Conjecture 1.1 *For $N > 2m$, $\alpha, \beta \geq 1$ and $\frac{1}{\alpha+1} + \frac{1}{\beta+1} = \frac{N-2m}{N}$, problem (1.7) has infinitely many sign changing solutions.*

The paper is organized as follows: Section 2 contains the construction of an approximation solution and the estimates of the error. While the Section 3 will devote to the detailed calculus and further thoughts on the gluing procedures and linearization of the problem. The proof of the theorem will be also given in this section.

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2 Approximation solution and the estimate of the error

In this section, we first construct an approximation solution for our problem (1.5). Then we give the precise estimate of the error.

As we mentioned in the introduction, it is well known that the equation

$$(-\Delta)^m u = u^{\frac{N+2m}{N-2m}} \tag{2.1}$$

has the following radial solution

$$U(x) = P_{m,N}^{\frac{N-2m}{4m}} (1+|x|^2)^{-\frac{N-2m}{2}},$$

with

$$P_{m,N} = \prod_{h=-m}^{m-1} (N + 2h).$$

Moreover this radial solution U is invariant under the Kelvin type transform :

$$\widehat{u}(y) = \left|y\right|^{2m-N} u(\frac{y}{\left|y\right|^2}). \tag{2.2}$$

That is, $\widehat{U}(y) = U(y)$(c.f.[6]).

We begin with

Lemma 2.1 *The equation* (2.1) *is invariant under the Kelvin transform* (2.2)*, namely,*

$$(-\Delta)^m \hat{u}(y) = \left|\widehat{u}(y)\right|^{m^*-2} \widehat{u}(y), \quad \textit{where} \quad m^* = \frac{2N}{N-2m}.$$

Proof. The result is known. For the sake of completeness, we give the proof. We first prove the case of $m = 2$. To simplify our proof, we make use of the spherical coordinates, then

$$\Delta u(r,\theta) = (\partial_r^2 + \frac{N-1}{r}\partial_r + \frac{\Delta_\theta}{r^2}) u(r,\theta).$$

Iterate the Laplace-Beltrami operators two times, we obtain

$$\begin{aligned}
\Delta^2 u(r,\theta) =& (\partial_r^2 + \frac{N-1}{r}\partial_r + \frac{\Delta_\theta}{r^2})^2 u(r,\theta) \\
=& \Big[\partial_r^4 + \frac{2(N-1)}{r}\partial_r^3 + \frac{(N-1)(N-3)}{r^2}\partial_r^2 - \frac{(N-1)(N-3)}{r^3}\partial_r \\
&+ \frac{8-2N}{r^4}\Delta_\theta + \frac{2N-6}{r^3}\Delta_\theta\partial_r + \frac{2}{r^2}\Delta_\theta\partial_r^2 + \frac{1}{r^4}\Delta_\theta^2\Big] u(r,\theta),
\end{aligned}$$

which gives the formula of the scalar transform of u as the following:

$$\begin{aligned}
\Delta^2 u(\rho,\theta)\Big|_{\rho=\frac{1}{r}} =& (\partial_\rho^2 + \frac{N-1}{\rho}\partial_\rho + \frac{\Delta_\theta}{\rho^2})^2 u(\rho,\theta)\Big|_{\rho=\frac{1}{r}} \\
=& \Big[\partial_r^4 + 2(N-1)r\partial_r^3 + (N-1)(N-3)r^2\partial_r^2 - (N-1)(N-3)r^3\partial_r \\
&+ (8-2N)r^4\Delta_\theta + (2N-6)r^3\Delta_\theta\partial_r + 2r^2\Delta_\theta\partial_r^2 + r^4\Delta_\theta^2\Big] u(\frac{1}{r},\theta).
\end{aligned} \tag{2.3}$$

For the same reason, we have for $\alpha > 0$,

$$\Delta\Big(r^\alpha u(\frac{1}{r},\theta)\Big) = \Big[\alpha(N+\alpha-2)r^{\alpha-2} + (3-N-2\alpha)r^{\alpha-3} + r^{\alpha-4}\partial_r^2 + r^{\alpha-2}\Delta_\theta\Big] u(\frac{1}{r},\theta), \tag{2.4}$$

and

$$\begin{aligned}
\Delta^2 r^\alpha u(\frac{1}{r},\theta) =& \Big\{\alpha(\alpha-2)(N+\alpha-2)(N+\alpha-4)r^{\alpha-4} + r^{\alpha-8}\partial_r^4 + (14-2N-4\alpha)r^{\alpha-7}\partial_r^3 \\
&+ \Big[\alpha(\alpha+N-2) + (3-N-2\alpha)(9-N-2\alpha) + (\alpha-4)(N+\alpha-6)\Big] r^{\alpha-6}\partial_r^2 \\
&+ \Big[\alpha(\alpha+N-2)(7-N-2\alpha) + (3-N-2\alpha)(\alpha-3)(\alpha+N-5)\Big] r^{\alpha-5}\partial_r \\
&+ \Big[\alpha(N+\alpha-2) + (\alpha-2)(N+\alpha-4)\Big] r^{\alpha-4}\Delta_\theta \\
&+ (10-2N-4\alpha)r^{\alpha-5}\Delta_\theta\partial_r + 2r^{\alpha-6}\Delta_\theta\partial_r^2 + r^{\alpha-4}\Delta_\theta^2\Big\} u(\frac{1}{r},\theta).
\end{aligned} \tag{2.5}$$

In order to avoid the possible u term and preserve the derivative terms, we set $\alpha = 4 - N$ in (2.5) above, by comparison with the equation (2.3), we can derive the following formula of the Δ^2 operator on the Kelvin type transform, namely,

$$\begin{aligned}\Delta^2 r^{4-N} u(\frac{1}{r}, \theta) =& r^{-(N+4)} \Big[\partial_r^4 + 2(N-1) r \partial_r^3 + (N-1)(N-3) r^2 \partial_r^2 - (N-1)(N-3) r^3 \partial_r \\ &+ (8-2N) r^4 \Delta_\theta + (2N-6) r^3 \Delta_\theta \partial_r + 2r^2 \Delta_\theta \partial_r^2 + r^4 \Delta_\theta^2\Big] u(\frac{1}{r}, \theta) \\ =& r^{-(N+4)} \Delta^2 u(\rho, \theta)\Big|_{\rho=\frac{1}{r}}.\end{aligned} \tag{2.6}$$

By using the formula (2.6), we have,

$$\begin{aligned}(-\Delta)^2 \widehat{u}(y) =& \Delta^2 \widehat{u}(y) = r^{-(N+4)} \Delta^2 u(\rho, \theta)\Big|_{\rho=\frac{1}{r}} \\ =& r^{-(N+4)} (-\Delta)^2 u(\rho, \theta)\Big|_{\rho=\frac{1}{r}} \\ =& r^{-(N+4)} |u|^{\frac{8}{N-4}} u(\rho, \theta)\Big|_{\rho=\frac{1}{r}} \\ =& |\widehat{u}|^{\frac{8}{N-4}} \widehat{u}(y).\end{aligned}$$

For any $m > 1$, to avoid the horrible details and inessential repeats, it is reasonable to give an induction to reveal the scheme of the proof in the case of $m \neq 2$. Indeed, for some fixed $\alpha > 0$,

$$\begin{aligned}\Delta^m (r^\alpha u(\frac{1}{r}, \theta)) =& \Delta\Big(\Delta^{m-1}(r^\alpha u(\frac{1}{r}, \theta))\Big) \\ =& \Delta\Big\{\prod_{h=0}^{m-2} \big[(\alpha - 2h)(N + \alpha - (h+1))\big] r^{\alpha - 2(m-1)} u(\frac{1}{r}, \theta) \\ &+ r^{\alpha - 4(m-1)} \partial_r^{2(m-1)} u(\frac{1}{r}, \theta) + \cdots\Big\} \\ =& \prod_{h=0}^{m-2} \big[(\alpha - 2h)(N + \alpha - (h+1))\big] \Delta\big(r^{\alpha-2(m-1)} u(\frac{1}{r}, \theta)\big) \\ &+ \Delta\big(r^{\alpha - 4(m-1)} \partial_r^{2(m-1)} u(\frac{1}{r}, \theta)\big) + \Delta \cdots \\ =& \Big\{\prod_{h=0}^{m-2} \big[(\alpha - 2h)(N + \alpha - (h+1))\big]\Big\} \cdot \big[(\alpha - 2(m-1))(\alpha + N - 2m)\big] \cdot \\ & r^{\alpha - 2m} u(\frac{1}{r}, \theta) + r^{\alpha - 4m} \Big[\partial_r^{2m} u(\frac{1}{r}, \theta) + \cdots\Big] \\ =& \prod_{h=0}^{m-1} \big[(\alpha - 2h)(N + \alpha - (h+1))\big] r^{\alpha - 2m} u(\frac{1}{r}, \theta) + r^{\alpha - 4m} \Delta^m u(\rho, \theta)\Big|_{\rho=\frac{1}{r}}.\end{aligned}$$

By using the same statement as that in the case of $m = 2$, we set $\alpha = 2m - N$, and the conformal invariance under the Kelvin type transform (2.2) holds. □

Let

$$w_\mu(y - \xi) = \mu^{-\frac{N-2m}{2}} U(\mu^{-1}(y - \xi)).$$

Then a simple algebra computation shows that:

Lemma 2.2 *$w_\mu(y-\xi)$ is invariant under the Kelvin type transform* (2.2) *if and only if* $\mu^2+|\xi|^2 = 1$.

Let k be a large positive integer and $\mu > 0$ be a small concentration parameter such that:

$$\begin{cases} \mu = \delta^{\frac{2}{N-2m}} k^{-2}, \quad N \geq 2m+2, \\ \mu = \delta^2 k^{-3} \log^{-2} k, \quad N = 2m+1, \end{cases}$$

where δ is a positive parameter that will be fixed later. Let

$$\xi_j = \sqrt{1-\mu^2}(e^{\frac{2\pi(j-1)}{k}\sqrt{-1}}), \quad j = 1, 2, \cdots, k,$$

be the points that are arranged symmetrically as the vertices of a planar regular polygon. Set

$$U_j(y) = w_\mu(y-\xi_j), j = 1, 2, ..., k$$

and

$$U_* = U - \sum_{j=1}^{k} U_j.$$

In order to find out sign-changing solutions for the problem (1.5). We follow the method of [24] and use the number of the bubble solutions U_j as a parameter. This was originally developed by Wei and Yan in [31] for the critical problems with the presence of weights. We will show that when the bubbles number k is large enough, the problem (1.5) admits a solution of the form:

$$u(y) = U_*(y) + \phi(y)$$

where ϕ is a function which is small when compared with U_*. With u being this form, the equation (1.5) can be restated as

$$(-\Delta)^m \phi - p|U_*|^{p-1}\phi + E - N(\phi) = 0 \tag{2.7}$$

where $p = m^* - 1$, and

$$E = (-\Delta)^m U_* - |U_*|^{p-1} U_*,$$

$$N(\phi) = |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\phi.$$

We expect that for k large, the error term E will be controlled small enough so that some asymptotic estimate holds. In order to get the better control on the error, for a fixed number $N > q > \frac{N}{2}$, we introduce the following weighted L^q norm:

$$\|h\|_{**} := \|(1+|y|)^{N+2m-\frac{2N}{q}} h(y)\|_{L^q(\mathbb{R}^N)} \tag{2.8}$$

and

$$\|\phi\|_* := \|(1+|y|^{N-2m})\phi(y)\|_{L^\infty(\mathbb{R}^n)}. \tag{2.9}$$

Proposition 2.3 *There exists an integer k_0 and a positive constant C such that for $\forall k \geq k_0$, the following estimates for the error term E hold true:*

$$\|E\|_{**} \leq \begin{cases} Ck^{1-\frac{N}{q}} & \text{if} \quad N \geq 2m+2; \\ C\log^{-1} k & \text{if} \quad N = 2m+1. \end{cases} \tag{2.10}$$

Proof. We estimate the error in two steps. In the first step, we estimate the error in the exterior region:

$$EXT := \bigcap_{j=1}^{k} B^c_{\xi_j}(\eta/k) := \bigcap_{j=1}^{k} \{|y-\xi_j| > \eta/k\}.$$

In the second step, we estimate the error in the interior:

$$INT = EXT^c = \bigcup_{j=1}^{k} \{|y-\xi_j| \leq \eta/k\} \quad \text{where } \eta << 1.$$

Step 1: (**The estimate for the exterior region** EXT). In order to use mean value theorem appropriately, we write the formula of E as the following:

$$\begin{aligned}
E =&(-\Delta)^m U_* - |U_*|^{p-1}U_* \\
=&(-\Delta)^m \big[U - \sum_{j=1}^{k} U_j\big] - \big|U - \sum_{j=1}^{k} U_j\big|^{p-1}\big(U - \sum_{j=1}^{k} U_j\big) \\
=&U^p - \sum_{j=1}^{k} U_j^p - \big|U - \sum_{j=1}^{k} U_j\big|^{p-1}\big(U - \sum_{j=1}^{k} U_j\big) \\
=&-\Big[|x|^{p-1}x\Big|_U^{U-\sum\limits_{j=1}^{k} U_j} + \sum_{j=1}^{k} U_j^p\Big] \\
=&-\Big[\big|U - s\sum_{j=1}^{k} U_j\big|^{p-1}\big(U - s\sum_{j=1}^{k} U_j\big) + \sum_{j=1}^{k} U_j^p\Big], \text{ for } s \in (0,1),
\end{aligned}$$

where $|x|^{p-1}x\Big|_U^{U-\sum\limits_{j=1}^{k} U_j} = |U - \sum\limits_{j=1}^{k} U_j|^{p-1}(U - \sum\limits_{j=1}^{k} U_j) - |U|^{p-1}U.$

We split the exterior region into two parts, namely:

$$I := \{y \big| |y| \geq 2\} \quad \text{and} \quad II := \{|y| < 2\} \bigcap \Big[\bigcap_{j=1}^{k} \{|y-\xi_j| > \eta/k\}\Big].$$

For $y \in I$, we have $\frac{1}{|y-\xi_j|} \sim \frac{1}{1+|y|}$. Thus

$$|E(y)| \leq C\Big\{\big(1+|y|^2\big)^{-2m} + \big[\sum_{j=1}^{k} \mu^{\frac{N-2m}{2}}\big(\mu^2+|y-\xi_j|^2\big)^{-\frac{N-2m}{2}}\big]^{\frac{4m}{N-2m}}\Big\}.$$

$$\big[\sum_{j=1}^{k} \mu^{\frac{N-2m}{2}}\big(\mu^2+|y-\xi_j|^2\big)^{-\frac{N-2m}{2}}\big]$$

$$\leq C\Big[\big(1+|y|^2\big)^{-2m} + \frac{\mu^{2m}k^{\frac{4m}{N-2m}}}{(1+|y|^2)^{2m}}\Big] \cdot \sum_{j=1}^{k} \frac{\mu^{\frac{N-2m}{2}}}{|y-\xi_j|^{N-2m}}$$

$$\leq C\frac{\mu^{\frac{N-2m}{2}}}{(1+|y|^2)^{2m}} \sum_{j=1}^{k} \frac{1}{|y-\xi_j|^{N-2m}};$$

For $y \in II$, we have two cases:

Case 1. There exists $i_0 \in \{1,2,3,\cdots,k\}$ such y is closest to ξ_{i_0}, but far away from all the other ξ_j's $(j \neq i_0)$ so that

$$|y-\xi_j| \geq \frac{1}{2}|\xi_j-\xi_i| \sim \frac{|j-i_0|}{k}.$$

Case 2. y is far from all ξ_i's , namely, $\exists C_0>0$ such that $|y-x_i| \geq C_0$ $(1 \leq i \leq k)$.

In both of the cases, we have

$$|E(y)| \leq C\Big\{\big(1+|y|^2\big)^{-2m} + \big[\sum_{j=1}^{k} \mu^{\frac{N-2m}{2}}\big(\mu^2+|y-\xi_j|^2\big)^{-\frac{N-2m}{2}}\big]^{\frac{4m}{N-2m}}\Big\}.$$

$$\big[\sum_{j=1}^{k} \mu^{\frac{N-2m}{2}}\big(\mu^2+|y-\xi_j|^2\big)^{-\frac{N-2m}{2}}\big]$$

$$\leq C\Big\{\big(1+|y|^2\big)^{-2m} + \big[\frac{\mu^{\frac{N-2m}{2}}}{|y-x_{i_0}|^{N-2m}} + \sum_{j\neq i_0} \frac{\mu^{\frac{N-2m}{2}}}{|y-x_j|^{N-2m}}\big]^{\frac{4m}{N-2m}}\Big\} \cdot \sum_{j=1}^{k} \frac{\mu^{\frac{N-2m}{2}}}{|y-\xi_j|^{N-2m}}$$

$$\leq C\Big\{\big(1+|y|^2\big)^{-2m} + \Big[\mu^{2m}k^{4m} + \max\{\sum_{j\neq x_{i_0}} \frac{\mu^{2m}k^{4m}}{|j-i_0|^4}, \quad k^{\frac{4m}{N-2m}}\mu^{2m}\}\Big]\Big\} \cdot \sum_{j=1}^{k} \frac{\mu^{\frac{N-2m}{2}}}{|y-\xi_j|^{N-2m}}$$

$$\leq C\frac{\mu^{\frac{N-2m}{2}}}{(1+|y|^2)^{2m}} \sum_{j=1}^{k} \frac{1}{|y-\xi_j|^{N-2m}}.$$

Hence, by combining the results above, we obtain the estimate for E in the exterior region as:

$$\begin{aligned}
\|E\|_{**(EXT)} =& \|(1+|y|)^{(N+2m)q-2N}E^q(y)\|_{L^q(EXT)} \\
\leq& C\mu^{\frac{N-2m}{2}} \sum_{j=1}^{k} \Big[\int_{B^c_{\xi_j}(\eta/k)} \frac{\big(1+|y|\big)^{(N+2m)q-2N}}{\big(1+|y|\big)^{4mq}|y-\xi_j|^{(N-2m)q}}\Big]^{1/q} \\
\leq& C\mu^{\frac{N-2m}{2}} k\Big[\int_{\eta/k}^{1} \frac{r^{N-1}dr}{r^{(N-2m)q}} + \int_{1}^{+\infty} r^{-(N+1)}dr\Big]^{1/q} \\
\leq& \begin{cases} C(k^{1-\frac{N}{q}} + k^{1+2m-N}) \leq Ck^{1-\frac{N}{q}}, & \text{if} \quad N \geq 2m+1; \\ C\log^{-1} k, & \text{if} \quad N = 2m+1. \end{cases}
\end{aligned}$$

Step 2: (**For the interior region INT.**) In this case, we see that for $\forall y \in INT$, there exists $j \in \{1,2,3,\cdots,k\}$, such that $|y-\xi_j| \le \eta/k$. In order to make the integral region restricted to the regular region centered at the origin, for this particular j, we define

$$\widetilde{E}_j(y) = \mu^{\frac{N+2m}{2}} E(\xi_j + \mu y).$$

Note that $\mu^{\frac{N-2m}{2}} U_j(\xi_j + \mu y) = U(y)$ and for $i \neq j$, $\mu^{\frac{N-2}{2}} U_i(\xi_j + \mu y) = U\big(y - \mu^{-1}(\xi_i - \xi_j)\big)$, where $\mu^{-1}\big|\xi_j - \xi_i\big| \sim \frac{|j-i|}{k\mu}$.

Since $\mu^2(1+|y|^4) < \mu^2(1+\eta\mu^{-4}k^{-4}) \le C$. We have

$$\begin{aligned}
&\big|\widetilde{E}_j(y)\big| \\
\le& C\Big|U(y) + \sum_{i\neq j}\frac{(k\mu)^{N-2m}}{|j-i|^{N-2m}} + \mu^{\frac{N-2m}{2}}U(\xi_j+\mu y)\Big|^{p-1}\cdot\big(\sum_{i\neq j}\frac{(k\mu)^{N-2m}}{|j-i|^{N-2m}} + \mu^{\frac{N-2m}{2}}U(\xi_j+\mu y)\big) \\
&+\sum_{i\neq j}\Big(\frac{(k\mu)^{N-2m}}{|j-i|^{N-2m}}\Big)^p + \mu^{\frac{N+2m}{2}}U^p(\xi_j+\mu y) \\
\le& C\Big|\big(\frac{1}{1+|y|^2}\big)^{\frac{N-2m}{2}} + \mu^{\frac{N-2m}{2}}\Big|^{p-1}\cdot\mu^{\frac{N-2m}{2}} + \mu^{\frac{N-2m}{2}p} + \mu^{\frac{N+2m}{2}} \\
\le& C\Big|\frac{\mu^{\frac{N-2m}{2}}}{1+|y|^{4m}} + \mu^{\frac{N+2m}{2}}\Big| \\
\le& C\frac{\mu^{\frac{N-2m}{2}}}{1+|y|^{4m}}.
\end{aligned}$$

Hence we get the estimate of the error E in one branch of the interior region as

$$\begin{aligned}
\|E\|_{**(|x-\xi_j|<\eta/k)} \le& C\Big[\int_{|y|\le\eta/(k\mu)}\Big|\mu^{\frac{N}{q}-\frac{N+2m}{2}}\widetilde{E}_j(y)\Big|^q dy\Big]^{1/q} \\
\le& C\Big[\mu^{N-2mq}\int_0^{\eta/(k\mu)}\frac{r^{N-1}}{1+r^{4mq}}dr\Big]^{1/q} \\
\le& C\mu^{2m}\cdot k^{-\frac{N}{q}+4m} \\
\le& \begin{cases} Ck^{-\frac{N}{q}}, & \text{if} \quad N\ge 2m+2; \\ Ck^{-\frac{N}{q}}\cdot\log^{-4m}k, & \text{if} \quad N=2m+1. \end{cases}
\end{aligned}$$

At last, by combining the estimates in the exterior region and interior region together, we get

$$\begin{aligned}
\|E\|_{**} \le& \|E\|_{**(EXT)} + \|E\|_{**(INT)} \le \|E\|_{**(EXT)} + \sum_{j=1}^k \|E\|_{**(|x-\xi_j|<\eta/k)} \\
\le& \begin{cases} Ck^{1-\frac{N}{q}}, & \text{if} \quad N\ge 2m+2; \\ C\big(\log^{-1}k + k^{1-\frac{N}{q}}\cdot\log^{-4m}k\big) \le C\log^{-1}k, & \text{if} \quad N=2m+1. \end{cases}
\end{aligned}$$

3 Linearization and gluing

In this section, we focus on the invertibility theory for a linearized equation and the proof of the main theorem follows from the obtained series of propositions and lemmas.

We consider the linear operator L_0 defined by

$$L_0(\phi) := \Big[(-\Delta)^m - pU^{p-1}\Big]\phi, \quad \text{with} \quad p = m^* - 1.$$

We consider the following linear equation

$$L_0(\phi) = h. \tag{3.1}$$

Then it is well known that (see [6]) the solution space for the corresponding homogeneous equation

$$L_0(\phi) = 0$$

is spanned by the following $N+1$ functions,

$$v_i = \partial_{y_i} U, \quad i = 1, 2, 3, \cdots, N; \quad v_{N+1} = x \cdot \nabla U + \frac{n-2m}{2} U.$$

We also consider the linear operator L_* of (2.7), that is

$$L_*(\phi) := \Big[(-\Delta)^m - p|U_*|^{p-1}\Big]\phi, \quad \text{with} \quad p = m^* - 1,$$

Since the region is scattered around the vertices of the regular k-polygonal, the direct calculus on this L_* is not convenient. We introduce the following gluing procedure to split the working space into the respective single branch by some cut-off functions, and the equation (1.5) will be splitted into $k+1$ equations with respective single branches or simple linear operator L_0.

Let $\zeta(s)$ be a smooth function satisfying

$$\zeta(s) = \begin{cases} 1, \quad 0 \le s < 1/2; \\ \\ \text{smooth}, \quad 1/2 \le s \le 1; \\ \\ 0, \quad s > 1. \end{cases}$$

We define the cut-off functions as

$$\zeta_j(y) = \begin{cases} \zeta\Big(k\eta^{-1}|y|^{-2} \cdot \big|y - |y|^2 \xi_k\big|\Big), \quad \text{if} \quad |y| \ge 1; \\ \zeta\Big(k\eta^{-1}\big|y - \xi_j\big|\Big), \quad \text{if} \quad |y| < 1, \end{cases}$$

such that

$$\zeta_j(y) = \zeta_j(\frac{y}{|y|^2}), \quad supp\{\zeta_j\} \subset \{y \big| |y - \xi_j| \le \eta/k\}, j = 1, 2, \cdots, k.$$

By means of the cut-off functions, we can split the equation (2.7) into a system comprised of $k+1$ equations.

Let $\phi = \sum\limits_{j=1}^{k} \widetilde{\phi}_j + \psi, \overline{y} = (y_1, y_2); \quad y' = (y_3, \cdots, y_N)$, we assume

$$\widetilde{\phi}_j(\overline{y}, y') = \widetilde{\phi}_1(e^{-\frac{2\pi(j-1)}{k}\sqrt{-1}}\overline{y}, y'), j = 1, \cdots, k, \tag{3.2}$$

$$\widetilde{\phi}_1(y) = \big|y\big|^{2m-N}\widetilde{\phi}_1(\frac{y}{|y|^2}), \tag{3.3}$$

$$\widetilde{\phi}_1(y_1, \cdots, y_s, \cdots, y_N) = \widetilde{\phi}_1(y_1, \cdots, -y_s, \cdots, y_N), \quad s = 2, 3, \cdots, N, \tag{3.4}$$

and

$$\|\phi_1\|_* \le \rho \text{ with } \rho << 1, \tag{3.5}$$

where $\phi_1(y) := \mu^{\frac{N-2m}{2}}\widetilde{\phi}_1(\xi_1 + \mu y)$.

Then the equation (2.7) can be splitted into the following system:

$$\begin{cases} (-\Delta)^m\widetilde{\phi}_j - p|U_*|^{p-1}\zeta_j\widetilde{\phi}_j + \zeta_j\Big[-p|U_*|^{p-1}\psi + E - N(\widetilde{\phi}_j + \sum\limits_{i\neq j}\widetilde{\phi}_i + \psi)\Big] = 0, \quad j = 1, \cdots, k; \\ (-\Delta)^m\psi - pU^{p-1}\psi + \Big[-p(|U_*|^{p-1} - U^{p-1})(1 - \sum\limits_{j=1}^{k}\zeta_j) + pU^{p-1}(\sum\limits_{j=1}^{k}\zeta_j)\Big]\psi \\ -p|U_*|^{p-1}\sum\limits_{j=1}^{k}(1-\zeta_j)\widetilde{\phi}_j + (1 - \sum\limits_{j=1}^{k}\zeta_j)\big(E - N(\sum\limits_{j=1}^{k}\widetilde{\phi}_j + \psi)\big) = 0. \end{cases} \tag{3.6}$$

3.1 The existence of ψ

In this subsection, we will focus on the existence of ψ in the second equation of the system (3.6). For this purpose, we first prove the following

Proposition 3.1 *Assume that $\frac{N}{2} < q < N$, let $h(y)$ be a function such that $\|h\|_{**} < +\infty$, and*

$$\int_{\mathbb{R}^N} v_l h = 0, \quad \forall l = 1, 2, \cdots, N+1.$$

Then the equation

$$L_0(\phi) = \Big[(-\Delta)^m - pU^{p-1}\Big]\phi = h, \tag{3.7}$$

has a unique solution ϕ satisfying $\|\phi\|_* < \infty$ and

$$\int_{\mathbb{R}^N} U^{p-1}Z_l\phi = 0, \quad \forall l = 1, 2, \cdots, N+1.$$

Moreover, there is a constant C depending only on q and N such that

$$\|\phi\|_* \le C\|h\|_{**}.$$

Proof. Let

$$H = \{\phi \in \mathcal{D}^{m,2}(\mathbb{R}^N)\Big| \int_{\mathbb{R}^n} U^{p-1}v_l\phi = 0, \quad \forall l = 1, 2, \cdots, N+1\}.$$

Then H is a Hilbert space endowed with the inner product:

$$< u, v >_H := \begin{cases} \int_{\mathbb{R}^n} \Delta^{m/2}u(y) \cdot \Delta^{m/2}v(y)dy & \text{if m is even;} \\ \int_{\mathbb{R}^n} \Big[\big(\nabla\Delta^{\frac{m-1}{2}}u(y)\big) \cdot \big(\nabla\Delta^{\frac{m-1}{2}}v(y)\big)\Big]dy & \text{if m is odd.} \end{cases}$$

Moreover, for any $\widetilde{\phi} \in H$

$$\begin{aligned}
\big(L_0\phi, \widetilde{\phi}\big) =& \big((-\Delta)^m\phi, \widetilde{\phi}\big) - p\big(U^{p-1}\phi, \widetilde{\phi}\big)\\
=& < \phi, \widetilde{\phi} >_H - p\big(U^{p-1}\phi, \widetilde{\phi}\big)\\
=& < \widetilde{\phi}, \phi >_H - \big(\widetilde{\phi}, pU^{p-1}\phi\big)\\
=& \ \big(\phi, L_0\widetilde{\phi}\big).
\end{aligned}$$

By the Sobolev inequality, L^p estimates (Caldéron-Zygmund inequality)(c.f.[13]), and the iteration by m times, we get

$$\begin{aligned}
\|L_0\phi\|_2^2 \leq & \begin{cases} C\Big[\|(-\Delta)^m\phi\|_2^2 + \|\Delta^{\frac{m}{2}}\phi\|_2^2 + \|\phi\|_2^2\Big], & \text{m is even;}\\ C\Big[\|(-\Delta)^m\phi\|_2^2 + \|\nabla\Delta^{\frac{m-1}{2}}\phi\|_2^2 + \|\phi\|_2^2\Big], & \text{m is odd,}\end{cases}\\
\leq & \begin{cases} C\Big[\|(-\Delta)^m\phi\|_2^2 + \|\Delta^{\frac{m}{2}}\phi\|_2^2 + \|\phi\|_2^2\Big], & \text{m is even;}\\ C\Big[\|(-\Delta)^m\phi\|_2^2 + \|\Delta^{\frac{m+1}{2}}\phi\|_2^2 + \|\phi\|_2^2\Big], & \text{m is odd,}\end{cases}\\
\leq & \begin{cases} C\Big[\|(-\Delta)^{m-1}\phi\|_2^2 + \|\Delta^{\frac{m}{2}-1}\phi\|_2^2 + \|\phi\|_2^2\Big], & \text{m is even;}\\ C\Big[\|(-\Delta)^{m-1}\phi\|_2^2 + \|\Delta^{\frac{m-1}{2}}\phi\|_2^2 + \|\phi\|_2^2\Big], & \text{m is odd,}\end{cases}\\
\leq & \cdots\\
\leq & C\|\phi\|_2^2.
\end{aligned}$$

Hence L_0 is a bounded, linear and self-adjoint operator in $\Big(H, (\cdot,\cdot)\Big)$. The *Fredholm* alternative in *Hilbert* space tells us that the closure of the range of the operator L_0 is the orthogonal complement of the null space of $L_0^* = L_0$. We have known from the beginning of this section, the null space is actually spanned by $\{v_1, v_2, \cdots, v_{n+1}\}$, therefore, the invertibility problem (3.1) has a weak solution if and only if

$$(h, v_i) = 0, \quad \text{for} \quad i = 1, 2, 3, \cdots, N+1,$$

which are exactly the assumption required in Proposition 3.1. It admits a weak solution ϕ.

Since $\|h\|_{**} < \infty$, we choose the pair $r = \frac{2N}{N+2m}, r' = \frac{2N}{N-2m} = p+1$, by Hölder inequality, we have

$$\begin{aligned}
\|h\|_r \leq & \Big[\int_{\mathbb{R}^N} |h|^q(1+|y|)^{(N+2m)q-2N}dy\Big]^{1/q} \cdot \Big[\int_{\mathbb{R}^N}(1+|y|)^{-2N}dy\Big]^{\frac{1}{r}-\frac{1}{q}}\\
\leq & C\|h\|_{**} < \infty,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
\|U^{p-1}\phi\|_r \leq & \Big(\int_{\mathbb{R}^N} |\phi|^{r\cdot\frac{N+2m}{N-2m}}\Big)^{\frac{N-2m}{(N+2m)r}} \Big(\int_{\mathbb{R}^N} U^{(p-1)r\cdot\frac{N+2m}{4m}}\Big)^{\frac{4m}{(N+2m)r}}\\
=& \|\phi\|_{p+1} \cdot \Big(\int_{\mathbb{R}^N} U^{\frac{2N}{N-2m}}\Big)^{\frac{2m}{N}}\\
\leq & C\|\phi\|_{p+1} = C\|\phi\|_{m^*} \leq C\|(\nabla)^m\phi\| = C\|\phi\|_H < \infty.
\end{aligned} \tag{3.9}$$

By (3.8) and (3.9), we have $f = pU^{p-1}\phi + h \in L^r$, and the weak solution can be represented by the following equations

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} \Delta^{\frac{m}{2}}\phi \cdot \Delta^{\frac{m}{2}}\psi + \int_{\mathbb{R}^N} f\psi = 0, & \text{m is even;} \\ \displaystyle\int_{\mathbb{R}^N} \Big[(\nabla\Delta^{\frac{m-1}{2}}\phi)\cdot(\nabla\Delta^{\frac{m-1}{2}}\psi)\Big] + \int_{\mathbb{R}^N} f\psi = 0, & \text{m is odd,} \end{cases} \quad \text{for } \forall\psi \in H. \tag{3.10}$$

Now we define the functional $A_f : H \to \mathbb{R}$ by

$$A_f(\psi) = -\int_{\mathbb{R}^N} f\psi,$$

then

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} \Delta^{\frac{m}{2}}\phi \cdot \Delta^{\frac{m}{2}}\psi = A_f(\psi), & \text{m is even;} \\ \displaystyle\int_{\mathbb{R}^N} \Big[(\nabla\Delta^{\frac{m-1}{2}}\phi)\cdot(\nabla\Delta^{\frac{m-1}{2}}\psi)\Big] = A_f(\psi), & \text{m is odd.} \end{cases}$$

Moreover, by Hölder inequality, we know that

$$\big|A_f(\psi)\big| \le \|f\|_r\|\psi\|_{p+1} \le C\|f\|_r\|\psi\|_H.$$

Thus A_f is a bounded linear functional on the Hilbert space $\Big(H, \big(\cdot,\cdot\big)\Big)$, by the Riesz's representation theorem, there exists a unique $\phi \in H$ such that

$$A_f(\psi) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Delta^{\frac{m}{2}}\phi \cdot \Delta^{\frac{m}{2}}\psi & \text{m is even,} \\ \displaystyle\int_{\mathbb{R}^N} \Big[(\nabla\Delta^{\frac{m-1}{2}}\phi)\cdot(\nabla\Delta^{\frac{m-1}{2}}\psi)\Big] & \text{m is odd.} \end{cases}$$

Consequently, we can define an operator $A : L^r \to H$, through the functional A_f, by

$$A(f) = \phi, \quad \text{and} \quad < A(f), \psi >_H = \big(f, \psi\big), \quad \forall\psi \in H.$$

As a result, the equation (3.1) can be equivalent to

$$\phi = A(h) + A(pU^{p-1}\phi) = A(h) + A(\tau(\phi)),$$

where $\tau : H \to L^r, \quad \phi \mapsto pU^{p-1}\phi$, is a compact mapping by the rapid decreasing rate of U^{p-1}.

Set $B = A \circ \tau$, then B is the operator from H to H. Moreover, it is easy to see that B is also compact since it is the composition of the bounded linear operator and the compact operator, hence the equation (3.1) can be rewritten as

$$(I - B)\phi = A(h).$$

Also it is natural to verify that B is also self-adjoint, and the Fredholm alternative applies. Hence $(I - B)\phi = A(h)$ has a solution if and only if

$$\forall v \in Ker(I - B), \quad (I - B)v = 0 = A(0),$$

since A is injective.

Then, we obtain $h \equiv 0$ with

$$A(0) \in R(I - B) = (Ker(I - B^*))^\perp = (Ker(I - B))^\perp.$$

Therefore the equation (3.1) is reduced to the homogeneous version, that is

$$L_0(v) = 0,$$

where v can be written as the sum of v_i's,

$$v(y) = \sum_{l=1}^{N+1} a_l \cdot v_l(y),$$

with constants $a_1, a_2, \cdots, a_{N+1}$.

Recall the constraint of the H is such that

$$0 = \int_{\mathbb{R}^N} U^{p-1} v_l \cdot v = a_l \int_{\mathbb{R}^N} U^{p-1}(y) v_l^2(y) dy,$$

which yields the vanishing components

$$a_l = 0, \quad l = 1, 2, \cdots, N+1, \quad \text{and} \quad v \equiv 0, \quad Ker(I - B) = \{0\}.$$

Hence, the orthogonal complement $R(I - B) = H$,

this shows the existence of ϕ by

$$(I - B)\phi = A(h),$$

and the uniqueness of ϕ by

$$Ker(I - B) = \{0\}.$$

In the following , we will show that

$$\|\phi\|_* \leq C\|h\|_{**}.$$

Set $\phi_0 = \phi$, the linearized equation (3.1) is equivalent to the following system:

$$\begin{cases} (-\Delta)\phi = \phi_1, \\ (-\Delta)\phi_1 = \phi_2, \\ \cdots \\ (-\Delta)\phi_{m-1} = pU^{p-1}\phi + h. \end{cases}$$

By the elliptic regularity, we know that $\{\phi_l, l = 0, 1, 2, \cdots, m-1\}$ are all bounded in L^∞ norm, and h is also bounded in the L^∞ norm. Moreover by the Local elliptic estimates, we have

$$\|D^2\phi_{m-1}\|_{L^q(B_1)} + \|D\phi_{m-1}\|_{L^q(B_1)} + \|\phi_{m-1}\|_{L^\infty(B_1)} \leq C\|h\|_{L^2(B_2)} \leq C\|h\|_r \leq C\|h\|_{**};$$

$$\|D^2\phi_l\|_{L^q(B_{\frac{l}{m}})}+\|D\phi_l\|_{L^q(B_{\frac{l}{m}})}+\|\phi_l\|_{L^\infty(B_{\frac{l}{m}})}\le C\|\phi_{l+1}\|_{L^2(B_{\frac{l+1}{m}})}\le C\|\phi_{l+1}\|_{L^\infty(B_{\frac{l+1}{m}})},$$

$$l=0,1,\cdots,m-2.$$

Hence

$$\|\phi\|_{L^\infty(B_{\frac{1}{m}})}\le C\|\phi_{m-1}\|\le C\|h\|_{**}.$$

Without loss of generality, we can write

$$\|\phi\|_{*(B_1)}=\|(1+|y|^{N-2m})\phi(y)\|_{L^\infty(B_1)}\le C\|\phi\|_{L^\infty(B_1)}\le C\|\phi\|_{**}.$$

To complete the estimate outside the unit ball, we make use of the Kelvin type transform $\widetilde{\phi}(y)=\big|y\big|^{2m-N}\phi(\frac{y}{|y|^2})$. And a simple algebra shows,

$$\begin{aligned}
&(-\Delta)^m\widetilde{\phi}(y)-pU^{p-1}(y)\widetilde{\phi}(y)\\
=&\big|y\big|^{-(2m+N)}(-\Delta)^m\phi(\frac{y}{|y|^2})-pU^{p-1}(y)\cdot\big|y\big|^{2m-N}\phi(\frac{y}{|y|^2})\\
=&\big|y\big|^{-(2m+N)}(-\Delta)^m\phi(\frac{y}{|y|^2})-p\big|y\big|^{(p-1)(2m-N)}U^{p-1}(\frac{y}{|y|^2})\cdot\big|y\big|^{2m-N}\phi(\frac{y}{|y|^2})\\
=&\big|y\big|^{-(2m+N)}\Big[(-\Delta)^m-pU^{p-1}\Big]\phi(\frac{y}{|y|^2})\\
=&\big|y\big|^{-(2m+N)}\cdot h(\frac{y}{|y|^2})=\widetilde{h}(y).
\end{aligned}$$

It turns out that

$$(-\Delta)^m\widetilde{\phi}-pu^{p-1}\widetilde{\phi}(y)=\widetilde{h}(y).$$

Similarly, by using the Hölder inequality and the local elliptic estimates, we have

$$\|\widetilde{h}\|_{L^q(B_2)}\le C\Big(\int_{B^c_{\frac{1}{2}}}\big|y\big|^{(N+2m)q-2N}\big|h^q(y)\big|dy\Big)^{1/q}\le C\|(1+|y|)^{N+2m-\frac{2N}{q}}h\|_q=C\|h\|_{**},$$

and

$$\|\phi\|_{*(B_1^c)}=\|(1+|y|^{N-2m})\phi\|_{L^\infty(B_1^c)}\le C\|\widetilde{\phi}\|_{L^\infty(B_1)}\le C\|\widetilde{h}\|_{L^q(B_2)}\le C\|h\|_{**}.$$

Therefore, we get the estimate for ϕ

$$\|\phi\|_*\le\|\phi\|_{*B_1}+\|\phi\|_{*B_1^c}\le C\|h\|_{**}.$$

This completes the proof of proposition 3.1. □

Now we return to the existence and uniqueness of solution ψ for the equation in (3.6), which can be simplified to

$$(-\Delta)^m\psi-pU^{p-1}\psi+(V_1+V_2)\cdot\psi+M(\psi)=0, \tag{3.11}$$

where

$$V_1=-p\Big(\big|U_*\big|^{p-1}-U^{p-1}\Big)\Big(1-\sum_{j=1}^k\zeta_j\Big),\quad V_2=pU^{p-1}\Big(\sum_{j=1}^k\zeta_j\Big),$$

$$M(\psi) = -p\big|U_*\big|^{p-1}\sum_{j=1}^{k}\big(1-\zeta_j\big)\phi_j + \big(1-\sum_{j=1}^{k}\zeta_j\big)\Big[E - N\big(\sum_{j=1}^{k}\widetilde{\phi}_j+\psi\big)\Big], \tag{3.12}$$

and

$$N(\phi) = \big|U_*+\phi\big|^{p-1}\big(U_*+\phi\big) - \big|U_*\big|^{p-1}U_* - p\big|U_*\big|^{p-1}\phi. \tag{3.13}$$

Proposition 3.2 *There exists some positive constants k_0, C, ρ_0, such that for $\forall k \geq k_0$, and $\widetilde{\phi}_j$ satisfying (3.2)–(3.5), with $\rho < \rho_0$. Then there exists a unique solution $\psi = \Psi(\phi_1)$ to (3.11) satisfying the symmetries:*

$$\begin{aligned}
\psi(\overline{y}, y_3, \cdots, y_l, \cdots, y_n) =& \psi(\overline{y}, y_3, \cdots, -y_l, \cdots, y_n);\\
\psi(\overline{y}, y') =& \psi(e^{\frac{2\pi j}{k}\sqrt{-1}}\overline{y}, y'), \quad j = 1, 2, \cdots, k-1;\\
\psi(y) =& \big|y\big|^{2m-N}\psi\big(\frac{y}{|y|^2}\big).
\end{aligned}$$

Moreover

$$\begin{cases}
\|\psi\|_* \leq C\Big[k^{1-\frac{N}{q}} + \|\phi_1\|_*^2\Big], & \text{if} \quad N \geq 2m+2;\\
\|\psi\|_* \leq C\Big[\log^{-1} k + \|\phi_1\|_*^2\Big], & \text{if} \quad N = 2m+1.
\end{cases}$$

And the operator Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1^1) - \Psi(\phi_1^2)\|_* \leq C\|\phi_1^1 - \phi_1^2\|_*.$$

Proof. Notice that

$$\Big[\big(V_1 + V_2\cdot\psi\big) + M\big(\psi\big)\Big]\big(y\big) = \big|y\big|^{-(N+2m)}\Big[\big(V_1 + V_2\cdot\psi\big) + M\big(\psi\big)\Big]\big(\frac{y}{|y|^2}\big),$$

we digress to a general problem for (3.7) in Proposition 3.1, where h satisfies

$$\begin{aligned}
h(\overline{y}, y_3, \cdots, y_l, \cdots, y_N) =& h(\overline{y}, y_3, \cdots, -y_l, \cdots, y_N);\\
h(\overline{y}, y') =& h(e^{\frac{2\pi j}{k}\sqrt{-1}}\overline{y}, y'), \quad j = 1, 2, \cdots, k-1;\\
h(y) =& \big|y\big|^{-(N+2m)}h\big(\frac{y}{|y|^2}\big).
\end{aligned}$$

We claim that (3.7) has a unique bounded solution $\psi = T(h)$ such that there is a constant C, depending on q and N satisfying

$$\|\phi\|_* \leq C\|h\|_*.$$

Thanks to the results in Proposition 3.1, it is sufficient to check

$$(h, v_l) = \int_{\mathbb{R}^n} h v_l = 0, \quad \forall \quad l = 1, 2, \cdots, N+1.$$

From the assumption that h is even with respect to $y_3, y_4, \cdots, y_n$, and the oddness of $v_l = \frac{\partial U}{\partial y_l}$, it is natural $(h, v_l) = 0$ for $l = 3, \cdots, N$.

For $l = 1, 2$, we consider the vector integral

$$I = \int_{\mathbb{R}^N} h \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = c_N \int_{\mathbb{R}^N} \frac{h(y)}{\left(1+|y|^2\right)^{\frac{N}{2}-1+m}} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} dy.$$

Let

$$(\overline{z}, z') = (e^{\frac{2\pi j}{k}\sqrt{-1}}\overline{y}, y'),$$

by the invariance of h and the integral I under this change of variables, we know that

$$\begin{aligned} e^{\frac{2\pi j}{k}\sqrt{-1}} \cdot I =& c_N \int_{\mathbb{R}^N} \frac{h(y)}{\left(1+|y|^2\right)^{\frac{N}{2}-1+m}} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \cdot e^{\frac{2\pi j}{k}\sqrt{-1}} dy \\ =& c_N \int_{\mathbb{R}^N} \frac{h(z)}{\left(1+|z|^2\right)^{\frac{N}{2}-1+m}} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} dz \\ =& I, \end{aligned}$$

which yields $I = 0$, since $e^{\frac{2\pi j}{k}\sqrt{-1}} \neq 0$ for $k \geq 2$.

For $l = N + 1$, we define a function $I(\lambda)$, for $\lambda > 0$, by

$$I(\lambda) = \lambda^{\frac{N-2m}{2}} \int_{\mathbb{R}^N} U(\lambda y) h(y) dy.$$

By changing the variables $y \mapsto z = \frac{y}{|y|^2}$, we have

$$\begin{aligned} I(\lambda) =& \lambda^{\frac{N-2m}{2}} \int_{\mathbb{R}^N} U(\lambda y) h(y) dy \\ =& \lambda^{\frac{N-2m}{2}} \int_{\mathbb{R}^N} U(\frac{\lambda y}{|y|^2}) h(\frac{y}{|y|^2}) d(\frac{y}{|y|^2}) \\ =& \left(\lambda^{-1}\right)^{\frac{N-2m}{2}} \int_{\mathbb{R}^N} U(\lambda^{-1} y) h(y) dy \\ =& I(\lambda^{-1}) := g(\lambda), \end{aligned}$$

which shows

$$I'(1) = g'(1) = -\frac{1}{\lambda^2} I'(\frac{1}{\lambda})\big|_{\lambda=1} = -I'(1).$$

Thus

$$0 = I'(1) = (h, v_{N+1}),$$

Up to now we have verified that all conditions of proposition 3.1 are satisfied, hence

$$T(h) = \|\psi\|_* \leq C\|h\|_{**},$$

and T is a bounded linear operator.

Now we return to our problem, take $h = (V_1 + V_2)\psi + M(\psi)$, then the unique existence of ψ is reduced to the survey of the fixed point of an operator $\mathcal{M}$ from the complete space X to itself, where X denotes the linear space with bounded norm $\|\cdot\|_*$ and all symmetries in Proposition 3.2. For this purpose, we consider the following fixed point problem:

$$\psi = -T\Big[\big(V_1 + V_2\big) + M(\psi)\Big] := \mathcal{M}(\psi), \quad \psi \in X.$$

By the uniqueness result of Proposition 3.1 and fact that

$$\psi_l(y) = \psi(\overline{y}, y_3, \cdots, -y_l, \cdots, y_N), \quad l = 3, 4, \cdots, N;$$

$$\psi_{j_2}(y) = \psi(e^{\frac{2\pi j}{k}\sqrt{-1}}\overline{y}, y'); \quad \psi_{N+1}(y) = \big|y\big|^{2m-N}\psi(\frac{y}{|y|^2})$$

satisfy the $\psi-$ equation in (3.7), we obtain that

$$\psi = \psi_l = \psi_{j_2} = \psi_{N+1},$$

which are exactly the symmetries required in Proposition 3.2.

Throughout the last part of this section, we will prove that $\mathcal{M}$ is a contraction mapping. This crucial conclusion is derived from a series of estimates of $V_1, \quad V_2, \quad M$ respectively.

Recall

$$V_1 = -p\Big(\big|U_*\big|^{p-1} - U^{p-1}\Big)\Big(1 - \sum_{j=1}^{k}\zeta_j\Big),$$

then the multiplier $\Big(1 - \sum\limits_{j=1}^{k}\zeta_j\Big)$ shows that $suppV_1 \subset EXT$.

By using the similar arguments as in the discussion of **Step 1** in Proposition 2.3, for $y \in EXT$, there exists $s \in (0,1)$ such that

$$\begin{aligned}
\big|V_1(y)\psi(y)\big| =& \Big|V_1(y)\psi(y)\big(1+|y|^{N-2m}\big)\cdot\frac{1}{1+|y|^{N-2m}}\Big| \\
\leq& C\Big|V_1(y)U(y)\Big|\cdot\Big|1+\big|y\big|^{N-2m}\psi(y)\Big| \\
\leq& C\|\psi\|_*\Big|V_1(y)U(y)\Big| \\
\leq& C\|\psi\|_*\Big|U^{p-1}(y) - \big|U_*\big|^{p-1}(y)\Big|U(y) \\
=& C\|\psi\|_*U(y)\Big|U(y) - s\sum_{j=1}^{k}U_j(y)\Big|^{p-2}\Big[\sum_{j=1}^{k}U_j(y)\Big].
\end{aligned}$$

Note that $EXT = I\bigcup II$. For $y \in I$, we have $\mu^2 + |y-\xi_j|^2 \sim 1+|y|^2$, hence

$$\begin{aligned}
\sum_{i=1}^{k}U_i(y) \leq& Ck\mu^{\frac{N-2m}{2}}U(y) \\
\leq& \begin{cases} Ck^{2m+1-N}U(y) \leq CU(y), & N \geq 2m+2; \\ Ck^{-\frac{1}{2}}\log^{-1}k\cdot U(y) \leq CU(y), & N = 2m+1. \end{cases}
\end{aligned}$$

For $y \in II$, we have

$$\begin{aligned}
\sum_{i=1}^{k}U_i(y) =& \sum_{i=1}^{k}\mu^{\frac{N-2m}{2}}\Big(\frac{1}{\mu^2+|y-\xi_i|^2}\Big)^{\frac{N-2m}{2}} \\
\leq& Ck\cdot k^{\frac{N-2m}{2}}\mu^{\frac{N-2m}{2}} \\
\leq& \begin{cases} Ck^{\frac{2m+2-N}{2}} \leq C \leq CU(2) \leq CU(y), & N \geq 2m+2; \\ C(\log k)^{-\frac{1}{2}} \leq CU(2) \leq CU(y), & N = 2m+1. \end{cases}
\end{aligned}$$

Combine the obtained results for I and II, we have

$$\sum_{i=1}^{k} U_i(y) \le CU(y), \quad \forall y \in EXT. \tag{3.14}$$

Put (3.14) into the estimate of $|V_1\psi|$, we obtain, for $y \in EXT$,

$$\begin{aligned} \big|V_1(y)\psi(y)\big| \le & C\|\psi\|_* U^{p-1}(y) \cdot \big(\sum_{i=1}^{k} U_i(y)\big) \\ \le & C\|\phi\|_* \big(\frac{1}{(1+|y|)^2}\big)^{2m} \sum_{i=1}^{k} \frac{\mu^{\frac{N-2m}{2}}}{|y-\xi_i|^{N-2m}}. \end{aligned}$$

At last by using the similar arguments as in the the **Step 2** of Proposition 2.3, we know that

$$\|V_1 \cdot \psi\|_{**} \le \begin{cases} Ck^{1-\frac{N}{q}}, & N \ge 2m+2; \\ C\log^{-1} k, & N = 2m+1. \end{cases}$$

Now we turn to the estimate of $V_2 \cdot \psi$. Recall

$$V_2 = pU^{p-1}\Big(\sum_{j=1}^{k} \zeta_j\Big).$$

The multiplier $\Big(\sum\limits_{j=1}^{k} \zeta_j\Big)$ shows that the support of $V_2 \cdot \psi$ lies in the annular region, namely,

$$suppV_2 \subset INT \subset \{y \Big| \big||y| - \sqrt{1-\mu^2}\big| \le \frac{\eta}{k}\} := ANN.$$

By the argument of measures, we obtain

$$\begin{aligned} \|V_2\psi\|_{**} \le & C\|\psi\|_* \sum_{j=1}^{k} \|U^p\zeta_j\|_{**(|y-\xi_j| \le \eta/k)} \\ \le & C\|\psi\|_* \cdot meas\big(ANN\big) \\ \le & C\|\psi\|_* \cdot k^{1-N} < C\|\psi\|_* k^{1-\frac{N}{q}}. \end{aligned}$$

The discussion of $M(\psi)$ will be more technical, since the operator M is not linear as V_1 and V_2 are. We introduce the following notations to simplify the presentation of $M(\psi)$.

Define

$$M_1 = -p\big|U_*\big|^{p-1} \sum_{j=1}^{k} (1-\zeta_j)\phi_j; \quad M_2 = (1 - \sum_{j=1}^{k} \zeta_j)E;$$

$$M_3(\psi) = -(1 - \sum_{j=1}^{k} \zeta_j) N(\sum_{j=1}^{k} \widetilde{\phi}_j + \psi).$$

Then the nonlinear operator $M(\psi)$ can be rewritten as

$$M(\psi) = M_1 + M_2 + M_3(\psi).$$

For M_1, apply the estimate of the exterior region, we have

$$
\begin{aligned}
\|M_1\|_{**} \leq & C\sum_{j=1}^{k} \||U_*|^{p-1}\phi_j\|_{**(|y-\xi_j|>\eta/k)} \\
\leq & \begin{cases} C\|\psi\|_* k^{1-\frac{N}{q}}, & N\geq 2m+2; \\ C\|\psi\|_* \log^{-1} k, & N=2m+1. \end{cases}
\end{aligned}
$$

The estimate for M_2 is the same as that for the error term E.

For $M_3(\psi)$, we have

$$
suppM_3(\psi)\subset EXT.
$$

Recall the formula of N, by means of the mean value theorem, there exist $s,t\in(0,1)$ such that

$$
\begin{aligned}
\big|N(\sum_{j=1}^{k}\widetilde{\phi}_j+\psi)\big| = & \Big||U_*+\sum_{j=1}^{k}\widetilde{\phi}_j+\psi|^{p-1}\cdot\big(U_*+\sum_{j=1}^{k}\widetilde{\phi}_j+\psi\big)-|U_*|^{p-1}\cdot U_*-p|U_*|^{p-1}\big(\sum_{j=1}^{k}\widetilde{\phi}_j+\psi\big)\Big| \\
= & p\Big||U_*+s(\sum_{j=1}^{k}\widetilde{\phi}_j+\psi)|^{p-1}-|U_*|^{p-1}\big(\sum_{j=1}^{k}\widetilde{\phi}_j+\psi\big)\Big| \\
= & p|U_*+ts(\sum_{j=1}^{k}\widetilde{\phi}_j+\psi)|^{p-2}|\sum_{j=1}^{k}\widetilde{\phi}_j+\psi|^2
\end{aligned}
$$

We restricted N to the exterior region, by the proof of **Step 1** in the Proposition 2.3, we have

$$
|\sum_{j=1}^{k}\widetilde{\psi}_j(y)|\leq \begin{cases} C\|\phi_1\|_* U(y); \\ \sum_{j=1}^{k}\frac{\mu^{\frac{N-2}{2}}}{|y-\xi_j|^{N-2}}. \end{cases}
$$

Hence

$$
\begin{aligned}
\|M_3(\psi)\|_{**} = & \|(1-\sum_{j=1}^{k}\zeta_j)N(\sum_{j=1}^{k}\widetilde{\phi}_j+\psi)\|_{**} \\
\leq & C\|N(\sum_{j=1}^{k}\widetilde{\phi}_j+\psi)\|_{**(EXT)} \\
\leq & C\|\big[|U_*|+|\sum_{j=1}^{k}\widetilde{\phi}_j|+|\psi|\big]^{p-1}\cdot\big[|\sum_{j=1}^{k}\widetilde{\phi}_j|^2+|\psi|^2\big]\|_{**(EXT)} \\
\leq & C\|\phi_1\|_*^2\|U^{p-1}\cdot\sum_{j=1}^{k}\frac{\mu^{\frac{N-2}{2}}}{|y-\xi_j|^{N-2}}\|_{**(EXT)}+C\|\psi\|_*^2\|U^p\|_{**(EXT)} \\
\leq & \begin{cases} C\|\phi_1\|_*^2 k^{1-\frac{N}{q}}+C\|\psi\|_*^2, & N\geq 2m+2 \\ C\|\phi_1\|_*^2\log^{-1}k+C\|\psi\|_*^2, & N=2m+1. \end{cases}
\end{aligned}
$$

Summing together the obtained estimates above, we have

$$\begin{aligned}\|M(\psi)\|_{**} \leq & \|M_1\|_{**} + \|M_2\|_{**} + \|M_3(\psi)\|_{**} \\ \leq & \begin{cases} C\big(k^{1-\frac{N}{q}} + \|\phi_1\|_*^2 k^{1-\frac{N}{q}} + \|\psi\|_*^2\big), & N \geq 2m+2 \\ C\big(\log^{-1} k + \|\phi\|_*^2 \log^{-1} k + \|\psi\|_*^2\big), & N = 2m+1. \end{cases}\end{aligned}$$

On the other hand, for any $\psi_1, \psi_2 \in B_{\rho/2} \subset X$, by the mean value theorem, there exists some $s, t \in (0,1)$ such that

$$\begin{aligned}\|M(\psi_1) - M(\psi_2)\|_{**} = & \|M_3(\psi_1) - M_3(\psi_2)\|_{**} \\ \leq & C\|N(\sum_{j=1}^{k} \widetilde{\phi}_j + \psi_1) - N(\sum_{j=1}^{k} \widetilde{\phi}_j + \psi_2)\|_{**(EXT)} \\ = & C\|p\big|U_* + \sum_{j=1}^{k} \widetilde{\phi}_j + s(\psi_1 - \psi_2)\big|^{p-1}(\psi_1 - \psi_2) - p\big|U_*\big|^{p-1}(\psi_1 - \psi_2)\|_{**(EXT)} \\ \leq & C\|\big|U_* + t\sum_{j=1}^{k} \widetilde{\phi}_j + ts(\psi_1 - \psi_2)\big|^{p-2} \cdot \big|\sum_{j=1}^{k} \widetilde{\phi}_j + s(\psi_1\psi_2)\big| \cdot \big|\psi_1 - \psi_2\big|\|_{**(EXT)} \\ \leq & C\big(\|\phi_1\|_* + \|\psi_1 - \psi_2\|_*\big)\|\psi_1 - \psi_2\|_* \cdot \|U^p\|_{**(EXT)} \\ \leq & C\rho\|\psi_1 - \psi_2\|_*.\end{aligned}$$

More generally, we have

$$\begin{aligned}\|\mathcal{M}(\psi_1 - \psi_2)\|_* = & \| - T\big[(V_1 + V_2) \cdot (\psi_1 - \psi_2) + \big(M(\psi_1 - \psi_2)\big)\big]\|_* \\ \leq & C\Big[\|\big(V_1 + V_2\big) \cdot (\psi_1 - \psi_2)\|_{**} + \|M_3(\psi_1) - M_3(\psi_2)\|_{**}\Big] \\ \leq & \begin{cases} C(k^{1-\frac{N}{q}} + \rho)\|\psi_1 - \psi_2\|_*, & N \geq 2m+2; \\ C(\log^{-1} k + \rho)\|\psi_1 - \psi_2\|_*, & N = 2m+1. \end{cases}\end{aligned}$$

Choosing k_0 large enough and ρ_0 small enough, then for any $k \geq k_0$ and $\rho \leq \rho_0$, it holds:

$$\begin{cases} C(k^{1-\frac{N}{q}} + \rho) < 1, \\ C(\log^{-1} k + \rho) < 1. \end{cases}$$

Hence $\mathcal{M}$ is a contraction mapping from the small ball in X to the ball itself. The Banach fixed point theorem gives the unique existence of ψ. □

3.2 The existence of $\tilde{\phi}_j$ in (3.6)

In this subsection, we will turn to study the first series of equations (3.6):

$$(-\Delta)^m \widetilde{\phi}_j - p|U_*|^{p-1}\zeta_j \widetilde{\phi}_j + \zeta_j\Big[- p|U_*|^{p-1}\psi + E - N(\widetilde{\phi}_j + \sum_{i \neq j} \widetilde{\phi}_i + \psi)\Big] = 0, \quad j = 1, \cdots, k;$$

Indeed, these equations can all be reduced to the ϕ_1- equation by means of the changing of the variables, that is the equation:

$$(-\Delta)^m\widetilde{\phi}_1-p|U_*|^{p-1}\zeta_1\widetilde{\phi}_1+\zeta_1\Big[-p|U_*|^{p-1}\psi+E-N(\widetilde{\phi}_1+\sum_{i\neq 1}\widetilde{\phi}_i+\psi)\Big]=0.$$

In order to simplify the horrible formula above, we introduce the following new notation $\mathcal{N}$, $\widetilde{h}$:

$$\mathcal{N}(\phi_1):=p\big(|U_1|^{p-1}-|U_*|^{p-1}\zeta_1\big)\widetilde{\phi}_1+\zeta_1\Big[-p|U_*|^{p-1}\psi+E-N(\widetilde{\phi}_1+\sum_{i\neq 1}\widetilde{\phi}_i+\psi)\Big];$$

$$\widetilde{h}:=\zeta_1E+\mathcal{N}(\phi_1);$$

where $\widetilde{h}$ is even with respect to each of the variables $y_2,y_3,\cdots,y_N$ and satisfies

$$\tilde{h}(y)=|y|^{-(N+2m)}\widetilde{h}(\frac{y}{|y|^2}).$$

Then the ϕ_1- equation can be simplified as

$$\Big[(-\Delta)^m-p\big|U_1\big|^{p-1}\zeta_1\Big]\widetilde{\phi}_1+\widetilde{h}=0. \tag{3.15}$$

Recall the definition of μ,

$$\begin{cases}\mu=\delta^{\frac{2}{N-2m}}k^{-2}, & N\geq 2m+2,\\ \mu=\delta^2k^{-3}\log^{-2}k, & N=2m+1.\end{cases}$$

We know that μ is relevant to δ, hence

$$c_{N+1}(\delta):=\frac{\int_{\mathbb{R}^N}(\zeta_1E+\mathcal{N}(\phi_1))\widetilde{v}_{N+1}}{\int_{\mathbb{R}^N}U_1^{p-1}\widetilde{v}_{N+1}^2}=\frac{\int_{\mathbb{R}^N}\widetilde{h}\widetilde{v}_{N+1}}{\int_{\mathbb{R}^N}U_1^{p-1}\widetilde{v}_{N+1}^2}$$

is also a variable relevant to δ.

By the argument of changing variables through translating and scaling, we can get the equivalence between equation (3.15) and equation (3.7).

Consider the result of Proposition 3.1, it is evident to assert that, the unique existence of $\widetilde{\phi}_1$ is equivalent to the verification of the following series of conditions

$$\int_{\mathbb{R}^N}\widetilde{h}\widetilde{v}_l=\int_{\mathbb{R}^N}hv_l=0,\quad l=1,2,3,\cdots,N+1.$$

From the formula of $c_{N+1}(\delta)$, we know one of the conditions holds

$$\int_{\mathbb{R}^N}\widetilde{h}\widetilde{v}_{N+1}=0\Leftrightarrow\int_{\mathbb{R}^N}hv_{N+1}=0\Leftrightarrow c_{N+1}(\delta)=0,$$

under a selected positive number δ.

And the existence of this particular δ is granted by the following lemma (see [24])

Lemma 3.3 *We can write the δ related $\int_{\mathbb{R}^N}\widetilde{h}\widetilde{v}_{N+1}$ in the following form as*

$$\int_{\mathbb{R}^N}\widetilde{h}\widetilde{v}_{N+1}=\begin{cases}A_N\dfrac{\delta}{k^{N-2m}}\big[\delta a_N-1\big]+\dfrac{1}{k^{N-m}}\Theta_k(\delta), & N\geq 2m+2;\\ A_3\dfrac{\delta}{k\log k}\big[\delta a_3-1\big]+\dfrac{1}{k^2\log^2 k}\Theta_k(\delta), & N=2m+1,\end{cases}$$

where $\Theta_k(\delta)$ is continuous w.r.t δ and uniformly bounded as $k \to \infty$, $A_N = p\int_{\mathbb{R}^N} U^{p-1}v_{N+1}$, with the positive number

$$
a_N = \begin{cases} 2^{\frac{N-2m}{2}} \lim\limits_{k\to\infty} \dfrac{1}{k^{N-2m}} \displaystyle\sum_{j=2}^{k} \dfrac{1}{|\xi_1 - \xi_j|^{N-2m}}, & N \geq 2m+2; \\ \sqrt{2} \lim\limits_{k\to\infty} \dfrac{1}{k\log k} \displaystyle\sum_{j=2}^{k} \dfrac{1}{|\xi_1 - \xi_j|}, & N = 2m+1. \end{cases}
$$

In fact, by Lemma 3.3, we can see that for δ small enough, $\int_{\mathbb{R}^N} \widetilde{h}\widetilde{v}_{N+1} < 0$, while for δ large enough, $\int_{\mathbb{R}^N} \widetilde{h}\widetilde{v}_{N+1} > 0$. By the continuity arguments with respect to δ, we can always find some $\delta_0 > 0$ such that $\int_{\mathbb{R}^N} \widetilde{h}\widetilde{v}_{N+1} = 0$.

For the simplified version (3.15), and for this particular δ_0, we give the following theorem to complete the unique existence problem of splitting system (3.6).

Proposition 3.4 *For $\widetilde{h}$ given above, assume in addition that*

$$
h(y) := \mu^{\frac{N+2m}{2}} \widetilde{h}(\xi_1 + \mu y) \quad \textit{satisfying} \quad \|h\|_{**} < \infty.
$$

Then the equation (3.15) has a unique solution $\widetilde{\phi} := \widetilde{T}(\widetilde{h})$ that is even with respect to each of the variables $y_2, y_3, \cdots, y_N$, and is invariant under the Kelvin type transform:

$$
\widetilde{\phi}(y) = |y|^{2m-N} \widetilde{\phi}\left(\frac{y}{|y|^2}\right),
$$

and

$$
\int_{\mathbb{R}^N} \phi U^{p-1} v_{N+1} = 0, \quad \textit{with} \quad \|\phi\|_* \leq C\|h\|_{**}.
$$

where $\phi(y) = \mu^{\frac{N-2m}{2}} \widetilde{\phi}(\xi_1 + \mu y)$.

Proof. By Lemma 3.3, we have

$$
\int_{\mathbb{R}^N} h v_{N+1} = 0.
$$

It follows from the oddness of $v_2, v_3, \cdots, v_N$ and the evenness of h that

$$
\int_{\mathbb{R}^N} h v_j = 0, \quad j = 2, 3, \cdots, N.
$$

In the following, we only need to prove that $\int_{\mathbb{R}^N} h v_1 = 0$. However, this needs some toiling work, since we do not have any symmetry of h with respect to the first component y_1.

Define an integral $I(t)$ and $w_\mu(y)$ as

$$
I(t) := \int_{\mathbb{R}^N} w_\mu(y - t\xi_1)\widetilde{h}(y)dy, \quad w_\mu(y) = \mu^{-\frac{N-2m}{2}} U(\mu^{-1}y).
$$

Then the direct calculus gives,

$$
\begin{aligned}
I'(1) = & -\int_{\mathbb{R}^N} \frac{\xi_1}{\mu} \cdot U(\frac{y-t\xi_1}{\mu}) \cdot h(\frac{y-\xi_1}{\mu}) \cdot \mu^{-\frac{N-2m}{2}} \cdot \mu^{-\frac{N+2m}{2}} dy\big|_{t=1} \\
= & -\frac{\sqrt{1-\mu^2}}{\mu} \int_{\mathbb{R}^N} U_1(y)h(y)dy \\
= & -\frac{\sqrt{1-\mu^2}}{\mu} \int_{\mathbb{R}^N} hv_1.
\end{aligned}
\tag{3.16}
$$

Changing the variable $y \mapsto z = \frac{y}{|y|^2}$, we obtain

$$
\begin{aligned}
I(t) = & \int_{\mathbb{R}^N} w_\mu(y-t\xi_1)|y|^{-(N+2m)}\widetilde{h}(\frac{y}{|y|^2})dy \\
= & \int_{\mathbb{R}^N} w_\mu(\frac{y}{|y|^2}-t\xi_1)\big|z\big|^{2m-N}\widetilde{h}(z)dz \\
= & \int_{\mathbb{R}^N} \Big(\frac{\mu}{\mu^2+t^2|\xi_1|^2}\Big)^{\frac{N-2m}{2}} P_{m,N}^{\frac{N-2m}{4m}} \cdot \Big[\big|y-\frac{t\xi_1}{\mu^2+t^2|\xi_1|^2}\big|^2 + \frac{\mu^2}{(\mu^2+t^2|\xi_1|^2)^2}\Big]^{-\frac{N-2m}{2}} \widetilde{h}(y)dy \\
= & \int_{\mathbb{R}^N} w_{\mu(t)}(y-s(t)\xi_1)\widetilde{h}(y)dy,
\end{aligned}
$$

where

$$
\mu(t) := \frac{\mu}{\mu^2+t^2|\xi_1|^2}, \quad s(t) = \frac{t}{\mu^2+t^2|\xi_1|^2}.
$$

Take the derivative on both sides of the formula of $I(t)$, we get

$$
\begin{aligned}
I'(1) = & \Big[\int_{\mathbb{R}^N} \partial_\mu w_\mu(y-s(t)\xi)\cdot\widetilde{h}(y)dy\cdot\mu'(t) - \xi_1^1\int_{\mathbb{R}^N} \partial_{y_1}\big(w_{\mu(t)}(y-s(t)\xi_1)\big)\widetilde{h}(y)dys'(t)\Big]\Big|_{t=1} \\
= & 2\mu^2\int_{\mathbb{R}^N} \widetilde{v}_{N+1}(y)\widetilde{h}(y)dy - \frac{\sqrt{1-\mu^2}}{\mu}(2\mu^2-1)\int_{\mathbb{R}^N} hv_1 \\
= & \frac{\sqrt{1-\mu^2}}{\mu}(2\mu^2-1)\int_{\mathbb{R}^N} hv_1.
\end{aligned}
\tag{3.17}
$$

Compare (3.16) and (3.17), we have

$$
-\frac{\sqrt{1-\mu^2}}{\mu}\int_{\mathbb{R}^N} hv_1 = I'(1) = \frac{\sqrt{1-\mu^2}}{\mu}(2\mu^2-1)\int_{\mathbb{R}^N} hv_1,
$$

and the equality holds if and only if $\int_{\mathbb{R}^N} hv_1 = 0$, this is what we need at last.

Apply Proposition 3.1, we see that if $\widetilde{h}$ is a general function satisfying all the symmetries in Proposition 3.2, then there exists some unique solution $\widetilde{\phi} := \widetilde{T}(\widetilde{h})$ that is even with respect to each of the variables $y_2, y_3, \cdots, y_N$ and

$$
\|\widetilde{\phi}\|_* = \|\widetilde{T}\widetilde{h}\|_* \leq C\|\widetilde{h}\|_{**}.
$$

However, this $\widetilde{h}$ in our problem (3.15) is not general, but relevant to ϕ_1 itself, which tells us the direct application of the Proposition 3.1 can not work. We need to emulate what we have done in Proposition 3.2, that is to construct a contraction mapping, then the Banach fixed point theorem will give the answer to our problem. Since $\widetilde{h} = \zeta_1 E + \mathcal{N}(\phi_1)$, we define an operator $\mathcal{M}$ by

$$\mathcal{M}(\phi_1) := \widetilde{T}\big(\zeta_1 E + \mathcal{N}(\phi_1)\big).$$

Then the unique existence of the solution ϕ_1 is reduced to the existence of a fixed point of a contraction mapping $\mathcal{M}$.

In the following, we split $\zeta_1 E + \mathcal{N}(\phi_1)$ into several shorter terms and estimate these terms one by one. Define

$$f_1 := p\zeta_1\big(U_1^{p-1} - |U_*|^{p-1}\big)\cdot \widetilde{\phi}_1; \quad f_2 := (1-\zeta_1)U_1^{p-1}\widetilde{\phi}_1;$$

$$f_3 := -p\zeta_1\big|U_*\big|^{p-1}\psi(\phi_1); \quad f_4 := \zeta_1 N\Big(\sum_{j=1}^{k}\widetilde{\phi}_j + \psi(\phi_1)\Big); \quad f_5 := \zeta_1 E,$$

and

$$\widetilde{f_i}(y) = \mu^{\frac{n+2m}{2}} f_i(\xi_1 + \mu y), i = 1,2,3,4,5.$$

Set

$$\widetilde{h} = \sum_{i=1}^{5} f_i.$$

Thanks to the multiplier ζ_1, we know that

$$suppf_j \subset \{y\big||y-\xi_1| < \eta/k\} =: INT_1 \subset INT, \quad j = 1,3,4,5.$$

For f_1, we have

$$\begin{aligned}
|\widetilde{f}_1(y)| \le & \Big|p\Big|U(y) + \sum_{j=2}^{k} U\big(y + \mu^{-1}(\xi_1-\xi_j)\big) - \mu^{\frac{N-2m}{2}} U(\xi_1+\mu y)\Big|^{p-1} - pU^{p-1}(y)\Big| \cdot \big|\phi_1(y)\big| \\
\le & C\Big|\sum_{j=2}^{k} U\big(y + \mu^{-1}(\xi_1-\xi_j)\big) + \mu^{\frac{N-2m}{2}} U(\xi_1+\mu y) + U(y)\Big|^{p-2} \\
& \cdot \big|\mu^{\frac{N-2m}{2}} U(\xi_1+\mu y) + U(y)\big| \cdot U(y) \cdot \|\phi_1\|_* \\
\le & C\|\phi_1\|_* U^{p-1}(y)\cdot \mu^{\frac{N-2m}{2}} \le C\|\phi_1\|_* \frac{\mu^{\frac{N-2m}{2}}}{1+|y|^{4m}},
\end{aligned}$$

Thus we can proceed the same discussion of **Step 2** in Proposition 2.3 and obtain

$$\|f_1\|_{**} = \|f_1\|_{**(INT_1)} \le \begin{cases} C\|\phi_1\|_* k^{-\frac{N}{q}}, & N \ge 2m+2, \\ C\|\phi_1\|_* \log^{-1} k, & N = 2m+1. \end{cases} \tag{3.18}$$

Similarly, with the application of the estimate of ψ in Proposition 3.2, we can get the estimate of f_3, f_4, for $y \in INT_1$:

$$\begin{aligned}
\big|\widetilde{f}_3(y)\big| \le & CU^{p-1}\mu^{\frac{N-2m}{2}}\|\psi(\phi_1)\|_\infty \\
\le & CU^{p-1}\mu^{\frac{N-2m}{2}}\|\psi(\phi_1)\|_* \\
\le & \begin{cases} C\mu^{\frac{N-2m}{2}}\big(k^{1-\frac{N}{q}} + \|\phi_1\|_*^2\big)\cdot \dfrac{1}{1+|y|^{4m}}, & N \ge 2m+2; \\ C\mu^{\frac{N-2m}{2}}\big(\log^{-1} k + \|\phi_1\|_*^2\big)\cdot \dfrac{1}{1+|y|^{4m}}, & N \ge 2m+1, \end{cases}
\end{aligned}$$

and

$$\|f_3\|_{**} \leq \begin{cases} C\big(k^{1-\frac{N}{q}} + \|\phi_1\|_*^2\big), & N \geq 2m+2; \\ C\big(\log^{-1} k + \|\phi_1\|_*^2\big), & N = 2m+1, \end{cases} \tag{3.19}$$

$$\begin{aligned} \|f_4\|_{**} &\leq \begin{cases} C\|\phi_1\|_*^2 k^{1-\frac{N}{q}} + C\|\psi\|_*^2, & N \geq 2m+2; \\ C\|\phi_1\|_*^2 \log^{-1} k + C\|\psi\|_*^2, & N \geq 2m+1, \end{cases} \\ &\leq \begin{cases} C\|\phi_1\|_*^2 k^{1-\frac{N}{q}} + C\big(\|\phi_1\|_*^2 + k^{1-\frac{N}{q}}\big)^2, & N \geq 2m+2; \\ C\|\phi_1\|_*^2 \log^{-1} k + C\big(\|\phi_1\|_*^2 + \log^{-1} k\big)^2, & N \geq 2m+1. \end{cases} \end{aligned} \tag{3.20}$$

The estimate of f_5 can be directly derived from the error term E

$$\|f_5\|_{**} \leq \begin{cases} Ck^{1-\frac{n}{q}}, & N \geq 2m+2; \\ C\log^{-1} k, & N = 2m+1. \end{cases} \tag{3.21}$$

For f_2, we know that

$$\big|\widetilde{f}_2(y)\big| = \big|\zeta_1(\mu y + \xi_1) - 1\big| \cdot U^{p-1} \cdot |\phi_1| \leq CU^p \|\phi_1\|_*,$$

hence

$$\begin{aligned} \|f_2\|_{**} \leq & C\Big[\int_{|y-\xi_1|>\eta/k} (1+|y|)^{(N+2m)q-2n} \mu^{-\frac{N+2m}{2}q} \big|\widetilde{f}_2^q\big(\frac{n-\xi_1}{\mu}\big)\big| dy\Big]^{1/q} \\ \leq & C\Big[\mu^{-\frac{N+2m}{2}q} \mu^{(N+2m)q-2N} \cdot \int_{\eta/(k\mu)}^{+\infty} r^{(N+2)q-2N-(N-2)pq} dr\Big]^{1/q} \\ \leq & C\mu^{\frac{N+2}{2}-\frac{1}{2q}} < Ck^{-\frac{N}{q}}. \end{aligned} \tag{3.22}$$

Sum all the estimates obtained above, we can see that for $\widehat{\phi}, \widehat{\phi}_1, \widehat{\phi}_2 \in B_\rho(0) \subset X$,

$$\|\mathcal{M}(\widehat{\phi})\|_* \leq C\sum_{i=1}^{5} \|f_i(\widehat{\phi})\|_{**} \leq \begin{cases} C\big(k^{1-\frac{N}{q}} + \|\widehat{\phi}\|_*\big), & N \geq 2m+2; \\ C\big(\log^{-1} k + \|\widehat{\phi}\|_*\big), & N \geq 2m+1. \end{cases}$$

and

$$\begin{aligned} \|\mathcal{M}(\widehat{\phi}_1) - \mathcal{M}(\widehat{\phi}_2)\|_* \leq & C\sum_{i=1}^{4} \|f_i(\widehat{\phi}_1) - f_i(\widehat{\phi}_2)\|_{**} \\ \leq & \begin{cases} C\big(k^{-\frac{N}{q}} + \|\widehat{\phi}_1\|_* + \|\widehat{\phi}_2\|_*\big)\|\widehat{\phi}_1 - \widehat{\phi}_2\|_*, & N \geq 2m+2; \\ C\big(k^{-\frac{N}{q}} + k^{-1}\log^{-1} k + \|\widehat{\phi}_1\|_* + \|\widehat{\phi}_2\|_*\big)\|\widehat{\phi}_1 - \widehat{\phi}_2\|_*, & N = 2m+1, \end{cases} \\ =: & \mathcal{J}\|\widehat{\phi}_1 - \widehat{\phi}_2\|_*, \quad \text{with} \quad \mathcal{J} < 1. \end{aligned}$$

Hence $\mathcal{M}$ is a contraction mapping from $B_\rho(0)$ to $B_\rho(0)$, for k is large enough and ρ is small enough. By the Banach fixed point theorem, there exists a unique solution $\widetilde{\phi}_1$ of the equation (3.15). □

3.3 The Proof of the Main theorem

Since we are looking for the solution with the form:

$$u(y) = U_*(y) + \phi(y),$$

with u being this form, our equation (1.5) can be restated as (2.7). Then for $\phi = \sum_{j=1}^{k} \tilde{\phi}_j + \psi$, by means of the cut-off functions, we split the equation (2.7) into a system equation of $\tilde{\phi}_j, j = 1, 2, ..., k$ and ψ (see (3.6)). In this way, the original problem is reduced to prove the existence of ψ and $\tilde{\phi}_j, j = 1, 2, ..., k$. These are done in Subsection 3.1 and Subsection 3.2 , respectively. Thus for any $k \geq k_0$, we get the sign-changing solution $u_k = U_* + \sum\limits_{j=1}^{k} \phi_j + \psi$ for the poly-harmonic equation (1.5). This completes the proof of the main theorem. $\square$

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