On Toda system with Cartan matrix G_2

Weiwei Ao * Chang-Shou Lin † Juncheng Wei‡

March 5, 2014

Abstract

We consider the following Toda system

$$\Delta u_i + \sum_{j=1}^{2} a_{ij} e^{u_j} = 4\pi\gamma_i\delta_0 \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^{u_i} dx < \infty, \quad \text{for } i = 1, 2,$$

where $\gamma_i > -1$, δ_0 is Dirac measure at 0, and the coefficients a_{ij} is one of the Cartan matrix of rank 2: $A_2, B_2(= C_2), G_2$. In [15] and [1], the authors have gotten the classification and non-degeneracy results of solutions for Cartan matrix A_2 and B_2. In this paper, we consider the G_2 case, we completely classify the solutions and obtain the quantization result as well as the non-degeneracy of solutions for G_2 Toda system.

1 Introduction

Let $A = (a_{ij})$ be a Cartan matrix of rank r. The the non-Abelian gauged nonlinear Schrödinger equations can be reduced to the following Toda system with sources

$$-\Delta u_i = \sum_{j=1}^{r} a_{ij} e^{u_j} + 4\pi \sum_{j=1}^{N_i} \delta_{p_{ij}} \text{ in } \mathbb{R}^2, i = 1, ..., r \tag{1.1}$$

and the non-Abelian Chern-Simons system becomes

$$\Delta u_i + \sum_{j=1}^{r} a_{ij} e^{u_j} = \sum_{k,l} a_{ik} e^{u_k} a_{kl} e^{u_l} + 4\pi \sum_{j=1}^{N_i} \delta_{p_{ij}} \text{ in } \mathbb{R}^2, i = 1, ..., r \tag{1.2}$$

We refer to Chapter 6 of the book [28] for backgrounds. In [15], Lin-Wei-Ye obtained the classification and non-degeneracy of solutions for $SU(n+1)$ Toda system with one single source

$$-\Delta u_i = \sum_{j=1}^{n} a_{ij} e^{u_j} + 4\pi\gamma_i\delta_0 \text{ in } \mathbb{R}^2, i = 1, ..., n \tag{1.3}$$

where $A = (a_{ij})$ is the Cartan matrix for $SU(n+1)$, given by

$$A := (a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & & -1 & 2 \end{pmatrix}. \tag{1.4}$$

*Center for Advanced Study in Theoretical Science, National Taiwan University, Taipei, Taiwan. *weiweiao@gmail.com*

†Taida Institute of Mathematics, Center for Advanced study in Theoretical Science, National Taiwan University, Taipei, Taiwan. *cslin@math.ntu.edu.tw*

‡Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2 and Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong. *jcwei@math.ubc.ca*

However for Cartan matrices of B_n, C_n, G_n as well as exceptional Lie groups $F_2, E_6 - E_8$, there are little results so far. In [1], we consider the Cartan matrix of B_2, and get the classification and non-degeneracy results. The purpose of this paper is to give the classification result for Cartan matrix G_2, in this case, we get a complete classification for Cartan matrix of rank 2. We consider the 2-dimensional (open) Toda system

$$\begin{cases} \triangle u_i + \sum_{j=1}^{2} a_{ij} e^{u_j} = 4\pi\gamma_i\delta_0 \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_i} dx < +\infty \end{cases} \tag{1.5}$$

for $i = 1, 2$, where $\gamma_i > -1$, and $A = (a_{ij})$ is given by

$$A = G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} . \tag{1.6}$$

The A_2 case has been considered by Lin-Wei-Ye (appendix of [15]). (In fact they considered general A_n case. Note that A_n is the $SU(n+1)$ matrix.) And the B_2 case has been considered in [1]. An important ingredient of the B_2 case is that the B_2 Toda system can be embedded into A_3 Toda system under some group action. So we shall concentrate on the case G_2. Note that it is the first exceptional Lie group.

2 Classification for G_2 Toda system

We consider

$$\begin{cases} \triangle u + 2e^u - e^v = 4\pi\gamma_1\delta_0 \quad \text{in } \mathbb{R}^2 \\ \triangle v + 2e^v - 3e^u = 4\pi\gamma_2\delta_0 \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^u < +\infty, \ \int_{\mathbb{R}^2} e^v < +\infty \end{cases} \tag{2.7}$$

An important observation is that Toda system with G_2 can be embedded into Toda system with A_6:

$$\begin{cases} \triangle u_1 + 2e^{u_1} - e^{u_2} = 4\pi\gamma_1^{'}\delta_0 \quad \text{in } \mathbb{R}^2 \\ \triangle u_2 + 2e^{u_2} - e^{u_1} - e^{u_3} = 4\pi\gamma_2^{'}\delta_0 \quad \text{in } \mathbb{R}^2 \\ \triangle u_3 + 2e^{u_3} - e^{u_2} - e^{u_4} = 4\pi\gamma_3^{'}\delta_0 \quad \text{in } \mathbb{R}^2 \\ \triangle u_4 + 2e^{u_4} - e^{u_3} - e^{u_5} = 4\pi\gamma_4^{'}\delta_0 \quad \text{in } \mathbb{R}^2 \\ \triangle u_5 + 2e^{u_5} - e^{u_4} - e^{u_6} = 4\pi\gamma_5^{'}\delta_0 \quad \text{in } \mathbb{R}^2 \\ \triangle u_6 + 2e^{u_6} - e^{u_5} = 4\pi\gamma_6^{'}\delta_0 \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_i} < +\infty, \ i = 1, ..., 6. \end{cases} \tag{2.8}$$

The transformation from (2.8) to (2.7) is the following:

$$\begin{cases} u_1 = u, u_2 = v, u_3 = u + \log 2, u_4 = u + \log 2, u_5 = v, u_6 = u \\ \gamma_1^{'} = \gamma_1, \gamma_2^{'} = \gamma_2, \gamma_3 = \gamma_1^{'}, \gamma_4 = \gamma_1^{'}, \gamma_5 = \gamma_2^{'}, \gamma_6 = \gamma_1^{'} \end{cases} \tag{2.9}$$

In other words, Toda system with G_2 corresponds to solutions of Toda system with A_6 under the following group action

$$u_3 = u_1 + \log 2, u_4 = u_1 + \log 2, u_5 = u_2, u_6 = u_1. \tag{2.10}$$

As a consequence, we just need to take the solutions of Lin-Wei-Ye [15] in A_6 case with $\gamma_3 = \gamma_1, \gamma_4 = \gamma_1, \gamma_5 = \gamma_2, \gamma_6 = \gamma_1$ and compute the solutions under the group action (2.10). Note that the maximal dimension of Toda system with A_6 is 48.

We define

$$\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = G_2^{-1} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.11}$$

Then the system (2.7) is transformed to

$$\begin{cases} \Delta\tilde{w}_1 + e^{2\tilde{w}_1 - \tilde{w}_2} = 4\pi\alpha_1\delta_0, \\ \Delta\tilde{w}_2 + e^{2\tilde{w}_2 - 3\tilde{w}_1} = 4\pi\alpha_2\delta_0, \\ \int_{\mathbb{R}^2} e^{2\tilde{w}_1 - \tilde{w}_2} < +\infty \ , \ \int_{\mathbb{R}^2} e^{2\tilde{w}_2 - 3\tilde{w}_1} < +\infty. \end{cases} \tag{2.12}$$

where $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = G_2^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$. We introduce the notation $(w_1, w_2, w_3, w_4, w_5, w_6)^t = A_6^{-1}(u_1, u_2, u_3, u_4, u_5, u_6)^t$, and $(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \alpha'_5, \alpha'_6)^t = A_6^{-1}(\gamma'_1, \gamma'_2, \gamma'_3, \gamma'_4, \gamma'_5, \gamma'_6)^t$. Then (2.8) is transformed to

$$\Delta w_i + e^{u_i} = 4\pi\alpha'_i\delta_0 \quad \text{in } \mathbb{R}^2, \quad \text{where } \alpha'_i = \sum_{j=1}^{3} a^{ij}\gamma'_j \tag{2.13}$$

for $i = 1, \cdots, 6$.

In order to find the solution of (2.12), we only need to find the solution of (2.13) under the action $w_1 = w_6, w_3 = 2w_1 + \lg 2$. And then

$$\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} \ln 2 \\ 2\ln 2 \end{pmatrix}. \tag{2.14}$$

We have the following classification result:

Theorem 2.1. *Let $\tilde{w}$ be a solution of (2.12), and w satisfies (2.14), then w_1 can be expressed as*

$$e^{-w_1} = |z|^{-2\alpha_1}(\lambda_0 + \sum_{i=1}^{6} \lambda_i |P_i(z)|^2), \tag{2.15}$$

where

$$P_i(z) = z^{\mu'_1 + \cdots + \mu'_i} + \sum_{j=0}^{i-1} c_{ij} z^{\mu'_1 + \cdots + \mu'_j}, \tag{2.16}$$

$\mu'_i = \gamma'_i + 1$, and c_{ij} are complex numbers and $\lambda_i > 0$, satisfy

$$\begin{aligned}
\lambda_0 &= \frac{1}{(2^{11-\frac{1}{2}}\mu_1^2\mu_2(\mu_1+\mu_2)^2(2\mu_1+\mu_2)^2(3\mu_1+\mu_2)(3\mu_1+2\mu_2))^2\lambda_4\lambda_5}, \\
\lambda_1 &= \frac{1}{(2^7\mu_1\mu_2(\mu_1+\mu_2)^2(2\mu_1+\mu_2)(3\mu_1+2\mu_2))^2\lambda_5}, \\
\lambda_2 &= \frac{1}{(2^7\mu_1^2\mu_2(\mu_1+\mu_2)(2\mu_1+\mu_2)(3\mu_1+\mu_2))^2\lambda_4}, \\
\lambda_3 &= \frac{1}{(2^3\mu_1(\mu_1+\mu_2)(2\mu_1+\mu_2))^2}, \\
\lambda_6 &= (2^{3+\frac{1}{2}}\mu_1\mu_2(\mu_1+\mu_2))^2\lambda_4\lambda_5, \\
c_{10} &= \frac{\mu_2(\mu_1+\mu_2)}{(2\mu_1+\mu_2)(3\mu_1+\mu_2)}c_{65}, \;\; c_{20} = \frac{\mu_1\mu_2}{(2\mu_1+\mu_2)(3\mu_1+2\mu_2)}(c_{54}c_{65} - c_{64}), \\
c_{21} &= \frac{\mu_1(3\mu_1+\mu_2)}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)}c_{54}, \\
c_{30} &= \frac{\mu_1(\mu_1+\mu_2)}{2(3\mu_1+\mu_2)(3\mu_1+2\mu_2)}(c_{63} - c_{43}c_{64} - c_{53}c_{65} + c_{43}c_{54}c_{65}), \\
c_{31} &= -\frac{\mu_1(2\mu_1+\mu_2)}{2\mu_2(3\mu_1+2\mu_2)}(c_{53} - c_{43}c_{54}), \;\; c_{32} = \frac{(\mu_1+\mu_2)(2\mu_1+\mu_2)}{2\mu_2(3\mu_1+\mu_2)}c_{43}, \\
c_{40} &= \frac{1}{(2\mu_1+\mu_2)(3\mu_1+2\mu_2)}(-c_{62} + c_{32}c_{63} + c_{42}c_{64} - c_{32}c_{43}c_{64} + c_{52}c_{65} - c_{32}c_{53}c_{65} - c_{42}c_{54}c_{65} + c_{32}c_{43}c_{54}c_{65}), \\
c_{41} &= \frac{\mu_1(3\mu_1+\mu_2)}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)}(c_{52} - c_{32}c_{53} - c_{42}c_{54} + c_{32}c_{43}c_{54}), \\
c_{42} &= \frac{1}{2}c_{32}c_{43}, \\
c_{50} &= \frac{\mu_2(\mu_1+\mu_2)}{(2\mu_1+\mu_2)(3\mu_1+\mu_2)}(c_{61} - c_{41}c_{64} - c_{51}c_{65} + c_{41}c_{54}c_{65} + c_{31}(-c_{63} + c_{53}c_{65} + c_{43}c_{64} - c_{43}c_{54}c_{65}) \\
&+ c_{21}(-c_{62} + c_{42}c_{64} + c_{52}c_{65} - c_{42}c_{54}c_{65} + c_{32}(c_{63} - c_{43}c_{64} - c_{53}c_{65} + c_{43}c_{54}c_{65}))),
\end{aligned} \tag{2.17}$$

$$c_{51} = \frac{1}{2}(c_{21}c_{52} + c_{31}c_{53} - c_{21}c_{32}c_{53} + c_{41}c_{54} - c_{21}c_{42}c_{54} - c_{31}c_{43}c_{54} + c_{21}c_{32}c_{43}c_{54}),$$

$$c_{60} = \frac{\mu_1(2\mu_1+\mu_2)(\mu_1+\mu_2)c_{63}^2 + 4\mu_2(\mu_1+\mu_2)(3\mu_1+2\mu_2)c_{61}c_{65} - 4\mu_2\mu_1(3\mu_1+\mu_2)c_{62}c_{64}}{4(2\mu_1+\mu_2)(3\mu_1+\mu_2)(3\mu_1+2\mu_2)},$$

$$c_{63} = \frac{1}{2\mu_2(\mu_1+\mu_2)}(-4\mu_2(3\mu_1+\mu_2)c_{52} + (2\mu_1+\mu_2)(\mu_1+\mu_2)c_{43}c_{53}),$$

$$c_{64} = -\frac{2\mu_1+\mu_2}{2\mu_2}(c_{53} - c_{43}c_{54}), \ c_{65} = \frac{2\mu_1+\mu_2}{2\mu_2}c_{43},$$

where $\mu_i = \gamma_i + 1$ *for* $i = 1, 2$ *and the solutions depend on* 14 *parameters* $\lambda_4, \lambda_5, c_{43}, c_{52}, c_{53}, c_{54}, c_{61}$ *and* c_{62}. *Moreover,*

- *if* $\mu_1, \mu_2 \in \mathbb{N}$, *the solution space is a fourteen dimensional smooth manifold;*
- *if* $\mu_1 \in \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_2 \in \mathbb{N}$, *then* $c_{52} = c_{54} = c_{62} = 0$, *the solution manifold is eight dimensional;*
- *if* $\mu_1 \in \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_2 \notin \mathbb{N}$, *then* $c_{52} = c_{54} = c_{61} = c_{62} = 0$, *the solution manifold is six dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \in \mathbb{N}$, *and* $2\mu_1 \in \mathbb{N}$, *then* $c_{43} = c_{61} = c_{62} = 0$, *the solution manifold is eight dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \in \mathbb{N}$, *and* $3\mu_1 \in \mathbb{N}$, *then* $c_{52} = c_{53} = c_{43} = 0$, *the solution manifold is eight dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \in \mathbb{N}$, *and* $2\mu_1 \notin \mathbb{N}, 3\mu_1 \notin \mathbb{N}$, *then* $c_{43} = c_{52} = c_{53} = c_{62} = c_{62} = 0$, *the solution manifold is four dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_1 \in \mathbb{N}, \mu_1 + \mu_2 \in \mathbb{N}$, *then* $c_{43} = c_{54} = c_{52} = c_{61} = 0$, *the solution manifold is six dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_1 \in \mathbb{N}, \mu_1 + 2\mu_2 \in \mathbb{N}$, *then* $c_{43} = c_{54} = c_{52} = c_{62} = 0$, *the solution manifold is six dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_1 \in \mathbb{N}, \mu_1 + \mu_2 \notin \mathbb{N}, \mu_1 + 2\mu_2 \notin \mathbb{N}$, *then* $c_{43} = c_{54} = c_{52} = c_{61} = c_{62} = 0$, *the solution manifold is four dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $3\mu_1 \in \mathbb{N}, 2\mu_2 \in \mathbb{N}$, *then* $c_{43} = c_{54} = c_{52} = c_{62} = c_{53} = 0$, *the solution manifold is four dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $3\mu_1 \in \mathbb{N}, 2\mu_2 \notin \mathbb{N}, \mu_1 + \mu_2 \in \mathbb{N}$, *then* $c_{43} = c_{54} = c_{61} = c_{62} = c_{53} = 0$, *the solution manifold is four dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $3\mu_1 \in \mathbb{N}, 2\mu_2 \notin \mathbb{N}, \mu_1 + \mu_2 \notin \mathbb{N}$, *then* $c_{43} = c_{54} = c_{61} = c_{62} = c_{53} = c_{52} = 0$, *the solution manifold is two dimensional. All the solutions must be radial;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_1 \notin \mathbb{N}, 3\mu_1 \notin \mathbb{N}, \mu_1 + \mu_2 \in \mathbb{N}$, *then* $c_{43} = c_{54} = c_{53} = c_{62} = c_{61} = c_{52} = 0$, *the solution manifold is two dimensional. All the solutions must be radial;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_1 \notin \mathbb{N}, 3\mu_1 \notin \mathbb{N}, \mu_1 + \mu_2 \notin \mathbb{N}, 3\mu_1 + \mu_2 \in \mathbb{N}$, *then* $c_{43} = c_{54} = c_{53} = c_{61} = c_{52} = 0$, *the solution manifold is four dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_1 \notin \mathbb{N}, 3\mu_1 \notin \mathbb{N}, \mu_1 + \mu_2 \notin \mathbb{N}, 3\mu_1 + \mu_2 \notin \mathbb{N}, 2\mu_1 + \mu_2 \in \mathbb{N}$, *then* $c_{43} = c_{54} = c_{53} = c_{62} = c_{61} = 0$, *the solution manifold is four dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_1 \notin \mathbb{N}, 3\mu_1 \notin \mathbb{N}, \mu_1 + \mu_2 \notin \mathbb{N}, 3\mu_1 + \mu_2 \notin \mathbb{N}, 2\mu_1 + \mu_2 \notin \mathbb{N}, 3\mu_1 + 2\mu_2 \in \mathbb{N}$, *then* $c_{43} = c_{54} = c_{53} = c_{62} = c_{52} = 0$, *the solution manifold is four dimensional;*
- *if* $\mu_1 \notin \mathbb{N}$, $\mu_2 \notin \mathbb{N}$, *and* $2\mu_1 \notin \mathbb{N}, 3\mu_1 \notin \mathbb{N}, \mu_1 + \mu_2 \notin \mathbb{N}, 3\mu_1 + \mu_2 \notin \mathbb{N}, 2\mu_1 + \mu_2 \notin \mathbb{N}, 3\mu_1 + 2\mu_2 \notin \mathbb{N}$, *then* $c_{43} = c_{54} = c_{53} = c_{62} = c_{52} = c_{61} = 0$, *the solution manifold is two dimensional. All the solutions are radial.*

Remark 2.2. *The maximal dimension of the space of the solutions is 14, which coincides with the dimension of the Lie algebra associated with* G_2.

Proof:

By Theorem 1.1 of Lin-Wei-Ye [15] with $n = 6$, the solution for (2.13) can be expressed as

$$e^{-w_1} = f = |z|^{-2\alpha_1'}(\lambda_0 + \sum_{i=1}^{6} \lambda_i |P_i(z)|^2), \tag{2.18}$$

where

$$P_i(z) = z^{\mu_1' + \cdots + \mu_i'} + \sum_{j=0}^{i-1} c_{ij} z^{\mu_1' + \cdots + \mu_j'}, \tag{2.19}$$

$\mu_i' = \gamma_i' + 1$, and c_{ij} are complex numbers and $\lambda_i > 0$, satisfy

$$\lambda_0 \cdots \lambda_6 = 2^{-6(6+1)} \Pi_{1 \le i \le j \le 6} (\sum_{k=i}^{j} \mu_k')^{-2}. \tag{2.20}$$

From the formula (5.16) in [15], if we denote by $L_i = \sqrt{\lambda_i}$, we have

$$e^{-w_k} = 2^{k(k-1)} det_k(f) \ \text{ for } 2 \le k \le 6. \tag{2.21}$$

So from $w_1 = w_6$, we can get the following:

$$
\begin{aligned}
L_0 &= \frac{1}{2^{11-\frac{1}{2}} \mu_1^2 \mu_2 (\mu_1+\mu_2)^2 (2\mu_1+\mu_2)^2 (3\mu_1+\mu_2)(3\mu_1+2\mu_2) L_4 L_5}, \\
L_1 &= \frac{1}{2^7 \mu_1 \mu_2 (\mu_1+\mu_2)^2 (2\mu_1+\mu_2)(3\mu_1+2\mu_2) L_5}, \\
L_2 &= \frac{1}{2^7 \mu_1^2 \mu_2 (\mu_1+\mu_2)(2\mu_1+\mu_2)(3\mu_1+\mu_2) L_4}, \\
L_3 &= \frac{1}{2^3 \mu_1 (\mu_1+\mu_2)(2\mu_1+\mu_2)}, \ L_6 = 2^{3+\frac{1}{2}} \mu_1 \mu_2 (\mu_1+\mu_2) L_4 L_5, \\
c_{10} &= \frac{\mu_2(\mu_1+\mu_2)}{(2\mu_1+\mu_2)(3\mu_1+\mu_2)} c_{65}, \ c_{20} = \frac{\mu_1 \mu_2}{(2\mu_1+\mu_2)(3\mu_1+2\mu_2)} (c_{54}c_{65} - c_{64}), \\
c_{21} &= \frac{\mu_1(3\mu_1+\mu_2)}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)} c_{54}, \\
c_{30} &= \frac{\mu_1(\mu_1+\mu_2)}{2(3\mu_1+\mu_2)(3\mu_1+2\mu_2)} (c_{63} - c_{43}c_{64} - c_{53}c_{65} + c_{43}c_{54}c_{65}), \\
c_{31} &= -\frac{\mu_1(2\mu_1+\mu_2)}{2\mu_2(3\mu_1+2\mu_2)} (c_{53} - c_{43}c_{54}), \ c_{32} = \frac{(\mu_1+\mu_2)(2\mu_1+\mu_2)}{2\mu_2(3\mu_1+\mu_2)} c_{43}, \\
c_{40} &= \frac{1}{(2\mu_1+\mu_2)(3\mu_1+2\mu_2)} (-c_{62} + c_{32}c_{63} + c_{42}c_{64} - c_{32}c_{43}c_{64} + c_{52}c_{65} - c_{32}c_{53}c_{65} - c_{42}c_{54}c_{65} + c_{32}c_{43}c_{54}c_{65}), \\
c_{41} &= \frac{\mu_1(3\mu_1+\mu_2)}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)} (c_{52} - c_{32}c_{53} - c_{42}c_{54} + c_{32}c_{43}c_{54}), \\
c_{42} &= \frac{1}{2} c_{32}c_{43}, \\
c_{50} &= \frac{\mu_2(\mu_1+\mu_2)}{(2\mu_1+\mu_2)(3\mu_1+\mu_2)} (c_{61} - c_{41}c_{64} - c_{51}c_{65} + c_{41}c_{54}c_{65} + c_{31}(-c_{63} + c_{53}c_{65} + c_{43}(c_{64} - c_{54}c_{65})) \\
&+ c_{21}(-c_{62} + c_{42}c_{64} + c_{52}c_{65} - c_{42}c_{54}c_{65} + c_{32}(c_{63} - c_{43}c_{64} - c_{53}c_{65} + c_{43}c_{54}c_{65}))), \\
c_{51} &= \frac{1}{2}(c_{21}c_{52} + c_{31}c_{53} - c_{21}c_{32}c_{53} + c_{41}c_{54} - c_{21}c_{42}c_{54} - c_{31}c_{43}c_{54} + c_{21}c_{32}c_{43}c_{54}), \\
c_{60} &= \frac{(\mu_1(2\mu_1+\mu_2)(\mu_1+\mu_2)c_{63}^2 + 4\mu_2(\mu_1+\mu_2)(3\mu_1+2\mu_2)c_{61}c_{65} - 4\mu_2\mu_1(3\mu_1+\mu_2)c_{62}c_{64})}{4(2\mu_1+\mu_2)(3\mu_1+\mu_2)(3\mu_1+2\mu_2)},
\end{aligned}
$$

and from $w_3 = 2w_1 + \ln 2$, one can get

$$c_{63} = \frac{1}{2\mu_2(\mu_1+\mu_2)}(-4c_{52}\mu_2(3\mu_1+\mu_2) + c_{43}c_{53}(2\mu_1+\mu_2)(\mu_1+\mu_2)),$$
$$c_{64} = -\frac{2\mu_1+\mu_2}{2\mu_2}(c_{53} - c_{43}c_{54}),\ c_{65} = \frac{2\mu_1+\mu_2}{2\mu_2}c_{43}.$$

So we can get that the solutions satisfy (2.15) and (2.17) and depend on 14 parameters $c_{43}, c_{52}, c_{53}, c_{54}, c_{61}, c_{62}, \lambda_4$ and λ_5. The other parts of the theorem follow from [15].

□

From Theorem 2.1,we can get that the solutions of (2.13) depend on 14 parameters $\lambda_4, \lambda_5, c_{43}, c_{52}, c_{53}, c_{54}, c_{61}$ and c_{62}. By formula (5.16) in [15], we get the radial solution of this system $(-w_{1,0}, -w_{2,0})$ can be written as

$$\begin{aligned}
\rho_{1,G}^{-1} &= r^{2\alpha_1'}e^{-w_{1,0}} \\
&= \lambda_0 + \lambda_1 r^{2\mu_1} + \lambda_2 r^{2(\mu_1+\mu_2)} + \lambda_3 r^{2(2\mu_1+\mu_2)} + \lambda_4 r^{2(3\mu_1+\mu_2)} + \lambda_5 r^{2(3\mu_1+2\mu_2)} + \lambda_6 r^{2(4\mu_1+2\mu_2)}, \\
\rho_{2,G}^{-1} &= r^{2\alpha_2'}e^{-w_{2,0}} \\
&= 4\Big[\lambda_0\mu_1^2\lambda_1 + r^{4(\mu_1+\mu_2)}(4r^{2\mu_1}(\lambda_0\lambda_6(2\mu_1+\mu_2)^2 + \lambda_1\lambda_5(\mu_1+\mu_2)^2) + \lambda_0\lambda_5(3\mu_1+2\mu_2)^2 \\
&\quad + r^{4\mu_1}(\lambda_1\lambda_6(3\mu_1+2\mu_2)^2 + \mu_1^2\lambda_3\lambda_4) + \mu_1^2\lambda_2(\lambda_3 + 4\lambda_4 r^{2\mu_1})) \\
&\quad + r^{2\mu_2}(\lambda_0(\lambda_2(\mu_1+\mu_2)^2 + \lambda_3(2\mu_1+\mu_2)^2 r^{2\mu_1} + \lambda_4(3\mu_1+\mu_2)^2 r^{4\mu_1}) \\
&\quad + \lambda_1 r^{2\mu_1}(\mu_2^2\lambda_2 + \lambda_3(\mu_1+\mu_2)^2 r^{2\mu_1} + \lambda_4(2\mu_1+\mu_2)^2 r^{4\mu_1})) \\
&\quad + r^{6(\mu_1+\mu_2)}(r^{2\mu_1}(\lambda_2\lambda_6(3\mu_1+\mu_2)^2 + \lambda_3\lambda_5(\mu_1+\mu_2)^2) + \lambda_2\lambda_5(2\mu_1+\mu_2)^2 \\
&\quad + r^{4\mu_1}(\lambda_3\lambda_6(2\mu_1+\mu_2)^2 + \mu_2^2\lambda_4\lambda_5) + \lambda_4\lambda_6(\mu_1+\mu_2)^2 r^{6\mu_1}) + \mu_1^2\lambda_5\lambda_6 r^{12\mu_1+8\mu_2}\Big],
\end{aligned}$$

where the parameters are defined in (2.17), and the radial solution of (2.12) can be expressed as $\begin{pmatrix} e^{-\tilde{w}_{1,0}} \\ e^{-\tilde{w}_{2,0}} \end{pmatrix} = \begin{pmatrix} 2e^{-w_{1,0}} \\ 4e^{-w_{2,0}} \end{pmatrix}$.

Corollary 2.3. *(Nondegeneracy) Assume $\gamma_1, \gamma_2 \in \mathbb{N}$. The set of solutions corresponding to the linearized operator of (2.12) is exactly fourteen dimensional. More precisely, if $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ satisfies $|\phi(z)| \le C(1+|z|)^\alpha$ for some $0 \le \alpha < 1$, and*

$$\begin{cases} \Delta\phi_1 + e^{2\tilde{w}_{1,0}-\tilde{w}_{2,0}}(2\phi_1 - \phi_2) = 0 \\ \Delta\phi_2 + e^{2\tilde{w}_{2,0}-3\tilde{w}_{1,0}}(2\phi_2 - 3\phi_1) = 0. \end{cases} \tag{2.22}$$

Then ϕ belongs to the following linear space $\mathcal{K}$: the span of

$$\{w_{\lambda_4}, w_{\lambda_5}, w_{c_{43,1}}, w_{c_{43,2}}, w_{c_{52,1}}, w_{c_{52,2}}, w_{c_{53,1}}, w_{c_{53,2}}, w_{c_{54,1}}, w_{c_{54,2}}, w_{c_{61,1}}, w_{c_{61,2}}, w_{c_{62,1}}, w_{c_{62,2}}\}, \tag{2.23}$$

where we denote by $w_X = \begin{pmatrix} \frac{\partial w_{1,0}}{\partial X} \\ \frac{\partial w_{2,0}}{\partial X} \end{pmatrix}$, and $X \in \{c_{43,i}, c_{52,i}, c_{53,i}, c_{54,i}, c_{61,i}, c_{62,i}, \lambda_4$ and $\lambda_5\}$ for $i = 1, 2$, and

$$\begin{aligned}
w_{1,\lambda_4} &= \rho_{1,G}\Big(r^{2(3\mu_1+\mu_2)} + 2^7\lambda_5 r^{4(2\mu_1+\mu_2)}\mu_1^2\mu_2^2(\mu_1+\mu_2)^2 - \frac{r^{2(\mu_1+\mu_2)}}{2^{14}\lambda_4^2\mu_1^4\mu_2^2(\mu_1+\mu_2)^2(2\mu_1+\mu_2)^2(3\mu_1+\mu_2)^2} \\
&\quad - \frac{1}{2^{21}\lambda_4^2\lambda_5\mu_1^4\mu_2^2(\mu_1+\mu_2)^4(2\mu_1+\mu_2)^4(3\mu_1+\mu_2)^2(3\mu_1+2\mu_2)^2}\Big), \\
w_{2,\lambda_4} &= \frac{4\rho_{2,G}}{2^{25}\mu_2^4(\mu_1+\mu_2)^8}\Big[- \frac{1}{\lambda_4^2\lambda_5^2(2\mu_1+\mu_2)^6(3\mu_1+\mu_2)^2(3\mu_1^2+2\mu_1\mu_2)^4} \\
&\quad + \frac{2^{21}\mu_2^2\lambda_5(\mu_1+\mu_2)^6 r^{6(\mu_1+\mu_2)}(2^{14}\mu_1^4\mu_2^4\lambda_4^2(\mu_1+\mu_2)^2(3\mu_1+\mu_2)^2 r^{4\mu_1}(256\mu_1^2\lambda_4(\mu_1+\mu_2)^4 r^{2\mu_1} + 3) - 1)}{\mu_1^4\lambda_4^2(3\mu_1+\mu_2)^2}
\end{aligned}$$

$$
+\frac{3\times 2^{14}\mu_2^2(\mu_1+\mu_2)^4 r^{4(\mu_1+\mu_2)}(2^{14}\mu_1^4\mu_2^2\lambda_4^2(\mu_1+\mu_2)^2(2\mu_1+\mu_2)^2(3\mu_1+\mu_2)^2 r^{4\mu_1}-1)}{\mu_1^4\lambda_4^2(2\mu_1+\mu_2)^4(3\mu_1+\mu_2)^2}
$$

$$
+\frac{2(\mu_1+\mu_2)^2 r^{2\mu_2}(-\frac{(\mu_1+\mu_2)^2}{\lambda_4^3(2\mu_1+\mu_2)^6(3\mu_1+\mu_2)^4}-\frac{192\mu_1^2\mu_2^2 r^{2\mu_1}}{\lambda_4^2(2\mu_1+\mu_2)^4(3\mu_1+\mu_2)^2}+2^{20}\mu_1^6\mu_2^2(\mu_1+\mu_2)^2 r^{6\mu_1})}{\mu_1^8\lambda_5(3\mu_1+2\mu_2)^2}
$$

$$
+2^{42}\mu_1^4\mu_2^6\lambda_5^2(\mu_1+\mu_2)^{10} r^{12\mu_1+8\mu_2}\Big],
$$

$$
w_{1,\lambda_5} = \rho_{1,G}\Big[r^{6\mu_1+4\mu_2}+2^7\lambda_4 r^{4(2\mu_1+\mu_2)}\mu_1^2\mu_2^2(\mu_1+\mu_2)^2-\frac{r^{2\mu_1}}{2^{14}\lambda_5^2\mu_1^2\mu_2^2(\mu_1+\mu_2)^4(2\mu_1+\mu_2)^2(3\mu_1+2\mu_2)^2}
$$

$$
-\frac{1}{2^{21}\lambda_4\lambda_5^2\mu_1^4\mu_2^2(\mu_1+\mu_2)^4(2\mu_1+\mu_2)^4(3\mu_1+\mu_2)^2(3\mu_1+2\mu_2)^2}\Big],
$$

$$
w_{2,\lambda_5} = 4\rho_{2,G}\Big[3\mu_2^2\lambda_4 r^{10\mu_1+6\mu_2}+2^8\mu_1^4\mu_2^2\lambda_4\lambda_5(\mu_1+\mu_2)^2 r^{12\mu_1+8\mu_2}+2^7\mu_1^2\mu_2^2\lambda_4^2(\mu_1+\mu_2)^4 r^{6(2\mu_1+\mu_2)}
$$

$$
+\frac{3r^{8\mu_1+6\mu_2}}{2^7\mu_1^2(2\mu_1+\mu_2)^2}-\frac{\lambda_4 r^{2(3\mu_1+\mu_2)}}{2^{14}\mu_1^2\mu_2^2\lambda_5^2(\mu_1+\mu_2)^4(3\mu_1+2\mu_2)^2}
$$

$$
-\frac{3r^{2(2\mu_1+\mu_2)}}{2^{21}\mu_1^4\mu_2^2\lambda_5^2(\mu_1+\mu_2)^4(2\mu_1+\mu_2)^4(3\mu_1+2\mu_2)^2}-\frac{r^{2\mu_2}}{2^{35}\mu_1^8\mu_2^4\lambda_4^2\lambda_5^2(\mu_1+\mu_2)^4(2\mu_1+\mu_2)^6(3\mu_1+\mu_2)^4(3\mu_1+2\mu_2)^2}
$$

$$
-\frac{1}{2^{34}\mu_1^6\mu_2^4\lambda_4(\mu_1+\mu_2)^8(3\mu_1+\mu_2)^2}\Big(\frac{\mu_1^2}{\lambda_5^3(2\mu_1+\mu_2)^6(3\mu_1+2\mu_2)^4}+\frac{3\ 2^6\mu_2^2(\mu_1+\mu_2)^2 r^{2(\mu_1+\mu_2)}}{\lambda_5^2(2\mu_1+\mu_2)^4(3\mu_1+2\mu_2)^2}
$$

$$
-2^{20}\mu_1^2\mu_2^2(\mu_1+\mu_2)^6 r^{6(\mu_1+\mu_2)}\Big)\Big],
$$

$$
w_{1,c_{43,1}} = \rho_{1,G}\Big(\lambda_4 r^{5\mu_1+2\mu_2}+\frac{\lambda_6(2\mu_1+\mu_2)r^{7\mu_1+4\mu_2}}{2\mu_2}+\frac{\lambda_1(\mu_1+\mu_2)r^{\mu_1}}{2(3\mu_1+\mu_2)}+\frac{\lambda_3(\mu_1+\mu_2)(2\mu_1+\mu_2)r^{3\mu_1+2\mu_2}}{2\mu_2(3\mu_1+\mu_2)}\Big)\cos\mu_1\theta,
$$

$$
w_{2,c_{43,1}} = \frac{4\rho_{2,G} r^{\mu_1+2\mu_2}}{2\mu_2(3\mu_1+\mu_2)}\Big[(\mu_1+\mu_2)^2(\lambda_0\lambda_3(2\mu_1+\mu_2)^2+\lambda_1\mu_2^2\lambda_2)+r^{4\mu_1}\Big(2r^{2\mu_2}(\lambda_0\lambda_6(3\mu_1+\mu_2)(3\mu_1+2\mu_2)(2\mu_1+\mu_2)^2
$$

$$
+\mu_2(\lambda_1\lambda_5(\mu_1+\mu_2)^2(3\mu_1+2\mu_2)+2\mu_1^2\lambda_2\lambda_4(3\mu_1+\mu_2)))+3\lambda_1\mu_2\lambda_4(\mu_1+\mu_2)(2\mu_1+\mu_2)(3\mu_1+\mu_2)\Big)
$$

$$
+2\mu_2(2\mu_1+\mu_2)r^{2\mu_1}(\lambda_0\lambda_4(3\mu_1+\mu_2)^2+\lambda_1\lambda_3(\mu_1+\mu_2)^2)+(2\mu_1+\mu_2)r^{2(3\mu_1+\mu_2)}\Big(2(\mu_1+\mu_2)(\lambda_1\lambda_6(3\mu_1+2\mu_2
$$

$$
+\mu_1^2\lambda_3\lambda_4)+(2\mu_1+\mu_2)r^{2\mu_2}(\lambda_2\lambda_6(3\mu_1+\mu_2)^2+\lambda_3\lambda_5(\mu_1+\mu_2)^2)\Big)
$$

$$
+2(\mu_1+\mu_2)(3\mu_1+\mu_2)r^{4(2\mu_1+\mu_2)}(\lambda_3\lambda_6(2\mu_1+\mu_2)^2+\mu_2^2\lambda_4\lambda_5)
$$

$$
+3\mu_2\lambda_4\lambda_6(\mu_1+\mu_2)(2\mu_1+\mu_2)(3\mu_1+\mu_2)r^{10\mu_1+4\mu_2}\Big]\cos\mu_1\theta,
$$

$$
w_{1,c_{52,1}} = \frac{\rho_{1,G} r^{2\mu_1+\mu_2}}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)}\Big[(r^{2\mu_1}(\mu_1\lambda_4(3\mu_1+\mu_2)+(3\mu_1+2\mu_2)r^{2\mu_2}(\lambda_5(\mu_1+\mu_2)
$$

$$
-2\lambda_6(3\mu_1+\mu_2)r^{2\mu_1}))-\mu_1\lambda_3(\mu_1+\mu_2))\Big]\cos(2\mu_1+\mu_2)\theta,
$$

$$
w_{2,c_{52,1}} = \frac{4\rho_{2,G} r^{2\mu_1+\mu_2}}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)}\Big[r^{2\mu_2}\Big(-2(3\mu_1+2\mu_2)r^{2\mu_1}(2\lambda_0\lambda_6(2\mu_1+\mu_2)^2(3\mu_1+\mu_2)-\lambda_1\mu_2\lambda_5(\mu_1+\mu_2)^2)
$$

$$
+\lambda_0\lambda_5(\mu_1+\mu_2)^2(3\mu_1+2\mu_2)^2-2(\mu_1+\mu_2)(3\mu_1+\mu_2)r^{4\mu_1}(\lambda_1\lambda_6(3\mu_1+2\mu_2)^2+\mu_1^2\lambda_3\lambda_4)
$$

$$
+\mu_1^2\lambda_2(\lambda_3(\mu_1+\mu_2)^2-2\mu_2\lambda_4(3\mu_1+\mu_2)r^{2\mu_1})\Big)+\mu_1^2(\lambda_0\lambda_4(3\mu_1+\mu_2)^2+\lambda_1\lambda_3(\mu_1+\mu_2)^2)
$$

$$
+\mu_1 r^{4(\mu_1+\mu_2)}\Big(-2(3\mu_1+2\mu_2)(\lambda_2\lambda_6(3\mu_1+\mu_2)^2+\lambda_3\lambda_5(\mu_1+\mu_2)^2)
$$

$$
-2(\mu_1+\mu_2)r^{2\mu_1}(\lambda_3\lambda_6(2\mu_1+\mu_2)^2+\mu_2^2\lambda_4\lambda_5)+3\lambda_4\lambda_6(\mu_1+\mu_2)(3\mu_1+\mu_2)(3\mu_1+2\mu_2)r^{4\mu_1}\Big)
$$

$$
+3\mu_1\lambda_5\lambda_6(\mu_1+\mu_2)(3\mu_1+\mu_2)(3\mu_1+2\mu_2)r^{8\mu_1+6\mu_2}\Big]\cos(2\mu_1+\mu_2)\theta,
$$

$$\begin{aligned}
w_{1,c_{53,1}} &= \rho_{1,G}\Big[\frac{\mu_1\lambda_2 r^{\mu_1+\mu_2}}{2(3\mu_1+2\mu_2)} - \frac{\mu_1\lambda_3(2\mu_1+\mu_2)r^{3\mu_1+\mu_2}}{2\mu_2(3\mu_1+2\mu_2)} + \lambda_5 r^{5\mu_1+3\mu_2} - \frac{\lambda_6(2\mu_1+\mu_2)r^{7\mu_1+3\mu_2}}{2\mu_2}\Big]\cos(\mu_1+\mu_2)\theta,\\
w_{2,c_{53,1}} &= \frac{4\rho_{2,G}}{2\mu_2(3\mu_1+2\mu_2)}\Big[r^{3(\mu_1+\mu_2)}\Big(2r^{2\mu_1}(\mu_2(2\lambda_1\lambda_5(\mu_1+\mu_2)^2(3\mu_1+2\mu_2)+\mu_1^2\lambda_2\lambda_4(3\mu_1+\mu_2))\\
&\quad -\lambda_0\lambda_6(2\mu_1+\mu_2)^2(3\mu_1+\mu_2)(3\mu_1+2\mu_2)) + 2\mu_2(2\mu_1+\mu_2)(\lambda_0\lambda_5(3\mu_1+2\mu_2)^2+\mu_1^2\lambda_2\lambda_3)\\
&\quad -(2\mu_1+\mu_2)^2 r^{4\mu_1}(\lambda_1\lambda_6(3\mu_1+2\mu_2)^2+\mu_1^2\lambda_3\lambda_4)\Big) - \mu_1^2 r^{\mu_1+\mu_2}(\lambda_0\lambda_3(2\mu_1+\mu_2)^2+\lambda_1\mu_2^2\lambda_2)\\
&\quad -\mu_1 r^{5(\mu_1+\mu_2)}(2(2\mu_1+\mu_2)r^{2\mu_1}(\lambda_2\lambda_6(3\mu_1+\mu_2)^2+\lambda_3\lambda_5(\mu_1+\mu_2)^2) - 3\mu_2\lambda_2\lambda_5(2\mu_1+\mu_2)(3\mu_1+2\mu_2)\\
&\quad +2(3\mu_1+2\mu_2)r^{4\mu_1}(\lambda_3\lambda_6(2\mu_1+\mu_2)^2+\mu_2^2\lambda_4\lambda_5)) + 3\mu_1\mu_2\lambda_5\lambda_6(2\mu_1+\mu_2)(3\mu_1+2\mu_2)r^{11\mu_1+7\mu_2}\Big]\cos(\mu_1+\mu_2)
\end{aligned}$$

$$\begin{aligned}
w_{1,c_{54,1}} &= \rho_{1,G}\Big[\lambda_5 r^{6\mu_1+3\mu_2} + \frac{\lambda_2 r^{2\mu_1+\mu_2}\mu_1(3\mu_1+\mu_2)}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)}\Big]\cos\mu_2\theta,\\
w_{2,c_{54,1}} &= \frac{4\rho_{2,G}r^{\mu_2}}{(\mu_1+\mu_2)(3\mu_1+2\mu_2)}\Big[\lambda_0(\mu_1+\mu_2)(3\mu_1+\mu_2)(\mu_1^2\lambda_2+\lambda_5(3\mu_1+2\mu_2)^2 r^{2(2\mu_1+\mu_2)})\\
&\quad +r^{2(2\mu_1+\mu_2)}\Big(\lambda_5(\mu_1+\mu_2)^2(3\mu_1+2\mu_2)r^{2\mu_1}(2\lambda_1(2\mu_1+\mu_2)+\mu_1 r^{2(\mu_1+\mu_2)}(\lambda_3+\lambda_6 r^{2(2\mu_1+\mu_2)}))\\
&\quad +\mu_1\lambda_2(\mu_1(3\mu_1+\mu_2)(\lambda_3(\mu_1+\mu_2)+2\lambda_4(2\mu_1+\mu_2)r^{2\mu_1})\\
&\quad +r^{2(\mu_1+\mu_2)}(6\lambda_5(\mu_1+\mu_2)(2\mu_1+\mu_2)^2+\lambda_6(3\mu_1+\mu_2)^2(3\mu_1+2\mu_2)r^{2\mu_1}))\Big)\Big]\cos\mu_2\theta,
\end{aligned}$$

$$\begin{aligned}
w_{1,c_{61,1}} &= \rho_{1,G}\Big[\lambda_6 r^{5\mu_1+2\mu_2} + \frac{\lambda_5 r^{3\mu_1+2\mu_2}\mu_2(\mu_1+\mu_2)}{(2\mu_1+\mu_2)(3\mu_1+\mu_2)}\Big]\cos(3\mu_1+2\mu_2)\theta,\\
w_{2,c_{61,1}} &= \frac{4\rho_{2,G}r^{3\mu_1+2\mu_2}}{(2\mu_1+\mu_2)(3\mu_1+\mu_2)}\Big[2\mu_1(\lambda_0\lambda_6(2\mu_1+\mu_2)^2(3\mu_1+\mu_2)-\lambda_1\mu_2\lambda_5(\mu_1+\mu_2)^2)\\
&\quad -r^{2\mu_2}\Big(\mu_2\lambda_2(2\mu_1+\mu_2)(\lambda_5(\mu_1+\mu_2)^2+\lambda_6(3\mu_1+\mu_2)^2 r^{2\mu_1})\\
&\quad +\lambda_3(\mu_1+\mu_2)(2\mu_1+\mu_2)r^{2\mu_1}(\mu_2\lambda_5(\mu_1+\mu_2)+\lambda_6(2\mu_1+\mu_2)(3\mu_1+\mu_2)r^{2\mu_1})\\
&\quad +\lambda_4(\mu_1+\mu_2)(3\mu_1+\mu_2)r^{4\mu_1}(\mu_2^2\lambda_5+\lambda_6(2\mu_1+\mu_2)^2 r^{2\mu_1})\Big)\\
&\quad -6\mu_1^2\lambda_5\lambda_6(\mu_1+\mu_2)(2\mu_1+\mu_2)r^{6\mu_1+4\mu_2}\Big]\cos(3\mu_1+2\mu_2)\theta,
\end{aligned}$$

$$\begin{aligned}
w_{1,c_{62,1}} &= \rho_{1,G}\Big[\lambda_6 r^{5\mu_1+3\mu_2} - \frac{\lambda_4 r^{3\mu_1+\mu_2}\mu_1\mu_2}{(2\mu_1+\mu_2)(3\mu_1+2\mu_2)}\Big]\cos(3\mu_1+\mu_2)\theta,\\
w_{2,c_{62,1}} &= \frac{4\rho_{2,G}r^{3\mu_1+\mu_2}}{(2\mu_1+\mu_2)(3\mu_1+2\mu_2)}\Big[r^{2\mu_2}(2(\mu_1+\mu_2)(\lambda_0\lambda_6(2\mu_1+\mu_2)^2(3\mu_1+2\mu_2)+\mu_1^2\mu_2\lambda_2\lambda_4)\\
&\quad +\mu_2(2\mu_1+\mu_2)r^{2\mu_1}(\lambda_1\lambda_6(3\mu_1+2\mu_2)^2+\mu_1^2\lambda_3\lambda_4))\\
&\quad +\mu_1^2\lambda_1\mu_2\lambda_4(2\mu_1+\mu_2) - \mu_1 r^{4(\mu_1+\mu_2)}((3\mu_1+2\mu_2)(\lambda_3\lambda_6(2\mu_1+\mu_2)^2+\mu_2^2\lambda_4\lambda_5)\\
&\quad +6\lambda_4\lambda_6(\mu_1+\mu_2)^2(2\mu_1+\mu_2)r^{2\mu_1}) - \mu_1\lambda_5\lambda_6(2\mu_1+\mu_2)^2(3\mu_1+2\mu_2)r^{6(\mu_1+\mu_2)}\Big]\cos(3\mu_1+\mu_2)\theta,
\end{aligned}$$

and by replacing the cos by sin , we get $w_{c_{jk,2}}$.

Finally, using Theorem 1.3 of [15], we have the following quantization result:

Corollary 2.4. *Suppose* (u,v) *is the solution of (2.7). Then the following hold:*

$$\int_{\mathbb{R}^2} e^u dx = 8\pi(2\gamma_1+\gamma_2+3),\ \int_{\mathbb{R}^2} e^v dx = 8\pi(3\gamma_1+2\gamma_2+5), \tag{2.24}$$

and $u(z) = -(4+2\gamma_1)\log|z| + O(1)$, $v(z) = -(4+2\gamma_2)\log|z| + O(1)$ *as* $|z|\to\infty$.

Acknowledgement

The third author is partially supported by NSERC of Canada.

References

[1] W.W. Ao, C.S. Lin and J.C. Wei, On Non-topological Solutions of the $\mathbf{A}_2$ and $\mathbf{B}_2$ Chern-Simons System, *Memoirs of Amer. Math. Soc.*, to appear..

[2] L. Caffarelli and Y.S. Yang, Vortex condensation in the Chern-Simons Higgs model: An existence theorem, *Commun. Math. Phys.* 168(1995), 321-336.

[3] D. Chae and O.Y. Imanuvilov, The existence of non-topological multi-vortex solutions in the relativistic self-dual Chern-Simons theory, *Comm. Math. Phys.* 215(2000), 119-142.

[4] H. Chan, C.C. Fu and C.S. Lin, Non-topological multi-vortex solution to the self-dual Chern-Simons-Higgs equation. *Comm. Math. Phys.*, 231(2002), 189-221.

[5] J.L. Chern, Z.-Y. Chen and C.S. Lin, Uniqueness of topological solutions and the structure of solutions for the Chern-Simons system with two Higgs particles, *Comm. Math. Phys.*, 296(2010), 323-351.

[6] K. Choe, N. Kim and C.S. Lin, Existence of self-dual non-topological solutions in the Chern-Simons-Higgs model, *Ann. Inst. H. Poincaŕe Anal. Non Linéaire*, 28(2011), 837-852.

[7] G. Dunne, Mass degeneracies in self-dual models. *Phys. Lett.*, B 345(1995), 452-457.

[8] G. Dunne, Self-dual Chern-Simons Theories. *Lecture Notes in Physics*, vol. m36(1995), Berlin-New York, Spring-Verlag.

[9] G. Dunne, Vacuum mass spectra for $SU(N)$ self-dual Chern-Simons-Higgs. *Nucl. Phys.*, B 433(1995), 333-348.

[10] J. Hong, Y. Kim and P.Y. Pac, Multivortex solutions of the Abelian Chern-Simons theory, *Phys. Rev. Lett.*, 64(1990), 2230-2233.

[11] R. Jackiw and E.J. Weinberg, Self-dual Chern-Simons vortices, *Phys. Rev. Lett.*, 64(1990), 2234-2237.

[12] H. Kao and K. Lee, Selfsual $SU(3)$ Chern-Simons-Higgs systems, *Phys. Rev.*, D 50(1994), 6626-6635.

[13] K. Lee, Relativistic nonabelian Chern-Simons systems, *Phys. Lett.*, B 225(1991), 381-384.

[14] K. Lee, Selfdual nonabelian Chern-Simons solitons, *Phys. Rev. Lett.*, 66(1991), 553-555.

[15] C.S. Lin, J. Wei and D. Ye, On classification and nondegeneracy of $SU(n)$ Toda system, *Invent. Math.* 190(2012), no.1, 169-207.

[16] C.S. Lin, S. Yan, Bubbling solutions for relativistic abelian Chern-Simons model on a torus, *Comm. Math. Phys.*, 297(2010), 733-758.

[17] C.S. Lin, S. Yan, Bubbling solutions for the $SU(3)$ Chern-Simons model on a torus, *Comm. Pure Appl. Math.*, to appear.

[18] M. Nolasco, G. Tarantello, Double vortex condensaties in the Chern-Simons-Higgs theory, *Calc. Var. Par. Diff. Eqns.*, 9(1999), 31-94.

[19] M. Nolasco, G. Tarantello, Vortex condensates for the $SU(3)$ Chern-Simons theory, *Comm. Math. Phys.*, 213(2000), 599-639.

[20] J. Prajapat and G. Tarantello, On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness resluts, *Proc. Royal Society Edin.* **131A** (2001), 967-985.

[21] J. Spruck and Y. Yang, Topological solutions in the self-dual Chern-Simons theory: Existence and approximation, *Ann. Inst. H. P*, 12(1995), 75-97.

[22] J. Spruck and Y. Yang, The existence of non-topological solitions in the self-dual Chern-Simons theory, *Comm. Math. Phys.*, 149(1992), 361-376.

[23] G. Tarantello, Multiple condensate solutions for the Chern-Simons Higgs theory, *J. Math. Phys.*, 37(1996), 3769-3796.

[24] G. Tarantello, Uniqueness of self-dual penodic Chern-Simons vortices of topological type, *Calc. Var. Par. Diff. Eqns.*, 29(2007), 191-217.

[25] R. Wang, The existence of Chern-Simons vortices, *Comm. Math. Phys*, 137(1991), 587-597.

[26] J. Wei, CY Zhao and F. Zhou, On non-degeneracy of solutions to $SU(3)$ Toda system, *CRAS.* 349(2011), no.3-4, 185-190.

[27] G. Wang and L. Zhang, Non-topological solutions of the relativistic SU(3) Chern-Simons Higgs model, *Comm. Math. Phys.*, 202(1999), no.3, 501-515.

[28] Y. Yang, *Solitons in field theory and nonlinear analysis.* Springer Monographs in Mathematics, Springer, New York, 2001.