INFINITELY MANY POSITIVE SOLUTIONS OF FRACTIONAL NONLINEAR SCHRÖDINGER EQUATIONS WITH NON-SYMMETRIC POTENTIALS

WEIWEI AO, JUNCHENG WEI, AND WEN YANG

ABSTRACT. We consider the fractional nonlinear Schrödinger equation

$$(-\Delta)^s u + V(x)u = u^p \text{ in } \mathbb{R}^N, \quad u \to 0 \text{ as } |x| \to +\infty,$$

where $V(x)$ is a uniformly positive potential and $p > 1$. Assuming that

$$V(x) = V_\infty + \frac{a}{|x|^m} + O\Big(\frac{1}{|x|^{m+\sigma}}\Big) \text{ as } |x| \to +\infty,$$

and p, m, σ, s satisfy certain conditions, we prove the existence of infinitely many positive solutions for $N = 2$. For $s = 1$, this corresponds to the multiplicity result given by Del Pino, Wei, and Yao [24] for the classical nonlinear Schrödinger equation.

1. INTRODUCTION

In this paper, we consider the fractional nonlinear Schrödinger equation

$$(-\Delta)^s u + V(x)u - u^p = 0 \quad \text{in } \mathbb{R}^2, \quad u \to 0 \text{ as } |x| \to +\infty, \tag{1.1}$$

where $(-\Delta)^s$, $0 < s < 1$, denotes the fractional Laplace operator, $V(x)$ is a nonnegative potential, and $1 < p < \frac{1+s}{1-s}$.

We are interested in the existence of infinitely many spike solutions to (1.1). The natural space on which to look for solutions of (1.1) is the space $H^{2s}(\mathbb{R}^2)$ of all functions $u \in L^2(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} (1 + |\xi|^{4s})(\hat{u}(\xi))^2 d\xi < \infty,$$

where $\hat{}$ denotes the Fourier transform. The fractional Laplacian $(-\Delta)^s u$ of a function $u \in H^{2s}(\mathbb{R}^2)$ is defined in terms of its Fourier transform by the relation

$$\widehat{(-\Delta)^s u} = |\xi|^{2s}\hat{u} \in L^2(\mathbb{R}^2).$$

Problem (1.1) arises from the study of standing waves $\varphi(x,t) = u(x)e^{iEt}$ for the following nonlinear Schrödinger equations:

$$i\varphi_t = (-\Delta)^s \varphi + (V(x) - E)\varphi - |\varphi|^{p-1}\varphi. \tag{1.2}$$

Equation (1.2) was introduced by Laskin [33] as an extension of the classical nonlinear Schrödinger equation for $s = 1$, where the Brownian motion of the quantum paths is replaced by a Lévy flight. Namely, if the path integral over Brownian trajectories leads to the well-known Schrödinger equation, then the path integral over Lévy trajectories leads to the fractional Shcrödinger equation. Here, $\varphi = \varphi(x,t)$ represents the quantum mechanical probability amplitude for a given unit-mass particle to have position x at time t (the corresponding probability density is $|\varphi|^2$), under a confinement resulting from the potential V. We refer the reader to [33, 34, 35] for detailed physical discussions and the motivation for equation (1.2).

Note that the fractional Schrödinger case exhibits interesting differences from to the classical case. For instance, the energy of a particle of unit mass is proportional to $|p|^{2s}$ (rather than $|p|^2$; see, e.g., [33]). Furthermore, space/time scaling of the process implies that the fractal dimension of Lévy paths is $2s$ (unlike in the classical Brownian case, where it is 2).

Lévy processes occur widely in physics, chemistry, and biology for instance in high energy Hamiltonians of relativistic theories and the Heisenberg uncertainty principle. See [9, 30] for further motivation of the fractional Laplacian in modern physics. Stable Lévy processes that give rise to equations with fractional Laplacians have recently attracted significant research interest, and there are many results in the literature regarding the existence of such solutions, for example in [2, 13, 12, 18, 28, 44] and references therein.

Let us come back to equation (1.1). When $s = 1$, i.e., $(-\Delta)^s$ reduces to the standard Laplacian $-\Delta$, equation (1.1) becomes

$$-\Delta u + V(x)u = u^p \text{ in } \mathbb{R}^2,\ u(x) \to 0 \text{ as } |x| \to +\infty. \tag{1.3}$$

Equation (1.3) has been extensively studied in the last thirty years. If

$$0 < \inf_{x\in\mathbb{R}^2} V(x) < V_\infty := \lim_{|x|\to\infty} V(x), \tag{1.4}$$

then one can show that (1.3) has a least energy (ground state) solution by using the concentration compactness principle (cf.[37, 38]). However, if (1.4) does not hold, then problem (1.3) may not admit a least energy solution, and one has to look for higher energy level solutions. Results in this direction are presented in [6, 7, 8], where a positive solution has been found using variational methods under a suitable decay condition on V at infinity.

Let us now consider the semi-classical limit case:

$$-\varepsilon^2\Delta u + W(x)u - u^p = 0 \text{ in } \mathbb{R}^N,\ u > 0,\ u \in H^1(\mathbb{R}^N), \tag{1.5}$$

where $\varepsilon > 0$ is a small parameter. Naturally, problem (1.5) is equivalent to (1.3) for $V(x) = W(\varepsilon x)$. It is known that as ε goes to zero, highly concentrated solutions can be found near critical points of the potential W [1, 10, 11, 19, 20, 21, 22, 29, 31, 42, 48] or near higher dimensional stationary sets of other auxiliary potentials [3, 23, 39]. The number of solutions of (1.5) may depend on the number or types of critical points of $W(x)$.

By assuming that $V = V(|x|)$ is radially symmetric, Wei and Yan [50] proved that problem (1.3) admits infinitely many positive non-radial solutions if there exist constants $V_\infty > 0, a > 0, m > 1$ and $\sigma > 0$ such that

$$V(r) = V_\infty + \frac{a}{r^m} + O\Big(\frac{1}{r^{m+\sigma}}\Big), \text{ as } r \to \infty. \tag{1.6}$$

The proof given in [50] relies heavily on the radial symmetry of the potential V. Inspired by this result, Wei-Yan proposed the following conjecture:

Wei-Yan's Conjecture: Problem (1.3) admits infinitely many positive solutions if there exist constants $V_\infty > 0, a > 0, m > 1$ and $\sigma > 0$ such that the potential $V(x)$ satisfies (1.6).

Recently, del Pino, Wei, and Yao developed an intermediate Lyapunov-Schmidt reduction method to solve this conjecture under some additional conditions on m, p, σ. The result is as follows for a general dimension N:

Theorem A. *Suppose that $V(x)$ is a locally Hölder continuous function, $V_0 = \inf_{x\in\mathbb{R}^2} V(x) > 0$ and (1.6) hold for some constants $V_\infty, a > 0$, and*

$$\min\left\{1, \frac{p-1}{2}\right\} m > 2,\ \sigma > 2. \tag{1.7}$$

Then, problem (1.3) admits infinitely many non-radial positive solutions, whose energy can be made arbitrarily large.

Regarding the existence of multiple spike solutions for more general non-symmetric potentials, in [14] Cerami, Passaseo, and Solimini proved the following existence result:

Theorem B. *Let the following assumptions hold:*
(h1) $V(x) \to V_\infty > 0$ as $|x| \to \infty$,
(h2) $V(x) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$,
(h3) $V \in L^{N/2}_{loc}(\mathbb{R}^N)$,
(h4) there exists $\bar{\eta} \in (0, \sqrt{V_\infty})$ such that $\lim_{|x|\to\infty} |V(x) - V_\infty| e^{\bar{\eta}|x|} = \infty$
is satisfied.
Then, there exists a positive constant $\mathcal{A} = \mathcal{A}(N, \bar{\eta}, V_\infty, V_0) \in \mathbb{R}$, such that when

$$|V(x) - V_\infty|_{N/2,loc} := \sup_{y\in\mathbb{R}^N} |V(x) - V_\infty|_{L^{N/2}(B_1(y))} < \mathcal{A},$$

equation (1.3) admits infinitely many positive solutions belonging to $H^1(\mathbb{R}^N)$.

For further results regarding the existence of (1.3) with non-symmetric potentials, we refer the reader to [5, 15, 16, 17] and references therein.

For the fractional case where $0 < s < 1$, Dávila, del Pino, and Wei [18] recently obtained the first result regarding multiple spikes for the corresponding fractional nonlinear Schrödinger equation with $1 < p < \frac{N+2s}{N-2s}$. Subsequently, Wang and Zhao [47] extended the result of Wei and Yan [50] (which was also achieved by Wei, Peng, and Yang [49] independently) to the fractional case, under the following assumption:

Assumption 1. V is positive and radially symmetric, i.e., $V(x) = V(|x|) > 0$ and there exist constants $a > 0$ and $V_\infty > 0$ such that

$$V(r) = V_\infty + \frac{a}{r^m} + o\Big(\frac{1}{r^m}\Big), \text{ as } r \to \infty,$$

where

$$\max\left\{0, (N+2s)\Big[1-(p-1)N - 2ps + \max\left\{s, p - \frac{N}{2}\right\}\Big]\right\} < m < N+2s.$$

A natural question is to ask whether or not we can obtain the multiplicity result of (1.1) for a potential $V(x)$ without the radially symmetric assumption adopted in the paper of [24] for the $s = 1$ case . In this paper, we will provide an affirmative answer to this question under the following assumption on the positive potential $V(x)$:

Assumption 2. There exist constants $m, a > 0$ and $V_\infty > 0$ such that

$$V(x) = V_\infty + \frac{a}{|x|^m} + O\Big(\frac{1}{|x|^{m+\sigma}}\Big), \text{ as } |x| \to \infty, \tag{1.8}$$

where m, σ, p, and s satisfy the following conditions:

$$\frac{2s+3}{2s+2} < p < \frac{1+s}{1-s}, \tag{1.9}$$

and there exists $1 < \mu < 2 + 2s$ such that

$$m + \frac{5}{2} - \frac{m}{4s+4} < \min\left\{\tau_1, \tau_2, \tau_3, \tau_4, m + \frac{3m}{2s+2}\right\}, \ 1 < m < 2+2s, \tag{1.10}$$

where

$$\tau_1 = \min\left\{m+3, m+\sigma, 2m+1, 2m + \frac{m}{s+1}, 4s+2+\frac{m}{2s+2}\right\},$$

$$\tau_2 = \min\{2m, pm\}, \ \tau_3 = \min\{m(p-1), m, 2s+1\} + \tau_0, \ \tau_4 = \min\{2\tau_0, p\tau_0\},$$

and

$$\tau_0 = \min\left\{pm - \frac{\mu m}{2s+2}, 2s+2-\mu, m\right\}.$$

Remark 1. It is not difficult to see that we can choose s sufficiently close to 1, p sufficiently large, m sufficiently close to $2+2s$, σ sufficiently large, and $\mu > 1$ sufficiently close to 1 such that (1.10) holds. Therefore, there exist parameters m, σ, p, and s satisfying Assumption 2. On the other hand, we can also obtain similar results for $N \geq 3$ under certain conditions on m, σ, s, and p if we assume further that

$$V(x) = V(x', x'') = V(x', -x''), \tag{1.11}$$

where $x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$. For the sake of the simplicity of the statement, we only consider the case with $N = 2$ in this paper.

Our main result in this paper is stated in the following theorem.

Theorem 1.1. *If Assumption* 2 *holds, then the problem (1.1) admits infinitely many non-radial positive solutions. Moreover, the energy of these solutions can be made arbitrarily large.*

Remark 2. We develop an intermediate Lyapunov-Schmidt reduction method for the fractional Laplacian Schr*ö*dinger equation (1.1). For $s = 1$, this has been carried out by Del Pino, Wei, and Yao in [24]. For $s = 1$, the spikes decays exponentially. As a result, each spike only interacts with neighboring spikes. Meanwhile, for $0 < s < 1$ the spikes decay algebraically and each spike interacts with all the other spikes, which makes the reduced problem more complicated than in the $s = 1$ case. A complicated matrix T is present in the reduced problem (see Lemma 5.2), and we must precisely obtain the decay rate for this matrix. Unlike in the case that $s = 1$, all of the entries in the matrix are nonzero for $0 < s < 1$, and we must carefully estimate the eigenvalues and determine the exact decay rate. This is a new result, and is the main contribution of this paper (see the final section). We believe that our technique can be employed in the construction of solutions for fractional Laplacian equation or the equation with a critical exponent in future work.

Throughout thispaper, we will employ the following notation and conventions:

- For quantities G_K and H_K, we write $G_K \sim H_K$ to indicate that there exists a positive constant C such that $\frac{1}{C} \le \frac{G_K}{H_K} \le C$ for sufficiently large K. Furthermore, $G_K = O(H_K)$ means that $\left|\frac{G_K}{H_K}\right|$ is uniformly bounded as K tends to infinity, and $G_K = o(H_K)$ denotes that $\left|\frac{G_K}{H_K}\right| \to 0$ as $K \to \infty$.
- For simplicity, the letter C denotes various generic constants that are independent of K. This is allowed to vary from line to line, as well as within the same formula.
- We shall employ the notation $|y| = \|y\|_2$ for the Euclidean norm in various Euclidean spaces $\mathbb{R}^N$ when no confusion can arise, and we always denote the inner product of a and b in $\mathbb{R}^N$ by $a \cdot b$.
- The transpose of a matrix A shall be denoted by A^T.

This paper is organized as follows. Some preliminary facts and estimates are explained in Section 2. In Section 3, we describe the procedure for our construction and describe the main ideas of each step. In Sections 4-6, we shall prove each of the steps outlined in Section 3, and then complete the proof of Theorem 1.1. We omit certain technical results in the final section.

2. Preliminaries

In this section, we study the fractional Laplacian operator and the ground state solution w of the following equation:

$$(-\Delta)^s w + w - w^p = 0 \text{ in } \mathbb{R}^N,\ w \to 0 \text{ as } |x| \to \infty. \tag{2.1}$$

Let $0 < s < 1$. Various definitions exist for the fractional Laplacian $(-\Delta)^s\phi$ of a function ϕ defined in $\mathbb{R}^N$, depending on its regularity and growth properties.

For $\phi \in H^{2s}(\mathbb{R}^N)$, the standard definition is given via the Fourier transform $\widehat{\ }$. $(-\Delta)^s\phi \in L^2(\mathbb{R}^N)$ is defined by the formula

$$|\xi|^{2s}\hat{\phi}(\xi) = \widehat{(-\Delta)^s\phi}. \tag{2.2}$$

When ϕ is additionally assumed to be sufficiently regular, we obtain the direct representation

$$(-\Delta)^s\phi(x) = d_{s,N}\int_{\mathbb{R}^N}\frac{\phi(x)-\phi(y)}{|x-y|^{N+2s}}\mathrm{d}y \tag{2.3}$$

for a suitable constant $d_{s,N}$, where the integral is to be understood in a principal value sense. This makes sense as a direct integral when $s < \frac{1}{2}$ and $\phi \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha > 2s$, or if $\phi \in C^{1,\alpha}(\mathbb{R}^N)$ with $1 + \alpha > 2s$. In the latter case, we can desingularize the representative integral in the following form:

$$(-\Delta)^s\phi(x) = d_{s,N}\int_{\mathbb{R}^N}\frac{\phi(x)-\phi(y)-\nabla\phi(x)(x-y)}{|x-y|^{N+2s}}\mathrm{d}y.$$

Another useful (local) representation, found by Caffarelli and Silverstre [13], is given via the following boundary value problem in the half space $\mathbb{R}^{N+1}_+ = \{(x,y) \mid$

$x \in \mathbb{R}^N, y > 0\}$:

$$\begin{cases} \nabla \cdot (y^{1-2s} \nabla \tilde{\phi}) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \tilde{\phi}(x,0) = \phi(x) & \text{on } \mathbb{R}^N. \end{cases}$$

Here, $\tilde{\phi}$ is the $s-$harmonic extension of ϕ, explicitly given as a convolution integral with the $s-$Poisson kernel $p_s(x,y)$,

$$\tilde{\phi}(x,y) = \int_{\mathbb{R}^N} p_s(x-z,y)\phi(z)\mathrm{d}z,$$

where

$$p_s(x,y) = C_{N,s} \frac{y^{4s-1}}{(|x|^2 + |y|^2)^{\frac{N-1+4s}{2}}}$$

and $C_{N,s}$ satisfies $\int_{\mathbb{R}^N} p(x,y) = 1$. Then, under suitable regularity conditions, $(-\Delta)^s \phi$ is the Dirichlet-to-Neumann map for this problem, namely,

$$(-\Delta)^s \phi(x) = \lim_{y \to 0^+} y^{1-2s} \partial_y \tilde{\phi}(x,y). \tag{2.4}$$

The characterizations (2.2)-(2.4) are all equivalent, for instance, in Schwartz's space of rapidly decreasing smooth functions.

For $m > 0$ and $g \in L^2(\mathbb{R}^N)$, let us consider now the equation

$$(-\Delta)^s \phi + m\phi = g \text{ in } \mathbb{R}^N.$$

In terms of the Fourier transform, for $\phi \in L^2$ this problem reads

$$(|\xi|^{2s} + m)\hat{\phi} = \hat{g},$$

and it admits a unique solution $\phi \in H^{2s}(\mathbb{R}^N)$ given by the convolution

$$\phi(x) = T_m[g] := \int_{\mathbb{R}^N} G(x-z)g(z)\mathrm{d}z, \tag{2.5}$$

where

$$\widehat{G}(\xi) = \frac{1}{|\xi|^{2s} + m}.$$

Using the characterization (2.4) written in a weak form, ϕ can be characterized by $\phi(x) = \tilde{\phi}(x,0)$ in a trace sense, where $\tilde{\phi} \in H$ is the unique solution of

$$\int\int_{\mathbb{R}^{N+1}_+} \nabla\tilde{\phi}\nabla\varphi y^{1-2s} + m\int_{\mathbb{R}^N} \phi\varphi = \int_{\mathbb{R}^N} g\varphi, \text{ for all } \varphi \in H. \tag{2.6}$$

Here, H is the Hilbert space of functions $\varphi \in H^1_{\text{loc}}(\mathbb{R}^{N+1}_+)$ such that

$$\|\varphi\|_H^2 := \int\int_{\mathbb{R}^{N+1}_+} |\nabla\varphi|^2 y^{1-2s} + m\int_{\mathbb{R}^N} |\varphi|^2 < +\infty,$$

or equivalently the closure of the set of all functions in $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ under this norm.

A useful fact for our purpose is the equivalence of the representations (2.5) and (2.6) for $g \in L^2(\mathbb{R}^N)$.

Lemma 2.1. *Let $g \in L^2(\mathbb{R}^N)$. Then, the unique solution $\tilde{\phi} \in H$ for the problem (2.6) is given by the $s-$harmonic extension of the function $\phi = T_m[g] = G * g$.*

For the proof, we refer the reader to Lemma 2.1 in [18]. Let us recall the main properties of the fundamental solution $G(x)$ in the representation (2.5), which are stated, for instance, in [28] and [27].

We have that G is radially symmetric and positive. That is , $G \in C^\infty(\mathbb{R}^N \setminus \{0\})$ satisfies:

-
$$|G(x)| + |x||\nabla G(x)| \leq \frac{C}{|x|^{N-2s}} \text{ for all } |x| \leq 1,$$
-
$$\lim_{|x|\to\infty} G(x)|x|^{N+2s} = \gamma > 0,$$
-
$$|x||\nabla G(x)| \leq \frac{C}{|x|^{N+2s}} \text{ for all } |x| \geq 1.$$

In order to consider an a priori estimate involving the fractional Laplacian operator, we require the following Lemmas (see [18]):

Lemma 2.2. *Let $0 \leq \mu < N + 2s$. Then, there exists a positive constant C such that*
$$\|(1+|x|)^\mu T_m(g)\|_{L^\infty(\mathbb{R}^N)} \leq C\|(1+|x|^\mu)g\|_{L^\infty(\mathbb{R}^N)}.$$

Lemma 2.3. *Assume that $g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then. the following holds. If $\phi = T_m[g]$, then there exists $C > 0$ such that*
$$\sup_{x\neq y} \frac{|\phi(x) - \phi(y)|}{|x-y|^\beta} \leq C\|g\|_{L^\infty(\mathbb{R}^N)}, \tag{2.7}$$
where $\beta = \min\{1, 2s\}$.

Lemma 2.4. *Let $\varphi \in H^{2s}$ be the solution of*
$$(-\Delta)^s \varphi + W\varphi = g \text{ in } \mathbb{R}^N \tag{2.8}$$
with a bounded potential W. If $\inf_{x\in\mathbb{R}^N} W(x) =: m > 0, g \geq 0$, then $\varphi \geq 0$ in $\mathbb{R}^N$.

The next lemma provides an a priori estimate for a solution $\phi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ of (2.8).

Lemma 2.5. *Let W be a continuous function, and assume that for k points $q_1, q_2, \cdots, q_k$ there exist $R > 0$ and $B = \cup_{j=1}^k B_R(q_j)$ such that*
$$\inf_{x\in\mathbb{R}^N\setminus B} W(x) =: m > 0.$$
Then, given any number $\frac{N}{2} < \mu < N + 2s$, there exists a uniform positive constant $C = C(\mu, R)$ independent of k such that for any $\varphi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and g satisfying (2.8) with
$$\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)} < +\infty, \quad \text{where } \rho(x) = \sum_{j=1}^k \frac{1}{(1+|x-q_j|)^\mu},$$
the following estimate is valid:
$$\|\rho^{-1}\varphi\|_{L^\infty(\mathbb{R}^N)} \leq C\big(\|\varphi\|_{L^\infty(\mathbb{R}^N)} + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}\big).$$

Proof. We write (2.8) as

$$(-\Delta)^s\varphi + \tilde{W}\varphi = \tilde{g},$$

where $\tilde{g} = (m - W)\chi_B\varphi + g$, $\tilde{W} = m\chi_B + W(1-\chi_B)$, and χ_B is the characteristic function on B. We observe that

$$|\tilde{g}(x)| \leq C\Big(\|\varphi\|_{L^\infty(B)} + \|\rho^{-1}g\|_{L^\infty(B)}\Big)\rho \leq M\rho,$$

where

$$\begin{aligned}
M &= C\|\varphi\|_{L^\infty(B)} \sup_{x\in B}\Big(\sum_{j=1}^k \frac{1}{(1+|x-q_j|)^\mu}\Big)^{-1} + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}\\
&\leq C\|\varphi\|_{L^\infty(B)} \max_{1\leq j\leq k}\sup_{x\in B_R(q_j)}\Big(\frac{1}{(1+|x-q_j|)^\mu}\Big)^{-1} + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}\\
&\leq C\|\varphi\|_{L^\infty(B)}(1+R^\mu) + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}\\
&\leq C(\mu,R)\big(\|\varphi\|_{L^\infty(B)} + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}\big).
\end{aligned}$$

From Lemma 2.2 with $0<\mu<N+2s$, the positive solution φ_0 to the problem

$$(-\Delta)^s\varphi_0 + m\varphi_0 = \frac{1}{(1+|x|)^\mu}$$

satisfies $\varphi_0 = O(|x|^{-\mu})$ as $|x|\to+\infty$. Because $\inf_{x\in\mathbb{R}^N}\tilde{W}(x)\geq m$, we clearly have that

$$\Big((-\Delta)^s + \tilde{W}\Big)\tilde{\varphi} \geq M\sum_{j=1}^k \frac{1}{(1+|x-q_j|)^\mu},$$

where $\tilde{\varphi}(x) = M\sum_{j=1}^k \varphi_0(x-q_j)$. Setting $\psi = \varphi - \tilde{\varphi}$, we find that

$$(-\Delta)^s\psi + \tilde{W}\psi = \tilde{g} \leq 0,$$

with $\tilde{g}\in L^2$. By using Lemma 2.4, we obtain that $\varphi\leq\tilde{\varphi}$. By a similar argument for $-\varphi$, we obtain that $|\varphi|\leq\tilde{\varphi}$. Then, it holds that

$$\begin{aligned}
\|\rho^{-1}\varphi\|_{L^\infty(\mathbb{R}^N)} \leq \|\rho^{-1}\tilde{\varphi}\|_{L^\infty(\mathbb{R}^N)} &= M\Big\|\Big(\sum_{j=1}^k\frac{1}{(1+|x-q_j|)^\mu}\Big)^{-1}\sum_{i=1}^k\varphi_0(x-q_i)\Big\|_{L^\infty(\mathbb{R}^N)}\\
&\leq CM\Big\|\Big(\sum_{j=1}^k\frac{1}{(1+|x-q_j|)^\mu}\Big)^{-1}\sum_{i=1}^k\frac{1}{(1+|x-q_i|)^\mu}\Big\|_{L^\infty(\mathbb{R}^N)}\\
&\leq CM.
\end{aligned}$$

The desired estimate follows immediately. $\square$

From the above Lemma, we can immediately obtain the following corollary.

Corollary 2.1. *Let $\rho(x)$ be defined as in the previous lemma. Assume that $\phi\in H^{2s}(\mathbb{R}^N)$ satisfies equation (2.7), and that*

$$\inf_{x\in\mathbb{R}^N} W(x) =: m > 0.$$

Then, we have that $\phi\in L^\infty(\mathbb{R}^N)$, and it satisfies

$$\|\rho^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \leq C\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}, \tag{2.9}$$

where $C = C(\mu)$ is independent of k.

A useful fact is that if $f, g \in L^2(\mathbb{R}^N)$ and $W = T_m(f)$, $Z = T_m(g)$, then the following holds:

$$\int_{\mathbb{R}^N} Z(-\Delta)^s W - \int_{\mathbb{R}^N} W(-\Delta)^s Z = \int_{\mathbb{R}^N} T_m[f]g - \int_{\mathbb{R}^N} T_m[g]f,$$

where we have used the fact that the Green kernel G is radially symmetric.

Next, we recall some basic and useful properties regarding the ground state solution w of (2.1).

Lemma 2.6. *Let $N \geq 1$, $s \in (0,1)$ and $1 < p < \frac{N+2s}{N-2s}$. Then, the following hold:*

(1). *There exists a nonnegative function $w \geq 0$ with $w \neq 0$ solving (2.1), and there exists some $x_0 \in \mathbb{R}^N$ such that $w(\cdot - x_0)$ is radial, positive, and strictly decreasing in $|x - x_0|$. Moreover, the function $w(x) = w(|x|)$ belongs to $H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$, and satisfies*

$$\frac{C_1}{1+|x|^{N+2s}} \leq w(x) \leq \frac{C_2}{1+|x|^{N+2s}} \text{ for } x \in \mathbb{R}^N, \tag{2.10}$$

with some constants $C_2 \geq C_1 > 0$.

(2). *The linearized operator $L = (-\Delta)^s + 1 - pw^{p-1}$ is non-degenerate, i.e., its kernel is given by*

$$\mathrm{Ker}\big((-\Delta)^s + 1 - pw^{p-1}\big) \cap L^\infty(\mathbb{R}^N) = \mathrm{Span}\Big\{\partial_{x_1} w, \dots, \partial_{x_N} w\Big\}.$$

Proof. One can find a proof of the above lemma in [27, Proposition 3.1, Theorem 3, and Theorem 4]. □

Remark 3. From the asymptotic behavior of the Green kernel G (see Lemma 2.1) and the above Lemma 2.6, we obtain the following more accurate description of the asymptotic behavior of w and w':

$$w(r) = \frac{A}{r^{N+2s}}(1 + o(1)),\ A > 0 \text{ as } r \to \infty, \tag{2.11}$$

and

$$w'(r) = \frac{B}{r^{N+2s+1}}(1 + o(1)),\ B < 0, \text{ as } r \to \infty, \tag{2.12}$$

under the assumption that $(N+2s)p > N + 2s + 1$, i.e., condition (1.9) for $N = 2$.

3. Description of the construction

In this section, we shall briefly describe the solutions to be constructed later, and will describe the main ideas of the construction.

First, without loss of generality we can assume by suitable scaling that $V_\infty = 1$. Following the developments in [50], we will use the loss of compactness to construct solutions. More precisely, we will construct solutions with large numbers of spikes whose inter-distances and distances from the origin are sufficiently large.

By the asymptotic behavior of V at infinity, the basic building block is the ground state solution w of (2.1). The solutions we construct will consist of small perturbations of sums of copies of w, centered at some carefully chosen points in $\mathbb{R}^2$.

Let $K \in \mathbb{N}_+$ be the number of spikes, whose locations are given by $Q_j \in \mathbb{R}^2, j = 1, 2, \cdots, K$. We define

$$w_j(x) = w(x - Q_j) \text{ and } U(x) = \sum_{j=1}^{K} w_j(x), \text{ and } x \in \mathbb{R}^2. \tag{3.1}$$

In order to further describe the configuration space of the Q_j's, we define

$$Q_j^0 = (R\cos\theta_j, R\sin\theta_j) \in \mathbb{R}^2, \text{ for } j = 1, 2, \cdots, K,$$

where

$$\theta_j = \alpha + (j-1)\frac{2\pi}{K} \in \mathbb{R}.$$

Here, α is a parameter representing the degeneracy resulting from rotations, and R is a positive constant to be determined later. Observe that each point Q_j^0 depends on α. Thus, we write $Q_j^0 = Q_j^0(\alpha)$. The number of spikes K and the radius R are related by the so-called balancing condition:

$$\sum_{j\neq 1} \Psi(2R\sin\frac{\theta_j}{2})\sin\frac{\theta_j}{2} - \Gamma R^{-m-1} = 0, \tag{3.2}$$

where $\Gamma = \frac{am}{2}\int_{\mathbb{R}^2} w^2(x)dx > 0$, and Ψ is the interaction function defined by

$$\Psi(t) = -\int_{\mathbb{R}^2} w(x - t\vec{\mathbf{e}}) div(w^p(x)\vec{\mathbf{e}})dx. \tag{3.3}$$

Here, $\vec{\mathbf{e}}$ can be any unit vector in $\mathbb{R}^2$. For $\Psi(t)$, we obtain the following expansion:

Lemma 3.1. *For large t, the following expansion holds for $\Psi(s)$:*

$$\Psi(t) = \frac{c_p}{t^{2s+3}}(1 + o(1)), \tag{3.4}$$

where

$$c_p = (\frac{p}{2}\int_{\mathbb{R}^2} w^{p-1}w'(|x|)|x|dx) \lim_{t\to\infty} w'(t)t^{2s+3} > 0. \tag{3.5}$$

Proof. By the definition of $\Psi(t)$ and the use of a Taylor expansion, we have that

$$\begin{aligned}\Psi(t) =& -\int_{\mathbb{R}^2} w(x - t\vec{\mathbf{e}}) div(w^p(x)\vec{\mathbf{e}})dx \\ =& -\int_{\mathbb{R}^2} pw(x)^{p-1}w'(|x|)\frac{x\cdot\vec{\mathbf{e}}}{r}[w(t) + w'(t)\vec{\mathbf{e}}\cdot(-x) + O(w''(t)|x|^2)] \\ =& w'(t)(\int_{\mathbb{R}^2} pw(x)^{p-1}w'(|x|)\frac{x\cdot\vec{\mathbf{e}}}{r}\vec{\mathbf{e}}\cdot x + o(1)) \\ =& c_p t^{-2s-3}(1 + o(1)).\end{aligned}$$

Thus, we have proved the Lemma.

$\square$

From the above balancing condition (3.2) and Lemma 3.1, we can obtain that

$$K = O(R^{1-\frac{m}{2+2s}}). \tag{3.6}$$

Because we require that $1 < m < 2 + 2s$, we have as a consequence of (3.6) that

$$|Q_j^0| = R \text{ and } \min_{j\neq l}\{|Q_j^0 - Q_l^0|\} = 2R\sin\frac{\pi}{K} \to +\infty, \text{ as } K \to \infty.$$

Next, we define a small neighborhood of $\mathbf{Q}^0 = (Q_1^0, \cdots, Q_K^0)$ on $(\mathbb{R}^2)^K$ and introduce an additional parameter. Let $f_j, g_j \in \mathbb{R},\ j = 1, 2, \cdots, K$. Then, we define

$$Q_j = Q_j^0 + f_j n_j + g_j t_j = (R + f_j) n_j + g_j t_j, \tag{3.7}$$

where

$$n_j = (\cos\theta_j, \sin\theta_j), \text{ and } t_j = (-\sin\theta_j, \cos\theta_j).$$

Note that f_j and g_j measure the displacement in the normal and tangential directions, respectively. Writing $Q_j = Q_j(\alpha), n_j = n_j(\alpha)$ and $t_j = t_j(\alpha)$, we note the following trivial but important fact:

$$Q_j(\alpha + 2\pi) = Q_j(\alpha),\ \forall \alpha \in \mathbb{R}, \text{ and } \forall j = 1, 2, \cdots, K. \tag{3.8}$$

We can now introduce an additional parameter $\mathbf{q}$ and define a suitable norm. Denote

$$\mathbf{q} = (f_1, \cdots, f_K, g_1, \cdots, g_K)^T \in \mathbb{R}^{2K}.$$

With this notation, we can define the configuration space of the Q_j's by

$$\Lambda_K = \left\{ (Q_1, \cdots, Q_K) \in (\mathbb{R}^2)^K \mid Q_j \text{ is defined in (3.7)and } \|\mathbf{q}\|_\infty \le 1 \right\}. \tag{3.9}$$

For any $(Q_1, Q_2, \cdots, Q_K) \in \Lambda_K$, an easy computation shows that for $j = 1, 2, \cdots, K$,

$$|Q_j| = R + f_j + O(R^{-1}),$$

and

$$|Q_{j+1} - Q_j| = d + 2(f_j + \dot{g}_j)\frac{\pi}{K} + O(K^{-2}).$$

We will prove Theorem 1.1 by demonstrating the following result.

Theorem 3.1. *Under the assumption of Theorem 1.1, there exists a positive integer K_0 such that for all integers $K \ge K_0$ there exist $\alpha \in [0, 2\pi)$ and $(Q_1, Q_2, \cdots, Q_K) \in \Lambda_K$ such that the problem (1.1) has two solutions of the form*

$$u(x) = \sum_{j=1}^{K} w(x - Q_j) + \phi(x), \tag{3.10}$$

where $\|\phi\|_{L^\infty(\mathbb{R}^2)} \to 0$ as $K \to +\infty$.

To prove Theorem 3.1, it is sufficient to show that for sufficiently large K there exist parameters α and $\mathbf{q}$ such that $U + \phi$ is a genuine solution for a small perturbation ϕ. To achieve this, we will adopt the techniques for the singularly perturbed problem. Unlike for the problem (1.5), there is no apparent parameter in this case . As stated in Theorem 3.1, we adopt the number of spikes as the $\varepsilon-$type parameter.

Let $u = U + \phi$. Then, solving $(-\Delta)^s u + V(x)u - u^p = 0$ is equivalent to solving

$$L[\phi] + E + N(\phi) = 0, \tag{3.11}$$

where

$$L[\phi] = (-\Delta)^s \phi + V(x)\phi - pU^{p-1}\phi,$$

$$E = (-\Delta)^s U + V(x)U - U^p,$$

and

$$N(\phi) = -(U + \phi)_+^p + U^p + pU^{p-1}\phi.$$

As mentioned in Remark 2 in the introduction, we will employ intermediate Lyapunov-Schmidt reduction to solve this. We will solve (3.11) through the following two steps.

Step 1: Solving the projected problem Let $\alpha \in \mathbb{R}$ and $\mathbf{q}$ satisfy (3.9). We shall first solve a projected version of (3.11). More precisely, we look for a function ϕ and some multiplier $\hat{\beta} \in \mathbb{R}^{2K}$ such that

$$\begin{cases} L[\phi] + E + N(\phi) = \hat{\beta} \cdot \frac{\partial U}{\partial \mathbf{q}}, \\ \int_{\mathbb{R}^2} \phi Z_{Q_j} dx = 0, \ \forall j = 1, \cdots, K, \end{cases} \tag{3.12}$$

where the vector field Z_{Q_j} is defined by

$$Z_{Q_j}(x) = \nabla w(x - Q_j). \tag{3.13}$$

By direct computation, we have that

$$\frac{\partial U}{\partial \mathbf{q}} = -(Z_{Q_1} \cdot n_1, \cdots, Z_{Q_1} \cdot n_K, Z_{Q_1} \cdot t_1, \cdots, Z_{Q_1} \cdot t_K)^T.$$

This constitutes the first step in the Lyapunov-Schmidt reduction. It is performed in Section 4 using an a priori estimate and the contraction mapping theorem. A required condition in this step is the non-degeneracy of w. It is worth noting that the function ϕ and the multiplier $\hat{\beta}$ found in Step 1 depend on the parameters α and $\mathbf{q}$.

Step 2: Solving the reduced problem

From Step 1, we know that $\hat{\beta}$ is small. However, it is not easy to solve $\hat{\beta}(\alpha, \mathbf{q}) = 0$ directly, because the linear part of the expansion of $\hat{\beta}$ in $\mathbf{q}$ is degenerate.

More precisely, let us write

$$\hat{\beta}(\alpha, \mathbf{q}) = \tilde{T}\mathbf{q} + \Phi(\alpha, \mathbf{q}),$$

where $\tilde{T}\mathbf{q}$ is the linear part and $\Phi(\alpha, \mathbf{q})$ denotes the remaining term. As we will see in Section 5, $\tilde{T}\mathbf{q}$ does not depend on α, and there is a unique vector (up to scalar)

$$\mathbf{q}_0 = (\underbrace{0, \cdots, 0}_{K}, \underbrace{1, \cdots, 1}_{K})^T \in \mathbb{R}^{2K},$$

such that $\tilde{T}\mathbf{q}_0 = 0$.

By applying Lyapunov-Schmidt reduction again (called the secondary Lyapunov-Schmidt reduction), the step of solving the reduced problem $\hat{\beta}(\alpha, \mathbf{q}) = 0$ can be divided into two steps. To write the projected problem of $\hat{\beta} = 0$ in a proper norm, note that

$$\frac{\partial U}{\partial \alpha} = R \sum_{j=1}^{K} \frac{\partial U}{\partial g_j} + \sum_{j=1}^{K} \Big(f_j \frac{\partial U}{\partial g_j} - g_j \frac{\partial U}{\partial f_j} \Big) = (R\mathbf{q}_0 + \mathbf{q}^{\perp}) \cdot \frac{\partial U}{\partial \mathbf{q}}, \tag{3.14}$$

where $\mathbf{q}^{\perp} = (-\vec{g}, \vec{f})$ for $\mathbf{q} = (\vec{f}, \vec{g})$. Hence, we define

$$\vec{\beta} = \hat{\beta} - \gamma(R\mathbf{q}_0 + \mathbf{q}^{\perp}), \text{ for every } \gamma \in \mathbb{R}. \tag{3.15}$$

With this notation,

$$\hat{\beta} \cdot \frac{\partial U}{\partial \mathbf{q}} = \vec{\beta} \frac{\partial U}{\partial \mathbf{q}} + \gamma \frac{\partial U}{\partial \alpha}.$$

Clearly, the multiplier $\vec{\beta}$ depends on the parameters $\alpha, \mathbf{q}$ and γ. Thus, we write $\vec{\beta} = \vec{\beta}(\alpha, \mathbf{q}, \gamma)$.

Step 2.A: Solving $\vec{\beta}(\alpha, \mathbf{q}, \gamma) = 0$ by adjusting γ and $\mathbf{q}$

In this step, for each $\alpha \in \mathbb{R}$ we will determine parameters $(\gamma, \mathbf{q})$ such that

$$\vec{\beta}(\alpha, \mathbf{q}, \gamma) = 0, \text{ and } \mathbf{q} \perp \mathbf{q}_0. \tag{3.16}$$

This can be viewed as the step of solving the projected problem in the secondary Lyapunov-Schmidt reduction. To achieve this, we will make use of condition (3.2). This step is performed in Section 5 by using various integral estimates and the contraction mapping theorem. A key condition in this step is the invertibility of a $2K \times 2K$ matrix, the proof of which is given in the final section. As mentioned before, the analysis of this large matrix is the key contribution of this paper, where the properties of circulant matrices and the Fourier series of the Bernoulli and Euler polynomials are used.

After Step 2.A is completed, we denote the unique solution of (3.16) by $(\gamma(\alpha), \mathbf{q}(\alpha))$. Then, the original problem (1.1) is reduced to the problem $\gamma(\alpha) = 0$ in one dimension.

Step 2.B: Solving $\gamma(\alpha) = 0$ by choosing α In this final step, we want to prove that there exists an α such that $\gamma(\alpha) = 0$. As a result, the function $u = U + \phi$ is a genuine solution of (1.1).

This constitutes the second step of solving the reduced problem in the secondary Lyapunov-Schmidt reduction. To achieve this step, note that by Step 2.A the function $\phi = \phi(x, \alpha, \mathbf{q}(\alpha))$ determined in Step 1 solves the following problem:

$$\begin{cases} L[\phi] + E + N(\phi) = \gamma(\alpha) \frac{\partial U}{\partial \alpha}, \\ \int_{\mathbb{R}^2} \phi Z_{Q_j} dx = 0, \ \forall j = 1, 2, \cdots, K, \end{cases} \tag{3.17}$$

where all of the quantities depending implicitly on $(\alpha, \mathbf{q})$ take values at $(\alpha, \mathbf{q}(\alpha))$. To solve $\gamma(\alpha) = 0$, we first apply the so-called variational reduction technique to show that finding a solution of the equation $\gamma(\alpha) = 0$ is equivalent to finding the critical point of the energy function $F(\alpha) = \mathcal{E}(U + \phi)$. Second, by using (3.8) it is not difficult to see that $F(\alpha) = 0$ is 2π periodic in α. Hence, it has at least two critical points. More details for this step are presented in Section 6.

In the following three sections, we shall complete Step 1, Step 2.A, and Step 2.B, respectively.

4. Lyapunov-Schmidt reduction

This section is devoted to completing the first step in the procedure of our construction.

Before stating the main result, we first introduce some notation. We define the weighted norm as follows:

$$\|h\|_* = \sup_{x \in \mathbb{R}^2} \Big| \Big(\sum_{j=1}^{K} \frac{1}{(1 + |x - Q_j|)^\mu} \Big)^{-1} h(x) \Big|, \tag{4.1}$$

where Q_j is defined in Section 3 and $\mu > 0$ will be chosen later. In the following, we assume that $(Q_1, \cdots, Q_K) \in \Lambda_k$, i.e., the parameter $\mathbf{q}$ satisfies (3.9).

We first claim that

$$\|h\|_{L^\infty(\mathbb{R}^2)} \le C\|h\|_*. \tag{4.2}$$

To prove (4.2), it suffices to show that $\sum_{j=1}^K \frac{1}{(1+|x-Q_j|)^\mu} < +\infty$. Indeed, for any $x \in \mathbb{R}^2$, suppose without loss of generality that Q_1 is the point such that

$$|x - Q_1| = \min_j \{|x - Q_j|,\ j = 1, 2, \cdots, K\}.$$

It is known that $|x-Q_1| + |x-Q_i| \ge |Q_1 - Q_i|$, and we can deduce that $|x - Q_i| \ge \frac{1}{2}|Q_1 - Q_i|$. Then,

$$\begin{aligned}\sum_{j=1}^K \frac{1}{(1+|x-Q_j|)^\mu} &\le C + C\sum_{j=2}^K \frac{1}{|Q_j - Q_1|^\mu} \le C + C\Big(\frac{1}{R}\Big)^\mu \sum_{j=2}^{[\frac{K}{2}]+1} \big(\sin(\frac{j-1}{K}\pi)\big)^{-\mu} \\ &\le C + C\Big(\frac{K}{R}\Big)^\mu \sum_{j=2}^{[\frac{K}{2}]+1} \frac{1}{j^\mu} \le C + C\Big(\frac{K}{R}\Big)^\mu,\end{aligned} \tag{4.3}$$

where we have used that

$$|Q_j - Q_1| \sim 2R\sin\Big(\frac{j-1}{K}\pi\Big) \text{ for } j \in \big[2, [\frac{K}{2}] + 1\big],$$

$$|Q_j - Q_1| \sim 2R\sin\Big(\frac{K-j+1}{K}\pi\Big) \text{ for } j \in \big[[\frac{K}{2}] + 2, K\big],$$

and $\sum_{j=1}^{[\frac{K}{2}]+1} \frac{1}{j^\mu} \le C$ for $\mu > 1$. Here, $[x]$ denotes the integer part of x. By using the relation $K = O(R^{1-\frac{m}{2s+2}})$, we can obtain that

$$\rho(x) = \sum_{j=1}^K \frac{1}{(1+|x-Q_j|)^\mu} \le C$$

from (4.3). Denote

$$\mathfrak{B}_* = \Big\{h \in L^\infty(\mathbb{R}^2) \Big| \|h\|_* < \infty\Big\}.$$

Then, $\mathfrak{B}_*$ is a Banach space with the norm $\|h\|_*$. To demonstrate its completeness, suppose that $\{h_n\}$ is a Cauchy sequence in $\mathfrak{B}_*$. By (4.2), $\{h_n\}$ is also a Cauchy sequence in $L^\infty(\mathbb{R}^2)$. Hence, h_n converges to a function h_∞ in $L^\infty(\mathbb{R}^2)$. It is easy to see that for any ε, there exists $n_0 \in \mathbb{N}$ such that

$$\Big(\sum_{j=1}^K \frac{1}{(1+|x-Q_j|)^\mu}\Big)^{-1} |h_n(x) - h_k(x)| \le \|h_n - h_k\|_* < \varepsilon,\ \forall x \in \mathbb{R}^2,\ \text{if } n, k \ge n_0.$$

By letting $k \to \infty$, we obtain that

$$\Big(\sum_{j=1}^K \frac{1}{(1+|x-Q_j|)^\mu}\Big)^{-1} |h_n(x) - h_\infty(x)| < \varepsilon,\ \forall x \in \mathbb{R}^2,\ \text{if } n \ge n_0,$$

which implies that $\|h_n - h_\infty\|_* \to 0$ as $n \to +\infty$.

In the following, we always assume that $1 < \mu < 2 + 2s$. Now, we can state our main result for this section.

Proposition 4.1. *Suppose that $V(x)$ satisfies (1.8) for constants $V_\infty > 0$, $\alpha \in \mathbb{R}$, $m \geq 1$, and $\sigma > 0$, given $1 < \mu < 2+2s$. Then, there exists a positive integer K_0 such that for all $K \geq K_0$, $\alpha \in \mathbb{R}$, and $\mathbf{q}$ satisfying (3.9), there exists a unique function $\phi \in H^{2s}(\mathbb{R}^2) \cap \mathcal{B}_K$ and a unique multiplier $\mathbf{c} \in \mathbb{R}^{2K}$ such that*

$$\begin{cases} L[\phi] + E + N(\phi) = \mathbf{c} \cdot \frac{\partial U}{\partial \mathbf{q}}, \\ \int_{\mathbb{R}^N} \phi Z_{Q_j} \mathrm{d}x = 0, \ \forall j = 1, , \cdots, K, \end{cases} \tag{4.4}$$

where

$$\mathcal{B}_K = \left\{ \phi \in L^\infty(\mathbb{R}^2) \mid \|\phi\|_* \leq \ CR^{-\tau_0}, \right.$$

and

$$\tau_0 = \min\left\{ pm - \frac{\mu m}{2s+2}, 2s+2-\mu, m \right\}.$$

Here, C is a positive constant that is independent of K. Moreover, $(\alpha, \mathbf{q}) \mapsto \phi(x; \alpha, \mathbf{q})$ is of class C^1, and

$$R^{-1}\Big\|\frac{\partial \phi}{\partial \alpha}\Big\|_* + \Big\|\frac{\partial \phi}{\partial \mathbf{q}}\Big\|_* \leq \ CR^{-\min\{p-1,1\}\tau_0}.$$

The proof of Proposition 4.1 is somewhat standard, and can be divided into two steps:

(i) Studying the invertibility of the linear operator.
(ii) Applying fixed point theorems.

Let M denotes a $2K \times 2K$ matrix defined by

$$M_{ij} = \int_{\mathbb{R}^2} \frac{\partial U}{\partial q_i} \frac{\partial U}{\partial q_i}, \ \forall \ i, j = 1, 2, \cdots, 2K. \tag{4.5}$$

Lemma 4.1. *Assume that $m \geq 1$. Then, for sufficiently large K, given any vector $\mathbf{b} \in \mathbb{R}^{2K}$ there exists a unique vector $\mathbf{c} \in \mathbb{R}^{2K}$ such that $M\mathbf{c} = \mathbf{b}$. Moreover,*

$$\|\mathbf{c}\|_\infty \leq C\|\mathbf{b}\|_\infty, \tag{4.6}$$

for some constant C that is independent of K.

Proof. To prove the existence, it is sufficient to prove the a priori estimate (4.6). Suppose that $|c_j| = \|\mathbf{c}\|$. Then, by the definition we have that

$$\sum_{i=1}^{K} M_{ji} c_i = b_j. \tag{4.7}$$

For the entries M_{ji}, a simple computation gives that

$$|M_{ji}| \sim \left| 2R \sin\left(\frac{j-i}{K}\pi\right) \right|^{-2s-3}, \ \forall \ i \neq j, \tag{4.8}$$

and

$$M_{jj} = \int_{\mathbb{R}^2} (\partial_{x_1} w)^2 \mathrm{d}x = c_0 > 0, \ \forall \ j = 1, 2, \cdots, 2K. \tag{4.9}$$

Hence, by (4.7)-(4.9) we have for sufficiently large K that

$$c_0 \|\mathbf{c}\|_\infty = c_0 |c_j| \leq \sum_{i \neq j} |M_{ji}||c_i| + |b_j| \leq \frac{c_0}{2} \|\mathbf{c}\|_\infty + \|\mathbf{b}\|_\infty,$$

from which the desired result follows. $\square$

Before we give the proof of Proposition 4.1, we first study the following linearized problem.

Proposition 4.2. *Under the assumption of Proposition 4.1, given* $1 < \mu < 2+2s$, *there exists a positive integer* K_0 *such that for all* $K \geq K_0$,, $\alpha \in \mathbb{R}$, $\mathbf{q}$ *satisfying (3.9), and* $h \in \mathfrak{B}_*$, *there exists a unique function* $\phi \in H^{2s}(\mathbb{R}^2) \cap \mathfrak{B}_*$ *and a unique multiplier* $\mathbf{c} \in \mathbb{R}^{2K}$ *such that*

$$\begin{cases} L[\phi] = h + \mathbf{c} \cdot \frac{\partial U}{\partial \mathbf{q}}, \\ \int_{\mathbb{R}^2} \phi Z_{Q_j} \mathrm{d}x = 0, \ \forall \ j = 1, 2, \cdots, K. \end{cases} \tag{4.10}$$

Moreover, we have that

$$\|\phi\|_* + \|\mathbf{c}\|_\infty \leq C \|h\|_*, \tag{4.11}$$

for some constant C *that is independent of* K.

Proof. By following the same process as in the proof of Proposition 4.1 in [18], it suffices to prove the a priori estimate (4.11).

To prove (4.11), we first multiply the equation (4.10) by $\frac{\partial u}{\partial \mathbf{q}}$ and integrate over $\mathbb{R}^2$, to obtain

$$M\mathbf{c} = \int_{\mathbb{R}^2} L[\phi] \frac{\partial U}{\partial \mathbf{q}} - \int_{\mathbb{R}^2} h \frac{\partial U}{\partial \mathbf{q}} \mathrm{d}x,$$

where M is a $2K \times 2K$ matrix as defined in (4.5).

Using integration by parts, we obtain that

$$\int_{\mathbb{R}^2} L[\phi] Z_{Q_i} \mathrm{d}x = \int \phi L[Z_{Q_i}] \mathrm{d}x.$$

Then, observe that

$$L[Z_{Q_i}] = (V(x) - 1)\nabla w_i - p(U^{p-1} - w_i^{p-1})\nabla w_i. \tag{4.12}$$

We begin by studying the first term on the right hand side of (4.12). For the sake of the simplicity of our argument, we shall discuss the first term on the right hand side of (4.12) for $i = 1$. The other terms can be treated in a similar manner. From Section 3, we have that $\rho(x)$ is uniformly bounded. Then,

$$\begin{aligned} &\Big| \int_{\mathbb{R}^2} (V(x) - 1)\nabla w_1 \phi \Big| \leq C\|\phi\|_* \int_{\mathbb{R}^2} \rho(x) \big| V(x) - 1 \big| \frac{1}{(1 + |x - Q_1|)^{2s+3}} \mathrm{d}x \\ &\leq C\|\phi\|_* \int_{|x| \leq \left|\frac{Q_1}{2}\right|} \rho(x) \big| V(x) - 1 \big| \frac{1}{(1 + |x - Q_1|)^{2s+3}} \mathrm{d}x \\ &\quad + C\|\phi\|_* \int_{|x| \geq \left|\frac{Q_1}{2}\right|} \rho(x) \big| V(x) - 1 \big| \frac{1}{(1 + |x - Q_1|)^{2s+3}} \mathrm{d}x \\ &\leq C\|\phi\|_* \frac{1}{R^{2s+1}} + C\|\phi\|_* \frac{1}{R^m} = o(1)\|\phi\|_* \text{ as } R \to \infty, \end{aligned}$$

Next, we consider the second term on the right hand side of (4.12). We divide $\mathbb{R}^2$ into K parts:

$$\mathbb{R}^2 = \bigcup_{j=1}^{K} \Omega_j,$$

where

$$\Omega_j = \{x \in \mathbb{R}^2 \big| |x - Q_j| = \min_i |x - Q_i|, \ i = 1, 2, \cdots, K\}.$$

When $1 < p \leq 2$, it holds that

$$\begin{aligned}\Big|\int_{\Omega_1}(w_1^{p-1}-U^{p-1})Z_{Q_1}\phi\Big| &\leq C\|\phi\|_*\int_{\Omega_1}\rho(x)\Big(\sum_{j=2}^{K}w_j\Big)^{p-1}|Z_{Q_1}| \\ &\leq C\|\phi\|_*\Big(\sum_{j=2}^{K}\frac{1}{|Q_1-Q_j|^{2s+2}}\Big)^{p-1}\int_{\Omega_1}\frac{1}{(1+|x-Q_1|)^{2s+3}} \\ &\leq C\|\phi\|_*\Big(\frac{K}{R}\Big)^{(2s+2)(p-1)},\end{aligned}$$

and for $\ell \neq 1$ we have that

$$\begin{aligned}&\Big|\int_{\Omega_\ell}(w_1^{p-1}-U^{p-1})Z_{Q_1}\phi\Big| \leq C\|\phi\|_*\int_{\Omega_\ell}\rho(x)\Big(\sum_{j=2}^{K}w_j\Big)^{p-1}|Z_{Q_1}|\mathrm{d}x \\ &\leq C\|\phi\|_*\int_{\Omega_\ell}\Big[\frac{1}{(1+|x-Q_\ell|)^{(2s+2)(p-1)}}+\Big(\frac{K}{R}\Big)^{(2s+2)(p-1)}\Big]\frac{1}{(1+|x-Q_1|)^{2s+3}}\mathrm{d}x \\ &\leq C\|\phi\|_*\Big[\Big(\frac{K}{R}\Big)^{(2s+2)(p-1)}\frac{1}{|Q_1-Q_\ell|^{2s+1}}+\frac{1}{|Q_1-Q_\ell|^{2s+3-(2-(2s+2)(p-1))_+}}\Big].\end{aligned}$$

Then,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2\setminus\Omega_1}(w_1^{p-1}-U^{p-1})Z_{Q_1}\phi\Big| \\ &\leq C\|\phi\|_*\sum_{\ell=2}^{K}\Big[\Big(\frac{K}{R}\Big)^{(2s+2)(p-1)}\frac{1}{|Q_1-Q_\ell|^{2s+1}}+\frac{1}{|Q_1-Q_\ell|^{2s+3-(2-(2s+2)(p-1))_+}}\Big] \\ &\leq C\|\phi\|_*\Big[\Big(\frac{K}{R}\Big)^{2s+1+(2s+2)(p-1)}+\Big(\frac{K}{R}\Big)^{2s+3-(2-(2s+2)(p-1))_+}\Big].\end{aligned}$$

Therefore, we obtain that

$$\Big|\int_{\Omega_1}(w_1^{p-1}-U^{p-1})Z_{Q_1}\phi\Big| = o(1)\|\phi\|_*.$$

For the case that $p > 2$, we have that

$$\begin{aligned}U^{p-1}-w_1^{p-1} &\leq C\Big(w_1^{p-2}\sum_{j=2}^{K}w_j+\Big(\sum_{j=2}^{K}w_j\Big)^{p-1}\Big) \\ &\leq C\Big[\Big(\frac{K}{R}\Big)^{2s+2}w_1^{p-2}+\Big(\frac{K}{R}\Big)^{(2s+2)(p-1)}\Big] \text{ in } \Omega_1,\end{aligned}$$

and

$$U^{p-1}-w_1^{p-1} \leq C\Big[w_\ell^{p-1}+\Big(\frac{K}{R}\Big)^{(p-1)(2s+2)}\Big] \text{ in } \Omega_\ell, \quad 2 \leq \ell \leq K.$$

As a consequence, we have that

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^2}(w_1^{p-1}-U^{p-1})Z_{Q_1}\phi\Big|\\
&\le C\|\phi\|_*\int_{\Omega_1}C\rho(x)\Big[\Big(\frac{K}{R}\Big)^{2s+2}w_1^{p-2}+\Big(\frac{K}{R}\Big)^{(2s+2)(p-1)}\Big]|Z_{Q_1}|\\
&\quad+C\|\phi\|_*\sum_{l=2}^{K}\int_{\Omega_\ell}\Big[\frac{1}{(1+|x-Q_\ell|)^{(2s+2)(p-1)}}+\Big(\frac{K}{R}\Big)^{(2s+2)(p-1)}\Big]|Z_{Q_1}|\mathrm{d}x\\
&\le C\|\phi\|_*\Big[\Big(\frac{K}{R}\Big)^{2s+2}+\Big(\frac{K}{R}\Big)^{2s+3-(2-(2s+2)(p-1))_+}\Big]=o(1)\|\phi\|_*.
\end{aligned}
$$

Therefore, we can conclude that

$$
\int_{\mathbb{R}^2}\phi L[Z_{Q_i}]\mathrm{d}x=o(1)\|\phi\|_*. \tag{4.13}
$$

On the other hand, it is easy to see that

$$
\int_{\mathbb{R}^2}hZ_{Q_i}\mathrm{d}x\le C\|h\|_*. \tag{4.14}
$$

By using (4.13), (4.14), and Lemma 4.1, we obtain that

$$
\mathbf{c}=o(1)\|\phi\|_*+O(1)\|h\|_*. \tag{4.15}
$$

Now, we prove the a priori estimate (4.11). We argue by contradiction. Suppose that there exist h_K and ϕ_K solving (4.10) for $h=h_K,\ R=R_K$, with $\|h_K\|_*\to 0$ and $\|\phi_K\|_*=1$ as $K\to\infty$. For simplicity, we drop the K in the subscript.

From the conditions on the potential V, it is obvious that $\inf_{\mathbb{R}^2}V>0$. On the other hand, from the equation satisfied by ϕ,

$$
(-\Delta)^s\phi+(V-pW^{p-1})\phi=h+\mathbf{c}\cdot\frac{\partial U}{\partial\mathbf{q}},
$$

we find that

$$
\begin{aligned}
V(x)-pW^{p-1}(x)\ge&V(x)-C\Big(\frac{1}{(1+|x-Q_1|)^{2s+2}}+\sum_{j=2}^{K}\frac{1}{|Q_j-Q_1|^{2s+2}}\Big)^{p-1}\\
\ge&V(x)-C\Big(\frac{1}{(1+|x-Q_1|)^{2s+2}}+\Big(\frac{K}{R}\Big)^{2s+2}\Big)^{p-1}\ge\frac{1}{2}V(x)
\end{aligned}
$$

for any $x\in\Omega_1\setminus B_r(Q_1)$, which leads to

$$
\inf_{\mathbb{R}^2\setminus\cup_{j=1}^K B_r(Q_j)}\big(V(x)-pW^{p-1}(x)\big)\ge\frac{1}{2}\inf_{\mathbb{R}^2}V(x)>0.
$$

Accordingly, from Lemma 2.5 and (4.13), it holds that

$$
\|\phi\|_*\le\ C\Big(\|\phi\|_{L^\infty(\cup_{j=1}^K B_r(q_j))}+\|h\|_*+|c|\Big\|\sum_{j=1}^{K}Z_j\Big\|_*\Big)\le C\|\phi\|_{L^\infty(\cup_{j=1}^K B_r(q_j))}+o(1),
$$

from which we may assume that, up to a subsequence,

$$
\|\phi\|_{L^\infty(B_r)(Q_1)}\ge\gamma>0. \tag{4.16}
$$

Let us set $\tilde{\phi}(x)=\phi(x+Q_1)$. Then, $\tilde{\phi}$ satisfies

$$
(-\Delta)^s\tilde{\phi}+V(x+Q_1)\tilde{\phi}-pw^{p-1}\tilde{\phi}=\tilde{g}, \tag{4.17}
$$

where

$$\begin{aligned}\tilde{g} =&g(x+Q_1)+\Big(c_1\cdot Z_1(x+Q_1)+\sum_{j=2}^{K}c_j\cdot Z_j(x+Q_1)\Big)\\&+p\Big[\Big(w(x)+\sum_{j=2}^{K}w(x+Q_1-Q_j)\Big)^{p-1}-w^{p-1}\Big]\tilde{\phi}.\end{aligned}\tag{4.18}$$

For any point x in an arbitrary compact set of $\mathbb{R}^2$, we have that

$$|g(x+Q_1)|\le\|g\|_*\rho(x+Q_1)\le C\|g\|_*=o(1).$$

It is easy to see that $V(x+Q_1)\to 1$,

$$\mathbf{c}=o(\|\phi\|_*)+O(\|g\|_*)\to 0,$$

and

$$\begin{aligned}\Big|Z_1(x+Q_1)+\sum_{j=2}^{K}Z_j(x+Q_1)\Big|\le&\frac{C}{(1+|x+Q_1|)^{2s+2}}+\sum_{j=2}^{K}\frac{C}{|x+Q_1-Q_j|^{2s+2}}\\\le&\,C+\sum_{j=2}^{K}\frac{C}{|Q_1-Q_j|^{2s+2}}\le C.\end{aligned}$$

For the last term in (4.18), as $1<p\le 2$ it holds that

$$\begin{aligned}&\Big|\Big[\Big(w(x)+\sum_{j=2}^{K}w(x+Q_1-Q_j)\Big)^{p-1}-w(x)^{p-1}\Big]\tilde{\phi}\Big|\\\le&\,C\Big(\sum_{j=2}^{K}w(x+Q_1-Q_j)\Big)^{p-1}\le C\Big(\sum_{j=2}^{K}\frac{1}{|x+Q_1-Q_j|^{2s+2}}\Big)^{p-1}\\\le&\,C\Big(\sum_{j=2}^{K}\frac{1}{|Q_1-Q_j|^{2s+2}}\Big)^{p-1}\le C\Big(\frac{K}{R}\Big)^{(2s+2)(p-1)},\end{aligned}$$

while for $p>2$ we have that

$$\begin{aligned}&\Big|\Big[\Big(w(x)+\sum_{j=2}^{K}w(x+Q_1-Q_j)\Big)^{p-1}-w^{p-1}(x)\Big]\tilde{\phi}\Big|\\\le&\,Cw^{p-2}\sum_{j=2}^{K}w(x+Q_1-Q_j)+C\Big(\sum_{j=2}^{K}w(x+Q_1-Q_j)\Big)^{p-1}\\\le&\sum_{j=2}^{K}\frac{C}{|x+Q_1-Q_j|^{2s+2}}+C\Big(\sum_{j=2}^{K}\frac{1}{|x+Q_1-Q_j|^{2s+2}}\Big)^{p-1}\\\le&\sum_{j=2}^{K}\frac{C}{|Q_1-Q_j|^{2s+2}}+C\Big(\sum_{j=2}^{K}\frac{1}{|Q_1-Q_j|^{2s+2}}\Big)^{p-1}\\\le&\,C\Big(\frac{K}{R}\Big)^{2s+2}+C\Big(\frac{K}{R}\Big)^{(2s+2)(p-1)}\le C\Big(\frac{K}{R}\Big)^{2s+2}.\end{aligned}$$

Hence, $\tilde{g}\to 0$ uniformly on any compact set of $\mathbb{R}^2$ as $K\to\infty$. Meanwhile, by considering

$$(-\Delta)^s\tilde{\phi}+\tilde{\phi}=(1-V(x+Q_1))\tilde{\phi}+pw^{p-1}\tilde{\phi}+\tilde{g},$$

and applying Lemma 2.3, we obtain that

$$\begin{aligned}\sup_{x\neq y}\frac{|\tilde\phi(x)-\tilde\phi(y)|}{|x-y|^{\beta}} &\le C\big(\|(1-V)\tilde\phi\|_{L^\infty}+\|w^{p-1}\tilde\phi\|_{L^\infty}+\|\tilde g\|_{L^\infty}\big)\\ &\le C(\|\phi\|_*+\|\tilde g\|_{L^\infty})\le C,\end{aligned}$$

where $\beta=\min\{1,2s\}$. Hence, up to a subsequence, we may assume that $\tilde\phi\to\phi_0$ uniformly on any compact set. It is easy to see that $\tilde\phi_0$ satisfies

$$\begin{cases}(-\Delta)^s\phi_0+\phi_0-pw^{p-1}\phi_0=0 \text{ in } \mathbb{R}^2,\\ \phi_0\in H^{2s},\quad \int_{\mathbb{R}^2}\nabla w\phi_0=0,\end{cases}$$

where $x=(x^1,x^2)$. Furthermore, by using the fact that $\mu>1$ we have that

$$\int_{B_R(0)}\phi_0^2\le\int_{B_R(0)}\tilde\phi_k^2=\int_{B_R(Q_1)}\phi_k^2\le\|\phi_k\|_*^2\int_{B_R(Q_1)}\rho^2\le C,$$

which means that $\phi_0\in L^2(\mathbb{R}^2)$. Then, the non-degeneracy result in [27] together with the orthogonality condition implies that $\phi_0\equiv 0$, which contradicts (4.16). Hence, have we proved (4.11). Once we obtain (4.11), we can follow the same steps as in [18] to obtain a solution (4.10). Then, the lemma will be proved. □

Before we give the complete proof of Proposition 4.1, we will estimate the error.

Lemma 4.2. *Given $(Q_1,Q_2,\cdots,Q_K)\in\Lambda_K$, for any fixed $\mu\in(1,2+2s)$ there exists a constant C (independent of K) such that*

$$\|E\|_*\le CR^{-\tau_0}. \tag{4.19}$$

Proof. By the definition of E, we have that

$$E=\sum_{j=1}^K(V(x)-1)w_j-\Big\{\Big(\sum_{j=1}^K w_j\Big)^p-\sum_{j=1}^K w_j^p\Big\}=E_1+E_2.$$

We simply assume that $x\in\Omega_1$ in the following proof, because the other parts can be treated similarly. We first consider E_1. If $|x|\ge\frac{|Q_1|}{2}$, then

$$V(x)-1=O(\frac{1}{|x|^m})=O(\frac{1}{R^m}),$$

and in this region we have that

$$\begin{aligned}\Big|\rho^{-1}\sum_{j=1}^K(1-V(x))w_j\Big| &\le\frac{C}{R^m}\Big(\sum_{i=1}^K\frac{1}{(1+|x-Q_i|)^\mu}\Big)^{-1}\sum_{j=1}^K w_j\\ &\le\frac{C}{R^m}\Big(\sum_{i=1}^K\frac{1}{(1+|x-Q_i|)^\mu}\Big)^{-1}\sum_{j=1}^K\frac{1}{(1+|x-Q_j|)^{2s+2}}\\ &\le\frac{C}{R^m}\Big(\sum_{i=1}^K\frac{1}{(1+|x-Q_i|)^\mu}\Big)^{-1}\sum_{j=1}^K\frac{1}{(1+|x-Q_j|)^{\mu}}\\ &\le\frac{C}{R^m}.\end{aligned}$$

Meanwhile, for $|x|\le\frac{R}{2}$, we have that

$$|x-Q_1|\ge|Q_1|-|x|\ge\frac{R}{2}\text{ and }|x-Q_j|\ge\frac{R}{2}\text{ for } j=2,\cdots,K.$$

Hence,

$$
\begin{aligned}
\Big|\rho^{-1}\sum_{j=1}^{K}(1-V(x))w_j\Big| &\le C\rho^{-1}\sum_{j=1}^{K}w_j \le\ C\rho^{-1}\sum_{j=1}^{K}\frac{1}{(1+|x-Q_j|)^{2s+2}}\\
&\le C\rho^{-1}\sum_{j=1}^{K}\frac{1}{(1+|x-Q_j|)^{\mu}}\cdot\frac{1}{(1+|x-Q_j|)^{2s+2-\mu}}\\
&\le C\Big(\sum_{i=1}^{K}\frac{1}{(1+|x-Q_i|)^{\mu}}\Big)^{-1}\sum_{j=1}^{K}\frac{1}{(1+|x-Q_j|)^{\mu}}\frac{1}{R^{2s+2-\mu}}\\
&\le \frac{C}{R^{2s+2-\mu}}.
\end{aligned}
$$

For E_2, we observe that

$$
\Big|\Big(\sum_{j=1}^{K}w_j\Big)^p-\sum_{j=1}^{K}w_j^p\Big|\le\ C\Big(w_1^{p-1}\sum_{j=2}^{K}w_j+\sum_{j=2}^{K}w_j^p+(\sum_{j=2}^{K}w_j)^p\Big).
$$

In the case that $\mu\le(2s+2)(p-1)$, it holds that

$$
\begin{aligned}
\rho^{-1}W_1^{p-1}\sum_{j=2}^{K}W_j &\le C\rho^{-1}\frac{1}{(1+|x-Q_1|)^{(2+2s)(p-1)}}\sum_{j=2}^{K}\frac{1}{(1+|x-Q_j|)^{2s+2}}\\
&\le C\frac{(1+|x-Q_1|)^{\mu}}{(1+|x-Q_1|)^{(2s+2)(p-1)}}\sum_{j=2}^{K}\frac{1}{(1+|x-Q_j|)^{2s+2}}\\
&\le C\sum_{j=2}^{K}\frac{1}{(1+|Q_1-Q_j|)^{2s+2}}\le\ C\Big(\frac{K}{R}\Big)^{2s+2}.
\end{aligned}
$$

Otherwise, if $\mu>(2+2s)(p-1)$ then

$$
\begin{aligned}
\rho^{-1}W_1^{p-1}\sum_{j=2}^{K}W_j &\le C\rho^{-1}\frac{1}{(1+|x-Q_1|)^{(2s+2)(p-1)}}\sum_{j=2}^{K}\frac{1}{(1+|x-Q_j|)^{2s+2}}\\
&\le C\rho^{-1}\frac{1}{(1+|x-Q_1|)^{\mu}}\sum_{j=2}^{K}\frac{1}{(1+|x-Q_j|)^{2s+2-\mu+(2s+2)(p-1)}}\\
&\le C\sum_{j=2}^{K}\frac{1}{|Q_j-Q_1|^{(2s+2)p-\mu}}\le C\Big(\frac{K}{R}\Big)^{(2s+2)p-\mu}.
\end{aligned}
$$

It is easy to deduce that

$$
\begin{aligned}
\rho^{-1}\sum_{j=2}^{K}W_j^p &\le C\rho^{-1}\sum_{j=2}^{K}\frac{1}{(1+|x-Q_j|)^{(2+2s)p-\mu}}\frac{1}{(1+|x-Q_1|)^{\mu}}\\
&\le C\sum_{j=2}^{K}\frac{1}{|Q_j-Q_1|^{(2s+2)p-\mu}}\le C\Big(\frac{K}{R}\Big)^{(2s+2)p-\mu}
\end{aligned}
$$

and

$$\rho^{-1}\Big(\sum_{j=2}^{K}W_j\Big)^p \le C\rho^{-1}\Big(\sum_{j=2}^{K}\frac{1}{(1+|x-Q_j|)^{2s+2-\frac{\mu}{p}}}\frac{1}{(1+|x-Q_1|)^{\frac{\mu}{p}}}\Big)^p$$
$$\le C\Big(\sum_{j=2}^{K}\frac{1}{|Q_j-Q_1|^{2s+2-\frac{\mu}{p}}}\Big)^p \le C\Big(\frac{K}{R}\Big)^{(2s+2)p-\mu}.$$

Thus, we obtain the desired result by combining the above estimates. □

Now, we are in the position to give the proof of Proposition 4.1.

Proof of Proposition 4.1. We write the problem (4.4) as a fixed point problem:

$$\phi = T(E+N(\phi)) =: \mathcal{A}\phi, \text{ for } \phi\in H^{2s}.$$

Let

$$\mathfrak{F} = \{\phi\in H^{2s} \mid \|\phi\|_* \le \delta\},$$

where $\delta>0$ is a small number to be determined later.

Let $\phi\in\mathfrak{F}$. Then for $1<p\le 2$ we have that

$$\|N(\phi)\|_* \le C\|\phi^p\|_* \le C\|\phi\|_*^p\|\rho^{p-1}\|_{L^\infty(\mathbb{R}^2)} \le C\|\phi\|_*^p,$$

or if $p>2$ we have that

$$\begin{aligned}\|N(\phi)\|_* &\le C\|\phi^2W^{p-2}\|_* + C\|\phi^p\|_*\\ &\le C\|\phi\|_*^2|\rho W^{p-2}|_{L^\infty(\mathbb{R}^2)} + C\|\phi\|_*^p\|\rho^{p-1}\|_{L^\infty(\mathbb{R}^2)}\\ &\le C\|\phi\|_*^2.\end{aligned}$$

By Proposition 4.2 and Lemma 4.2, we have that

$$\|\mathcal{A}(\phi)\|_* \le C(\|E\|_* + \|N(\phi)\|_*) \le C\|E\|_* + C(\|\phi\|_* + \|\phi\|_*^{p-1})\|\phi\|_* \le \delta.$$

Then, we can choose $C(\delta+\delta^{p-1})\le\frac{1}{2}$ and K sufficiently large such that

$$C\|E\|_* \le CR^{-\tau_0} \le \frac{1}{2}\delta.$$

On the other hand, for any $\phi_i\in H^{2s},\ i=1,2$, we have that

$$|N(\phi_1)-N(\phi_2)| = |N'(\xi)(\phi_1-\phi_2)|,$$

where ξ lies between ϕ_1 and ϕ_2.

For $1<p\le 2$, it holds that $N'(t)\le Ct^{p-1}\le C(|\phi_1|^{p-1}+|\phi_2|^{p-1})$, which tells us that

$$\begin{aligned}\|N(\phi_1)-N(\phi_2)\|_* &\le C\|\phi_1-\phi_2\|_*(\|\phi_1\|_{L^\infty(\mathbb{R}^2)}^{p-1} + \|\phi_2\|_{L^\infty(\mathbb{R}^2)}^{p-1})\\ &\le C\|\phi_1-\phi_2\|_*(\|\phi_1\|_*^{p-1}+\|\phi_2\|_*^{p-1})\|\rho\|_{L^\infty(\mathbb{R}^2)}^{p-1}\\ &\le C\|\phi_1-\phi_2\|_*(\|\phi_1\|_*^{p-1}+\|\phi_2\|_*^{p-1})\\ &\le C\delta^{p-1}\|\phi_1-\phi_2\|_* \le \frac{1}{2}\|\phi_1-\phi_2\|_*\end{aligned}$$

Provided that δ is small enough. For $p > 2$, $N'(t) \leq C(W^{p-2}|t| + |t|^{p-1})$, from which we can deduce that

$$
\begin{aligned}
&\|N(\phi_1) - N(\phi_2)\|_* \\
&\leq C\|\phi_1 - \phi_2\|_* \Big[\|\rho W^{p-2}\|_{L^\infty(\mathbb{R}^2)} (\|\phi_1\|_* + \|\phi_2\|_*) + (\|\phi_1\|_*^{p-1} + \|\phi_2\|_*^{p-1}) \|\rho\|_{L^\infty(\mathbb{R}^2)}^{p-1} \Big] \\
&\leq C\|\phi_1 - \phi_2\|_* (\|\phi_1\|_* + \|\phi_2\|_* + \|\phi_1\|_*^{p-1} + \|\phi_2\|_*^{p-1}) \\
&\leq C(\delta + \delta^{p-1}) \|\phi_1 - \phi_2\|_* \leq \frac{1}{2} \|\phi_1 - \phi_2\|_*
\end{aligned}
$$

Provided that δ is small enough. Thus, we obtain that $\mathcal{A}$ is a contraction mapping, and the problem (4.4) admits a unique solution ϕ. Clearly, Lemma 4.2 implies that

$$\|\phi\|_* \leq CR^{-\tau_0}.$$

Now, we will consider the differentiability of $\phi(x; \alpha, \mathbf{q})$ as function of $(\alpha, \mathbf{q})$. Consider the map

$$\mathcal{T} : \mathbb{R} \times \mathbb{R}^{2K} \times \mathfrak{B} \times \mathbb{R}^{2K} \to \mathfrak{B} \times \mathbb{R}^{2K}$$

of class C^1 given by:

$$
\mathcal{T}(\alpha, \mathbf{q}, \phi, \mathbf{c}) = \begin{pmatrix} \big((-\Delta)^s + 1\big)^{-1} S(U + \phi) - \mathbf{c}\big((-\Delta)^s + 1\big)^{-1} \frac{\partial U}{\partial \mathbf{q}} \\ \int_{\mathbb{R}^2} \phi Z_{Q_1} \mathrm{d}x \\ \vdots \\ \int_{\mathbb{R}^2} \phi Z_{Q_K} \mathrm{d}x \end{pmatrix},
$$

where $\mathfrak{B} = H^{2s}(\mathbb{R}^2) \cap \mathfrak{B}_*$. Equation (4.4) is equivalent to $\mathcal{T}(\alpha, \mathbf{q}, \phi, \mathbf{c}) = 0$. By the above argument, we know that given $\alpha \in \mathbb{R}$ and $\mathbf{q}$ satisfying (3.9), there exists a unique local solution $(\phi(\alpha, \mathbf{q}), \mathbf{c}(\alpha, \mathbf{q}))$. In the following, we write $(\phi, \mathbf{c}) = (\phi(\alpha, \mathbf{q}), \mathbf{c}(\alpha, \mathbf{q}))$ for simplicity. We claim that the linear operator

$$\frac{\partial \mathcal{T}(\alpha, \mathbf{q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\Big|_{(\alpha, \mathbf{q}, \phi, \mathbf{c})} : \mathfrak{B} \times \mathbb{R}^{2K} \to \mathfrak{B} \times \mathbb{R}^{2k}$$

is invertible for large K. Then, the C^1- regularity of $(\alpha, \mathbf{q}) \mapsto (\phi, \mathbf{c})$ follows from the implicit function theorem. Indeed, we have that

$$
\frac{\partial \mathcal{T}(\alpha, \mathbf{q}, \phi, \mathbf{c})}{\partial(\phi, \mathbf{c})}\Big|_{(\alpha, \mathbf{q}, \phi, \mathbf{c})} [\varphi, \vec{\xi}] = \begin{pmatrix} \big((-\Delta)^s + 1\big)^{-1} S'(U + \phi)[\varphi] - \vec{\xi}\big((-\Delta)^s + 1\big)^{-1} \frac{\partial U}{\partial \mathbf{q}} \\ \int_{\mathbb{R}^2} \varphi Z_{Q_1} \mathrm{d}x \\ \vdots \\ \int_{\mathbb{R}^2} \varphi Z_{Q_K} \mathrm{d}x \end{pmatrix}.
$$

Next, we study the dependence of ϕ on $(\alpha, \mathbf{q})$. Assume that we have two solutions corresponding to two sets of parameters. Let one of these be denoted by

$$L[\phi] + E + N(\phi) = \mathbf{c} \nabla_{\mathbf{q}} U,$$

corresponding to the parameters α and $\mathbf{q}$, and the other by

$$\mathring{L}[\mathring{\phi}] + \mathring{E} + \mathring{N}(\mathring{\phi}) = \mathring{\mathbf{c}} \mathring{\nabla}_{\mathbf{q}} U,$$

corresponding to the parameters $\mathring{\alpha}$ and $\mathring{\mathbf{q}}$. Observe that ϕ is L^2- orthogonal to $\nabla_{\mathbf{q}} U$, while $\mathring{\phi}$ is L^2- orthogonal to $\mathring{\nabla}_{\mathbf{q}} U$. To compare $\mathring{\phi}$ and ϕ, we first choose a vector $\vec{\omega}$ such that

$$\mathring{\phi}_\omega = \mathring{\phi} + \vec{\omega}\nabla_{\mathbf{q}} U$$

satisfies the same orthogonality condition as ϕ. Moreover, by the equation for $\mathring{\phi}$, the function $\mathring{\phi}_\omega$ satisfies the equation

$$L[\mathring{\phi}_\omega] + (\mathring{L} - L)[\mathring{\phi}] - \vec{\omega} \cdot L[\nabla_{\mathbf{q}} U] + \mathring{E} + \mathring{N}(\mathring{\phi}) + \mathring{\mathbf{c}}(\nabla_{\mathbf{q}} U - \mathring{\nabla}_{\mathbf{q}} U) = \mathring{\mathbf{c}} \cdot \nabla_{\mathbf{q}} U.$$

By taking the difference with the equation satisfied by ϕ, we obtain that

$$\begin{aligned} L[\mathring{\phi}_\omega - \phi] =& (L - \mathring{L})[\mathring{\phi}] + \vec{\omega} \cdot L[\nabla_{\mathbf{q}} U] + (E - \mathring{E}) + (N(\phi) - \mathring{N}(\mathring{\phi})) \\ & - \mathring{\mathbf{c}}(\nabla_{\mathbf{q}} U - \mathring{\nabla}_{\mathbf{q}} U) + (\mathring{\mathbf{c}} - \mathbf{c}) \cdot \nabla_{\mathbf{q}} U. \end{aligned}$$

For $j = 1, 2, \cdots, K$, we have that

$$|\mathring{Q}_j - Q_j| \le C(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty).$$

Assuming that $(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty) \le \frac{1}{2}$, we then have that

$$\|(\mathring{L} - L)[\mathring{\phi}]\|_* \le CR^{-\tau_0}(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty),$$

$$\|\vec{\omega} \cdot L[\nabla_{\mathbf{q}} U]\|_* \le CR^{-\tau_0}\|\vec{\omega}\|_\infty,$$

$$\|E - \mathring{E}\|_* \le CR^{-\tau_0}(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty),$$

$$\begin{aligned} \|N(\phi) - \mathring{N}(\mathring{\phi})\|_* \le & CR^{-\min\{p-1,1\}\tau_0}(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty) \\ & + CR^{-(p-1)\tau_0}\|\phi - \mathring{\phi}\|_*, \end{aligned}$$

and

$$\|\mathring{\mathbf{c}}(\nabla_{\mathbf{q}} U - \mathring{\nabla}_{\mathbf{q}} U)\|_* \le CR^{-\tau_0}(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty).$$

Hence, by Lemma 3.1 we have that

$$\begin{aligned} \|\mathring{\phi}_\omega - \phi\|_* + \|\mathring{\mathbf{c}} - \mathbf{c}\|_\infty \le & CR^{-\min\{p-1,1\}\tau_0}(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty) \\ & + CR^{-\tau_0}\|\vec{\omega}\|_\infty + CR^{-(p-1)\tau_0}\|\phi - \mathring{\phi}\|_*. \end{aligned}$$

On the other hand, from the definition of $\mathring{\phi}_\omega$ we have that

$$\begin{aligned} \|\vec{\omega}\|_\infty & \le C\|\mathring{\phi}\|_*(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty) \\ & \le CR^{-\tau_0}(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty). \end{aligned}$$

Hence,

$$\|\mathring{\phi} - \phi\|_* + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty \le CR^{-\min\{p-1,1\}\tau_0}(R|\mathring{\alpha} - \alpha| + \|\mathring{\mathbf{q}} - \mathbf{q}\|_\infty)$$

Therefore, we have completed the proof of Proposition 4.1. $\square$

5. A FURTHER REDUCTION PROCESS

The main purpose of this section is to achieve Step 2.A. We will define

$$\vec{\beta} = \hat{\beta} - \gamma(R\mathbf{q}_0 + \mathbf{q}^{\perp}), \forall\, \gamma \in \mathbb{R}. \tag{5.1}$$

The equation (4.4) then becomes

$$L(\phi) + E + N(\phi) = \vec{\beta} \cdot \frac{\partial U}{\partial \mathbf{q}} + \gamma \frac{\partial U}{\partial \alpha}. \tag{5.2}$$

Note that ϕ does not depend on γ, but $\vec{\beta}$ depends on the parameters $\alpha, \mathbf{q}$, and γ. We write $\vec{\beta} = \vec{\beta}(\alpha, \mathbf{q}, \gamma)$.

In this section, we will solve $\vec{\beta} = 0$ for each $\alpha \in \mathbb{R}$ by adjusting $\mathbf{q}$ and γ. By multiplying (5.2) by $\frac{\partial U}{\partial \mathbf{q}}$ and integrating over $\mathbb{R}^2$, we obtain that

$$\int_{\mathbb{R}^2} (L(\phi) + E + N(\phi)) \frac{\partial U}{\partial \mathbf{q}} dx = M\vec{\beta} + \gamma \int_{\mathbb{R}^2} \frac{\partial U}{\partial \alpha} \frac{\partial U}{\partial \mathbf{q}} dx.$$

Because the matrix M is invertible, solving $\vec{\beta} = 0$ is equivalent to solving

$$\int_{\mathbb{R}^2} (L(\phi) + E + N(\phi)) \frac{\partial U}{\partial \mathbf{q}} dx = \gamma \int_{\mathbb{R}^2} \frac{\partial U}{\partial \alpha} \frac{\partial U}{\partial \mathbf{q}} dx. \tag{5.3}$$

In the following subsection, we will compute the projections of the error and the terms involving ϕ.

5.1. **Projections.** We first compute $\int_{\mathbb{R}^2} E \frac{\partial U}{\partial \mathbf{q}} dx$. We begin with the following lemma.

Lemma 5.1. *The following expansion holds:*

$$\begin{aligned}\int_{\mathbb{R}^2} EZ_{Q_k} dx = \Gamma |Q_k|^{-m-1} \frac{Q_k}{|Q_k|} &+ \sum_{j \neq k} \Psi(|Q_j - Q_k|) \frac{Q_j - Q_k}{|Q_j - Q_k|} \\ &+ R^{-\tau_1} \Pi_{1,k}(\alpha, \mathbf{q}) + R^{-\tau_2} \Pi_{2,k}(\alpha, \mathbf{q}),\end{aligned}$$

where $\Gamma = \frac{am}{2} \int_{\mathbb{R}^2} w^2(x) dx$; τ_1 and τ_2 as given in (5.4) and (5.9) are numbers depending on p, m, and s only; and $\Pi_{ik}(\alpha, \mathbf{q})$, $i = 1, 2$, are smooth vector valued functions that are uniformly bounded as $K \to \infty$.

Proof. By the definition of the error E, we know that

$$\begin{aligned}\int_{\mathbb{R}^2} EZ_{Q_k} dx &= \sum_{j=1}^{k} \int_{\mathbb{R}^2} (V(x) - 1) w_j \nabla w_k dx - \int_{\mathbb{R}^2} \Big[\Big(\sum_{j=1}^{K} w_j \Big)^p - \sum_{j=1}^{K} w_j^p \Big] \nabla w_k dx \\ &= I_1 - I_2.\end{aligned}$$

We first estimate I_1:

$$\begin{aligned}I_1 &= \sum_{j=1}^{K} \int_{\mathbb{R}^2} (V(x) - 1) w_j \nabla w_k dx \\ &= \int_{\mathbb{R}^2} (V(x) - 1) w_k \nabla w_k dx + \sum_{j \neq k} \int_{\mathbb{R}^2} (V(x) - 1) w_j \nabla w_k dx \\ &= I_{11} + I_{12}.\end{aligned}$$

$$
\begin{aligned}
I_{11} &= \int_{|x|\leq\frac{R}{2}} (V(x)-1) w_{Q_k} \nabla w_{Q_k} dx + \int_{|x|\geq\frac{R}{2}} (V(x)-1) w_{Q_k} \nabla w_{Q_k} dx \\
&= \int_{|x|\geq\frac{R}{2}} \frac{a}{|x|^m} w_{Q_k} \nabla w_{Q_k} dx + O(R^{-4s-3}) + O(R^{-m-\sigma}) \\
&= \int_{|y+Q_k|\geq\frac{R}{2}} \frac{a}{|y+Q_k|^m} w(y) \nabla w(y) dy + O(R^{-4s-3}) + O(R^{-m-\sigma}) \\
&= \frac{am}{2} \frac{Q_k}{|Q_k|^{m+2}} \int_{\mathbb{R}^N} w^2(x) dx + O(R^{-4s-3}) + O(R^{-m-\sigma}) + O(R^{-m-3}),
\end{aligned}
$$

where in the last equality we use the following identities:

$$
\int_{\mathbb{R}^2} w(y) \nabla w(y) dy = 0, \ \int_{\mathbb{R}^2} y_j y_k w(y) \nabla w(y) dy = 0, \ \forall\, j,k = 1,2,
$$

and

$$
\int_{\mathbb{R}^2} (y^T \cdot \vec{e}) w(y) \nabla w(y) dy = -\frac{1}{2} \int_{\mathbb{R}^2} w^2 dy \cdot \vec{e}, \quad \forall\, \vec{e} \in \mathbb{R}^2.
$$

For the term I_{12}, we have that

$$
\begin{aligned}
I_{12} &= \sum_{j\neq k} \int_{\mathbb{R}^2} (V(x)-1) w_{Q_j} \nabla w_{Q_k} dx \\
&= \sum_{j\neq k} \int_{|x|\leq\frac{R}{2}} (V(x)-1) w_{Q_j} \nabla w_{Q_k} dx + \sum_{j\neq k} \int_{|x|>\frac{R}{2}} (V(x)-1) w_{Q_j} \nabla w_{Q_k} dx \\
&= \sum_{j\neq k} \int_{|y+Q_k|>\frac{R}{2}} \frac{a}{|y+Q_k|^m} w(y+Q_k-Q_j) \nabla w(y) dy \\
&\quad + O(R^{-m-\sigma}) \sum_{j\neq k} \frac{1}{|Q_j-Q_k|^{2s+2}} + O(KR^{-4s-3}) \\
&= O(R^{-m}) \left[\sum_{j\neq k} \frac{1}{|Q_j-Q_k|^{2s+3}} \Big(\frac{Q_j-Q_k}{|Q_j-Q_k|} \Big) + \sum_{j\neq k} \frac{1}{|Q_j-Q_k|^{2s+4}} \right] \\
&\quad + O(R^{-m-\sigma}) \sum_{j\neq k} \frac{1}{|Q_j-Q_k|^{2s+2}} + O(KR^{-4s-3}) \\
&= O(R^{-2m-1}) + O(R^{-2m-\frac{m}{s+1}}) + O(R^{-4s-2-\frac{m}{2s+2}}) + O(R^{-2m-\sigma}),
\end{aligned}
$$

where we have used $\frac{Q_{k-j}-Q_k}{|Q_{k-j}-Q_k|} + \frac{Q_{k+j}-Q_k}{|Q_{k+j}-Q_k|} = 2\sin\frac{j\pi}{K}$ in the last step. Thus, we have that

$$
I_1 = \frac{am}{2} \frac{Q_k}{|Q_k|^{m+2}} \int_{\mathbb{R}^2} w^2(x) dx + O(R^{-\tau_1}),
$$

where

$$
\tau_1 = \min\left\{ m+3,\ m+\sigma,\ 2m+1,\ 2m+\frac{m}{s+1},\ 4s+2+\frac{m}{2s+2} \right\}. \tag{5.4}
$$

Next, we consider I_2.

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^2} \left(\Big(\sum_{j=1}^{K} w_{Q_j} \Big)^p - \sum_{j=1}^{K} w_{Q_j}^p \right) \nabla w_{Q_k} dx \\
&= \int_{\mathbb{R}^2} p w_{Q_k}^{p-1} (\sum_{j \neq k} w_{Q_j}) \nabla w_{Q_k} dx \\
&\quad + \int_{\mathbb{R}^2} \left(\Big(\sum_{j=1}^{k} w_{Q_j} \Big)^p - w_{Q_k}^p - p w_{Q_k}^{p-1} \Big(\sum_{j \neq k} w_{Q_j} \Big) - \sum_{j \neq k} w_{Q_j}^p \right) \nabla w_{Q_k} dx \\
&= I_{21} + I_{22}.
\end{aligned}
$$

By the definition of the interaction function Ψ, we have that

$$
I_{21} = - \sum_{j \neq k} \Psi(|Q_j - Q_k|) \frac{Q_j - Q_k}{|Q_j - Q_k|}. \tag{5.5}
$$

For the term I_{22}, we divide our discussion into two cases. If $1 < p < 2$, we write

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \left(\Big(\sum_{j=1}^{K} w_{Q_j} \Big)^p - w_{Q_k}^p - p w_{Q_k}^{p-1} \Big(\sum_{j \neq k} w_{Q_j} \Big) - \sum_{j \neq k} w_{Q_j}^p \right) \nabla w_{Q_k} dx \\
= &\int_{\Omega_k} \left(\Big(\sum_{j=1}^{K} w_{Q_j} \Big)^p - w_{Q_k}^p - p w_{Q_k}^{p-1} \Big(\sum_{j \neq k} w_{Q_j} \Big) - \sum_{j \neq k} w_{Q_j}^p \right) \nabla w_{Q_k} dx \\
+ &\int_{\mathbb{R}^2 \setminus \Omega_k} \left(\Big(\sum_{j=1}^{K} w_{Q_j} \Big)^p - w_{Q_k}^p - p w_{Q_k}^{p-1} \Big(\sum_{j \neq k} w_{Q_j} \Big) - \sum_{j \neq k} w_{Q_j}^p \right) \nabla w_{Q_k} dx.
\end{aligned} \tag{5.6}
$$

For the first term on the right hand side of (5.6), we have that

$$
\begin{aligned}
&\left| \int_{\Omega_k} \left(\Big(\sum_{j=1}^{K} w_{Q_j} \Big)^p - w_{Q_k}^p - p w_{Q_k}^{p-1} \Big(\sum_{j \neq k} w_{Q_j} \Big) - \sum_{j \neq k} w_{Q_j}^p \right) \nabla w_{Q_k} dx \right| \\
&\leq C \int_{\Omega_k} \Big(\sum_{j \neq k} w_{Q_j} \Big)^p |\nabla w_{Q_k}| + \Big(\sum_{j \neq k} w_{Q_j}^p \Big) |\nabla w_{Q_k}| dx \\
&\leq C \Big[\Big(\sum_{j \neq k} \frac{1}{|Q_j - Q_k|^{(2s+2)}} \Big)^p + \sum_{j \neq k} \frac{1}{|Q_j - Q_k|^{2p(s+1)}} \Big] \leq C R^{-pm},
\end{aligned}
$$

where we have used that $(1+x)^p \le 1 + px + Cx^p$ for small x, and C is a constant that depends on p only. Similarly,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2 \setminus \Omega_k} \left(\Big(\sum_{j=1}^{K} w_{Q_j} \Big)^p - w_{Q_k}^p - p w_{Q_k}^{p-1} \Big(\sum_{j \neq k} w_{Q_j} \Big) - \sum_{j \neq k} w_{Q_j}^p \right) \nabla w_{Q_k} dx \right| \\
\le& \sum_{\ell \neq k} \int_{\Omega_\ell} C w_{Q_\ell}^{p-1} \Big(\sum_{j \neq \ell} w_{Q_j} \Big) |\nabla w_{Q_k}| dx + \sum_{\ell \neq k} \int_{\Omega_\ell} \sum_{j \neq \ell} w_{Q_j}^p |\nabla w_{Q_k}| dx \\
&+ \sum_{\ell \neq k} \int_{\Omega_\ell} \left(p w_{Q_k}^{p-1} w_{Q_\ell} |\nabla w_{Q_k}| + p w_{Q_k}^{p-1} \Big(\sum_{j \neq \ell, k} w_{Q_j} \Big) |\nabla w_{Q_k}| \right) dx \\
\le& C \sum_{\ell \neq k} \Big(\frac{1}{R^m} \frac{1}{|Q_l - Q_k|^{2s+3-(2-2(p-1)(s+1))_+}} + \frac{1}{|Q_l - Q_k|^{2p(s+1)}} \Big) \le C R^{-pm}.
\end{aligned} \tag{5.7}$$

Therefore, we have for $1 < p < 2$ that

$$I_{22} \le C R^{-pm}. \tag{5.8}$$

If $p > 2$, then

$$\begin{aligned}
&\left| \int_{\Omega_k} \left(\Big(\sum_{j=1}^{K} w_{Q_j} \Big)^p - w_{Q_k}^p - p w_{Q_k}^{p-1} \Big(\sum_{j \neq k} w_{Q_j} \Big) - \sum_{j \neq k} w_{Q_j}^p \right) \nabla w_{Q_k} dx \right| \\
\le& C \int_{\Omega_k} w_{Q_k}^{p-2} \Big(\sum_{j \neq k} w_{Q_j} \Big)^2 |\nabla w_{Q_k}| + \Big(\sum_{j \neq k} w_{Q_j} \Big)^p |\nabla w_{Q_k}| dx \\
\le& C \Big(\sum_{j \neq k} \frac{1}{|Q_j - Q_k|^{4s+4}} + \sum_{j \neq k} \frac{1}{|Q_j - Q_k|^{2p(s+1)}} \Big) \\
\le& C R^{-2m},
\end{aligned}$$

and similarly to (5.7) we have that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2 / \Omega_k} \left(\Big(\sum_{j=1}^{K} w_{Q_j} \Big)^p - w_{Q_k}^p - p w_{Q_k}^{p-1} \Big(\sum_{j \neq k} w_{Q_j} \Big) - \sum_{j \neq k} w_{Q_j}^p \right) \nabla w_{Q_k} dx \right| \\
\le& \sum_{\ell \neq k} \int_{\Omega_\ell} C w_{Q_\ell}^{p-1} \Big(\sum_{j \neq \ell} w_{Q_j} \Big) |\nabla w_{Q_k}| dx + \sum_{\ell \neq k} \int_{\Omega_\ell} \sum_{j \neq \ell} w_{Q_j}^p |\nabla w_{Q_k}| dx \\
&+ \sum_{\ell \neq k} \int_{\Omega_\ell} \left(p w_{Q_k}^{p-1} w_{Q_\ell} |\nabla w_{Q_k}| + p w_{Q_k}^{p-1} \Big(\sum_{j \neq \ell, k} w_{Q_j} \Big) |\nabla w_{Q_k}| \right) dx \\
\le& C R^{-2m}.
\end{aligned}$$

In summary, we have that

$$I_{22} = R^{-\tau_2},$$

where

$$\tau_2 = \min \{2m, pm\}. \tag{5.9}$$

By combining the above estimates, we complete the proof of Lemma 5.1. □

Now, we can analyze $\int_{\mathbb{R}^2} E\frac{\partial U}{\partial \mathbf{q}}dx$. Before we start, we define

$$d_j = \frac{2R\sin\frac{j\pi}{K}\Psi'(2R\sin\frac{j\pi}{K})}{\Psi(2R\sin\frac{j\pi}{K})}. \tag{5.10}$$

By considering to the asymptotic behavior of Ψ with its derivative, it is not difficult to see that $d_j = -(2s+3) + o(1)$.

Lemma 5.2. *The following expansion holds:*

$$\int_{\mathbb{R}^2} E\frac{\partial U}{\partial \mathbf{q}}dx = T\mathbf{q} + R^{-\tau_1}\Pi_1(\alpha,\mathbf{q}) + R^{-\tau_2}\Pi_2(\alpha,\mathbf{q}) + R^{-m-\frac{3m}{2s+2}}\Pi_3(\alpha,\mathbf{q}),$$

where

$$\Pi_i(\alpha,\mathbf{q}) = \big(\Pi_{i1},\Pi_{i2},\cdots,\Pi_{iK}\big),\ i=1,2,$$

and $\Pi_3(\alpha,\mathbf{q})$ are uniformly bounded vector valued functions, with $\Pi_3(\alpha,\mathbf{q})$ given in the following proof of Lemma 5.2. Here, T is a $2K\times 2K$ matrix defined by

$$T = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \tag{5.11}$$

and $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ are all $K\times K$ circulant matrices given as follows:

The matrix $\mathcal{A}$:

$$\mathcal{A} = \mathrm{Circ}\{A_1,A_2,\cdots,A_K\}, \tag{5.12}$$

where

$$a_1 = \sum_{j\neq 1} F_j\big[(1-d_j)\sin^2\frac{\theta_j}{2} - 1\big] - (m+1)\Gamma R^{-m-2},$$

$$a_j = F_j\big(\cos^2\frac{\theta_j}{2} - d_j\sin^2\frac{\theta_j}{2}\big),\ \textit{for } j=2,\cdots,K,$$

and

$$F_j = \frac{\Psi(2R\sin\frac{\theta_j}{2})}{2R\sin\frac{\theta_j}{2}},\ \textit{for } j=2,\cdots,K.$$

The matrix $\mathcal{B}$:

$$\mathcal{B} = \mathrm{Circ}\{B_1,B_2,\cdots,B_K\}, \tag{5.13}$$

where

$$B_1 = -\frac{1}{2}\sum_{j\neq 1} F_j\sin\theta_j(1-d_j),$$

and

$$B_j = -\frac{1}{2}F_j\sin\theta_j(1+d_j),\ \textit{for } j=2,\cdots,K.$$

The matrix $\mathcal{C}$:

$$\mathcal{C} = \mathrm{Circ}\{C_1,C_2,\cdots,C_K\}, \tag{5.14}$$

where

$$C_1 = -\frac{1}{2}\sum_{j\neq 1} F_j\sin\theta_j(1-d_j),$$

and

$$C_j = \frac{1}{2}F_j\sin\theta_j(1+d_j),\ \textit{for } j=2,\cdots,K.$$

The matrix $\mathcal{D}$:

$$\mathcal{D} = \text{Circ}\left\{D_1, D_2, \cdots, D_K\right\}, \tag{5.15}$$

where

$$D_1 = \sum_{j\neq 1} F_j\big[(1-d_j)\cos^2\frac{\theta_j}{2} - 1\big] + \Gamma R^{-m-2},$$

$$D_j = F_j\big(-\sin^2\frac{\theta_j}{2} + d_j\cos^2\frac{\theta_j}{2}\big), \textit{ for } j = 2, \cdots, K.$$

Proof. First, we recall a useful expansion that will be used several times in the following proof:

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{\vec{a}}{|\vec{a}|} + \frac{\vec{b}}{|\vec{a}|} - \Big(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\Big)\frac{\vec{a}}{|\vec{a}|} + O\Big(\frac{|\vec{b}|^2}{|\vec{a}|^2}\Big), \text{ if } |\vec{b}| << |\vec{a}|. \tag{5.16}$$

By considering (5.16) and performing a simple computation, we can obtain that

$$\frac{Q_k}{|Q_k|^{m+2}} = R^{-m-1}n_k + R^{-m-2}\big(g_k t_k - (m+1)f_k n_k\big) + O(R^{-m-3}), \tag{5.17}$$

$$\begin{aligned}\frac{Q_j - Q_1}{|Q_j - Q_1|} &= \frac{Q_j^0 - Q_1^0}{|Q_j^0 - Q_1^0|} + \frac{1}{|Q_j^0 - Q_1^0|}\big(f_j n_j + g_j t_j - f_1 n_1 - g_1 t_1\big)\\ &\quad - \Big(\frac{Q_j^0 - Q_1^0}{|Q_j^0 - Q_1^0|}\cdot\frac{f_j n_j + g_j t_j - f_1 n_1 - g_1 t_1}{|Q_j^0 - Q_1^0|}\Big)\frac{Q_j^0 - Q_1^0}{|Q_j^0 - Q_1^0|} + O\Big(\frac{|\mathbf{q}|_\infty^2}{|Q_j^0 - Q_1^0|^2}\Big),\end{aligned}$$

and

$$\begin{aligned}\Psi(|Q_j - Q_1|) &= \Psi(|Q_j^0 - Q_1^0|) + \Psi'(|Q_j^0 - Q_1^0|)\frac{Q_j^0 - Q_1^0}{|Q_j^0 - Q_1^0|}\cdot\big(f_j n_j + g_j t_j - f_1 n_1 - g_1 t_1\big)\\ &\quad + O\Big(\frac{|\mathbf{q}|_\infty^2}{|Q_j^0 - Q_1^0|^{2s+5}}\Big).\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&\sum_{j\neq 1}\Psi(|Q_j - Q_1|)\frac{Q_j - Q_1}{|Q_j - Q_1|}\\ &= \sum_{j\neq 1}\Psi(|Q_j^0 - Q_1^0|)\frac{Q_j^0 - Q_1^0}{|Q_j^0 - Q_1^0|} + \Psi(|Q_j^0 - Q_1^0|)\frac{f_j n_j + g_j t_j - f_1 n_1 - g_1 t_1}{|Q_j^0 - Q_1^0|}\\ &\quad - \Psi(|Q_j^0 - Q_1^0|)\Big(\frac{Q_j^0 - Q_1^0}{|Q_j^0 - Q_1^0|}\cdot\frac{f_j n_j + g_j t_j - f_1 n_1 - g_1 t_1}{|Q_j^0 - Q_1^0|}\Big)\frac{Q_j^0 - Q_1^0}{|Q_j^0 - Q_1^0|}\\ &\quad + \Psi'(|Q_j^0 - Q_1^0|)\frac{(Q_j^0 - Q_1^0)\cdot\big(f_j n_j + g_j t_j - f_1 n_1 - g_1 t_1\big)}{|Q_j^0 - Q_1^0|}\frac{Q_j^0 - Q_1^0}{|Q_j^0 - Q_1^0|}\\ &\quad + O\Big(\sum_{j\neq 1}\frac{|\mathbf{q}|_\infty^2}{|Q_j^0 - Q_1^0|^{2s+5}}\Big)\end{aligned}$$

By noting that $n_j = \cos\theta_j n_1 + \sin\theta_j t_1$ and $t_j = -\sin\theta_j n_1 + \cos\theta_j t_1$, we can further write that

$$
\begin{aligned}
&\sum_{j\neq 1}\Psi(|Q_j-Q_1|)\frac{Q_j-Q_1}{|Q_j-Q_1|}\\
&=-\sum_{j\neq 1}\Psi(2R\sin\frac{\theta_j}{2})\sin\frac{\theta_j}{2}n_1+\sum_{j\neq 1}F_j\Big\{\Big[\frac{1}{2}f_1(d_j-1)\sin\theta_j+g_1\big((1-d_j)\cos^2\frac{\theta_j}{2}-1\big)\\
&\quad+\frac{1}{2}f_j(1+d_j)\sin\theta_j+g_j\big(d_j\cos^2\frac{\theta_j}{2}-\sin^2\frac{\theta_j}{2}\big)\Big]t_1+\Big[f_1\big((1-d_j)\sin^2\frac{\theta_j}{2}-1\big)\\
&\quad+\frac{1}{2}g_1(d_j-1)\sin\theta_j+f_j\big(\cos^2\frac{\theta_j}{2}-d_j\sin^2\frac{\theta_j}{2}\big)-\frac{1}{2}g_j(1+d_j)\sin\theta_j\Big]n_1\Big\}\\
&\quad+O\Big(\sum_{j\neq 1}\frac{|\mathbf{q}|_\infty^2}{|Q_j^0-Q_1^0|^{2s+5}}\Big) \qquad (5.18)
\end{aligned}
$$

By combining the above estimates with Lemma 5.1 and using the balance relation

$$
\sum_{j\neq 1}\Psi(2R\sin\frac{\theta_j}{2})\sin\frac{\theta_j}{2}-\Gamma R^{-m-1}=0, \qquad (5.19)
$$

we get that

$$
\begin{aligned}
\int_{\mathbb{R}^2}EZ_{Q_1}dx=&\sum_{j\neq 1}F_j\Big\{\Big[\frac{1}{2}f_1(d_j-1)\sin\theta_j+g_1\big((1-d_j)\cos^2\frac{\theta_j}{2}-1\big)\\
&+\frac{1}{2}f_j(1+d_j)\sin\theta_j+g_j\big(d_j\cos^2\frac{\theta_j}{2}-\sin^2\frac{\theta_j}{2}\big)\Big]t_1\\
&+\Big[f_1\big((1-d_j)\sin^2\frac{\theta_j}{2}-1\big)+\frac{1}{2}g_1(d_j-1)\sin\theta_j\\
&+f_j\big(\cos^2\frac{\theta_j}{2}-d_j\sin^2\frac{\theta_j}{2}\big)-\frac{1}{2}g_j(1+d_j)\sin\theta_j\Big]n_1\Big\}\\
&+\Gamma R^{-m-2}(g_1t_1-(m+1)f_1n_1)\\
&+R^{-\tau_1}\Pi_1(\alpha,\mathbf{q})+R^{-\tau_2}\Pi_2(\alpha,\mathbf{q})+R^{-m-\frac{3m}{2s+2}}\Pi_3(\alpha,\mathbf{q}),
\end{aligned}
$$

where $\Pi_3(\alpha,\mathbf{q})$ is a smooth vector valued function that defines the remainder term appearing in (5.17) and (5.18) and is uniformly bounded as $R\to\infty$. □

Next, we compute the projections involving ϕ.

Lemma 5.3. *For sufficiently large K, the following expansions hold:*

$$
\int_{\mathbb{R}^2}L[\phi]\frac{\partial U}{\partial\mathbf{q}}dx=R^{-\tau_3}\Pi_4(\alpha,\mathbf{q}),
$$

and

$$
\int_{\mathbb{R}^2}N(\phi)\frac{\partial U}{\partial\mathbf{q}}dx=R^{-\tau_4}\Pi_5(\alpha,\mathbf{q})
$$

where $\Pi_4(\alpha,\mathbf{q}),\Pi_5(\alpha,\mathbf{q})$ are uniformly bounded smooth vector valued functions and τ_3,τ_4 are defined in the following proof.

Proof. From the proof of Proposition 4.2, we have that

$$\Big|\int_{\mathbb{R}^2} L[\phi]Z_{Q_k}dx\Big| = \Big|\int_{\mathbb{R}^2} L(Z_{Q_k})\phi dx\Big| \le C\|\phi\|_*\big[R^{-m(p-1)}+R^{-m}+R^{-2s-1}\big] = O(R^{-\tau_3}),$$

where $\tau_3=\tau_0+\min\{m(p-1),m,2s+1\}$. By direct computation, we obtain that

$$\int_{\mathbb{R}^2} N(\phi)Z_{Q_k}dx = \|\phi\|_*^{\min\{p,2\}}\int_{\mathbb{R}^2}\rho|Z_{Q_k}|dx = O(R^{-\min\{p,2\}\tau_0}) = O(R^{-\tau_4}).$$

□

5.2. **The invertibility of** T. In this subsection, we study the linear problem $T\mathbf{q}=\mathbf{b}$ and obtain the following result, the proof of which is deferred to the Appendix.

Lemma 5.4. *There exists $R_0>0$ such that for $R>R_0$ and every $\mathbf{b}\in\mathbb{R}^{2K}$, there exists a unique vector $\mathbf{q}\in\mathbb{R}^{2K}$ and a unique constant $\gamma\in\mathbb{R}$ such that*

$$T\mathbf{q}=\mathbf{b}+\gamma\mathbf{q}_1,\ \mathbf{q}\perp\mathbf{q}_0, \tag{5.20}$$

where $\mathbf{q}_1$ is defined by (5.26) below. Moreover, there exists a positive constant $C>0$ that is independent of R such that

$$\|\mathbf{q}\|_\infty \le CR^{m+\frac{5}{2}-\frac{m}{4s+4}}\|\mathbf{b}\|_\infty. \tag{5.21}$$

5.3. **Reduction to one dimension.** Now, we state the main result of the section.

Proposition 5.1. *Under the assumption of Theorem 1.1, there exists $R_0>0$ such that for all $R>R_0$ and every $\alpha\in\mathbb{R}$, there exists a unique $(\mathbf{q},\gamma)=(\mathbf{q}(\alpha),\gamma(\alpha))$ such that $\vec{\beta}=0$. As a result, $\phi(x,\alpha,\mathbf{q}(\alpha))$ and $\gamma(\alpha)$ satisfy the following equation:*

$$\begin{cases} L[\phi]+E+N(\phi)=\gamma\frac{\partial U}{\partial\alpha},\\ \int_{\mathbb{R}^2}\phi Z_{Q_k}dx=0,\ \forall k=1,\cdots,K. \end{cases} \tag{5.22}$$

Moreover, the function $\phi(x,\alpha,\mathbf{q}(\alpha))$ is of class C^1 in α, and we have that

$$\|\phi\|_*\le CR^{-\tau_0},\ \|\mathbf{q}\|_\infty+R^{-1}\|\partial_\alpha\mathbf{q}\|_\infty\le CR^{-(\tau_5-m-\frac{5}{2}+\frac{m}{4s+4})}, \tag{5.23}$$

$$\tau_5=\left\{\tau_1,\tau_2,\tau_3,\tau_4,m+\frac{3m}{2s+2}\right\}. \tag{5.24}$$

To prove Proposition 5.1, it suffices to solve $\vec{\beta}=0$ for each α. From the results in previous subsections, we can rewrite the equation as follows:

Lemma 5.5. *For every $\alpha\in\mathbb{R}$, the equation $\vec{\beta}=0$ is equivalent to*

$$T\mathbf{q}+\Phi(\alpha,\mathbf{q})=\gamma\mathbf{q}_1 \tag{5.25}$$

where T is the $2K\times 2K$ matrix defined in (5.11), Φ denotes the remainder term, and

$$\mathbf{q}_1=\int_{\mathbb{R}^2}\frac{\partial U}{\partial\alpha}\frac{\partial U}{\partial\mathbf{q}}dx = M\big(R\mathbf{q}_0+\mathbf{q}^\perp\big). \tag{5.26}$$

Using Lemma 5.2 and Lemma 5.3, we obtain the following estimate on $\Phi(\alpha,\mathbf{q})$.

Lemma 5.6. *The following expansion holds:*

$$\begin{aligned}\Phi(\alpha,\mathbf{q}) =& R^{-\tau_1}\Pi_1(\alpha,\mathbf{q}) + R^{-\tau_2}\Pi_2(\alpha,\mathbf{q}) + R^{-m-\frac{3m}{2s+2}}\Pi_3(\alpha,\mathbf{q}) \\ &+ R^{-\tau_3}\Pi_4(\alpha,\mathbf{q}) + R^{-\tau_4}\Pi_5(\alpha,\mathbf{q})\end{aligned}$$

where $\Pi_i(\alpha,\mathbf{q})$ are smooth vector valued functions that are uniformly bounded as $K\to\infty$.

Now, we are going to solve (5.25) and complete the proof of Proposition 5.1.

Proof of Proposition 5.1. By Lemma 5.4, equation (5.25) is equivalent to

$$\mathbf{q} = T_{\mathbf{q}}^{-1}[\Phi(\alpha,\mathbf{q})] = \mathcal{F}(\mathbf{q}).$$

By Lemma 5.6, the estimate (5.23), and the estimate (1.10) in **Assumption** 2, we know that

$$\Phi(\alpha,\mathbf{q}) = R^{-m-\frac{5}{2}+\frac{m}{4s+4}-\eta}\Pi(\alpha,\mathbf{q}),$$

where $\eta = \tau_5 - m - \frac{5}{2} - \frac{m}{4s+4}$ and $\Pi(\alpha,\mathbf{q})$ is a smooth vector valued function that is uniformly bounded as $K\to\infty$. Then, by Lemma 5.4, for $\|\mathbf{q}\|_\infty < 1$ we have that

$$\|\mathcal{F}(\mathbf{q})\|_\infty \le CR^{-\eta}$$

and

$$\|\mathcal{F}(\mathbf{q}) - \mathcal{F}(\bar{\mathbf{q}})\|_\infty \le CR^{-\eta}\|\mathbf{q}-\bar{\mathbf{q}}\|_\infty \le \frac{1}{2}\|\mathbf{q}-\bar{\mathbf{q}}\|_\infty.$$

On the other hand,

$$\|(T_{\mathbf{q}}^{-1} - T_{\bar{q}}^{-1})\mathbf{b}\|_\infty \le CR^{m+\frac{3}{2}-\frac{m}{2(N+2s)}}\|\mathbf{b}\|_\infty\|\mathbf{q}-\bar{\mathbf{q}}\|_\infty,$$

which shows that $\mathcal{F}$ is a contraction map. By the Banach fixed point theorem, we obtain the existence of a solution ϕ to (5.22).

To show the differentiability of $\mathbf{q}(\alpha)$, consider the map

$$\mathcal{T}(\alpha,\mathbf{q}) = \mathbf{q} - \mathcal{F}(\alpha,\mathbf{q}) : \mathbb{R}\times\mathbb{R}^{2K} \to \mathbb{R}^{2K}.$$

Because $\frac{\partial\mathcal{F}}{\partial\mathbf{q}} = O(R^{-\eta})$, we have that $\frac{\partial\mathcal{T}}{\partial\mathbf{q}}|_{\alpha,\mathbf{q}(\alpha)} = I - \frac{\partial\mathcal{F}}{\partial\mathbf{q}}$ is invertible. Thus, we obtain the differentiability of $\mathbf{q}(\alpha)$.

Next, we study the dependence of $\mathbf{q}$ on α. Assume that we have two solutions corresponding to two sets of parameters. Let one of these be denoted by

$$\mathbf{q} = T_{\mathbf{q}}^{-1}[\Phi(\alpha,\mathbf{q})],$$

corresponding to α, and the other denoted by

$$\bar{\mathbf{q}} = T_{\bar{\mathbf{q}}}^{-1}[\Phi(\bar{\alpha},\bar{\mathbf{q}})],$$

corresponding to $\bar{\alpha}$. Assume that $R|\bar{\alpha}-\alpha| < \frac{1}{2}$. Then, by direct computation and Lemma 5.4, we have that

$$\|\bar{\mathbf{q}} - \mathbf{q}\|_\infty \le CR^{-\eta}(R|\bar{\alpha}-\alpha|).$$

Hence, we have proved the Proposition.

6. Proof of Theorem 1.1: Variational Reduction

In this section, we complete the proof of Theorem 1.1. To solve $\gamma(\alpha) = 0$, we will apply variational reduction. To do this, we first introduce some notation. Let $\alpha \in \mathbb{R}$, and let $\phi = \phi(\alpha, \mathbf{q}(\alpha))$ be the function given by Proposition 5.1. Then, we define the following energy functional:

$$F(\alpha) = J(U + \phi) : \mathbb{R} \to \mathbb{R}, \tag{6.1}$$

where $U = U(x, \alpha, \mathbf{q}(\alpha))$.

Both U and ϕ are 2π periodic in α. Hence, by Proposition 5.1 the reduced energy functional $F(\alpha)$ has the following property.

Lemma 6.1. *The function $F(\alpha)$ is of class C^1, and satisfies $F(\alpha + 2\pi) = F(\alpha)$ for every $\alpha \in \mathbb{R}$.*

Lemma 6.2. *Let $F(\alpha) = J(U + \phi)$. Then, $\gamma(\alpha) = 0$ if and only if $F'(\alpha) = 0$.*

Proof. Assume that $\tilde{u}$ is the unique s-harmonic extension of $u = U + \phi$. Then, the well-known computation by Caffarelli and Silvestre [13] shows that

$$F(\alpha) = \frac{1}{2}\int_{\mathbb{R}^3_+} |\nabla \tilde{u}|^2 y^{1-2s} dxdy + \frac{1}{2}\int_{\mathbb{R}^2} V(x)u^2 - \frac{1}{p+1}\int_{\mathbb{R}^2} u_+^{p+1} dx.$$

By Proposition 5.1, for large K and every $\alpha \in \mathbb{R}$, $u = U + \phi$ satisfies the equation

$$(-\Delta)^s u + V(x)u - u^p = \gamma(\alpha)\frac{\partial U}{\partial \alpha}. \tag{6.2}$$

Thus, we obtain that

$$\begin{aligned}
F'(\alpha) &= \int_{\mathbb{R}^3_+} \nabla\tilde{u} \cdot \nabla(\partial_\alpha \tilde{u}) y^{1-2s} dxdy + \int_{\mathbb{R}^2} V(x)\tilde{u}\partial_\alpha \tilde{u} - \int_{\mathbb{R}^2} u_+^p \partial_\alpha u \\
&= \int_{\mathbb{R}^2} [(-\Delta)^s u + V(x)u - u^p]\partial_\alpha u dx \\
&= \gamma(\alpha)\int_{\mathbb{R}^2} \frac{\partial U}{\partial \alpha}(\partial_\alpha U + \partial_\alpha \phi) dx \\
&= \gamma(\alpha)\int_{\mathbb{R}^2} (R\mathbf{q}_0 + \mathbf{q}^\perp) \cdot \frac{\partial U}{\partial \mathbf{q}}\left(\frac{\partial U}{\partial \alpha} + \frac{\partial U}{\partial \mathbf{q}} \cdot \partial_\alpha \mathbf{q} + \frac{\partial \phi}{\partial \alpha} + \frac{\partial \phi}{\partial \mathbf{q}} \cdot \partial_\alpha \mathbf{q}\right) dx \\
&= KR^2(1 + o(1))\gamma(\alpha)\int_{\mathbb{R}^2}\left(\frac{\partial w}{\partial x_1}\right)^2 dx.
\end{aligned}$$

Hence, we find that $F'(\alpha) = 0$ if and only if $\gamma(\alpha) = 0$. Thus, we have proved the Lemma. $\square$

Proof of Theorem 1.1. By Lemma 6.1, $F(\alpha)$ is 2π periodic and of class C^1. Hence, it has at least two critical points (maximum and minimum points) in $[0, 2\pi]$. Therefore, Theorem 1.1 follows from Lemma 6.2. $\square$

7. Appendix: Circulant matrices and the proof of Lemma 5.4

In this section, we introduce circulant matrices, which play an important role in this paper. For further details regarding circulant matrices, we refer the reader to [32].

A circulant matrix $\mathcal{M}$ of dimension $K \times K$ has the form

$$\mathcal{M} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{K-2} & x_{K-1} \\ x_{K-1} & x_0 & x_1 & \cdots & x_{K-2} \\ \cdots & x_{K-1} & x_0 & x_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & x_1 \\ x_1 & \cdots & \cdots & x_{K-1} & x_0 \end{pmatrix}, \tag{7.1}$$

or equivalently if $M_{ij}, i, j = 1, \cdots, K$ are the entries of the matrix $\mathcal{M}$, then

$$M_{ij} = M_{1,|i-j|+1}.$$

In particular, in order to determine a circulant matrix it is sufficient to know the components of the first row. We denote the circulant matrix mentioned above by

$$\mathcal{M} = \mathrm{Cir}\{(x_0, x_1, \cdots, x_{K-1})\}. \tag{7.2}$$

The eigenvalues of a circulant matrix X are given by the explicit formula

$$\lambda_j = \sum_{l=0}^{K-1} x_l e^{\frac{2\pi i}{K} lj},\ j = 0, \cdots, K-1, \tag{7.3}$$

with corresponding normalized eigenvectors defined by

$$E_j = \frac{1}{\sqrt{K}} \left(1, e^{\frac{2\pi i}{K} j}, e^{\frac{4\pi i}{K} j}, \cdots, e^{\frac{2(K-1)\pi i}{K} j}\right)^T. \tag{7.4}$$

Observe that any circulant matrix X can be diagonalized:

$$\mathcal{M} = P D_{\mathcal{M}} P^t$$

where $D_{\mathcal{M}}$ is the diagonal matrix

$$D_{\mathcal{M}} = \mathrm{diag}\,(\eta_0, \eta_1, \cdots, \eta_{K-1})$$

and P is the $K \times K$ matrix defined by

$$P = \Big[\ E_0 \mid E_1 \mid\ \cdots \mid E_{K-1}\Big]. \tag{7.5}$$

Next, we analyze the matrix T. We have the following result.

Lemma 7.1. *There exists $K_0 > 0$ such that for all $K > K_0$ and every $\mathbf{b} \in \mathbb{R}^{2K}$, there exists a unique $\mathbf{q} \in \mathbb{R}^{2K}$ and a unique $\gamma \in \mathbb{R}$ such that*

$$T\mathbf{q} = \mathbf{b} + \gamma \mathbf{q}_0,\ \mathbf{q} \perp \mathbf{q}_0. \tag{7.6}$$

Moreover, there exists a positive constant C, which is independent of K, such that

$$\|\mathbf{q}\|_\infty \le C R^{m+\frac{5}{2}-\frac{m}{4s+4}} \|\mathbf{b}\|_\infty. \tag{7.7}$$

Furthermore, the number of zero (resp. negative, positive) eigenvalues of T is given by 1 (resp. $K-1$, K).

Proof. Recall that T is given by (5.11):

$$T = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}$$

and the entries $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are given by (5.12)-(5.15). Thus, for $\ell = 0, \cdots, K-1$ we have that the ℓ-th eigenvalues of the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are given by the following:

$$\begin{aligned}\lambda_{\mathcal{A},\ell} = & \sum_{j\neq 1} F_j \Big[(1-d_j)\sin^2\frac{\theta_j}{2} - 1\Big] - (m+1)\Gamma R^{-m-2} \\ & + \sum_{j\neq 1} F_j\Big(\cos^2\frac{\theta_j}{2} - d_j \sin^2\frac{\theta_j}{2}\Big) e^{\frac{2\ell\pi i}{K}(j-1)},\end{aligned} \tag{7.8}$$

$$\lambda_{\mathcal{B},\ell} = -\frac{1}{2}\sum_{j\neq 1}\Big[F_j \sin\theta_j (1-d_j) + F_j \sin\theta_j (1+d_j) e^{\frac{2\ell\pi i}{K}(j-1)}\Big], \tag{7.9}$$

$$\lambda_{\mathcal{C},\ell} = -\frac{1}{2}\sum_{j\neq 1}\Big[F_j \sin\theta_j (1-d_j) - F_j \sin\theta_j (1+d_j) e^{\frac{2\ell\pi i}{K}(j-1)}\Big], \tag{7.10}$$

and

$$\begin{aligned}\lambda_{\mathcal{D},\ell} = & \sum_{j\neq 1} F_j \Big[(1-d_j)\cos^2\frac{\theta_j}{2} - 1\Big] + \Gamma R^{-m-2} \\ & + \sum_{j\neq 1} F_j\Big(d_j\cos^2\frac{\theta_j}{2} - \sin^2\frac{\theta_j}{2}\Big) e^{\frac{2\ell\pi i}{K}(j-1)}.\end{aligned} \tag{7.11}$$

Now, define

$$\mathcal{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \tag{7.12}$$

where P is defined in (7.5). A simple algebraic manipulation gives that

$$T = \mathcal{P}\mathcal{G}\mathcal{P}^t, \tag{7.13}$$

where

$$\mathcal{G} = \begin{pmatrix} D_{\mathcal{A}} & D_{\mathcal{B}} \\ D_{\mathcal{C}} & D_{\mathcal{D}} \end{pmatrix} \tag{7.14}$$

and D_X denotes the diagonal matrix of the $K \times K$ circulant matrix X.

Because the matrix T is real and symmetric, all of its eigenvalues are real and satisfy the following equations:

$$\Lambda^2 - \Big(\lambda_{\mathcal{A},\ell} + \lambda_{\mathcal{D},\ell}\Big)\Lambda + \Big(\lambda_{\mathcal{A},\ell}\lambda_{\mathcal{D},\ell} - \lambda_{\mathcal{B},\ell}\lambda_{\mathcal{C},\ell}\Big) = 0 \tag{7.15}$$

for $\ell = 0, \cdots, K-1$.

Denote the solutions of (7.15) by $\Lambda_{1,\ell}$ and $\Lambda_{2,\ell}$, with $\Lambda_{1,\ell} \leq \Lambda_{2,\ell}$. Then,

$$\Lambda_{1,\ell} = \frac{\lambda_{\mathcal{A},\ell} + \lambda_{\mathcal{D},\ell}}{2}\left(1 - \sqrt{1 - \frac{4(\lambda_{\mathcal{A},\ell}\lambda_{\mathcal{D},\ell} - \lambda_{\mathcal{B},\ell}\lambda_{\mathcal{C},\ell})}{(\lambda_{\mathcal{A},\ell} + \lambda_{\mathcal{D},\ell})^2}}\right), \tag{7.16}$$

and

$$\Lambda_{2,\ell} = \frac{\lambda_{\mathcal{A},\ell} + \lambda_{\mathcal{D},\ell}}{2}\left(1 + \sqrt{1 - \frac{4(\lambda_{\mathcal{A},\ell}\lambda_{\mathcal{D},\ell} - \lambda_{\mathcal{B},\ell}\lambda_{\mathcal{C},\ell})}{(\lambda_{\mathcal{A},\ell} + \lambda_{\mathcal{D},\ell})^2}}\right). \tag{7.17}$$

In order to analyze the eigenvalues, we need to compute the terms $\lambda_{\mathcal{A},\ell} + \lambda_{\mathcal{D},\ell}$ and $\lambda_{\mathcal{A},\ell}\lambda_{\mathcal{D},\ell} - \lambda_{\mathcal{B},\ell}\lambda_{\mathcal{C},\ell}$.

When $\ell = 0$, we note that $\lambda_{\mathcal{B},0} = \lambda_{\mathcal{C},0} = \lambda_{\mathcal{D},0} = 0$. As a consequence, $\Lambda_{1,0} = 0$, and

$$
\begin{aligned}
\Lambda_{2,0} = \lambda_{A,0} &= -2\sum_{j\neq 1} d_j F_j \sin^2\frac{\theta_j}{2} - (m+1)\Gamma R^{-m-2} \\
&= \frac{1}{R}\sum_{j\neq 1} d_j \Psi(2R\sin\frac{\theta_j}{2})\sin\frac{\theta_j}{2} - (m+1)\Gamma R^{-m-2} \\
&= \frac{1}{R}\sum_{j\neq 1}(2s+3)\Psi(2R\sin\frac{\theta_j}{2})\sin\frac{\theta_j}{2}\big(1+o(1)\big) - (m+1)\Gamma R^{-m-2} \\
&= \big(2s+2-m+o(1)\big)\Gamma R^{-m-2} > 0,
\end{aligned}
$$

where we have used (5.10).

For $\ell \geq 1$, we use the idea introduced in [40]. First, we introduce the following functions. For any integer i, we define

$$
P_i(x) = \sum_{l=1}^{\infty}\frac{\cos(lx)}{l^i} \text{ and } Q_i(x) = \sum_{l=1}^{\infty}\frac{\sin(lx)}{l^i}. \tag{7.18}
$$

When n is even, up to a normalization constant, P_n and Q_n are related to the Fourier series of the Bernoulli polynomial $B_n(x)$, and when n is odd P_n and Q_n are related to the Fourier series of the Euler polynomial $E_n(x)$. Let

$$
g(x) = \sum_{j=1}^{\infty}\frac{1-\cos(jx)}{j^{2s+4}},\ 0 \leq x \leq \pi. \tag{7.19}
$$

Using $P_i(x)$, we can write $g(x)$ as

$$
g(x) = P_{2s+4}(0) - P_{2s+4}(x).
$$

We observe that

$$
g(x) = Q_{2s+3}(x),\ g''(x) = P_{2s+2}(x)
$$

and

$$
g(0) = g'(0) = 0,\ g''(0) > 0.
$$

With these functions in hand, and using the balancing condition (5.19), we set

$$
\begin{aligned}
F_j &= \frac{A}{|Q_j - Q_1|^{2s+4}}\big(1+o(1)\big) \\
&= \frac{A}{2^{s+2}R^{2s+4}}\frac{1}{(1-\cos\theta_j)^{s+2}}\big(1+o(1)\big) \\
&=: \frac{f_R}{(1-\cos\theta_j)^{s+2}}.
\end{aligned}
$$

Then, we have the following:

$$
\begin{aligned}
\lambda_{\mathcal{A},\ell} = & \sum_{j\neq 1} F_j\Big[(1-d_j)\sin^2\frac{\theta_j}{2}-1\Big]-(m+1)\sum_{j\neq 1}F_j\big(1-\cos\theta_j\big)\\
& +\sum_{j\neq 1}F_j\Big(\cos^2\frac{\theta_j}{2}-d_j\sin^2\frac{\theta_j}{2}\Big)\cos(\ell\theta_j)\\
= & \sum_{j\neq 1}F_j\big(1-\cos\theta_j\big)\Big[\frac{1-d_j}{2}-m-1\Big]-\sum_{j\neq 1}F_j\big(1-\cos(\ell\theta_j)\big)\\
& -\sum_{j\neq 1}\frac{1+d_j}{2}F_j\big(1-\cos\theta_j\big)\cos(\ell\theta_j)\\
= & f_R\Big(\sum_{j\neq 1}\frac{1}{(1-\cos\theta_j)^{s+1}}\Big[\frac{1-d_j}{2}-m-1\Big]-\sum_{j\neq 1}\frac{1-\cos(\ell\theta_j)}{(1-\cos\theta_j)^{s+2}}\\
& -\sum_{j\neq 1}\frac{1+d_j}{2}\frac{\cos(\ell\theta_j)}{(1-\cos\theta_j)^{s+1}}\Big)\\
= & f_R\Big\{\Big[\frac{1-d}{2}-m-1\Big]\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+2}g''(0)-\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+4}g(\frac{2\pi\ell}{K})\\
& -\frac{1+d}{2}\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+2}g''(\frac{2\pi\ell}{K})\Big\}\big(1+o(1)\big),
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{\mathcal{B},\ell} = & -\frac{i}{2}\sum_{j\neq 1}F_j(1+d_j)\sin\theta_j\sin(\ell\theta_j) = -\frac{if_R}{2}\sum_{j\neq 1}\frac{\sin\theta_j\sin(\ell\theta_j)}{(1-\cos\theta_j)^{s+2}}(1+d_j)\\
= & -\frac{if_R(1+d)\sqrt{2}}{2}\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+3}g'(\frac{2\pi\ell}{K})\big(1+o(1)\big),
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{\mathcal{C},\ell} = & \frac{i}{2}\sum_{j\neq 1}F_j(1+d_j)\sin\theta_j\sin(\ell\theta_j) = \frac{if_R}{2}\sum_{j\neq 1}\frac{\sin\theta_j\sin(\ell\theta_j)}{(1-\cos\theta_j)^{s+2}}(1+d_j)\\
= & \frac{if_R(1+d)}{2}\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+3}\sqrt{2}g'(\frac{2\pi\ell}{K})\big(1+o(1)\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{\mathcal{D},\ell} =& \sum_{j\neq 1} F_j\Big[\big(1-d_j\big)\cos^2\frac{\theta_j}{2}-1\Big]+\sum_{j\neq 1}F_j(1-\cos\theta_j)\\
&+\sum_{j\neq 1}F_j\Big(d_j\cos^2\frac{\theta_j}{2}-\sin^2\frac{\theta_j}{2}\Big)\cos(\ell\theta_j)\\
=&\sum_{j\neq 1}\frac{1+d_j}{2}F_j(1-\cos\theta_j)-\sum_{j\neq 1}d_jF_j(1-\cos(\ell\theta_j))\\
&-\sum_{j\neq 1}\frac{1+d_j}{2}F_j(1-\cos\theta_j)\cos(\ell\theta_j)\\
=&f_R\Big(\sum_{j\neq 1}\frac{1+d_j}{2}\frac{1}{(1-\cos\theta_j)^{s+1}}-\sum_{j\neq 1}d_j\frac{1-\cos(\ell\theta_j)}{(1-\cos\theta_j)^{s+2}}\\
&-\sum_{j\neq 1}\frac{1+d_j}{2}\frac{\cos(\ell\theta_j)}{(1-\cos\theta_j)^{s+1}}\Big)\\
=&f_R\Big\{\frac{1+d}{2}\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+2}g''(0)-d\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+4}g(\frac{2\pi\ell}{K})\\
&-\frac{1+d}{2}\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+2}g''(\frac{2\pi\ell}{K})\Big\}\big(1+o(1)\big).
\end{aligned}
$$

When $\frac{\ell}{K}=o(1)$, we can employ a Taylor expansion on g, g', and g'' to obtain that

$$
\lambda_{\mathcal{A},\ell}=\Big(\big(-d-m-1\big)-\ell^2\Big)\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+2}g''(0)f_R\big(1+o(1)\big),
$$

$$
\lambda_{\mathcal{B},\ell}=-i(1+d)\ell f_R\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+2}g''(0)\big(1+o(1)\big),
$$

$$
\lambda_{\mathcal{C},\ell}=i(1+d)\ell f_R\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+2}g''(0)\big(1+o(1)\big),
$$

and

$$
\lambda_{\mathcal{D},\ell}=-d\ell^2 f_R\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+2}g''(0)\big(1+o(1)\big).
$$

Then, we obtain that

$$
\lambda_{\mathcal{A},\ell}+\lambda_{\mathcal{D},\ell}=f_Rg''(0)\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+2}\Big[(2s+2)\ell^2+2s+2-m\Big]\big(1+o(1)\big) \tag{7.20}
$$

and

$$
\begin{aligned}
&\lambda_{\mathcal{A},\ell}\lambda_{\mathcal{D},\ell}-\lambda_{\mathcal{B},\ell}\lambda_{\mathcal{C},\ell}\\
=&-f_R^2(g''(0))^2\Big(\frac{K}{\sqrt{2}\pi}\Big)^{4s+4}\ell^2\Big[\big(2s+3\big)\big(\ell^2+m-1\big)+1\Big]\big(1+o(1)\big).
\end{aligned} \tag{7.21}
$$

We note that $\lambda_{\mathcal{A},\ell}\lambda_{\mathcal{D},\ell}-\lambda_{\mathcal{B},\ell}\lambda_{\mathcal{C},\ell}<0$.

Now, assume that $\frac{\ell}{K} \geq c$ for some positive constant c. In this case, because $g(x) > 0$ if $x > 0$, in the expressions of $\lambda_{A,\ell}$ to $\lambda_{D,\ell}$, the leading order terms in $\lambda_{\mathcal{A},\ell}, \lambda_{\mathcal{B},\ell}, \lambda_{\mathcal{C},\ell}, \lambda_{\mathcal{D},\ell}$ are given by the terms containing $g(\frac{2\pi\ell}{K})$. Thus, we have that

$$\begin{aligned}\lambda_{A,\ell} + \lambda_{D,\ell} =& (2s+2) f_R\Big(\frac{K}{\sqrt{2}\pi}\Big)^{2s+4} g\big(\frac{2\pi\ell}{K}\big)(1+o(1))\\ =& O\Big(\frac{K}{R}\Big)^{2s+4},\end{aligned}$$

and

$$\begin{aligned}\lambda_{A,\ell}\lambda_{D,\ell} - \lambda_{B,\ell}\lambda_{C,\ell} =& -(2s+3) f_R^2\Big(\frac{K}{\sqrt{2}\pi}\Big)^{4s+8}\Big(g\big(\frac{2\pi\ell}{K}\big)\Big)^2(1+o(1))\\ =& \, O\Big(\frac{K}{R}\Big)^{4s+8} < 0.\end{aligned}$$

By combining the above results, for $\ell \geq 1$ we obtain that

$$-\Lambda_{1,\ell} \geq C\frac{\ell^2 K^{2s+2}}{R^{2s+4}} \geq CR^{-m-2}, \tag{7.22}$$

and

$$\Lambda_{2,\ell} \geq C\frac{\ell^2 K^{2s+2}}{R^{2s+4}} \geq CR^{-m-2}. \tag{7.23}$$

From the above analysis, we know that the number of zero (resp. negative, positive) eigenvalues of T is 1 (resp. $K-1$, K). The eigenvector corresponding to the zero eigenvalue is

$$\mathbf{q}_0 = (0, \cdots, 0, 1, \cdots, 1). \tag{7.24}$$

Moreover, for the solution $(\mathbf{q}, \gamma)$ to (7.6), we have the following estimate:

$$\|\mathbf{q}\|_2 \leq CR^{m+2}\|\mathbf{b}\|_2.$$

Because

$$\|\mathbf{q}\|_\infty \leq \|\mathbf{q}\|_2 \leq CR^{m+2}\|\mathbf{b}\|_2 \leq CR^{m+2}\sqrt{K}\|\mathbf{b}\|_\infty,$$

we have that

$$\|\mathbf{q}\|_\infty \leq CR^{m+\frac{5}{2}-\frac{m}{4s+4}}\|\mathbf{b}\|_\infty.$$

□

Proof of Lemma 5.4. To prove Lemma 5.4, it suffices to verify the a priori estimate (5.21). Let

$$\gamma = -\frac{\mathbf{b}\cdot\mathbf{q}_0}{\mathbf{q}_1\cdot\mathbf{q}_0}.$$

By Lemma 7.1, we have that

$$\|\mathbf{q}\|_\infty \leq CR^{m+\frac{5}{2}-\frac{m}{4s+4}}(\|\mathbf{b}\|_\infty + |\gamma|).$$

Because

$$\mathbf{q}_1\cdot\mathbf{q}_0 = M(R\mathbf{q}_0 + \mathbf{q}^\perp)\cdot\mathbf{q}_0 = c_0 KR(1+o(1)),$$

and

$$|\mathbf{b}\cdot\mathbf{q}_0| \leq CK\|\mathbf{b}\|_\infty,$$

we have that

$$|\gamma| \leq CR^{-1}\|\mathbf{b}\|_\infty.$$

Thus, we can obtain that

$$\|\mathbf{q}\|_\infty \leq CR^{m+\frac{5}{2}-\frac{m}{4s+4}}\|\mathbf{b}\|_\infty.$$

REFERENCES

[1] A. Ambrosetti, M. Badiale and S. Cingolani, Semiclassical states of nonlinear Schrödinger equations. *Arch. Ration. Mech. Anal.* 140(3), (1997) 285C-300.

[2] L. Abdelouhab, J. L. Bona, M. Felland, J.-C. Saut, Nonlocal models for nonlinear, dispersive waves. *Phys. D* 40 (1989) 360C-392.

[3] A. Ambrosetti, M. Malchiodi and W. M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. I. *Commun. Math. Phys.* 235(3), (2003) 427C466.

[4] A. Ambrosetti, M. Malchiodi and W. M. Ni, Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres. II. *Indiana Univ. Math. J.* 53(2004) 297C-329.

[5] W.W. Ao and J.C. Wei, Infinitely many positive solutions for nonlinear equations with non-symmetric potential. *Calc. Var. Partial Differ. Equ.* 51(2014), no.3-4, 761-798.

[6] A. Bahri, Y. Y. Li, On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}$. *Rev. Mat. Iberoamericana* 6 no.1-2, 1990, 1-15.

[7] A. Bahri, P.L. Lions, On the existence of a positive solution of semilinear elliptic equations in unbounded domains. *Ann. Inst. H. Poincaré Anal. Non Linaire*, 14, no.3, 1997, 365-413.

[8] D. Cao, Positive solution and bifurcation from the essential spectrum of a semilinear elliptic equation on $\mathbb{R}$ *Nonlinear Anal.* 15, no. 11, 1990, 1045-1052.

[9] W. Chen, Soft matter and fractional mathematics: insights into mesoscopic quantum and time-space structures, Preprint. http://arxiv.org/abs/1305.4426.

[10] D. Cao, E. Noussair and S. Yan, Existence and uniqueness results on single-peaked solutions of a semilinear problem. *Ann. Inst. H. Poincaré Anal. Non Lineaire* 15(1), (1998) 73C-111.

[11] D. Cao, E. Noussair and S. Yan, Solutions with multiple peaks for nonlinear elliptic equations. *Proc. R. Soc. Edinb. Sect. A* 129(2), (1999) 235-C264.

[12] X. Cabré, J. G. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian. *Adv. Math.* 224(2010), no.5, 2052-2093.

[13] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* 32 (7-9) (2007) 1245-1260.

[14] G. Cerami, D. Passaseo, S. Solimini, Infinitely many positive solutions to some scalar field equations with nonsymmetric coefficients, *Comm. Pure Appl. Math.* 66, no. 3, 372-413, 2013.

[15] G. Cerami, R. Molle and D. Passaseo, Multiplicity of positive and nodal solutions for scalar field equations. *J. Differential Eqn.* 257 (2014), no. 10, 3554-3606.

[16] G. Cerami, D. Passaseo and S. Solimini, Nonlinear scalar field equations: existence of a positive solution with infinitely many bumps. *Ann. Inst. H. Poincar? Anal. Non Lin?aire* 32 (2015), no. 1, 23-40.

[17] G. Cerami, G. Devillanova and S. Solimini, Infinitely many bound states for some nonlinear scalar field equations. *Calc. Var. Partial Differential Equations* 23 (2005), no. 2, 139-168

[18] J. Dávila, M. del Pino and J. Wei, Concentrating standing waves for the fractional nonlinear Schrödinger equation. *J. Differential Equations* 256 (2014), no. 2, 858-C892.

[19] M. del Pino, P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains. *Calc. Var. Partial Differential Equations.* 4 (1996), 121-137.

[20] M. del Pino, P. Felmer, Multi-peak bound states of nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. NonLineaire* 15 (1998), 127-149.

[21] M. del Pino, P. Felmer, Semi-classical states for nonlinear Schrödinger equations. *J. Funct. Anal.* 149 (1997), 245-265.

[22] M. del Pino and P. Felmer, Semi-classical states for nonlinear Schrödinger equations: a variational reduction method. *Math. Ann.* 324 (2002), 1-32.

[23] Concentration on curves for nonlinear Schr?odinger equations. *Comm. Pure Appl. Math.* 70, no. 1, 113-146, 2007.

[24] M. Del Pino, J. Wei and W. Yao, Intermediate reduction method and infinitely many positive solutions of nonlinear Schrödinger equations with non-symmetric potentials. *Cal.Var. PDE*, 53(2015), no.1-2, 473-523
[25] W. Y. Ding, W. M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation. *Arch. Ration. Mech. Anal.* 91(4), 283-308 (1986).
[26] R. Frank, E. Lenzmann, Uniqueness of non-linear ground states for fractional Laplacians in $\mathbb{R}$. *Acta Math.* 210 (2013), no.2, 261-318.
[27] R. Frank, E. Lenzmann and L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, *Comm. Pure. Appl. Math.* (2015) online first.
[28] P. Felmer, A. Quaas and J. G. Tan, Positive solutions of the nonlinear Schr?dinger equation with the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A* 142 (2012), no. 6, 1237C1262.
[29] A. Floer, M. Weinstein, Nonspreading wave packets for the cubic Schr?dinger equations with a bounded potential. *J. Funct. Anal.* 69(1986), 397C408.
[30] D. Giulini, That strange procedure called quantisation, Quantum gravity, Lecture Notes in Phys., vol. 631, Springer, Berlin, 2003, 17-40.
[31] X.S. Kang and J.C. Wei, On interacting bumps of semi-classical states of nonlinear Schrödinger equations. *Adv. Diff. Eqn.* 5 (2000), 899-928.
[32] I. Kra and S. R. Sinmanca, On circulant matrices. *Notices AMS* (59)(2012), no.3 , 368-377 .
[33] N. Laskin, Fractional quantum mechanics. *Phys. Rev. E* 62(2000) 31C-35.
[34] N. Laskin, Fractional quantum mechanics and Levy path integrals. *Phys. Lett. A* 268(2000) 298-C305.
[35] N. Laskin, Fractional Schr?dinger equation. *Phys. Rev. E* 66(2002) 56C-108.
[36] Y. Li, W. M. Ni, On conformal scalar curvature equations in $\mathbb{R}^n$. *Duke Math, J.* 57 (1988), no.3, 895-924.
[37] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Lineaire* 1(1984), 109-145.
[38] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II. *Ann. Inst. H. Poincaré Anal. Non Lineaire* 1(1984), 223-283.
[39] F. Mahmoudi, A. Malchiodi, M. Montenegro, Solutions to the nonlinear Schrödinger equation carrying momentum along a curve. *Comm. Pure Appl. Math.* 62, no.9, 1155-1264, 2009.
[40] M. Musso and J.C.Wei, Nondegeneracy of nonlinear nodal solutions to Yamabe problem, *Comm. Math. Phy.* (340)2015, no. 3, 1049-1107.
[41] E. S. Noussair, S. S. Yan, On positive multi-peak solutions of a nonlinear elliptic problem. *J. London Math. Soc.* 62(2000), 213-277.
[42] Y. J. Oh, On positive multi-lump bound states nonlinear Schrödinger equations under multiple well potential. *Comm. Math. Phys.* 131 (1990), 223-253.
[43] X. Ros-Oton, J. Serra, The Pohozaev identity for the fractional Laplacian. *Arch. Ration. Mech. Anal.* 213 (2014), no. 2, 587C628.
[44] Y. Sire, E. Valdinoci, Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result. *J. Funct. Anal.* 256 (2009) 1842C1864.
[45] L. P. Wang, J. C. Wei and S. S. Yan, A Neumann problem with critical exponent in non-convex domains and Lin-Nis conjecture. *Tran. American Math. Society* 362(2010), no.9, 4581-4615.
[46] L. P. Wang, C. Y. Zhao, Infinitely many solutions for the prescribed boundary mean curvature problem in $\mathbb{R}^N$. *Canad. J. Math.* 65 (2013), no.4, 927-960.
[47] L. P. Wang, C. Y. Zhao, Infinitely many solutions to a fractional nonlinear Schrödinger equation. arXiv: 1403.0042v1.
[48] X. Wang, On concentration of positive bound states of nonlinear Schrödinger equations. *Comm. Math. Phys.* 153 (1993), 229-243.
[49] L. Wei, S. J. Peng and J. Yang, Infinitely many positive solutions for nonlinear fractional Schr?dinger equations. arXiv: 1402.1902v1.
[50] J. C. Wei, S. S. Yan, Infinitely many positive solutions for the nonlinear Schrödinger equations in $\mathbb{R}^N$. *Calc. Var. Partial Differ. Equ.* 37 (2010), no.3-4, 423-439.

WEIWEI AO, SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN, 430072, P.R. CHINA
E-mail address: wwao@whu.edu.cn

JUNCHENG WEI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BC V6T 1Z2, CANADA
E-mail address: jcwei@math.ubc.ca

WEN YANG, CENTER FOR ADVANCED STUDY IN THEORETICAL SCIENCES (CASTS), NATIONAL TAIWAN UNIVERSITY, TAIPEI 10617, TAIWAN
E-mail address: math.yangwen@gmail.com