FINITE TIME BLOW-UP FOR THE FRACTIONAL CRITICAL HEAT EQUATION IN $\mathbb{R}^n$

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Abstract. We consider the fractional critical heat equation

$$\begin{cases} u_t + (-\Delta)^s u = u^{\frac{n+2s}{n-2s}}, & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n, \end{cases}$$

where $4s < n < 6s$, $0 < s < 1$. For sufficiently small $T > 0$ and given k distinct points $q_1, \cdots, q_k$, we show that there exists an initial condition u_0 such that the solution $u(x,t)$ blows up at these k distinct points as $t \to T$. More precisely, the blow-up profile around each concentration point takes the form of sharply scaled bubble and

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \sim (T-t)^{-\frac{n-2s}{6s-n}}$$

as $t \to T$.

1. Introduction

Semilinear heat equation of form

$$\begin{cases} u_t = \Delta u + u^p, & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u(\cdot,0) = u_0, & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

with $p > 1$ has attracted much attention since Fujita's celebrated work [26]. Many works have been devoted to studying this problem about the blow-up rates, sets and profiles. See, for example, [28], [27], [30], [35], [45], [36], [37], [5] and references therein. The finite time blow-up is said to be of type I if

$$\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty < +\infty$$

and of type II if

$$\limsup_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_\infty = +\infty.$$

Type I blow-up is like that of the ODE $u_t = u^p$ and type II blow-up is much harder to detect. Many studies have predicted that blow-up phenomena in problem (1.1) are very sensitive to the values of the exponent p. It was first proved by Giga and Kohn [28] that for $1 < p < \frac{n+2}{n-2}$, only type I blow-up can occur for the case of convex domain. This result was generalized to general domain by Poon in [41]. For the critical case $p = \frac{n+2}{n-2}$, this is also the case for radial solutions [24] or the domain is star-shaped [3]. The critical case $p = \frac{n+2}{n-2}$ is special in many ways. For the subcritical case $p < \frac{n+2}{n-2}$, in [39] Merle and Zaag found multiple-point, finite time type I blow-up solution and studied its stability. For the supercritical case $p > \frac{n+2}{n-2}$, Matano and Merle classified the radial blow-up solutions in [37] and they found that for $\frac{n+2}{n-2} < p < p_{JL}(n)$ with the Joseph-Lundgren exponent defined as

$$p_{JL}(n) := \begin{cases} \infty & \text{if } \ 3 \le n \le 10 \\ 1 + \frac{4}{n-4-2\sqrt{n-1}} & \text{if } \ n \ge 11 \end{cases}$$

no type II blow-up can occur in radially symmetric class. In [19], del Pino, Musso and Wei constructed non-radial type II blow-up solution in the Matano-Merle rage $\frac{n+2}{n-2} < p < p_{JL}(n)$, where $p = \frac{n+1}{n-3}$ is the second critical exponent and the solution blows up along a certain curve with axial symmetry in the sense that the energy density approaches to the Dirac measure along the curve. For the critical case $p = \frac{n+2}{n-2}$, Collot, Merle and Raphaël classified the dynamics near the ground state of the energy critical heat equation in $\mathbb{R}^n$ with $n \geq 7$ in [4]. In [42], by using the energy method, Schweyer constructed the radial, type II finite time blow-up solution to the energy critical heat equation in $\mathbb{R}^4$. In [21], del Pino, Musso and Wei found the existence of finite time type II blow-up solution for the energy critical heat equation in $\mathbb{R}^5$. Concerning the infinite time blow-up, in a very interesting paper [23] Fila and King studied problem (1.1) with $p = \frac{n+2}{n-2}$ and provided insight on the question of infinite time blow-up in the case of a radially symmetric, positive initial condition with an exact power decay rate. Using formal matching asymptotic analysis, they demonstrated that the power decay determines the blow-up rate in a precise manner. Intriguingly enough, their analysis leads them to conjecture that infinite time blow-up should only happen in low dimensions 3 and 4, see Conjecture 1.1 in [23]. Recently, this is confirmed and rigorously proved in [20] for dimension 3. Bubbling phenomena triggered by criticality are present in many other contexts, for example, Keller-Segel chemotaxis system, harmonic map heat flow, Schrödinger map and geodesic flows. We refer the readers for instance to [7,9,12,13,29,32,33,38] and the references therein.

In [6], Cortázar, del Pino and Musso investigated the following critical heat equation in bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 5)$

$$\begin{cases} u_t = \Delta u + u^{\frac{n+2}{n-2}} & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases}$$

and they showed the existence of infinite time blow-up whose blow-up profile takes the form of sharply scaled bubble, and the blow-up points are determined by the Green's function and its regular part in Ω. In [40], the existence of infinite time blow-up has been proved for the fractional case

$$\begin{cases} u_t = -(-\Delta)^s u + u^{\frac{n+2s}{n-2s}}, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } (\mathbb{R}^n \setminus \Omega) \times (0,\infty), \\ u(\cdot,0) = u_0, & \text{in } \mathbb{R}^n \end{cases}$$

with $0 < s < 1$ and $n > 4s$. Here, for any point $x \in \mathbb{R}^n$, the fractional Laplacian $(-\Delta)^s u(x)$ is defined as

$$(-\Delta)^s u(x) := c_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy$$

with a suitable positive normalizing constant $c_{n,s} = \frac{2^{2s} s \Gamma(\frac{n+2s}{2})}{\Gamma(1-s)\pi^{\frac{n}{2}}}$. We refer to [22] for an introduction to the fractional Laplacian and to the appendix of [8] for a heuristic physical motivation in nonlocal quantum mechanics of the fractional operator considered here.

In this paper, we consider the fractional heat equation with the critical exponent

$$\begin{cases} u_t + (-\Delta)^s u = u^{\frac{n+2s}{n-2s}}, & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2}$$

Throughout the paper, we assume that $4s < n < 6s$, $0 < s < 1$ and a function $Z_0^* \in C_0^\infty(\mathbb{R}^n)$ is chosen such that

$$Z_0^*(q_j) < 0,$$

where q_j $(j = 1, \cdots, k)$ are distinct k points. It is well known that

$$U(y) = \alpha_{n,s} \left(\frac{1}{1+|y|^2} \right)^{\frac{n-2s}{2}}$$

is the bubble solving the fractional Yamabe problem

$$(-\Delta)^s U = U^{\frac{n+2s}{n-2s}} \quad \text{in} \quad \mathbb{R}^n,$$

where $\alpha_{n,s}$ is a constant depending only on n and s. See, for instance [2] and [34]. The scaled bubble is defined as

$$U_{\lambda,\xi} := \lambda^{-\frac{n-2s}{2}}(t)U\left(\frac{x-\xi(t)}{\lambda(t)}\right). \tag{1.3}$$

We show the existence of finite time blow-up for the fractional critical heat equation (1.2) and the main Theorem is stated as follows.

Theorem 1.1. *Assume that $4s < n < 6s$, $0 < s < 1$ and $Z_0^*(q_j) < 0$ for k distinct points q_j ($j = 1, \cdots, k$). For T sufficiently small, there exists an initial condition u_0 such that the solution $u(x,t)$ to problem (1.2) blows up at $q_1, \cdots, q_k$ at finite time T. Furthermore, the solution takes the form*

$$u(x,t) = \sum_{j=1}^{k} U_{\lambda_j(t),\xi_j(t)}(x) + Z_0^*(x) + \Theta(x,t),$$

where $U_{\lambda_j(t),\xi_j(t)}(x)$ is the scaled bubble defined in (1.3),

$$\lambda_j(t) \to 0,\ \xi_j(t) \to q_j \quad as \quad t \to T,$$

$\|\Theta\|_{L^\infty} \le T^c$ for some constant $c > 0$. More precisely,

$$\lambda_j(t) = \kappa_j(T-t)^{\frac{2}{6s-n}}(1+o(1))$$

for some positive constants $\kappa_j > 0$, $j = 1, \cdots, k$.

Remark 1.1. *Theorem 1.1 implies that*

- *For $n = 4$, finite time blow-up takes place for $s \in (2/3, 1)$*
- *For $n = 5$, finite time blow-up takes place for $s \in (5/6, 1)$*

which is a continuation of the local cases $n = 4$, $s = 1$ in [42] and $n = 5$, $s = 1$ in [21]. Also, our construction suggests that no finite time blow-up of this type should exist in higher dimension case $n \ge 6$, $s \in (0,1)$.

The proof of Theorem 1.1 is mainly based on *inner-outer gluing method*, which is well developed, for the higher dimensional concentration in elliptic settings, in [15], [16], [18], [17] for example. Recently, the *parabolic gluing method* is developed and has been successfully applied to the construction of solutions to various parabolic problems, such as the singular formation for the harmonic map flow from $\mathbb{R}^2$ to $\mathbb{S}^2$ [12], the infinite time blow-up in energy critical heat equation [6], [20], type II finite time blow-up along a circle for supercritical heat equation [19], type II ancient solution for the Yamabe flow [7], infinite time blow-up for the half-harmonic map flow [44], the vortex dynamics in Euler flows [9], blow-up for the 3-dimensional harmonic map flow along a curve with axial symmetry [10] and type II finite time blow-up for the energy critical heat equation in $\mathbb{R}^5$ [21]. We refer the readers to the survey by del Pino [14] for more results in parabolic settings.

The proof of Theorem 1.1 is close in spirit to [21]. However, in a central step that the linear theory for the associated linear problem of the inner problem is required, the ODE techniques are no longer applicable in the fractional setting. Instead, we shall develop a fractional linear theory by using a blow-up argument inspired by [12, Lemma 4.5]. See Section 5 for full details.

By our construction, finite time blow-up also exists on the bounded domain $\Omega \subset \mathbb{R}^n$. Suppose a smooth function $\mathcal{Z}_0^* \in L^\infty(\Omega)$ satisfies

$$\mathcal{Z}_0^*(q_j) < 0$$

for given k distinct points $q_1, \cdots, q_k$. For the fractional critical heat equation on bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} u_t + (-\Delta)^s u = u^{\frac{n+2s}{n-2s}}, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } (\mathbb{R}^n \setminus \Omega) \times (0,\infty), \\ u(\cdot,0) = u_0, & \text{in } \mathbb{R}^n \end{cases} \tag{1.4}$$

with $u_0(x) = 0$ in $\mathbb{R}^n \backslash \Omega$, $0 < s < 1, 4s < n < 6s$, finite time blow-up exists and we have the following Theorem.

Theorem 1.2. *Assume that $4s < n < 6s$, $0 < s < 1$ and $\mathcal{Z}_0^*(q_j) < 0$ for k distinct points q_j ($j = 1, \cdots, k$). For T sufficiently small, there exists an initial condition u_0 such that the solution $u(x,t)$ to problem (1.4) blows up at $q_1, \cdots, q_k$ at finite time T. Moreover, at main order, the solution takes the form*

$$u(x,t) = \sum_{j=1}^{k} U_{\lambda_j(t),\xi_j(t)}(x) + \mathcal{Z}_0^*(x) + \Upsilon(x,t),$$

where $U_{\lambda_j(t),\xi_j(t)}(x)$ is the scaled bubble defined in (1.3),

$$\lambda_j(t) \to 0, \ \xi_j(t) \to q_j \quad as \quad t \to T,$$

$\|\Upsilon\|_{L^\infty} \le T^c$ for some constant $c > 0$. More precisely,

$$\lambda_j(t) = \upsilon_j (T-t)^{\frac{2}{6s-n}}(1 + o(1))$$

for some positive constants $\upsilon_j > 0$, $j = 1, \cdots, k$.

The proof of Theorem 1.2 can be carried out similarly as that of Theorem 1.1. So we shall only prove Theorem 1.1 in this paper.

The paper is organized as follows. In Section 2, we construct an approximate solution and compute its error. In Section 3, the main parts of the parameters λ and ξ are given. In Section 4, we develop a linear theory for the outer problem. In Section 5, we develop a new fractional linear theory for the inner problem. Finally, we shall prove Theorem 1.1 in Section 6.

Notation. In the sequel, we shall use the symbol " $\lesssim$ " to denote " $\le C$" for a positive constant C independent of t and T, and C may change from line to line.

2. Approximate solution and error estimate

In this section, we shall choose the approximate solution and compute its error. For simplicity, we consider one bubble case. The multiple-bubble case is similar up to some minor modifications which we will point out if necessary.

Our first approximate solution is

$$w = U_{\lambda,\xi} + Z^*,$$

where

$$U_{\lambda,\xi} = \lambda^{-\frac{n-2s}{2}}(t) U\left(\frac{x-\xi(t)}{\lambda(t)}\right),$$

and Z^* is the solution to the fractional heat equation

$$\begin{cases} Z_t^* + (-\Delta)^s Z^* = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ Z^*(x,0) = Z_0^*(x), & \text{in } \mathbb{R}^n. \end{cases}$$

Here,

$$\lambda(t) = \lambda_0(t) + \lambda_1(t), \ \xi(t) = \xi_0(t) + \xi_1(t), \tag{2.1}$$

where $\lambda_0(t)$ and $\xi_0(t)$ are the main order terms of $\lambda(t)$ and $\xi(t)$ respectively, and $\lambda_1(t)$ and $\xi_1(t)$ are the reminder terms which are comparatively smaller than $\lambda_0(t)$ and $\xi_0(t)$ respectively. Define the error

$$S(u) = -u_t - (-\Delta)^s u + u^p,$$

where $p := \frac{n+2s}{n-2s}$. Direct computations imply that

$$\begin{aligned} S(w) &= -(U_{\lambda,\xi})_t + (U_{\lambda,\xi} + Z^*)^p - U_{\lambda,\xi}^p \\ &= \lambda^{-\frac{n-2s}{2}-1}\dot{\lambda} Z_{n+1}(y) + \lambda^{-\frac{n-2s}{2}-1}\dot{\xi}\cdot\nabla U(y) + (U_{\lambda,\xi} + Z^*)^p - U_{\lambda,\xi}^p, \end{aligned}$$

where $y = \frac{x-\xi}{\lambda}$, $Z_{n+1} = \frac{n-2s}{2}U(y) + y\cdot\nabla U(y)$.

We look for a solution of the following form

$$u = U_{\lambda,\xi} + Z^* + \varphi.$$

Then $S(u)=0$ yields that

$$S(u)=-\varphi_t-(-\Delta)^s\varphi+pU_{\lambda,\xi}^{p-1}(y)(\varphi+Z^*)+\lambda^{-\frac{n+2s}{2}}\mathcal{E}+N(\varphi+Z^*)=0, \tag{2.2}$$

where

$$\mathcal{E}(y,t)=\lambda^{2s-1}\dot\lambda Z_{n+1}(y)+\lambda^{2s-1}\dot\xi\cdot\nabla U \tag{2.3}$$

and

$$N(\varphi+Z^*)=(U_{\lambda,\xi}+\varphi+Z^*)^p-U_{\lambda,\xi}^p-pU_{\lambda,\xi}^{p-1}(\varphi+Z^*). \tag{2.4}$$

We look for perturbation consisting of inner and outer parts

$$\varphi(x,t)=\lambda^{-\frac{n-2s}{2}}(t)\eta_R(y)\phi(y,t)+\psi(x,t), \tag{2.5}$$

where $R>0$, η is a smooth cut-off function such that

$$\eta(s)=\begin{cases}1, & s<1,\\ 0, & s>2,\end{cases}$$

and $\eta_R=\eta(|y|/R)$. Then we can express (2.2) in terms of ϕ and ψ

$$\begin{aligned}&-\psi_t-(-\Delta)_x^s\psi+p\lambda^{-2}(1-\eta_R)U^{p-1}(y)(\psi+Z^*)+\mathcal{C}(\phi)+\mathcal{R}(\phi)+\lambda^{-\frac{n+2s}{2}}\mathcal{E}(1-\eta_R)+N(\varphi+Z^*)\\&+\eta_R\lambda^{-\frac{n+2s}{2}}\left(-\lambda^{2s}\phi_t-(-\Delta)_y^s\phi+pU^{p-1}(y)(\phi+\lambda^{\frac{n-2s}{2}}(\psi+Z^*))+\mathcal{E}\right)=0,\end{aligned}$$

where

$$\mathcal{C}(\phi):=\lambda^{-\frac{n-2s}{2}}\left[(-(-\Delta)_x^s-\partial_t)\eta_R(y)\phi+[-(-\Delta)_x^{s/2}\eta_R(y),-(-\Delta)_x^{s/2}\phi(y)]\right] \tag{2.6}$$

and

$$\mathcal{R}(\phi):=\lambda^{-\frac{n-2s}{2}-1}\left[\eta_R\dot\lambda\left(\frac{n-2s}{2}\phi+y\cdot\nabla_y\phi\right)+\eta_R\dot\xi\cdot\nabla_y\phi+\phi\left(\dot\lambda y\cdot\nabla_y\eta_R+\dot\xi\cdot\nabla_y\eta_R\right)\right]. \tag{2.7}$$

Here

$$[-(-\Delta)_x^{s/2}f(x),-(-\Delta)_x^{s/2}g(x)]:=c_{n,s}\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{[f(y)-f(x)][g(x)-g(y)]}{|x-y|^{n+2s}}dy \tag{2.8}$$

with $c_{n,s}=\frac{2^{2s}s\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)\pi^{\frac{n}{2}}}$. Therefore, $u=U_{\lambda,\xi}+Z^*+\lambda^{-\frac{n-2s}{2}}(t)\eta_R(y)\phi(y,t)+\psi(x,t)$ solves (1.2) if $(\phi(y,t),\psi(x,t))$ solves the so-called **inner-outer gluing system**

$$\lambda^{2s}\phi_t=-(-\Delta)_y^s\phi+pU^{p-1}(y)\phi+\mathcal{H}(\phi,\psi,\lambda,\xi)\ \text{ in }\ B_{2R}(0)\times(0,T) \tag{2.9}$$

$$\begin{cases}\psi_t=-(-\Delta)_x^s\psi+\mathcal{G}(\phi,\psi,\lambda,\xi), & \text{in }\ \mathbb{R}^n\times(0,T)\\ \psi(x,0)=0, & \text{in }\ \mathbb{R}^n\end{cases} \tag{2.10}$$

where

$$\mathcal{H}(\phi,\psi,\lambda,\xi):=\lambda^{\frac{n-2s}{2}}pU^{p-1}(y)(\psi(\lambda y+\xi,t)+Z^*(\lambda y+\xi,t))+\mathcal{E}(y,t) \tag{2.11}$$

and

$$\mathcal{G}(\phi,\psi,\lambda,\xi):=p\lambda^{-2s}(1-\eta_R)U^{p-1}(y)(\psi+Z^*)+\mathcal{C}(\phi)+\mathcal{R}(\phi)+\lambda^{-\frac{n+2s}{2}}\mathcal{E}(1-\eta_R)+N(\varphi+Z^*). \tag{2.12}$$

3. Choices of $\lambda_0(t)$ and $\xi_0(t)$

We shall choose $\lambda_0(t)$ and $\xi_0(t)$ defined in (2.1) in this section. Basically, the inner problem (2.9) will determine the parameter functions λ and ξ at main order. By the fractional linear theory developed in Section 5, the inner problem (2.9) will be solved under the orthogonality conditions

$$\int_{B_{2R}} \mathcal{H}(\phi,\psi,\lambda,\xi)Z_j(y)dy = 0, \quad \text{for all} \quad t\in(0,T), \ j=1,\cdots,n+1,$$

where R is fixed sufficiently large and

$$Z_j(y) = \partial_{y_j}U(y), \ j=1,\cdots,n, \ Z_{n+1}(y) = \frac{n-2}{2}U(y) + y\cdot\nabla U(y). \tag{3.1}$$

The leading term of $\mathcal{H}$ is

$$\begin{aligned} h[\lambda_0,\xi_0] = {} & \lambda_0^{2s-1}\dot{\lambda}_0 Z_{n+1}(y) + \lambda_0^{2s-1}\dot{\xi}_0\cdot\nabla U + \lambda_0^{\frac{n-2s}{2}} pU^{p-1}(y)Z_0^*(q) \\ & + \lambda_0^{\frac{n-2s}{2}+1} pU^{p-1}(y)\nabla Z_0^*(q)\cdot y. \end{aligned}$$

It is reasonable to choose $\lambda_0(t)$, $\xi_0(t)$ such that

$$\int_{\mathbb{R}^n} h[\lambda_0,\xi_0]Z_j(y)dy = 0 \ \text{ for all } \ t\in(0,T), \ j=1,\cdots,n+1$$

are satisfied. For $j=n+1$, we have

$$\int_{\mathbb{R}^n} h[\lambda_0,\xi_0]Z_{n+1}(y)dy = \lambda_0^{2s-1}\dot{\lambda}_0\int_{\mathbb{R}^n} Z_{n+1}^2(y)dy + \lambda_0^{\frac{n-2s}{2}} pZ_0^*(q)\int_{\mathbb{R}^n} U^{p-1}(y)Z_{n+1}(y)dy$$

and thus

$$c_0\dot{\lambda}_0 + c_1 Z_0^*(q)\lambda_0^{\frac{n-6s+2}{2}} = 0,$$

where

$$c_0 = \int_{\mathbb{R}^n} Z_{n+1}^2(y)dy, \ c_1 = p\int_{\mathbb{R}^n} U^{p-1}(y)Z_{n+1}(y)dy.$$

Observer that c_0 is well-defined thanks to the assumption $n>4s$ and $c_2<0$. Therefore, in order that $\lambda(t)\to 0$ as $t\to T$, we suppose

$$Z_0^*(q) < 0$$

and then we obtain the main order

$$\lambda_0(t) = a(T-t)^{\frac{2}{6s-n}} \tag{3.2}$$

with

$$a = \left(\frac{2c_0}{c_1 Z_0^*(q)(6s-n)}\right)^{\frac{2}{n-6s}}.$$

Similarly, we consider the case $j=1,\cdots,n$ and get

$$\int_{\mathbb{R}^n} h[\lambda_0,\xi_0]Z_j(y)dy = \lambda_0^{2s-1}\int_{\mathbb{R}^n}\dot{\xi}_0\cdot\nabla U(y)Z_j(y)dy + \lambda_0^{\frac{n-2s}{2}+1}p\int_{\mathbb{R}^n} U^{p-1}(y)Z_j(y)\nabla Z_0^*(q)\cdot y dy.$$

So we can write $\dot{\xi}_0(t) = \lambda_0^{\frac{n-6s}{2}+2}\vec{v}$ for some vector $\vec{v}$. Hence, by (3.2), we obtain

$$\xi_0(t) = q + O(T-t)^{\frac{4}{6s-n}}\vec{v}$$

for some vector $\vec{v}$. The remainders $\lambda_1(t)$ and $\xi_1(t)$ defined in (2.1) will be chosen when we finally solve the inner-outer gluing system in Section 6.

4. The outer problem

In this section, we shall get proper a priori estimates of the associated linear problem of the outer problem (2.10). We consider the linear problem

$$\begin{cases} \partial_t \psi(x,t) = -(-\Delta)_x^s \psi(x,t) + f(x,t), & \text{in } \mathbb{R}^n \times (0,T) \\ \psi(x,0) = 0, & \text{in } \mathbb{R}^n. \end{cases} \tag{4.1}$$

A solution to the Cauchy problem (4.1) is guaranteed. Recall that the heat kernel to the fractional heat operator $\partial_t + (-\Delta)^s$ is given by

$$K_s(x,t) \asymp \frac{t}{(t^{\frac{1}{s}} + |x|^2)^{\frac{n+2s}{2}}}. \tag{4.2}$$

Then by Duhamel's formula

$$\psi(x,t) = \int_0^t \int_{\mathbb{R}^n} K_s(x-z, t-r) f(z,r) dz dr$$

is the solution to (4.1).

Define the norms

$$\|\psi\|_* := \sup_{(x,t)\in\mathbb{R}^n\times(0,T)} \frac{|\psi(x,t)|}{\rho_*}, \tag{4.3}$$

$$\|f\|_{**} := \sup_{(x,t)\in\mathbb{R}^n\times(0,T)} \frac{|f(x,t)|}{\rho_{**}}, \tag{4.4}$$

where

$$\rho_* := 1 + \frac{1}{1 + \left|\frac{x-q}{\lambda_0(t)}\right|^{n-4s}} \quad \text{and} \quad \rho_{**} := 1 + \frac{\lambda_0^{-2s}(t)}{1 + \left|\frac{x-q}{\lambda_0(t)}\right|^{n-2s}}.$$

We have the following lemma.

Lemma 4.1. *Assume that $\|f\|_{**} < +\infty$. For sufficiently small $T > 0$, the solution ψ to problem (4.1) satisfies*

$$\|\psi\|_* \lesssim \|f\|_{**}.$$

Proof. First, we compute the asymptotic behavior of $\psi(x,t)$ as $|x| \to +\infty$ provided $\|f\|_{**} < +\infty$.

$$\begin{aligned} |\psi(x,t)| \lesssim & \left| \int_0^t \int_{\mathbb{R}^n} K_s(x-z, t-r) f(z,r) dz dr \right| \\ \lesssim & \|f\|_{**} \int_0^t \int_{\mathbb{R}^n} \frac{t-r}{((t-r)^{1/s} + |x-z|^2)^{\frac{n+2s}{2}}} \left(1 + \frac{\lambda_0^{-2s}(r)}{1 + \left|\frac{z-q}{\lambda_0(r)}\right|^{n-2s}} \right) dz dr \\ = & \|f\|_{**} \int_0^t \int_{\mathbb{R}^n} \frac{1}{(1+|X|^2)^{\frac{n+2s}{2}}} \left(1 + \frac{\lambda_0^{-2s}(r)}{1 + \left|\frac{x-(t-r)^{\frac{1}{2s}}X - q}{\lambda_0(r)}\right|^{n-2s}} \right) dX dr, \end{aligned} \tag{4.5}$$

where we have performed the change of variable

$$X = \frac{x-z}{(t-r)^{\frac{1}{2s}}}.$$

For the first integral in (4.5), we have

$$\|f\|_{**} \int_0^t \int_{\mathbb{R}^n} \frac{1}{(1+|X|^2)^{\frac{n+2s}{2}}} dX dr \lesssim t\|f\|_{**}. \tag{4.6}$$

We decompose the second integral in (4.5) as follows

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} \frac{1}{(1+|X|^2)^{\frac{n+2s}{2}}} \frac{\lambda_0^{-2s}(r)}{1+\left|\frac{x-(t-r)^{\frac{1}{2s}}X-q}{\lambda_0(r)}\right|^{n-2s}} dX dr \\
=&\int_0^t \left(\int_{D_1}+\int_{D_2}+\int_{D_3}\right) \frac{1}{(1+|X|^2)^{\frac{n+2s}{2}}} \frac{\lambda_0^{-2s}(r)}{1+\left|\frac{x-(t-r)^{\frac{1}{2s}}X-q}{\lambda_0(r)}\right|^{n-2s}} dX dr \\
:=&J_1+J_2+J_3,
\end{aligned}$$

where

$$D_1 := \left\{X\in\mathbb{R}^n : \left|x-(t-r)^{\frac{1}{2s}}X-q\right| \geq 2|x-q|\right\},$$

$$D_2 := \left\{X\in\mathbb{R}^n : 2|x-q| \geq \left|x-(t-r)^{\frac{1}{2s}}X-q\right| \geq \frac{|x-q|}{2}\right\}$$

and

$$D_3 := \left\{X\in\mathbb{R}^n : \frac{|x-q|}{2} \geq \left|x-(t-r)^{\frac{1}{2s}}X-q\right|\right\}.$$

Then we estimate term by term by recalling that $\lambda_0(t)\sim (T-t)^{\frac{2}{6s-n}}$ in (3.2)

$$\begin{aligned}
J_1 \lesssim& \int_0^t \int_{|x-q|(t-r)^{-\frac{1}{2s}}}^{+\infty} \frac{|X|^{n-1}}{1+|X|^{n+2s}} \frac{\lambda_0^{-2s}(r)}{1+\left|\frac{x-q}{\lambda_0(r)}\right|^{n-2s}} d|X| dr \\
\lesssim& \int_0^t \frac{t-r}{|x-q|^{2s}} \frac{\lambda_0^{-2s}(r)}{1+\left|\frac{x-q}{\lambda_0(r)}\right|^{n-2s}} dr \lesssim \frac{1}{|x-q|^n}\int_0^t (T-r)^{1+\frac{2(n-4s)}{6s-n}} dr \\
\lesssim& \frac{1}{|x-q|^n} T^{\frac{4s}{6s-n}},
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
J_2 \lesssim& \int_0^t \int_0^{\frac{3}{2}|x-q|(t-r)^{-\frac{1}{2s}}} \frac{|X|^{n-1}}{1+|X|^{n+2s}} \frac{\lambda_0^{-2s}(r)}{1+\left|\frac{x-q}{\lambda_0(r)}\right|^{n-2s}} d|X| dr \\
\lesssim& \int_0^t \frac{\lambda_0^{-2s}(r)}{1+\left|\frac{x-q}{\lambda_0(r)}\right|^{n-2s}} dr \lesssim \frac{1}{|x-q|^{n-2s}} T^{\frac{n-2s}{6s-n}},
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
J_3 \lesssim& \int_0^t \int_{\frac{1}{2}|x-q|(t-r)^{-\frac{1}{2s}}}^{\frac{3}{2}|x-q|(t-r)^{-\frac{1}{2s}}} \frac{|X|^{n-1}}{1+|X|^{n+2s}} \frac{\lambda_0^{n-4s}(r)}{|(t-r)^{\frac{1}{2s}}X|^{n-2s}} d|X| dr \\
\lesssim& \frac{1}{|x-q|^n}\int_0^t \frac{\lambda_0^{n-4s}(r)(t-r)^{\frac{n}{2s}}}{(t-r)^{\frac{n-2s}{2s}}} dr \\
\lesssim& \frac{1}{|x-q|^n} T^{\frac{4s}{6s-n}}.
\end{aligned} \tag{4.9}$$

Therefore, we conclude from (4.6)–(4.9) that $|\psi(x,t)|$ is bounded as $|x|\to+\infty$.

Now we build a supersolution to problem (4.1) with $\|f\|_{**}<+\infty$. Let $p(y)$ be the solution to

$$(-\Delta)_y^s p(y) = \frac{1}{1+|y|^{n-2s}},$$

where $y=\frac{x-q}{\lambda_0(t)}$. By the Riesz potential, it is direct to see that

$$p(y)\sim \frac{1}{1+|y|^{n-4s}} \quad \text{as} \quad |y|\to\infty.$$

We let $\psi_1(x,t) = 2\|f\|_{**}p(y)$ and compute

$$\begin{aligned}\partial_t\psi_1 + (-\Delta)^s\psi_1 &= -2\|f\|_{**}\dot{\lambda}_0\lambda_0^{-1}y\cdot\nabla p(y) + \frac{2\|f\|_{**}\lambda_0^{-2s}}{1+|y|^{n-2s}} \\ &\geq |f| + \frac{\|f\|_{**}\lambda_0^{-2s}}{1+|y|^{n-2s}} - 2\|f\|_{**}\dot{\lambda}_0\lambda_0^{-1}y\cdot\nabla p(y) - \|f\|_{**}.\end{aligned} \tag{4.10}$$

Observe that for some constants $\gamma, c > 0$ independent of T and t

$$2\|f\|_{**}\dot{\lambda}_0\lambda_0^{-1}y\cdot\nabla p(y) - \frac{\|f\|_{**}\lambda_0^{-2s}}{1+|y|^{n-2s}} \leq \begin{cases}0, & \text{for } |x-q| \leq c(T-t)^{\frac{1}{2s}} \\ \gamma(T-t)^{\beta-1}, & \text{for } |x-q| \geq c(T-t)^{\frac{1}{2s}}\end{cases} \tag{4.11}$$

where $\beta = \frac{(n-2s)(n-4s)}{2s(6s-n)}$. We take

$$\psi_2 = C\|f\|_{**}t + \gamma\beta^{-1}[T^\beta - (T-t)^\beta],\ \bar{\psi} = \psi_1 + \psi_2, \tag{4.12}$$

where C is a sufficiently large constant. Combining (4.10)-(4.12), we conclude that $\bar{\psi}$ is a supersolution to problem (4.1) and the estimate

$$\|\psi\|_* \lesssim \|f\|_{**}$$

follows immediately from the definititions of the norms (4.3) and (4.4). □

Remark 4.1. *For arbitrary $T' < T$, we have*

$$\|f\|_\infty \lesssim \lambda_0^{-2s}(T')\|f\|_{**}.$$

Then fractional parabolic estimates (see [31] and the references therein) imply the following Hölder estimate

$$[\psi]_{\alpha,T'} \lesssim \lambda_0^{-2s}(T')\|f\|_{**} \tag{4.13}$$

for some $0 < \alpha < 1$. Here the space-time Hölder semi-norm is defined as

$$[\psi]_{\alpha,T'} := \sup_{\substack{x_1,x_2\in\mathbb{R}^n \\ t_q,t_2\in[0,T']}} \frac{|\psi(x_1,t_1)-\psi(x_2,t_2)|}{|x_1-x_2|^\alpha + |t_1-t_2|^{\alpha/2}}.$$

5. The linear theory

In this section, we shall develop a fractional linear theory motivated by [12, Lemma 4.5] for the inner problem (2.9). Since the ODE techniques are no longer applicable in the fractional setting, we shall use the blow-up argument instead.

In order to solve the inner problem (2.9), we consider the associated linear problem

$$\lambda^{2s}\phi_t = -(-\Delta)_y^s\phi + pU^{p-1}(y)\phi + h(y,t) \quad \text{in} \quad B_{2R}(0)\times(0,T). \tag{5.1}$$

Recall that the linearized operator

$$L_0 := -(-\Delta)^s + pU^{p-1}$$

has a only positive eigenvalue μ_0 such that

$$L_0(Z_0) = \mu_0 Z_0,\ Z_0 \in L^\infty(\mathbb{R}^n),$$

where the corresponding eigenfunction Z_0 is radially symmetric with the asymptotic behavior

$$Z_0(y) \sim |y|^{-n-2s} \quad \text{as} \quad |y| \to +\infty, \tag{5.2}$$

see [25] for instance. Multiplying equation (5.1) by Z_0 and integrating over $\mathbb{R}^n$, we obtain that

$$\lambda^2(t)\dot{p}(t) - \mu_0 p(t) = q(t),$$

where

$$p(t) = \int_{\mathbb{R}^n} \phi(y,t)Z_0(y)dy \quad \text{and} \quad q(t) = \int_{\mathbb{R}^n} h(y,t)Z_0(y)dy.$$

Then we have

$$p(t) = e^{\int_0^t \mu_0 \lambda^{-2s}(r)dr}\left(p(0) + \int_0^t q(\eta)\lambda^{-2s}(\eta)e^{-\int_0^\eta \mu_0\lambda^{-2s}(r)dr}d\eta\right).$$

In order to get a decaying solution, the initial condition

$$p(0) = -\int_0^T q(\eta)\lambda^{-2s}(\eta)e^{-\int_0^\eta \mu_0\lambda^{-2s}(r)dr}d\eta$$

is required. The above formal argument suggests that a linear constraint should be imposed on the initial value $\phi(y,0)$. Therefore, we consider the associated linear Cauchy problem of the inner problem (2.9)

$$\begin{cases}\phi_\tau = -(-\Delta)_y^s\phi + pU^{p-1}(y)\phi + h(y,\tau), & \text{in } B_{2R}(0)\times(\tau_0,\infty)\\ \phi(y,\tau_0) = e_0 Z_0(y), & \text{in } B_{2R}(0)\end{cases} \tag{5.3}$$

where $R>0$ is fixed sufficiently large, and we have performed the following change of variables

$$\frac{d\tau}{dt} = \lambda^{-2s}(t).$$

Let $\nu>0$, $a>0$ such that

$$\tau^{-\nu}\sim\lambda^{\frac{n-2s}{2}}.$$

Define

$$\|h\|_{a,\nu,\eta} := \sup_{\substack{y\in B_{2R}\\ \tau>\tau_0}} \tau^\nu(1+|y|^a)\left(|h(y,\tau)| + (1+|y|^\eta)\chi_{B_{2R}}[h(\cdot,\tau)]_{\eta,B_{2R}}\right), \tag{5.4}$$

where the Hölder semi-norm is defined by

$$[h(\cdot,\tau)]_{\eta,B_{2R}} := \sup_{x,y\in B_{2R}}\frac{|h(x,\tau)-h(y,\tau)|}{|x-y|^\eta}$$

for $0<\eta<1$. In the sequel, we consider $h=h(y,\tau)$ as a function in the whole space $\mathbb{R}^n$ with zero extension outside of B_{2R} for all $\tau>\tau_0$. The main result of this section is stated as follows.

Proposition 5.1. *Assume $2s<a<n-2s$, $\nu>0$, $\|h\|_{2s+a,\nu,\eta}<+\infty$ and*

$$\int_{B_{2R}} h(y,\tau)Z_j(y)dy = 0, \quad \forall\tau\in(\tau_0,\,\infty),\ j=1,\cdots,n+1.$$

For sufficiently large R, there exist $\phi=\phi[h](y,\tau)$ and $e_0=e_0[h](\tau)$ solving (5.3) with

$$(1+|y|^s)\left(\int_{\mathbb{R}^n}\frac{[\phi(y,\tau)-\phi(x,\tau)]^2}{|y-x|^{n+2s}}dx\right)^{\frac12} + |\phi(y,\tau)| \lesssim \tau^{-\nu}(1+|y|)^{-a}\|h\|_{2s+a,\nu,\eta} \tag{5.5}$$

and

$$|e_0[h]|\lesssim\|h\|_{2s+a,\nu,\eta}, \tag{5.6}$$

for $(y,\tau)\in\mathbb{R}^n\times(\tau_0,\infty)$.

To prove Proposition 5.1, we consider the following problem

$$\begin{cases}\phi_\tau = -(-\Delta)^s\phi + pU^{p-1}(y)\phi + h(y,\tau) - c(\tau)Z_0(y), & \text{in } \mathbb{R}^n\times(\tau_0,\infty),\\ \phi(y,\tau_0)=0, & \text{in } \mathbb{R}^n,\\ \int_{\mathbb{R}^n}\phi(y,\tau)Z_0(y)dy = 0, \quad \text{for all } \tau\in(\tau_0,+\infty). & \end{cases} \tag{5.7}$$

Note that the orthogonality condition $\int_{\mathbb{R}^n}\phi(y,\tau)Z_0(y)dy=0$ is well-defined because Z_0 is of sufficiently fast decay. For problem (5.7), we have the following lemma.

Lemma 5.1. *Assume $2s < a < n-2s$, $\nu > 0$, $\|h\|_{2s+a,\nu,\eta} < +\infty$ and*

$$\int_{B_{2R}} h(y,\tau)Z_j(y)dy = 0, \quad \forall \tau \in (\tau_0,\, \infty),\ j = 1, \cdots, n+1.$$

Then for sufficiently large $\tau_1 > 0$, there exists solution $(\phi(y,\tau), c(\tau))$ to problem (5.7) satisfying

$$\|\phi\|_{a,\tau_1} \lesssim \|h\|_{2s+a,\tau_1} \tag{5.8}$$

and

$$|c(\tau)| \lesssim \tau^{-\nu} R^a \|h\|_{2s+a,\tau_1},\ \forall \tau \in (\tau_0, \tau_1). \tag{5.9}$$

Here the norm $\|\cdot\|_{b,\tau_1}$ is defined by

$$\|h\|_{b,\tau_1} := \sup_{\tau\in(\tau_0,\tau_1)} \tau^{\nu} \|(1+|y|^b)h\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. It is direct to see that problem (5.7) is equivalent to

$$\begin{cases} \phi_\tau = -(-\Delta)^s\phi + pU^{p-1}(y)\phi + h(y,\tau) - c(\tau)Z_0(y), & \text{in } \mathbb{R}^n \times (\tau_0, \infty), \\ \phi(y,\tau_0) = 0, & \text{in } \mathbb{R}^n, \end{cases} \tag{5.10}$$

for

$$c(\tau) = \frac{\int_{\mathbb{R}^n} h(y,\tau)Z_0(y)dy}{\int_{\mathbb{R}^n} |Z_0(y)|^2 dy}. \tag{5.11}$$

Some direct computations yield that

$$|c(\tau)| \lesssim \tau^{-\nu} R^a \|h\|_{2s+a,\tau_1},\ \forall \tau \in (\tau_0, \tau_1).$$

Next, we prove (5.8) by using the blow-up argument inspired by [12, Lemma 4.5].
We claim that given $\tau_1 > \tau_0$, we have

$$\|\phi\|_{a,\tau_1} < +\infty.$$

Indeed, by the fractional parabolic theory (see e.g. [31]), given $R_0 > 0$, there is a $K = K(R_0, \tau_1)$ such that

$$|\phi(y,\tau)| \le K \quad \text{in } B_{R_0} \times (\tau_0, \tau_1].$$

For fixed R_0, choosing K_1 sufficiently large, we get that $K_1\rho^{-a}$ is a supersolution to (5.10) for $\rho > R_0$. Then we get that

$$|\phi| \lesssim K_1\rho^{-a} \ \text{ and thus } \ \|\phi\|_{a,\tau_1} < +\infty.$$

By the definition (5.11), one has

$$\int_{\mathbb{R}^n} \phi(y,\tau)Z_0(y)dy = 0, \quad \forall \tau \in (\tau_0,\, \tau_1),$$

and we claim that

$$\int_{\mathbb{R}^n} \phi(y,\tau)Z_j(y)dy = 0, \quad \forall \tau \in (\tau_0,\, \tau_1),\ j = 1, \cdots, n+1. \tag{5.12}$$

Indeed, we test problem (5.10) against $\eta(|y|/R_1)Z_j(y)$, where R_1 is positive and the smooth cut-off function is defined by

$$\eta(r) = \begin{cases} 1 & \text{for } r < 1 \\ 0 & \text{for } r > 2 \end{cases}$$

It then follows that

$$\int_{\mathbb{R}^n} \phi(y,\tau)\eta(|y|/R_1)Z_j(y)dy = \int_0^\tau dr \int_{\mathbb{R}^n} \phi(y,r)\left(L_0(\eta Z_j) + h\eta Z_j - c(r)Z_0\eta Z_j\right)dy$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} \phi\left(L_0(\eta Z_j) + hZ_j\eta - c(r)Z_0Z_j\eta\right) &= \int_{\mathbb{R}^n} \phi\cdot\Big(Z_j(-(-\Delta)^s)\eta + \big[-(-\Delta)^{\frac{s}{2}}\eta, -(-\Delta)^{\frac{s}{2}}Z_j\big]\Big)\\ &\quad - h\cdot Z_j(1-\eta) + c(r)Z_0Z_j(1-\eta)\\ &= O(R_1^{-\varepsilon})\end{aligned}$$

for some small positive number ε uniformly on $\tau\in(\tau_0,\tau_1)$. Then (5.12) holds by letting $R_1\to+\infty$.

We prove (5.8) by contradiction. Suppose that there exists a sequence $\tau_1^k\to+\infty$ and ϕ_k, h_k, c_k satisfying

$$\begin{cases}\partial_\tau\phi_k = -(-\Delta)^s\phi_k + pU^{p-1}(y)\phi_k + h_k(y,\tau) - c_k(\tau)Z_0(y), & \text{in } \mathbb{R}^n\times(\tau_0,\infty),\\ \phi_k(y,\tau_0)=0, & \text{in } \mathbb{R}^n,\end{cases}$$

with

$$\|\phi_k\|_{a,\tau_1^k}=1,\quad \|h_k\|_{2s+a,\tau_1^k}\to 0. \tag{5.13}$$

From (5.13) and (5.9), it holds that for any fixed $R>0$,

$$\sup_{\tau\in(\tau_0,\tau_1^k)} \tau^\nu c_k(\tau)\to 0. \tag{5.14}$$

We first prove that

$$\sup_{\tau_0<\tau<\tau_1^k} \tau^\nu|\phi_k(y,\tau)|\to 0 \tag{5.15}$$

uniformly on any compact subset of $\mathbb{R}^n$. We prove (5.15) by contradiction.

Case 1. Suppose there exist some $|y_k|\le R$ and $\tau_0<\tau_2^k<\tau_1^k$, such that

$$(\tau_2^k)^\nu|\phi_k(y_k,\tau_2^k)|\ge\frac{1}{2}.$$

By (5.13), we then have $\tau_2^k\to+\infty$. Define

$$\bar\phi_k(y,\tau)=(\tau_2^k)^\nu\phi_k(y,\tau_2^k+\tau),\ \bar h_k(y,\tau)=(\tau_2^k)^\nu h_k(y,\tau_2^k+\tau),\ \bar c_k(\tau)=(\tau_2^k)^\nu c_k(\tau_2^k+\tau).$$

Then

$$\partial_\tau\bar\phi_k = L_0[\bar\phi_k]+\bar h_k-\bar c_k(\tau)Z_0(y)\quad\text{in } (\tau_0-\tau_2^k,0].$$

Here $\bar h_k\to 0$ uniformly in any compact subset of $\mathbb{R}^n\times(-\infty,0]$ and $\bar c_k\to0$ uniformly in any compact subset of $(-\infty,0]$. Furthermore, it holds that

$$|\bar\phi_k(y,\tau)|\le\frac{1}{1+|y|^a}\qquad\text{in } \mathbb{R}^n\times(\tau_0-\tau_2^k,0].$$

Thus by fractional parabolic regularity theory, we find that, up to a subsequence, $\bar\phi_k\to\bar\phi$ uniformly in any compact subset of $\mathbb{R}^n\times(-\infty,0]$ with $\bar\phi\ne0$ and $\bar\phi$ satisfies

$$\begin{cases}\partial_\tau\bar\phi = -(-\Delta)^s\bar\phi + pU^{p-1}\bar\phi & \text{in } \mathbb{R}^n\times(-\infty,0],\\ \int_{\mathbb{R}^n}\bar\phi(y,\tau)Z_j(y)dy=0 & \text{for all } \tau\in(-\infty,0],\ j=0,1,\cdots,n+1,\\ |\bar\phi(y,\tau)|\le\frac{1}{1+|y|^a} & \text{in } \mathbb{R}^n\times(-\infty,0],\\ \bar\phi(y,-\infty)=0, & \text{in } \mathbb{R}^n.\end{cases} \tag{5.16}$$

We now prove that $\bar\phi=0$ which yields a contradiction. In fact, fractional parabolic regularity yields that $\bar\phi$ is smooth and a scaling argument shows that

$$(1+|y|^s)|(-\Delta)^{\frac{s}{2}}\bar\phi|+|\bar\phi_\tau|+|(-\Delta)^s\bar\phi|\lesssim(1+|y|)^{-2s-a}. \tag{5.17}$$

Differentiating (5.16), we get $\partial_\tau\bar\phi_\tau=-(-\Delta)^s\bar\phi_\tau+pU^{p-1}(y)\bar\phi_\tau$ and

$$(1+|y|^s)|(-\Delta)^{\frac{s}{2}}\bar\phi_\tau|+|\bar\phi_{\tau\tau}|+|(-\Delta)^s\bar\phi_\tau|\lesssim(1+|y|)^{-4s-a}. \tag{5.18}$$

Define
$$Q(\xi,\xi) = -\int_{\mathbb{R}^n} L_0(\xi)\xi.$$
Here ξ satisfies certain asymptotic conditions at infinity. In particular, from (5.17), (5.18) and the fractional parabolic regularity theory ([31]), it holds that
$$|Q(\bar\phi,\bar\phi)| < +\infty, \quad |Q(\bar\phi_\tau,\bar\phi_\tau)| < +\infty.$$
It then follows that
$$\frac{1}{2}\partial_\tau \int_{\mathbb{R}^n} |\bar\phi_\tau|^2 + Q(\bar\phi_\tau,\bar\phi_\tau) = 0. \tag{5.19}$$
Recall that $\int_{\mathbb{R}^n} \bar\phi(y,\tau)Z_j(y)dy = 0$ for $\tau \in (-\infty,0]$, $j = 0,1,\cdots,n+1$. Hence, we get
$$\int_{\mathbb{R}^n} \bar\phi_\tau(y,\tau)Z_j(y)dy = 0$$
and
$$Q(\bar\phi_\tau,\bar\phi_\tau) \geq 0.$$
So from (5.19) we have
$$\frac{1}{2}\partial_\tau \int_{\mathbb{R}^n} |\bar\phi_\tau|^2 \leq 0.$$
On the other hand, multiplying (5.16) by $\bar\phi_\tau$ and integrating over $\mathbb{R}^n$, we obtain
$$\int_{\mathbb{R}^n} |\bar\phi_\tau|^2 = -\frac{1}{2}\partial_\tau Q(\bar\phi,\bar\phi).$$
Therefore, we find that
$$\partial_\tau \int_{\mathbb{R}^n} |\bar\phi_\tau|^2 \leq 0, \ \int_{-\infty}^0 d\tau \int_{\mathbb{R}^n} |\bar\phi_\tau|^2 < +\infty.$$
Hence, one has
$$\bar\phi_\tau = 0,$$
namely that $\bar\phi$ is independent of τ and $L_0[\bar\phi] = 0$. Since $\bar\phi$ is bounded, by the nondegeneracy of the linearized operator L_0 (see [11]), $\bar\phi$ is a linear combination of Z_j, $j = 0,1,\cdots,n+1$. Then by the orthogonality condition
$$\int_{\mathbb{R}^n} \bar\phi Z_j = 0,$$
we conclude that $\bar\phi = 0$, a contradiction. Hence (5.15) is proved.

Case 2. Suppose there exists a sequence of y_k with $|y_k| \to +\infty$ such that
$$(\tau_2^k)^\nu(1+|y_k|)^a|\phi_k(y_k,\tau_2^k)| \geq \frac{1}{2}.$$
Define
$$\tilde\phi_k(z,\tau) := (\tau_2^k)^\nu |y_k|^a \phi_k(y_k + |y_k|z, |y_k|^{2s}\tau + \tau_2^k).$$
Direct computations yield that
$$\partial_\tau\tilde\phi_k = -(-\Delta)^s\tilde\phi_k + a_k\tilde\phi_k + \tilde h_k,$$
where
$$a_k = pU^{p-1}(y_k + |y_k|z)$$
and
$$\tilde h_k(z,\tau) = (\tau_2^k)^\nu|y_k|^{2s+a}h_k(y_k+|y_k|z,|y_k|^{2s}\tau+\tau_2^k) - (\tau_2^k)^\nu|y_k|^{2s+a}c(|y_k|^{2s}\tau+\tau_2^k)Z_0(y_k+|y_k|z).$$

By (5.13), (5.14) and (5.2), we obtain that

$$|\tilde{h}_k(z,\tau)| \lesssim o(1)|\hat{y}_k + z|^{-2s-a}((\tau_2^k)^{-1}|y_k|^{2s}\tau + 1)^{-\nu}$$

where

$$\hat{y}_k = \frac{y_k}{|y_k|} \to -\hat{e}.$$

Hence $\tilde{h}_k \to 0$ uniformly in any compact subset of $\mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]$ and $a_k \to 0$ uniformly in any compact subset of $\mathbb{R}^n$. Note that $\tilde{\phi}_k(0,0) \geq \frac{1}{2}$ and

$$|\tilde{\phi}_k(z,\tau)| \lesssim |\hat{y}_k + z|^{-a}((\tau_2^k)^{-1}|y_k|^{2s}\tau + 1)^{-\nu}.$$

Then, up to a subsequence, $\tilde{\phi}_k \to \tilde{\phi} \neq 0$ uniformly in any subset of $\mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]$. Moreover,

$$\tilde{\phi}_\tau = -(-\Delta)^s\tilde{\phi} \quad \text{in } \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0], \tag{5.20}$$

with

$$|\tilde{\phi}| \leq |z - \hat{e}|^{-a} \quad \text{in } \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]. \tag{5.21}$$

For problem (5.20)–(5.21), we now claim a Liouville type result $\tilde{\phi} \equiv 0$.

Indeed, without loss of generality, we consider

$$\begin{cases} \tilde{\phi}_\tau = -(-\Delta)^s\tilde{\phi} & \text{in } (\mathbb{R}^n \setminus \{0\}) \times (-\infty, 0], \\ |\tilde{\phi}| \leq |z|^{-a} & \text{in } (\mathbb{R}^n \setminus \{0\}) \times (-\infty, 0]. \end{cases} \tag{5.22}$$

Inspired by [1], we shall construct a supersolution to problem (5.22). Let $\delta > 0$ be an arbitrary fixed constant and

$$u_0(x) := \begin{cases} |x|^{-a}, & |x| \geq \varepsilon, \\ \varepsilon^{-a}, & |x| \leq \varepsilon. \end{cases}$$

Here $\varepsilon > 0$ is small enough such that $\delta > \varepsilon^{n-2s-a}$. Define

$$\bar{u}(x,t) := \int_{\mathbb{R}^n} K_s(x-y,t)u_0(y)dy + \frac{\delta}{|x|^{n-2s}},$$

where $K_s(x,t)$ is the heat kernel given in (4.2). It is clear that

$$\bar{u}(r^2, 0) \geq r^{-a}.$$

Then for all $M > 0$, $\bar{u}(r^2, \tau + M)$ is a supersolution to

$$\begin{cases} \tilde{\phi}_\tau = -(-\Delta)^s\tilde{\phi} & \text{in } \mathbb{R}^n \setminus \{0\} \times [-M, 0], \\ |\tilde{\phi}| \leq |z|^{-a} & \text{in } \mathbb{R}^n \setminus \{0\} \times [-M, 0]. \end{cases}$$

Now for $t > 0$ large, we estimate

$$|\bar{u}(x,t)| \lesssim \int_{\mathbb{R}^n} K_s(x-y,t)u_0(y)dy + \frac{\delta}{|x|^{n-2s}}.$$

Direct computations yield that for any fixed $x \neq 0$,

$$\begin{aligned} \left| \int_{\mathbb{R}^n} K_s(x-y,t)u_0(y)dy \right| &\leq \int_{\mathbb{R}^n} \frac{t}{(t^{\frac{1}{s}} + |x-y|^2)^{\frac{n+2s}{2}}} \frac{1}{|y|^a} dy \\ &\leq \left(\int_{B_{2|x|}} + \int_{\mathbb{R}^n \setminus B_{2|x|}} \right) \frac{t}{(t^{\frac{1}{s}} + |x-y|^2)^{\frac{n+2s}{2}}} \frac{1}{|y|^a} dy \\ &\lesssim t^{-\frac{n}{2s}}|x|^{n-a} + t^{-\frac{a}{2s}}. \end{aligned}$$

For any fixed $(x,\tau) \in \mathbb{R}^n \times (-\infty, 0]$, we have that

$$|\tilde{\phi}(x,\tau)| \lesssim (\tau + M)^{-\frac{n}{2s}}|x|^{n-a} + (\tau + M)^{-\frac{a}{2s}} + \frac{\delta}{|x|^{n-2s}}, \quad \forall M > 0.$$

Letting $M \to +\infty$, we obtain that

$$|\tilde{\phi}(x,\tau)| \lesssim \frac{\delta}{|x|^{n-2s}}.$$

Since $\delta > 0$ is arbitrary, it holds that

$$\tilde{\phi}(x,\tau) = 0.$$

The proof is complete. □

Proof of Proposition 5.1. First, we consider the problem

$$\begin{cases} \partial_\tau \phi = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y,\tau) - c(\tau)Z_0, & \text{in } \mathbb{R}^n \times (\tau_0, +\infty) \\ \phi(y,\tau_0) = 0, & \text{in } \mathbb{R}^n. \end{cases}$$

Let $(\phi(y,\tau), c(\tau))$ be the solution of the initial value problem (5.7). From Lemma 5.1, for any $\tau_1 > \tau_0$, we have

$$|\phi(y,\tau)| \lesssim \tau^{-\nu}(1+|y|)^{-a}\|h\|_{2s+a,\tau_1} \text{ for all } \tau \in (\tau_0,\tau_1),\ y \in \mathbb{R}^n$$

and

$$|c(\tau)| \le \tau^{-\nu} R^a \|h\|_{2s+a,\tau_1} \text{ for all } \tau \in (\tau_0,\tau_1).$$

By assumption, $\|h\|_{2s+a,\nu,\eta} < +\infty$ and $\|h\|_{2s+a,\tau_1} \le \|h\|_{2s+a,\nu,\eta}$ for an arbitrary τ_1. It follows that

$$|\phi(y,\tau)| \lesssim \tau^{-\nu}(1+|y|)^{-a}\|h\|_{2s+a,\nu,\eta} \text{ for all } \tau \in (\tau_0,\tau_1),\ y \in \mathbb{R}^n$$

and

$$|c(\tau)| \le \tau^{-\nu} R^a \|h\|_{2s+a,\nu,\eta} \text{ for all } \tau \in (\tau_0,\tau_1).$$

By the arbitrariness of τ_1, we have

$$|\phi(y,\tau)| \lesssim \tau^{-\nu}(1+|y|)^{-a}\|h\|_{2s+a,\nu,\eta} \text{ for all } \tau \in (\tau_0,+\infty),\ y \in \mathbb{R}^n$$

and

$$|c(\tau)| \le \tau^{-\nu} R^a \|h\|_{2s+a,\nu,\eta} \text{ for all } \tau \in (\tau_0,+\infty).$$

By the regularity theory of [43] and a scaling argument, we get the validity of (5.5) and (5.6). □

Remark 5.1. *In the inner problem (2.9), $\mathcal{H}$ behaves like*

$$\mathcal{H} \lesssim \lambda^{\frac{n-2s}{2}} \left(\frac{1}{1+|y|^{n-2s}} + \frac{1}{1+|y|^{4s}} \right).$$

Recall that $4s < n < 6s$. For $a > 2s$, we get

$$\mathcal{H} \lesssim \frac{\lambda^{\frac{n-2s}{2}}}{1+|y|^{a+2s}} \left(R^{a+4s-n} + R^{a-2s} \right) \lesssim \frac{\lambda^{\frac{n-2s}{2}}}{1+|y|^{a+2s}} R^{a+4s-n}.$$

Define the $\|\cdot\|_B$ norm as

$$\|\phi\|_B := \sup_{\substack{t\in(0,T)\\ y\in B_{2R}}} \frac{\lambda^{-\frac{n-2s}{2}}(t)(1+|y|^a)}{R^{a+4s-n}} \left[|\phi(y,t)| + (1+|y|^s) \left(\int_{\mathbb{R}^n} \frac{[\phi(y,t)-\phi(x,t)]^2}{|y-x|^{n+2s}} dx \right)^{\frac{1}{2}} \right]. \tag{5.23}$$

Then by Proposition 5.1, we obtain

$$\|\phi\|_B \lesssim \|\mathcal{H}\|_{n-2s,\nu,\eta}.$$

6. Solving the inner-outer gluing system

In this section, we shall solve the inner-outer gluing system

$$\lambda^{2s}\phi_t = -(-\Delta)_y^s\phi + pU^{p-1}(y)\phi + \mathcal{H}(\phi,\psi,\lambda,\xi) \quad \text{in} \quad B_{2R}(0)\times(0,T)$$

$$\begin{cases} \psi_t = -(-\Delta)_x^s\psi + \mathcal{G}(\phi,\psi,\lambda,\xi), & \text{in} \quad \mathbb{R}^n\times(0,T) \\ \psi(x,0)=0, & \text{in} \quad \mathbb{R}^n \end{cases}$$

where $\mathcal{H}$ and $\mathcal{G}$ are defined in (2.11) and (2.12) respectively. We shall solve the inner-outer gluing system as a fixed piont problem for $\vec{p}=(\phi,\psi,\lambda,\xi)$ in a proper Banach space.

We define

$$c_j[\mathcal{H}(\phi,\psi,\lambda,\xi)] := \frac{\int_{B_{2R}}\mathcal{H}(\phi,\psi,\lambda,\xi)Z_j(y)dy}{\int_{B_{2R}}|Z_j(y)|^2dy}$$

and

$$\bar{\mathcal{H}}(\phi,\psi,\lambda,\xi) := \mathcal{H}(\phi,\psi,\lambda,\xi) - \sum_{j=1}^{n+1} c_j[\mathcal{H}(\phi,\psi,\lambda,\xi)]Z_j.$$

Then the linear theory is automatically applicable to the following problem

$$\begin{cases} \lambda^2\phi_t = -(-\Delta)_y^s\phi + pU^{p-1}(y)\phi + \bar{\mathcal{H}}(\phi,\psi,\lambda,\xi), & \text{in} \quad B_{2R}\times(0,T) \\ \phi(x,0) = eZ_0(x), & \text{in} \quad B_{2R} \end{cases} \tag{6.1}$$

Problem (6.1) can be formulated as the following fixed point problem

$$\phi = \mathcal{T}_\lambda^{in}[\bar{\mathcal{H}}(\phi,\psi,\lambda,\xi)] := \mathcal{F}_1(\phi,\psi,\lambda,\xi). \tag{6.2}$$

If in addition we have

$$c_j[\mathcal{H}(\phi,\psi,\lambda,\xi)] = 0 \quad \text{for all} \quad j=1,\cdots,n+1, \tag{6.3}$$

we get a true solution to the real inner problem. Similarly, for the outer problem, we look for a fixed point of

$$\psi = \mathcal{T}^{out}[\mathcal{G}(\phi,\psi,\lambda,\xi)] := \mathcal{F}_2(\phi,\psi,\lambda,\xi) \tag{6.4}$$

Therefore, the inner-outer gluing system is now reduced to the system (6.2)–(6.4). We shall solve the system by Leray-Schauder degree theory. For $\theta\in[0,1]$, we define the homotopy class

$$\begin{aligned} \mathcal{H}_\theta(\psi,\lambda,\xi)(y,t) = {} & \lambda^{2s-1}\dot{\lambda}Z_{n+1}(y) + \lambda^{2s-1}\sum_{j=1}^n \dot{\xi}_j Z_j(y) \\ & + \lambda^{\frac{n-2s}{2}}pU^{p-1}(y)Z_0^*(q) + \lambda^{\frac{n-2s}{2}+1}pU^{p-1}(y)\nabla Z_0^*(q)\cdot y \\ & + \theta\lambda^{\frac{n-2s}{2}}pU^{p-1}(y)[Z^*(\lambda y+\xi,t) - Z_0^*(q) - \lambda y\cdot\nabla Z_0^*(q) + \psi(\lambda y+\xi,t)]. \end{aligned}$$

Consider the following system

$$\begin{cases} \phi = \mathcal{T}_\lambda^{in}\left[\mathcal{H}_\theta(\phi,\psi,\lambda,\xi) - \sum_{j=1}^{n+1}c_j[\mathcal{H}_\theta(\phi,\psi,\lambda,\xi)]Z_j\right] \\ c_j[\mathcal{H}_\theta(\phi,\psi,\lambda,\xi)] = 0, \quad \text{for all} \quad j=1,\cdots,n+1 \\ \psi = \mathcal{T}^{out}[\theta\mathcal{G}(\phi,\psi,\lambda,\xi)] \end{cases} \tag{6.5}$$

The case $\theta=1$ corresponds to the original system that we need to solve.

We write

$$\lambda(t) = \lambda_0(t)+\lambda_1(t),\ \xi(t) = q+\xi_1(t),\ t\in[0,T],$$

where $\lambda_0(t)$ is defined in (3.2) and $\lambda_1(T)=0,\ \xi_1(T)=0$.

Suppose that we have a solution $(\phi,\psi,\lambda_1,\xi_1)$ to system (6.5) satisfying the constraints

$$|\dot{\lambda}_1(t)| + |\dot{\xi}_1(t)| \le \delta_0,\ \ \|\phi\|_B + \|\psi\|_\infty \le \delta_1, \tag{6.6}$$

where δ_0 and δ_1 are small positive constants to be determined later and the norm $\|\cdot\|_B$ is defined in (5.23). We also assume that $\|Z^*\|_\infty \ll 1$ independent of T.

From section 3, the function $\lambda_0(t)$ solves the equation

$$\dot{\lambda}_0(t)\int_{\mathbb{R}^n} Z_{n+1}^2 dy + \lambda_0(t)^{\frac{n-6s+2}{2}} Z_0^*(q)\int_{\mathbb{R}^n} pU^{p-1}Z_{n+1}dy = 0. \tag{6.7}$$

The equation

$$c_{n+1}(H_\theta(\psi,\lambda_0+\lambda_1,\xi))(t) = 0 \quad t\in[0,T) \tag{6.8}$$

which corresponds to

$$\begin{aligned}0 &= \dot{\lambda}(t)\Big(\int_{B_{2R}} Z_{n+1}^2 dy\Big) + \lambda(t)^{\frac{n-6s}{2}+1} Z_0^*(q)\int_{B_{2R}} pU^{p-1}Z_{n+1}dy \\ &+ \theta\lambda(t)^{\frac{n-6s}{2}+1}\int_{B_{2R}} pU(y)^{p-1}(Z^*(\xi+\lambda y,t) - Z_0^*(q) - \lambda y\cdot\nabla Z_0^*(q) + \psi(\xi+\lambda y,t))Z_{n+1}(y)dy\end{aligned}$$

can be written as

$$\dot{\lambda}(t) + \beta\lambda(t)^{\frac{n-6s}{2}+1} = \lambda(t)^{\frac{n-6s}{2}+1}(\delta_R + \theta\pi(\psi,\xi,\lambda_1)) \tag{6.9}$$

for a suitable number $\beta>0$, $\delta_R = O(R^{-2s})$ and the operator π satisfies, for some absolute constant C,

$$|\pi(\psi,\xi,\lambda_1)| \lesssim T + \|\psi\|_\infty.$$

By (6.7), for a suitable $\gamma>0$, equation (6.9) can be written in the linearized form

$$\dot{\lambda}_1 + \frac{\gamma}{T-t}\lambda_1 = (T-t)^{\frac{2}{6s-n}-1} g_0(\psi,\lambda_1,\xi,\theta)$$

with

$$|g_0(\psi,\lambda_1,\xi,\theta)(t)| \lesssim \|\psi\|_\infty + T + R^{-2s}.$$

The linear problem

$$\dot{\lambda}_1 + \frac{\gamma}{T-t}\lambda_1 = (T-t)^{\frac{2}{6s-n}-1} g_0(t), \quad \lambda_1(T) = 0$$

can be uniquely solved by the operator in g_0,

$$\lambda_1(t) = \mathcal{T}^{(0)}[g_0](t) := -(T-t)^\gamma\int_t^T (T-s)^{\frac{2}{6s-n}-1-\gamma} g_0(s)ds.$$

It defines a linear operator on g_0 with estimates

$$\|(T-t)^{\frac{2}{n-6s}+1}\dot{\lambda}_1\|_\infty + \|(T-t)^{\frac{2}{n-6s}}\lambda_1\|_\infty \lesssim \|g_0\|_\infty.$$

Equation (6.8) then becomes

$$\lambda_1(t) = \mathcal{T}^{(0)}[g_0(\psi,\lambda_1,\xi,\theta)](t) \ \text{ for all } \ t\in[0,T)$$

and we get

$$\|(T-t)^{\frac{2}{n-6s}+1}\dot{\lambda}_1\|_\infty + \|(T-t)^{\frac{2}{n-6s}}\lambda_1\|_\infty \lesssim \|\psi\|_\infty + T + R^{-2s} \tag{6.10}$$

Similarly, equations

$$c_j[\mathcal{H}_\theta(\psi,\lambda,\xi)] = 0 \ \text{ for all } \ j = 1,\dots,n,$$

can be written in vector form as

$$\xi_1(t) = \mathcal{T}^{(1)}[g_1(\psi,\lambda_1,\xi_1,\theta)](t) \ \text{ for all } \ t\in[0,T). \tag{6.11}$$

where

$$\mathcal{T}^{(1)}[g_1] := \int_t^T (T-s)^{\frac{2}{6s-n}-1} g_1(s)ds$$

and

$$|g_1(\psi,\lambda_1,\xi_1,\theta)(t)| \lesssim \|\psi\|_\infty + T.$$

From equation (6.11), we have

$$\|(T-t)^{\frac{2}{n-6s}+1}\dot{\xi}_1\|_\infty + \|(T-t)^{\frac{2}{n-6s}}\xi_1\|_\infty \lesssim \|\psi\|_\infty + T \tag{6.12}$$

On the other hand, we have

$$|\mathcal{H}(\phi,\psi,\lambda,\xi)(y,t)| \lesssim \frac{\lambda(t)^{\frac{n-2s}{2}}}{1+|y|^{4s}}(\|\psi\|_\infty + \|Z^*\|_\infty) + \frac{\lambda^{2s-1}\dot{\lambda}}{1+|y|^{n-2s}} + \frac{\lambda^{2s-1}|\dot{\xi}|}{1+|y|^{n-2s+1}}$$

hence for $a > 2s$, we have

$$|\mathcal{H}(\phi,\psi,\lambda,\xi)(y,t)| \lesssim \frac{\lambda_0(t)^{\frac{n-2s}{2}}}{1+|y|^{2s+a}} R^{a+4s-n}\big(\|\psi\|_\infty + \|Z^*\|_\infty\big).$$

and from (6.5) and Proposition 5.1 we obtain that

$$\|\phi\|_B \lesssim \|\psi\|_\infty + \|Z^*\|_\infty, \tag{6.13}$$

where the norm $\|\cdot\|_B$ is defined in (5.23). Next we consider the outer problem in (6.5). We recall that the outer problem is

$$\begin{cases} \psi_t = -(-\Delta)_x^s \psi + \mathcal{G}(\phi,\psi,\lambda,\xi), & \text{in } \mathbb{R}^n \times (0,T) \\ \psi(x,0) = 0, & \text{in } \mathbb{R}^n \end{cases}$$

where

$$\begin{aligned}
&\mathcal{G}(\phi,\psi,\lambda,\xi) := p\lambda^{-2s}(1-\eta_R)U^{p-1}(y)(\psi+Z^*) + \lambda^{-\frac{n+2s}{2}}\mathcal{E}(1-\eta_R) + \mathcal{C}(\phi) + \mathcal{R}(\phi) + N(\varphi+Z^*) \\
&\mathcal{E}(y,t) = \lambda^{2s-1}\dot{\lambda} Z_{n+1}(y) + \lambda^{2s-1}\dot{\xi}\cdot\nabla U \\
&\mathcal{C}(\phi) := \lambda^{-\frac{n-2s}{2}}\Big[(-(-\Delta)_x^s - \partial_t)\eta_R(y)\phi + [-(-\Delta)_x^{s/2}\eta_R(y), -(-\Delta)_x^{s/2}\phi(y)]\Big] \\
&\mathcal{R}(\phi) := \lambda^{-\frac{n-2s}{2}-1}\left[\eta_R\dot{\lambda}\left(\frac{n-2s}{2}\phi + y\cdot\nabla_y\phi\right) + \eta_R\dot{\xi}\cdot\nabla_y\phi + \phi\left(\dot{\lambda}y\cdot\nabla_y\eta_R + \dot{\xi}\cdot\nabla_y\eta_R\right)\right].
\end{aligned}$$

Here

$$[-(-\Delta)_x^{s/2}f(x), -(-\Delta)_x^{s/2}g(x)] := c_{n,s}\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{[f(y)-f(x)][g(x)-g(y)]}{|x-y|^{n+2s}}dy$$

with $c_{n,s} = \frac{2^{2s}s\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)\pi^{\frac{n}{2}}}$.

We estimate term by term. It is direct to see that

$$|p\lambda^{-2s}(1-\eta_R)U^{p-1}(y)(\psi+Z^*)| \lesssim \frac{\lambda^{-2s}}{R^{2s-\sigma}}\frac{1}{1+|y|^{2s+\sigma}}(\|Z^*\|_\infty + \|\psi\|_\infty) \tag{6.14}$$

and

$$\begin{aligned}
|\lambda^{-\frac{n+2s}{2}}\mathcal{E}(1-\eta_R)| &\lesssim \frac{1}{\lambda^{2s}}\left[\frac{1}{1+|y|^{n-2s}}\lambda^{-\frac{n-2s}{2}}(|\lambda^{2s-1}\dot{\lambda}| + |\lambda^{2s-1}\dot{\xi}|)\right]\Bigg|_{|y|>2R} \\
&\lesssim \frac{\lambda^{-2s}}{R^{n-4s-\sigma}}\frac{1}{1+|y|^{2s+\sigma}}.
\end{aligned} \tag{6.15}$$

Let us now estimate the term $\mathcal{C}(\phi)$. Let us choose $0<\sigma<n-4s$. Then we have

$$
\begin{aligned}
&\lambda^{-\frac{n-2s}{2}}\left|\left(\left[-(-\Delta)^{s/2}\eta_R,-(-\Delta)^{s/2}\phi\right]\right)(x,t)\right|\\
&\lesssim \lambda^{-\frac{n-2s}{2}}\left[\int_{\mathbb{R}^n}\left(\frac{\eta_R(x)-\eta_R(y)}{|x-y|^{\frac{n}{2}+s}}\right)^2dy\right]^{\frac12}\left[\int_{\mathbb{R}^n}\left(\frac{\phi(x)-\phi(y)}{|x-y|^{\frac{n}{2}+s}}\right)^2dy\right]^{\frac12}\\
&\lesssim \lambda^{-\frac{n-2s}{2}}\frac{1}{R^s\lambda^s}\left[\int_{\mathbb{R}^n}\left(\frac{\eta(|\frac{x-\xi}{R\lambda}|)-\eta(|\frac{y-\xi}{R\lambda}|)}{|\frac{x-y}{R\lambda}|^{\frac{n}{2}+s}}\right)^2d\left(\frac{y-\xi}{R\lambda}\right)\right]^{\frac12}\\
&\quad\times\frac{1}{\lambda^s}\left[\int_{\mathbb{R}^n}\left(\frac{\phi(\frac{x-\xi}{\lambda},t)-\phi(\frac{y-\xi}{\lambda},t)}{|\frac{x-y}{\lambda}|^{\frac{n}{2}+s}}\right)^2d\left(\frac{y-\xi}{\lambda}\right)\right]^{\frac12}\\
&\lesssim \frac{1}{R^s\lambda^{2s}}\left[\int_{\mathbb{R}^n}\left(\frac{\eta(|\frac{x-\xi}{R\lambda}|)-\eta(|\frac{y-\xi}{R\lambda}|)}{|\frac{x-y}{R\lambda}|^{\frac{n}{2}+s}}\right)^2d\left(\frac{y-\xi}{R\lambda}\right)\right]^{\frac12}\frac{R^{a+4s-n}}{(1+|y|^{s+a})}\|\phi\|_B\\
&\lesssim \lambda^{-2s}\frac{R^{\sigma+4s-n}}{1+|y|^{2s+\sigma}}\|\phi\|_B.
\end{aligned}
$$

Similar computations yield that

$$
\begin{aligned}
\left|\mathcal{C}(\phi)\right| &\lesssim \lambda^{-2s}\frac{1}{R^{2s}}\frac{R^{a+4s-n}}{1+|y|^a}\lambda^{-\frac{n-2s}{2}}\|\phi\|_B\\
&\lesssim \lambda^{-2s}\frac{R^{\sigma+4s-n}}{1+|y|^{2s+\sigma}}\|\phi\|_B.
\end{aligned}
\tag{6.16}
$$

Also, we have

$$
\left|\mathcal{R}(\phi)\right|\lesssim \lambda^{-2s}\frac{\lambda^{\frac{n-2s}{2}}R^{\sigma+6s-n}}{1+|y|^{2s+\sigma}}\|\phi\|_B. \tag{6.17}
$$

Now for some $\sigma\in(0,n-4s)$, we have

$$
\begin{aligned}
\left|N(Z^*+\lambda^{-\frac{n-2s}{2}}\eta_R\phi+\psi)\right| &\lesssim \left(\lambda^{-\frac{n-2s}{2}}\eta_R\phi\right)^p+(Z^*+\psi)^p\\
&\lesssim \lambda^{-2s}\frac{\lambda^{2s}R^{(4s-n)p+2s+\sigma}}{1+|y|^{2s+\sigma}}\|\phi\|_B^p+(\|Z^*\|_\infty+\|\psi\|_\infty)^p.
\end{aligned}
\tag{6.18}
$$

Collecting the above estimates (6.14)–(6.18), we get by using Lemma 4.1 that

$$
\|\psi\|_\infty\lesssim T^{\sigma'}\|Z^*\|_\infty+R^{-\sigma'}\|\phi\|_B \tag{6.19}
$$

for some positive constant σ'. By (6.10)–(6.13) and (6.19), we obtain

$$
\begin{cases}
\|\psi\|_\infty\lesssim T^{\sigma'}\|Z^*\|_\infty\\
\|\phi\|_B\lesssim \|Z^*\|_\infty\\
\|(T-t)^{\frac{2}{n-6s}+1}\dot\lambda_1\|_\infty+\|(T-t)^{\frac{2}{n-6s}}\lambda_1\|_\infty\lesssim T^{\sigma'}(\|Z^*\|_\infty+1)\\
\|(T-t)^{\frac{2}{n-6s}+1}\dot\xi_1\|_\infty+\|(T-t)^{\frac{2}{n-6s}}\xi_1\|_\infty\lesssim T^{\sigma'}(\|Z^*\|_\infty+1)+R^{-2s}
\end{cases}
\tag{6.20}
$$

Then the inner-outer gluing system (6.5) can be written in the form

$$
\begin{cases}
\phi=\mathcal{T}_\lambda^{in}[\bar{\mathcal{H}}_\theta(\mathcal{T}^{out}[\theta\mathcal{G}(\phi,\psi,\lambda,\xi)],\lambda,\xi)]\\
\psi=\mathcal{T}^{out}[\theta\mathcal{G}(\phi,\psi,\lambda,\xi)]\\
\lambda_1=\mathcal{T}^{(0)}[\tilde g_0(\psi,\lambda_1,\xi_1,\theta)]\\
\xi_1=\mathcal{T}^{(1)}[\tilde g_1(\psi,\lambda_1,\xi_1,\theta)]
\end{cases}
$$

where $\tilde g_0$ and $\tilde g_1$ can be expressed as

$$\begin{aligned}\tilde g_0(\psi,\lambda_1,\xi_1,\theta) =& c_R^1\int_{B_{2R}}\mathcal{H}_\theta(\mathcal{T}^{out}[\,\theta\mathcal{G}(\phi,\psi,\lambda,\xi)],\lambda,\xi)Z_{n+1}(y)dy\\ \tilde g_1(\psi,\lambda_1,\xi_1,\theta) =& c_R^2\int_{B_{2R}}\mathcal{H}_\theta(\mathcal{T}^{out}[\,\theta\mathcal{G}(\phi,\psi,\lambda,\xi)],\lambda,\xi)\nabla U(y)dy\end{aligned}$$

for suitable positive constants c_R^0 and c_R^1. For $\epsilon>0$ fixed sufficiently small, we consider the following problem defined only up to time $t=T-\epsilon$

$$\begin{cases}\phi=\mathcal{T}^{in}_\lambda[\,\bar{\mathcal{H}}_\theta(\mathcal{T}^{out}[\theta\mathcal{G}(\phi,\psi,\lambda,\xi)],\lambda,\xi)], & (y,t)\in\bar B_{2R}\times[0,T-\epsilon]\\ \psi=\mathcal{T}^{out}[\theta\mathcal{G}(\phi,\psi,\lambda,\xi)\,], & (x,t)\in\mathbb{R}^n\times[0,T-\epsilon]\\ \lambda_1=\mathcal{T}^{(0)}_\epsilon[\,\tilde g_0(\psi,\lambda_1,\xi_1,\theta)\,], & t\in[0,T-\epsilon]\\ \xi_1=\mathcal{T}^{(1)}_\epsilon[\,\tilde g_1(\psi,\lambda_1,\xi_1,\theta)\,], & t\in[0,T-\epsilon]\end{cases} \tag{6.21}$$

where

$$\mathcal{T}^{(0)}_\epsilon[g](t):=-(T-t)^\gamma\int_t^{T-\epsilon}(T-s)^{\frac{2}{6s-n}-\gamma-1}g(s)\,ds,\quad \mathcal{T}^{(1)}_\epsilon[g]:=\int_t^{T-\epsilon}(T-s)^{\frac{2}{6s-n}-1}g(s)\,ds.$$

We consider problem (6.21) in the space of functions

$$(\phi,\psi,\lambda_1,\xi_1)\in X_1\times X_2\times X_3\times X_4$$

where X_ℓ $(\ell=1,\cdots,4)$ and corresponding norms are defined as

$$\begin{aligned} X^1 =& \{\phi:\ \phi\in C(\bar B_{2R}\times[0,T-\epsilon]),\ \nabla^s_y\phi\in C(\bar B_{2R}\times[0,T-\epsilon])\},\quad \|\phi\|_{X_1}=\|\phi\|_\infty+\|\nabla_y^s\phi\|_\infty\\ X^2 =& \{\psi:\ \phi\in C(\mathbb{R}^n\times[0,T-\epsilon])\},\quad \|\psi\|_{X_2}=\|\psi\|_\infty\\ X^3 =& \{\lambda_1:\ \lambda_1\in C^1[0,T-\epsilon]\},\quad \|\lambda_1\|_{X_3}=\|\lambda_1\|_\infty+\|\dot\lambda_1\|_\infty\\ X^4 =& \{\xi_1:\ \xi_1\in C^1[0,T-\epsilon]\},\quad \|\xi_1\|_{X_4}=\|\xi_1\|_\infty+\|\dot\xi_1\|_\infty\end{aligned}$$

where

$$\nabla^s_y\phi:=\left(\int_{\mathbb{R}^n}\frac{[\phi(y,t)-\phi(x,t)]^2}{|y-x|^{n+2s}}dx\right)^{1/2}.$$

As a direct consequence of Arzelà–Ascoli's theorem, compactness on bounded sets of all the operators involved in the above expression (6.21) follows from the Hölder estimate (4.13) for the operator $\mathcal{T}^{out}$. On the other hand, the a priori estimates we obtained for $\epsilon=0$ holds equally well, uniformly for arbitrary small $\epsilon>0$, and for a solution of (6.21).

We now apply Leray–Schauder degree theory in a suitable ball $\mathcal{B}$ containing the origin which is slightly larger than the one defined by relations (6.20), which amounts to a choice of the parameters δ_0 and δ_1 in (6.6). The homotopy connects with the identity at $\theta=0$, and hence the total degree in the region defined by relations (6.20) is equal to 1. Hence we have the existence of a solution to the approximate problem satisfying bounds (6.20). Finally, by a standard diagonal argument, we find a solution to the original problem for $k=1$ with the desired size.

The multiple-bubble case of k distinct points $q_1,\ldots,q_k$ is actually identical. In this case, we have k inner problems and one outer problem with similar properties. We want to find a solution of the form

$$u(x,t)=\sum_{j=1}^k U_{\lambda_j,\xi_j}(x)+Z^*(x,t)+\lambda_j^{-\frac{n-2}{2}}\phi(y_j,t)\eta_R(y_j)+\psi(x,t),\quad y_j=\frac{x-\xi_j}{\lambda_j} \tag{6.22}$$

where Z^* solves heat equation with initial condition Z_0^* which is chosen such that $Z_0^*(q_j)<0$, and $\xi_j(T)=q_j,\ \lambda_j(T)=0$ for $j=1,\cdots,k$. Then by solving a series of fixed point problems similar as the one bubble case, we obtain a solution of form (6.22). Hence we omit the details here. $\square$

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