STABILITY OF CLUSTER SOLUTIONS IN A COOPERATIVE CONSUMER CHAIN MODEL

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Abstract. We study a cooperative consumer chain model which consists of one producer and two consumers. It is an extension of the Schnakenberg model suggested in [9, 22] for which there is only one producer and one consumer. In this consumer chain model there is a middle component which plays a hybrid role: it acts both as consumer and as producer. It is assumed that the producer diffuses much faster than the first consumer and the first consumer much faster than the second consumer. The system also serves as a model for a sequence of irreversible autocatalytic reactions in a container which is in contact with a well-stirred reservoir.

In the small diffusion limit we construct cluster solutions in an interval which have the following properties: The spatial profile of the third component is a spike. The profile for the middle component is that of two partial spikes connected by a thin transition layer. The first component in leading order is given by a Green's function. In this profile multiple scales are involved: The spikes for the middle component are on the small scale, the spike for the third on the very small scale, the width of the transition layer for the middle component is between the small and the very small scale. The first component acts on the large scale. To the best of our knowledge, this type of spiky pattern has never before been studied rigorously. It is shown that, if the feedrates are small enough, there exist two such patterns which differ by their amplitudes.

We also study the stability properties of these cluster solutions. We use a rigorous analysis to investigate the linearized operator around cluster solutions which is based on nonlocal eigenvalue problems and rigorous asymptotic analysis. The following result is established: If the time-relaxation constants are small enough, one cluster solution is stable and and the other one is unstable. The instability arises through large eigenvalues of order $O(1)$. Further, there are small eigenvalues of order $o(1)$ which do not cause any instabilities.

Our approach requires some new ideas:

(i) The analysis of the large eigenvalues of order $O(1)$ leads to a novel system of non-local eigenvalue problems with inhomogeneous Robin boundary conditions whose stability properties have been investigated rigorously.

(ii) The analysis of the small eigenvalues of order $o(1)$ needs a careful study of the interaction of two small length scales and is based on a suitable inner/outer expansion with rigorous error analysis. It is found that the order of these small eigenvalues is given by the smallest diffusion constant ϵ_2^2.

Pattern Formation, Reaction-Diffusion System, Consumer Chain Model, Cluster Solutions, Stability.

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1. Chain models in biology and other sciences

Models involving a chain of components play an important role in biology, chemistry, social sciences and many other fields. Well-known examples include food chains, consumer chains, genetic signaling pathways and autocatalytic chemical reactions or nuclear chain reactions. For food chains it is commonly assumed that there is only limited supply of resources which leads to a saturation effect. On the other hand, for autocatalytic chemical or nuclear chain reactions the chain has a self-enforcing effect and after an initial cue the concentration of the system components are able to grow by themselves. Consumer chains include food chains but are more general and consumption of different commodities are also taken into account such as water, energy, raw materials, technology and information. An advanced consumer chain model considers both the limited amount of resources and the cooperation of consumers. Depending on the specific circumstances both properties play a role or one of them dominates. For example if the consumption rate is small and resources are plentiful their limited amount is not felt and it can be ignored in a realistic model. If consumers cooperate they will be able to utilize other constituents of the chain very efficiently with increasing concentration and some of the nonlinear terms in system may be superlinear.

In this respect, it is interesting to consider the work of Bettencourt and West [3] who collected extensive empirical data on typical activities in cities such as scientific publications or patents, GDP or the number of educational institutions but also crime, traffic congestion or certain diseases indicating that they grow at a superlinear rate. They established a universal growth rate which applies to most of the activities in major cities independent of geographic location or ethnicity of the population and cultural background. In our model we consider this situation: the limited amount of resources is not felt and consumers cooperative to utilize nutrients and other supplies very efficiently.

In biology consumers and suppliers are often called predator and prey. For background on predator-prey models we refer to [16]. Our system also serves as a model for a sequence of irreversible autocatalytic reactions in a container which is in contact with a well-stirred reservoir and similar models have been suggested before, see e.g. Chapter 8 of [25] and the references therein.

Although we do not consider genetic signalling pathways in this publication it is generally understood that their typical behaviour includes activator and inhibitor feedback loops between different components. Some work has been done on modelling their dynamics including stochastic approaches. On the other hand, in the vast majority of studies their spatial components are ignored. However, they are important for many settings, e.g. for the Wnt signaling pathway which describes interaction of cells and passing signals from the surface of the cell to its nucleus via a complex signaling pathway. It plays a role in embryonic development, cell differentiation and cell polarity generation. For further information we refer to the excellent review article [13]. A reaction-diffusion model for planar cell polarity is introduced and treated numerically in [1]. We plan to address these issues in future work by targeting our chain models more closely to the genetic signaling framework and including further typical ingredients stemming from genetic interaction such as typical lengthscales, mutual switch-on and switch-off mechanisms of genetic components, strength of interaction classified into ranges of growth and plateau levels, interaction of neighbouring cells resulting in spatio-temporal patterns at cellular or intracellular level, complex combinations of activator and inhibitor loops. In our example we just consider a chain of two activators which could be a first step in the modeling of complex signaling pathways and mathematical analysis of spatio-temporal structures for genetic signaling pathways which generally are represented by complex networks of multiple activators and inhibitors.

2. A cooperative consumer chain model

We consider a reaction-diffusion system which serves as a cooperative consumer chain model. It considers the interaction of three components, one producer and two consumers, which supply each other in a sequence. It is an extension of the Schnakenberg model suggested in [9, 22] for which there is only one producer and one consumer. In this consumer chain model there is a middle component which plays a hybrid role: it acts both as consumer and producer. It is assumed that the producer diffuses much faster than the first consumer and the first consumer much faster than the second consumer.

This system can be written as follows:

$$\begin{cases} \tau\dfrac{\partial S}{\partial t} = D\Delta S + 1 - \dfrac{a_1}{\epsilon_1} S u_1^2, \quad x\in\Omega,\ t>0, \\ \tau_1\dfrac{\partial u_1}{\partial t} = \epsilon_1^2\Delta u_1 - u_1 + S u_1^2 - a_2\dfrac{\epsilon_1}{\epsilon_2}u_1 u_2^2, \quad x\in\Omega,\ t>0, \\ \dfrac{\partial u_2}{\partial t} = \epsilon_2^2\Delta u_2 - u_2 + u_1 u_2^2, \quad x\in\Omega,\ t>0, \end{cases} \tag{2.1}$$

where S and u_i denote the concentrations of producer and the two consumers, respectively. Here $0<\epsilon_2^2 \ll \epsilon_1^2 \ll 1$ and $0<D$ are three positive diffusion constants. There are two small parameters: The diffusion constant ϵ_1^2 and the ratio of the two small diffusion constants $\frac{\epsilon_2^2}{\epsilon_1^2}$. These two small parameters will play an important role throughout the paper. We also set

$$\epsilon = \max\left\{\epsilon_1, \frac{\epsilon_2}{\epsilon_1}\right\} \tag{2.2}$$

and will consider the limit $\epsilon\to 0$ which means that both $\epsilon_1\to 0$ and $\frac{\epsilon_2}{\epsilon_1}\to 0$. The constants a_1, a_2 (positive) for the feed rates and τ, τ_1 (nonnegative) for the time relaxation constants will be treated as parameters and their choices will determine existence and stability properties of steady-state cluster solutions. Note that the overall supply rates $\frac{a_1}{\epsilon_1}$ and $a_2\frac{\epsilon_1}{\epsilon_2}$ are large, and they increase as the two small parameters decrease.

The system will be considered on the interval $\Omega=(-1,1)$ with Neumann boundary conditions for $t>0$:

$$\frac{dS}{dx}(-1,t)=\frac{dS}{dx}(1,t)=0, \quad \frac{du_1}{dx}(-1,t)=\frac{du_1}{dx}(1,t)=0, \quad \frac{du_2}{dx}(-1,t)=\frac{du_2}{dx}(1,t)=0. \tag{2.3}$$

The model (2.1) represents a consumer chain under the assumption that different social groups interact on vastly different scales, e.g. worldwide – national – regional, or national

– regional – local. Typically the spatial scales of consumers are much smaller than those of the resources they use. For example major cities can only exist by importing most of their natural resources, e.g. materials, energy, food as well as consumer products through a complex logistic supply chain. Even within cities there are often certain areas such as central business districts or residential neighborhoods which on the one hand interact very efficiently within themselves but can only be sustained by receiving services and products from many other areas of the city on a sustained regular basis.

In cooperative and interconnected societies superlinear growth of consumption is frequently observed. For example Bettencourt and West showed empirically based on large data sets that in cities economic and socio-cultural activities grow superlinearly with size, whereby an increase in population of 100 % results in growth of typical activities by approximately 115 % [3]. For simplicity and easy mathematical treatability we have chosen quadratic nonlinearities in our model.

These observations provide a strong motivation to consider this consumer chain model with quadratic nonlinearities and diffusion coefficients having vastly different sizes. We consider the limit of two small diffusion constants which converge to zero at two different rates. This results in two different small spatial scales. Thus the model truly has the multiscale property.

We first prove the existence of cluster solutions in an interval for which the profile of the third component is that of the commonly observed spike in the Schnakenberg model. However, for the middle component a new cluster-type profile is observed which comes from the fact that it acts as producer and consumer simultaneously. Its profile is that of two partial spikes connected by a thin transition layer. In this profile different scales are involved: The spikes for the middle component are on the small scale, the spike for the last component on the very small scale, the width of the transition layer for the middle component is between the small and the very small scale. The first component is in leading order given by a Green's function and acts on the large scale. To our knowledge this type of cluster solution has never been studied before rigourously. It is shown that if the feed rates are small enough, more precisely if the combination $a_1^2 a_2$ is below a certain threshold which has been characterized and computed explicitly, there exist two such cluster patterns which differ by their size.

We study the stability properties of this solution in terms of the system parameter using a rigorous approach to analyze the linearized operator around cluster solutions based on nonlocal eigenvalue problems and rigorous asymptotic analysis. The following result is established: If the time-relaxation constants are small enough, one cluster solutions is stable and and the other one is unstable. The instability comes through large eigenvalues of order $O(1)$. Further, there are small eigenvalues of order $o(1)$ which do not cause any instabilities.

The analysis uses some novel ideas:

(i) The consideration of the large eigenvalues leads to a new system of nonlocal eigenvalue problems which are coupled by an inhomogeneous Robin boundary condition. It is stated in (5.5), (5.6). Its stability properties system are determined by a rigorous approach.

(ii) The investigation of the small eigenvalues uses some new ideas to deal with the interaction of two small scales which is based on a suitable inner/outer expansion with rigorous error analysis. Remarkably, the order of the small eigenvalues is determined by the smallest diffusion constant ϵ_2^2 and is intimately connected to the smallest length scale.

These results are generalizations of similar but much easier findings for the Schnakenberg model. Therefore, before stating our main results, let us briefly recall some previous studies for the Schnakenberg model or the related Gray-Scott model. In [12, 26] the existence and stability of spiky patterns on bounded intervals is established. In [34] similar results are shown for two-dimensional domains. In [2] it is shown how the degeneracy of the Turing bifurcation can be lifted using spatially varying diffusion coefficients. In [17, 18, 19] spikes are considered rigorously for the shadow system.

For the Gray-Scott model introduced in [10, 11], some of the results are the following. In [5, 6, 7] the existence and stability of spiky patterns on the real line is proved. In [20, 21] a skeleton structure and separators for the Gray-Scott model are established.

Other "large" reaction diffusion systems (more than two components) with concentrated patterns include the hypercycle of Eigen and Schuster [8, 30, 32], and Meinhardt and Gierer's model of mutual exclusion and segmentation [15, 35].

The structure of this paper is as follows:

In Section 2, we state and explain the main theorems on existence and stability.

In Section 3 and Appendix A, we prove the main existence result, Theorem 3.1. In Section 3, we compute the amplitudes of the spikes. In Appendix A, we give a rigorous existence proof.

In Section 4 and Appendix B, we prove the main stability result, Theorem 3.2. In Section 4, we derive a nonlocal eigenvalue problem (NLEP) and determine the stability of the $O(1)$ eigenvalues. In Appendix B, we study the stability of the $o(1)$ eigenvalues.

Finally, in Appendix C, we derive two Green's functions which are needed throughout the paper.

Throughout this paper, the letter C will denote various generic constants which are independent of ϵ, for ϵ sufficiently small. The notation $A \sim B$ means that $\lim_{\epsilon\to 0}\frac{A}{B}=1$; and $A=O(|B|)$ is defined as $|A|\le C|B|$ for some $C>0$; $A=o(|B|)$ means that $\frac{|A|}{|B|}\to 0$.

3. Main Results: Existence and Stability

In this section we state the main results of this paper on existence and stability of cluster solutions. But we first need to introduce some notations and assumptions. We will construct stationary cluster solutions to (2.1), i.e. cluster solutions to the system

$$\begin{cases} D\Delta S + 1 - \dfrac{a_1}{\epsilon_1} S u_1^2 = 0, \quad x\in\Omega,\\ \epsilon_1^2\Delta u_1 - u_1 + S u_1^2 - a_2\dfrac{\epsilon_1}{\epsilon_2}u_1u_2^2 = 0, \quad x\in\Omega,\\ \epsilon_2^2\Delta u_2 - u_2 + u_1u_2^2 = 0, \quad x\in\Omega \end{cases} \tag{3.1}$$

with the Neumann boundary conditions given in (2.3). The solutions of (3.1) will be even functions:

$$S(|x|), \qquad \text{where } S\in H_N^2(\Omega),$$

$$u_1(|x|), \quad \text{where } (1-\chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right) u_1 \in H_N^2(\Omega_{\epsilon_1}), \quad \chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right) u_1 \in H_N^2(\Omega_{\epsilon_2}),$$
$$u_2(|x|), \quad \text{where } u_2 \in H_N^1(\Omega_{\epsilon_2}) \tag{3.2}$$

with

$$H_N^2(-1,1) = \left\{v \in H^2(-1,1) \,:\, v'(-1) = v'(1) = 0\right\}$$

and

$$\Omega_{\epsilon_i} = \left(-\frac{1}{\epsilon_i}, \frac{1}{\epsilon_i}\right), \quad i = 1, 2.$$

These solutions are bounded in their respective norms as $\epsilon = \max\left\{\epsilon_1, \frac{\epsilon_2}{\epsilon_2}\right\} \to 0$. Here χ is a smooth cutoff function which satisfies the following properties:

$$\chi \in C_0^\infty(-1,1), \quad \chi(x) = 1 \text{ for } |x| \le \frac{5}{8}, \quad \chi(x) = 0 \text{ for } |x| \ge \frac{3}{4} \tag{3.3}$$

and

$$r_\epsilon = 10\frac{\epsilon_2}{\epsilon_1}\log\frac{\epsilon_1}{\epsilon_2} \tag{3.4}$$

Let w be the unique solution of the problem

$$\begin{cases} w_{yy} - w + w^2 = 0, \quad w > 0 \text{ in } \mathbb{R}, \\ w(0) = \max_{y\in\mathbb{R}} w(y), \quad w(y) \to 0 \text{ as } |y| \to +\infty \end{cases} \tag{3.5}$$

which is given by

$$w(y) = \frac{3}{2\cosh^2 \frac{y}{2}}. \tag{3.6}$$

Before stating our main results, let us formally discuss how to derive the cluster solution by considering the interaction of the three scales. We set

$$y_i = \frac{x}{\epsilon_i}, \quad i = 1, 2,$$

and consider the limit $\epsilon \to 0$. From now on, we often drop the subscript ϵ if this does not cause confusion.

The third equation of (3.1) in leading order is given by

$$\Delta_{y_2} u_2 - u_2 + u_1(0)u_2^2 \sim 0$$

and u_2 satisfies

$$u_2(y_2) \sim \frac{1}{u_1(0)} w(y_2). \tag{3.7}$$

The middle equation of (3.1) in leading order is given by

$$\Delta_{y_1} u_1 - u_1 + S(0)u_1^2 \sim 0$$

and u_1 satisfies

$$u_1(y_1) \sim \frac{1}{S(0)} w(|y_1| - L_0) \quad \text{for } y_1 > 0, \tag{3.8}$$

where the constant $L_0 > 0$ has to be determined. Integrating the last term in the middle equation of (3.1) results in a jump condition at $y_1 = 0$:

$$u_{1,y_1}(0^+) - u_{1,y_1}(0^-) \sim a_2 u_1(0) d_1,$$

where lower variables denote derivatives. The constant d_1 has to be worked out from the asymptotic behavior of u_2 given in (3.7). For u_1 we use the asymptotic behavior of u_1 given in (3.8). This implies our first solvability condition.

Finally, the first equation of (3.1) in leading order is given by

$$D\Delta S + 1 - S(0)\delta_0 d_0 \sim 0,$$

where δ_0 is the Dirac distribution. The constant d_0 has to be worked out using the asymptotic behavior of u_1 given in (3.8). Then S can be computed using a Green's function which will be defined in (9.1). Here, to have a solution for S, the integral of the last two terms must vanish. This implies our second solvability condition.

Introducing the notation $S_0 = S(0)$, $z = \tanh\left(\frac{L_0}{2}\right)$, the two solvability conditions can be written as follows:

$$S_0 = a_1(6 + 9z - 3z^3), \tag{3.9}$$

$$S_0 = \frac{\sqrt{3}}{2\sqrt{a_2}}\sqrt{z}(1 - z^2). \tag{3.10}$$

We will show that, if $a_1^2 a_2$ is small enough, there are two solutions (S_0, L_0) such that

$$0 < L_0^s < L_0^m < L_0^l,$$

where $L_0^m \approx 0.6380$. Otherwise there are no solutions (S_0, L_0). (At the threshold value for $a_1^2 a_2$ there is exactly one solution (S_0, L_0).)

Using S_0 and L_0, the other properties of the cluster solution can now be worked out easily.

Actually, a finer analysis of the behavior of u_1 near zero is required on the y_2 scale. Basically, we obtain u_1 in that inner region by integrating the last term in the middle equation of (3.1) on the y_2 scale using the profile of u_2. The details will be given below (see the function $u_{1a,\epsilon}$ which will be introduced in (4.8)).

The rigorous mathematical statement is given in the following main existence result.

Theorem 3.1. *Assume that*

$$\epsilon = \max\left\{\epsilon_1, \frac{\epsilon_2}{\epsilon_1}\right\} \ll 1, \quad D = const. \tag{3.11}$$

and

$$a_1^2 a_2 < c_0, \tag{3.12}$$

where

$$c_0 = \max_{0<z<1} \frac{z(z-1)^2}{9(z-2)^2(z+1)^2} \approx 0.0025. \tag{3.13}$$

Then problem (3.1) admits two "cluster" solutions $(S_\epsilon^s, u_{1,\epsilon}^s, u_{2,\epsilon}^s)$ and $(S_\epsilon^l, u_{1,\epsilon}^l, u_{2,\epsilon}^l)$ with the following properties:

(1) all components are even functions.

(2) $S_\epsilon(0)$ satisfies

$$S_\epsilon(0) = S_0 + O\left(\epsilon \log \tfrac{1}{\epsilon}\right). \tag{3.14}$$

(3)

$$u_{1,\epsilon}(x) = \xi_1^\epsilon \Bigg[w\left(\frac{|x|}{\epsilon_1} - L_0 - r_\epsilon\right)\left(1 - \chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right)\right)$$

$$+ \left(u_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1} u_{1a,\epsilon}\left(\frac{|x|}{\epsilon_2}\right)\right)\chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right)\Bigg] + O\left(\epsilon \log \frac{1}{\epsilon}\right), \tag{3.15}$$

$$u_{2,\epsilon}(x) = \xi_2^\epsilon w\left(\frac{|x|}{\epsilon_2}\right) + O\left(\epsilon \log \frac{1}{\epsilon}\right), \tag{3.16}$$

where χ is a smooth cutoff function given in (3.3),

$$\xi_1^\epsilon = \frac{1}{S_\epsilon(\epsilon_1 r_\epsilon)} = \frac{1}{S_0} + O(\epsilon_1 r_\epsilon), \quad \xi_2^\epsilon = \frac{1}{u_{1,\epsilon}(0)} = \frac{1}{\xi_1^\epsilon w(-L_0)} + O(r_\epsilon) = \frac{S_0}{w(-L_0)} + O(r_\epsilon), \tag{3.17}$$

r_ϵ has been defined in (3.4) and w is the unique solution of (3.5), The estimates are in the sense of the norms given in (3.2). The function $u_{1a,\epsilon}$ describing $u_{1,\epsilon}$ in the inner region will be introduced in (4.8).

(4) L_0 and S_0 are determined by solving the system (3.9), (3.10) which has two solutions (S_0^s, L_0^s), (S_0^l, L_0^l), where

$$0 < L_0^s < L_0^m < L_0^l$$

with $L_0^m \approx 0.6380$.

Finally, if $a_1^2 a_2 > c_0$, then for ϵ small enough there are no cluster solutions which satisfy (1) – (4).

Theorem 3.1 will be proved in Section 3 and Appendix A.

Remarks.

1.Note that

$$\epsilon_2 \ll \epsilon_1 r_\epsilon \ll \epsilon_1,$$

i.e. the scale of $\epsilon_1 r_\epsilon$ is between the very small scale and the small scale.

2. If $L_0^s < L_0^l$ then, by varying a_1 and a_2, it is possible for the corresponding value of S_0 to satisfy either $S_0^s < S_0^l$ or $S_0^l < S_0^s$. This means that the cluster in the larger interval could have larger or smaller amplitude.

3. Expressed more precisely, (3.11) means that ϵ_1 and $\frac{\epsilon_2}{\epsilon_1}$ are small enough; (3.12) means the following: for every $\delta_0 > 0$ there exists and $\epsilon_0 > 0$ such that for all ϵ_1, ϵ_2 which satisfy $0 < \epsilon_1 < \epsilon_0$ and $0 < \frac{\epsilon_2}{\epsilon_1} < \epsilon_0$ we have $a_1^2 a_2 < c_0 - \delta_0$. The assumption in the last sentence of the theorem is to be understood in the same way.

4. We remark that using spaces of even functions will make the existence proof easier since the single spike for u_2 must be located at the center and translations of it are automatically excluded.

The second main result of this paper concerns the stability properties of the cluster solutions constructed in Theorem 3.1 and can be stated as follows:

Theorem 3.2. *Assume that (3.11) and (3.12) are satisfied.*

Let $(S_\epsilon^l, u_{1,\epsilon}^l, u_{2,\epsilon}^l)$ and $(S_\epsilon^s, u_{1,\epsilon}^s, u_{2,\epsilon}^s)$ be the cluster solutions given in Theorem 3.1.

Then we have the following:

(1) (Stability) Suppose that $0 \le \tau < \tau_0$*, and* $0 \le \tau_1 < \tau_{1,0}$*, where* $\tau_0 > 0$ *and* $\tau_{1,0} > 0$ *are suitable constants which may be chosen independently of* ϵ_1 *and* $\frac{\epsilon_2}{\epsilon_1}$*. Then* $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$ *is linearly stable.*

(2) (Instability) The solution $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ *is linearly unstable for all* $\tau \ge 0$ *and* $\tau_1 \ge 0$*.*

Theorem 3.2 implies that, in agreement with the Schnakenberg model, the small cluster solutions with $L_0 \sim L^s_0$ are always linearly unstable [28, 29]. The large solutions with $L_0 \sim L^l_0$ can be linearly stable or unstable, depending on certain conditions for the parameters of the system (2.1). To elucidate this issue, we will rigorously investigate their stability behavior in detail. Theorem 3.2 will be proved in Section 4 and Appendix B.

4. Existence I: Formal computation of the amplitudes

In this section and Appendix A, we will show the existence of cluster solutions to system (3.1) and prove Theorem 3.1. In this section we determine S and L_0.

We choose the approximate solution to (3.1) as follows:

$$u_{1,\epsilon}(x) = \xi^\epsilon_1 \left[w\left(\frac{|x|}{\epsilon_1} - L_0 - r_\epsilon\right)\left(1 - \chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right)\right)\chi(|x|) + \left(u_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1} u_{1a,\epsilon}\left(\frac{|x|}{\epsilon_2}\right)\right)\chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right)\right],$$

$$u_{2,\epsilon}(x) = \xi^\epsilon_2\, w\left(\frac{|x|}{\epsilon_2}\right)\chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right), \tag{4.1}$$

where ξ_i, $i = 1, 2$, are positive constants to be determined. They will follow from the solution (S_0, L_0) computed in this section.

Substituting (3.16) into the last equation of (3.1) and using (3.5), we compute

$$\xi^\epsilon_2 = \frac{1}{u_{1,\epsilon}(0)}. \tag{4.2}$$

Substituting (3.1) into the second equation of (3.1) and using (3.5), we get

$$\xi^\epsilon_1 = \frac{1}{S_\epsilon(\epsilon_1 r_\epsilon)} = \frac{1}{S_\epsilon(0)} + O(\epsilon_1 r_\epsilon). \tag{4.3}$$

Next we will derive the two solvability conditions which determine the limiting amplitude S_0 of the source and half of the middle spike distance L_0 (or equivalently, for $z := \tanh\left(\frac{L_0}{2}\right)$) which have been stated in (3.9), (3.10). Then we will use these two conditions to determine S_0 and L_0.

We begin by substituting (4.1) with (4.2) and (4.3) in (3.1).

Integrating the first equation in (3.1), using the Neumann boundary condition and balancing the last two terms, we get

$$1 = 2a_1\left(\frac{1}{S_\epsilon(0)}\int_{-L_0}^{\infty} w^2(y_1)\,dy_1 + S_\epsilon(0)\int_0^{r_\epsilon}(u^2_{1,\epsilon}(0) + O(r_\epsilon))\,dy_1\right)(1 + O(\epsilon_1))$$

$$= \frac{2a_1}{S_\epsilon(0)}\left(\int_{-L_0}^{\infty} w^2(y_1)\,dy_1\right)\left(1 + O\left(\epsilon\log\frac{1}{\epsilon}\right)\right) \tag{4.4}$$

using

$$S_\epsilon(\epsilon_1 y_1) = S_\epsilon(0) + O(\epsilon_1 |y_1|).$$

Denoting

$$\rho(L) := \int_0^L w^2(y)\,dy = \frac{3}{2}\tanh\left(\frac{L}{2}\right)\left(3 - \tanh^2\left(\frac{L}{2}\right)\right),$$

(see equation (2.3) in [36]), which implies as a special case

$$\int_{-\infty}^{\infty} w^2(y)\,dy = 6,$$

we can rewrite (4.4) as

$$S_\epsilon(0) = a_1(6 + 9z - 3z^3) + O(r_\epsilon).$$

Taking the limit $\epsilon \to 0$ implies our first condition for S_0 and z given in (3.9):

$$S_0 = a_1(6 + 9z - 3z^3).$$

Using (3.1) and (4.3), we get

$$u_{1,\epsilon}(r_\epsilon) = \frac{1}{S_\epsilon(0)} w(-L_0) + O(\epsilon_1 r_\epsilon) \tag{4.5}$$

which implies

$$u_{1,\epsilon}(0) = u_{1,\epsilon}(r_\epsilon) + O(r_\epsilon) = \frac{1}{S_\epsilon(0)} w(-L_0) + O(r_\epsilon). \tag{4.6}$$

Using (3.16) and (4.2), we derive

$$u_{2,\epsilon}(0) = \frac{1}{u_{1,\epsilon}(0)} w(0) = \frac{S_\epsilon(0)}{w(-L_0)} w(0) + O(r_\epsilon), \tag{4.7}$$

where $S_\epsilon(0)$ and L_0 are unknown constants to be determined.

On the ϵ_2 scale, we compute from the second equation of (3.1) for $|y_2| \le \frac{\epsilon_1}{\epsilon_2} r_\epsilon = \log \frac{\epsilon_1}{\epsilon_2}$

$$\begin{aligned} u''_{1a,\epsilon}(y_2) &= a_2 u_{1,\epsilon}(0) u^2_{2,\epsilon}(y_2)\,(1 + O(r_\epsilon)) \\ &= a_2 \frac{1}{u_{1,\epsilon}(0)} w^2(y_2)\,(1 + O(r_\epsilon)). \end{aligned}$$

Integrating gives

$$u'_{1a,\epsilon}(y_2) = a_2 \frac{1}{u_{1,\epsilon}(0)} \rho(y_2)\,(1 + O(r_\epsilon))$$

and

$$u_{1a,\epsilon}(y_2) = c + a_2 \frac{1}{u_{1,\epsilon}(0)} \int_0^{y_2} \rho(s)\,ds\,(1 + O(r_\epsilon)). \tag{4.8}$$

This implies

$$\lim_{y_2 \to \pm\infty} u'_{1a,\epsilon}(y_2) = a_2 \frac{1}{u_{1,\epsilon}(0)} \lim_{y_2 \to \pm\infty} \rho(y_2)\,(1 + O(r_\epsilon)) = a_2 \frac{1}{u_1(0)} (\pm 3)\,(1 + O(r_\epsilon)). \tag{4.9}$$

Considering even functions, matching

$$\lim_{y_2 \to \pm\infty} u'_{1a,\epsilon}(y_2) = u'_{1,\epsilon}(\pm r_\epsilon) \tag{4.10}$$

and using (4.6), (4.9), we get

$$2u'_{1,\epsilon}(r_\epsilon) = a_2 \frac{6S_\epsilon(0)}{w(-L_0)} + O(r_\epsilon). \tag{4.11}$$

From (3.6), we compute

$$w(-L_0) = \frac{3}{2}(1 - z^2)$$

and

$$\begin{aligned} 2u'_{1,\epsilon}(r_\epsilon) &= \frac{2}{S_\epsilon(0)}(w'(-L_0))\,(1 + O(\epsilon)) \\ &= \frac{2}{S_0}\tanh\left(\frac{L_0}{2}\right) w(-L_0)\,(1 + O(\epsilon)) \\ &= \frac{3z(1-z^2)}{S_0} \end{aligned}$$

Taking limits in (4.11) implies our second condition for S_0 and z given in (3.10):

$$S_0 = \frac{\sqrt{3}}{2\sqrt{a_2}}\sqrt{z}(1 - z^2).$$

Next we determine S_0 and z from (3.9) and (3.10). Equating (3.9) and (3.10) gives

$$\frac{\sqrt{z}(z-1)}{3(z-2)(z+1)} = \frac{2a_1\sqrt{a_2}}{\sqrt{3}}. \tag{4.12}$$

Elementary computations show that the function on the l.h.s. of (4.12) vanishes for $z = 0$ or $z = 1$ and it has a unique maximum for z in the interval $(0, 1)$. This maximum is reached for $z^m \approx 0.30851$ which corresponds to $L_0^m \approx 0.6380$. The value of the maximum of l.h.s. is ≈ 0.0578.

Now in Figure 1 we plot the l.h.s. of (4.12) with $z = \tanh\left(\frac{L_0}{2}\right)$ versus $L_0 =$.

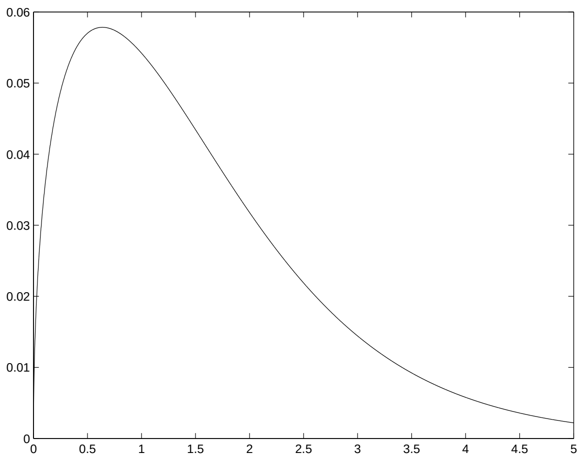

Figure 1. This graph shows the l.h.s. of (4.12) versus L_0. The maximum of l.h.s. of (4.12) is reached for $L_0^m \approx 0.6380$, the value reached is ≈ 0.0578.

This implies that under the condition

$$a_1^2\, a_2 < c_0, \qquad \text{where} \quad c_0 \approx 0.0025,$$

there are two solutions (S_0, L_0) for which L_0 satisfies

$$0 < L_0^s < L_0^m < L_0^l.$$

If

$$a_1^2 a_2 > c_0,$$

there are no such solutions. Trivially, the large z corresponds to the large L_0.

This result can be interpreted as follows: To have this type of cluster solution, the feed rates for both a_1 and a_2 must be small enough. Otherwise the producers S and u_1 will not be able to sustain the consumers u_1 and u_2, respectively. Instead, among others, the following behaviors are possible:

(i) The component u_2 will die out and a spike for the Schnakenberg model remains which involves only the components S and u_1 with $u_2 = 0$.

(ii) The component u_2 will die out and u_1, S will both approach positive constants. It can easily be seen that

$$u_1 = \frac{\epsilon_1}{a_1}, \quad S = \frac{a_1}{\epsilon_1}.$$

(iii) The components approach the positive homogeneous steady state

$$S = \frac{\epsilon_1}{a_1 u_1^2}, \quad u_1^2 - \frac{\epsilon_1}{a_1} u_1 + a_2 \frac{\epsilon_1}{\epsilon_2} = 0, \quad u_2 = \frac{1}{u_1}.$$

Figure 2 shows the spatial profiles of the steady states S, u_1, u_2 for parameters $D = 10$, $\epsilon_1^2 = 10^{-4}$, $\epsilon_2^2 = 10^{-8}$, $\frac{a_1}{\epsilon} = 10$, $a_2 \frac{\epsilon_1}{\epsilon_2} = 1$. Note that the small space variables are $y_1 = \epsilon_1 x = 10^{-2} x$ (scale of the two spikes for u_1) and $y_2 = \epsilon_2 x = 10^{-4} x$ (scale of the spike for u_2).

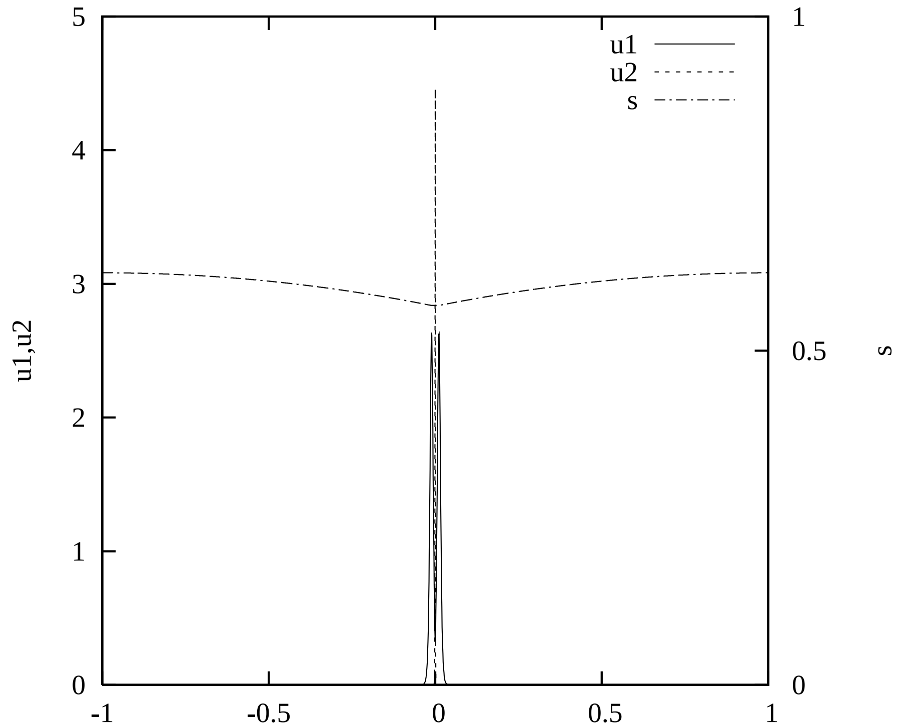

Figure 2a. The spatial profiles of the steady states S, u_1, u_2 for parameters $D = 10$, $\epsilon_1^2 = 10^{-4}$, $\epsilon_2^2 = 10^{-8}$, $\frac{a_1}{\epsilon} = 10$, $a_2 \frac{\epsilon_1}{\epsilon_2} = 1$.

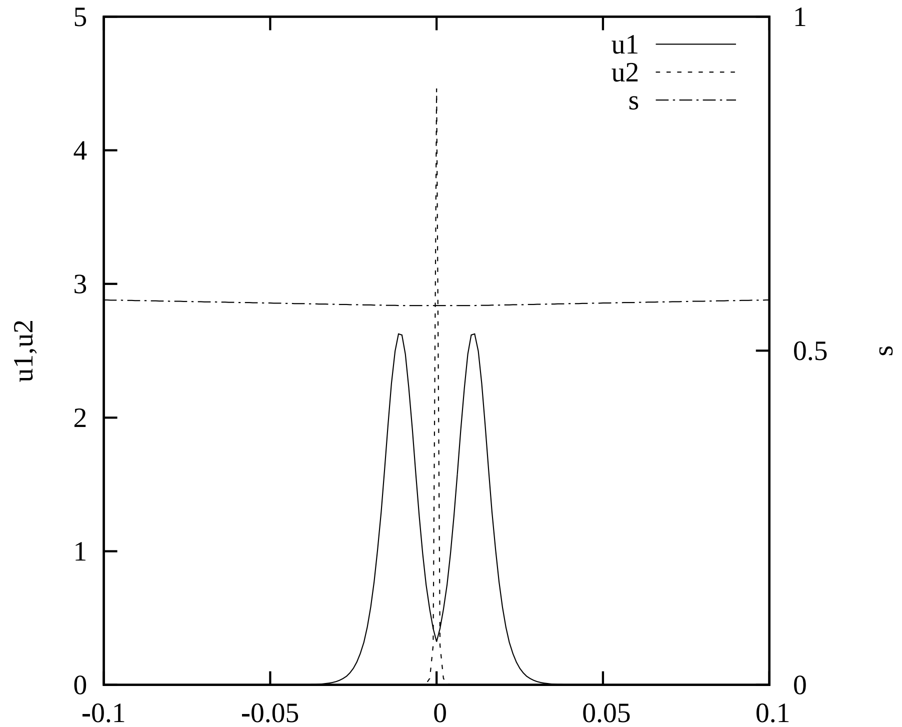

Figure 2b. Same as Figure 2a, but zoomed in for the spatial scale.

In the next section we analyze the stability properties of the cluster steady states.

5. Stability I: Derivation, rigorous deduction and analysis of a NLEP

We linearize (2.1) around the cluster solution $S_\epsilon + \psi_\epsilon e^{\lambda t}$, $u_{\epsilon,i} + \phi_{\epsilon,i} e^{\lambda t}$, $i = 1, 2$, and study the eigenvalue problem of the resulting linearized operator:

$$\mathcal{L}_\epsilon \begin{pmatrix} \psi_\epsilon \\ \phi_{1,\epsilon} \\ \phi_{2,\epsilon} \end{pmatrix} = \begin{pmatrix} \tau \lambda_\epsilon \psi_\epsilon \\ \tau_1 \lambda_\epsilon \phi_{1,\epsilon} \\ \lambda_\epsilon \phi_{2,\epsilon} \end{pmatrix}, \tag{5.1}$$

where $\mathcal{L}_\epsilon$ denotes the linearized operator.

We assume that the domain of the operator $\mathcal{L}_\epsilon$ is $H^2_N(\Omega) \times H^2_N(\Omega_{\epsilon_1}) \times H^2_N(\Omega_{\epsilon_2})$ and that the eigenvalue satisfies $\lambda_\epsilon \in \mathbb{C}$, the set of complex numbers.

We say that a cluster solution is **linearly stable** if the spectrum $\sigma(\mathcal{L}_\epsilon)$ of $\mathcal{L}_\epsilon$ lies in a left half plane $\{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) \le -c_0\}$ for some $c_0 > 0$. A cluster solution is called **linearly unstable** if there exists an eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ with $\mathrm{Re}(\lambda_\epsilon) > 0$.

Next we write down $\mathcal{L}_\epsilon$ explicitly. First we express $\psi_\epsilon = T'[\phi_{1,\epsilon}]$ using the Green's function $G_{D,\tau\lambda_\epsilon}$ defined in (9.7), where $\psi_\epsilon = T'[\phi_{1,\epsilon}]$ is the unique solution of the equation

$$D\Delta\psi_\epsilon - \frac{a_1}{\epsilon_1}\left(\psi_\epsilon u^2_{1,\epsilon} + 2S_\epsilon u_{1,\epsilon}\phi_{1,\epsilon}\right) = \tau\lambda_\epsilon\psi_\epsilon \quad x \in \Omega. \tag{5.2}$$

Note that ψ_ϵ depends on $\phi_{1,\epsilon}$ but not on $\phi_{2,\epsilon}$.

Then we can rewrite (5.1) as follows:

$$\begin{cases} \epsilon_1^2\Delta\phi_{1,\epsilon} - \phi_{1,\epsilon} + 2S_\epsilon u_{1,\epsilon}\phi_{1,\epsilon} + T'[\phi_{1,\epsilon}]u_{1,\epsilon}^2 - a_2\dfrac{\epsilon_1}{\epsilon_2}\phi_{1,\epsilon}u_{2,\epsilon}^2 - 2a_2\dfrac{\epsilon_1}{\epsilon_2}u_{1,\epsilon}u_{2,\epsilon}\phi_{2,\epsilon} = \tau_1\lambda_\epsilon\phi_{1,\epsilon}, \\ \epsilon_2^2\Delta\phi_{2,\epsilon} - \phi_{2,\epsilon} + \phi_{1,\epsilon}u_{2,\epsilon}^2 + 2u_{1,\epsilon}u_{2,\epsilon}\phi_{2,\epsilon} = \lambda_\epsilon\phi_{2,\epsilon}. \end{cases} \tag{5.3}$$

We decompose

$$\phi_{1,\epsilon}(y_1) = \phi_{1a,\epsilon}\left(|y_1| - L_0\right)\left(1 - \chi\left(\frac{|y_1|}{r_\epsilon}\right)\right) + \frac{\epsilon_2}{\epsilon_1}\phi_{1b,\epsilon}\,\chi\left(\frac{y_1}{r_\epsilon}\right)$$

and assume that

$$|||(\phi_{1,\epsilon}, \phi_{2,\epsilon}|||_{L^2}^2 = \|\chi\phi_1\|_{L^2(\Omega_{\epsilon_2})}^2 + \|(1-\chi)\phi_1\|_{L^2(\Omega_{\epsilon_1})}^2 + \|\phi_2\|_{L^2(\Omega_{\epsilon_2})}^2 = 1.$$

Then, arguing as in the proof of Proposition 7.1 below, this sequence has a converging subsequence. We derive an eigenvalue problem for the limit. (Since we consider even eigenfunctions, it is enough to restrict our attention to the positive real axis.)

Now we first derive the limiting eigenvalue problem for the limit (ϕ_1, ϕ_2) as $\epsilon \to 0$. Second we reduce it to a NLEP for ϕ_2 only. In these computation we set $\tau = \tau_1 = 0$. At the end of the section we will show that considering small $\tau \geq 0$ and small $\tau_1 \geq 0$ will only introduce a small perturbation.

For $\tau = 0$, integrating the first equation in (5.3) gives

$$\int_{\mathbb{R}^+} 2S_\epsilon u_{1,\epsilon}\phi_{1,\epsilon}\,dx + \int_{\mathbb{R}^+} \psi_\epsilon u_{1,\epsilon}^2\,dx = 0.$$

Taking the limit $\epsilon \to 0$ gives

$$\frac{\psi(0)}{S_0^2(0)} = -\frac{2\int_{-L_0}^\infty w\phi_1\,dy}{\int_{-L_0}^\infty w^2\,dy}. \tag{5.4}$$

Using (5.4), then the first equation in (5.3), after taking the limit $\epsilon \to 0$, leads to

$$\Delta_{y_1}\phi_1 - \phi_1 + 2w\phi_1 - \frac{2\int_{-L_0}^\infty w\phi_1}{\int_{-L_0}^\infty w^2}w^2 = 0, \quad y > -L_0. \tag{5.5}$$

The jump condition at $y = 0$ translates into the boundary condition

$$\phi_1'(-L_0) - c\phi_1(-L_0) - c(u_1(0))^2\frac{2\int_{\mathbb{R}} w\phi_2}{\int_{\mathbb{R}} w^2} = 0, \tag{5.6}$$

where

$$c = a_2\frac{S_0^2(0)}{w^2(-L_0)}3.$$

Note that this is an inhomogeneous boundary condition of Robin type. Here we have used that

$$u_1(0) = \frac{w(-L_0)}{S_0(0)}, \quad u_2(0) = \frac{1}{u_1(0)}.$$

Similarly, the second equation in (5.3), after taking the limit $\epsilon \to 0$, leads to the eigenvalue problem

$$\Delta_{y_2}\phi_2 - \phi_2 + 2w\phi_2 + \frac{\phi_1(-L_0)}{(u_1(0))^2}w^2 = \lambda\phi_2, \quad y > 0, \tag{5.7}$$

subject to the boundary condition

$$\phi_2'(0) = 0. \tag{5.8}$$

For later use, we compute

$$c = a_2 \frac{3}{4a_2} z(1-z^2)^2 \frac{1}{\frac{9}{4}(1-z^2)^2} 3 = z. \tag{5.9}$$

We summarize this result as follows: Taking the limit $\epsilon \to 0$ in (5.3), we get

$$\begin{cases} \Delta_{y_1}\phi_1 - \phi_1 + 2w\phi_1 - \dfrac{2\int_{-L_0}^{\infty} w\phi_1}{\int_{-L_0}^{\infty} w^2} w^2 = 0, \quad y_1 > -L_0, \\ \Delta_{y_2}\phi_2 - \phi_2 + 2w\phi_2 + \dfrac{\phi_1(-L_0)}{(u_1(0))^2} w^2 = \lambda\phi_2, \quad y_2 \in \mathbb{R}, \end{cases} \tag{5.10}$$

with the boundary conditions

$$\phi_1'(-L_0) - c\phi_1(-L_0) - c(u_1(0))^2 \frac{2\int_{\mathbb{R}} w\phi_2}{\int_{\mathbb{R}} w^2} = 0, \quad \phi_2'(0) = 0. \tag{5.11}$$

Although the derivations given above are formal, we can rigorously prove the following separation of eigenvalues.

Theorem 5.1. *Let λ_ϵ be an eigenvalue of (5.3) for which $Re(\lambda_\epsilon) > -a_0$ with some suitable constant a_0 independent of ϵ.*

(1) Suppose that (for suitable sequences $\epsilon_n \to 0$) we have $\lambda_{\epsilon_n} \to \lambda_0 \neq 0$. Then λ_0 is an eigenvalue of NLEP (5.10) with boundary condition (5.11).

(2) Let $\lambda_0 \neq 0$ be an eigenvalue of NLEP (5.10) with boundary condition (5.11). Then, for ϵ sufficiently small, there is an eigenvalue λ_ϵ of (5.3) with $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$.

Remark: *From Theorem 5.1 we see rigorously that the eigenvalue problem (5.3) is reduced to the study of the NLEP (5.10) with boundary conditions (5.11).*

Now we prove Theorem 5.1.

Proof of Theorem 5.1:

Part (1) follows by an asymptotic analysis combined with passing to the limit as $\epsilon \to 0$ which is similar to the proof of Proposition 7.1 given below.

Part (2) follows from a compactness argument by Dancer introduced in Section 2 of [4]. It was applied in ([33]) to a related situation, therefore we omit the details.

□

The stability or instability of the large eigenvalues follows from the following results:

Theorem 5.2. [27]*: Consider the following nonlocal eigenvalue problem*

$$\phi'' - \phi + 2w\phi - \gamma \frac{\int_{\mathbb{R}} w\phi}{\int_{\mathbb{R}} w^2} w^2 = \alpha\phi. \tag{5.12}$$

(1) If $\gamma < 1$, then there is a positive eigenvalue to (5.12).

(2) If $\gamma > 1$, then for any nonzero eigenvalue α of (5.12), we have

$$Re(\alpha) \leq -c_0 < 0.$$

(3) If $\gamma \neq 1$ and $\alpha = 0$, then $\phi = c_0 w'$ for some constant c_0.

In our applications to the case when $\tau > 0$ or $\tau_1 > 0$, we need to handle the situation when the coefficient γ is a complex function of $\tau\alpha$. Let us suppose that

$$\gamma(0) \in \mathbb{R}, \quad |\gamma(\tau\alpha)| \leq C \text{ for } \alpha_R \geq 0,\, \tau \geq 0, \tag{5.13}$$

where C is a generic constant independent of τ, α.

Now we have

Theorem 5.3. *(Theorem 3.2 of* [33].*)*

Consider the following nonlocal eigenvalue problem

$$\phi'' - \phi + 2w\phi - \gamma(\tau\alpha)\frac{\int_{\mathbb{R}} w\phi}{\int_{\mathbb{R}} w^2} w^2 = \alpha\phi, \tag{5.14}$$

where $\gamma(\tau\alpha)$ satisfies (5.13). Then there is a small number $\tau_0 > 0$ such that for $\tau < \tau_0$,
(1) if $\gamma(0) < 1$, then there is a positive eigenvalue to (5.12);
(2) if $\gamma(0) > 1$, then for any nonzero eigenvalue α of (5.14), we have

$$Re(\alpha) \leq -c_0 < 0.$$

Now we consider the NLEP (5.10) and show the following result:

Lemma 5.1. *The nonlocal eigenvalue problem (5.10) with boundary conditions (5.11) is stable for $L_0 > L_0^m$ and unstable for $L_0 < L_0^m$. For $L_0 \neq L_0^m$ it does not have zero eigenvalue.*

Proof of Lemma 5.1: Using Lemma 2.1 of [14] together with the Fredholm alternative, it follows that there is a unique solution ϕ_1 of the problem (5.5) with boundary condition (5.6). We now compute this solution. It is easy to see that

$$\phi_1(y) = \alpha w(y) + \beta w'(y), \quad y > -L_0, \tag{5.15}$$

for some real constants α and β.

Plugging the ansatz (5.15) into (5.5), we compute

$$\alpha \underbrace{(w'' - w + 2w^2)}_{=w^2} - \alpha\frac{2\int_{-L_0}^{\infty} w^2}{\int_{-L_0}^{\infty} w^2} w^2$$

$$+\beta \underbrace{(w''' - w' + 2ww')}_{=0} - \beta\frac{2\int_{-L_0}^{\infty} ww'}{\int_{-L_0}^{\infty} w^2} w^2 = 0, \quad y > -L_0.$$

This implies

$$\alpha - 2\alpha - 2\beta\frac{\int_{-L_0}^{\infty} ww'}{\int_{-L_0}^{\infty} w^2} = 0$$

and finally

$$\alpha = \beta\frac{w^2(-L_0)}{\rho(L_0) + 3}$$

since

$$\int_{-L_0}^{\infty} ww' = -\frac{1}{2}w^2(-L_0).$$

Using the boundary condition (5.6), we derive

$$\alpha w'(-L_0) + \beta w''(-L_0) - c(\alpha w(-L_0) + \beta w'(-L_0)) - c u_1(0) \frac{2\int_{\mathbb{R}} w\phi_2}{\int_{\mathbb{R}} w^2} = 0.$$

Elementary computations give

$$\begin{aligned}\frac{\beta}{u_1^2(0)} &= \frac{c}{w''(-L_0) - cw'(-L_0)} \frac{2\int w\phi_2}{\int w^2} \\ &= \frac{c}{-cw'(-L_0) + w(-L_0) - w^2(-L_0)} \frac{2\int w\phi_2}{\int w^2}.\end{aligned}$$

Then by Theorem 5.2 the NLEP (5.10) is stable if

$$\gamma = -\frac{\phi_1(-L_0)}{u_1^2(0)} \left(\frac{\int w\phi_2}{\int w^2}\right)^{-1} > 1.$$

We compute, using (5.15),

$$\begin{aligned}\gamma = -\frac{\phi_1(-L_0)}{u_1^2(0)} \left(\frac{\int w\phi_2}{\int w^2}\right)^{-1} &= -\frac{\beta}{u_1^2(0)} \left(\frac{\alpha}{\beta} w(-L_0) + w'(-L_0)\right) \left(\frac{\int w\phi_2}{\int w^2}\right)^{-1} \\ &= -\frac{\beta}{u_1^2(0)} \left(\frac{w^3(-L_0)}{\rho(L_0)+3} + w'(-L_0)\right) \left(\frac{\int w\phi_2}{\int w^2}\right)^{-1} \\ &= \frac{c\left(\frac{w^3(-L_0)}{\rho(L_0)+3} + w'(-L_0)\right)}{cw'(-L_0) - w(-L_0) + w^2(-L_0)} 2.\end{aligned}$$

Using

$$z = \tanh\left(\frac{L_0}{2}\right) \quad (\text{and } c = z,\ w'(-L_0) = zw,\ w(-L_0) = \frac{3}{2}(1-z^2)\ ), \tag{5.16}$$

the condition $\gamma > 1$ can be rewritten as

$$\frac{z\frac{\frac{27}{8}(1-z^2)^3}{3+\frac{9}{2}z-\frac{3}{2}z^3} + \frac{3}{2}z^2(1-z^2)}{\frac{3}{2}z^2(1-z^2) - \frac{3}{2}(1-z^2) + \frac{9}{4}(1-z^2)^2} > \frac{1}{2}.$$

This is equivalent to

$$\frac{z\frac{27}{8}(1-z^2)^3}{3+\frac{9}{2}z-\frac{3}{2}z^3} > \frac{3}{8}\left(1-5z^2\right)\left(1-z^2\right)$$

and to

$$6z(z-1)^2 > (5z^2-1)(z-2). \tag{5.17}$$

We compare this result with Figure 1. Taking the derivative of l.h.s. of (4.12) w.r.t. z, we compute

$$\begin{aligned}&\frac{d}{dz}\frac{\sqrt{z}(z-1)}{3(z-2)(z+1)} \\ &= \frac{(5z^2-1)(z-2) - 6z(z-1)^2}{6\sqrt{z}(z-2)^2(z+1)^2} < 0\end{aligned}$$

which is equivalent to the condition (5.17). This means that we have linearized stability for the large eigenvalues at the decreasing part of the graph in Figure 1 for which L_0 and z

are large. On the other hand, we have linearized instability for the large eigenvalues at the increasing solution branch, for which L_0 and z are small.

Finally, we continue to consider the stability problem for the linearized operator by extending our approach to the case $\tau \geq 0$ small and $\tau_1 \geq 0$ small.

First note that ψ is continuous in $\tau\lambda$ which follows by using Green's function to solve (5.2) for $\psi_\epsilon = T'[\phi_{1,\epsilon}]$.

Second, using (5.15) in a perturbed version of (5.5), we derive that now

$$\phi_1 = \alpha w + \beta w' + O((\tau + \tau_1)|\lambda|)$$

with the same values for α and β as before.

Third, integrating the second equation in (5.3) near zero, taking the limit $\epsilon \to 0$ and using the perturbed boundary condition (5.6), now we get

$$\gamma = -\frac{\phi_1(-L_0)}{u_1(0)} \left(\frac{\int w\phi_2}{\int w^2} \right)^{-1} + O((\tau + \tau_1)|\lambda|) =$$

$$= \frac{c\left(\frac{w^3(-L_0)}{\rho(L_0)+3} + w'(-L_0)\right)}{cw'(-L_0) - w(-L_0) + w^2(-L_0)} 2 + O((\tau + \tau_1)|\lambda|).$$

Finally, multiplying the eigenvalue problem (5.3) by the eigenfunction and using quadratic forms, it can be shown that $|\lambda|$ is bounded for τ and τ_1 small enough. This argument is given in detail in [31] and we refer to that work for further information. This implies that small values of $\tau \geq 0$ and $\tau_1 \geq 0$ introduce only small perturbations to the eigenvalue problem for large eigenvalues λ_ϵ of order $O(1)$ and Theorem 3.2 continues to hold in this case.

□

For the existence proof in Appendix A we will need to know that the adjoint operator $\mathcal{L}_\epsilon^*$ to the linear operator $\mathcal{L}_\epsilon$ is invertible. This is the issue of our next result.

Expressing $\mathcal{L}_\epsilon^*$ explicitly, we can rewrite the adjoint eigenvalue problem as follows:

$$\begin{cases} D\Delta\psi_\epsilon + \dfrac{1}{\epsilon_1}(\phi_{1,\epsilon} - a_1\psi_\epsilon)u_{1,\epsilon}^2 = \tau_1\lambda_\epsilon\psi_\epsilon, \\ \epsilon_1^2\Delta\phi_{1,\epsilon} - \phi_{1,\epsilon} + 2S_\epsilon u_{1,\epsilon}(\phi_{1,\epsilon} - a_1\psi_\epsilon) + \dfrac{\epsilon_1}{\epsilon_2}(\phi_{2,\epsilon} - a_2\phi_{1,\epsilon})u_{2,\epsilon}^2 = \tau_1\lambda_\epsilon\phi_{1,\epsilon}, \\ \epsilon_2^2\Delta\phi_{2,\epsilon} - \phi_{2,\epsilon} + 2u_{1,\epsilon}u_{2,\epsilon}(\phi_{2,\epsilon} - a_2\phi_{1,\epsilon}) = \lambda_\epsilon\phi_{2,\epsilon}. \end{cases} \tag{5.18}$$

Taking the limit $\epsilon \to 0$, we get the following limiting adjoint eigenvalue problem

$$\begin{cases} \Delta_{y_1}\phi_1 - \phi_1 + 2w\phi_1 - 2\dfrac{\int_{-L_0}^\infty w^2\phi_1}{\int_{-L_0}^\infty w^2} w = \tau_1\lambda_\epsilon\phi_1, \\ \Delta_{y_2}\phi_2 - \phi_2 + 2w\phi_2 - 2a_2\phi_1(0)w = \lambda_\epsilon\phi_2, \\ \phi_1'(-L_0) - c\phi_1(-L_0) + \dfrac{c}{a_2}u_1(0)\dfrac{\int_{\mathbb{R}} w^2\phi_2}{\int_{\mathbb{R}} w^2} = 0, \\ \phi_2'(0) = 0. \end{cases} \tag{5.19}$$

We are now going to show that this limit of the adjoint operator has only the trivial kernel and prove the following lemma:

Lemma 5.2. *The limiting adjoint eigenvalue problem given in (5.19) has only the trivial kernel.*

Proof of Lemma 5.2:
We introduce the notation $\mathcal{L}\phi := \Delta\phi - \phi + 2w\phi, y > -L_0, \quad \phi'(-L_0) - c\phi(-L_0) = 0.$
It is easy to see that

$$\phi_1(y) = \alpha(\mathcal{L}^{-1}w)(y) + \beta w'(y), \quad y > -L_0, \tag{5.20}$$

for some real constants α and β.
From Lemma 2.2 in [14], we get

$$\mathcal{L}^{-1}w(y) = w + \frac{1}{2}(y + L_0)w' + \frac{z}{1-z^2}w'. \tag{5.21}$$

Plugging (5.20) into (5.19), we compute

$$\alpha \underbrace{\mathcal{L}(\mathcal{L}^{-1}w)}_{=w} - \alpha \frac{2\int_{-L_0}^{\infty} w^2(\mathcal{L}^{-1}w)\,dy}{\int_{-L_0}^{\infty} w^2\,dy} w$$
$$+\beta \underbrace{\mathcal{L}w'}_{=0} - \beta \frac{2\int_{-L_0}^{\infty} w^2 w'\,dy}{\int_{-L_0}^{\infty} w^2\,dy} w = 0, \quad y > -L_0.$$

Using

$$\begin{aligned}\int_{-L_0}^{\infty} w^2\mathcal{L}^{-1}w\,dy &= \frac{5}{6}\int_{-L_0}^{\infty} w^3\,dy + \frac{z}{1-z^2}\int_{-L_0}^{\infty} w^2 w'\,dy \\ &= 3 + \frac{45}{8}\int_0^z (1-t^2)^2\,dt + \frac{z}{1-z^2}\left(-\frac{1}{3}w^3(-L_0)\right) \\ &= 3 + \frac{45}{8}\left(z - \frac{2}{3}z^3 + \frac{z^5}{5}\right) - \frac{9}{8}z(1-z^2)^2,\end{aligned}$$

this implies

$$\alpha - 2\alpha \frac{3 + \frac{45}{8}\left(z - \frac{2}{3}z^3 + \frac{z^5}{5}\right) - \frac{9}{8}z(1-z^2)^2}{3 + \frac{3}{2}z(3-z^2)} + \beta \frac{\frac{9}{4}(1-z^2)^3}{3 + \frac{3}{2}z(3-z^2)} = 0$$

and finally

$$\alpha = \frac{3(1-z^2)^3}{4 - 2z(3-z^2) + 15(z - \frac{2}{3}z^3 + \frac{z^5}{5}) - 3z(1-z^2)^2}\beta. \tag{5.22}$$

Substituting (5.20) into the boundary condition in (5.19), we derive

$$\phi_1(-L_0) = \alpha(\mathcal{L}^{-1}w)'(-L_0) + \beta w''(-L_0) - c(\alpha(\mathcal{L}^{-1}w)(-L_0) + \beta w'(-L_0)) + \frac{c}{a_2}\frac{\int_{\mathbb{R}} w^2\phi_2}{\int_{\mathbb{R}} w^2} = 0.$$

Elementary computations give

$$-\frac{3}{4}\beta(1-z^2)^2 + \frac{c}{a_2}\frac{\int_{\mathbb{R}} w^2\phi_2\,dy}{\int w^2\,dy} = 0$$

which implies

$$\beta = \frac{4}{3}\frac{1}{(1-z^2)^2}\frac{c}{a_2}\frac{\int w^2\phi_2\,dy}{\int w^2\,dy}. \tag{5.23}$$

Substituting (5.23) into (5.22), we get

$$\alpha = \frac{4(1-z^2)}{4-2z(3-z^2)+15(z-\frac{2}{3}z^3+\frac{z^5}{5})-3z(1-z^2)^2}\frac{c}{a_2}\frac{\int w^2\phi_2\,dy}{\int w^2\,dy} \tag{5.24}$$

We now compute the term $2a_2\phi_1(0)$ needed in the second equation of (5.19). We decompose

$$2a_2\phi_1(0) = \gamma\frac{\int w\phi_2\,dy}{\int w^2\,dy}.$$

Then (5.19) has only the trivial kernel if

$$\gamma \neq 1.$$

(This follows after multiplying NLEP by w, integrating, and using a standard result on a local eigenvalue problem. The details of the argument are shown in Section 3 of [33] and for brevity we skip it here.) We compute, using (5.20),

$$\begin{aligned}\gamma &= 2a_2\left(\alpha(\mathcal{L}^{-1}w)(-L_0)+\beta w'(-L_0)\right)\left(\frac{\int w^2\phi_2\,dy}{\int w^2\,dy}\right)^{-1}\\ &= 2a_2\left(\frac{3}{2}\alpha+\frac{3}{2}z(1-z^2)\beta\right)\\ &= 2z\left(\frac{6(1-z^2)}{4-2z(3-z^2)+15(z-\frac{2}{3}z^3+\frac{z^5}{5})-3z(1-z^2)^2}+\frac{2z}{1-z^2}\right).\end{aligned}$$

Now the condition $\gamma \neq 1$ is equivalent to

$$6z(z-1)^2 \neq (5z^2-1)(z-2).$$

This is the same condition as for the operator $\mathcal{L}$ and the monotonicity of the solution graph (compare (5.17)).

□

6. Discussion

Let us finally discuss the biological implications of our results. The patterns observed in our model combine different length scales for different components and are (linearly) stable. This shows that within our model efficient cooperation over different length-scales is possible and the consumer chain can be established in a stable and reliable manner. It confirms that the flow of resources, e.g. raw materials from worldwide mining activities, refinement on a worldwide or national level, production of end-product on a worldwide or national level, finally end-design and sales on a local level can be combined efficiently and reliably.

They will also be important in a biological context to understand spatiotemporal structures in genetic signaling pathways which combine different lengthscales, e.g. at intercellular, intracellular and nucleus level. Morphogens are able to spread information at the various levels by diffusion processes, where smaller diffusivity will lead to smaller lengthscales. Our problem is a simple prototype of such a process in the case of merely two activators.

7. Appendix A – Existence II: Rigorous proof

Completion of the Proof of Theorem 3.1:

Now we consider the approximate cluster solution which has been introduced in (4.1) and is given by

$$\tilde{u}_{1,\epsilon}(x) = \xi_1^\epsilon \left[w\left(\frac{|x|}{\epsilon_1} - L_0 - r_\epsilon\right)\left(1 - \chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right)\right)\chi(|x|) \right.$$

$$\left. + \left(\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1} u_{1a,\epsilon}\left(\frac{|x|}{\epsilon_2}\right)\right)\chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right)\right]$$

$$= \tilde{u}_{1out,\epsilon}\,\chi(|x|) + \left(\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1} u_{1a,\epsilon}\left(\frac{|x|}{\epsilon_2}\right)\right)\chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right)\Bigg],$$

$$\tilde{u}_{2,\epsilon}(x) = \xi_2^\epsilon\, w\left(\frac{|x|}{\epsilon_2}\right)\chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right),$$

where the amplitudes ξ_i, $i = 1, 2$, satisfy (4.3) and (4.2), respectively, r_ϵ is given in (3.4), χ has been introduced in (3.3), and $u_{1a,\epsilon}$ is defined in (4.8).

We remark that we use cutoff functions with two different scalings: $\chi(|x|)$ serves to produce an approximate solution which vanishes at the boundary exactly, whereas the role of $\chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right)$ is to separate the two small scales. In the following we will sometimes drop the argument of the χ functions, and, to distinguish the two types we will use the shorthands $\chi_0 = \chi(|x|)$ and $\chi_1 = \chi\left(\frac{|x|}{\epsilon_1 r_\epsilon}\right)$. Note that the perturbation caused by χ_0 is exponentially small (of order $e^{-C/\epsilon}$ for any $0 < C < 1$ as $\epsilon \to 0$) in $L^2_(\Omega_{\epsilon_1})$.

We first compute the error of the approximate cluster solution in system (3.1).

We compute the first component $\tilde{S}_\epsilon$ by $\tilde{S}_\epsilon = T[\tilde{u}_{1,\epsilon}]$. and so the first equation of (3.1) is solved exactly.

The second equation of (3.1) at $(\tilde{S}_\epsilon, \tilde{u}_{1,\epsilon}, \tilde{u}_{2,\epsilon})$ is calculated as follows:

$$(\tilde{u}_{1,\epsilon})'' - \tilde{u}_{1,\epsilon} + \tilde{S}_\epsilon \tilde{u}_{1,\epsilon}^2 - a_2 \frac{\epsilon_1}{\epsilon_2} \tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}^2$$

$$= (1 - \chi_1)\left[\tilde{u}_{1,\epsilon}'' - \tilde{u}_{1,\epsilon} + \tilde{S}_\epsilon(0)\tilde{u}_{1,\epsilon}^2 - a_2\frac{\epsilon_1}{\epsilon_2}\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}^2\right]$$

$$+ (1 - \chi_1)[\tilde{S}_\epsilon - \tilde{S}_\epsilon(0)]\tilde{u}_{1,\epsilon}^2$$

$$+ \chi_1\left[\frac{\epsilon_1^2}{\epsilon_2^2}\Delta_{y_2}\left(\frac{\epsilon_2}{\epsilon_1}u_{1a,\epsilon}\right) - \left(\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1}u_{1a,\epsilon}\right)\right.$$

$$\left. + \tilde{S}_\epsilon\left(\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1}u_{1a,\epsilon}\right)^2 - a_2\left(\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1}u_{1a,\epsilon}\right)\tilde{u}_{2,\epsilon}^2\right]$$

$$+ 2\chi_1'\left(u_{1a,\epsilon,y_1} - \tilde{u}_{1,\epsilon}'\right) + \chi_1''\left(\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1}u_{1a,\epsilon} - \tilde{u}_{1,\epsilon}\right)$$

$$=: E_1 + E_2 + E_3 + E_4.$$

Here $'$ denotes derivative w.r.t. the variable of the corresponding function, i.e. it means derivative w.r.t. x for S_ϵ, y_1 for u_1 and y_2 for u_2. We divide $E_1 = E_{1a} + E_{1b}$ into two parts and estimate

$$\begin{aligned} E_{1a} &= (1-\chi_1)[\tilde{u}''_{1,\epsilon} - \tilde{u}_{1,\epsilon} + \tilde{S}_\epsilon(0)\tilde{u}^2_{1,\epsilon}] \\ &= O(\epsilon) \quad \text{ in } L^2(\Omega_{\epsilon_1}). \end{aligned}$$

Here we have used that $\tilde{u}_{1,\epsilon}(y_1) = (1+O(\epsilon))\frac{1}{\tilde{S}_\epsilon(0)} w(|y_1| - L_0 - r_\epsilon)$ for $|y_1| > 0.5 r_\epsilon$, and (3.5). The second part of E_1 is estimated by

$$E_{1b} = \left| (1-\chi_1) a_2 \frac{\epsilon_1}{\epsilon_2} \tilde{u}_{1,\epsilon}\tilde{u}^2_{2,\epsilon} \right| \le C\frac{\epsilon_1}{\epsilon_2} w^2\left(0.5\frac{\epsilon_1}{\epsilon_2} r_\epsilon\right) = O\left(\left(\frac{\epsilon_2}{\epsilon_1}\right)^9\right).$$

Note for later use that the definition of r_ϵ (see (3.4)) implies

$$\int_0^\infty w^2(y_2)\,dy_2 - \int_0^{r_\epsilon \epsilon_1/\epsilon_2} w^2(y_2)\,dy_2 \sim \int_{r_\epsilon\epsilon_1/\epsilon_2}^\infty e^{-2y_2}\,dy_2 \sim e^{-2r_\epsilon\epsilon_1/\epsilon_2} \sim \left(\frac{\epsilon_2}{\epsilon_1}\right)^{20}.$$

Computing $\tilde{S}_\epsilon(x)$ using the Green's function G_D introduced in (9.1), we derive the following estimate:

$$\begin{aligned} E_2 &= [\tilde{S}_\epsilon(\epsilon_1 y_1) - \tilde{S}_\epsilon(0)]\tilde{u}^2_{1,\epsilon} \\ &= \tilde{u}^2_{1,\epsilon} a_1 \int_{-1/\epsilon_1}^{1/\epsilon_1} [G_D(\epsilon_1 y_1, \epsilon_1 z) - G_D(0, \epsilon_1 z)]\tilde{S}_\epsilon(z)\tilde{u}^2_{1,\epsilon}(z)\,dz\,(1+O(\epsilon_1)) \\ &= a_1 \frac{\tilde{u}^2_{1,\epsilon}}{\tilde{S}_\epsilon(0)}\epsilon_1 \int_{\mathbb{R}} \left(\frac{1}{2D}|y_1 - z_1| - \frac{1}{2D}|z_1|\right) w^2(|z| - L_0)\,dz\,(1 + O(r_\epsilon + \epsilon_1|y_1|)) \\ &\quad + a_1 \frac{\tilde{u}^2_{1,\epsilon}}{\tilde{S}_\epsilon(0)} \epsilon_1^2 y_1^2 \nabla^2 H_D(0,0)(6 + 2\rho(L_0))\,(1 + O(r_\epsilon + \epsilon_1|y_1|)) \\ &= O(\epsilon_1|y_1|\tilde{u}^2_{1,\epsilon}) = O(\epsilon) \quad \text{ in } L^2(\Omega_{\epsilon_1}). \end{aligned}$$

Note that $\nabla H_D(0,0) = 0$ by symmetry. (See the computation of H_D in the appendix).

On the small scale, we estimate

$$\begin{aligned} E_3 &= \chi_1 \left[\frac{\epsilon_1^2}{\epsilon_2^2}\Delta_{y_2}\frac{\epsilon_2}{\epsilon_1}u_{1a,\epsilon} - \left(\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1}u_{1a,\epsilon}\right) \right. \\ &\quad \left. + \tilde{S}_\epsilon\left(\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1}u_{1a,\epsilon}\right)^2 - a_2\left(\frac{\epsilon_1}{\epsilon_2}\tilde{u}_{1,\epsilon}(0) + u_{1a,\epsilon}\right)\tilde{u}^2_{2,\epsilon} \right] \\ &= \chi_1 \frac{\epsilon_1}{\epsilon_2}[\Delta_{y_2} u_{1a,\epsilon} - a_2 \tilde{u}_{1,\epsilon}(0)\tilde{u}^2_{2,\epsilon}] \\ &\quad - \tilde{u}_{1,\epsilon}(r_\epsilon) + \tilde{S}_\epsilon \tilde{u}^2_{1,\epsilon}(r_\epsilon) + O(\epsilon) \\ &= O(\epsilon) \quad \text{ in } L^2(\Omega_{\epsilon_2}) \end{aligned}$$

by the definition of $u_{1a,\epsilon}$ given in (4.8) and due to the relation between $\xi_1 = \tilde{u}_{1,\epsilon}(0)$ and $\tilde{S}_\epsilon(r_\epsilon)$ given in (4.3).

Finally, we estimate the error on the overlapping region between inner and outer scale for which $\chi_1' \neq 0$ and is contained in $0.5 r_\epsilon < y_1 < r_\epsilon$.

The exact matching conditions at $x = r_\epsilon$ are

$$\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1} u_{1a,\epsilon}\left(\frac{\epsilon_1}{\epsilon_2} r_\epsilon\right) - \tilde{u}_{1out,\epsilon}(r_\epsilon) = 0 \tag{7.1}$$

and

$$u_{1a,\epsilon,y_2}\left(\frac{\epsilon_1}{\epsilon_2} r_\epsilon\right) - \tilde{u}'_{1out,\epsilon}(r_\epsilon) = 0. \tag{7.2}$$

Note that the system (7.1), (7.2) in leading order is given by

$$c_{1,\epsilon}\frac{\epsilon_1}{\epsilon_2} r_\epsilon + c_{2,\epsilon} e^{-r_\epsilon \epsilon_1/\epsilon_2} + c_{4,\epsilon} = c_{3,\epsilon} w(-L_0)$$

$$c_{1,\epsilon} - c_{2,\epsilon} e^{-r_\epsilon \epsilon_1/\epsilon_2} = c_{3,\epsilon} w_{y_1}(-L_0),$$

where $c_{1,\epsilon}$, $c_{2,\epsilon}$, $c_{3,\epsilon}$ are given constants which have limits as $\epsilon \to 0$, r_ϵ is given by (3.4), and the unknowns are $c_{4,\epsilon}$ and L_0. The system (7.1), (7.2) can be solved exactly and it has a unique solution.

This implies

$$\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1} u_{1a,\epsilon}\left(\frac{\epsilon_1}{\epsilon_2} y_1\right) - \tilde{u}_{1out,\epsilon}(y_1) = O((y_1 - r_\epsilon)^2), \tag{7.3}$$

and

$$u_{1a,\epsilon,y_2}\left(\frac{\epsilon_1}{\epsilon_2} y_1\right) - \tilde{u}'_{1out,\epsilon} = O(|y_1 - r_\epsilon|). \tag{7.4}$$

Combining this with

$$|\chi_1| \le 1, \quad |\chi_1'| \le \frac{C}{r_\epsilon}, \quad |\chi_1''| \le \frac{C}{r_\epsilon^2},$$

we derive

$$|\chi_1' \tilde{u}'_{1,\epsilon}| \le C, \quad |\chi_1'' \tilde{u}_{1,\epsilon}| \le C.$$

With this knowledge in hand, we can now easily estimate

$$\begin{aligned} E_4 &= 2\chi_1'\left(u_{1a,\epsilon,y_1} - \tilde{u}'_{1,\epsilon}\right) + \chi_1''\left(\tilde{u}_{1,\epsilon}(0) + \frac{\epsilon_2}{\epsilon_1} u_{1a,\epsilon} - \tilde{u}_{1,\epsilon}\right) \\ &= O(\epsilon) \quad \text{in } L^2(\Omega_{\epsilon_2}). \end{aligned}$$

The third equation in (3.1) becomes

$$\begin{aligned} &\Delta_{y_2}\tilde{u}_{2,\epsilon} - \tilde{u}_{2,\epsilon} + \tilde{u}_{1,\epsilon}\tilde{u}^2_{2\,\epsilon} \\ &= \Delta_{y_2}\tilde{u}_{2,\epsilon} - \tilde{u}_{2,\epsilon} + \tilde{u}_{1,\epsilon}(0)\tilde{u}^2_{2\,\epsilon} \\ &\quad + [\tilde{u}_{1,\epsilon} - \tilde{u}_{1,\epsilon}(0)]\tilde{u}^2_{2\,\epsilon} \\ &= O(\epsilon) + O\left(\frac{\epsilon_2}{\epsilon_1}|y_2|\tilde{u}^2_{2\,\epsilon}\right) \\ &= O(\epsilon) \quad \text{in } L^2(\Omega_{\epsilon_2}) \end{aligned}$$

by using Taylor expansion of $\tilde{u}_{1,\epsilon}$ at 0.

We introduce the following vectorial L^2 norm

$$|||(\phi_1, \phi_2)|||^2_{L^2} = \|\chi_1\phi_1\|^2_{L^2(\Omega_{\epsilon_2})} + \|(1-\chi_1)\phi_1\|^2_{L^2(\Omega_{\epsilon_1})} + \|\phi_2\|^2_{L^2(\Omega_{\epsilon_2})}. \tag{7.5}$$

Similarly, for the H^2 norm we define

$$|||(\phi_1,\phi_2)|||_{H^2}^2 = \|\chi_1\phi_1\|_{H^2(\Omega_{\epsilon_2})}^2 + \|(1-\chi_1)\phi_1\|_{H^2(\Omega_{\epsilon_1})}^2 + \|\phi_2\|_{H^2(\Omega_{\epsilon_2})}^2. \tag{7.6}$$

Writing the system (3.1) as $R_\epsilon(S,u_1,u_2)=0$, we have now shown the estimate

$$|||R_\epsilon(T[\tilde{u}_{1,\epsilon}],\tilde{u}_{1,\epsilon},\tilde{u}_{2,\epsilon})|||_{L^2} = O(\epsilon). \tag{7.7}$$

(Note that the first equation is solved exactly and the first component does not enter in the definition of the norm.)

Next study the linearized operator $\tilde{\mathcal{L}}_\epsilon$ around the approximate solution $(\tilde{S}_\epsilon,\, \tilde{u}_{\epsilon,1},\, \tilde{u}_{\epsilon,2})$. It is defined as follows

$$\tilde{\mathcal{L}}_\epsilon : (H_N^2(\Omega))^3 \to (L^2(\Omega))^3, \quad \tilde{\mathcal{L}}_\epsilon \begin{pmatrix} \psi_\epsilon \\ \phi_{1,\epsilon} \\ \phi_{2,\epsilon} \end{pmatrix} = \tag{7.8}$$

$$= \begin{pmatrix} D\Delta\psi_\epsilon - 2\frac{a_1}{\epsilon_1}\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\phi_{1,\epsilon} - \frac{a_1}{\epsilon_1}\psi_\epsilon\tilde{u}_{1,\epsilon}^2 \\ \epsilon_1^2\Delta\phi_{1,\epsilon} - \phi_{1,\epsilon} + 2\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\phi_{1,\epsilon} + \psi_\epsilon\tilde{u}_{1,\epsilon}^2 - a_2\frac{\epsilon_1}{\epsilon_2}\phi_{1,\epsilon}\tilde{u}_{2,\epsilon}^2 - 2a_2\frac{\epsilon_1}{\epsilon_2}\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\phi_{2,\epsilon} \\ \epsilon_2^2\Delta\phi_{2,\epsilon} - \phi_{2,\epsilon} + \phi_{1,\epsilon}\tilde{u}_{2,\epsilon}^2 + 2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\phi_{2,\epsilon} \end{pmatrix}.$$

When discussing the kernel of $\tilde{\mathcal{L}}_\epsilon$ we first determine $\psi = T'[\phi_1]$ using the Green's function $G_{D,\tau\lambda_\epsilon}$ defined in (9.7). Therefore we study instead the following operator $\bar{\mathcal{L}}_\epsilon$ which is applied to the second and third components. Further, to have uniform invertibility we have to introduce suitable approximate kernel and co-kernel given by

$$\mathcal{K}_\epsilon = \mathrm{span}\{\tilde{u}_{2,\epsilon}'\} \subset H_N^2(\Omega_{\epsilon_2}),$$

$$\mathcal{C}_\epsilon = \mathrm{span}\{\tilde{u}_{2,\epsilon}'\} \subset L^2(\Omega_{\epsilon_2}).$$

Then the linear operator $\bar{\mathcal{L}}_\epsilon$ is defined by

$$\bar{\mathcal{L}}_\epsilon : H_N^2(\Omega_{\epsilon_1}) \oplus \mathcal{K}_\epsilon^\perp \to L^2(\Omega_{\epsilon_1}) \oplus \mathcal{C}_\epsilon^\perp, \tag{7.9}$$

$$\bar{\mathcal{L}}_\epsilon \begin{pmatrix} \phi_{1,\epsilon} \\ \phi_{2,\epsilon} \end{pmatrix} = \begin{pmatrix} \Delta_{y_1}\phi_{1,\epsilon} - \phi_{1,\epsilon} + 2\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\phi_{1,\epsilon} + T'[\phi_{1,\epsilon}]\tilde{u}_{1,\epsilon}^2 - a_2\frac{\epsilon_1}{\epsilon_2}\phi_{1,\epsilon}\tilde{u}_{2,\epsilon}^2 - 2a_2\frac{\epsilon_1}{\epsilon_2}\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\phi_{2,\epsilon} \\ \Delta_{y_2}\phi_{2,\epsilon} - \phi_{2,\epsilon} + \phi_{1,\epsilon}\tilde{u}_{2,\epsilon}^2 + 2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\phi_{2,\epsilon} \end{pmatrix}.$$

where $\perp$ means perpendicular in L^2 sense. Now we show that this operator is uniformly invertible for ϵ small enough. In fact, we have the following result:

Proposition 7.1. *There exist positive constants $\bar{\epsilon}$, λ such that for all $\epsilon \in (0,\bar{\epsilon})$,*

$$|||\bar{\mathcal{L}}_\epsilon(\phi_1,\phi_2)|||_{L_2} \ge \lambda|||(\phi_1,\phi_2)|||_{H^2} \quad \text{for all } (\phi_1,\phi_2) \in H_N^2(\Omega_{\epsilon_1}) \oplus \mathcal{K}_\epsilon^\perp. \tag{7.10}$$

Further, the linear operator $\bar{\mathcal{L}}_\epsilon$ is surjective with the norms introduced in (7.5), (7.6).

Proof of Proposition 7.1. We give an indirect proof. Suppose (7.10) is false. Using the notation $\Phi_\epsilon = \begin{pmatrix} \phi_{1,\epsilon} \\ \phi_{2,\epsilon} \end{pmatrix}$ and employing the norms denoted $|||\Phi|||$ and introduced in (7.5), (7.6), then there exist sequences $\{\epsilon_k\}$, $\{\Phi^k\}$ with $\epsilon_k \to 0$, $\Phi^k = \Phi_{\epsilon_k}$, $k = 1, 2, \ldots$ such that

$$|||\bar{\mathcal{L}}_{\epsilon_k}\Phi^k|||_{L^2} \to 0, \quad \text{as } k \to \infty, \tag{7.11}$$

$$|||\Phi^k|||_{H^2} = 1, \qquad k = 1,\, 2,\, \ldots\,. \tag{7.12}$$

Using the cutoff function defined in (3.3), we introduce the following functions:

$$\phi_{1a,\epsilon}(y_1) = \phi_{1,\epsilon}(y_1), \quad r_\epsilon \le |y_1| \le \frac{1}{\epsilon_1}. \tag{7.13}$$

$$\phi_{1b,\epsilon}(y_2) = \frac{\epsilon_1}{\epsilon_2}\left(\phi_{1,\epsilon}\left(\frac{\epsilon_2}{\epsilon_1}y_2\right) - \phi_{1,\epsilon}(0)\right), \quad |y_2| \le \frac{\epsilon_1}{\epsilon_2}r_\epsilon.$$

Because of (7.1), (7.2) we have

$$\phi_{1b,\epsilon}\left(\frac{\epsilon_1}{\epsilon_2}r_\epsilon\right) - \phi_{1a,\epsilon}(r_\epsilon) = 0 \tag{7.14}$$

and

$$\phi_{1b,\epsilon,y_2}\left(\frac{\epsilon_1}{\epsilon_2}r_\epsilon\right) - \phi_{1a,\epsilon,y_2}(r_\epsilon) = 0. \tag{7.15}$$

At first (after rescaling) the functions $\phi_{1a,\epsilon}$, $\phi_{1b,\epsilon}$, $\phi_{2,\epsilon}$ are only defined in for $r_\epsilon \le |y_1| \le \frac{1}{\epsilon_1}$, $|y_2| \le \frac{\epsilon_1}{\epsilon_2}r_\epsilon$ and $|y_2| \le \frac{1}{\epsilon_2}$, respectively. However, by a standard result, $\phi_{1a,\epsilon}$ can be extended to $\mathbb{R} \setminus \{0\}$ and $\phi_{1b,\epsilon}$, $\phi_{2,\epsilon}$ can be extended to $\mathbb{R}$ such that the norms of $\phi_{1a,\epsilon}$ in $H^2(\mathbb{R} \setminus \{0\})$, $\phi_{1b,\epsilon}$ locally in $H^2(\mathbb{R})$ (on any bounded domain) and $\phi_{2,\epsilon}$ in $H^2(\mathbb{R})$, respectively, are bounded by a constant independent of ϵ for all ϵ small enough. In the following we will study this extension. For simplicity of notation we keep the same notation for the extension. Since for $i = 1b, 2$ each sequence $\{\phi_i^k\} := \{\phi_{i,\epsilon_k}\}$ $(k = 1, 2, \ldots)$ is bounded in $H^2_{loc}(\mathbb{R})$ it has a weak limit in $H^2_{loc}(\mathbb{R})$, and therefore also a strong limit in $L^2_{loc}(\mathbb{R})$ and $L^\infty_{loc}(\mathbb{R})$. (For $i = 1a$, the same argument holds with $\mathbb{R}$ replaced by $\mathbb{R} \setminus \{0\}$.) Call these limits ϕ_i. Then, taking the limit $\epsilon \to 0$ in (7.9), we derive that $\Phi = \begin{pmatrix} \phi_{1a} \\ \phi_{1b} \\ \phi_2 \end{pmatrix}$ satisfies

$$\int_{\mathbb{R}} \phi_2 w_{y_2}\, dy_2 = 0$$

and solves the system

$$\mathcal{L}\Phi = 0, \tag{7.16}$$

where the operator $\mathcal{L}$ is defined as follows

$$\mathcal{L}\begin{pmatrix}\phi_{1a}\\ \phi_{1b}\\ \phi_2\end{pmatrix}=\begin{pmatrix}\Delta_{y_1}\phi_{1a}-\phi_{1a}+2w_{L_0}\phi_{1a}-\dfrac{2\int_0^\infty w_{L_0}\phi_1}{\int_0^\infty w_{L_0}^2}w_{L_0}^2, \quad y_1\neq 0\\ \phi_{1b}(y_2)-a_2\dfrac{\phi_{1b}(0)}{u_1^2(0)}\displaystyle\int_0^{y_2}\int_0^t w^2\,dt\,dy_2-2a_2\int_0^{y_2}\int_0^t w\phi_2\,dt\,dy_2\\ \Delta_{y_2}\phi_2-\phi_2+2w\phi_2+\dfrac{\phi_{1b}(0)}{(u_1(0))^2}w^2, \quad y_2\in\mathbb{R}\end{pmatrix}$$

with $w_{L_0}(|y|)=w(|y|-L_0)$.

The matching condition

$$\phi_{1b}'(\pm\infty)=\phi_{1a}'(\pm 0) \tag{7.17}$$

follows from (7.14) and it implies the boundary condition (5.6).

The matching condition

$$\phi_{1b}(\pm\infty)=\phi_{1a}(0) \tag{7.18}$$

follows from (7.15) and it determines $\phi_{1b}(0)$.

By Lemma 5.1, it follows for $L_0\neq L_0^m$ that $\Phi=0$.

By elliptic estimates we get $\|\phi_{i,\epsilon_k}\|_{H^2(\mathbb{R})}\to 0$ as $k\to\infty$. for $i=1b,\,2$ and $\|\phi_{1a,\epsilon_k}\|_{H^2(\mathbb{R}\setminus\{0\})}\to 0$ as $k\to\infty$.

This contradicts $|||\Phi^k|||_{H^2}=1$. To complete the proof of Proposition 7.1, we need to show that the adjoint operator to L_ϵ (denoted by L_ϵ^*) is injective from H^2 to L^2.

The limiting process as $\epsilon\to 0$ for the adjoint operator $\bar{\mathcal{L}}_\epsilon^*$ follows exactly along the same lines as the proof for $\bar{\mathcal{L}}_\epsilon$ and is therefore omitted. By Lemma 5.2, the limiting adjoint operator $\mathcal{L}^*$ has only the trivial kernel.

□

Finally, we solve the system (3.1), which we write as

$$R_\epsilon(\tilde{S}_\epsilon+\psi,\tilde{u}_{1,\epsilon}+\phi_1,\tilde{u}_{2,\epsilon}+\phi_2)=R_\epsilon(U_\epsilon+\Phi)=0, \tag{7.19}$$

using the notation $U_\epsilon=\left(\tilde{S}_\epsilon,\tilde{u}_{1,\epsilon},\tilde{u}_{2,\epsilon}\right)$. By Proposition 7.1, for ϵ small enough we can write (7.19) as follows:

$$\Phi=-\mathcal{L}_\epsilon^{-1}R_\epsilon(U_\epsilon)-\mathcal{L}_\epsilon^{-1}N_\epsilon(\Phi)=:M_\epsilon(\Phi), \tag{7.20}$$

where

$$N_\epsilon(\Phi)=R_\epsilon(U_\epsilon+\Phi)-R_\epsilon(U_\epsilon)-R_\epsilon'(U_\epsilon)\Phi \tag{7.21}$$

and the operator M_ϵ defined by (7.20) is a mapping from H^2 into itself. We are going to show that the operator M_ϵ is a contraction on

$$B_{\epsilon,\delta}\equiv\{\phi\in H^2\,:\,|||\phi|||_{H^2}<\delta\}$$

if δ and ϵ are suitably chosen. By (7.7) and Proposition 7.1 we have

$$\begin{aligned}|||M_\epsilon(\Phi)|||_{H^2}&\leq\lambda^{-1}\Big(|||N_\epsilon(\Phi)|||_{L^2}+|||R_\epsilon(U_\epsilon)|||_{L^2}\Big)\\&\leq\lambda^{-1}C(c(\delta)\delta+\epsilon),\end{aligned}$$

where $\lambda > 0$ is independent of $\delta > 0$, $\epsilon > 0$ and $c(\delta) \to 0$ as $\delta \to 0$. Similarly we show

$$|||M_\epsilon(\Phi_1) - M_\epsilon(\Phi_2)|||_{H^2} \leq \lambda^{-1} C(c(\delta)\delta)|||\Phi_1 - \Phi_2|||_{H^2},$$

where $c(\delta) \to 0$ as $\delta \to 0$. If we choose $\delta = c_3\epsilon$, then, for suitable $c_3 > 0$ and ϵ small enough, M_ϵ is a contraction on $B_{\epsilon,\delta}$. The existence of a fixed point Φ_ϵ now follows from the standard contraction mapping principle, and Φ_ϵ is a solution of (7.20).

We have thus proved

Lemma 7.1. *There exists $\bar{\epsilon} > 0$ such that for every ϵ with $0 < \epsilon < \bar{\epsilon}$ there is a unique $\Phi_\epsilon \in H^2$ satisfying $R_\epsilon(U_\epsilon + \Phi_\epsilon) = 0$. Furthermore, we have the estimate*

$$|||\Phi_\epsilon|||_{H^2} \leq C\epsilon. \tag{7.22}$$

This completes the proof of Theorem 3.1.

□

In this section we have constructed an exact cluster solution of the form $U_\epsilon + \Phi_\epsilon = (S_\epsilon, u_{\epsilon,1}, u_{\epsilon,2})$. In the next section we are going to study its stability.

8. Appendix B – Stability II: Computation of the small eigenvalues

Completion of the Proof of Theorem 3.2:

We compute the small eigenvalues of the eigenvalue problem (5.3), i.e. we determine the eigenvalues assuming that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. We will prove that they satisfy $\lambda_\epsilon = O(\epsilon_2^2)$. Let us define

$$\tilde{u}_{i,\epsilon}(x) = \chi(|x|)u_{i,\epsilon}(x). \tag{8.1}$$

Then it follows easily that $\tilde{u}_{i,\epsilon}$ in $H_N^2(\Omega_{\epsilon_i})$ and

$$u_{i,\epsilon}(x) = \tilde{u}_{i,\epsilon}(x) + \text{e.s.t.} \quad \text{in } H^2(\Omega_{\epsilon_i}). \tag{8.2}$$

Taking the derivative of the system (3.1) w.r.t. y_2, we compute

$$\begin{cases} \epsilon_1^2 \Delta \frac{\epsilon_2}{\epsilon_1}\tilde{u}'_{1,\epsilon} - \frac{\epsilon_2}{\epsilon_1}\tilde{u}'_{1,\epsilon} + 2S_\epsilon u_{1,\epsilon}\frac{\epsilon_2}{\epsilon_1}\tilde{u}'_{1,\epsilon} + \epsilon_2 S'_\epsilon u_{1,\epsilon}^2 - a_2\frac{\epsilon_1}{\epsilon_2}\frac{\epsilon_2}{\epsilon_1}\tilde{u}'_{1,\epsilon}u_{2,\epsilon}^2 - 2a_2\frac{\epsilon_1}{\epsilon_2}u_{1,\epsilon}u_{2,\epsilon}\tilde{u}'_{2,\epsilon} = \text{e.s.t.}, \\ \epsilon_2^2\Delta\tilde{u}'_{2,\epsilon} - \tilde{u}'_{2,\epsilon} + \frac{\epsilon_2}{\epsilon_1}\tilde{u}'_{1,\epsilon}u_{2,\epsilon}^2 + 2u_{1,\epsilon}u_{2,\epsilon}\tilde{u}'_{2,\epsilon} = \text{e.s.t.} \end{cases} \tag{8.3}$$

Let us now decompose the eigenfunction $(\psi_\epsilon,\ \phi_{1,\epsilon},\ \phi_{2,\epsilon})$ as follows:

$$\phi_{1,\epsilon} = a^\epsilon \frac{\epsilon_2}{\epsilon_1}\tilde{u}'_{1,\epsilon} + \bar{\phi}_{1,\epsilon} \tag{8.4}$$

where a^ϵ is a complex number to be determined and

$$\bar{\phi}_{1,\epsilon} \in H_N^2(\Omega_{\epsilon_1});$$

$$\phi_{2,\epsilon} = a^\epsilon\tilde{u}'_{2,\epsilon} + \bar{\phi}_{2,\epsilon} \sim \frac{a^\epsilon}{u_{1,\epsilon}(0)}w'_L\chi_0 + \bar{\phi}_{2,\epsilon}, \tag{8.5}$$

where

$$\bar{\phi}_{2,\epsilon} = \bar{\phi}_{2a,\epsilon} + \phi_{2,\epsilon}^\perp,$$

$$\bar{\phi}_{2a,\epsilon} \in H^2_N(\Omega_{\epsilon_2}), \quad \phi^{\perp}_{2,\epsilon} \perp \mathcal{K}_\epsilon = \text{span}\,\{\tilde{u}'_{2,\epsilon}\} \subset H^2_N(\Omega_{\epsilon_2}).$$

We decompose the eigenfunction ψ_ϵ as follows:

$$\psi_\epsilon = a^\epsilon \psi_{1,\epsilon} + \bar{\psi}_\epsilon,$$

where $\psi_{1,\epsilon}$ satisfies

$$\begin{cases} D\Delta\psi_{1,\epsilon} - \frac{a_1}{\epsilon_1}\psi_{1,\epsilon}u^2_{1,\epsilon} - 2\frac{a_1}{\epsilon_1}S_\epsilon u_{1,\epsilon}\frac{\epsilon_2}{\epsilon_1}\tilde{u}'_{1,\epsilon} = \tau\lambda_\epsilon\psi_{1,\epsilon}, \\ \psi'_{1,\epsilon}(\pm 1) = 0 \end{cases} \tag{8.6}$$

and $\bar{\psi}_\epsilon$ is given by

$$\begin{cases} D\Delta\bar{\psi}_\epsilon - \frac{a_1}{\epsilon_1}\bar{\psi}_\epsilon u^2_{1,\epsilon} - 2\frac{a_1}{\epsilon_1}S_\epsilon u_{1,\epsilon}\bar{\phi}_{1,\epsilon} = \tau\lambda_\epsilon\bar{\psi}_\epsilon, \\ (\bar{\psi}_\epsilon)'(\pm 1) = 0. \end{cases} \tag{8.7}$$

Note that ψ_ϵ can be uniquely expressed in terms of $\phi_{1,\epsilon}$ using the Green's function $G_{D,\tau\lambda_\epsilon}$ given in (9.7):

$$\psi_\epsilon = a^\epsilon\psi_{1,\epsilon} + \bar{\psi}_\epsilon = a^\epsilon T'_{\tau\lambda_\epsilon}[\frac{\epsilon_2}{\epsilon_1}\tilde{u}'_{1,\epsilon}] + T'_{\tau\lambda_\epsilon}[\bar{\phi}_{1,\epsilon}]. \tag{8.8}$$

The derivative S'_ϵ satisfies

$$D(S'_\epsilon)'' - \frac{a_1}{\epsilon_1}S'_\epsilon u^2_{1,\epsilon} - 2\frac{a_1}{\epsilon_1}S_\epsilon u_{1,\epsilon}\frac{1}{\epsilon_1}\tilde{u}'_{1,\epsilon} = \text{e.s.t.}, \quad S'_\epsilon \in H^2(-1,1),\ S'_\epsilon(-1) = S'_\epsilon(1) = 0.$$

Using the following Green's function for Dirichlet boundary conditions

$$DG''_p = \delta_z,\ x \in (-1,1), \quad G_p(-1) = G_p(1) = 0$$

which is given by

$$G_p(x,z) = \frac{1}{2D}|x-z| + \frac{1}{2D}(xz-1)$$

we compute S'_ϵ near zero. We get

$$\begin{aligned} &S'_\epsilon(\epsilon_1 y_1) - S'_\epsilon(0) \\ &= a_1\int_{-1/\epsilon_1}^{1/\epsilon_1}[G_p(\epsilon_1 y_1,\epsilon_1 z) - G_p(0,\epsilon_1 z)]\left(S'_\epsilon u^2_{1,\epsilon} + 2S_\epsilon u_{1,\epsilon}\frac{1}{\epsilon_1}\tilde{u}'_{1,\epsilon}\right)dz_1\,(1+O(\epsilon_1)) \\ &= a_1\int_{-1/\epsilon_1}^{1/\epsilon_1}\left[\frac{1}{2D}\epsilon_1(|y_1-z_1|-|z_1|) + \frac{1}{2D}(\epsilon_1^2 y_1 z_1)\right]\left(S'_\epsilon u^2_{1,\epsilon} + 2S_\epsilon u_{1,\epsilon}\frac{1}{\epsilon_1}\tilde{u}'_{1,\epsilon}\right)dz_1\,(1+O(\epsilon_1)) \\ &= \frac{a_1}{D}\left[\int_{-1/\epsilon_1}^{1/\epsilon_1}(|y_1-z_1|-|z_1|)S_\epsilon u_{1,\epsilon}\frac{1}{\epsilon_1}\tilde{u}'_{1,\epsilon}\,dz_1 + \frac{\epsilon_1}{S_\epsilon(0)}y_1\int_{-L_0-r_\epsilon}^{\infty}z_1 2ww'\,dz_1\right](1+O(\epsilon)) \\ &= \frac{a_1}{D}\left[\int_{-1/\epsilon_1}^{1/\epsilon_1}(|y_1-z_1|-|z_1|)S_\epsilon u_{1,\epsilon}\frac{1}{\epsilon}\tilde{u}'_{1,\epsilon}\,dz_1 - \frac{\epsilon_1}{S_\epsilon(0)}(1+O(r_\epsilon))y_1(6+2\rho(L_0))\right](1+O(\epsilon)). \end{aligned} \tag{8.9}$$

Here we have used that by symmetry

$$S'_\epsilon(0) = 0 \text{ and } S'_\epsilon(\epsilon_1 y_1) = \epsilon_1 y_1 S''_\epsilon(0) + O(\epsilon_1^2 y_1^2). \tag{8.10}$$

Similarly, we compute, using the Green's function $G_{D,\tau\lambda_\epsilon}$ (see (9.7)), that

$$\psi_{1,\epsilon}(\epsilon_1 y_1) - \psi_{1,\epsilon}(0)$$

$$= a_1 \int_{\Omega_{\epsilon_1}} [G_{D,\tau\lambda_\epsilon}(\epsilon_1 y_1, \epsilon_1 z_1) - G_{D,\tau\lambda_\epsilon}(0, \epsilon_1 z_1)] 2S_\epsilon(\epsilon_1 z_1) u_{1,\epsilon}(z_1) \frac{\epsilon_2}{\epsilon_1} \tilde{u}'_{1,\epsilon}(z_1)\, dz_1\, (1 + O(\epsilon_1 |y_1|))$$

$$= a_1 \left(1 + O\left(\epsilon \log \frac{1}{\epsilon}\right)\right) \left[\epsilon_2 \int_{-1/\epsilon_1}^{1/\epsilon_1} \frac{1}{D} (|y_1 - z_1| - |z_1|) S_\epsilon u_{1,\epsilon} \frac{1}{\epsilon_1} \tilde{u}'_{1,\epsilon}\, dz_1 \right.$$

$$\left. + \epsilon_1 \epsilon_2 \frac{2}{S_\epsilon(0)} \underbrace{\nabla_x \nabla_z H_D(x,z)|_{x=y=0}}_{=0} y_1 \int_{-L_0 - r_\epsilon}^{\infty} z_1 2 w w'\, dz_1 \right] (1 + O((\tau + \tau_1)|\lambda_\epsilon|)). \tag{8.11}$$

Note that from (8.6), we derive, using (9.7), that

$$\psi_{1,\epsilon}(0) = O(\epsilon + (\tau + \tau_1)|\lambda_\epsilon|). \tag{8.12}$$

Adding (8.9) and (8.11), we get

$$\frac{d}{dx} [S_\epsilon(\epsilon_1 y_1) - S_\epsilon(0)] - [\psi_\epsilon(\epsilon_1 y_1) - \psi_\epsilon(0)]$$

$$= -\frac{\epsilon_1}{D} \frac{a_1}{S_\epsilon(0)} y_1 (6 + 2\rho(L_0)) \left(1 + O\left(\epsilon \log \frac{1}{\epsilon} + (\tau + \tau_1)|\lambda_\epsilon|\right)\right). \tag{8.13}$$

Suppose that $(\phi_{1,\epsilon}, \phi_{2,\epsilon})$ satisfies $|||(\phi_{1,\epsilon}, \phi_{2,\epsilon})|||_{H^2} = 1$ for the norms $|||\cdot|||$ introduced in (7.5), (7.6). Then $|a^\epsilon| \le C$.

Substituting the decompositions of ψ_ϵ, $\phi_{1,\epsilon}$ and $\phi_{2,\epsilon}$ into (5.3) and subtracting (8.3), we have

$$\tau_1 \lambda_\epsilon \left(a^\epsilon \frac{\epsilon_2}{\epsilon_1} \tilde{u}'_{1,\epsilon} \right)$$

$$= a^\epsilon \tilde{u}^2_{1,\epsilon} \left(\psi_{1,\epsilon} - \epsilon_2 S'_\epsilon \right)$$

$$+ (\bar{\phi}_{1,\epsilon})'' - \bar{\phi}_{1,\epsilon} + 2 u_{1,\epsilon} S_\epsilon \bar{\phi}_{1,\epsilon} + u^2_{1,\epsilon} \bar{\psi}_\epsilon - \tau_1 \lambda_\epsilon \bar{\phi}_{1,\epsilon}$$

$$- a_2 \frac{\epsilon_1}{\epsilon_2} \bar{\phi}_{1,\epsilon} u^2_{2,\epsilon} - 2 a_2 \frac{\epsilon_1}{\epsilon_2} u_{1,\epsilon} u_{2,\epsilon} \bar{\phi}_{2,\epsilon} + \text{e.s.t.}$$

$$=: I_1 + I_2 + I_3. \tag{8.14}$$

Let us first compute, using (8.10), (8.12) and (8.13),

$$I_1 = a^\epsilon \tilde{u}^2_{1,\epsilon} \left(\psi_{1,\epsilon} - \epsilon_2 S'_\epsilon \right)$$

$$= \epsilon_1 \epsilon_2 a^\epsilon \frac{a_1}{D S_\epsilon(0)} y_1 u^2_{1,\epsilon} (6 + 2\rho(L_0)) \left(1 + O\left(\epsilon \log \frac{1}{\epsilon} + (\tau + \tau_1)|\lambda_\epsilon|\right)\right)$$

$$= c^\epsilon \epsilon_2^2 y_2 u^2_{1,\epsilon} \left(1 + O\left(\epsilon \log \frac{1}{\epsilon} + (\tau + \tau_1)|\lambda_\epsilon|\right)\right), \tag{8.15}$$

where

$$c^\epsilon = a^\epsilon \frac{a_1}{D S_\epsilon(0)} (6 + 2\rho(L_0)). \tag{8.16}$$

Since we will see that $|\lambda_\epsilon| = O(\epsilon_2^2)$, the second part $(\tau + \tau_1)|\lambda_\epsilon|$ in the error of (8.15) is dominated by its first part $\epsilon \log \frac{1}{\epsilon}$. The same remark applies for the rest of this proof. For brevity, the second part is omitted from now on.

Next we estimate I_2. To this end, we decompose $\bar{\phi}_{1,\epsilon}$ into three parts:

$$\bar{\phi}_{1,\epsilon} = -c^\epsilon \epsilon_1 \epsilon_2 y_1 \tilde{u}_{1,\epsilon} + a^\epsilon \left(\bar{\phi}_{1a,\epsilon} - \frac{\epsilon_2}{\epsilon_1} \tilde{u}'_{1,\epsilon} \right) + \bar{\phi}_{1b,\epsilon}.$$

We will show that $-c^\epsilon \epsilon_1 \epsilon_2 y_1 \tilde{u}_{1,\epsilon}$ is the leading part which gives the main contribution to the small eigenvalues. Further, $\bar{\phi}_{1a,\epsilon}$ and $\bar{\phi}_{1b,\epsilon}$ will be introduced and estimated below.

The leading part $-c^\epsilon \epsilon_1 \epsilon_2 y_1 \tilde{u}_{1,\epsilon}$ of $\bar{\phi}_{1,\epsilon}$ satisfies

$$(-c^\epsilon \epsilon_1 \epsilon_2 y_1 \tilde{u}_{1,\epsilon})'' - (-c^\epsilon \epsilon_1 \epsilon_2 y_1 \tilde{u}_{1,\epsilon}) + 2\tilde{u}_{1,\epsilon} S_\epsilon (-c^\epsilon \epsilon_1 \epsilon_2 y_1 \tilde{u}_{1,\epsilon})$$
$$+c^\epsilon \epsilon_1^2 \frac{\epsilon_2}{\epsilon_1} y_1 \tilde{u}_{1,\epsilon}^2 + 2c^\epsilon \epsilon_1 \epsilon_2 \tilde{u}'_{1,\epsilon} = O(\epsilon_1^2 \epsilon_2 y_1^2) \quad \text{in } L^2(\Omega_{\epsilon_1}).$$

Thus it introduces the extra term $2c^\epsilon \epsilon_1 \epsilon_2 \tilde{u}'_{1,\epsilon}$ into I_2.

We have to cancel this extra term by adding a correction to $\frac{\epsilon_2}{\epsilon_1} \tilde{u}'_{1,\epsilon}$. This is done as follows: Let $\bar{\phi}_{1a,\epsilon}$ for $|y_2| \le r_\epsilon \frac{\epsilon_1}{\epsilon_2}$ be defined by

$$(\bar{\phi}_{1a,\epsilon})_{y_2,y_2} - \frac{\epsilon_2^2}{\epsilon_1^2}(1+2c^\epsilon \epsilon_1^2)\bar{\phi}_{1a,\epsilon} + \frac{\epsilon_2^2}{\epsilon_1^2} 2u_{1,\epsilon} S_\epsilon \bar{\phi}_{1a,\epsilon} - a_2 \frac{\epsilon_2}{\epsilon_1}[2u_{1,\epsilon} u_{2,\epsilon} \tilde{u}'_{2,\epsilon} + \bar{\phi}_{1a,\epsilon} u_{2,\epsilon}^2] = o(\epsilon_2^2). \quad (8.17)$$

Note that $\frac{\epsilon_2}{\epsilon_1} \tilde{u}'_{1,\epsilon}$ solves the equation (8.17) except for the term $2c^\epsilon \epsilon_2^2 \bar{\phi}_{1a,\epsilon}$ (and higher order terms dominated by it). Comparing $\frac{\epsilon_2}{\epsilon_1} \tilde{u}'_{1,\epsilon}$ and $\bar{\phi}_{1a,\epsilon}$, we get

$$\left(\bar{\phi}_{1a,\epsilon} - \tilde{u}_{1,\epsilon,y_2}\right)_{y_2,y_2} - 2c^\epsilon \epsilon_2^2 \tilde{u}_{1,\epsilon,y_2} = o(\epsilon_2^2), \quad |y_2| \le r_\epsilon \frac{\epsilon_1}{\epsilon_2}. \quad (8.18)$$

Integrating the first equation of (8.3) w.r.t. y_2, we derive

$$|\tilde{u}_{1,\epsilon,y_2}| = O\left(\frac{\epsilon_2}{\epsilon_1}\right), \quad |y_2| \le r_\epsilon \frac{\epsilon_1}{\epsilon_2}.$$

Inserting into (8.18) and integrating, we get

$$\left|\bar{\phi}_{1a,\epsilon} - \tilde{u}_{1,\epsilon,y_2}\right| \le |c^\epsilon|\, O\left(\frac{\epsilon_2^3}{\epsilon_1} y_2^2\right), \quad |y_2| \le r_\epsilon \frac{\epsilon_1}{\epsilon_2}.$$

By Proposition 7.1 we derive the error estimate

$$\|\|(\bar{\phi}_{1b,\epsilon}, \phi_{2,\epsilon}^\perp)\|\|_{H^2} = o(\epsilon^2). \quad (8.19)$$

Using the estimate

$$|\bar{\psi}_\epsilon(0)| = O(\epsilon_1^2 \epsilon_2)$$

which follows from (8.7) and (9.7) since $\bar{\phi}_{1,\epsilon}$ is an odd function, we get

$$I_2 = o(\epsilon_2^2) \quad \text{in } L^2(\Omega_{\epsilon_1}).$$

Finally, from (8.19) we get

$$I_3 = o(\epsilon_2^2) \quad \text{in } L^2(\Omega_{\epsilon_2}).$$

Therefore we derive

$$(\bar{\phi}'_{1a,\epsilon}(0) - \tilde{u}_{1,\epsilon,y_2}(0)) \frac{\epsilon_2}{\epsilon_1} y_2 w^2 = o(\epsilon_2^2) c^\epsilon u_{1,\epsilon}(0) y_2 w^2,$$

$$\bar{\phi}'_{1b,\epsilon}(0) \frac{\epsilon_2}{\epsilon_1} y_2 w^2 = o(\epsilon_2^2) c^\epsilon u_{1,\epsilon}(0) y_2 w^2$$

and so

$$\bar{\phi}'_{1,\epsilon}(0)\frac{\epsilon_2}{\epsilon_1}y_2w^2 = \epsilon_2^2 c^\epsilon u_{1,\epsilon}(0)y_2w^2(1+o(1)). \tag{8.20}$$

Multiplying the second component of the eigenvalue problem (5.3) by w' and integrating, we get

$$\begin{aligned}
\text{l.h.s.} &= \epsilon_2^2 c^\epsilon u_{1,\epsilon}(0)\left(1+O\left(\epsilon\log\frac{1}{\epsilon}\right)\right)\int_{\mathbb{R}} y_2w^2w'\,dy_2 \\
&= -\epsilon_2^2 c^\epsilon u_{1,\epsilon}(0)\left(1+O\left(\epsilon\log\frac{1}{\epsilon}\right)\right)\int_{\mathbb{R}} \frac{w^3}{3}\,dy_2 \\
&= -2.4\epsilon_2^2 c^\epsilon u_{1,\epsilon}(0)\left(1+O\left(\epsilon\log\frac{1}{\epsilon}\right)\right) \\
&= -4.8\epsilon_2^2 a^\epsilon\frac{a_1}{DS_\epsilon(0)}u_{1,\epsilon}(0)(3+\rho(L_0))\left(1+O\left(\epsilon\log\frac{1}{\epsilon}\right)\right) \\
&= -4.8\epsilon_2^2 a^\epsilon\frac{a_1}{DS_\epsilon^2(0)}w(-L_0)(3+\rho(L_0))\left(1+o(1)\right)
\end{aligned}$$

and

$$\begin{aligned}
\text{r.h.s.} &= \lambda_\epsilon a^\epsilon\int_{\mathbb{R}}(w')^2\,dy_2\left(1+O\left(\epsilon\log\frac{1}{\epsilon}\right)\right) \\
&= 1.2a^\epsilon\lambda_\epsilon\left(1+o(1)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\lambda_\epsilon &= -4\epsilon_2^2\frac{a_1}{DS_\epsilon^2(0)}w(-L_0)(3+\rho(L_0))+o(\epsilon_2^2) \\
&= -12\epsilon_2^2\frac{a_1a_2}{D}\frac{(z+1)(2-z)}{z(1-z)}+o(\epsilon_2^2).
\end{aligned}$$

We summarize our result on the small eigenvalues in the following theorem.

Theorem 8.1. *The eigenvalues of (5.1) with $\lambda_\epsilon \to 0$ satisfy*

$$\lambda_\epsilon = -12\epsilon_2^2\frac{a_1a_2}{D}\frac{(z+1)(2-z)}{z(1-z)}+o(\epsilon_2^2), \tag{8.21}$$

where z has been introduced in (5.16). In particular these eigenvalues are stable.

This completes the proof of Theorem 3.2.

□

9. Appendix C: Two Green's functions

Let $G_D(x,z)$ be the Green's function of the Laplace operator with Neumann boundary conditions:

$$\begin{cases} DG_D''(x,z)+\frac{1}{2}-\delta_z=0 & \text{in } (-1,1), \\ \int_{-1}^1 G_D(x,z)\,dx=0, G_D'(-1,z)=G_D'(1,z)=0. \end{cases} \tag{9.1}$$

We can decompose $G_D(x,z)$ as follows

$$G_D(x,z)=\frac{1}{2D}|x-z|+H_D(x,z) \tag{9.2}$$

where H_D is the regular part of G_D.

Written explicitly, we have

$$G_D(x,z)=\begin{cases} \frac{1}{D}\left[\frac{1}{3}-\frac{(x+1)^2}{4}-\frac{(1-z)^2}{4}\right], & -1<x\le z, \\ \frac{1}{D}\left[\frac{1}{3}-\frac{(z+1)^2}{4}-\frac{(1-x)^2}{4}\right], & z\le x<1. \end{cases} \tag{9.3}$$

By simple computations,

$$H_D(x,z)=\frac{1}{2D}\left[-\frac{1}{3}-\frac{x^2}{2}-\frac{z^2}{2}\right]. \tag{9.4}$$

For $x\neq z$ we calculate

$$\nabla_x\nabla_z G_D(x,z)=0, \quad \nabla_x G_D(x,z)=\begin{cases} -\frac{x+1}{2D} & \text{if } x\le z \\ -\frac{x-1}{2D} & \text{if } z\le x. \end{cases} \tag{9.5}$$

We further have

$$\nabla_x G_D(x,z)|_{x=z}=\nabla_x H_D(x,z)|_{x=z}=-\frac{z}{2D} \tag{9.6}$$

(see [12]).

Similarly, let $G_{D,\tau\lambda_\epsilon}(x,z)$ be Green's function of

$$\begin{cases} DG_{D,\tau\lambda_\epsilon}''(x,z)-\tau\lambda_\epsilon G_{D,\tau\lambda_\epsilon}(x,z)-\delta_z=0 & \text{in } (-1,1), \\ G_{D,\tau\lambda_\epsilon}'(-1,z)=G_{D,\tau\lambda_\epsilon}'(1,z)=0. \end{cases} \tag{9.7}$$

We can decompose $G_{D,\tau\lambda_\epsilon}(x,z)$ as follows

$$G_{D,\tau\lambda_\epsilon}(x,z)=\frac{1}{2D}|x-z|+H_{D,\tau\lambda_\epsilon}(x,z) \tag{9.8}$$

where $H_{D,\tau\lambda_\epsilon}$ is the regular part of $G_{D,\tau\lambda_\epsilon}$.

An elementary computation shows that

$$|H_D(x,z)-H_{D,\tau\lambda}(x,z)|\le C|\tau\lambda_\epsilon|$$

uniformly for all $(x,z)\in\Omega\times\Omega$.

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