Semiclassical states for nonlinear Schrödinger equations with sign-changing potentials

Yanheng Ding[†)*] and Juncheng Wei[‡) †]

†) Institute of Mathematics, AMSS, Chinese Academy of Sciences
Beijing 100080, P. R. China. Email: dingyh@math.ac.cn

‡) Department of Mathematics, The Chinese University of Hong Kong
Shatin, Hong Kong. Email: wei@math.cuhk.edu.hk.

Abstract

We establish the existence and multiplicity of semiclassical bound states of the following nonlinear Schrödinger equation

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

where V *changes sign* and g is super linear with critical or supercritical growth as $|u| \to \infty$.

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1 Introduction and main results

We consider the existence and multiplicity of semiclassical bound states to the following nonlinear Schrödinger equation

$$(\mathcal{P}_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

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†Corresponding author. Supported by an Earmarked Grant from RGC of HK

where $0 < \varepsilon << 1$. We are interested in the case where the potential V *changes sign* and the nonlinearity g is super linear with *critical or supercritical* growth as $|u| \to \infty$. Here we say that V *changes sign* if $V(x_1) < 0 < V(x_2)$ for some $x_1, x_2 \in \mathbb{R}^N$; g *is super linear* if $|g(x,u)|/|u| \to \infty$ as $|u| \to \infty$; and g *is critical or supercritical* if $N \geq 3$ and $c_1|u|^{2^*-1} \leq |g(x,u)| \leq c_2|u|^{2^*-1}$ or only $c_1|u|^{2^*-1} \leq |g(x,u)|$ with $2^* = 2N/(N-2)$ for all large $|u|$. The motivation of such a study is two-fold. On one hand, it is expected that $(\mathcal{P}_\varepsilon)$ has solutions $u \in H^1(\mathbb{R}^N)$ provided, roughly speaking, $\liminf_{|x|\to\infty} V(x) > 0$ (whether or not it changes sign). It is known that by variational arguments the Dirichlet problem on smooth bounded domain $\Omega \subset \mathbb{R}^N$:

$$-\Delta u + V(x)u = |u|^{p-2}u, \;\; \text{in } \Omega, \;\; p \in (2, 2^*), \;\; u = 0 \;\; \text{on } \partial\Omega$$

always possesses solutions $u \in H_0^1(\Omega)$ without any restriction on the sign of $V(x)$. For the Schrödinger equation $(\mathcal{P}_\varepsilon)$, the condition $\liminf_{|x|\to\infty} V(x) > 0$ guarantees the embedding $\|u\|_{H^1}^2 \leq c_\varepsilon \int_{\mathbb{R}^N}(\varepsilon^2|\nabla u|^2 + V^+(x)u^2)$, $V^+(x) := \max\{0, V(x)\}$, hence the variational argument for Dirichlet problem should work. On the other hand, when V changes sign the energy functional associated to the equation is indefinite and consequently has no mountain-pass structure, which stimulates the development of new methods.

Problem $(\mathcal{P}_\varepsilon)$ arises in finding standing wave solutions of the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\varphi + W(x)\varphi - f(x, |\varphi|)\varphi. \tag{1.1}$$

A standing wave solution of (1.1) is a solution of the form $\varphi(x,t) = u(x)e^{-\frac{iEt}{\hbar}}$. Then $\varphi(x,t)$ solves (1.1) if and only if $u(x)$ solves $(\mathcal{P}_\varepsilon)$ with $V(x) = W(x) - E, \varepsilon^2 = \frac{\hbar^2}{2m}$ and $g(x,u) = f(x,|u|)u$.

Equation $(\mathcal{P}_\varepsilon)$ has being extensively investigated in the literatures based on various assumptions on the potential $V(x)$ and the nonlinearity $g(x,u)$. See for example [1, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25] and the references therein. We summarize the findings in the following three cases:

a) $\inf V > 0$ *and g is super linear and subcritical.* Most of the papers deal with this case. Floer and Weinstein in [15] considered $N = 1$, $g(u) = u^3$ and studied firstly the existence of single and multiple spike solutions based on a Lyapunov-Schmidt reductions. This result was extended in higher dimension and for $g(u) = |u|^{p-2}u$ in Oh [21, 22]. A mountain-pass reduction method has been subsequently applying to finding solutions of $(\mathcal{P}_\varepsilon)$. In [1] Ambrosetti, Badiale and Cingolani studied concentration phenomena of the solutions at isolated local minima and maxima of V with polynomial degeneracy. See also Grossi [16], Li [20] and Pistoia [23] for related results. In Kang and

Wei [19] the authors establish the existence of positive solutions with any prescribed number of spikes clustering around a given local maximum point of V. Without assumption of non-degeneracy on critical points of V, the existence of (positive) solutions was handled in del Pino and Felmer [12, 13] and Jeanjean and Tanaka [18]. For concentrations on higher dimensional sets, we refer to Ambrosetti, Malchiodi and Ni [2], and M. del Pino, M. Kowalczyk and Wei [14].

b) $\min V = 0$. Only a few papers investigated this case. Among the results, Byeon and Wang [6] considered the case with $g(u)$ independent of x, and Ding and Lin [9] handle the case with $g(x,u)$ is of critically growth.

c) $\inf V < 0$ *and* V *changes sign.* Recently, Ding and Szulkin [10] studied this case. They assumed that the potential V satisfies

(V_0) $V \in C(\mathbb{R}^N)$, and there exists $b > 0$ such that the set $\mathcal{V}_b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite measure,

and the nonlinearity $g(x,u)$ is subcritical (together with some technical conditions of course). They showed that there is a sequence $\varepsilon_k \to 0$ such that each $(\mathcal{P}_{\varepsilon_k})$ has at least one solution; and if additionally the set $V^{-1}(0)$ has nonempty interior then for all ε small sufficiently $(\mathcal{P}_\varepsilon)$ has at least one solution.

In this paper we study the case different from these *a)-c)* above. As mentioned at the beginning, we assume V changes sign, however, without the condition $\mathrm{int}V^{-1}(0) \neq \emptyset$. Moreover, we allow that the nonlinearity g grows critically or supercritically as $|u| \to \infty$.

Let $G(x,u) = \int_0^u g(x,s)\,ds$. We first consider the subcritical case, hence make the following assumptions:

(V_1) The set $\mathcal{V}_- := \{x \in \mathbb{R}^N : V(x) \leq 0\}$ is nonempty and bounded;

(G_0) $g_1)$ $g \in C(\mathbb{R}^N \times \mathbb{R})$ and $g(x,u) = o(|u|)$ uniformly in x as $u \to 0$;

$g_2)$ there are $c_0 > 0$ and $q < 2^*$ such that $|g(x,u)| \leq c_0\,(1 + |u|^{q-1})$ for all (x,u);

$g_3)$ there are $a_0 > 0, \mu \geq p > 2$ such that $G(x,u) \geq a_0|u|^p$ and $\mu G(x,u) \leq g(x,u)u$ for all (x,u).

For a solution u_ε of $(\mathcal{P}_\varepsilon)$ we denote its energy by

$$E(u_\varepsilon) := \int_{\mathbb{R}^N} \left(\frac{1}{2}\left(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)|u_\varepsilon|^2\right) - G(x,u_\varepsilon)\right) dx.$$

Theorem 1.1. *Let* (V_0), (V_1) *and* (G_0) *be satisfied.*

(1) *For any* $\sigma > 0$ *there is* $\mathcal{E}_\sigma > 0$ *such that* $(\mathcal{P}_\varepsilon)$ *has at least one solution* u_ε *with* $0 < E(u_\varepsilon) \le \sigma\varepsilon^N$ *whenever* $\varepsilon \le \mathcal{E}_\sigma$.

(2) *Assuming additionally that* $g(x,u)$ *is odd in* u, *for any* $m \in \mathbb{N}$ *and* $\sigma > 0$ *there is* $\mathcal{E}_{m\sigma} > 0$ *such that* $(\mathcal{P}_\varepsilon)$ *has at least* m *pairs of solutions* u_ε *with* $0 < E(u_\varepsilon) \le \sigma\varepsilon^N$ *whenever* $\varepsilon \le \mathcal{E}_{m\sigma}$.

We next deal with the critical case which will be re-stated for distinction as the following (as considered by Ding and Lin [9])

$$(\mathcal{Q}_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u = g(x,u) + K(x)|u|^{2^*-2}u & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty\,. \end{cases}$$

Assume that the function $K(x)$ is continuous and bounded:

(K_0) $K \in C(\mathbb{R}^N)$ with $0 < \inf K \le \sup K < \infty$.

We denote similarly the energy of a solution u_ε of $(\mathcal{Q}_\varepsilon)$ by

$$E_*(u_\varepsilon) := \int_{\mathbb{R}^N} \left(\frac{1}{2}\left(\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)|u_\varepsilon|^2\right) - G(x,u_\varepsilon) - \frac{1}{2^*}K(x)|u_\varepsilon|^{2^*} \right) dx.$$

Then we have the following corollary

Theorem 1.2. *Let* (V_0), (V_1), (G_0) *and* (K_0) *be satisfied. Then the conclusions (1) and (2) of Theorem* 1.1 *both remain true with* $(\mathcal{P}_\varepsilon)$ *replaced by* $(\mathcal{Q}_\varepsilon)$ *and* $E(u_\varepsilon)$ *by* $E_*(u_\varepsilon)$.

If the numbers μ and p in $g_3)$ are equal then the restriction assumption $g_2)$ on g at infinity can be removed. More precisely, assume g satisfies

(H_0) $h_1)$ $g \in C(\mathbb{R}^N \times \mathbb{R})$ and $g(x,u) = o(|u|)$ uniformly in x as $u \to 0$;

$h_2)$ There is $p \in (2, 2^*)$ and $a_0 > 0$ such that $a_0|u|^p \le pG(x,u) \le g(x,u)u$ for all (x,u).

Then we have the following result:

Theorem 1.3. *Let* (V_0), (V_1) *and* (H_0) *be satisfied. All the conclusions of Theorem* 1.1 *remain true.*

We note that condition $h_2)$ simply implies that there exists $p \in (2, 2^*)$ such that

$$(1.2) \qquad a_0|u|^p \le G(x,u) \le a_1|u|^p \text{ for } |u| \le 1, \quad G(x,u) \ge a_0|u|^p \text{ for } |u| \ge 1.$$

Thus $g(x,u)$ can grow supercritically as $|u| \to +\infty$. As examples one may take the following problems:

$$(i)_\varepsilon \qquad -\varepsilon^2\Delta u + V(x)u = au + |u|^{p-2}u$$

and

$$(ii)_\varepsilon \qquad -\varepsilon^2\Delta u + V(x)u = au + |u|^{p-2}u + |u|^{q-2}u,$$

where V satisfies (V_0), $p \in (2, 2^*), p < q$ and $a \in \mathbb{R}$. Defining according to (V_0) the number $b_{\max} := \sup\{b > 0 : \ |\mathcal{V}_b| < \infty\}$, one sees from Theorems 1.1 and 1.3 that, fixed arbitrarily $\inf V \le a < b_{\max}$, for any $m \in \mathbb{N}$, there is $\mathcal{E}_m > 0$ such that, if $\varepsilon \le \mathcal{E}_m$ then $(i)_\varepsilon$, respectively $(ii)_\varepsilon$, has m pairs of solutions u_ε with $\|u_\varepsilon\|_{H^1} \le c\varepsilon^{N-2}$.

We point out that the boundedness assumption (V_1) can be replaced by a slightly general geometric condition, see Remark 2.7.

Our argument is variational. To outline it we consider the subcritical case. Observe that defining $v(x) = u(\varepsilon x)$ the equation $(\mathcal{P}_\varepsilon)$ is equivalent to

$$-\Delta v + V(\varepsilon x)v = g(\varepsilon x, v). \tag{1.3}$$

If $a = \inf V \ge 0$ and $g(x,u) = g(u)$ is independent of x, (1.3) possesses the limiting equation

$$-\Delta v + av = g(v)$$

which has well-known nice properties, for example, it possesses, up to a translation, a radially symmetric solution, and such a solution is exponentially decreasing as $|x| \to \infty$. This helps one to study firstly a cut-off equation with energy functional satisfying the mountain-pass structure and the Palais-Smale condition to obtain solutions of the cut-off problem, then, by virtue of the properties of the limiting equation as well as an elliptic estimate, to show that, for $\varepsilon > 0$ small enough, the solution is in fact a solution of the original equation (1.3). However, this process does not go in our present situation because V changes sign and $g(x,u)$ depends on x. Therefore, we will consider another equivalent problem as in [9, 10]:

$$-\Delta u + \lambda V(x)u = \lambda g(x,u) \tag{1.4}$$

with $\lambda = \varepsilon^{-2} \to \infty$. The relative functional can be normally written as

$$\Phi_\lambda(u) = \frac{1}{2}\left(\|u^+\|_\lambda^2 - \|u^-\|_\lambda^2\right) - \lambda \int_{\mathbb{R}^N} G(x,u)$$

defined on the Hilbert space $E = E_\lambda^- \oplus E_\lambda^0 \oplus E_\lambda^+$ with $u = u^- + u^0 + u^+$. We verify that the functional Φ_λ possesses the linking geometry and satisfies the $(PS)_c$ condition for $c < \alpha_0 \lambda^{1-\frac{N}{2}}$ with $\alpha_0 > 0$ fixed independent of λ. We then decompose the space $E = F_\lambda^- \oplus F_\lambda^0 \oplus F_\lambda^+$ so as to get certain minimax level $\sigma\lambda^{1-\frac{N}{2}}$ with $0 < \sigma < \alpha_0$ for all λ sufficiently large, say $\lambda \geq \Lambda_\sigma$. Now the standard linking theorem applies and yields the desired solutions.

2 Variational setting and preliminaries

Setting $\lambda = \varepsilon^{-2}$, $(\mathcal{P}_\varepsilon)$ is equivalent to the following problem

$$(\mathcal{P}_\lambda) \qquad \begin{cases} -\Delta u + \lambda V(x) u = \lambda g(x,u) & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

and $(\mathcal{Q}_\varepsilon)$ is equivalent to

$$(\mathcal{Q}_\lambda) \qquad \begin{cases} -\Delta u + \lambda V(x) u = \lambda g(x,u) + \lambda K(x)|u|^{2^*-2}u & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

for $\lambda \to \infty$. We are going to prove the following result:

Theorem 2.1. *Let (V_0), (V_1) and (G_0) be satisfied.*

(1) *For any $\sigma > 0$ there is $\Lambda_\sigma > 0$ such that $(\mathcal{P}_\lambda)$ has at least one solution u_λ for each $\lambda \geq \Lambda_\sigma$ satisfying $0 < \hat{E}(u_\lambda) \leq \sigma\lambda^{1-\frac{N}{2}}$ where*

$$\hat{E}(u_\lambda) := \int_{\mathbb{R}^N} \left(\frac{1}{2}\Big(|\nabla u_\lambda|^2 + \lambda V(x)|u_\lambda|^2\Big) - \lambda G(x, u_\lambda) \right) dx.$$

(2) *Assuming additionally that $g(x,u)$ is odd in u, for any $m \in \mathbb{N}$ and $\sigma > 0$ there is $\Lambda_{m\sigma} > 0$ such that $(\mathcal{P}_\lambda)$ has at least m pairs of solutions u_λ with $0 < \hat{E}(u_\lambda) \leq \sigma\lambda^{1-\frac{N}{2}}$ whenever $\lambda \geq \Lambda_{m\sigma}$.*

Theorem 2.2. *Let (V_0), (V_1), (G_0) and (K_0) be satisfied. Then the conclusions (1) and (2) both remain true with $(\mathcal{P}_\lambda)$ replaced by $(\mathcal{Q}_\lambda)$ and $\hat{E}(u_\lambda)$ by $\hat{E}_*(u_\lambda) := \hat{E}(u_\lambda) - \frac{\lambda}{2^*}\int_{\mathbb{R}^N} K(x)|u_\lambda|^{2^*}$.*

Theorem 2.3. *Let (V_0), (V_1) and (H_0) be satisfied. Then the conclusions (1) and (2) of Theorem 2.1 both remain true for $(\mathcal{P}_\lambda)$.*

Denote $V^\pm(x) := \max\{\pm V(x), 0\}$ and let E denote the Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V^+(x) u^2 < \infty \right\}$$

equipped with the inner product

$$(u,v)_E := \int_{\mathbb{R}^N} (\nabla u \nabla v + V^+(x)uv)$$

and the associated norm $\|u\|_E^2 = (u,u)_E$. By (V_0), E embeds continuously in $H^1(\mathbb{R}^N)$. On E we define the bilinear form

$$a_\lambda(u,v) := \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V(x)uv)$$

with the associated quadrature denoted by $a_\lambda(u) = a_\lambda(u,u)$.

Consider the functionals

$$\begin{aligned} \Phi_\lambda(u) &:= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x)u^2) - \lambda \int_{\mathbb{R}^N} G(x,u) \\ &= \frac{1}{2} a_\lambda(u) - \lambda \int_{\mathbb{R}^N} G(x,u). \end{aligned} \tag{2.1}$$

and

$$\Psi_\lambda(u) := \frac{1}{2} a_\lambda(u) - \lambda \int_{\mathbb{R}^N} \left(G(x,u) + \frac{1}{2^*} K(x)|u|^{2^*} \right). \tag{2.2}$$

Then Φ_λ and $\Psi_\lambda \in \mathcal{C}^1(E,\mathbb{R})$, and critical points of Φ_λ (resp. Ψ_λ) are solutions of $(\mathcal{P}_\lambda)$ (resp. $(\mathcal{Q}_\lambda)$).

For convenience we will use some direct sum decompositions of E described below. Throughout by $(\cdot,\cdot)_{L^2}$ we denote the usual L^2-inner product, and $|\cdot|_s$ the usual L^s-norm.

2.1 $E = E_\lambda^- \oplus E_\lambda^0 \oplus E_\lambda^+$ and $E = E_\lambda^d \oplus E_\lambda^e$

Let $A_\lambda := -\Delta + \lambda V$ denote the selfadjoint operator in $L^2(\mathbb{R}^N)$. By $\sigma(A_\lambda)$, $\sigma_e(A_\lambda)$ and $\sigma_d(A_\lambda)$ we denote the spectrum, the essential spectrum and the eigenvalues of A_λ below $\lambda_e := \inf \sigma_e(A_\lambda)$, respectively. Note that each $\mu \in \sigma_d(A_\lambda)$ is of finite multiplicity. Moreover, it is possible that $\sigma(A_\lambda) = \sigma_d(A_\lambda)$, for example, it occurs when $V(x) \to \infty$ as $|x| \to \infty$. If this is the case we set $\lambda_e = \infty$.

Lemma 2.4. *Suppose (V_0) holds. Then $\lambda_e \geq \lambda b$.*

Proof. Set $W_\lambda(x) = \lambda(V(x) - b)$, $W_\lambda^\pm = \max\{\pm W_\lambda, 0\}$ and $D_\lambda = -\Delta + \lambda b + W_\lambda^+$. By (V_0), the multiplicity operator W_λ^- is compact relative to D_λ, hence $\sigma_e(A_\lambda) \subset \sigma_e(D_\lambda) \subset [\lambda b, \infty)$. □

Remark 2.5. If $\lambda_e = \infty$, then a standard argument shows that E embeds compactly into L^2 hence L^p for all $p \in [2, 2^*)$ (cf. [26]). In this case the proofs of Theorems 2.1 and 2.2 are simpler (in fact one can prove the existence and multiplicity results for any $\lambda > 0$). If $\lambda_e < \infty$ and A_λ has infinitely many eigenvalues below λ_e, then λ_e is the unique cluster of $\sigma_d(A_\lambda)$. Taking into account this possibility, we fix in the following a number b' closing b with

$$0 < b' < b .$$

Let k_λ^-, k_λ^0 and k_λ^+ be the numbers of the negative, zero and positive eigenvalues which $\le \lambda b'$. We write $\eta_{\lambda i}^{\#}$ and $h_{\lambda i}^{\#}\, (1 \le i \le k_\lambda^{\#})$, $\# \in \{-, 0, +\}$, for the eigenvalues and eigenfunctions. Setting

$$L_\lambda^- := \text{span}\{h_{\lambda 1}^-, \cdots, h_{\lambda k_\lambda^-}^-\} \quad \text{and} \quad L_\lambda^0 := \text{span}\{h_{\lambda 1}^0, \cdots, h_{\lambda k_\lambda^0}^0\},$$

we have the orthogonal decomposition

$$L^2 = L_\lambda^- \oplus L_\lambda^0 \oplus L_\lambda^+, \quad u = u^- + u^0 + u^+.$$

On E we introduce another inner product

$$(u, v)_\lambda := (|A_\lambda|^{1/2} u,\ |A_\lambda|^{1/2} v)_{L^2} + (u^0, v^0)_{L^2}$$

and norm $\|\cdot\|_\lambda := (\cdot, \cdot)_\lambda^{1/2}$. It is easy to see that $\|\cdot\|_E$ and $\|\cdot\|_\lambda$ are equivalent for each $\lambda > 0$. There holds the decomposition:

$$E = E_\lambda^- \oplus E_\lambda^0 \oplus E_\lambda^+ \quad \text{with } E_\lambda^{\#} = E_\lambda \cap L_\lambda^{\#}, \tag{2.3}$$

orthogonal with respect to both the inner product $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)_\lambda$. With the decomposition (2.3) one has

$$a_\lambda(u) = \|u^+\|_\lambda^2 - \|u^-\|_\lambda^2 \ \text{ for all } u \in E. \tag{2.4}$$

Set

$$L_\lambda^d = L_\lambda^- \oplus L_\lambda^0 \oplus \text{span}\{h_{\lambda 1}^+, \cdots, h_{\lambda k_\lambda^+}^+\}.$$

We will also use the following orthogonal decomposition

$$L^2 = L_\lambda^d \oplus L_\lambda^e, \quad u = u^d + u^e.$$

Correspondingly, one has

$$E = E_\lambda^d \oplus E_\lambda^e \quad \text{with } E_\lambda^d = L_\lambda^d \text{ and } E_\lambda^e = L_\lambda^e \cap E \tag{2.5}$$

orthogonal with respect to $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)_\lambda$. Remark that

$$\lambda b'|u|_2^2 \le \|u\|_\lambda^2 \quad \text{for all } u \in E_\lambda^e \tag{2.6}$$

Let S denote the best Sobolev constant if $N \ge 3$:

$$S|u|_{2^*}^2 \le \int_{\mathbb{R}^N} |\nabla u|^2.$$

From now on we always assume that $N \ge 3$ (the proofs of the main results for $N = 1, 2$ are similar).

Lemma 2.6. *For each $s \in [2, 2^*]$, there is $c_s > 0$ such that*

$$c_s\, \lambda^{\frac{2^*-s}{2^*-2}} |u|_s^s \le \|u\|_\lambda^s \quad \textit{for all } u \in E_\lambda^e.$$

Proof. Observe that $\|u\|_\lambda^2 = a_\lambda(u)$ for all $u \in E_\lambda^e$. By (2.6) one sees that

$$\begin{aligned} S|u|_{2^*}^2 &\le \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda V|u|^2 - \lambda \int_{\mathbb{R}^N} V|u|^2 \\ &\le \|u\|_\lambda^2 + \frac{-\inf V}{b'} \lambda b'|u|_2^2 \\ &\le \left(1 + \frac{-\inf V}{b'}\right) \|u\|_\lambda^2, \end{aligned}$$

hence

$$c_{2^*}|u|_{2^*}^{2^*} \le \|u\|_\lambda^{2^*} \quad \text{for all } u \in E_\lambda^e.$$

For $s \in (2, 2^*)$, by the Hölder inequality and (2.6),

$$\begin{aligned} |u|_s^s &\le \left(\int_{\mathbb{R}^N} |u|^2\right)^{\frac{2^*-s}{2^*-2}} \left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{\frac{s-2}{2^*-2}} \\ &\le \left((b'\lambda)^{-1}\|u\|_\lambda^2\right)^{\frac{2^*-s}{2^*-2}} \left(c_{2^*}^{-1}\|u\|_\lambda^{2^*}\right)^{\frac{s-2}{2^*-2}} \end{aligned}$$

for all $u \in E_\lambda^e$, one gets the desired conclusion. □

2.2 $E = F_\lambda^- \oplus F_\lambda^0 \oplus F_\lambda^+$ and $F_\lambda^+ = X_\lambda \oplus Y$

Let $\mathcal{V}_+ := \mathbb{R}^N \setminus \mathcal{V}_-$. Observe that the assumption (V_1) implies that

$(\tilde{V}_1)$ There are $x_0 \in \mathbb{R}^N$ and $\ell_0 \in \mathbb{R}^N$ with $V(x_0) = 0$ and $\{x + x_0 : x \in \mathbb{R}^N,\ \ell_0 \cdot x \ge 0\} \subset \{x : x \in \mathbb{R}^N, V(x) \ge 0\}$.

This means that there is a hyperplane $\mathcal{L} = \{x \in \mathbb{R}^N : \ell_0(x) = 0\} + x_0$ with $V(x_0) = 0$ such that $\mathcal{L}^+ := \{x + x_0 : \ell_0(x) \geq 0\} \subset \overline{\mathcal{V}}_+$. For example, let $x_0 \in \mathbb{R}^N$ be such that $|x_0| = \max\{|x| : x \in \partial\mathcal{V}_-\}$ (recalling that $\mathcal{V}_-$ is bounded). Then the tangent plane of ∂B_r at x_0 is a choice of such a hyperplane. *From now on we assume without loss of generality that $x_0 = 0$, that is,*

$$\mathcal{L}^+ = \{x \in \mathbb{R}^N : \ell_0(x) \geq 0\} \subset \overline{\mathcal{V}}_+. \tag{2.7}$$

Indeed, otherwise, $\tilde{V}(\cdot) = V(\cdot + x_0)$, and $\tilde{g}(\cdot, u) = g(\cdot + x_0, u)$ satisfy the same assumptions on V and g with additionally $\tilde{V}(0) = 0$. If $\tilde{u}(x)$ is a solution of $-\varepsilon^2\Delta\tilde{u} + \tilde{V}(x)\tilde{u} = \tilde{g}(x, \tilde{u})$ then $u(x) = \tilde{u}(x - x_0)$ is a solution of $(\mathcal{P}_\varepsilon)$. Likely, setting $\tilde{K}(\cdot) = K(\cdot + x_0)$, if $\tilde{u}$ solves $-\varepsilon^2\Delta\tilde{u} + \tilde{V}(x)\tilde{u} = \tilde{g}(x, \tilde{u}) + \tilde{K}(x)|\tilde{u}|^{2^*-2}\tilde{u}$ then $u(x) = \tilde{u}(x - x_0)$ solves $(\mathcal{Q}_\varepsilon)$.

Remark 2.7. We mention that throughout the paper the boundedness assumption (V_1) on $\mathcal{V}_-$ is used only for checking $(\tilde{V}_1)$. Thus the main results (Theorems 1.1 and 1.2) keep true if (V_1) is replaced by $(\tilde{V}_1)$.

Let $A_{\lambda+} := -\Delta + \lambda V^+$, a positive selfadjoint operator in L^2, with domain $\mathcal{D}_\lambda := \mathcal{D}(A_{\lambda+}) \subset L^2$. $\mathcal{D}_\lambda$ is a Hilbert space equipped with the inner product $(u, v)_{\mathcal{D}_\lambda} = (A_{\lambda+}u, A_{\lambda+}v)_{L^2}$. $A_{\lambda+}$ has a bounded inverse $A_{\lambda+}^{-1} : L^2 \to \mathcal{D}_\lambda$. On E we introduce the inner product

$$\langle u, v\rangle_\lambda := (A_{\lambda+}^{1/2}u, A_{\lambda+}^{1/2}v)_{L^2} = \int_{\mathbb{R}^N} \nabla u\nabla v + \lambda V^+(x)uv$$

with the associated norm $\|u\|_\lambda = \langle u, u\rangle_\lambda^{1/2}$. It is clear that $\|\cdot\|_\lambda$ is equivalent to $\|\cdot\|_E$. Let $\hat{E}_\lambda$ denote the Hilbert space of E with norm $\|\cdot\|_\lambda$. Define the map $B_\lambda := \lambda A_{\lambda+}^{-1}V^- : \hat{E}_\lambda \to \hat{E}_\lambda$ by

$$\hat{E}_\lambda \xrightarrow{\lambda V^-} L^2 \xrightarrow{A_{\lambda+}^{-1}} \mathcal{D}_\lambda \xrightarrow{\iota} \hat{E}_\lambda$$

(where ι is the embedding map). Obviously B_λ is non-negative. Since the measure $|\mathcal{V}_-| < \infty$, it is easy to see that B_λ is compact and $\mathcal{C}_0^\infty(\mathcal{V}_+) \subset \ker(B_\lambda)$. Therefore, $\sigma(B_\lambda)$ consists of 0 and positive eigenvalues denoted by $\nu_{\lambda 1} \geq \nu_{\lambda 2} \geq \cdots$ with $\nu_{\lambda j} \to 0$ as $j \to \infty$. Let $(f_{\lambda j})_{j\in\mathbb{N}}$ denote the associated eigenfunctions: $B_\lambda f_{\lambda j} = \nu_{\lambda j} f_{\lambda j}$, and set $\mu_{\lambda j} = \nu_{\lambda j}^{-1}$. Then $\mu_{\lambda j}$ and $f_{\lambda j}$ satisfy

$$-\Delta f_{\lambda j} + \lambda V^+ f_{\lambda j} = \mu_{\lambda j}\lambda V^- f_{\lambda j}.$$

There exists also a minimax characterization of such eigenvalues, see [10]. We have the decomposition

$$E = F_\lambda^- \oplus F_\lambda^0 \oplus F_\lambda^+, \quad w = w^- + w^0 + w^+ \tag{2.8}$$

orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_\lambda$ and the bilinear form $a_\lambda(\cdot,\cdot)$, where

$$F_\lambda^- := \mathrm{span}(\{f_{\lambda j} : \mu_{\lambda j} < 1\}) \quad \text{and} \quad F_\lambda^0 := \mathrm{span}(\{f_{\lambda j} : \mu_{\lambda j} = 1\}).$$

Remark that, for each $\lambda > 0$, the spaces F_λ^- and F_λ^0 are finite dimensional. Observe also that one has the $\langle\cdot,\cdot\rangle_\lambda$ orthogonal decomposition

$$F_\lambda^+ = X_\lambda \oplus Y_\lambda \quad \text{where} \quad Y_\lambda = \ker(B_\lambda).$$

Therefore, we have the $\langle\cdot,\cdot\rangle_\lambda$ orthogonal decomposition

$$E = Z_\lambda \oplus Y_\lambda \quad \text{where} \quad Z_\lambda = F_\lambda^- \oplus F_\lambda^0 \oplus X_\lambda. \tag{2.9}$$

Clearly, Z_λ is the closure of $\mathrm{span}\{f_{\lambda j} : j \in \mathbb{N}\}$ with respect to the norm $\|\cdot\|_\lambda$. Note that for $u \in \ker(B_\lambda)$ one has $A_{\lambda+}^{-1} V^- u = 0$, and since $A_{\lambda+}^{-1}$ is injective, $V^- u$ is the zero of L^2 which implies $u(x) = 0$ a.e. in $\mathcal{V}_-$. Thus, as a linear space,

$$Y_\lambda = Y = \{u \in E : u(x) = 0 \text{ a.e. in } \mathcal{V}_-\} \quad \text{is independent of } \lambda. \tag{2.10}$$

It is clear that $\{f_{\lambda j}\}_{j\in\mathbb{N}}$ is a base of Z_λ. Moreover, since $\mu_{\lambda j} \to \infty$ as $j \to \infty$, it is easy to check that Z_λ embeds compactly into L^s for any $s \in [2, 2^*)$.

For any $s > 1$ let $M_{\lambda s}$ be the closure of Z_λ in L^s. Then $\{f_{\lambda j}\}$ is a base of $M_{\lambda s}$ and each element $u \in M_{\lambda s}$ has the representation $u = \sum_j c_j f_{\lambda j}$ in L^s. Let N_s be the closure of Y in L^s. Then $M_{\lambda s} \cap N_s = \{0\}$. Indeed, let $u \in M_{\lambda s} \cap N_s$ with $u = \sum_j c_j f_{\lambda j}$. Then, since $u \in N_s$,

$$u(x) = \sum_j c_j f_{\lambda j}(x) = 0 \quad \text{for } x \in \mathcal{V}_-.$$

One has, for every $k \in \mathbb{N}$,

$$0 = \lambda\mu_{\lambda k} V^-(x) f_{\lambda k} u(x) = \sum_j c_j f_{\lambda j} \lambda\mu_{\lambda k} V^-(x) f_{\lambda k},$$

hence

$$\begin{aligned} 0 &= \sum_j \int_{\mathbb{R}^N} c_j f_{\lambda j} \lambda\mu_{\lambda k} V^-(x) f_{\lambda k} \\ &= c_k (A_\lambda f_{\lambda k},\ f_{\lambda k})_2 \\ &= c_k \lambda\mu_{\lambda k} \int_{\mathbb{R}^N} V^-(x)|f_{\lambda k}|^2 \\ &= c_k \lambda\mu_{\lambda k} \end{aligned}$$

This implies $c_k = 0$, hence $u(x) \equiv 0$. In addition, since $\mathcal{C}_0^\infty \subset E \subset L^s$ with dense embeddings and $E = Z_\lambda \oplus Y$, one has $L^s = M_{\lambda s} \oplus N_s$. Let P denote the projection onto N_s along $M_{\lambda s}$. Since N_s is independent of λ (see (2.10)), there is $c_s > 0$ such that $|Pu|_s \le c_s|u|_s$ for all $u \in L^s$. In particular, for any $u = u_1 + u_2 \in Z_\lambda \oplus Y$,

$$|u_2|_s \le c_s|u|_s\,. \tag{2.11}$$

Letting $\mu_\lambda^- := \max\{\mu_{\lambda j} :\ \mu_{\lambda j} < 1\}$, $\mu_\lambda^+ := \min\{\mu_{\lambda j} :\ \mu_{\lambda j} > 1\}$ and

$$a_\lambda^- = \frac{1}{\mu_\lambda^-} - 1, \quad a_\lambda^+ = 1 - \frac{1}{\mu_\lambda^+},$$

one has

$$a_\lambda(u) \begin{cases} \le\ -a_\lambda^-\|u\|_\lambda^2 & \text{for all } u \in F_\lambda^-, \\ =\ 0 & \text{for all } u \in F_\lambda^0, \\ \ge\ a_\lambda^+\|u\|_\lambda^2 & \text{for all } u \in X_\lambda, \\ =\ \|u\|_\lambda^2 & \text{for all } u \in Y. \end{cases} \tag{2.12}$$

and

$$a_\lambda(u) = a_\lambda(w^+) + a_\lambda(w^-) \tag{2.13}$$

for all $u = w^- + w^0 + w^+ \in F_\lambda^- \oplus F_\lambda^0 \oplus F_\lambda^+$.

3 The subcritical case: (PS)-conditions

In this and the next sections we deal with the subcritical case.

Invoking the decomposition (2.3), Φ_λ has the representation

$$\Phi_\lambda(u) = \frac{1}{2}\left(\|u^+\|_\lambda^2 - \|u^-\|_\lambda^2\right) - \lambda\int_{\mathbb{R}^N} G(x,u) \ \text{ for } u = u^- + u^0 + u^+.$$

Set

$$\mathcal{G}(x,u) = \frac{1}{2}g(x,u)u - G(x,u).$$

The assumptions (G_0) implies that

$$\mathcal{G}(x,u) \ge \frac{\mu-2}{2\mu}g(x,u)u \ge \frac{\mu-2}{2}G(x,u)$$

and moreover, for any $\delta > 0$, there exist $\rho_\delta > 0$ and $c_\delta > 0$ such that

$$\frac{g(x,u)}{u} \le \delta \text{ if } |u| \le \rho_\delta \quad \text{and} \quad \left(\frac{g(x,u)}{u}\right)^\tau \le c_\delta g(x,u)u \text{ if } |u| \ge \rho_\delta \tag{3.1}$$

where $\tau = q/(q-2)$.

Recall that a sequence $(u_m) \subset E$ is a Palais-Smale sequence at level $c \in \mathbb{R}$ ($(PS)_c$ sequence for short) for Φ_λ if it satisfies $\Phi_\lambda(u_m) \to c$ and $\Phi'_\lambda(u_m) \to 0$. Φ_λ is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence has a convergent subsequence.

Lemma 3.1. *Assume that* (V_0) *and* (G_0) *are satisfied. Then any* $(PS)_c$*-sequence for* Φ_λ *is bounded.*

Proof. Let $(u_m) \subset E$ be a $(PS)_c$-sequence:

$$\Phi_\lambda(u_m) \to c \quad \text{and} \quad \Phi'_\lambda(u_m) \to 0.$$

By assumptions $g_2)$ and $g_3)$ (a type of Ambrosetti-Rabinowitz conditions) and the embedding property of E into L^s for $s \in [2, 2^*]$, a stand argument shows that (u_m) is bounded. The details are hence omitted. □

By the above lemma we may assume without loss of generality $u_m \rightharpoonup u$ in E, $u_m \to u$ in L^s_{loc} for $1 \le s < 2^*$, and $u_m(x) \to u(x)$ a.e. for $x \in \mathbb{R}^N$. Clearly u is a critical point of Φ_λ.

Lemma 3.2. *Let* $s \in [2, 2^*)$*. There is a subsequence* (u_{m_j}) *such that for each* $\varepsilon > 0$*, there exists* $r_\varepsilon > 0$ *with*

$$\limsup_{j\to\infty} \int_{B_j \setminus B_r} |u_{m_j}|^s \le \varepsilon$$

for all $r \ge r_\varepsilon$*, where* $B_j := \{x \in \mathbb{R}^N : |x| \le j\}$.

Proof. See [9]. For each $j \in \mathbb{N}$, $\int_{B_j} |u_m|^s \to \int_{B_j} |u|^s$ as $m \to \infty$. There exists $\hat{m}_j \in \mathbb{N}$ such that

$$\int_{B_j} (|u_m|^s - |u|^s) < \frac{1}{j} \quad \text{for all } m = \hat{m}_j + i,\ i = 1, 2, 3, \ldots.$$

Without loss of generality we can assume $\hat{m}_{j+1} \ge \hat{m}_j$. In particular, for $m_j = \hat{m}_j + j$ we have

$$\int_{B_j} \left(|u_{m_j}|^s - |u|^s\right) < \frac{1}{j}\,.$$

Observe that there is r_ε satisfying

$$\int_{\mathbb{R}^N \setminus B_r} |u|^s < \varepsilon \tag{3.2}$$

for all $r \geq r_\varepsilon$. Since

$$\int_{B_j\setminus B_r} |u_{m_j}|^s = \int_{B_j} \left(|u_{m_j}|^s - |u|^s\right) + \int_{B_j\setminus B_r} |u|^s + \int_{B_r} \left(|u|^s - |u_{m_j}|^s\right)$$
$$\leq \frac{1}{j} + \int_{\mathbb{R}^N\setminus B_r} |u|^s + \int_{B_r} \left(|u|^s - |u_{m_j}|^s\right),$$

the lemma follows. □

Recall that, by g_1) and g_2), $|g(x,u)| \leq c_1(|u| + |u|^{q-1})$ for all (x,u). Let firstly $(u_{m_j})_{j\in\mathbb{N}}$ be a subsequence of $(u_m)_{m\in\mathbb{N}}$ such that Lemma 3.2 holds for $s = 2$. Repeating the argument we can then find a subsequence $(u_{m_{j_i}})_{i\in\mathbb{N}}$ of $(u_{m_j})_{j\in\mathbb{N}}$ such that Lemma 3.2 holds for $s = q$. Therefore, for notational convenience, we assume in the following that Lemma 3.2 holds for both $s = 2$ and $s = q$ with the same subsequence.

Let $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \leq 1$, $\eta(t) = 0$ if $t \geq 2$. Define $\tilde{u}_j(x) = \eta(2|x|/j)u(x)$. Clearly,

$$\|u - \tilde{u}_j\|_E \to 0 \quad \text{as } j \to \infty. \tag{3.3}$$

Lemma 3.3. *We have*

$$\sup_{\|\varphi\|\leq 1} \left| \int_{\mathbb{R}^N} \big(g(x,u_{m_j}) - g(x,u_{m_j} - \tilde{u}_j) - g(x,u)\big)\,\varphi \right| \to 0.$$

Proof. Note that (3.3) and the local compactness of Sobolev embedding imply that, for any $r > 0$,

$$\lim_{j\to\infty} \left| \int_{B_r} \big(g(x,u_{m_j}) - g(x,u_{m_j} - \tilde{u}_j) - g(x,\tilde{u}_j)\big)\,\varphi \right| = 0$$

uniformly in $\|\varphi\| \leq 1$. For any $\varepsilon > 0$ it follows from (3.2) and (3.3) that

$$\limsup_{j\to\infty} \int_{B_j\setminus B_r} |\tilde{u}_j|^s \leq \int_{\mathbb{R}^N\setminus B_r} |u|^s \leq \varepsilon$$

for all $r \geq r_\varepsilon$. Using Lemma 3.2 for $s = 2, q$ we get

$$\begin{aligned}
&\limsup_{j\to\infty} \left| \int_{\mathbb{R}^N} \left(g(x, u_{m_j}) - g(x, u_{m_j} - \tilde{u}_j) - g(x, \tilde{u}_j)\right) \varphi \right| \\
&= \limsup_{j\to\infty} \left| \int_{B_j \setminus B_r} \left(g(x, u_{m_j}) - g(x, u_{m_j} - \tilde{u}_j) - g(x, \tilde{u}_j)\right) \varphi \right| \\
&\leq c_1 \limsup_{j\to\infty} \int_{B_j \setminus B_r} \left(|u_{m_j}| + |\tilde{u}_j|\right) |\varphi| \\
&\quad + c_2 \limsup_{j\to\infty} \int_{B_j \setminus B_r} \left(|u_{m_j}|^{q-1} + |\tilde{u}_j|^{q-1}\right) |\varphi| \\
&\leq c_1 \limsup_{j\to\infty} \left(|u_{m_j}|_{L^2(B_j\setminus B_r)} + |\tilde{u}_j|_{L^2(B_j\setminus B_r)}\right) |\varphi|_2 \\
&\quad + c_2 \limsup_{j\to\infty} \left(|u_{m_j}|^{q-1}_{L^q(B_j\setminus B_r)} + |\tilde{u}_j|^{q-1}_{L^q(B_j\setminus B_r)}\right) |\varphi|_p \\
&\leq c_3 \varepsilon^{1/2} + c_4 \varepsilon^{(q-1)/q},
\end{aligned}$$

which implies the conclusion. □

Lemma 3.4. *We have:*

1) $\Phi_\lambda(u_{m_j} - \tilde{u}_j) \to c - \Phi_\lambda(u)$;

2) $\Phi'_\lambda(u_{m_j} - \tilde{u}_j) \to 0$.

Proof. One has

$$\begin{aligned}
\Phi_\lambda(u_{m_j} - \tilde{u}_j) &= \Phi_\lambda(u_{m_j}) - \Phi_\lambda(\tilde{u}_j) \\
&\quad + \lambda \int_{\mathbb{R}^N} \left(G(x, u_{m_j}) - G(x, u_{m_j} - \tilde{u}_j) - G(x, \tilde{u}_j)\right).
\end{aligned}$$

It follows from (3.3) that $\Phi_\lambda(\tilde{u}_j) \to \Phi_\lambda(u)$. Using (3.3) again and following an argument (likely for the Brézis-Lieb lemma, see e.g. [27]), it is not difficult to check that

$$\int_{\mathbb{R}^N} \left(G(x, u_{m_j}) - G(x, u_{m_j} - \tilde{u}_j) - G(x, \tilde{u}_j)\right) \to 0.$$

We thus get 1).

To verify 2), observe that, for any $\varphi \in E$,

$$\begin{aligned}
\Phi'_\lambda(u_{m_j} - \tilde{u}_j)\varphi &= \Phi'_\lambda(u_{m_j})\varphi - \Phi'_\lambda(\tilde{u}_j)\varphi \\
&\quad + \lambda \int_{\mathbb{R}^N} \Big(g(x, u_{m_j}) - g(x, u_{m_j} - \tilde{u}_j) - g(x, \tilde{u}_j)\Big)\varphi.
\end{aligned}$$

By Lemma 3.3 we get

$$\lim_{m\to\infty}\int_{\mathbb{R}^N}\Big(g(x,u_{m_j})-g(x,u_{m_j}-\tilde u_j)-g(x,\tilde u_j)\Big)\varphi=0$$

uniformly in $\|\varphi\|\le 1$, proving 2). □

In the following we will utilize the decomposition (2.5): $E=E_\lambda^d\oplus E_\lambda^e$. Recall that $\dim(E_\lambda^d)<\infty$. Write

$$u_{m_j}^1:=u_{m_j}-\tilde u_j=v_j^d+v_j^e$$

Then $v_j^d=(u_{m_j}^d-u^d)+(u^d-\tilde u_j^d)\to 0$ and, by Lemma 3.4, $\Phi_\lambda(u_{m_j}^1)\to c-\Phi_\lambda(u)$, $\Phi_\lambda'(u_{m_j}^1)\to 0$. It follows from

$$\Phi_\lambda(u_{m_j}^1)-\frac12\Phi_\lambda'(u_{m_j}^1)u_{m_j}^1=\lambda\int_{\mathbb{R}^N}\mathcal{G}(x,u_{m_j}^1)$$

that

$$\lambda\int_{\mathbb{R}^N}\mathcal{G}(x,u_{m_j}^1)\to c-\Phi_\lambda(u).$$

Using (g_3),

$$\Phi_\lambda(u_{m_j}^1)-\frac1\mu\Phi_\lambda'(u_{m_j}^1)u_{m_j}^1\ge\left(\frac12-\frac1\mu\right)\|v_j^e\|_\lambda^2+o(1),$$

hence

$$\limsup_{j\to\infty}\|v_j^e\|_\lambda^2\le\frac{2\mu(c-\Phi_\lambda(u))}{\mu-2}\tag{3.4}$$

Moreover, by (3.1) and the Hölder inequality,

$$\begin{aligned}\|v_j^e\|_\lambda^2+o(1)&=\lambda\int_{\mathbb{R}^N}g(x,u_{m_j}^1)u_{m_j}^1=\lambda\int_{\mathbb{R}^N}\frac{g(x,u_{m_j}^1)}{u_{m_j}^1}|u_{m_j}^1|^2\\&\le\lambda\delta|u_{m_j}^1|_2^2+\lambda c_\delta\left(\int_{|u_{m_j}^1|\ge\rho_\delta}\left(\frac{g(x,u_{m_j}^1)}{u_{m_j}^1}\right)^\tau\right)^{1/\tau}|u_{m_j}^1|_{2\tau'}^2\\&\le\lambda\delta|u_{m_j}^1|_2^2+\lambda c_\delta\left(\frac{c-\Phi_\lambda(u)+o(1)}{\lambda}\right)^{1/\tau}|u_{m_j}^1|_{2\tau'}^2\end{aligned}\tag{3.5}$$

where $\tau'=\tau/(\tau-1)=q/2$. Remark that $u_{m_j}-u=u_{m_j}^1+(\tilde u_j-u)$, hence it follows from (3.3) that

$$u_{m_j}-u\to 0\ \text{ if and only if }\ u_{m_j}^1\to 0.$$

Lemma 3.5. *Under the assumptions of Lemma* 3.1, *there is* $\alpha_0 > 0$ *independent of* λ *such that, for any* $(PS)_c$*-sequence* (u_m) *for* Φ_λ *with* $u_m \rightharpoonup u$, *either* $u_m \to u$ *along a subsequence or*

$$\alpha_0 \lambda^{1-\frac{N}{2}} \le c - \Phi_\lambda(u).$$

Proof. Assume (u_m) has no convergent subsequences. Then using the above notations $\liminf_{j\to\infty} \|u^1_{m_j}\|_\lambda > 0$. Since $v^d_j \to 0$, we may assume $|u^1_j|_2^2 \le 2|v^e_j|_2^2$ and $|u^1_j|^2_{2\tau'} \le 2|v^e_j|^2_{2\tau'}$. Choosing $\delta = b'/4$ (see (2.6)), it follows from (3.5) and Lemma 2.6 that

$$\begin{aligned}\|v^e_j\|^2_\lambda + o(1) &\le \frac{\lambda b'}{2}|v^e_j|^2_2 + \lambda 2c_\delta \left(\frac{c - \Phi_\lambda(u) + o(1)}{\lambda}\right)^{1/\tau} |v^e_j|^2_{2\tau'} \\ &\le \frac{1}{2}\|v^e_j\|^2_\lambda + c_1 \lambda^{1-\frac{1}{\tau}-\frac{2^*-2\tau'}{(2^*-2)\tau'}} (c - \Phi_\lambda(u) + o(1))^{1/\tau} \|v^e_j\|^2_\lambda\end{aligned}$$

hence,

$$\|v^e_j\|^2_\lambda + o(1) \le 2c_1 \lambda^{1-\frac{1}{\tau}-\frac{2^*-2\tau'}{(2^*-2)\tau'}} (c - \Phi_\lambda(u))^{1/\tau} \|v^e_j\|^2_\lambda$$

or equivalently

$$1 + o(1) \le 2c_1 \lambda^{1-\frac{1}{\tau}-\frac{2^*-2\tau'}{(2^*-2)\tau'}} (c - \Phi_\lambda(u))^{1/\tau}.$$

We thus get

$$\alpha_0 \lambda^{1-\frac{N}{2}} \le c - \Phi_\lambda(u)$$

with $\alpha_0 > 0$ independent of λ, proving the lemma. □

As a consequence, we obtain the following

Lemma 3.6. *Under the assumptions of Lemma* 3.1, Φ_λ *satisfies the* $(PS)_c$ *condition for all* $c < \alpha_0 \lambda^{1-\frac{N}{2}}$.

4 The subcritical case: proof of Theorem 2.1

We use the decomposition of (2.8), thus the functional has the form

$$\Phi_\lambda(u) = \frac{1}{2} a_\lambda(u^+) + \frac{1}{2} a_\lambda(u^-) - \lambda \int_{\mathbb{R}^N} G(x, u)$$

for $u = u^- + u^0 + u^+ \in F^-_\lambda \oplus F^0_\lambda \oplus F^+_\lambda$, see (2.12) and (2.13). The following two lemmas are standard, which imply that Φ_λ possesses the linking structure.

Lemma 4.1. *Assume* (V_0), (V_1) *and* (G_0) *are satisfied. There exist* α_λ, $\rho_\lambda > 0$ *such that* $\Phi_\lambda(u) > 0$ *if* $u \in B^+_{\rho_\lambda} \setminus \{0\}$ *and* $\Phi_\lambda(u) \geq \alpha_\lambda$ *if* $u \in \partial B^+_{\rho_\lambda}$, *where* $B^+_{\rho_\lambda} = \{u \in F^+_\lambda : \|u\|_\lambda \leq \rho_\lambda\}$.

Lemma 4.2. *Under the assumptions of Lemma* 4.1, *for any* $\lambda > 0$ *and any finite-dimensional subspace* $F \subset E$, *there is* $R = R(\lambda, F) > 0$ *such that* $\Phi_\lambda(u) \leq 0$ *for all* $u \in F$ *with* $\|u\|_\lambda \geq R$.

By virtue of the above lemmas, if Φ_λ satisfies the $(PS)_c$ for all $c > 0$, then Theorem 2.1 follows from standard critical point theory. Unfortunately, in general, Φ_λ does not satisfy $(PS)_c$ for all $c > 0$. However, by Lemma 3.6, for λ large enough and c_λ small sufficiently, Φ_λ satisfies $(PS)_{c_\lambda}$. Thus, in the following we will find special finite-dimensional subspaces by which we construct sufficiently small minimax levels.

In order to construct such levels we let $-\Delta_D$ denote the unique selfadjoint operator on $\mathcal{V}_+$ associated to the form $(u, v)_V := \int_{\mathbb{R}^N} \nabla u \nabla v$, with the form domain

$$H^1_D(\mathcal{V}_+) := \left\{ u \in H^1(\mathbb{R}^N) : \; u(x) = 0 \text{ for a.e. } x \in \mathbb{R}^N \setminus \mathcal{V}_+ \right\}.$$

Note that when $\partial\mathcal{V}_+$ is smooth $H^1_D(\mathcal{V}_+)$ coincides with $H^1_0(\mathcal{V}_+)$, and $-\Delta_D = -\Delta$ with Dirichlet 0-boundary condition on $\partial\mathcal{V}_+$.

Let $\{P^V_\gamma : \gamma \geq 0\}$ denote the spectrum family of $-\Delta_D$ and $F^V_\gamma = P^V_\gamma L^2$.

Proposition 4.3. $\dim F^V_\gamma = \infty$ *for any* $\gamma > 0$.

Proof. It suffices to show that $0 \in \sigma_e(-\Delta_D)$. Indeed, since $\sigma(-\Delta) = \sigma_c(-\Delta) = [0, \infty)$, there is a sequence $\varphi_j \in \mathcal{C}^\infty_0(\mathbb{R}^N)$ with $|\varphi_j|_2 = 1$ and $|\Delta\varphi_j|_2 \to 0$. Since $\mathcal{V}_-$ is bounded, for each j, we can choose $x_j \in \mathbb{R}^N$ so that $\tilde{\varphi}_j := \varphi_j(\cdot + x_j) \in \mathcal{C}^\infty_0(\mathcal{V}_+)$. Then $|\tilde{\varphi}_j|_2 = 1$ and $|\nabla\tilde{\varphi}_j|_2 = |\nabla\varphi_j|_2 \to 0$. Thus $0 \in \sigma_e(-\Delta_D)$. □

Proposition 4.4. *For any* $s \in [2, 2^*)$, *there is a sequence* $\varphi_j \in \mathcal{C}^\infty_0(\mathcal{V}_+)$ *with* $|\varphi_j|_s = 1$ *and* $|\nabla\varphi_j|_2 \to 0$.

Proof. It is known that

$$\inf \left\{ \int_{\mathbb{R}^N} |\nabla\psi|^2 : \; \psi \in \mathcal{C}^\infty_0(\mathbb{R}^N), \; |\psi|_s = 1 \right\} = 0.$$

Since $\mathcal{V}_-$ is bounded, for each j, we can choose $x_j \in \mathbb{R}^N$ so that $\varphi_j := \psi_j(\cdot + x_j) \in \mathcal{C}^\infty_0(\mathcal{V}_+)$. Then $|\varphi_j|_s = |\psi_j|_s = 1$ and $|\nabla\varphi_j|_s = |\nabla\psi_j|_s \to 0$. □

Define

$$\hat{F}_\lambda := F_\lambda^- \oplus F_\lambda^0 \oplus Y.$$

By g_3) and (2.11), for $u = u^- + u^0 + u^+ \in \hat{F}_\lambda$,

$$\int_{\mathbb{R}^N} G(x,u) \geq c_0|u|_p^p \geq c_0|u^+|_p^p. \tag{4.1}$$

Hence, for $u = u^- + u^0 + u^+ \in \hat{F}_\lambda$

$$\begin{aligned}
\Phi_\lambda(u) &= \frac{1}{2}a_\lambda(u) - \lambda\int_{\mathbb{R}^N} G(x,u) \\
&= \frac{1}{2}a_\lambda(u^+) - \frac{1}{2}a_\lambda(u^-) - \lambda\int_{\mathbb{R}^N} G(x,u) \\
&\leq \frac{1}{2}a_\lambda(u^+) - \frac{1}{2}a_\lambda^-\|u^-\|_{\lambda+}^2 - \lambda c_0|u^+|_p^p \\
&= J_\lambda(u^+) - \frac{1}{2}a_\lambda^-\|u^-\|_{\lambda+}^2
\end{aligned}$$

where $J_\lambda \in \mathcal{C}^1(E,\mathbb{R})$ defined by

$$J_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + \lambda V(x)v^2\right) - \lambda c_0\int_{\mathbb{R}^N}|v|^p.$$

For any $\delta > 0$ one can choose $\varphi_\delta \in \mathcal{C}_0^\infty(\mathcal{V}_+)$ with $|\varphi_\delta|_p = 1$ and $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0) \cap \mathcal{L}^+$ (see (2.7)) so that $|\nabla\varphi_\delta|_2^2 < \delta$. Set

$$e_\lambda(x) := \varphi_\delta(\lambda^{1/2}x). \tag{4.2}$$

Then

$$\operatorname{supp} e_\lambda \subset B_{\lambda^{-1/2}r_\delta}(0) \cap \mathcal{L}^+,$$

hence $e_\lambda \in Y$. Remark that for $t \geq 0$,

$$\begin{aligned}
J_\lambda(te_\lambda) &= \frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla e_\lambda|^2 + \lambda V(x)|e_\lambda|^2 - \lambda c_0 t^p\int_{\mathbb{R}^N}|e_\lambda|^p \\
&= \lambda^{1-\frac{N}{2}}\left(\frac{t^2}{2}\int_{\mathbb{R}^N}|\nabla\varphi_\delta|^2 + V\left(\lambda^{-1/2}x\right)|\varphi_\delta|^2 - c_0t^p\int_{\mathbb{R}^N}|\varphi_\delta|^p\right) \\
&= \lambda^{1-\frac{N}{2}}I_\lambda(t\varphi_\delta)
\end{aligned}$$

where $I_\lambda \in \mathcal{C}^1(E,\mathbb{R})$ defined by

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2 + V\left(\lambda^{-1/2}x\right)|u|^2 - c_0\int_{\mathbb{R}^N}|u|^p.$$

Plainly,

$$\max_{t\geq 0} I_\lambda(t\varphi_\delta) = \frac{p-2}{2p(pc_0)^{2/(p-2)}}\left(\int_{\mathbb{R}^N}|\nabla\varphi_\delta|^2 + V(\lambda^{-1/2}x)|\varphi_\delta|^2\right)^{p/(p-2)}.$$

Since $V(0)=0$ and note that $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0)$, there is $\hat{\Lambda}_\delta > 0$ such that

$$V(\lambda^{-1/2}x) \leq \frac{\delta}{|\varphi_\delta|_2^2} \quad \text{for all } |x|\leq r_\delta \text{ and } \lambda \geq \hat{\Lambda}_\delta.$$

This implies that

$$\max_{t\geq 0} I_\lambda(t\varphi_\delta) \leq \frac{p-2}{2p(pc_0)^{2/(p-2)}}(2\delta)^{p/(p-2)}. \tag{4.3}$$

Therefore, for all $\lambda \geq \hat{\Lambda}_\delta$,

$$\max_{t\geq 0} \Phi_\lambda(t\varphi_\delta) \leq \frac{p-2}{2p(pc_0)^{2/(p-2)}}(2\delta)^{p/(p-2)}\lambda^{1-\frac{N}{2}}. \tag{4.4}$$

Therefore, we have

Lemma 4.5. *Under the assumptions of Lemma 4.1, for any $\sigma > 0$ there exists $\Lambda_\sigma > 0$, such that, for each $\lambda \geq \Lambda_\sigma$, there is $\bar{e}_\lambda \in E$ with $\|\bar{e}_\lambda\|_\lambda > \rho_\lambda$ and*

$$\max_{u\in F_\lambda} \Phi_\lambda(u) \leq \sigma\,\lambda^{1-\frac{N}{2}},$$

where ρ_λ is from Lemma 4.1 and $F_\lambda = F_\lambda^- \oplus F_\lambda^0 \oplus \mathbb{R}\bar{e}_\lambda$.

Proof. Choose $\delta > 0$ so small that

$$\frac{p-2}{2p(pc_0)^{2/(p-2)}}(2\delta)^{p/(p-2)} \leq \sigma,$$

and let $e_\lambda \in E$ be the function defined by (4.2). Take $\Lambda_\sigma = \hat{\Lambda}_\delta$. Let $\bar{t}_\lambda > 0$ be such that $\bar{t}_\lambda\|e_\lambda\|_\lambda > \rho_\lambda$ and $\Phi_\lambda(te_\lambda) \leq 0$ for all $t \geq \bar{t}_\lambda$. Then by (4.4), $\bar{e}_\lambda := \bar{t}_\lambda e_\lambda$ satisfies the requirements. □

In general, for any $m \in \mathbb{N}$, one can choose m functions $\varphi_\delta^j \in \mathcal{C}_0^\infty(\mathcal{V}_+)$ such that $\operatorname{supp}\varphi_\delta^i \cap \operatorname{supp}\varphi_\delta^k = \emptyset$ if $i \neq k$, $|\varphi_\delta^j|_p = 1$ and $|\nabla\varphi_\delta^j|_2^2 < \delta$. Let $r_\delta^m > 0$ be such that $\operatorname{supp}\varphi_\delta^j \subset B_{r_\delta^m}(0)$ for $j = 1, ..., m$. Set

$$e_\lambda^j(x) = \varphi_\delta^j(\lambda^{1/2}x) \quad \text{for } j = 1, ..., m$$

and

$$H_{\lambda\delta}^m = \operatorname{span}\{e_\lambda^1, ..., e_\lambda^m\}.$$

Observe that for each $u = \sum_{j=1}^m c_j e_\lambda^j \in H_{\lambda\delta}^m$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \sum_{j=1}^m |c_j|^2 \int_{\mathbb{R}^N} |\nabla e_\lambda^j|^2,$$

$$\int_{\mathbb{R}^N} V(x)|u|^2 = \sum_{j=1}^m |c_j|^2 \int_{\mathbb{R}^N} V(x)|e_\lambda^j|^2$$

and

$$\int_{\mathbb{R}^N} G(x,u) = \sum_{j=1}^m \int_{\mathbb{R}^N} G(x, c_j e_\lambda^j).$$

Hence

$$\Phi_\lambda(u) = \sum_{j=1}^m \Phi_\lambda(c_j e_\lambda^j)$$

and as before

$$\Phi_\lambda(c_j e_\lambda^j) \le \lambda^{1-\frac{N}{2}} I_\lambda(|c_j| e_\lambda^j).$$

Set

$$\beta_\delta := \max\{|\varphi_\delta^j|_2^2 : \ j = 1, ..., m\},$$

and choose $\hat{\Lambda}_{m\delta}$ so that

$$V(\lambda^{-1/2}x) \le \frac{\delta}{\beta_\delta} \quad \text{for all } |x| \le r_\delta^m \text{ and } \lambda \ge \hat{\Lambda}_{m\delta}.$$

As before, one obtains easily the following

$$\sup_{u \in H_{\lambda\delta}^m} \Phi_\lambda(u) \le \frac{m(p-2)}{2p(pc_0)^{2/(p-2)}} (2\delta)^{p/(p-2)} \lambda^{1-\frac{N}{2}} \tag{4.5}$$

for all $\lambda \ge \hat{\Lambda}_{m\delta}$.

Using this estimate we can prove easily the following

Lemma 4.6. *Under the assumptions of Lemma 4.1, for any $m \in \mathbb{N}$ and $\sigma > 0$ there exist $\Lambda_{m\sigma} > 0$, such that, for each $\lambda \ge \Lambda_{m\sigma}$, there exists an m-dimensional subspace $H_{\lambda m}$ satisfying*

$$\sup_{u \in F_{\lambda m}} \Phi_\lambda(u) \le \sigma \lambda^{1-\frac{N}{2}}$$

where $F_{\lambda m} = F_\lambda^- \oplus F_\lambda^0 \oplus H_{\lambda m}$

Proof. Choose $\delta > 0$ small so that

$$\frac{m(p-2)}{2p(pc_0)^{2/(p-2)}}(2\delta)^{p/(p-2)} \le \sigma,$$

and take $H_{\lambda m} = H^m_{\lambda\delta}$. Then (4.5) yields the conclusion as required. □

Proof of Theorem 2.1. First we prove the existence. Invoking Lemma 4.5 set

$$Q_\lambda := \left\{u = y + t\bar{e}_\lambda : \; y \in F^-_\lambda \oplus F^0_\lambda, t \ge 0, \|u\|_\lambda \le R\right\}$$

with $R = R(\lambda, F_\lambda) > 0$ where F_λ is the finite dimensional subspace from Lemma 4.5 and R is the associated number from Lemma 4.2. Then

$$\max \Phi_\lambda(Q_\lambda) \le \sigma\lambda^{1-\frac{N}{2}},$$

and, by Lemma 4.2 and the fact that $\Phi_\lambda|_{F^-_\lambda \oplus F^0_\lambda} \le 0$,

$$\Phi_\lambda(u) \le 0 \quad \text{for } u \in \partial Q_\lambda.$$

The standard linking argument yields a $(PS)_{c_\lambda}$-sequence with $\alpha_\lambda \le c_\lambda \le \sigma\lambda^{1-\frac{N}{2}}$. Since by Lemma 3.6 Φ_λ satisfies the $(PS)_{c_\lambda}$-condition, there is $u_\lambda \in E$ such that $\Phi'_\lambda(u_\lambda) = 0$ and $\Phi_\lambda(u_\lambda) = c_\lambda$, hence the existence is proved.

Next we establish the multiplicity. By virtue of Lemma 4.6, for any $m \in \mathbb{N}$, we can choose a m-dimensional subspace $H_{\lambda m}$ such that $\max \Phi_\lambda(F_{\lambda m}) \le \sigma\lambda^{1-\frac{N}{2}}$ if $\lambda \ge \Lambda_{m\sigma}$. By Lemma 4.2, there is $R > 0$ (depending on λ and m) such that $\Phi_\lambda(u) \le 0$ for all $u \in F_{\lambda m} \setminus B_R$.

Denote the set of all symmetric (in the sense that $-A = A$) and closed subsets of E by Σ, for each $A \in \Sigma$ let $\text{gen}(A)$ be the Krasnoselski genus and

$$i(A) := \min_{h \in \Gamma_m} \text{gen}(h(A) \cap \partial B^+_{\rho_\lambda}),$$

where Γ_m is the set of all odd homeomorphisms $h \in C(E, E)$, and $B^+_{\rho_\lambda}$ was defined in Lemma 4.1. Then i is a version of Benci's pseudoindex [4]. Let

$$c_j := \inf_{i(A) \ge j} \sup_{u \in A} \Phi_\lambda(u), \quad 1 \le j \le m.$$

Since $\Phi_\lambda(u) \ge \alpha_\lambda$ for all $u \in \partial B^+_{\rho_\lambda}$ (see Lemma 4.1) and since $i(F_{\lambda m}) = \dim F_{\lambda m} = m$,

$$\alpha_\lambda \le c_1 \le \ldots \le c_m \le \sup_{u \in H_{\lambda m}} \Phi_\lambda(u) \le \sigma\lambda^{1-\frac{N}{2}}.$$

It follows from Lemma 3.6 that Φ_λ satisfies the $(PS)_c$-condition at all levels c_j. By the usual critical point theory, all c_j are critical levels and Φ_λ has at least m pairs of nontrivial critical points. □

5 The critical case: proof of Theorem 2.2

We now turn to the critical case, that is, to prove Theorem 2.2 hence Theorem 1.2. We will consider the functional Ψ_λ along the way as before. Let

$$f(x,u) := g(x,u) + K(x)|u|^{2^*-2}u,$$

$$F(x,u) := \int_0^u f(x,s)ds = G(x,u) + \frac{1}{2^*}K(x)|u|^{2^*}$$

and

$$\mathcal{F}(x,u) := \frac{1}{2}f(x,u)u - F(x,u) = \mathcal{G}(x,u) + \frac{1}{N}K(x)|u|^{2^*}.$$

Then for any $\delta > 0$ there exist $\rho_\delta > 0$ and $c_\delta > 0$ such that

$$\text{(5.1)} \qquad \frac{f(x,u)}{u} \le \delta \text{ if } |u| \le \rho_\delta \quad \text{and} \quad \left(\frac{f(x,u)}{u}\right)^{N/2} \le c_\delta \mathcal{F}(x,u) \text{ if } |u| \ge \rho_\delta$$

First of all we prove the following lemma.

Lemma 5.1. *Assume that $(V_0), (V_1), (G_0)$ and (K_0) are satisfied. There is $\alpha_0 > 0$ independent of λ such that any $(PS)_c$ sequence with $c < \alpha_0 \lambda^{1-\frac{N}{2}}$ contains a convergent subsequence.*

Proof. Let (u_m) be a $(PS)_c$ sequence:

$$\Psi_\lambda(u_m) \to c \quad \text{and} \quad \Psi'_\lambda(u_m) \to 0.$$

A standard argument shows that (u_m) is bounded.

We may assume without loss of generality that $u_m \rightharpoonup u$ in E. Similarly to Lemma 3.2 one checks that, for $s \in [2, 2^*)$ and any $\varepsilon > 0$, along a subsequence (u_{m_j}), there exists $r_\varepsilon > 0$ such that

$$\limsup_{j\to\infty} \int_{B_j \setminus B_r} |u_{m_j}|^s \le \varepsilon$$

for all $r \ge r_\varepsilon$, where $B_j := \{x \in \mathbb{R}^N : |x| \le j\}$. Define $\tilde{u}_j(x) = \eta(2|x|/j)u(x)$ where $\eta : [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ if $t \ge 2$. We claim

$$\text{(5.2)} \qquad \Psi_\lambda(u_{m_j} - \tilde{u}_j) \to c - \Psi_\lambda(u)$$

and

$$\text{(5.3)} \qquad \Psi'_\lambda(z_{m_j} - \tilde{u}_j) \to 0.$$

In fact, observe that

$$\begin{aligned}\Psi_\lambda(u_{m_j}-\tilde u_j) &= \Psi_\lambda(u_{m_j})-\Psi_\lambda(\tilde u_j)\\ &\quad+\lambda\int_{\mathbb{R}^N}\left(G(x,u_{m_j})-G(x,u_{m_j}-\tilde u_j)-G(x,\tilde u_j)\right)\\ &\quad+\frac{\lambda}{2^*}\int_{\mathbb{R}^N}K(x)\left(|u_{m_j}|^{2^*}-|u_{m_j}-\tilde u_j|^{2^*}-|\tilde u_j|^{2^*}\right).\end{aligned}$$

Along the lines in proving the Brézis-Lieb lemma, it is not difficult to check that the second and the third terms on the right-hand side above tends to 0 as $j\to\infty$, respectively. Thus $\Psi_\lambda(u_{m_j}-\tilde u_j)\to c-\Psi_\lambda(u)$ and one gets (5.2). Similarly, observe that for any $\varphi\in E$,

$$\begin{aligned}&\Psi'_\lambda(u_{m_j}-\tilde u_j)\varphi\\ &=\Psi'_\lambda(u_{m_j})\varphi-\Psi'_\lambda(\tilde u_j)\varphi\\ &\quad+\lambda\int_{\mathbb{R}^N}\Big(g(x,u_{m_j})-g(x,u_{m_j}-\tilde u_j)-g(x,\tilde u_j)\Big)\varphi\\ &\quad+\lambda\int_{\mathbb{R}^N}K(x)\Big(|u_{m_j}|^{2^*-2}u_{m_j}-|u_{m_j}-\tilde u_j|^{2^*-2}(u_{m_j}-\tilde u_j)-|\tilde u_j|^{2^*-2}\tilde u_j\Big)\varphi.\end{aligned}$$

As the proof of Lemma 3.3 the second term on the right-hand side above goes to 0 uniformly in $\|\varphi\|_\lambda\le 1$, and so does the third term by a standard argument. This proves (5.3), that is, $\Psi'_\lambda(u_{m_j}-\tilde u_j)\to 0$.

Now using the decomposition (2.5) we write $y_j:=u_{m_j}-\tilde u_j=y_j^d+y_j^e\in E_\lambda^d\oplus E_\lambda^e$. Then by (5.2) and (5.3) one has $\Psi_\lambda(y_j)\to c-\Psi_\lambda(u)$ and $\Psi'_\lambda(y_j)\to 0$. It follows then that

$$\lambda\int_{\mathbb{R}^N}\mathcal{F}(x,y_j)\to c-\Psi_\lambda(u).$$

Noting that $y_j^d\to 0$ and using (5.1), for any $\delta>0$,

$$\begin{aligned}\|y_j^e\|_\lambda^2+o(1) &= \lambda\int_{\mathbb{R}^N}f(x,y_j)y_j=\lambda\int_{\mathbb{R}^N}\frac{f(x,y_j)}{y_j}|y_j|^2\\ &\le \lambda\delta|y_j|_2^2+\lambda c_\delta\left(\int_{|y_j|\ge\rho_\delta}\left(\frac{f(x,y_j)}{y_j}\right)^{N/2}\right)^{2/N}|y_j|_{2^*}^2\\ &\le o(1)+\lambda\delta|y_j^e|_2^2+\lambda c'_\delta\left(\frac{c-\Psi_\lambda(u)}{\lambda}\right)^{2/N}|y_j^e|_{2^*}^2\\ &\le o(1)+\frac{\delta}{b'}\|y_j^e\|_\lambda^2+C_\delta\lambda^{1-\frac{2}{N}}\left(c-\Psi_\lambda(u)\right)^{2/N}\|y_j^e\|_\lambda^2.\end{aligned}\tag{5.4}$$

Remark that $u_{m_j} - u = y_j + (\tilde{u}_j - u)$, hence $u_{m_j} - u \to 0$ if and only if $y_j^e \to 0$. Assume $u_{m_j} \not\to u$. Then $\liminf_{j\to\infty} \|y_j^e\|_\lambda > 0$ and $c - \Psi_\lambda(u) > 0$. Choosing $\delta = b'/4$, it follows from (5.4) that

$$\frac{3}{4}\|y_j^e\|_\lambda^2 \le o(1) + c_1\lambda^{1-\frac{2}{N}} \left(c - \Psi_\lambda(u)\right)^{2/N} \|y_j^e\|_\lambda^2.$$

This implies that

$$1 \le c_2\lambda^{\frac{N}{2}-1} \left(c - \Psi_\lambda(u)\right).$$

The proof is hereby completed. □

In the following we will use the decomposition (2.8). Plainly we have

Lemma 5.2. *Assume that* $(V_0), (V_1), (G_0)$ *and* (K_0) *are satisfied.*

(1°) *For each* $\lambda \ge 1$, *there exists* $\rho_\lambda > 0$ *such that* $\kappa_\lambda := \inf \Psi_\lambda(\partial B_{\rho_\lambda}^+) > 0$ *where* $B_{\rho_\lambda}^+ = \{u \in F_\lambda^+ : \ \|u\|_\lambda \le \rho_\lambda\}$.

(2°) *For any* $0 \ne e \in F_\lambda^+$ *there is* $R > \rho_\lambda$ *such that* $(\Psi_\lambda)|_{\partial Q} \le 0$ *where* $Q := \{u = u^- + u^0 + se : \ u^- + u^0 \in F_\lambda^- \oplus F_\lambda^0, s \ge 0, \|u\|_\lambda \le R\}$.

(3°) *For any finite dimensional subspace* $F \subset F_\lambda^+$, *there is* $R_F > \rho_\lambda$ *such that* $\Psi_\lambda(u) < \inf \Psi_\lambda(B_{\rho_\lambda}^+)$ *for all* $u \in F_\lambda^- \oplus F_\lambda^0 \oplus F$ *with* $\|u\|_\lambda \ge R_F$.

Lemma 5.3. *Under the assumptions of Lemma 5.1, for any* $\sigma > 0$ *there exists* $\Lambda_\sigma > 0$, *such that, for each* $\lambda \ge \Lambda_\sigma$, *there is* $e_\lambda \in F_\lambda^+ \setminus \{0\}$ *such that*

$$\max_{u \in F_{\sigma\lambda}} \Psi_\lambda(u) \le \sigma\, \lambda^{1-\frac{N}{2}},$$

where $F_{\sigma\lambda} := F_\lambda^- \oplus F_\lambda^0 \oplus \mathbb{R}e_\lambda$.

Proof. This follows from Lemma 4.5 and that

$$\Psi_\lambda(z) = \Phi_\lambda - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|u|^{2^*}. \tag{5.5}$$

□

Lemma 5.4. *Under the assumptions of Lemma 5.1, for any* $m \in \mathbb{N}$ *and* $\sigma > 0$ *there exist* $\Lambda_{m\sigma} > 0$, *such that, for each* $\lambda \ge \Lambda_{m\sigma}$, *there exists an* m*-dimensional subspace* $H_{\lambda m} \subset F_\lambda^+$ *satisfying*

$$\sup_{u \in F_{\lambda m}} \Psi_\lambda(u) \le \sigma\, \lambda^{1-\frac{N}{2}}$$

where $F_{\lambda m} = F_\lambda^- \oplus F_\lambda^0 \oplus H_{\lambda m}$.

Proof. It follows from (5.5) and Lemma 4.6. □

Proof of Theorem 2.2. With Lemma 5.1 to Lemma 5.4, repeating the corresponding arguments of the proof of Theorem 2.1 one gets easily the desired results. □

6 The case $\mu = p$: proof of Theorem 2.3

Proof of Theorem 2.3. By h_2) we define

$$\tilde{g}(x,u) := \begin{cases} g(x,u) & \text{for } -1 \le u \le 1 \\ g(x,1)|u|^{p-2}u & \text{for } u \ge 1 \\ -g(x,-1)|u|^{p-2}u & \text{for } u \le -1 \end{cases}$$

and

$$\tilde{G}(x,u) = \int_0^u \tilde{g}(x,s)ds.$$

Then $\tilde{g}(x,u)$ satisfies (G_0), and moreover

$$c_1|u|^{p-1} \le |\tilde{g}(x,u)| \le c_2|u|^{p-1} \quad \text{and} \quad c_1|u|^p \le \tilde{G}(x,u) \le c_2|u|^p$$

for all (x,u).

Consider the equation

$$(\mathcal{R}_\lambda) \qquad \begin{cases} -\Delta u + \lambda V(x)u = \lambda \tilde{g}(x,u) & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where V satisfies $(V_0) - (V_1)$. Let u_λ be the solutions of $(\mathcal{R}_\lambda)$ given by Theorem 2.1. Then u_λ satisfies

$$\tilde{\Phi}_\lambda(u_\lambda) \le \sigma\lambda^{1-\frac{N}{2}}, \quad \tilde{\Phi}'_\lambda(u_\lambda) = 0,$$

where

$$\tilde{\Phi}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda V(x)|u|^2) - \lambda\int_{\mathbb{R}^N}\tilde{G}(x,u),$$

and

$$-\Delta u_\lambda + \lambda V^+(x)u_\lambda = \lambda V^-(x)u_\lambda + \lambda\tilde{g}(x,u_\lambda),$$

$$\sigma\lambda^{1-\frac{N}{2}} \ge \frac{(p-2)\lambda}{2p}\int_{\mathbb{R}^N}\tilde{g}(x,u_\lambda)u_\lambda \ge \frac{(p-2)\lambda}{2}\int_{\mathbb{R}^N}\tilde{G}(x,u_\lambda) \ge c_1\lambda|u_\lambda|_p^p,$$

hence

$$|u_\lambda|_p^p \le c_2\sigma\lambda^{-\frac{N}{2}}.$$

Using the Hölder inequality and the fact that $|\mathcal{V}_-| < \infty$,

$$\lambda\int_{\mathbb{R}^N}V^-(x)|u_\lambda|^2 \le c_3\sigma\lambda^{1-\frac{N}{p}}.$$

Therefore,

$$\|u_\lambda\|_\lambda^2 \le c_4 \sigma \lambda^{1-\frac{N}{p}},$$

$$\|u_\lambda\|_{H^1}^2 \le c_4 \sigma \lambda^{1-\frac{N}{p}}$$

and, for $p_0 = 2^*/(p-1)$,

$$|\tilde{g}(\cdot, u_\lambda)|_{p_0}^{p_0} \le c_5 \int_{\mathbb{R}^N} |u_\lambda|^{2^*} \le c_6 \sigma^{\frac{N}{N-2}} \lambda^{-\frac{N(N-p)}{p(N-2)}}.$$

As in the proof of [3, Theorem 1.3] one has

$$|u_\lambda(x)| \le C_1 \int_{B_1(x)} |u_\lambda(y)|\, dy$$

with $C_1 > 0$ independent of λ and $B_1(x) := \{y \in \mathbb{R}^N : |y-x| \le 1\}$. Hence

$$|u_\lambda(x)| \le C_2 \sigma^{\frac{1}{p}} \lambda^{-\frac{N}{2p}}.$$

Therefore, for λ large sufficiently, $|u_\lambda|_\infty < 1$ and u_λ is the solution of $(\mathcal{P}_\lambda)$. The proof is complete. □

References

[1] Ambrosetti, A.; Badiale, M.; Cingolani, S., Semiclassical states of nonlinear Schödinger equations. *Arch. Rat. Mech. Anal.* **140** (1997), 285–300.

[2] Ambrosetti, A., Malchiodi A., Ni, W.-M., Singularly perturbed elliptic equations with symmetry: existence of solutions concentrating on spheres, Part I, *Comm. Math. Phys.* 235 (2003), 427-466.

[3] Bartsch, T.; Pankov, A., Wang, Z.Q., Nonlinear Schrödinger equations with steep potential well. *Commun. Contemp. Math.* **3** (2001), 549–569.

[4] Benci, V., On critical point theory of indefinite functionals in the presence of symmetries. *Trans. Amer. Math. Soc.* **274** (1982), 533–572.

[5] Brézis, H.; Nirenberg, L., Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.* **36** (1983), 437–477.

[6] Byeon, J.; Wang, Z.Q., Standing waves with a critical frequency for nonlinear Schrödinger equations, II, Calc. Var. PDE 18 (2003), 207–219.

[7] Coti-Zelati, V.; Rabinowitz, P., Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^n$. *Comm. Pure Appl. Math.* **46** (1992), 1217-1269.

[8] Ding, W. Y.; Ni, W. M., On the existence of positive entire solutions of a semilinear elliptic equation. *Arch. Rat. Mech. Anal.* **91** (1986), 283–308.

[9] Ding, Y. H.; Lin, F. H., Solutions of perturbed Schrödinger equations with critical nonlinearity. *Calc. Var. Partial Differential Equations* **30** (2007), 231–249.

[10] Ding, Y. H.; Szulkin, A., Bound states for semilinear Schrödinger equations with sign-changing potential. *Calc. Var. Partial Differential Equations* **29** (2007), 397–419.

[11] Ding, Y. H.; Szulkin, A., Existence and number of solutions for a class of semilinear Schrödinger equations. *Progr. Nonlinear Differential Equations Appl.*, **66** (2006), 221–231.

[12] del Pino, M.; Felmer, P., Multipeak bound states of nonlinear Schrödinger equations. *Ann. IHP, Analyse Nonlineaire* **15** (1998), 127–149.

[13] del Pino, M.; Felmer, P., Semi-classical states of nonlinear Schrödinger equations: a variational reduction method. *Math. Ann.* **324** (2002), 1–32.

[14] del Pino, M., Kowalczyk, K., Wei, J., Nonlinear Schrödinger equations: concentration on weighted geodesics in the semi-classical limit, *Comm. Pure Appl. Math.* 60(2007), no. 1, 113-146.

[15] Floer, A.; Weinstein, A., Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. *J. Funct. Anal.* **69** (1986), 397–408.

[16] Grossi, M., Some results on a class of nonlinear Schrödinger equations. *Math. Z.* **235** (2000), 687–705.

[17] Gui, C., Existence of milti-bump solutions for nonlinear Schrödinger equations via variational method. *Comm. Part. Diff. Eqs.* **21** (1996), 787–820.

[18] Jeanjean, L.; Tanaka, K., Singularly perturbed elliptic problems with superlinear or asymptotically linear nonlinearities. *Calc. Var. PDE* **21** (2004), 287–318.

[19] Kang, X.; Wei, J., On interacting bumps of semi-classical states of nonlinear Schrödinger equations. *Adv. Diff. Eqs.* **5** (2000), 899–928.

[20] Li, Y. Y., On singularly perturbed elliptic equation. *Adv. Diff. Eqs.* **2** (1997), 955–980.

[21] Oh, Y. G., Existence of semiclassical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_\alpha$. *Comm. Part. Diff. Eqs.* **13** (1988), 1499–1519.

[22] Oh, Y. G., On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential. *Comm. Math. Phys.* **131** (1990), 223–253.

[23] Pistoia, A., Multi-peak solutions for a class of nonlinear Schödinger equations. *NoDEA Nonlinear Diff. Eq. Appl.* **9** (2002), 69–91.

[24] Sirakov, B., Standing wave solutions of the nonlinear Schrödinger equations in $\mathbb{R}^N$. *Annali di Matematica* **183** (2002), 73–83.

[25] Wang, X., On concentration of positive bound states of nonlinear Schrödinger equations. *Comm. Math. Phys.* **153** (1993), 229–244.

[26] Reed, M.; Simon, B., *Methods of Modern Mathematical Physics, IV Analysis of Operators,* Academic Press, 1978.

[27] Willem, M., *Minimax Theorems*, Birkhäuser, Boston, 1996.