2π-periodic self-similar solutions for the anisotropic affine curve shortening problem

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Abstract

We study the existence of 2π-periodic positive solutions of the equation

$$u_{\theta\theta} + u = \frac{a(\theta)}{u^3},$$

where $a(\theta)$ is a positive smooth 2π-periodic function. A priori estimates and sufficient conditions for the existence of solutions of the equation are established.

1 Introduction and statement of the results

We are concerned in this paper with the equation

$$u_{\theta\theta} + u = \frac{a(\theta)}{u^3}, \quad \theta \in \mathbb{S}^1, \tag{1.1}$$

where $a(\theta)$ is a positive smooth function on $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Equation (1.1) arises from the study of the generalized curve shortening problem, which can be derived as follows. Consider the following generalized curve shortening problem

$$\frac{\partial \gamma}{\partial t} = \Phi(\theta)|k|^{\sigma-1}kN, \qquad \sigma > 0, \quad \theta \in \mathbb{S}^1, \tag{1.2}$$

where $\gamma(\cdot, t)$ is a planar curve, $k(\cdot, t)$ is its curvature with respect to the unit normal N, and Φ is a positive function depending on the normal angle θ of the curve. This

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problem has been extensively studied in the last two decades. See [1-7, 15, 16, 18-19, 21-25, 32]. Assuming that $\gamma(\cdot, t)$ is convex, then we can use the normal angle θ to parameterize γ, and equation (1.2) is equivalent to

$$\frac{\partial w}{\partial t} = \frac{-\Phi(\theta)}{(w_{\theta\theta} + w)^{\sigma}}, \qquad \theta \in \mathbb{S}^1, \tag{1.3}$$

where $w(\theta, t)$ is the support function of $\gamma(\cdot, t)$. A self-similar solution is of the form $w(\theta, t) = \xi(t)u(\theta)$, which means that the shape of the curves does not change during the evolution governed by (1.2). Such solutions are important in understanding the long time behaviors and the structure of singularities of (1.2). It is rather easy to see that $\xi(t)u(\theta)$ is a self-similar solution if and only if u satisfies

$$u_{\theta\theta} + u = \frac{a(\theta)}{u^{p+1}}, \quad \theta \in \mathbb{S}^1 \tag{1.4}$$

with $a(\theta) = \Phi^{\frac{1}{\sigma}}(\theta)$, $p + 1 = \frac{1}{\sigma}$ and $|\xi(t)|^{\sigma-1}\xi(t)\xi'(t) = -C$, where C is a positive constant. The case $\sigma = \frac{1}{3}$ or equivalently, $p = 2$, (1.2) is called the affine curve shortening problem, and equation (1.4) becomes (1.1). Thus a solution of equation (1.1) is a self-similar solution of the anisotropic affine curve shortening problem. Note that in general $a(\theta)$ can only be assumed to be 2π-periodic. Equation (1.4) also appears in image processing [32], 2-dimensional L^p-Minkowski problem [8, 27] and other problems [25].

Equation (1.1) is a special case of

$$u'' + f(\theta, u) = 0 \tag{1.5}$$

where $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is continuous, $T-$periodic in the first variable and has a singularity of repulsive type near the origin. The existence of periodic solutions of (1.5) had been studied by many people. Using the Poincaré-Birkhoff fixed point theorem, the following result was proved by del Pino, Manásevich and Montero in [17]. Let $\{\mu_n\}_{n=0}^{\infty}$ be the eigenvalues of

$$u'' + \mu u = 0$$

with T-periodic boundary conditions:

$$u(0) = u(T), \quad u'(0) = u'(T),$$

that is, $\mu_n = \left(\frac{2\pi n}{T}\right)^2$, $n = 0, 1, \cdots$. If f satisfies

$$\frac{c'}{s^{\nu}} \leq -f(\theta, s) \leq \frac{c}{s^{\nu}}, \qquad \forall\ s \in (0, \delta), \tag{$f1$}$$

for some positive constants c, c', δ and $\nu \geq 1$, and there exists a nonnegative integer n such that

$$\frac{\mu_n}{4} < \liminf_{s\to+\infty} \frac{f(\theta, s)}{s} \leq \limsup_{s\to+\infty} \frac{f(\theta, s)}{s} < \frac{\mu_{n+1}}{4} \tag{$f2$}$$

uniformly in $\theta \in [0,T]$, then problem (1.5) possesses at least one T-periodic positive solution. This result gives a Fredholm alternative-like result for the problem

$$u'' + \lambda u = \frac{a(\theta)}{u^\nu}, \tag{1.6}$$

which means that (1.6) possesses a $T-$periodic solution if $\lambda \neq \frac{\mu_n}{4}$ for all $n = 0, 1, \cdots$. Thus for $T = 2\pi$, problem (1.6) has at least one 2π-periodic solution for λ satisfying $\frac{n^2}{4} < \lambda < \frac{(n+1)^2}{4}$, $n = 0, 1, \cdots$.

Equation (1.4) with $p \neq 2$ has been studied by many authors. When $a \equiv 1$, all solutions of (1.4) can be classified. See Abresh and Langer [1] in the case of $p = 0$ and B. Andrews [5] in the case of general p. When a is 2π-periodic, Dohmen and Giga [18], Dohmen, Giga and Mizoguchi [19] studied the $p \leq 1$ case. Matano and Wei [29] proved that (1.4) is solvable if $0 \leq p < 7, p \neq 2$ and a is 2π-periodic.

These results do not cover the affine case, that is, equation (1.1). Indeed, the situation for the affine case is quite different. It is known that there are some obstructions for the existence and one can't get a priori estimates for the solutions of (1.1) without additional assumptions on a due to the invariance of the problem. To see this, let us consider its simplest form

$$u_{\theta\theta} + u = \frac{1}{u^3}, \quad \theta \in \mathbb{S}^1. \tag{1.7}$$

Equation (1.7) is invariant under the action of the special linear group $SL(2,\mathbb{R})$. For any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL(2,\mathbb{R})$, we have an "affine diffeomorphism" on $\mathbb{S}^1$ given by

$$(\cos\theta, \sin\theta) \longmapsto \frac{(\cos\theta, \sin\theta)A^T}{\|(\cos\theta, \sin\theta)A^T\|} = (\cos\tilde\theta, \sin\tilde\theta),$$

that is,

$$\tan\tilde\theta = \frac{c + d\tan\theta}{a + b\tan\theta}, \qquad ad - bc = 1.$$

For any function u in $\mathbb{S}^1$, define

$$u_A(\tilde\theta) = \cos\tilde\theta\sqrt{(a\tan\tilde\theta - c)^2 + (b\tan\tilde\theta - d)^2}u(\theta).$$

Then $u_A(\tilde\theta)$ is a solution of (1.7) if and only if $u(\theta)$ is a solution of (1.7). Starting with the trivial solution $u \equiv 1$, one can show that all solutions of (1.7) are given by a 2-parameter family of functions

$$u_{\varepsilon,t}(\theta) = \left(\varepsilon^2\cos^2(\theta - t) + \varepsilon^{-2}\sin^2(\theta - t)\right)^{\frac{1}{2}}, \tag{1.8}$$

for $(\varepsilon, t) \in (0,1] \times [0,\pi)$. Thus the solutions of (1.7) are not bounded. The group $SL(2,\mathbb{R})$ plays an important role as that of the conformal group $Conf(\mathbb{S}^2)$ in the Nirenberg's problem in geometry, which has been extensively studied and many significant results have been obtained. See [9-14, 25, 27, 32, 34] and the references therein.

In [2], Ai, Chou and Wei considered the solvability of problem (1.1) under the assumption that *a is π-periodic*. After scaling, in this case the problem is equivalent to the equation (1.6) with $T = \pi$, $\lambda = \frac{\mu_1}{4} = 1$ and $\nu = 3$. Let

$$
\begin{aligned}
B(\theta) &= \int_0^{\pi} \frac{a(\theta+t) - a(\theta) - 2^{-1}a'(\theta)\sin 2t}{\sin^2 t} dt \\
&= \int_0^{\pi} \frac{(a'(\theta+t) - a'(\theta))\sin 2t}{\sin^2 t} dt.
\end{aligned}
$$

A function a is called *B-nondegenerate* if $B(\theta)$ never vanishes at any critical point θ of the function a. They proved that if a is a positive, B-nondegenerate and C^2-function of period π, then one can get a priori estimates for all π-periodic solutions of (1.1). Moreover, if the winding number of the map G around the origin is not equal to -1 , where

$$
G(\theta) = (-B(\theta), a'(\theta)) \qquad \theta \in [0,\pi),
$$

then problem (1.1) has a π-periodic solution.

In this paper we study equation (1.1) for 2π-periodic function a, which is more interesting and natural from geometric point of view. We will consider a slightly general form, that is, for a fixed $n \geq 2$, the existence of $n\pi$-periodic solutions of

$$
u_{\theta\theta} + u = \frac{a(\theta)}{u^3} \tag{1.9}
$$

where a is $n\pi$-periodic. It is the same as the equation (1.6) with $T = \pi$, $\nu = 3$ and $\lambda = \frac{\mu_n}{4} = n^2$ by scaling.

Before we state our results, we comment on the difficulties in studying (1.9). A major problem is to study possible blow-ups. When $a(\theta)$ is $\pi-$periodic, since the blow-up sequence $u_{\varepsilon,t}$ (defined at (1.8)) is $\pi-$periodic, only single blow-up can occur. However, when a is $n\pi$-periodic, there are n possible blow-ups. We have to analyze the interaction between different blow-ups.

To state our main results, for any positive C^2-function $a(\theta)$, we define

$$
A_n(\theta) = \sum_{j=1}^{n} \frac{a'(\theta + (j-1)\pi)}{\sqrt{a(\theta + (j-1)\pi)}} \tag{1.10}
$$

and

$$
B_n(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\sum_{j=1}^n a'(\theta + t + (j-1)\pi) - \sum_{j=1}^n a'(\theta + (j-1)\pi)\right)\sin 2t}{\sin^2 t} dt. \tag{1.11}
$$

Following [2], a function a is called B_n-nondegenerate if $B_n(\theta)$ does not vanish whenever $A_n(\theta)=0$. Note that by definition, $A_n(\theta)$ and $B_n(\theta)$ are $\pi-$periodic.

Our first result is the a priori estimates.

Theorem 1.1 *Let a be a positive, C^2 and $n\pi$-periodic function. Suppose that $a(\theta)$ is B_n-nondegenerate. Then there exists a constant C depending on a only such that*

$$\frac{1}{C} \le u \le C \tag{1.12}$$

for any $n\pi$-periodic solution u of (1.9).

As for the existence we have

Theorem 1.2 *Let a be a positive, C^2 and $n\pi$-periodic function. Suppose that*

$$\min_{A_n(\theta)=0} B_n(\theta) > 0, \quad \text{or} \ \max_{A_n(\theta)=0} B_n(\theta) < 0, \tag{1.13}$$

then equation (1.9) has an $n\pi$-periodic solution.

An example of a satisfying (1.13) is

$$a(\theta) = (1 + b_1 \cos\theta + b_2 \cos 2\theta)^2, \tag{1.14}$$

where b_1 and b_2 are some constants in $(-\frac{1}{2}, \frac{1}{2})$ to be determined later. It is easy to see that $A_2(\theta) = -8b_2 \sin 2\theta$. Hence $A_2(\theta)=0$ if and only if $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and

$$a(\theta+t+\pi) + a(\theta+t) = \begin{cases} 2(1+b_2\cos 2t)^2 + 2(b_1 \cos t)^2, & \theta = 0, \pi; \\ 2(1-b_2\cos 2t)^2 + 2(b_1 \sin t)^2, & \theta = \frac{\pi}{2}, \frac{3\pi}{2}. \end{cases}$$

Then

$$\big(a(\theta+t+\pi)+a(\theta+t)\big)' = \begin{cases} 4(1+b_2\cos 2t)(-2b_2 \sin 2t) - 2b_1^2 \sin 2t, & \theta = 0, \pi; \\ 4(1-b_2\cos 2t)(2b_2 \sin 2t) + 2b_1^2 \sin 2t, & \theta = \frac{\pi}{2}, \frac{3\pi}{2} \end{cases}$$

and

$$B_2(\theta) = \begin{cases} -4(b_1^2 + 4b_2)\pi - 8b_2^2\pi, & \theta = 0, \pi; \\ 4(b_1^2+4b_2)\pi - 8b_2^2\pi, & \theta = \frac{\pi}{2}, \frac{3\pi}{2}. \end{cases}$$

Thus (1.13) is satisfied if $b_1^2 + 4b_2 = 0$ and $b_2 \neq 0$.

Let

$$G_n(\theta) = \Big(- B_n(\theta), A_n(\theta) \Big) : \mathbb{S}^1 \to \mathbb{R}^2,$$

$$\hat{G}_n(\theta) = \frac{G_n(\theta)}{|G_n(\theta)|} : \mathbb{S}^1 \to \mathbb{S}^1$$

whose Brouwer degree $deg(\hat{G}_n, \mathbb{S}^1)$ is well defined if a is B_n-nondegenerate. See the Appendix or [20] for definitions and properties. With the condition (1.13), we see that $deg(\hat{G}_n, \mathbb{S}^1) = 0$. (See (i) of the Appendix.)

The following result concerns with the case $deg(\hat{G}_n, \mathbb{S}^1) \neq 0$. For any fixed C^2-function $a(\theta)$, let

$$\overline{A}_n(\theta) = \sum_{j=1}^{n} a'(\theta + (j-1)\pi), \tag{1.15}$$

$$\overline{B}_n(\theta) = \int_0^\pi \frac{\left(\sum_{j=1}^n a'(\theta + t + (j-1)\pi) - \sum_{j=1}^n a'(\theta + (j-1)\pi)\right) \sin 2t}{\sin^2 t} dt, \tag{1.16}$$

$$\overline{G}_n(\theta) = (-\overline{B}_n(\theta), \overline{A}_n(\theta)). \tag{1.17}$$

For $a = 1 + \varepsilon a$ we have $A_n(\varepsilon)(\theta)$ and $B_n(\varepsilon)(\theta)$ given by (1.10) and (1.11). Then

$$(-B_n(\varepsilon)(\theta), A_n(\varepsilon)(\theta)) = (-\overline{B}_n(\theta), \overline{A}_n(\theta))\varepsilon + O(\varepsilon^2).$$

Thus for small ε, $1 + \varepsilon a$ is B_n-nondegenerate if $\overline{G}_n(\theta) := (-\overline{B}_n(\theta), \overline{A}_n(\theta)) \neq (0,0)$. By homotopy invariance of degree, $deg(\hat{G}_n, \mathbb{S}^1) = deg(\frac{\overline{G}_n}{|\overline{G}_n|}, \mathbb{S}^1)$.

Theorem 1.3 *Let a be a positive, C^2 and $n\pi$-periodic function such that for all θ, $\overline{G}_n(\theta) = (-\overline{B}_n(\theta), \overline{A}_n(\theta)) \neq (0,0)$ and $deg(\frac{\overline{G}_n}{|\overline{G}_n|}, \mathbb{S}^1) \neq -2$. Then the equation*

$$u_{\theta\theta} + u = \frac{1 + \varepsilon a(\theta)}{u^3} \tag{1.18}$$

has an $n\pi$-periodic solution if ε is small.

It is not difficult to see that if $n = 1$, the map $\overline{G}_n(\theta)$ is the same as the map G in [2] and it is $\pi-$peridic. In this case, we fix $\varepsilon << 1$ and consider the homotopy of $a_s(\theta) = (1-s)(1 + \varepsilon a(\theta)) + s a(\theta)$. For $s \in [0,1]$, the function a_s is always $B-$nondegenerate, and thus one can get uniform a priori estimates for all π-periodic solutions of

$$u_{\theta\theta} + u = \frac{a_s(\theta)}{u^3}. \tag{1.19}$$

Using the degree argument, one can solve the problem up to $s = 1$ if $deg(\frac{G}{|G|}, \mathbb{R}/\pi\mathbb{Z}) =$ winding number of $G+1 \neq 0$. However, for $n \geq 2$, we do not know how to construct such a homotopy. The degree argument can only be used for small ε. The general a case remains open.

We briefly sketch the idea of the proofs of our results. We only consider the case $n = 2$ since that of $n \geq 3$ is the same. To prove Theorems 1.1 and 1.2, we take a sequence λ_k such that $\forall\ k, \lambda_k \neq 1, \lambda_k \to 1$ as $k \to \infty$ and consider the equation

$$u_{\theta\theta} + \lambda_k u = \frac{a(\theta)}{u^3}, \qquad \theta \in \mathbb{S}^1. \tag{1.20}$$

According to [17], there is a 2π-periodic solution u_k. By careful analysis of blow-up, we can get asymptotic estimates of u_k and $\lambda_k - 1$. Under the condition (1.13), these estimates ensure that u_k converges to a solution of (1.1) if $\lambda_k \nearrow 1$ or $\lambda_k \searrow 1$ as $k \to \infty$. The proof of Theorem 1.3 follows from the Liapunov-Schmidt reduction and degree argument.

The paper is organized as follows. In section 2, some asymptotical estimates are given based on blow-up analysis, and in section 3 we give the necessary condition for the existence and sharp estimates. Theorem 1.1 and Theorem 1.2 are proved in section 4, and in section 5 we provide a proof of Theorem 1.3.

2 Preliminary Blow-up Analysis

Let a be a positive 2π-periodic function, λ_k be a sequence such that $\forall\ k, \lambda_k \neq 1, \lambda_k \to 1$ as $k \to \infty$ and u_k be a 2π-periodic solution of

$$u_{\theta\theta} + \lambda_k u = \frac{a(\theta)}{u^3}, \qquad \theta \in [0, 2\pi]. \tag{2.1}$$

The aim of this section is to derive some asymptotical estimates of u_k and $\lambda_k - 1$, which will lead to sharp estimates in the next section.

Here and after, unless otherwise stated, the letter C will always denote various generic constants which are independent of k. We denote $A \sim B$ if there exist two positive uniform constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$. $C_k = o(1)$ means that $\lim_{k\to+\infty} C_k = 0$.

We start with the following simple but useful lemma.

Lemma 2.1 *For any solution u of problem (2.1), defining $F_k(\theta) = u_{k,\theta}^2 + \lambda_k u_k^2 + \dfrac{a(\theta)}{u_k^2}$, then there is a constant C independent of k such that*

$$C^{-1} F_k(\theta_2) \leq F_k(\theta_1) \leq C F_k(\theta_2), \qquad \forall\ \theta_1, \theta_2 \in [0, 2\pi].$$

Proof. By equation (2.1), we can easily get that

$$F_k'(\theta) = \frac{a'(\theta)}{u_k^2},$$

which implies that

$$|F_k'(\theta)| \leq C|F_k(\theta)|$$

since $a(\theta)$ is smooth and positive. Thus

$$|(\log F_k)'| \leq C,$$

which leads to

$$C^{-1} \leq \frac{F_k(\theta_1)}{F_k(\theta_2)} \leq C, \qquad \forall\ \theta_1, \theta_2 \in [0, 2\pi].$$

Thus we finish the proof. $\square$

From the above lemma, we know that u_k is bounded from above if and only if it is bounded from below. Without loss of generality, we assume that $\min_{\theta\in[0,2\pi]} u_k(\theta) \to 0$. Then $F_k(\theta) \to +\infty$ uniformly in θ as $k \to \infty$. This in particular implies that, as $k \to \infty$,

$$\text{either} \quad u_k(\theta) \quad \text{or} \quad \frac{1}{u_k(\theta)} \to +\infty \quad \text{whenever} \quad u_k'(\theta) = 0.$$

If $u_k'(\tau_k) = 0$ and $u_k(\tau_k) \to 0$, then by the equation (2.1) we see that $u_k''(\tau_k) = \frac{a(\tau_k)}{(u_k(\tau_k))^3} - \lambda_k u_k(\tau_k) > 0$ and hence τ_k is a local minimum point. Similarly, if $u_k'(\theta_k) = 0$ and $u_k(\theta_k) \to +\infty$, then $u_k''(\theta_k) < 0$ and θ_k is a local maximum point. This also implies that the values of u_k at any local maximum point of u_k must approach $+\infty$, and the value of u_k at any local minimum point of u_k must approach 0.

Hence for $k >> 1$, the local minimum and maximum points of u_k are isolated. Therefore they appear alternatively and satisfy

$$u_k(\tau_k) \sim \frac{1}{u_k(\theta_k)},$$

where τ_k and θ_k are local minimum and maximum points of u_k, respectively.

Let $\tau_k^1 < \tau_k^2$ be two consecutive local minimum points of u_k and let $\theta_k \in [\tau_k^1, \tau_k^2]$ satisfy $u_k(\theta_k) = M_k = \max_{\theta\in[\tau_k^1,\tau_k^2]} u_k(\theta)$. Then $u_k(\tau_k^1), u_k(\tau_k^2) \to 0$ and $M_k \to \infty$ as $k \to \infty$. We have the following convergence result.

Lemma 2.2 *For $k \to \infty$ we have*

$$\theta_k - \tau_k^1 \to \frac{\pi}{2}, \tag{2.2}$$

$$\tau_k^2 - \tau_k^1 \to \pi, \tag{2.3}$$

$$\frac{u_k(\frac{\tau_k^2-\tau_k^1}{\pi}\theta + \tau_k^1)}{M_k} \to \sin\theta \quad \textit{uniformly in} \quad [0,\pi]. \tag{2.4}$$

In particular, u_k has two minimum and two maximum points in $[0, 2\pi)$.

Proof. We first prove that $\theta_k - \tau_k^1 \le \frac{\pi}{2}$. The idea is to compare with the case when a is constant.

Let $m_k = u(\tau_k^1)$. Then $m_k \to 0$ and $M_k \sim \frac{1}{m_k}$. On the interval $[\tau_k^1, \theta_k]$, u_k satisfies

$$\frac{C_1}{2u_k^3} \le u_k'' + \lambda_k u_k \le \frac{C_2}{2u_k^3}, \;\; u_k' > 0 \text{ in } (\tau_k^1, \theta_k). \tag{2.5}$$

Set

$$H_C(u_k) = \lambda_k u_k^2 + \frac{C}{u_k^2}\ .$$

Then $(u_k')^2 + H_{C_1}(u_k)$ is strictly increasing over $[\tau_k^1, \theta_k]$ while $(u_k')^2 + H_{C_2}(u_k)$ is strictly decreasing over $[\tau_k^1, \theta_k]$.

Let $H_{C_2}(\hat{m}_k) = H_{C_2}(M_k)$ where $\hat{m}_k < 1$. Then since $H_{C_2}(m_k) > H_{C_2}(M_k) = H_{C_2}(\hat{m}_k)$, we see that $\hat{m}_k > m_k$ and $\hat{m}_k \sim M_k^{-1} \sim m_k$. So there exists a unique $\hat{\tau}_k^1 \in (\tau_k^1, \theta_k)$ such that $u_k(\hat{\tau}_k^1) = \hat{m}_k$.

Over $[\tau_k^1, \hat{\tau}_k^1]$, by equation (2.1), $u_k^{''}(\theta) \sim m_k^{-3}$. Hence $u_k^{'}(\theta) \sim m_k^{-3}(\theta - \tau_k^1)$ for $\theta \in [\tau_k^1, \hat{\tau}_k^1]$. Therefore

$$u_k(\hat{\tau}_k^1) - u_k(\tau_k^1) = \int_{\tau_k^1}^{\hat{\tau}_k^1} u^{'}(\theta)d\theta \sim m_k^{-3}(\hat{\tau}_k^1 - \tau_k^1)^2$$

which implies that

$$\hat{\tau}_k^1 - \tau_k^1 = O(m_k^2). \tag{2.6}$$

On the other hand, we have

$$(u_k^{'}(\theta))^2 \geq H_{C_2}(M_k) - H_{C_2}(u_k), \quad \forall\, \theta \in [\tau_k^1, \theta_k].$$

Hence

$$\theta_k - \hat{\tau}_k^1 \leq \int_{\hat{m}_k}^{M_k} \frac{du}{\sqrt{H_{C_2}(M_k) - H_{C_2}(u)}}. \tag{2.7}$$

Let $T_k = \int_{\hat{m}_k}^{M_k} \frac{du}{\sqrt{H_{C_2}(M_k) - H_{C_2}(u)}}$. Then we note that T_k is the length between consecutive minimum and maximum points of the equation

$$v^{''} + \lambda_k v = \frac{C_2}{v^3}, v(0) = \hat{m}_k, v(T_k) = M_k, v^{'}(\theta) > 0 \text{ for } \theta \in (0, T_k) \tag{2.8}$$

since $v^{'}(\theta) = \sqrt{H_{C_2}(M_k) - H_{C_2}(v)}$. On the other hand, by rescaling of $\tilde{v}(\tilde{\theta}) = (\frac{C_2}{\lambda_k})^{-\frac{1}{4}} v(\frac{\tilde{\theta}}{\sqrt{\lambda_k}})$, $\tilde{v}$ satisfies

$$\tilde{v}^{''} + \tilde{v} = \frac{1}{\tilde{v}^3}. \tag{2.9}$$

By (1.8), all solutions of (2.9) have period π. This implies that

$$T_k = \int_{\hat{m}_k}^{M_k} \frac{du}{\sqrt{H_{C_2}(M_k) - H_{C_2}(u)}} = \frac{\pi}{2\sqrt{\lambda_k}}. \tag{2.10}$$

So

$$\lim_{k\to+\infty} (\theta_k - \tau_k^1) = \lim_{k\to+\infty} (\theta_k - \hat{\tau}_k^1 + \hat{\tau}_k^1 - \tau_k^1) \leq \frac{\pi}{2}. \tag{2.11}$$

Similarly, we can prove that

$$\lim_{k\to+\infty} (\tau_k^2 - \theta_k) \leq \frac{\pi}{2}, \quad \lim_{k\to+\infty} (\tau_k^2 - \tau_k^1) \leq \pi. \tag{2.12}$$

Now we let $\tilde{u}_k = \frac{u_k(\frac{\tau_k^2 - \tau_k^1}{\pi}\theta + \tau_k^1)}{M_k}$. Then it follows from equation (2.1) that $\tilde{u}_k$ satisfies

$$\tilde{u}_{k,\theta\theta} + (\frac{\tau_k^2 - \tau_k^1}{\pi})^2 \lambda_k \tilde{u}_k = (\frac{\tau_k^2 - \tau_k^1}{\pi M_k^2})^2 \frac{a(\frac{\tau_k^2 - \tau_k^1}{\pi}\theta + \tau_k^1)}{\tilde{u}_k^3}. \tag{2.13}$$

Integration by parts gives that

$$(\frac{\tau_k^2-\tau_k^1}{\pi})^2\lambda_k\int_0^\pi \tilde{u}_k^2 d\theta - \int_0^\pi \tilde{u}_{k,\theta}^2 d\theta = (\frac{\tau_k^2-\tau_k^1}{\pi M_k^2})^2\int_0^\pi \frac{a(\frac{\tau_k^2-\tau_k^1}{\pi}\theta+\tau_k^1)}{\tilde{u}_k^2}d\theta \geq 0$$

since $\tilde{u}_{k,\theta}(0)=\tilde{u}_{k,\theta}(\pi)=0$. Hence

$$\int_0^\pi \tilde{u}_{k,\theta}^2 d\theta \leq (\frac{\tau_k^2-\tau_k^1}{\pi})^2\lambda_k\int_0^\pi \tilde{u}_k^2 d\theta. \tag{2.14}$$

Using the fact that $0<\tilde{u}_k(\theta)\leq \max\limits_{\theta\in[0,\pi]}\tilde{u}_k(\theta)=1$ and (2.14), we deduce that $\tilde{u}_k$ is bounded in $H^1([0,\pi])$. Thus we can assume $\tilde{u}_k \rightharpoonup \tilde{u}$ weakly in $H^1([0,\pi])$ and $\tau=\lim\limits_{k\to\infty}(\tau_k^2-\tau_k^1)$. Letting $k\to\infty$, one gets

$$\int_0^\pi \tilde{u}_\theta^2 d\theta \leq \frac{\tau^2}{\pi^2}\int_0^\pi \tilde{u}^2 d\theta. \tag{2.15}$$

By the embedding theorem, $\tilde{u}_k \to \tilde{u}$ in $C([0,\pi])$, and so

$$\tilde{u}(0)=\lim_{k\to\infty}\frac{u_k(\tau_k^1)}{M_k}=0, \quad \tilde{u}(\pi)=\lim_{k\to\infty}\frac{u_k(\tau_k^2)}{M_k}=0$$

and $\tilde{u}\in H_0^1([0,\pi])$.

On the other hand, by Wirtinger's Inequality, we know that

$$\int_0^\pi \tilde{u}^2 d\theta \leq \int_0^\pi \tilde{u}_\theta^2 d\theta, \tag{2.16}$$

and equality holds if and only if $\tilde{u}=C\sin\theta$.

Combining (2.15) and (2.16), we see that $\tau\geq\pi$. But according to the previous argument, $\tau\leq\pi$ and hence $\tau=\pi$ and the equality at (2.16) holds. Therefore, $\tilde{u}=C\sin\theta$. The assumption that $\max_{\theta\in[0,\pi]}\tilde{u}_k(\theta)=1$ yields that $\max_{\theta\in[0,\pi]}\tilde{u}(\theta)=1$. Hence $\tilde{u}=\sin\theta$ and $\tilde{u}_k\to\sin\theta$ in $C^{\frac{1}{2}}[0,\pi]$. This proves (2.4). The proof of Lemma 2.2 is completed. □

The following so-called Pohozaev's Identity will be used frequently in the rest of the paper.

Lemma 2.3 *Let a be a positive, C^2 and 2π-periodic function. Then we have*

$$\int_0^{2\pi}\frac{a'(\theta)+4(1-\lambda_k)u^3u'}{u^2}(1+\cos 2\theta)d\theta=0 \tag{2.17}$$

and

$$\int_0^{2\pi}\frac{a'(\theta)+4(1-\lambda_k)u^3u'}{u^2}\sin 2\theta d\theta=0 \tag{2.18}$$

for any 2π-periodic solution u of (2.1).

Proof. For any solution u of (2.1), we have

$$(\frac{u^2}{2})''' + 4(\frac{u^2}{2})' = \frac{\big(a(\theta) + (1-\lambda_k)u^4\big)'}{u^2} \tag{2.19}$$

and the lemma follows from an integration over $[0, 2\pi]$. $\square$

Let $\varepsilon_k = u_k(\tau_k) = \min_{\theta\in[0,2\pi]} u_k(\theta) \to 0$ and let θ_k be the next local maximum point of u_k. Then $M_k = u_k(\theta_k) \to \infty$. The above lemmas lead to the following estimates.

Proposition 2.4 *There exists a uniform positive constant C such that*

$$|\lambda_k - 1| \leq C\varepsilon_k^2. \tag{2.20}$$

Proof. We first prove

$$\int_0^{2\pi} \frac{1}{u_k^2} d\theta \leq C. \tag{2.21}$$

To this end, let $\overline{u}_k(\theta) = u_k\big(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k\big), \theta \in [0, \frac{\pi}{2}]$ and $\overline{\lambda}_k = \big(\frac{2(\theta_k-\tau_k)}{\pi}\big)^2\lambda_k$. Then $\overline{u}_k$ satisfies

$$\overline{u}_{k,\theta\theta} + \overline{\lambda}_k\overline{u}_k = \big(\frac{2(\theta_k-\tau_k)}{\pi}\big)^2 \frac{a\big(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k\big)}{\overline{u}_k^3} \tag{2.22}$$

and

$$(\frac{\overline{u}_k^2}{2})''' + 4(\frac{\overline{u}_k^2}{2})' = \big(\frac{2(\theta_k-\tau_k)}{\pi}\big)^3 \frac{a'\big(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k\big)}{\overline{u}_k^2} + 4(1-\overline{\lambda}_k)\overline{u}_k\overline{u}_k'. \tag{2.23}$$

By virtue of $\overline{u}_k'(0) = 0$ and $\overline{u}_k'(\frac{\pi}{2}) = 0$ we obtain

$$\int_0^{\frac{\pi}{2}} \big((\frac{\overline{u}_k^2}{2})''' + 4(\frac{\overline{u}_k^2}{2})'\big)\sin 2\theta d\theta = 0. \tag{2.24}$$

Consequently, from (2.23) and (2.24) we deduce that

$$\begin{aligned}\big(\frac{2(\theta_k-\tau_k)}{\pi}\big)^3 \int_0^{\frac{\pi}{2}} \frac{a'\big(\frac{2(\theta_k-\tau_k)}{\pi}\theta + \tau_k\big)}{\overline{u}_k^2}\sin 2\theta d\theta &= 4(\overline{\lambda}_k - 1)\int_0^{\frac{\pi}{2}} \overline{u}_k\overline{u}_k'\sin 2\theta d\theta \\ &= -4(\overline{\lambda}_k - 1)\int_0^{\frac{\pi}{2}} \overline{u}_k{}^2\cos 2\theta d\theta.\end{aligned} \tag{2.25}$$

It follows from Lemma 2.2 that $\exists\, C > 0$ such that

$$|\int_0^{\frac{\pi}{2}} \overline{u}_k{}^2 d\theta| \leq C|\int_0^{\frac{\pi}{2}} \overline{u}_k{}^2\cos 2\theta d\theta|.$$

Hence for small δ,

$$\begin{aligned}
|\overline{\lambda}_k-1|\int_0^{\frac{\pi}{2}}\overline{u}_k^2d\theta &\le C\Big|\int_0^{\frac{\pi}{2}}\frac{a'(\frac{2(\theta_k-\tau_k)}{\pi}\theta+\tau_k)\sin 2\theta}{\overline{u}_k^2}d\theta\Big|\\
&\le C(|\int_0^{\delta}f_k(\theta)d\theta|+|\int_{\delta}^{\frac{\pi}{2}-\delta}f_k(\theta)d\theta|+|\int_{\frac{\pi}{2}-\delta}^{\frac{\pi}{2}}f_k(\theta)d\theta|) \qquad (2.26)\\
&\le C\delta\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta+C(\delta)M_k^{-2}
\end{aligned}$$

where $f_k(\theta)=\frac{a'(\frac{2(\theta_k-\tau_k)}{\pi}\theta+\tau_k)\sin 2\theta}{\overline{u}_k^2}$, $C(\delta)$ is a constant depending on δ and in the last inequality we have used (2.4).

On the other hand, from (2.22) and (2.26) we get

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}}(\overline{u}_k^2-\overline{u}_{k,\theta}^2)d\theta\\
=&(\frac{2(\theta_k-\tau_k)}{\pi})^2\int_0^{\frac{\pi}{2}}\frac{a(\frac{2(\theta_k-\tau_k)}{\pi}\theta+\tau_k)}{\overline{u}_k^2}d\theta+(1-\overline{\lambda}_k)\int_0^{\frac{\pi}{2}}\overline{u}_k^2d\theta\\
\ge&(\frac{2(\theta_k-\tau_k)}{\pi})^2\int_0^{\frac{\pi}{2}}\frac{a(\frac{2(\theta_k-\tau_k)}{\pi}\theta+\tau_k)}{\overline{u}_k^2}d\theta-C\delta\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta-C(\delta)M_k^{-2}\\
\ge&\frac{\min_{\theta\in\mathbb{S}^1}a(\theta)}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta-C
\end{aligned} \qquad (2.27)$$

since δ is small. The left hand side of (2.27) can be estimated as follows. Reflecting the function $\overline{u}_k$ with respect to $\theta=\frac{\pi}{2}$, we get a new function defined on $[0,\pi]$, still denoted by $\overline{u}_k$. Then $\overline{u}_k(0)=\overline{u}_k(\pi)$. By Sobolev Inequality (See Proposition 1.3 of [2]) we have that

$$\int_0^{\pi}\frac{1}{\overline{u}_k^2}d\theta\big(\int_0^{\pi}(\overline{u}_k^2-\overline{u}_{k,\theta}^2)d\theta\big)\le\pi^2.$$

Since $\overline{u}_k$ is symmetric with respect to $\theta=\frac{\pi}{2}$, we see that

$$\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta\big(\int_0^{\frac{\pi}{2}}(\overline{u}_k^2-\overline{u}_{k,\theta}^2)d\theta\big)\le\frac{\pi^2}{4}. \qquad (2.28)$$

Combining (2.27) with (2.28) we obtain

$$\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta\Big(\frac{\min_{\theta\in\mathbb{S}^1}a(\theta)}{2}\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta-C\Big)\le\frac{\pi^2}{4}. \qquad (2.29)$$

Consequently,

$$\int_0^{\frac{\pi}{2}}\frac{1}{\overline{u}_k^2}d\theta\le C \qquad (2.30)$$

and

$$\int_{\tau_k}^{\theta_k} \frac{1}{u_k^2} d\theta \le \frac{2(\theta_k - \tau_k)}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\overline{u}_k^2} d\theta \le C. \tag{2.31}$$

Similarly, let τ_k' be the local minimum point of u_k next to θ_k, we have

$$\int_{\theta_k}^{\tau_k'} \frac{1}{u_k^2} d\theta \le C. \tag{2.32}$$

It follows from (2.31) and (2.32) that

$$\int_{\tau_k}^{\tau_k'} \frac{1}{u_k^2} d\theta \le C. \tag{2.33}$$

The same argument yields

$$\int_{\tau_k'}^{2\pi+\tau_k} \frac{1}{u_k^2} d\theta \le C. \tag{2.34}$$

(2.21) follows from (2.33) and (2.34) since u_k has two local minimum and maximum points on $[0, 2\pi)$.

Now we can prove the estimate (2.20). Using the identity (2.18) we see that

$$4|\lambda_k - 1|| \int_0^{2\pi} u_k(\theta) u_k'(\theta) \sin 2\theta d\theta| \le \left| \int_0^{2\pi} \frac{a'(\theta) \sin 2\theta}{u_k^2} \right| \le C. \tag{2.35}$$

It is easy to see from Lemma 2.2 that

$$| \int_0^{2\pi} u_k(\theta) u_k'(\theta) \sin 2\theta d\theta| = | \int_0^{2\pi} u_k^2(\theta) \cos 2\theta d\theta| \ge C M_k^2. \tag{2.36}$$

Inserting (2.36) into (2.35), we lead to

$$|\lambda_k - 1| \le \frac{C}{M_k^2} \le C\varepsilon_k^2.$$

Thus (2.20) is proved. □

Let $\varepsilon_k = \min_{\theta \in [0,2\pi]} u_k(\theta) = u_k(\tau_k) \to 0$ and

$$U_\varepsilon(\theta) = \left(\varepsilon^2 \cos^2 \theta + \varepsilon^{-2} \sin^2 \theta\right)^{\frac{1}{2}} .$$

We define a transformation

$$\theta = \tau_k + \psi_k(y) = \tau_k + \int_0^y \frac{1}{U^2_{\varepsilon_k'^{-1}}(\tau)} d\tau, \tag{2.37}$$

where $\varepsilon_k' = (a(\tau_k))^{-\frac{1}{4}} \varepsilon_k$. It induces a rule of transformation of the equation (2.1) as follows: let

$$v_k(y) = U_{\varepsilon_k'^{-1}}(y) u_k(\tau_k + \psi_k(y)) = U^{-1}_{\varepsilon_k'}(\theta - \tau_k) u_k(\theta),$$

using

$$\frac{dy}{d\theta} = U_{\varepsilon_k'}^{-2}(\theta - \tau_k), \qquad \text{and}$$

$$\frac{d^2 y}{d\theta^2} = \left({\varepsilon_k'}^2 - {\varepsilon_k'}^{-2}\right) \sin 2(\theta - \tau_k) U_{\varepsilon_k'}^{-4}(\theta - \tau_k),$$

one can verify that

$$\frac{d^2 v_k}{dy^2} + v_k = \frac{a(\tau_k + \psi_k)}{v_k^3} + (1 - \lambda_k) \frac{v_k}{U^4_{{\varepsilon_k'}^{-1}}(y)}. \tag{2.38}$$

Using the estimate (2.20), we can prove the following result on the asymptotical behavior of u_k.

Proposition 2.5 *Let a be a positive, C^2 and 2π-periodic function and let u_k be a solution of (2.1) such that $\varepsilon_k = \min_{\theta \in [0,2\pi]} u_k(\theta) = u_k(\tau_k) \to 0$ and $\tau_k \to \theta_0$. Then the functions $v_k, v_{k,y}, \frac{1}{v_k}$ are bounded. That is, $\exists\, C > 0$ such that*

$$\frac{1}{C} \le v_k(y) \le C, \quad |v_{k,y}(y)| \le C, \quad y \in [0, 2\pi], \tag{2.39}$$

which implies

$$C_1 U_{\varepsilon_k'}(\theta - \tau_k) \le u_k(\theta) \le C_2 U_{\varepsilon_k'}(\theta - \tau_k), \quad \theta \in [0, 2\pi]. \tag{2.40}$$

Moreover, for any $0 < \alpha < 1$

$$v_k \to v^\infty \quad in \quad C^{1,\alpha}([0, 2\pi]), \quad k \to \infty, \tag{2.41}$$

where the function v^∞ is given by

$$v^\infty(y) = \begin{cases} a^{\frac{1}{4}}(\theta_0), & y \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ \left(a^{\frac{1}{2}}(\theta_0) \sin^2 y + a(\theta_0 + \pi) a^{-\frac{1}{2}}(\theta_0) \cos^2 y\right)^{\frac{1}{2}}, & y \in [\frac{\pi}{2}, \frac{3\pi}{2}]. \end{cases} \tag{2.42}$$

Proof. For simplicity of notations we assume $\tau_k = 0$ and $\theta_0 = 0$. Let

$$\tilde{F}_k(y) = \frac{1}{2}\left(v_{k,y}^2 + v_k^2 + \frac{a(\psi_k(y))}{v_k^2} \right).$$

Then

$$\begin{aligned} \frac{d\tilde{F}_k}{dy} &= v_{k,y}\left(v_{k,yy} + v_k - \frac{a(\psi_k)}{v_k^3} \right) + \frac{\big(a(\psi_k)\big)_y}{2v_k^2} \\ &= (1 - \lambda_k) \frac{v_k v_{k,y}}{U^4_{{\varepsilon_k'}^{-1}}(y)} + \frac{a_\theta(\psi_k(y))}{2 v_k^2 U^2_{{\varepsilon_k'}^{-1}}(y)}. \end{aligned} \tag{2.43}$$

Using Proposition 2.4, we have

$$\left|\frac{1-\lambda_k}{U^2_{\varepsilon_k'^{-1}}(y)}\right| \le C\frac{\varepsilon_k^2}{{\varepsilon_k'}^{-2}\cos^2 y+{\varepsilon_k'}^2\sin^2 y} \le C. \tag{2.44}$$

Combining (2.43) and (2.44) we get that

$$\begin{aligned}\left|\frac{(1-\lambda_k)v_k v_{k,y}}{U^4_{\varepsilon_k'^{-1}}(y)}\right| &\le \frac{C|v_k v_{k,y}|}{{\varepsilon_k'}^{-2}\cos^2 y+{\varepsilon_k'}^2\sin^2 y}\\ &\le \frac{C|\tilde F_k|}{{\varepsilon_k'}^{-2}\cos^2 y+{\varepsilon_k'}^2\sin^2 y}\end{aligned}$$

and

$$\left|\frac{a_\theta(\psi_k(y))}{2v_k^2 U^2_{\varepsilon_k'^{-1}}}\right| \le \frac{Ca(\psi_k(y))}{v_k^2({\varepsilon_k'}^{-2}\cos^2 y+{\varepsilon_k'}^2\sin^2 y)} \le \frac{C|\tilde F_k|}{{\varepsilon_k'}^{-2}\cos^2 y+{\varepsilon_k'}^2\sin^2 y}.$$

Thus

$$\left|\frac{d\tilde F_k}{dy}\right| \le C\tilde F_k\cdot\frac{1}{{\varepsilon_k'}^{-2}\cos^2 y+{\varepsilon_k'}^2\sin^2 y}. \tag{2.45}$$

Since

$$\int_0^{2\pi}\frac{dy}{{\varepsilon_k'}^{-2}\cos^2 y+{\varepsilon_k'}^2\sin^2 y} = \int_0^{2\pi} d\theta = 2\pi,$$

from (2.45) we derive that

$$C^{-1}\tilde F_k(y_2) \le \tilde F_k(y_1) \le C\tilde F_k(y_2), \qquad \text{for any } y_1, y_2.$$

In particular, since $\tilde F_k(0)\le C$, this implies that

$$C^{-1} \le \tilde F_k(y) \le C,$$

that is,

$$v_k(y)\le C, \qquad |v_{k,y}(y)|\le C \quad \text{and} \quad \frac{1}{v_k}\le C, \tag{2.46}$$

which concludes (2.39) and (2.40).

From (2.39) a better estimate for λ_k-1 can be derived:

$$|\lambda_k-1| \le C\varepsilon_k^4\log\frac{1}{\varepsilon_k}. \tag{2.47}$$

In fact, from (2.35), we have that

$$\begin{aligned}|\lambda_k-1|\int_0^{2\pi}u_k^2 d\theta \le& \left|\int_0^{2\pi}\frac{a'(\theta)\sin 2\theta}{u_k^2}\right| d\theta\\ \le& C\int_0^{2\pi}\frac{|\sin 2\theta|}{u_k^2}d\theta \le C\int_0^{2\pi}\frac{|\sin 2\theta|}{U^2_{\varepsilon_k'}(\theta)}d\theta\\ \le& C{\varepsilon_k'}^2\log\frac{1}{\varepsilon_k'} \le C\varepsilon_k^2\log\frac{1}{\varepsilon_k},\end{aligned}$$

which proves (2.47).

Let $f_k = (1-\lambda_k)\frac{v_k}{U^4_{\varepsilon_k'^{-1}}(y)}$. From (2.47), we see that for $p > 1$,

$$\int_0^{2\pi} |f_k|^p dy \le C|1-\lambda_k|^p \int_0^{2\pi} \frac{1}{U^{4p}_{\varepsilon_k'^{-1}}} dy \le C(\varepsilon_k^4 \log \frac{1}{\varepsilon_k})^p \varepsilon_k^{2-4p} \to 0 \tag{2.48}$$

where we have used the following estimate

$$\int_0^{2\pi} \frac{1}{(\varepsilon_k'^{-2}\cos^2 y + \varepsilon_k'^2 \sin^2 y)^{2p}} dy \le C\varepsilon_k^{2-4p}.$$

Standard regularity shows that $\{v_k\}$ converges to some v^∞ in $C^{1,\alpha}$-norm for any $\alpha > 0$. Away from $y = \frac{\pi}{2}, \frac{3\pi}{2}$, $U^4_{\varepsilon_k'^{-1}}(y)$ is bounded from below, so v^∞ is C^2 if $y \ne \frac{\pi}{2}, \frac{3\pi}{2}$ and satisfies

$$v^\infty_{yy} + v^\infty = \frac{a(0)}{(v^\infty)^3}, \quad y \in (-\frac{\pi}{2}, \frac{\pi}{2}), \tag{2.49}$$

and

$$v^\infty_{yy} + v^\infty = \frac{a(\pi)}{(v^\infty)^3}, \quad y \in (\frac{\pi}{2}, \frac{3\pi}{2}). \tag{2.50}$$

Hence

$$v^\infty(y) = a^{\frac{1}{4}}(0) \quad if \quad y \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

since $v^\infty(0) = \lim_{k\to\infty} v_k(0) = a^{\frac{1}{4}}(0)$ and $(v^\infty)'(0) = \lim_{k\to\infty} v_k'(0) = 0$. It follows that

$$v^\infty(\frac{\pi}{2}) = v^\infty(0) = a^{\frac{1}{4}}(0), \quad (v^\infty)'(\frac{\pi}{2}) = (v^\infty)'(0) = 0.$$

Therefore, from (2.50) we get

$$v^\infty(y) = \left(a^{\frac{1}{2}}(0)\sin^2 y + a(\pi)a^{-\frac{1}{2}}(0)\cos^2 y\right)^{\frac{1}{2}}, \quad y \in [\frac{\pi}{2}, \frac{3\pi}{2}].$$

□

Remark: Under the same conditions as those of Proposition 2.5, the following estimates can be obtained:

$$\|v_k - v_k^\infty\|_{C^1([0,2\pi])} \le C\varepsilon_k^2 |\log \varepsilon_k|, \tag{2.51}$$

where the function v_k^∞ is given by

$$v_k^\infty(y) = \begin{cases} a^{\frac{1}{4}}(\tau_k), & y \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ \left(a^{\frac{1}{2}}(\tau_k)\sin^2 y + a(\tau_k+\pi)a^{-\frac{1}{2}}(\tau_k)\cos^2 y\right)^{\frac{1}{2}}, & y \in [\frac{\pi}{2}, \frac{3\pi}{2}]. \end{cases} \tag{2.52}$$

By definition, $v_k^\infty \in C^1([-\frac{\pi}{2}, \frac{3\pi}{2}])$ and is uniformly bounded from above and below.

Observe that, v_k^∞ satisfies

$$\frac{d^2 v_k^\infty}{dy^2} + v_k^\infty = \begin{cases} \dfrac{a(\tau_k)}{(v_k^\infty)^3}, & \text{for } y \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ \\ \dfrac{a(\tau_k + \pi)}{(v_k^\infty)^3}, & \text{for } y \in [\frac{\pi}{2}, \frac{3\pi}{2}]. \end{cases} \tag{2.53}$$

Combining (2.38) and (2.53), we have for $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\begin{aligned} \frac{d^2(v_k - v_k^\infty)}{dy^2} + (v_k - v_k^\infty) &= \frac{a(\tau_k + \psi_k) - a(\tau_k)}{v_k^3} + \frac{a(\tau_k)}{v_k^3} - \frac{a(\tau_k)}{(v_k^\infty)^3} + \frac{(1-\lambda_k)v_k}{U^4_{\varepsilon_k'^{-1}}(y)} \\ &= \frac{a(\tau_k + \psi_k) - a(\tau_k)}{v_k^3} + c_k(y)(v_k - v_k^\infty) + \frac{(1-\lambda_k)v_k}{U^4_{\varepsilon_k'^{-1}}(y)}, \end{aligned} \tag{2.54}$$

where $\varepsilon_k' = (a(\tau_k))^{-\frac{1}{4}}\varepsilon_k$, and c_k is a bounded function due to (2.39) and the definition of v_k^∞. In fact, $c_k(y) = -3 + o(1)$. Thus, the function $\varphi(y) := v_k - v_k^\infty$ satisfies

$$\frac{d^2\varphi}{dy^2} + \tilde{c}_k\varphi = \tilde{f}, \ \varphi(-\frac{\pi}{2}) = \varphi'(-\frac{\pi}{2}) = 0 \tag{2.55}$$

where $\tilde{c}_k$ is a bounded function and

$$\tilde{f} = \frac{a(\tau_k + \psi_k) - a(\tau_k)}{v_k^3} + \frac{(1-\lambda_k)v_k}{U^4_{\varepsilon_k'^{-1}}(y)}.$$

Let $\tilde{h}_i, i = 1, 2$ be the two fundamental solutions of the linear equation $\frac{d^2\varphi}{dy^2} + \tilde{c}_k\varphi = 0$ such that $\tilde{h}_1(0) = 1, \tilde{h}_1'(0) = 0, \tilde{h}_2(0) = 0, \tilde{h}_2'(0) = 1$. Since $\tilde{c}_k$ is bounded, it is easy to see that $\|\tilde{h}_i\|_{C^1([-\frac{\pi}{2}, \frac{\pi}{2}])} \leq C, i = 1, 2$. By the method of variation of parameters (since $\varphi(\frac{\pi}{2}) = \varphi'(\frac{\pi}{2}) = 0$), we have

$$\varphi(y) = -\tilde{h}_1(y)\int_{-\frac{\pi}{2}}^{y} \tilde{h}_2\tilde{f} + \tilde{h}_2(y)\int_{-\frac{\pi}{2}}^{y} \tilde{h}_1\tilde{f} \tag{2.56}$$

and hence

$$\|v_k - v_k^\infty\|_{C^1([-\frac{\pi}{2}, \frac{\pi}{2}])} \leq C\|\tilde{f}\|_{L^1([-\frac{\pi}{2}, \frac{\pi}{2}])}. \tag{2.57}$$

Note that

$$
\begin{aligned}
&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |a(\tau_k+\psi_k(y))-a(\tau_k)|dy \\
=&\int_{-\frac{\pi}{2}+\tau_k}^{\frac{\pi}{2}+\tau_k} |a(\theta)-a(\tau_k)|\frac{a^{\frac{1}{2}}(\tau_k)\varepsilon_k^2}{a(\tau_k)\sin^2(\theta-\tau_k)+\varepsilon_k^4\cos^2(\theta-\tau_k)}d\theta \\
=&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |a(\theta+\tau_k)-a(\tau_k)|\frac{a^{\frac{1}{2}}(\tau_k)\varepsilon_k^2}{a(\tau_k)\sin^2\theta+\varepsilon_k^4\cos^2\theta}d\theta \\
=&a^{\frac{1}{2}}(\tau_k)\varepsilon_k^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{|a(\theta+\tau_k)-a(\tau_k)|}{|\sin\theta|}\frac{|\sin\theta|}{a(\tau_k)\sin^2\theta+\varepsilon_k^4\cos^2\theta}d\theta \\
\leq&C\varepsilon_k^2|\log\varepsilon_k|.
\end{aligned}
\tag{2.58}
$$

On the other hand, similar to (2.48), it is easy to see that

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\frac{(1-\lambda_k)v_k}{U^4_{\varepsilon_k'^{-1}}(y)}|dy \leq C\varepsilon_k^2|\log\varepsilon_k|. \tag{2.59}
$$

Combining (2.58) and (2.59), we obtain

$$
\|v_k-v_k^\infty\|_{C^1([-\frac{\pi}{2},\frac{\pi}{2}])} \leq C\varepsilon_k^2|\log\varepsilon_k|. \tag{2.60}
$$

For $y\in[\frac{\pi}{2},\frac{3\pi}{2}]$, v_k^∞ satisfies

$$
\frac{d^2v_k^\infty}{dy^2}+v_k^\infty=\frac{a(\tau_k+\pi)}{(v_k^\infty)^3}.
$$

Using the similar arguments as before, we can obtain

$$
\|v_k-v_k^\infty\|_{C^1([\frac{\pi}{2},\frac{3\pi}{2}])} \leq C\varepsilon_k^2|\log\varepsilon_k|. \tag{2.61}
$$

Now the estimate (2.51) is a consequence of (2.60) and (2.61).

3 Sharp blow-up estimates

In this section we will use Pohozaev Identities to get a sharp estimate of λ_k-1. Let $a(\theta)$ be a 2π-periodic positive C^2 function and $\lambda_k\to 1$, and let u_k be a 2π-periodic solution of

$$
u_{k,\theta\theta}+\lambda_k u_k=\frac{a(\theta)}{u_k^3}, \quad \theta\in[0,2\pi]. \tag{3.1}
$$

The main result of this section is

Proposition 3.1 *Assume that* $\min_{\theta\in[0,2\pi]} u_k(\theta) = u_k(\tau_k) = \varepsilon_k \to 0$ *and* $\tau_k \to \theta_0$. *Then*

$$A_2(\theta_0) = \frac{a'(\theta_0)}{\sqrt{a(\theta_0)}} + \frac{a'(\pi+\theta_0)}{\sqrt{a(\pi+\theta_0)}} = 0, \tag{3.2}$$

and

$$\begin{aligned}
&\lambda_k - 1 \\
&= \frac{\varepsilon_k^4}{2\pi a^2(0)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\big(a'(\theta+\theta_0) + a'(\theta+\pi+\theta_0) - a'(\theta_0) - a'(\pi+\theta_0)\big)\sin 2\theta}{\sin^2\theta} d\theta + o(\varepsilon_k^4) \\
&= \frac{\varepsilon_k^4}{2\pi a^2(0)} B_2(\theta_0) + o(\varepsilon_k^4).
\end{aligned} \tag{3.3}$$

Proof. For simplicity of notations, we assume $\tau_k = 0$. This can be achieved by translation. Then $\theta_0 = 0$. By (2.17) and Proposition 2.4 we get that

$$\begin{aligned}
\int_0^{2\pi} \frac{a'(\theta)}{u_k^2}(\cos 2\theta + 1) d\theta &= 4\int_0^{2\pi} (\lambda_k - 1) u_k u_k' (\cos 2\theta + 1) d\theta \\
&= 4(\lambda_k - 1) \int_0^{2\pi} u_k^2 \sin 2\theta d\theta \\
&= 4(\lambda_k - 1) \int_0^{2\pi} \big(\frac{u_k}{M_k}\big)^2 \sin 2\theta d\theta \cdot M_k^2 \\
&= 4(\lambda_k - 1) \Big(\int_0^{2\pi} \sin^2\theta \sin 2\theta d\theta + o(1) \Big) M_k^2 \\
&= o\big(|1 - \lambda_k| M_k^2\big) = o(1).
\end{aligned} \tag{3.4}$$

On the other hand, as $k \to +\infty$, using the change of variable $\theta = \psi_k(y)$ we have

$$\begin{aligned}
\int_0^{2\pi} \frac{a'(\theta)}{u_k^2(\theta)}(\cos 2\theta + 1) d\theta &= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)}{u_k^2(\theta)}(\cos 2\theta + 1) d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\psi_k(y))}{v_k^2(y)} \frac{2\cos^2 y}{\cos^2 y + a^{-1}(\tau_k)\varepsilon_k^4 \sin^2 y} dy \\
&= 2\Big(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(0)}{(v^\infty)^2(y)} dy + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\pi)}{(v^\infty)^2(y)} dy\Big) + o(1) \\
&= 2\pi\Big(\frac{a'(0)}{\sqrt{a(0)}} + \frac{a'(\pi)}{\sqrt{a(\pi)}}\Big) + o(1)
\end{aligned} \tag{3.5}$$

by Lebesgue Dominated Convergence Theorem and Proposition 2.5. This proves (3.2).

The proof of (3.3) is more involved.

By (2.18) we see that

$$\int_0^{2\pi} \frac{a'(\theta)}{u_k^2(\theta)} \sin 2\theta d\theta = -4(\lambda_k - 1) \int_0^{2\pi} u_k^2(\theta) \cos 2\theta d\theta. \tag{3.6}$$

From $u_k(\psi_k(y))({\varepsilon'_k}^{-2} \cos^2 y + {\varepsilon'_k}^2 \sin^2 y)^{\frac{1}{2}} \to v^\infty(y)$ in C^1 we have

$$\begin{aligned} &|u_k(\psi_k(y))({\varepsilon'_k}^{-2} \cos^2 y + {\varepsilon'_k}^2 \sin^2 y)^{\frac{1}{2}} - v^\infty(y)| \\ =&|u_k(\theta)({\varepsilon'_k}^2 \cos^2 \theta + {\varepsilon'_k}^{-2} \sin^2 \theta)^{-\frac{1}{2}} - v^\infty(\psi_k^{-1}(\theta))| = o(1). \end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned} \int_0^{2\pi} u_k^2(\theta) \cos 2\theta d\theta =& \int_0^{2\pi} \big((v^\infty)^2(\psi_k^{-1}(\theta)) + o(1)\big)({\varepsilon'_k}^2 \cos^2 \theta + {\varepsilon'_k}^{-2} \sin^2 \theta) \cos 2\theta d\theta \\ =&\varepsilon_k^{-2} a^{\frac{1}{2}}(\tau_k) \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} (v^\infty)^2(\psi_k^{-1}(\theta)) \sin^2 \theta \cos 2\theta d\theta + o(\varepsilon_k^{-2}) \\ =&\varepsilon_k^{-2} a^{\frac{1}{2}}(0) \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} (v^\infty)^2(\psi_k^{-1}(\theta)) \sin^2 \theta \cos 2\theta d\theta + o(\varepsilon_k^{-2}). \end{aligned} \tag{3.8}$$

For $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $\psi_k^{-1}(\theta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, so $(v^\infty)^2(\psi_k^{-1}(\theta)) = a^{\frac{1}{2}}(0)$ and

$$a^{\frac{1}{2}}(0) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (v^\infty)^2(\psi_k^{-1}(\theta)) \sin^2 \theta \cos 2\theta d\theta = -\frac{\pi}{4} a(0). \tag{3.9}$$

Similarly, if $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, we have $\psi_k^{-1}(\theta) \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and

$$\begin{aligned} (v^\infty)^2(\psi_k^{-1}(\theta)) =& a^{\frac{1}{2}}(0) \sin^2 \psi_k^{-1}(\theta) + a(\pi) a^{-\frac{1}{2}}(0) \cos^2 \psi_k^{-1}(\theta) \\ =& \frac{{\varepsilon'}_k^4 a(\pi) a^{-\frac{1}{2}}(0) \cos^2 \theta + a^{\frac{1}{2}}(0) \sin^2 \theta}{{\varepsilon'}_k^4 \cos^2 \theta + \sin^2 \theta}. \end{aligned}$$

Then

$$\begin{aligned} & a^{\frac{1}{2}}(0) \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (v^\infty)^2(\psi_k^{-1}(\theta)) \sin^2 \theta \cos 2\theta d\theta \\ =& a^{\frac{1}{2}}(0) \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{{\varepsilon'}_k^4 a(\pi) a^{-\frac{1}{2}}(0) \cos^2 \theta + a^{\frac{1}{2}}(0) \sin^2 \theta}{{\varepsilon'}_k^4 \cos^2 \theta + \sin^2 \theta} \sin^2 \theta \cos 2\theta d\theta \\ =& a(0) \int_{\pi}^{2\pi} \sin^2 \theta \cos 2\theta d\theta + o(1) \\ =& -\frac{\pi}{4} a(0) + o(1). \end{aligned} \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we get

$$\int_0^{2\pi} u_k^2(\theta)\cos 2\theta d\theta = -\frac{\pi}{2}a(0)\varepsilon_k^{-2} + o(\varepsilon_k^{-2}). \tag{3.11}$$

Now we estimate the left hand side of (3.6). It follows from (3.7) that

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)\sin 2\theta}{u_k^2(\theta)}d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)}{u_k^2(\theta)({\varepsilon'_k}^2\cos^2\theta + {\varepsilon'_k}^{-2}\sin^2\theta)^{-1}} \frac{\sin 2\theta d\theta}{({\varepsilon'_k}^2\cos^2\theta + {\varepsilon'_k}^{-2}\sin^2\theta)} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{(v_k^\infty)^2(\psi_k^{-1}(\theta))} + O(\varepsilon_k^2|\log\varepsilon_k|)\right)\frac{{\varepsilon'_k}^2 a'(\theta)\sin 2\theta}{{\varepsilon'_k}^4\cos^2\theta + \sin^2\theta}d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{(v_k^\infty)^2(\psi_k^{-1}(\theta))}\frac{{\varepsilon'_k}^2 a'(\theta)\sin 2\theta}{{\varepsilon'_k}^4\cos^2\theta + \sin^2\theta}d\theta + o(\varepsilon_k^2) \\
&= \frac{{\varepsilon'_k}^2}{a^{\frac{1}{2}}(\tau_k)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)\sin 2\theta}{{\varepsilon'_k}^4\cos^2\theta + \sin^2\theta}d\theta + o(\varepsilon_k^2) \\
&= \frac{{\varepsilon'_k}^2}{a^{\frac{1}{2}}(\tau_k)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)-a'(0)}{\sin\theta}\frac{\sin\theta\sin 2\theta}{{\varepsilon'_k}^4\cos^2\theta + \sin^2\theta}d\theta + o(\varepsilon_k^2) \\
&= \frac{\varepsilon_k^2}{a(\tau_k)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)-a'(0)}{\sin^2\theta}\sin 2\theta d\theta + o(\varepsilon_k^2) \\
&= \frac{\varepsilon_k^2}{a(0)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)-a'(0)}{\sin^2\theta}\sin 2\theta d\theta + o(\varepsilon_k^2),
\end{aligned} \tag{3.12}$$

where we have used (2.51).

Similarly, we have

$$\begin{aligned}
&\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)\sin 2\theta}{u_k^2}d\theta \\
= &\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{(v_k^\infty)^2(\psi_k^{-1}(\theta))}\frac{{\varepsilon'_k}^2 a'(\theta)\sin 2\theta}{{\varepsilon'_k}^4\cos^2\theta + \sin^2\theta}d\theta + o(\varepsilon_k^2) \\
= &\,{\varepsilon'}_k^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)\sin 2\theta}{{\varepsilon'}_k^4 a(\pi+\tau_k)a^{-\frac{1}{2}}(\tau_k)\cos^2\theta + a^{\frac{1}{2}}(\tau_k)\sin^2\theta}d\theta + o(\varepsilon_k^2) \\
= &\,{\varepsilon'}_k^2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)-a'(\pi)}{\sin\theta}\frac{\sin\theta\sin 2\theta}{{\varepsilon'}_k^4 a(\pi+\tau_k)a^{-\frac{1}{2}}(\tau_k)\cos^2\theta + a^{\frac{1}{2}}(\tau_k)\sin^2\theta}d\theta + o(\varepsilon_k^2) \\
= &\,\frac{\varepsilon_k^2}{a(\tau_k)}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{a'(\theta)-a'(\pi)}{\sin^2\theta}\sin 2\theta d\theta + o(\varepsilon_k^2) \\
= &\,\frac{\varepsilon_k^2}{a(0)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta+\pi)-a'(\pi)}{\sin^2\theta}\sin 2\theta d\theta + o(\varepsilon_k^2).
\end{aligned} \tag{3.13}$$

Combining (3.6) and (3.13) we can obtain that

$$\int_0^{2\pi} \frac{a'(\theta)\sin 2\theta}{u_k^2} d\theta = \frac{\varepsilon_k^2}{a(0)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta) - a'(0) + a'(\theta+\pi) - a'(\pi)}{\sin^2\theta} \sin 2\theta d\theta + o(\varepsilon_k^2).$$

This completes the proof of Proposition 3.1. □

4 Proofs of Theorem 1.1 and Theorem 1.2

In this section, we prove Theorem 1.1 and Theorem 1.2. We only discuss the case $n = 2$ since for $n > 2$, the proof is the same.

Proof of Theorem 1.1: This is an immediate consequence of Proposition 3.1. Indeed, if there is a sequence u_k of

$$u_{k,\theta\theta} + u_k = \frac{a(\theta)}{u_k^3}, \quad \theta \in \mathbb{S}^1 \tag{4.1}$$

such that $\varepsilon_k = \min_{\theta\in\mathbb{S}^1} u_k(\theta) = u_k(\tau_k) \to 0$. Let $\tau_k \to \theta_0$, then by Proposition 3.1, we have

$$A_2(\theta_0) = \frac{a'(\theta_0)}{\sqrt{a(\theta_0)}} + \frac{a'(\pi+\theta_0)}{\sqrt{a(\pi+\theta_0)}} = 0 \tag{4.2}$$

and

$$\begin{aligned} &\frac{\varepsilon_k^4}{2\pi a^2(\theta_0)} \int_0^{\pi} \frac{\big(a'(\theta+\theta_0) + a'(\theta+\theta_0+\pi) - a'(\theta_0) - a'(\theta_0+\pi)\big)\sin 2\theta}{\sin^2\theta} d\theta + o(\varepsilon_k^4) \\ =&\frac{\varepsilon_k^4}{2\pi a^2(\theta_0)} B_2(\theta_0) + o(\varepsilon_k^4) = 0. \end{aligned}$$

Hence

$$B_2(\theta_0) = 0. \tag{4.3}$$

This contradicts the assumption that a is B_2-nondegenerate and proves Theorem 1.1. □

Proof of Theorem 1.2: Without loss of generality we suppose $\min_{A_2(\theta)=0} B_2(\theta) > 0$. We take a sequence $\lambda_k < 1$ and $\lambda_k \to 1$. According to [17], there is a 2π-periodic solution u_k of

$$u_{k,\theta\theta} + \lambda_k u_k = \frac{a(\theta)}{u_k^3}. \tag{4.4}$$

If $\varepsilon_k = \min_{\theta\in\mathbb{S}^1} u_k(\theta) = u_k(\tau_k) \to 0, \tau_k \to \theta_0$, then by Proposition 3.1 we have $A_2(\theta_0) = 0$ and

$$\lambda_k - 1 = \frac{\varepsilon_k^4}{2\pi a^2(\theta_0)} B_2(\theta_0) + o(\varepsilon_k^4). \tag{4.5}$$

So $B_2(\theta_0) < 0$, which is impossible. Thus the sequence u_k are uniformly bounded from below and above. By taking a limit, we obtain a solution u of (1.1). □

5 Proof of Theorem 1.3

In this section, we will use Liapunov-Schmidt reduction and degree theory to prove Theorem 1.3. Similar approach has been used by Rey-Wei ([30], [31]) and Wei-Xu ([34]). In particular, we shall follow the argument in Section 4 of [34]. For simplicity we only give the proof of 2π-periodic case and that of $n > 2$ is similar and hence it is omitted.

Let ε be a small positive number and

$$S[u] := u_{\theta\theta} + u - \frac{1+\varepsilon a(\theta)}{u^3}. \tag{5.1}$$

In order to prove Theorem 1.3, we need to find a solution of $S[u] = 0$. In the following we consider $S[u] = 0$ as a perturbation of

$$u_{\theta\theta} + u = \frac{1}{u^3}. \tag{5.2}$$

It is known that all solutions of (5.2) are given by

$$U_{\Lambda,t}(\theta) = \left(\Lambda^2\cos^2(\theta-t) + \Lambda^{-2}\sin^2(\theta-t)\right)^{\frac{1}{2}},$$

where $(\Lambda, t) \in (0,1] \times \mathbb{S}^1$. Note that when $\Lambda = 1, U_{\Lambda,t} \equiv 1$ which is independent of t.

We are going to find a 2π-periodic solution u of $S[u] = 0$ having the form

$$u(\theta) = U_{\Lambda,t}(\theta) + \phi(\theta), \tag{5.3}$$

where $\phi(\theta)$ is relatively small.

Substituting (5.3) into the equation (5.1), we obtain

$$S[U_{\Lambda,t} + \phi] = S[U_{\Lambda,t}] + L_{\Lambda,t}[\phi] + N[\phi], \tag{5.4}$$

where

$$S[U_{\Lambda,t}] = \frac{d^2U_{\Lambda,t}}{d\theta^2} + U_{\Lambda,t} - \frac{1+\varepsilon a(\theta)}{U_{\Lambda,t}^3} = -\frac{\varepsilon a(\theta)}{U_{\Lambda,t}^3},$$

$$L_{\Lambda,t}[\phi] = \frac{d^2\phi}{d\theta^2} + \phi + \frac{3\phi}{U_{\Lambda,t}^4},$$

and

$$N[\phi] = -\left(\frac{1+\varepsilon a(\theta)}{(U_{\Lambda,t}+\phi)^3} - \frac{1+\varepsilon a(\theta)}{U_{\Lambda,t}^3} + \frac{3\phi}{U_{\Lambda,t}^4}\right).$$

Since $L_{\Lambda,t}$ is a second order ODE operator, the kernel of L is two-dimensional. It is easy to see that

$$Z_1 = \frac{\Lambda\cos^2(\theta-t) - \Lambda^{-3}\sin^2(\theta-t)}{U_{\Lambda,t}}, \quad Z_2 = \frac{\sin(2(\theta-t))}{U_{\Lambda,t}} \tag{5.5}$$

are orthogonal and satisfy $L_{\Lambda,t}[Z_1] = L_{\Lambda,t}[Z_2] = 0$. Thus we deduce that

$$\{\phi \in C^2(\mathbb{S}^1) | L_{\Lambda,t}[\phi] = 0\} = span\{Z_1, Z_2\}.$$

For later purpose, we need to consider another two kernels:

$$\bar{Z}_1 = \cos(2t) a_\Lambda Z_1 - \sin(2t) b_\Lambda Z_2, \quad \bar{Z}_2 = \cos(2t) b_\Lambda Z_2 + \sin(2t) a_\Lambda Z_1 \tag{5.6}$$

where $a_\Lambda = \frac{1}{\sqrt{\int_0^{2\pi} Z_1^2}}, b_\Lambda = \frac{1}{\sqrt{\int_0^{2\pi} Z_2^2}}$.

It is easy to see that $\int_0^{2\pi} \bar{Z}_1^2 = 1, \int_0^{2\pi} \bar{Z}_2^2 = 1, \int_0^{2\pi} \bar{Z}_1 \bar{Z}_2 = 0$, and $span\{Z_1, Z_2\} = span\{\bar{Z}_1, \bar{Z}_2\}$. The reason for choosing $\bar{Z}_1, \bar{Z}_2$ instead of Z_1, Z_2 will be clear later. But we note that when $\Lambda = 1, \bar{Z}_1 = \frac{1}{\sqrt{\pi}} \cos(2\theta), \bar{Z}_2 = \frac{1}{\sqrt{\pi}} \sin(2\theta)$ (which are independent of t).

For $h \in C(\mathbb{S}^1)$, we consider the following linear problem:

$$\begin{cases} L_{\Lambda,t}[\phi] = h + c_1 \bar{Z}_1 + c_2 \bar{Z}_2, \quad \theta \in \mathbb{S}^1, \\ \\ \displaystyle\int_0^{2\pi} \phi(\theta) \bar{Z}_1 d\theta = \int_0^{2\pi} \phi(\theta) \bar{Z}_2 d\theta = 0, \end{cases} \tag{5.7}$$

where $(c_1, c_2) \in \mathbb{R}^2$. By Fredholm Alternative theorem, we know that (5.7) is solvable if and only if (c_1, c_2) satisfies

$$\begin{cases} \displaystyle\int_0^{2\pi} h \bar{Z}_1 d\theta + c_1 \int_0^{2\pi} \bar{Z}_1^2 d\theta + c_2 \int_0^{2\pi} \bar{Z}_1 \bar{Z}_2 d\theta = 0, \\ \displaystyle\int_0^{2\pi} h \bar{Z}_2 d\theta + c_1 \int_0^{2\pi} \bar{Z}_1 \bar{Z}_2 d\theta + c_2 \int_0^{2\pi} Z_2^2 d\theta = 0. \end{cases} \tag{5.8}$$

Since $\int_0^{2\pi} \bar{Z}_1 \bar{Z}_2 d\theta = 0$, (c_1, c_2) is uniquely determined by the following

$$c_1 = \int_0^{2\pi} h \bar{Z}_1 d\theta, \qquad c_2 = \int_0^{2\pi} h \bar{Z}_2 d\theta. \tag{5.9}$$

Moreover, if (5.8) is satisfied, then the solution is unique and there is a positive constant C which depends on the lower bound of Λ only such that

$$\|\phi\|_{C(\mathbb{S}^1)} \le C \|h\|_{C(\mathbb{S}^1)} \tag{5.10}$$

and

$$|c_1| + |c_2| \le C \|h\|_{C(\mathbb{S}^1)}. \tag{5.11}$$

The estimate (5.10) is a consequence of the fact that $h \to \phi$ is a bounded linear operator from $C(\mathbb{S}^1) \to C(\mathbb{S}^1)$, and (5.11) follows from (5.8).

After solving the linear problem, now we can solve the nonlinear problem:

$$\begin{cases} L_{\Lambda,t}[\phi] = -S[U_{\Lambda,t}] - N[\phi] + c_1 \bar{Z}_1 + c_2 \bar{Z}_2, \qquad \theta \in \mathbb{S}^1, \\ \displaystyle\int_0^{2\pi} \phi(\theta) \bar{Z}_1 d\theta = \int_0^{2\pi} \phi(\theta) \bar{Z}_2 d\theta = 0, \end{cases} \tag{5.12}$$

for some coefficients c_1 and c_2. Namely we have

Lemma 5.1 *For $\Lambda_0 > 0$, there exist $\varepsilon_0 > 0$ and C which is independent of ε such that for any $\varepsilon < \varepsilon_0$, $\Lambda_0 \le \Lambda \le 1$ and $t \in \mathbb{S}^1$, problem (5.12) has a unique solution $(\phi, c_1, c_2) = (\phi_{\Lambda,t,\varepsilon}, c_{\Lambda,t,\varepsilon}, c_{\Lambda,t,\varepsilon})$ satisfying*

$$\|\phi_{\Lambda,t,\varepsilon}\|_{C(\mathbb{S}^1)} \le C\varepsilon. \tag{5.13}$$

Moreover, the maps $(\Lambda, t) \to \phi_{\Lambda,t,\varepsilon}$ and $(\Lambda, t) \to (c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon))$ are continuous.

When $\Lambda = 1$, $\phi_{\Lambda,t,\varepsilon}$ and $(c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon))$ are independent of t.

Proof. We will use the contraction mapping principle to prove the lemma. To this end, we write the first equation of problem (5.12) in its equivalent form:

$$\phi = A\big(- S[U_{\Lambda,t}] - N[\phi]\big) := B[\phi]. \tag{5.14}$$

For a positive constant $\varepsilon_1 \le \frac{\Lambda_0}{2}$, define a convex set in $C(\mathbb{S}^1)$ by

$$\mathcal{C} := \left\{\phi \;\middle|\; \begin{array}{l} \phi \text{ is } 2\pi-\text{periodic}, \ \|\phi\|_{C(\mathbb{S}^1)} \le \varepsilon_1, \\ \displaystyle\int_0^{2\pi} \phi(\theta)\bar{Z}_1 d\theta = \int_0^{2\pi} \phi(\theta)\bar{Z}_2 d\theta = 0 \end{array}\right\}.$$

It follows from the mean value theorem that

$$\|N[\phi]\|_{C(\mathbb{S}^1)} \le C(\Lambda_0)(\|\frac{\varepsilon a \phi}{U_{\Lambda,t}^4}\|_{C(\mathbb{S}^1)} + \|\frac{\phi^2}{U_{\Lambda,t}^5}\|_{C(\mathbb{S}^1)}) \le C(\Lambda_0)(\varepsilon_1\varepsilon + \varepsilon_1^2), \quad \forall\ \phi \in \mathcal{C}. \tag{5.15}$$

Since $S[U_{\Lambda,t}] = -\frac{\varepsilon a}{U_{\Lambda,t}^3}$, we have that for $\phi, \phi_1 \in \mathcal{C}$,

$$\begin{aligned} \|B[\phi]\|_{C(\mathbb{S}^1)} &\le C(\Lambda_0)(\|S[U_{\Lambda,t}]\|_{C(\mathbb{S}^1)} + \|N[\phi]\|_{C(\mathbb{S}^1)}) \\ &\le C(\Lambda_0)(\varepsilon + \varepsilon\varepsilon_1 + \varepsilon_1^2) \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} \|B[\phi_1] - B[\phi]\|_{C(\mathbb{S}^1)} &\le C(\Lambda_0)\|N[\phi_1] - N[\phi]\|_{C(\mathbb{S}^1)} \\ &\le C(\Lambda_0)(\varepsilon + \|\phi_1\|_{C(\mathbb{S}^1)} + \|\phi\|_{C(\mathbb{S}^1)})\|\phi_1 - \phi\|_{C(\mathbb{S}^1)} \\ &\le C(\Lambda_0)(\varepsilon + 2\varepsilon_1)\|\phi_1 - \phi\|_{C(\mathbb{S}^1)}. \end{aligned} \tag{5.17}$$

Letting $\varepsilon_0 = \frac{\varepsilon_1}{4C(\Lambda_0)}$, $\varepsilon_1 < \frac{1}{4C(\Lambda_0)}$, then (5.16) and (5.17) imply that the operator B is a contraction mapping from $\mathcal{C}$ to $\mathcal{C}$. Hence $\mathcal{C}$ has a unique fixed point $\phi_{\Lambda,t,\varepsilon} \in \mathcal{C}$ and

$$\|\phi_{\Lambda,t,\varepsilon}\|_{C(\mathbb{S}^1)} = \|B[\phi_{\Lambda,t,\varepsilon}]\|_{C(\mathbb{S}^1)} \le 2C(\Lambda_0)\varepsilon + C(\Lambda_0)\|\phi_{\Lambda,t,\varepsilon}\|^2_{C(\mathbb{S}^1)},$$

that is

$$\|\phi_{\Lambda,t,\varepsilon}\|_{C(\mathbb{S}^1)} \leq C\varepsilon.$$

The continuity of $\phi_{\Lambda,t,\varepsilon}$, $(c_1(\Lambda,t,\varepsilon), c_2(\Lambda,t,\varepsilon))$ on parameters Λ, t also follows from the contraction mapping theorem.

When $\Lambda = 1, U_{\Lambda,t} = 1, \bar{Z}_1 = \frac{1}{\sqrt{\pi}}\cos(2\theta), \bar{Z}_2 = \frac{1}{\sqrt{\pi}}\sin(2\theta)$ are all independent of t and hence $\phi_{1,t,\varepsilon}, c_1(1,t,\varepsilon), c_2(1,t,\varepsilon)$ are also independent of t. Hence Lemma 5.1 holds. □

The proof of Theorem 1.3 will be finished if for $\varepsilon \leq \varepsilon_0$ we can find some $(\Lambda, t) \in (0,1] \times \mathbb{S}^1$ such that $(c_1(\Lambda,t,\varepsilon), c_2(\Lambda,t,\varepsilon)) = 0$ in problem (5.12). This will be accomplished by degree theory. In order to use degree theory we need the asymptotic expansions of $c_1(\Lambda,t,\varepsilon)$ and $c_2(\Lambda,t,\varepsilon)$ as $\Lambda \to 0$.

According to (5.9), $c_1(\Lambda,t,\varepsilon)$ and $c_2(\Lambda,t,\varepsilon)$ satisfy

$$\begin{cases} c_1(\Lambda,t,\varepsilon) = \int_0^{2\pi}(S[U_{\Lambda,t}] + N[\phi])\bar{Z}_1 d\theta, \\ c_2(\Lambda,t,\varepsilon) = \int_0^{2\pi}(S[U_{\Lambda,t}] + N[\phi])\bar{Z}_2 d\theta. \end{cases} \tag{5.18}$$

Let

$$\begin{cases} \overline{c}_1(\Lambda,t,\varepsilon) = \frac{\int_0^{2\pi} S[U_{\Lambda,t}]Z_1 d\theta}{\int_0^{2\pi} Z_1^2 d\theta}, \\ \overline{c}_2(\Lambda,t,\varepsilon) = \frac{\int_0^{2\pi} S[U_{\Lambda,t}]Z_2 d\theta}{\int_0^{2\pi} Z_2^2 d\theta}. \end{cases} \tag{5.19}$$

Then we have

Lemma 5.2 *For $\Lambda \to 0$ we have*

$$\tilde{c}_1(\Lambda,t,\varepsilon) = \varepsilon(\frac{\overline{B}_2(t)}{2\pi}\Lambda^5 + o(\Lambda^5)) \tag{5.20}$$

and

$$\tilde{c}_2(\Lambda,t,\varepsilon) = \varepsilon(-\frac{\overline{A}_2(t)}{4} + +o(1)) \tag{5.21}$$

where $\overline{A}_2$ and $\overline{B}_2$ are defined in (1.15) and (1.16) respectively.

Proof. From the definition of $U_{\Lambda,t}$, we have

$$\begin{aligned} \int_0^{2\pi} Z_1^2 d\theta &= \int_0^{2\pi} \frac{(\Lambda\cos^2(\theta-t) - \Lambda^{-3}\sin^2(\theta-t))^2}{\Lambda^2\cos^2(\theta-t) + \Lambda^{-2}\sin^2(\theta-t)} d\theta \\ &= \Lambda^{-2}\int_0^{2\pi}(\Lambda^2\cos^2\theta + \Lambda^{-2}\sin^2\theta)d\theta \\ &\quad + 4\Lambda^{-2}\int_0^{2\pi}\frac{\cos^4\theta - \cos^2\theta}{\Lambda^2\cos^2\theta + \Lambda^{-2}\sin^2\theta}d\theta = \pi\Lambda^{-4} + O(1) \end{aligned} \tag{5.22}$$

and

$$\frac{1}{\varepsilon}\int_0^{2\pi} S[U_{\Lambda,t}]Z_1 d\theta = -\int_0^{2\pi} a(\theta+t)\frac{\Lambda\cos^2\theta - \Lambda^{-3}\sin^2\theta}{(\Lambda^2\cos^2\theta + \Lambda^{-2}\sin^2\theta)^2}d\theta. \tag{5.23}$$

Let
$$\theta=\psi_\Lambda(y)=\int_0^y\frac{d\tau}{\Lambda^{-2}\cos^2\tau+\Lambda^2\sin^2\tau}.$$
Then $\psi:\mathbb{S}^1\to\mathbb{S}^1$ is a diffeomorphism and
$$\frac{\cos^2\theta}{(\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta)^2}d\theta=\frac{1}{\Lambda^2}\cos^2 ydy,\tag{5.24}$$
$$\frac{\sin^2\theta}{(\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta)^2}d\theta=\Lambda^2\sin^2 ydy,\tag{5.25}$$
$$\frac{\sin 2\theta}{\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta}d\theta=\frac{\sin 2y}{\Lambda^{-2}\cos^2 y+\Lambda^2\sin^2 y}dy.\tag{5.26}$$
Inserting (5.24) and (5.25) into (5.23), as $\Lambda\to 0$, we get
$$\begin{aligned}\frac{1}{\varepsilon}\int_0^{2\pi}S[U_{\Lambda,t}]Z_1d\theta=&-\frac{1}{\Lambda}\int_0^{2\pi}a(\psi_\Lambda(y)+t)\cos 2ydy\\
=&\frac{1}{2\Lambda}\int_0^{2\pi}a'(\psi_\Lambda(y)+t)\psi_\Lambda'(y)\sin 2ydy\\
=&\frac{\Lambda}{2}\int_0^{2\pi}a'(\theta+t)\frac{\sin 2\theta}{\Lambda^4\cos^2\theta+\sin^2\theta}d\theta\\
=&\frac{\Lambda}{2}\int_0^{\pi}[a'(\theta+t)-a'(t)]\frac{\sin 2\theta}{\Lambda^4\cos^2\theta+\sin^2\theta}d\theta\\
&+\frac{\Lambda}{2}\int_\pi^{2\pi}[a'(\theta+t)-a'(t+\pi)]\frac{\sin 2\theta}{\Lambda^4\cos^2\theta+\sin^2\theta}d\theta\\
=&\frac{\Lambda}{2}\int_0^{\pi}\frac{[a'(\theta+t)-a'(t)]}{\sin\theta}\frac{\sin\theta\sin 2\theta}{\sin^2\theta+\Lambda^4\cos^2\theta}d\theta\\
&+\frac{\Lambda}{2}\int_0^{\pi}\frac{[a'(\theta+t+\pi)-a'(t+\pi)]}{\sin\theta}\frac{\sin\theta\sin 2\theta}{\sin^2\theta+\Lambda^4\cos^2\theta}d\theta\\
=&\frac{\Lambda}{2}\overline{B}_2(t)+o(\Lambda)\end{aligned}\tag{5.27}$$
where we have used (5.26), $\int_0^\pi\frac{\sin 2\theta}{\Lambda^4\cos^2\theta+\sin^2\theta}d\theta=0$ and the dominated convergence theorem. Hence it follows from (5.19), (5.22) and (5.27) that
$$\overline{c}_1(\Lambda,t,\varepsilon)=\varepsilon\left(\frac{\overline{B}_2(t)}{2\pi}\Lambda^5+o(\Lambda^5)\right).\tag{5.28}$$

Similarly,

$$
\begin{aligned}
\int_0^{2\pi} Z_2^2 d\theta &= \int_0^{2\pi} \frac{(\sin 2(\theta-t))^2}{\Lambda^2\cos^2(\theta-t)+\Lambda^{-2}\sin^2(\theta-t)}d\theta \\
&= \int_0^{2\pi} \frac{4\cos^2\theta\sin^2\theta}{\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta}d\theta \\
&= 4\Lambda^2\int_0^{2\pi}(\cos^2\theta-\frac{\Lambda^2\cos^4\theta}{\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta})d\theta \\
&= 4\Lambda^2\int_0^{2\pi}\cos^2\theta d\theta+O(\Lambda^4) \\
&= 4\pi\Lambda^2+O(\Lambda^4)
\end{aligned}
\tag{5.29}
$$

and

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_0^{2\pi} S[U_{\Lambda,t}]Z_2 d\theta =& -\int_0^{2\pi} a(\theta+t)\frac{\sin(2\theta)}{(\Lambda^2\cos^2\theta+\Lambda^{-2}\sin^2\theta)^2}d\theta \\
=& -\int_0^{2\pi} a(\psi_\Lambda(y)+t)\sin 2y dy \\
=& -\frac{1}{2}\int_0^{2\pi}\frac{da(\psi_\Lambda(y)+t)}{dy}(\cos 2y+1)dy \\
=& -\frac{1}{2}\int_0^{2\pi} a'(\psi_\Lambda(y)+t)\frac{\cos 2y+1}{\Lambda^{-2}\cos^2 y+\Lambda^2\sin^2 y}dy \\
=& -\Lambda^2\Big[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a'(\psi_\Lambda(y)+t)\frac{\cos^2 y}{\cos^2 y+\Lambda^4\sin^2 y}dy \\
&+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} a'(\psi_\Lambda(y)+t)\frac{\cos^2 y}{\cos^2 y+\Lambda^4\sin^2 y}dy\Big] \\
=& -\pi\Lambda^2[a'(t)+a'(\pi+t)+o(1)] \\
=&\pi\Lambda^2[-\overline{A}_2(t)+o(1)]
\end{aligned}
\tag{5.30}
$$

provided by $\psi_\Lambda(y)\to 0$ for $y\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\psi_\Lambda(y)\to\pi$ for $y\in(\frac{\pi}{2},\frac{3\pi}{2})$. So from (5.19), (5.29) and (5.30) we have

$$
\overline{c}_2(\Lambda,t,\varepsilon)=\varepsilon(-\frac{\overline{A}_2(t)}{4}+o(1)).
\tag{5.31}
$$

This completes the proof of Lemma 5.2. $\square$

Now we fix $\Lambda_0<\Lambda_1$ small enough such that for all $\Lambda_0\le\Lambda\le\Lambda_1$

$$
|(\overline{c}_1(\Lambda,t,\varepsilon),\overline{c}_2(\Lambda,t,\varepsilon)|\ge\frac{\varepsilon}{16\pi^2}[\overline{A}_2{}^2(t)+\overline{B}_2{}^2(t)\Lambda^{10}]^{\frac{1}{2}},\quad\forall\, t\in\mathbb{S}^1.
$$

Then for $s\in[0,1]$,

$$
(1-s)\frac{(\overline{c}_1(\Lambda,t,\varepsilon),\overline{c}_2(\Lambda,t,\varepsilon))}{\varepsilon}+s(\frac{\overline{B}_2(t)}{2\pi}\Lambda^5,-\frac{\overline{A}_2(t)}{4})\neq(0,0).
\tag{5.32}
$$

Lemma 5.3 *For $\varepsilon \to 0$, $\Lambda_0 \le \Lambda$, we have*

$$c_1(\Lambda, t, \varepsilon) = \cos(2t)\overline{c}_1(\Lambda, t, \varepsilon) - \sin(2t)\overline{c}_2(\Lambda, t, \varepsilon) + O(\varepsilon^2) \tag{5.33}$$

and

$$c_2(\Lambda, t, \varepsilon) = \cos(2t)\overline{c}_2(\Lambda, t, \varepsilon) + \sin(2t)\overline{c}_1(\Lambda, t, \varepsilon) + O(\varepsilon^2). \tag{5.34}$$

Proof. From (5.15) and the definitions of $c_1(\Lambda, t, \varepsilon)$ we have

$$\begin{aligned}|\frac{\int_0^{2\pi}(S(U_{\Lambda,t} + N[\phi])Z_1 d\theta}{\int_0^{2\pi} Z_1^2 d\theta} - \overline{c}_1(\Lambda, t, \varepsilon)| &= \left|\frac{\int_0^{2\pi} N[\phi]Z_1 d\theta}{\int_0^{2\pi} Z_1^2 d\theta}\right| \\ &\le C\varepsilon^2\end{aligned}$$

for some constant C depending on Λ_0. The proof of (5.34) is the same. $\square$

Now we choose an ε_0 such that for $s \in [0,1]$ and $0 < \varepsilon < \varepsilon_0$, $\Lambda_0 \le \Lambda \le \Lambda_1$,

$$(1-s)(\frac{c_1(\Lambda, t, \varepsilon)}{\varepsilon}, \frac{c_2(\Lambda, t, \varepsilon)}{\varepsilon}) + se^{2it}(\frac{\overline{c}_1(\Lambda, t, \varepsilon)}{\varepsilon}, \frac{\overline{c}_2(\Lambda, t, \varepsilon)}{\varepsilon}) \neq (0,0). \tag{5.35}$$

This is possible by Lemma 5.3. Here we denote $e^{i\theta}(x,y) = ((\cos\theta)x - (\sin\theta)y, (\cos\theta)y + (\sin\theta)x)$.

Set $\Lambda = 1 - \lambda$. We see that $\lambda \to 1$ as $\Lambda \to 0$ and $\lambda = 0$ if and only if $\Lambda = 1$. Let $\Lambda_0 = 1 - \lambda_0$. Then we have continuous maps from $D(\lambda_0) = \{(X,Y) \in \mathbb{R}^2 | X^2 + Y^2 \le \lambda_0^2\}$ to $\mathbb{R}^2$ given by

$$G_\varepsilon(X,Y) = \frac{1}{\varepsilon}(c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon)), \quad \overline{G}_\varepsilon(X,Y) = \frac{1}{\varepsilon}e^{2it}(\overline{c}_1(\Lambda, t, \varepsilon), \overline{c}_2(\Lambda, t, \varepsilon)),$$

where $(X,Y) = (\lambda\cos t, \lambda\sin t)$.

Note that $G_\varepsilon(0,0)$ and $\bar{G}_\varepsilon(0,0)$ are well-defined since when $\lambda = 0$, c_1, c_2 are independent of t.

Proof of Theorem 1.3: For ε small enough, we have by Lemma 5.2

$$\bar{G}_\varepsilon(X,Y) = e^{2it}(\frac{\overline{B}_2(t)}{2\pi}\Lambda^5 + o(\Lambda^5), -\frac{\overline{A}_2(t)}{4} + o(1)) : \mathbb{B}^2_{1-\Lambda_0} \to \mathbb{R}^2$$

where $\mathbb{B}^2_{1-\Lambda_0} = \{x^2 + y^2 \le (1-\Lambda_0)^2\}$.

Set $\delta = \frac{2\Lambda^5}{\pi}$. We obtain

$$\bar{G}_\varepsilon(X,Y) = \frac{1}{4}e^{2it}(\delta\bar{B}_2(t) + o(\delta), -\bar{A}_2(t) + o(1)). \tag{5.36}$$

By non-degeneracy assumption on the function a, we see that if δ is sufficiently small, $|\bar{G}_\varepsilon(X,Y)| > 0$ on $\partial\mathbb{B}^2_{1-\delta}(0) = \mathbb{S}^1_{1-\delta}$. By simple property of the degree theory (see Proposition 1.27 of [20]), we have

$$deg(\bar{G}_\varepsilon, \mathbb{B}^2_{1-\delta}(0), 0) = \deg(\bar{G}_\varepsilon/|\bar{G}_\varepsilon|, \mathbb{S}^1_{1-\delta}).$$

For any real numbers $s > 0$ and $\eta > 0$, we define the maps:

$$G_{s,\eta}(X,Y) = e^{2it}(-s\bar{B}_2(t), \eta\bar{A}_2(t)).$$

Since these maps never vanish for all $(X,Y) \in \mathbb{S}^1$, their degrees are well defined and they all have the same degrees which implies that

$$\begin{aligned} deg(G_\varepsilon, D(\lambda_0), 0) &= deg(\bar{G}_\varepsilon, D(\lambda_0), 0) = deg(\bar{G}_\varepsilon/|\bar{G}_\varepsilon|, \mathbb{S}^1_{1-\delta}) \\ &= deg(G_{1-r_0,1}/|G_{1-r_0,1}|, \mathbb{S}^1_{1-\delta}) \\ &= deg(G_{1,1}/|G_{1,1}|, \mathbb{S}^1_{1-\delta}) \\ &= deg(e^{2it}\bar{G}_2/|\bar{G}_2|, \mathbb{S}^1) \\ &= 2 + deg(\bar{G}_2/|\bar{G}_2|, \mathbb{S}^1) \neq 0, \end{aligned}$$

where we have identified the domains $\mathbb{S}^1$ and $\mathbb{S}^1_{1-\delta}$ for our map $\bar{G}_2$ which is clearly true since they give the same values for $\bar{G}_2$.

By the properties of the degree (see the Appendix), there exists $(\lambda, t) \in D(\lambda_0)$ such that $(c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon)) = (0,0)$. This completes the proof of Theorem 1.3. $\square$

Appendix: Definition and Properties of $d(\psi, \mathbb{S}^{N-1})$

In this appendix, we give the definition of the degree $d(\psi, \mathbb{S}^{N-1})$ (where $N \geq 2$) and collect some basic properties of the degree. All the materials in this appendix can be found in the book by Fonseca and Gambo [20].

Let $\mathbb{B}^N$ denote the unit ball $\{|x| < 1\} \subset \mathbb{R}^N$ and $\mathbb{S}^{N-1} = \partial\mathbb{B}^N$ be the unit sphere in $\mathbb{R}^N$. Let $\phi : \mathbb{B}^N \to \mathbb{R}^N$ be a continuous function such that $\phi(\mathbb{S}^{N-1}) \in \mathbb{R}^N\backslash\{0\}$. Then the Brouwer degree $d(\phi, \mathbb{B}^N, 0)$ can be defined in the usual way (Definition 1.26 of [20]). Now we define

$$\psi = \frac{\phi(x)}{|\phi(x)|} : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}.$$

By [Proposition 1.27, p.24] of [20], we have

$$(A1) \quad d(\phi, \mathbb{B}^N, 0) = d(\psi, \mathbb{S}^{N-1}, p) =: d(\psi, \mathbb{S}^{N-1}), \quad \forall\ p \in \mathbb{S}^{N-1}.$$

We recall the following properties of $d(\psi, \mathbb{S}^{N-1})$ (Corollary 2.3 of [20]): Let $\psi : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ be a continuous mapping. Then the following assertions are equivalent:

(i) ψ is not homotopic to a constant.
(ii) Every continuous extension ϕ of ψ: $\bar{\mathbb{B}}^N \to \mathbb{R}^N$ admits a zero.
(iii) Every continuous extension ϕ of ψ: $\bar{\mathbb{B}}^N \to \mathbb{R}^N$ verifies $d(\phi, \mathbb{B}^N, 0) = d(\psi, \mathbb{S}^{N-1}) \neq 0$.

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