MULTIPLE CLUSTERED LAYER SOLUTIONS FOR SEMILINEAR ELLIPTIC PROBLEMS ON $\mathbf{S}^n$

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Abstract. We consider the following superlinear elliptic equation on $\mathbf{S}^n$

$$\begin{cases} \varepsilon^2 \Delta_{\mathbf{S}^n} u - u + f(u) = 0 \ \text{ in } D; \\ u > 0 \ \text{ in } \ D \text{ and } u = 0 \ \text{ on } \partial D, \end{cases}$$

where D is a geodesic ball on $\mathbf{S}^n$ with geodesic radius θ_1, and $\Delta_{\mathbf{S}^n}$ is the Laplace-Beltrami operator on $\mathbf{S}^n$. We prove that for any $\theta \in (\frac{\pi}{2}, \pi)$ and for any positive integer $N \geq 1$, there exist at least $2N+1$ solutions to the above problem for ε sufficiently small. Moreover, the asymptotic behavior of such solutions is also characterized. We then apply this result to the Brezis-Nirenberg problem and establish the existence of solutions which are not minimizers of the associated energy.

1. Introduction

Let D be a geodesic ball in the n-dimensional sphere $\mathbf{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, centered at the North pole with geodesic radius θ_1 and let $\Delta_{\mathbf{S}^n}$ be the Laplace-Beltrami operator on $\mathbf{S}^n$. We consider the following problem

$$\begin{cases} \varepsilon^2 \Delta_{\mathbf{S}^n} u - u + f(u) = 0, u > 0 \ \text{ in } D; \\ u = 0 \ \text{ on } \partial D, \end{cases} \tag{1.1}$$

where ε is a small parameter and $f(t)$ is positive for $t > 0$ and vanishes for $t \leq 0$. The class of nonlinearities we are interested in, includes powers of the type $f(u) = u^p$, $p > 1$ and more generally $f(u) = u^p + \sum_{i=1}^{l} a_i u^{q_i}, 1 < q_i < p < \infty$.

Of particular interest is the case of $\theta_1 \in (\frac{\pi}{2}, \pi)$. (Note that when $\theta_1 = \frac{\pi}{2}$, this corresponds to the upper half sphere; while when $\theta_1 = \pi$, this is the full sphere.)

The analogous problem in $\mathbb{R}^n$ with power nonlinearity

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0, u > 0 \ \text{ in } \Omega; \\ u = 0 \ \text{ on } \partial \Omega, \end{cases} \tag{1.2}$$

has attracted a lot of attention in recent years. For $p < \frac{n+2}{n-2}$, and ε small, problem (1.2) admits solutions with spike layers concentrating at (local or global) maximum points of the distance function. See [10], [11], [18], [22], [24], and the references therein.

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The aim of this paper is to prove the existence of solutions which concentrate on whole spheres. For this purpose *we have to assume that D contains the upper hemisphere that is*:

$$\theta_1 > \pi/2.$$

Moreover we shall impose on f the following conditions:

(f1) $f(t) \equiv 0$ for $t < 0, f(0) = f'(0) = 0$ and $f \in C^{1+\sigma}[0, \infty) \cap C^2(0, \infty)$,

(f2) the following ODE has a unique solution

$$w^{''} - w + f(w) = 0 \ \text{ in } \mathbb{R}, w(0) = \max_{y \in \mathbb{R}} w(y), w(y) \to 0 \text{ as } |y| \to +\infty. \tag{1.3}$$

Assumption (f2) implies that the only solution of the linearization of (1.3) at w

$$v'' - v + f'(w)v = 0 \ \text{ in } \mathbb{R}, v(y) \to 0 \text{ for } |y| \to +\infty \tag{1.4}$$

is a multiple of $w^{'}$. In fact, problem $v^{''} - v + f^{'}(w)v = 0$ is a second order linear ODE and admits two linearly independent solutions. Since $w^{'}$ is a solution to (1.4), another solution to $v^{''} - v + f(w)v = 0$ must grow exponentially as $|y| \to +\infty$. Notice that when $f(u) = u^p$, the function $w(y)$ can be computed explicitely and has the form

$$w(y) = (\frac{p+1}{2})^{\frac{1}{p+1}} \left(\cosh(\frac{p-1}{2} y) \right)^{-\frac{2}{p-1}}.$$

We will show that problem (1.1) possesses three types of solutions:

(1) type I solutions with a boundary layer

(2) type II solutions with clustered layers on spheres near the equator

(3) type III solutions with clustered layers both on spheres near the equator and a boundary layer.

Our main result in this paper is the following.

Theorem 1.1. *Let $N > 0$ be a fixed positive integer. Then there exists $\varepsilon_N > 0$ such that for all $\varepsilon < \varepsilon_N$, problem (1.1) admits three radially symmetric solution $u_\varepsilon^1(\theta_1), u_\varepsilon^2(\theta_1), u_\varepsilon^3(\theta_1)$ with the following properties*

(1) (Type I) u_ε^1 concentrates at $\{\theta_1 = \arccos r_0^\varepsilon\}$ with

$$r_0^\varepsilon + R = \frac{1}{2}\sqrt{1-R^2}\varepsilon \log \frac{1}{\varepsilon} + O(\varepsilon) \tag{1.5}$$

More precisely, we have $u_\varepsilon^1(\arccos r_0^\varepsilon) \to w(0)$, where $w(y)$ is the unique solution of (1.3), and there exist two constants a and b such that

$$u_\varepsilon^1(\theta_1) \leq a e^{-b|\cos\theta_1 - r_0^\varepsilon|/\varepsilon}. \tag{1.6}$$

(2) (Type II) $u_\varepsilon^2(\theta_1)$ concentrates at N spheres $\{\theta_1 = \arccos r_j^\varepsilon, j = 1, ..., N\}$ with

$$r_j^\varepsilon = (j - \frac{N+1}{2})\varepsilon\rho_\varepsilon + O(\varepsilon), j = 1, ..., N \tag{1.7}$$

where ρ_ε satisfies

$$e^{-\rho_\varepsilon} = A_0\varepsilon^2\rho_\varepsilon, \rho_\varepsilon > 1 \tag{1.8}$$

where A_0 is some generic constant to be given in (2.9).

(3) (Type III) $u_\varepsilon^3(\theta_1)$ concentrates at $(N+1)$-sphere $\{\theta_1 = \arccos r_j^\varepsilon, j = 0, 1, ..., N\}$ with

$$r_0^\varepsilon + R = \frac{1}{2}\sqrt{1-R^2}\varepsilon\log\frac{1}{\varepsilon} + O(\varepsilon), \quad r_j^\varepsilon = (j - \frac{N+1}{2})\varepsilon\rho_\varepsilon + O(\varepsilon), j = 1, ..., N. \tag{1.9}$$

More precisely, we have $u_\varepsilon^3(\arccos r_j^\varepsilon) \to w(0)$, and there exist two constants a and b such that

$$u_\varepsilon^2(\theta_1) \leq ae^{-b\min_{j=0,\ldots,N}|\cos\theta_1 - r_j^\varepsilon|/\varepsilon}. \tag{1.10}$$

As a consequence, for each $N \geq 1$, there exists at least $2N+1$ solutions for ε sufficiently small.

Type II solutions are studied for the following singularly perturbed problem

$$\begin{cases} \varepsilon^2\Delta u - u + f(u) = 0, u > 0 \text{ in } \Omega; \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \tag{1.11}$$

where $\Omega = B_1(0)$. See [1], [2], and [20]. In particular, we mention the results of [20] which state that for any positive integer $N \geq 1$, there exists a layered solution u_ε to (1.11) with the property that u_ε concentrates on N spheres $r_1^\varepsilon > ... > r_N^\varepsilon$ satisfying $1 - r_1^\varepsilon = \varepsilon\log\frac{1}{\varepsilon} + O(\varepsilon), r_{j-1}^\varepsilon - r_j^\varepsilon = \varepsilon\log\frac{1}{\varepsilon} + O(\varepsilon), j = 2, ..., N$.

Here the type II and type III solutions have clustered layers at an interior sphere. Moreover type III solution is new in the sense it has a boundary layer and an interior clustered layer.

This study was motivated by the Brezis- Nirenberg problem

$$\triangle_{\mathbf{S}^n} u + \lambda u + u^{\frac{n+2}{n-2}} = 0, \quad u > 0 \text{ in } D, \quad u = 0 \text{ on } \partial D.$$

In contrast to the classical problem in $\mathbb{R}^n$ where no solutions exist for negative values of λ (cf. [6] and [19]), numerical computations showed that for large balls with $\theta_1 \in (\frac{\pi}{2}, \pi)$, there are more and more radially symmetric solutions as $\lambda \to +\infty$, (cf. [3] and [23]). In the last part of this paper we give a rigorous proof of such a phenomenon and improve some recent results of Brezis and Peletier [7] for $\mathbf{S^3}$. It turns out that the appearance of more and more solutions as λ decreases is not specific to the critical power $p = \frac{n+2}{n-2}$. The results are true for any power nonlinearity $f(u) = u^p$ with $p > 1$ and any dimension $n \geq 2$.

Our approach mainly relies upon a finite dimensional reduction procedure. Such a method has been used successfully in many papers, see e.g. [1], [2], [12], [13], [14], [20]. In particular, we shall follow the one used in [20].

In the rest of this section, we introduce some notation for later use.

Since we only consider radially symmetric solutions, i.e., solutions that only depend on the geodesic distance θ, equation (1.1) can be written in a more convenient form (with $x = \cos\theta$ and $-R = \cos\theta_1$, $-R < 0$ since $\theta_1 > \pi/2$), namely

$$\begin{cases} \varepsilon^2(1-x^2)^{-\frac{n-2}{2}}((1-x^2)^{\frac{n}{2}}u_x)_x - u + f(u) = 0, u > 0, \ -R < x < 1, \\ u'(1) \ \text{ exists}, \ \ u(-R) = 0. \end{cases} \tag{1.12}$$

By the following scaling $x = \varepsilon z$, problem (1.12) is reduced to the ODE

$$\begin{cases} \Delta' u - u + f(u) = 0, z \in (-\frac{R}{\varepsilon}, \frac{1}{\varepsilon}), \\ u_z(\frac{1}{\varepsilon}) \ \text{ exists}, \ \ u(-\frac{R}{\varepsilon}) = 0, \quad u(z) > 0. \end{cases} \tag{1.13}$$

where

$$\Delta' = (1-\varepsilon^2 z^2)u_{zz} - n\varepsilon^2 z u_z. \tag{1.14}$$

We also define the operator

$$\mathcal{S}_\varepsilon[u] := \Delta' u - u + f(u) = (1-\varepsilon^2 z^2)u_{zz} - n\varepsilon^2 z u_z - u + f(u). \tag{1.15}$$

From now on, we shall work with (1.13).

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small. The notation $A_\varepsilon = 0(B_\varepsilon)$ means that $|\frac{A_\varepsilon}{B_\varepsilon}| \leq C$, while $A_\varepsilon = o(B_\varepsilon)$ means that $\lim_{\varepsilon\to 0} \frac{|A_\varepsilon|}{|B_\varepsilon|} = 0$.

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2. Approximate Solutions

In this section we introduce a family of approximate solutions to (1.13) and derive some useful estimates. Since the construction of type III is the most complicated, we shall focus on the existence of u_ε^3 only. The proof of existence of $u_\varepsilon^1, u_\varepsilon^2$ can be modified accordingly.

Let w be the unique solution of (1.3), (see assumption (f2)), and let $t \in (-R, 1)$. Using ODE analysis, it is standard to see that

$$w(y) = A_n e^{-y} + O(e^{-(1+\sigma)y}), \quad w'(y) = -A_n e^{-y} + O(e^{-(1+\sigma)y}), \quad \text{for } \ y \geq 1, \tag{2.1}$$

where $A_n > 0$ is a fixed constant and $\sigma > 0$ is given in (f1). For simplicity of notation, let

$$I_\varepsilon = \left[-\frac{R}{\varepsilon}, \frac{1}{\varepsilon} \right],$$

and

$$w_t(z) := w(\frac{\varepsilon z - t}{\varepsilon \sqrt{1 - t^2}}), \ z \in I_\varepsilon. \tag{2.2}$$

We start with the approximation of a type II function concentrating at $z = t/\varepsilon$.

For $t \in (-\frac{R}{4}, \frac{1}{4})$, we define

$$w_{\varepsilon,t}(z) = w(\frac{\varepsilon z - t}{\varepsilon \sqrt{1 - t^2}}) \eta_2(\varepsilon z), \quad z \in I_\varepsilon, \tag{2.3}$$

where

$$\eta_2(x) = \begin{cases} 1 \text{ for } -\frac{R}{2} < x < \frac{1}{2}; \\ 0 \text{ for } x > \frac{3}{4} \text{ or } x < -\frac{3}{4}R. \end{cases} \tag{2.4}$$

The approximation of the type I solution with a boundary layer is more complicated.

For $t_0 \in (-R, -\frac{R}{4})$, we have to define w_{ε,t_0} differently. First we set

$$\rho_\varepsilon(t_0) = w(\frac{-R - t_0}{\varepsilon \sqrt{1 - t_0^2}}), \qquad \beta_\varepsilon(z) = e^{-\frac{\varepsilon z + R}{\varepsilon \sqrt{1 - t_0^2}}}, \quad z \in I_\varepsilon. \tag{2.5}$$

(2.1) implies that for $\frac{R+t_0}{\varepsilon} >> 1$

$$\rho_\varepsilon(t_0) = A_n e^{-\frac{R+t_0}{\varepsilon\sqrt{1-t_0^2}}} + O(e^{-(1+\sigma)\frac{-R-t_0}{\varepsilon\sqrt{1-t_0^2}}}). \tag{2.6}$$

A first ansatz for type I solutions is

$$\tilde{w}(z) = \left(w(\frac{\varepsilon z - t_0}{\varepsilon \sqrt{1 - t_0^2}}) - \rho_\varepsilon(t_0) \beta_\varepsilon(z) \right) \eta_1(\varepsilon z),$$

where η_1 is a cutoff function

$$\eta_1(x) = \begin{cases} 1 \text{ for } -1 < x < -\frac{R}{2}; \\ 0 \text{ for } -\frac{R}{4} < x. \end{cases} \tag{2.7}$$

If we compute $S_\varepsilon[\tilde{w}]$ first order terms in ε remain (cf. Section 6.2, in particular (6.18)). The correction which takes care of these terms is described next.

Let $\Psi_0(y)$ be the unique solution of the problem

$$\begin{cases} \Psi_0'' - \Psi_0 + f'(w)\Psi_0 = (-\frac{R}{\sqrt{1-R^2}})(2y w_{yy} + n w_y) - A_n e^{-2c_1} f'(w) e^{-y}, \\ \int_R \Psi_0 w'(y) dy = 0 \end{cases} \tag{2.8}$$

where A_0 and c_1 are defined by

$$A_0 = \frac{(n-1)\int_{\mathbb{R}}(w^{'})^2 dy}{A_n \int_{\mathbb{R}} f(w)e^{-y}dy} \text{ and } e^{-2c_1} = \frac{R}{\sqrt{1-R^2}}A_0. \tag{2.9}$$

(Observe that by (2.9), the right hand of (2.8) is perpendicular to $w^{'}(y)$. Hence by the assumption (**f2**) concerning (1.4), there exists a unique solution to (2.8).)

Since Ψ_0 does not satisfy the Dirichlet boundary conditions we have to modify it as follows: let

$$\hat{\Psi}_0(z) = \Psi_0(\frac{\varepsilon z - t_0}{\varepsilon\sqrt{1-t_0^2}}) - \Psi_0(\frac{-R-t_0}{\varepsilon\sqrt{1-t_0^2}})\beta_\varepsilon(z). \tag{2.10}$$

The approximate solution of type I assumes now the form:

$$w_{\varepsilon,t_0}(z) = \left(w(\frac{\varepsilon z - t_0}{\varepsilon\sqrt{1-t_0^2}}) - \rho_\varepsilon(t_0)\beta_\varepsilon(z) - \varepsilon\hat{\Psi}_0(z)\right)\eta_1(\varepsilon z), \tag{2.11}$$

where $t_0 \in (-R, -\frac{R}{4})$.

Note that for $z \geq \frac{1}{4\varepsilon}$, we have

$$|w_{\varepsilon,t}(z)| + |w'_{\varepsilon,t}(z)| + |w^{''}_{\varepsilon,t}(z)| \leq e^{-\frac{1}{C\varepsilon}}. \tag{2.12}$$

Observe also that, by construction, $w_{\varepsilon,t}$ satisfies the Dirichlet boundary condition, i.e., $w_{\varepsilon,t}(-\frac{R}{\varepsilon}) = 0$. Furthermore, $w_{\varepsilon,t}$ depends smoothly on t as a map with values in $C^2\left([-\frac{R}{\varepsilon}, \frac{1}{\varepsilon}]\right)$.

Next we describe the approximate location of the concentration points t. Let t_0^ε be such that

$$t_0^\varepsilon = -R + \frac{\sqrt{1-R^2}}{2}\varepsilon \log\frac{1}{\varepsilon} + \sqrt{1-R^2}c_1\varepsilon, \tag{2.13}$$

where c_1 is defined at (2.9).

To define $t_1^\varepsilon < ... < t_N^\varepsilon$, we have to consider the auxiliary functional

$$E_\varepsilon(\mathbf{t}) = \frac{A_0}{2}\sum_{j=1}^N t_j^2 + \sum_{j=2}^N e^{-\frac{|t_j - t_{j-1}|}{\varepsilon}}. \tag{2.14}$$

Then we have the following result, the proof of which is carried out in Section 6.1.

Lemma 2.1. *The functional $E_\varepsilon(\mathbf{t})$ has a unique minimizer $\mathbf{t}^\varepsilon$ in the set*

$$\{\mathbf{t} = (t_1, ..., t_N) | t_j - t_{j-1} > \varepsilon, j = 2, ..., N\}.$$

Moreover, we have

$$t_j^\varepsilon = (j - \frac{N+1}{2})\varepsilon\rho_\varepsilon + O(\varepsilon) \tag{2.15}$$

where ρ_ε is the unique solution of

$$e^{-\rho_\varepsilon} = A_0\varepsilon^2\rho_\varepsilon, \rho_\varepsilon > 1. \tag{2.16}$$

Furthermore, the smallest eigenvalue of the matrix

$$\mathcal{M} = (\frac{\partial^2 E_\varepsilon}{\partial t_i \partial t_j})$$

is greater than or equal to A_0*. As a consequence, we have that*

$$|\mathcal{M}^{-1}\mathbf{x}| \leq C|\mathbf{x}|. \tag{2.17}$$

Remark: Note that for any $0 < \delta < 1$ there exists $\varepsilon_0 > 0$ such that.

$$(2-\delta)\log\frac{1}{\varepsilon} \leq \rho_\varepsilon \leq 2\log\frac{1}{\varepsilon} \text{ for all } \varepsilon \leq \varepsilon_0. \tag{2.18}$$

We introduce the following set

$$\Lambda = \left\{ (t_0, \mathbf{t}) = (t_0, t_1, ..., t_N) \middle| \begin{array}{l} |t_0 - t_0^\varepsilon| \leq \varepsilon^{1+\tau_0}, \\ |t_j - t_j^\varepsilon| \leq \varepsilon, j = 1, ..., N \end{array} \right\}, \tag{2.19}$$

where $0 < \tau_0 < \frac{\sigma}{4}$.

For $(t_0, \mathbf{t}) \in \Lambda$, we define

$$t_0 = t_0^\varepsilon + \varepsilon^{\tau_0}\hat{t}_0, t_j = t_j^\varepsilon + \varepsilon\hat{t}_j, \; w_{\varepsilon,t_0,\mathbf{t}}(r) = \sum_{j=0}^N w_{\varepsilon,t_j}(r). \tag{2.20}$$

Then we have

$$(t_0, t_1, ..., t_N) \in \Lambda \;\text{ iff }\; |\hat{t}_j| < 1, j = 0, 1, ..., N$$

and

$$\rho_\varepsilon(t_0) = O(\sqrt{\varepsilon}), \; t_j = O(\varepsilon|\ln\varepsilon|), j = 1, ..., N, \; |t_i - t_j| \geq 2|i-j|\varepsilon\log\frac{1}{\varepsilon}. \tag{2.21}$$

The choice of the approximated location of the concentration points comes from the computations carried out in the proof of formula (5.1).

Finally we state the following important lemma on the error estimates. The proof of it is delayed to Section 6.2.

Lemma 2.2. *Let* $(t_0, \mathbf{t}) \in \mathbf{\Lambda}$ *and let* ε *be sufficiently small. Then*

(i) if $\varepsilon z = t_0 + \varepsilon\sqrt{1-t_0^2}y$*, we have*

$$\mathcal{S}_\varepsilon[w_{\varepsilon,t_0,\mathbf{t}}](z) = -\frac{2A_n}{\sqrt{1-R^2}}\varepsilon^{1+\tau_0}\hat{t}_0 e^{-2c_1} f'(w)e^{-y} + O(\varepsilon^{1+\frac{\sigma}{2}}), \tag{2.22}$$

where σ *is defined in the condition* (**f1**).

(ii) if $\varepsilon z = t_j + \varepsilon\sqrt{1-t_j^2}y, j = 1, ..., N$*, we have*

$$\mathcal{S}_\varepsilon[w_{\varepsilon,t_0,\mathbf{t}}](z) = -t_j\varepsilon(2yw_{yy} + nw_y) + \varepsilon^2(y^2w_{yy} - nyw_y) + f'(w(y))(w_{t_{j+1}} + w_{t_{j-1}}) + O(\varepsilon^{2+\frac{\sigma}{2}}). \tag{2.23}$$

As a consequence, we obtain

$$\|\mathcal{S}_\varepsilon[w_{\varepsilon,t_0,\mathbf{t}}]\|_{L^\infty(I_\varepsilon)} \leq C\varepsilon^{1+\tau_0}. \tag{2.24}$$

3. An Auxiliary Linear Problem

In this section we study a theory for linear operators which allows us to perform the finite-dimensional reduction procedure.

Fix $(t_0, \mathbf{t}) \in \Lambda$. We define two norms:

$$(u,v)_\varepsilon = \int_{I_\varepsilon} \left[(1-\varepsilon^2 z^2)^{\frac{n}{2}} u^{'} v^{'} + (1-\varepsilon^2 z^2)^{\frac{n-2}{2}} uv\right] dz, \quad < u, v >_\varepsilon = \int_{I_\varepsilon} (1-\varepsilon^2 z^2)^{\frac{n-2}{2}} uv. \tag{3.1}$$

Integration by parts implies that if $u(-\frac{R}{\varepsilon}) = 0$ and if $v'(\frac{1}{\varepsilon})$ exists then

$$(u, v)_\varepsilon = - < u, \triangle' v - v >_\varepsilon .$$

Define

$$z_{\varepsilon,t} = \frac{\partial w_{\varepsilon,t}}{\partial t}, \qquad Z_{\varepsilon,t} = \Delta^{'} z_{\varepsilon,t} - z_{\varepsilon,t}, \tag{3.2}$$

and

$$\mathcal{H} = \left\{ (u,u)_\varepsilon < +\infty : \ u(-\frac{R}{\varepsilon}) = 0, \ u'(\frac{1}{\varepsilon}) \text{ exists and } \quad (u, z_{\varepsilon,t_j})_\varepsilon = 0, \ \ j = 0, \ldots, N \right\}.$$

Note that, integrating by parts, one has

$$u \in \mathcal{H} \qquad \text{if and only if} \qquad < u, Z_{\varepsilon,t_j} >_\varepsilon = -(u, z_{\varepsilon,t_j})_\varepsilon = 0, \quad j = 0, 1, ..., N.$$

Let us consider the following linear problem: for given $h \in L^\infty(I_\varepsilon)$ find $\phi \in \mathcal{H}$ such that

$$(\phi, \psi)_\varepsilon - < f'(w_{\varepsilon,t_0,\mathbf{t}})\phi, \psi >_\varepsilon = < h, \psi >_\varepsilon, \ \forall \psi \in \mathcal{H}.$$

This equation can be rewritten as a differential equation

$$\begin{cases} L_\varepsilon[\phi] := (1-\varepsilon^2 z^2)\phi_{zz} - n\varepsilon^2 z \phi_z - \phi + f'(w_{\varepsilon,t_0,\mathbf{t}})\phi = h + \sum_{j=0}^N c_j Z_{\varepsilon,t_j}; \\ \phi_z(\frac{1}{\varepsilon}) \text{ exists}, \quad \phi(-\frac{R}{\varepsilon}) = 0; \quad < \phi, Z_{\varepsilon,t_j} >_\varepsilon = 0, \quad j = 0, 1, ..., N, \end{cases} \tag{3.3}$$

for some constants $c_j, j = 0, 1, ..., N$ or in an abstract form as

$$\phi + \mathcal{S}(\phi) = \bar{h} \qquad \text{in } \mathcal{H}, \tag{3.4}$$

where $\bar{h}$ is defined by duality and $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ is a linear compact operator. Using Fredholm's alternative, showing that equation (3.4) has a unique solution for each $\bar{h}$, is equivalent to showing that the equation has a unique solution for $\bar{h} = 0$. Next it will be shown that this is the case when ε is sufficiently small.

In order to derive an a priori bound for ϕ in terms of h we need the asymptotic behaviour of $z_{\varepsilon,t}$ and $Z_{\varepsilon,t}$ in ε. By elementary computations we obtain, setting $y=\frac{\varepsilon z-t}{\varepsilon\sqrt{1-t^2}}$

$$z_{\varepsilon,t} = -\frac{1}{\varepsilon\sqrt{1-t^2}}w'(\frac{\varepsilon z-t}{\varepsilon\sqrt{1-t^2}})\eta(\varepsilon z)+w'(\frac{\varepsilon z-t}{\varepsilon\sqrt{1-t^2}})(\frac{\varepsilon z-t}{\varepsilon\sqrt{1-t^2}})t(1-t^2)^{-1}\eta(\varepsilon z) = -\frac{1}{\varepsilon\sqrt{1-t^2}}w'(y)\eta(\varepsilon z)+R_1(y) \tag{3.5}$$

where $R_1(y)$ is bounded and integrable over $\mathbb{R}$.

Moreover we have, keeping in mind that $w'''-w'+f'(w)w'=0$

$$\begin{aligned} Z_{\varepsilon,t} = & (1-\varepsilon^2z^2)\frac{d^2z_{\varepsilon,t}}{dz^2}-z_{\varepsilon,t}+R_2(y) \\ &= \frac{f(w)w'(1-\varepsilon^2z^2)}{\varepsilon(1-t^2)^{5/2}}\eta+\frac{w'(\varepsilon^2z^2-t^2)}{\varepsilon(1-t^2)^{3/2}}\eta+R_3(y) \\ &= \frac{f'(w(y))w'(y)}{\sqrt{1-t^2}}\eta(\varepsilon z)+R_4(y), \end{aligned} \tag{3.6}$$

where $R_i(y)$ $i=2,3,4$ are bounded and integrable over $\mathbb{R}$.

Let us define the norm

$$\|\phi\|_* = \sup_{r\in(-\frac{R}{\varepsilon},\frac{1}{\varepsilon})}|\phi(z)|. \tag{3.7}$$

We have the following result.

Proposition 3.1. *Let ϕ satisfy (3.3). Then for ε sufficiently small, we have*

$$\|\phi\|_*\le C\|h\|_* \tag{3.8}$$

where C is a positive constant independent of ε and $(t_0,\mathbf{t})\in\Lambda$.

Proof. The proof of this proposition is similar to that of Proposition 3.1 of [20]. For the sake of completeness, we include it here.

Arguing by contradiction, assume that

$$\|\phi\|_*=1; \qquad \|h\|_*=o(1). \tag{3.9}$$

We multiply (3.3) by $(1-\varepsilon^2z^2)^{\frac{n-2}{2}}z_{\varepsilon,t_j}$ and integrate over I_ε to obtain

$$\begin{aligned} \sum_{i=0}^N c_i < Z_{\varepsilon,t_i}, z_{\varepsilon,t_j} >_\varepsilon &= -< h, z_{\varepsilon,t_j} >_\varepsilon \\ &+ < \Delta'\phi-\phi+f'(w_{\varepsilon,t_0,\mathbf{t}})\phi, z_{\varepsilon,t_j} >_\varepsilon . \end{aligned} \tag{3.10}$$

From the exponential decay of w one finds

$$< h, z_{\varepsilon,t_j} >_\varepsilon = \int_{I_\varepsilon}(1-\varepsilon^2z^2)^{\frac{n-2}{2}}hz_{\varepsilon,t_j} = O(\|h\|_*\varepsilon^{-1}).$$

Moreover, integrating by parts, using (3.5) and (3.6) we deduce

$$< \Delta^{'} - \phi + f'(w_{\varepsilon,t_0,\mathbf{t}})\phi, z_{\varepsilon,t_j} >_\varepsilon = < Z_{\varepsilon,t_j} + f'(w_{\varepsilon,t_0,\mathbf{t}}) z_{\varepsilon,t_j}, \phi >_\varepsilon =$$
$$< (f'(w_{\varepsilon,t_0,\mathbf{t}}) - f^{'}(w_{t_j})) z_{\varepsilon,t_j} + R_5, \phi >_\varepsilon$$
$$= o(\varepsilon^{-1}\|\phi\|_*).$$

In the last statement we have used the integrability of R_5 and the property that $f'(w_{\varepsilon,t_0,\mathbf{t}}) - f^{'}(w_{t_j}) = o(1)$ as $\varepsilon \to 0$. This observation becomes clear if we introduce the new variable $y = \frac{z - t_i/\varepsilon}{\sqrt{1-t^2}}$ and notice that for $i \neq j$, $w_{\varepsilon,t_i} = w(y + \frac{t_i - t_j}{\varepsilon\sqrt{1-t^2}})\eta \to 0$ as $\varepsilon \to 0$, for all $y \neq -\frac{t_i - t_j}{\varepsilon\sqrt{1-t^2}}$.

The same argument together with (3.5) and (3.6), implies that

$$< Z_{\varepsilon,t_i}, z_{\varepsilon,t_j} >_\varepsilon = -\varepsilon^{-2}\left(\delta_{ij}\int_{\mathbb{R}} f'(w)(w')^2 + o(1)\right), \tag{3.11}$$

where δ_{ij} denotes the Kronecker symbol. Note that, using the equation $w''' - w' + f'(w)w' = 0$ we find

$$\int_{\mathbb{R}} f'(w)(w')^2 = \int_{\mathbb{R}} ((w^{''})^2 + (w')^2) > 0.$$

This shows that in the left hand side of the equation (3.10) the terms with $i = j$ dominate, and hence by (3.9) we have

$$c_i = O(\varepsilon\|h\|_*) + o(\varepsilon\|\phi\|_*) = o(\varepsilon), \qquad i = 0, 1, ..., N. \tag{3.12}$$

Also, since we are assuming that $\|h\|_* = o(1)$ and since $\|Z_{\varepsilon,t_j}\|_* = O\left(\frac{1}{\varepsilon}\right)$, there holds

$$\|h + \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}\|_* = o(1). \tag{3.13}$$

Thus (3.3) yields

$$\begin{cases} \Delta^{'}\phi - \phi + f'(w_{\varepsilon,t_0,\mathbf{t}})\phi = o(1); \\ \phi^{'}(\frac{1}{\varepsilon}) \ \text{ exists}, \ \ \phi(-\frac{R}{\varepsilon}) = 0; \quad < \phi, Z_{\varepsilon,t_j} >_\varepsilon = 0, \quad j = 0, 1, ..., N, \end{cases} \tag{3.14}$$

We show that (3.14) is incompatible with our assumption $\|\phi\|_* = 1$. First we claim that, for arbitrary, fixed $R_0 > 0$, there holds

$$|\phi| \to 0 \quad \text{ on } \quad y \in \bigcup_{j=1}^{N}\left(\frac{t_j}{\varepsilon} - R_0, \frac{t_j}{\varepsilon} + R_0\right) \qquad \text{as } \ \varepsilon \to 0. \tag{3.15}$$

Indeed, assuming the contrary, there exist $\delta_0 > 0$, $j \in \{0, 1, \ldots, N\}$ and sequences $\varepsilon_k, \phi_k, y_k \in \left(\frac{t_j}{\varepsilon_k} - R_0, \frac{t_j}{\varepsilon_k} + R_0\right)$ such that ϕ_k satisfies (3.3) and

$$|\phi_k(y_k)| \geq \delta_0. \tag{3.16}$$

Let $\tilde{\phi}_k = \phi_k(y - \frac{t_j}{\varepsilon_k})$. Then using (3.14) and $\|\phi\|_* = 1$, as $\varepsilon_k \to 0$ $\tilde{\phi}_k$ converges weakly in $H^2_{loc}(\mathbb{R})$ and strongly in $C^1_{loc}(\mathbb{R})$ to a bounded function ϕ_0 which satisfies

$$\phi_0^{''} - \phi_0 + f'(w)\phi_0 = 0 \qquad \text{in } \mathbb{R}.$$

Hence ϕ_0 must tend to zero at infinity and so, by (1.4), $\phi_0 = cw'$ for some c. Since $\tilde{\phi}_k \perp Z_{\varepsilon,t_j}$ in $\mathcal{H}$, we conclude that $\int_{\mathbb{R}} \phi_0 f'(w)w'(y) = 0$, which yields $c = 0$. Hence $\phi_0 = 0$ and $\tilde{\phi}_k \to 0$ in $B_{2R}(0)$. This contradicts (3.16), so (3.15) holds true.

Given $\delta > 0$, the decay of w and (3.15) (with R_0 sufficiently large) imply

$$\|f'(w_{\varepsilon,t_0,\mathbf{t}})\phi\|_* \leq \delta + \frac{1}{2}\|\phi\|_*. \tag{3.17}$$

Using (3.14) and the Maximum Principle one finds

$$\begin{aligned}\|\phi\|_* &\leq \|f'(w_{\varepsilon,t_0,\mathbf{t}})\phi\|_* + \sum_{j=1}^{N}|c_j|\|Z_{\varepsilon,t_j}\|_* + \|h\|_* \\ &\leq 2\delta + \frac{1}{2}\|\phi\|_*,\end{aligned}$$

and hence

$$\|\phi\|_* \leq 4\delta < 1$$

if we choose $\delta < \frac{1}{4}$. This contradicts (3.9). □

The main result of this section can now be summarized in

Lemma 3.2. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ the following property holds true. Given $h \in L^\infty(I_\varepsilon)$, there exists a unique pair $(\phi, c_0, c_1, ..., c_N)$ such that*

$$L_\varepsilon[\phi] = h + \sum_{j=0}^{N} c_j Z_{\varepsilon,t_j}, \tag{3.18}$$

$$\phi^{'}(\frac{1}{\varepsilon}) \text{ exists, } \phi(-\frac{R}{\varepsilon}) = 0; \quad <\phi, Z_{\varepsilon,t_j}>_\varepsilon = 0, \quad j = 0, 1, ..., N. \tag{3.19}$$

Moreover we have

$$\|\phi\|_* \leq C\|h\|_* \tag{3.20}$$

for some positive constant C.

In the following we set for the unique solution ϕ given in Lemma 3.2,

$$\phi = \mathcal{A}_\varepsilon(h). \tag{3.21}$$

Note that (3.20) implies

$$\|\mathcal{A}_\varepsilon(h)\|_* \leq C\|h\|_*. \tag{3.22}$$

4. Finite-dimensional Reduction

In this section we reduce problem (1.13) to a finite-dimensional one. This amounts to finding a function ϕ such that for some constants $c_j, j = 0, 1, ..., N$, the following equation holds true

$$\begin{cases} \Delta^{'}(w_{\varepsilon,t_0,\mathbf{t}} + \phi) - (w_{\varepsilon,t_0,\mathbf{t}} + \phi) + f(w_{\varepsilon,t_0,\mathbf{t}} + \phi) = \sum_{j=0}^{N} c_j Z_{\varepsilon,t_j} \text{ in } \Omega_\varepsilon, \\ \phi^{'}(\frac{1}{\varepsilon}) \text{ exists}, \quad \phi(-\frac{R}{\varepsilon}) = 0, \ <\phi, Z_{\varepsilon,t_j}>_\varepsilon = 0, j = 0, 1, ..., N. \end{cases} \tag{4.1}$$

The first equation in (4.1) can be written as

$$\Delta^{'}\phi - \phi + f'(w_{\varepsilon,t_0,\mathbf{t}})\phi = (-\mathcal{S}_\varepsilon[w_{\varepsilon,t_0,\mathbf{t}}]) + N_\varepsilon[\phi] + \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j},$$

where

$$N_\varepsilon[\phi] = -\Bigg[f(w_{\varepsilon,t_0,\mathbf{t}} + \phi) - f(w_{\varepsilon,t_0,\mathbf{t}}) - f'(w_{\varepsilon,t_0,\mathbf{t}})\phi \Bigg]. \tag{4.2}$$

Lemma 4.1. *For $(t_0, \mathbf{t}) \in \Lambda$ and ε sufficiently small, we have for $\|\phi\|_* + \|\phi_1\|_* + \|\phi_2\|_* \le 1$,*

$$\|N_\varepsilon[\phi]\|_* \le C\|\phi\|_*^{1+\sigma}; \tag{4.3}$$

$$\|N_\varepsilon[\phi_1] - N_\varepsilon[\phi_2]\|_* \le C(\|\phi_1\|_*^\sigma + \|\phi_2\|_*^\sigma)\|\phi_1 - \phi_2\|_*, \tag{4.4}$$

where σ is defined in (f1).

Proof. Inequality (4.3) follows from the mean-value theorem. In fact, for every point in $\left[-\frac{R}{\varepsilon}, \frac{1}{\varepsilon}\right]$ there holds

$$f(w_{\varepsilon,t_0,\mathbf{t}} + \phi) - f(w_{\varepsilon,t_0,\mathbf{t}}) = f'(w_{\varepsilon,t_0,\mathbf{t}} + \theta\phi)\phi, \qquad \theta \in [0, 1].$$

Since f' is Holder continuous with exponent σ, we deduce

$$|f(w_{\varepsilon,t_0,\mathbf{t}} + \phi) - f(w_{\varepsilon,t_0,\mathbf{t}}) - f'(w_{\varepsilon,t_0,\mathbf{t}})\phi| \le C|\phi|^{1+\sigma},$$

which implies (4.3). The proof of (4.4) goes along the same way. □

Proposition 4.2. *For $(t_0, \mathbf{t}) \in \Lambda$ and ε sufficiently small, there exists a unique $(\phi, c) = (\phi_{\varepsilon,t_0,\mathbf{t}}, c_\varepsilon(t_0, \mathbf{t}))$ such that (4.1) holds. Moreover, the map $(t_0, \mathbf{t}) \mapsto (\phi_{\varepsilon,t_0,\mathbf{t}}, c_\varepsilon(t_0, \mathbf{t}))$ is of class C^0, and we have*

$$\|\phi_{\varepsilon,t_0,\mathbf{t}}\|_* \le C\varepsilon^{1+\tau_0}. \tag{4.5}$$

Proof. Let $\mathcal{A}_\varepsilon$ be as defined in (3.21). Then (4.1) can be written as

$$\phi = \mathcal{A}_\varepsilon \Bigg[(-\mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}]) + N_\varepsilon[\phi] \Bigg]. \tag{4.6}$$

Let r be a positive (large) number, and set

$$\mathcal{F}_r = \left\{ \phi \in \mathcal{H}, \quad \|\phi\|_* \le r\varepsilon^{1+\tau_0} \right\}.$$

Define now the map $\mathcal{B}_\varepsilon : \mathcal{F}_r \to \mathcal{H}$ as

$$\mathcal{B}_\varepsilon(\phi) = \mathcal{A}_\varepsilon \left[(-\mathcal{S}_\varepsilon[w_{\varepsilon,t_0,\mathbf{t}}]) + N_\varepsilon[\phi] \right].$$

Solving (4.1) is equivalent to finding a fixed point for $\mathcal{B}_\varepsilon$. By Lemma 4.1, for ε sufficiently small and r large we have

$$\begin{aligned} \|\mathcal{B}_\varepsilon[\phi]\|_* &\leq C\|\mathcal{S}_\varepsilon[w_{\varepsilon,t_0,\mathbf{t}}]\|_* + C\|N_\varepsilon[\phi]\|_* \\ &< r\varepsilon^{1+\tau_0}; \end{aligned}$$

$$\|\mathcal{B}_\varepsilon[\phi_1] - \mathcal{B}_\varepsilon[\phi_2]\|_* \leq C\|N_\varepsilon[\phi_1] - N_\varepsilon[\phi_2]\|_* < \frac{1}{2}\|\phi_1 - \phi_2\|_*,$$

which shows that $\mathcal{B}_\varepsilon$ is a contraction mapping on $\mathcal{F}_r$. Hence there exists a unique $\phi \in \mathcal{F}_r$ such that (4.1) holds.

The continuity of $\phi_{\varepsilon,t_0,\mathbf{t}}$ follows from the uniqueness.

This conclues the proof of Proposition 4.2.

□

5. Proof of Theorem 1.1

From (4.1), we see that, to prove the existence of Type III solutions of Theorem 1.1, it is enough to find a zero of the vector $\mathbf{c}_\varepsilon(t_0, \mathbf{t}) = (c_{0,\varepsilon}(t_0, \mathbf{t}), c_{1,\varepsilon}(t_0, \mathbf{t}), ..., c_{N,\varepsilon}(t_0, \mathbf{t}))^T$.

The next Proposition computes the asymptotic formula for $\mathbf{c}_\varepsilon(t_0, \mathbf{t})$:

Proposition 5.1. *For ε sufficiently small, we have the following asymptotic expansion*

$$\frac{1}{\varepsilon^{2+\tau_0} \int_{\mathbb{R}} (f'(w)(w')^2)} c_{0,\varepsilon}(t_0, \mathbf{t}) = d_0 \hat{t}_0 + \beta_{0,\varepsilon}(\hat{t}_0, \hat{\mathbf{t}}), \tag{5.1}$$

$$\frac{1}{\varepsilon^3 \int_{\mathbb{R}} (f'(w)(w')^2)} c_{j,\varepsilon}(t_0, \mathbf{t}) = d_j (\mathcal{M}\hat{\mathbf{t}})_j + \beta_{j,\varepsilon}(\hat{t}_0, \hat{\mathbf{t}}), j = 1, ..., N \tag{5.2}$$

where $d_j \neq 0, j = 0, 1, ..., N$ are positive constants, and $\beta_{j,\varepsilon}(\hat{t}_0, \hat{\mathbf{t}}), j = 0, 1, ..., N$ are continuous functions in $(\hat{t}_0, \hat{\mathbf{t}})$ with

$$\beta_{j,\varepsilon}(t_0, \mathbf{t}) = O(\varepsilon^{\tau_0} + |\hat{t}_0|^2 + |\hat{\mathbf{t}}|^2), j = 0, ..., N. \tag{5.3}$$

We delay the proof of the proposition at the end of the section. Let us now use it to prove Theorem 1.1.

Proof of Theorem 1.1: To find a zero of $\mathbf{c}_\varepsilon(t_0, \mathbf{t})$, it is enough to solve the following systems of equations

$$d_0 \hat{t}_0 + \beta_{0,\varepsilon}(\hat{t}_0, \hat{\mathbf{t}}) = 0, \;\; d_j (\mathcal{M}\hat{\mathbf{t}})_j + \beta_{j,\varepsilon}(\hat{t}_0, \hat{\mathbf{t}}) = 0, j = 1, ..., N. \tag{5.4}$$

By Lemma 2.1, the matrix $\mathcal{M}$ is invertible with uniform bound, (5.4) is equivalent to

$$(\hat{t}_0, \hat{\mathbf{t}}) = \hat{\beta}_\varepsilon(\hat{t}_0, \hat{\mathbf{t}}) \tag{5.5}$$

where $\hat{\beta}_\varepsilon(\hat{t}_0, \hat{\mathbf{t}})$ is a continuous function in $(\hat{t}_0, \hat{\mathbf{t}})$ satisfying

$$\hat{\beta}_\varepsilon(\hat{t}_0, \hat{\mathbf{t}}) = O(\varepsilon^{\tau_0} + \hat{t}_0^2 + |\hat{\mathbf{t}}|^2). \tag{5.6}$$

Let $\mathbf{B} = \{(\hat{t}_0, \hat{\mathbf{t}}) | \, |(\hat{t}_0, \hat{\mathbf{t}})| < \varepsilon^{\frac{\tau_0}{2}}\}$. Then Brouwer's fixed point theorem gives a solution in $\mathbf{B}$, called $(\hat{t}_0^\varepsilon, \hat{\mathbf{t}}^\varepsilon)$, to (5.5), which in turn, gives a solution

$$u_\varepsilon^3 = w_{\varepsilon, t_0 + \varepsilon^{\tau_0} \ \hat{t}_0^\varepsilon, \ \mathbf{t}_0^\varepsilon + \varepsilon \hat{\mathbf{t}}^\varepsilon} + \phi_{\varepsilon, t_0 + \varepsilon^{\tau_0} \hat{t}_0^\varepsilon, \ \mathbf{t}_0^\varepsilon + \varepsilon \hat{\mathbf{t}}^\varepsilon}$$

to equation (1.12). It is easy to see that u_ε^3 satisfies all the properties listed in Theorem 1.1.

□

Now we are ready to prove (5.1).

Proof of (5.1):

Multiplying equation (4.1) by $(1-\varepsilon^2 z^2)^{\frac{n-2}{2}} z_{\varepsilon, t_j}$, we obtain, using Lemma 2.2 and Proposition 4.2,

$$\sum_{l=0}^N c_{l,\varepsilon}(t_0, \mathbf{t}) < Z_{\varepsilon, t_l}, z_{\varepsilon, t_j} >_\varepsilon = \int_{I_\varepsilon} [(1-\varepsilon^2 z^2)^{\frac{n-2}{2}}] \mathcal{S}_\varepsilon[w_{\varepsilon, t_0, \mathbf{t}} + \phi_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j} \tag{5.7}$$

$$= \int_{I_\varepsilon} [(1-\varepsilon^2 z^2)^{\frac{n-2}{2}}] \mathcal{S}_\varepsilon[w_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j} + \int_{I_\varepsilon} [(1-\varepsilon^2 z^2)^{\frac{n-2}{2}}] L_\varepsilon[\phi_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j} + \int_{I_\varepsilon} (1-\varepsilon^2 z^2)^{\frac{n-2}{2}} N[\phi_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j}$$

$$= \int_{I_\varepsilon} [(1-\varepsilon^2 z^2)^{\frac{n-2}{2}}] \mathcal{S}_\varepsilon[w_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j} + \int_{I_\varepsilon} (1-\varepsilon^2 z^2)^{\frac{n-2}{2}} L_\varepsilon[z_{\varepsilon, t_j}] \phi_{\varepsilon, t_0, \mathbf{t}} + O(\varepsilon^{1+\tau_0})$$

$$= \int_{I_\varepsilon} \mathcal{S}_\varepsilon[w_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j} + \int_{I_\varepsilon} [(1-\varepsilon^2 z^2)^{\frac{n-2}{2}} - 1] \mathcal{S}_\varepsilon[w_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j} + O(\varepsilon^{1+\tau_0})$$

$$= \int_{I_\varepsilon} \mathcal{S}_\varepsilon[w_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j} + O(\varepsilon^{1+\tau_0}). \tag{5.8}$$

For $j = 0, 1, ..., N$, we make use of (3.5) and deduce that

$$\int_{I_\varepsilon} \mathcal{S}_\varepsilon[w_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j} = -\frac{1}{\varepsilon \sqrt{1-t_j^2}} \int_{I_{\varepsilon, t_j}} \mathcal{S}_\varepsilon[w_{\varepsilon, t_0, \mathbf{t}}] w' + O(\varepsilon^{1+\tau_0}), \tag{5.9}$$

where

$$I_{\varepsilon, t} = \left[-\frac{R+t}{\varepsilon}, \frac{1-t}{\varepsilon} \right].$$

For $j = 0$, we have, using (2.22),

$$\int_{I_{\varepsilon, t_0}} \mathcal{S}_\varepsilon[w_{\varepsilon, t_0, \mathbf{t}}] z_{\varepsilon, t_j} = \frac{2A_n}{1-R^2} \varepsilon^{\tau_0} \hat{t}_0 e^{-2c_1} \int_{\mathbb{R}} f'(w) w' e^{-y} dy + O(\varepsilon^{\frac{\sigma}{2}})$$

$$= d_0 \varepsilon^{\tau_0} \hat{t}_0 + O(\varepsilon^{2\tau_0}) \tag{5.10}$$

where

$$d_0 = \frac{2A_n}{1-R^2} e^{2c_1} \int_{\mathbb{R}} f^{'}(w) w^{'} e^{-y} = -\frac{2A_n}{1-R^2} e^{2c_1} \int_{\mathbb{R}} f(w) e^{-y} \neq 0.$$

For $j = 1, 2, ..., N$, we have, using (5.9) and (2.23),

$$\int_{I_{\varepsilon,t_j}} \mathcal{S}_\varepsilon[w_{\varepsilon,t_0,\mathbf{t}}] z_{\varepsilon,t_j} dz = t_j \int_{\mathbb{R}} (2y w_{yy} + n w_y) w_y - \frac{1}{\varepsilon} \int_{\mathbb{R}} f^{'}(w(y))(w_{t_{j+1}} + w_{t_{j-1}}) w^{'} + O(\varepsilon^{1+\tau_0})$$

$$= t_j \int_{\mathbb{R}} (n-1)(w^{'})^2 - \frac{A_n}{\varepsilon} \int_{\mathbb{R}} f^{'}(w(y)) w^{'} \left[e^{-|\frac{t_{j-1}-t_j}{\varepsilon} - y|} + e^{-|\frac{t_{j+1}-t_j}{\varepsilon} - y|} \right] + O(\varepsilon^{1+\tau_0})$$

$$= t_j \int_{\mathbb{R}} (n-1)(w^{'})^2 - \frac{A_n}{\varepsilon} \int_{\mathbb{R}} f^{'}(w(y)) w^{'} \left[e^{-|\frac{t_{j-1}-t_j}{\varepsilon}|} e^{-y} + e^{-|\frac{t_{j+1}-t_j}{\varepsilon}|} e^{y} \right] + O(\varepsilon^{1+\tau_0})$$

$$= B_0 \left[A_0 t_j - \frac{1}{\varepsilon} e^{-|\frac{t_{j-1}-t_j}{\varepsilon}|} + \frac{1}{\varepsilon} e^{-|\frac{t_{j+1}-t_j}{\varepsilon}|} \right] + O(\varepsilon^{1+\tau_0})$$

where

$$B_0 = A_n \int_{\mathbb{R}} f(w) e^{-y}.$$

So we have for $j = 1, ..., N$,

$$\int_{I_\varepsilon} \mathcal{S}_\varepsilon[w_{\varepsilon,t_0,\mathbf{t}}] z_{\varepsilon,t_j} dz = B_0 \frac{\partial E_\varepsilon(\mathbf{t})}{\partial t_j} + O(\varepsilon^{1+\tau_0})$$

$$= B_0 \frac{\partial E_\varepsilon}{\partial t_j}|_{\mathbf{t}=\mathbf{t}^\varepsilon} + B_0 \Big(\mathcal{M}(\mathbf{t} - \mathbf{t}^\varepsilon) \Big)_j + O(\varepsilon^{1+\tau_0})$$

$$= d_j \varepsilon \Big(\mathcal{M} \hat{t} \Big)_j + O(\varepsilon^{1+\tau_0}) \tag{5.11}$$

where $d_j = B_0, j = 1, ..., N$.

Since

$$< Z_{\varepsilon,t_l}, z_{\varepsilon,t_j} >_\varepsilon = \frac{1}{\varepsilon^2(1-t_j^2)} (\delta_{lj} \int_{\mathbb{R}} f^{'}(w)(w^{'})^2 + O(\varepsilon)) \tag{5.12}$$

we derive Proposition 5.1 from (5.7), (5.8), (5.10) and (5.11). (The fact that all the error terms are continuous in $(\hat{t}, \hat{\mathbf{t}})$ follows from the continuity of $\phi_{\varepsilon,t_0,\mathbf{t}}$ in $(t_0, \mathbf{t})$.)

□

6. Proof of two lemmas

6.1. **Proof of Lemma 2.1.** It is easy to see that a global minimizer of $E_\varepsilon(\mathbf{t})$ exists since $E_\varepsilon(\mathbf{t})$ is convex. Let us denote it by $\mathbf{t}^\varepsilon = (t_1^\varepsilon, ..., t_N^\varepsilon)$ which is unique. Setting $t_j^\varepsilon = \varepsilon(j - \frac{N+1}{2})\rho_\varepsilon + \varepsilon s_j^\varepsilon$, where ρ_ε satisfies

$$A_0 \varepsilon^2 \rho_\varepsilon = e^{-\rho_\varepsilon},$$

then s_j^ε satisfies

$$A_0(j - \frac{N+1}{2}) + A_0 \varepsilon s_j^\varepsilon + e^{-(s_j^\varepsilon - s_{j-1}^\varepsilon)} - e^{-(s_{j+1}^\varepsilon - s_j^\varepsilon)} = 0, j = 1, ..., N \tag{6.13}$$

which admits a unique solution $\mathbf{s}^\varepsilon = (s_1^\varepsilon, ..., s_N^\varepsilon) = O(1)$.

Let $\mathcal{M} = (\frac{\partial^2 E_\varepsilon(\mathbf{t})}{\partial t_i \partial t_j})$. We show that the smallest eigenvalue of $\mathcal{M}$ is uniformly bounded from below. In fact, let $\eta = (\eta_1, ..., \eta_N)^T$ and we compute

$$\sum_{i,j} \mathcal{M}_{ij}\eta_i\eta_j = A_0 \sum_{j=1}^N \eta_j^2 + \frac{1}{\varepsilon^2}\sum_{j=2}^N e^{-\frac{|t_j^\varepsilon - t_{j-1}^\varepsilon|}{\varepsilon}}(\eta_j - \eta_{j-1})^2 \geq A_0|\eta|^2 \tag{6.14}$$

which implies that the smallest eigenvalue of $\mathcal{M}$, denoted by λ_1, satisfies

$$\lambda_1 \geq A_0 > 0. \tag{6.15}$$

Now we consider

$$|\mathcal{M}^{-1}\eta|^2 = \eta^t \mathcal{M}^{-2}\eta \leq \lambda_1^{-2}|\eta|^2$$

which proves (2.17).

□

6.2. **Proof of Lemma 2.2.** Using (1.3) it is easy to see that

$$\mathcal{S}_\varepsilon[w_{\varepsilon,t_0,\mathbf{t}}] = \mathcal{S}_\varepsilon[w_{\varepsilon,t_0}] + \mathcal{S}_\varepsilon[\sum_{j=1}^N w_{\varepsilon,t_j}] + O(e^{-\frac{1}{C\varepsilon}})$$

$$= \mathcal{S}_\varepsilon[w_{\varepsilon,t_0}] + \sum_{j=1}^N \mathcal{S}_\varepsilon[w_{\varepsilon,t_j}] + f(\sum_{j=1}^N w_{\varepsilon,t_j}) - \sum_{j=1}^N f(w_{\varepsilon,t_j}) + O(e^{-\frac{1}{C\varepsilon}})$$

$$= \mathcal{S}_\varepsilon[w_{\varepsilon,t_0}] + \sum_{j=1}^N \mathcal{S}_\varepsilon[w_{t_j}] + f(\sum_{j=1}^N w_{t_j}) - \sum_{j=1}^N f(w_{t_j}) + O(e^{-\frac{1}{C\varepsilon}}). \tag{6.16}$$

Let us compute each term in the right hand side of (6.16): to this end, we first compute

$$\Delta' w_t = (1-\varepsilon^2 z^2)w_t'' - n\varepsilon^2 z w_t' = \frac{(1-\varepsilon^2 z^2)}{1-t^2}w'' - n\frac{\varepsilon^2 z}{\sqrt{1-t^2}}w'.$$

Let $\varepsilon z = t + \varepsilon\sqrt{1-t^2}y$. We have

$$(1-\varepsilon^2 z^2)w_t'' - n\varepsilon^2 z w_t'$$
$$= w_{yy} - \frac{\varepsilon\, t}{\sqrt{1-t^2}}\left[2yw_{yy} + nw_y\right] + \varepsilon^2(y^2 w_{yy} - nyw_y).$$

For $j = 0$, we have as before, letting $\varepsilon z = t_0 + \varepsilon\sqrt{1-t_0^2}y$,

$$\mathcal{S}_\varepsilon[w_{t_0}] = -\frac{\varepsilon\, t_0}{\sqrt{1-t_0^2}}\left[2yw_{yy} + nw_y\right] + O(\varepsilon^2)$$

and

$$(1-\varepsilon^2 z^2)\beta_\varepsilon'' - n\varepsilon^2 z\beta_\varepsilon' - \beta_\varepsilon$$
$$= \left[\frac{1-\varepsilon^2 z^2}{1-t_0^2} + n\varepsilon^2 z\frac{1}{\sqrt{1-t_0^2}} - 1\right]\beta_\varepsilon = O(\varepsilon)\beta_\varepsilon$$

and hence

$$\begin{aligned}\mathcal{S}_\varepsilon[w_{t_0}-\rho_\varepsilon(t_0)\beta_\varepsilon] &= f(w-\rho_\varepsilon(t_0)\beta_\varepsilon)-f(w)\\ &\quad -\frac{\varepsilon\ t_0}{\sqrt{1-t_0^2}}\left[2yw_{yy}+nw_y\right]+O(\varepsilon^{\frac{3}{2}})\\ &= -\rho_\varepsilon(t_0)e^{-\frac{R+t_0}{\sqrt{1-t_0^2}}}f^{'}(w)e^{-y}-\frac{\varepsilon\ R}{\sqrt{1-R^2}}(2yw_{yy}+nw_y)+O(\varepsilon^{1+\frac{\sigma}{2}})\\ &= -A_n e^{-\frac{2(R+t_0)}{\sqrt{1-R^2}}}f^{'}(w)e^{-y}-\frac{\varepsilon\ R}{\sqrt{1-R^2}}(2yw_{yy}+nw_y)+O(\varepsilon^{1+\frac{\sigma}{2}})\end{aligned}$$

Since

$$e^{-\frac{2(R+t_0)}{\sqrt{1-R^2}}}=e^{-\frac{2(R+t_0^\varepsilon)}{\sqrt{1-R^2}}}e^{-\frac{2\varepsilon^{\tau_0}\hat{t}_0}{\sqrt{1-R^2}}}=\varepsilon e^{-2c_1}e^{-\frac{2\varepsilon^{\tau_0}\hat{t}_0}{\sqrt{1-R^2}}},$$

we deduce that

$$\mathcal{S}_\varepsilon[w_{\varepsilon,t_0}]=\mathcal{S}_\varepsilon[w_{t_0}-\rho_\varepsilon(t_0)\beta_\varepsilon-\varepsilon\hat{\Psi}_0]=-\frac{2A_n}{\sqrt{1-R^2}}\varepsilon^{1+\tau_0}\hat{t}_0e^{-2c_1}f^{'}(w)e^{-y}+O(\varepsilon^{1+\frac{\sigma}{2}}) \tag{6.17}$$

which is just (2.22).

Next we have for $\varepsilon z=t_j+\varepsilon\sqrt{1-t_j^2}y, j=1,...,N$

$$\mathcal{S}_\varepsilon[w_{t_j}]=-\frac{\varepsilon\ t_j}{\sqrt{1-t_j^2}}\left[2yw_{yy}+nw_y\right]+\varepsilon^2(y^2w_{yy}-nyw_y)=O((\varepsilon t+\varepsilon^2)e^{|\varepsilon z-t|}). \tag{6.18}$$

On the other hand, the interaction terms can be estimated as follows: for $\varepsilon z=t_j+\varepsilon\sqrt{1-t_j^2}y, |l-j|\geq 2$

$$w_{t_l}(z)=O(e^{-\frac{|t_j-t_l-\varepsilon\sqrt{1-t_j^2}y|}{\varepsilon\sqrt{1-t_l^2}}})=O(\varepsilon^4|\ln\varepsilon|^4)$$

Therefore

$$f(\sum_{l=1}^N w_{t_l})-\sum_{j=1}^N f(w_{t_l})=f^{'}(w(y))\sum_{l=j-1,j+1}w_{t_l}+O(\varepsilon^{2+\sigma}). \tag{6.19}$$

Combining (6.18) and (6.19), we obtain (2.23). (2.24) follows from (2.22) and (2.23). □

7. The Brezis-Nirenberg problem

The Brezis-Nirenberg problem in $\mathbf{S}^n$ is of the form

$$\triangle_{\mathbf{S}^n}u-\lambda u+u^{\frac{n+2}{n-2}}=0,\quad u>0 \text{ in } D\subset\mathbf{S}^n, u=0 \text{ on } \partial D. \tag{7.1}$$

As in the previous sections we shall assume that D is a geodesic ball, centered at the North pole with geodesic radius θ_1. It is well-known (cf. [3], [5]) that there exists an interval (λ^*,λ_1), λ_1 being the first Dirichlet eigenvalue of $\triangle_{\mathbf{S}^n}$ in D, such that (7.1) has a solution for any $\lambda\in(\lambda^*,\lambda_1)$. This solution is a minimizer of the variational problem

$$S_\lambda(D) = \inf_v \int_D |\nabla v|^2_{\mathbf{S}^n} d\mu - \lambda \int_D v^2 d\mu, \text{ where } v \in W_0^{1,2}(D) \text{ and } \int_D |v|^{2^*} d\mu = 1. \tag{7.2}$$

Here $d\mu$ stands for the volume element and $|\nabla u|^2_{\mathbf{S}^n}$ for the first Beltrami operator in $\mathbf{S}^n$, and $2^* := \frac{n+2}{n-2}$ is the critical Sobolev exponent. Moreover from Pohozaev type arguments it follows that if $\lambda \notin (\lambda^*, \lambda_1)$ no solution of (7.2) exists. The value of λ^* is given by

$$\lambda^* = \begin{cases} \frac{\pi^2 - 4\theta_1^2}{4\theta_1^2} & \text{if } n = 3, \\ -\frac{n(n-2)}{4} & \text{if } n > 3. \end{cases}$$

Brock and Prajapat [8] have shown by means of the moving plane method that for $\lambda > -\frac{n(n-2)}{4}$ all solutions of (7.1) are radial in the sense that they depend only on the geodesic distance to the origin. It was observed in [23] that the uniqueness criterion of Kwong and Li [16] applies to (7.2), which implies that for $\lambda > -\frac{n(n-2}{4}$ problem (7.1) has at most one solution.

In contrast to the case of balls in the Euclidean space where all solutions of (7.1) are minimizers of (7.2), it was conjectured in [3] that for large balls in $\mathbf{S}^3$ containing the hemisphere ($\theta_1 > \pi/2$), non-minimizing solutions might exist for $\lambda < -\frac{3}{4}$. This conjecture was supported by numerical computations in [23] which indicate that there is a strong evidence that the result should be true in arbitrary dimensions provided $\lambda < -\frac{n(n-2)}{4}$.

In two recent papers Chen and Wei [9] and Brezis and Peletier [7] establish the existence of non-minimizing solutions in $\mathbf{S}^3$.

Numerical computations carried out in [23] yield the following solution diagrams s. Figure 1 for a fixed ball of radius $\theta_1 > \pi/2$. The horizontal axis denotes the values of λ and the vertical axis corresponds to the value at the center of the ball $u_0 = u(0)$, or in the second picture to $I = \frac{1}{2}\int_{B(\theta_1)} u^2 \, d\mu$.

If we set $\lambda = -\varepsilon^{-2}$ and $u = |\lambda|^{\frac{N-2}{4}} u_\varepsilon$ then (7.1) assumes the form

$$\varepsilon^2 \triangle_{\mathbf{S}^n} u_\varepsilon - u_\varepsilon + u_\varepsilon^{\frac{n+2}{n-2}} = 0, u_\varepsilon > 0 \text{ in } D, u_\varepsilon = 0 \text{ on } \partial D, \tag{7.3}$$

which is just a special case of (1.1). The function $w(y)$ in (**f2**) is

$$w(y) = \left(\frac{N}{(N-2)\cosh^2 \frac{2y}{N-2}} \right)^{\frac{N-2}{4}}.$$

From our main theorem in Section 1 it follows that, for any fixed ball of radius $\theta_1 > \pi/2$ and sufficiently small negative λ or equivalently for small $\varepsilon > 0$, there exist solutions of type (I)-(III). In terms of the variable $x = \cos\theta$ they are of the form

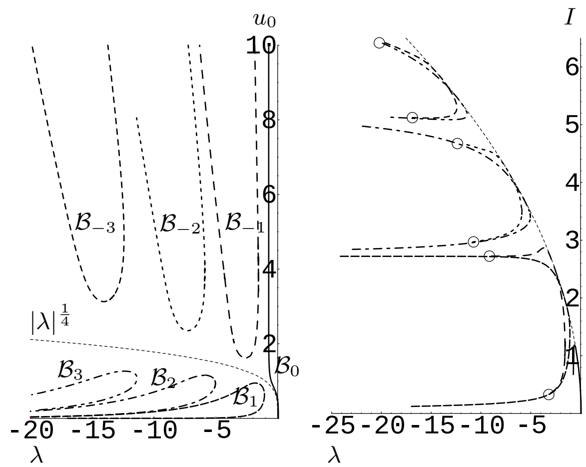

FIGURE 1. $n = 3$

$$u^1(x) = \varepsilon^{-\frac{N-2}{2}}\{w_{\varepsilon,t_0}(\frac{x}{\varepsilon}) + \phi_1\}, \quad \phi_1 = O(\varepsilon) : \tag{7.4}$$

solution with boundary layer

$$u^2(x) = \varepsilon^{-\frac{N-2}{2}}\left\{\sum_1^N w_{\varepsilon,t_i}(\frac{x}{\varepsilon}) + \phi_2\right\}, \quad \phi_2 = O(\varepsilon) :$$

solutions with clustered layers near the equator

$$u^3(x) = \varepsilon^{-\frac{N-2}{2}}\{w_{\varepsilon,t_0,\mathbf{t}}(\frac{x}{\varepsilon}) + \phi_3\}, \quad \phi_3 = O(\varepsilon) :$$

solutions with clustered layers near the equator and boundary layer ,

where $w_{\varepsilon,t}$ is defined in (2.3) and (2.11). The radii of the spheres where those solutions have clustered layers are $\theta_i = \arccos t_i$, $i = 0, \dots, N$ where t_i is given in (2.19). Observe that near the boundary there is at most one such sphere whereas near the equator there can be arbitrarily many.

Let us now relate the above solutions with those of diagram Fig.1 with those in Fig.2 . From the construction it follows that for small ε, the value of u^i, $i = 1, 2, 3$ at the origin is below the constant solution $u_c = \varepsilon^{-\frac{N-2}{2}} = |\lambda|^{\frac{4}{N-2}}$. Consequently they belong to the solution branches $\mathcal{B}_k$, $k = 1, 2, 3, \dots$.

For fixed $\lambda(\varepsilon)$ denote by $u_{1,k}$ the solution on the lower part of $\mathcal{B}_k$ and by $u_{2,k}$ the one on the upper part. They both satisfy (7.1). The following argument shows that the number of peaks of the solutions on $\mathcal{B}_k$ is constant and equal to k. Indeed suppose that $u_{1,k}(\theta; u_0^1)$ has k peaks and $u_{1,k}(\theta; u_0^2)$ has $k+1$ peaks for some value $u_0^2 > u_0^1$. Then by continuity there exists a value $u_0^1 < \tilde{u}_0 < u_0^2$ where a local minimum and a local maximum collide. This can only happen at $u_{1,k}(\theta; \tilde{u}_0) = |\lambda|^{\frac{4}{N-2}}$. By the uniqueness theorem we must have $u_{1,k}(\theta; \tilde{u}_0) = |\lambda|^{\frac{4}{N-2}}$ everywhere.

This observation together with the comparison of the values at the origin or the values of the L^2-norms implies for small values of λ:

$$u_{1,1} = u^1 \text{ and } u_{2,1} = u^2 : \text{ 1- layer solutions on } \mathcal{B}_1$$

$$u_{1,k} = u^3 \text{ and } u_{2,k} = u^2 : \text{ k -layer solutions on } \mathcal{B}_k \ .$$

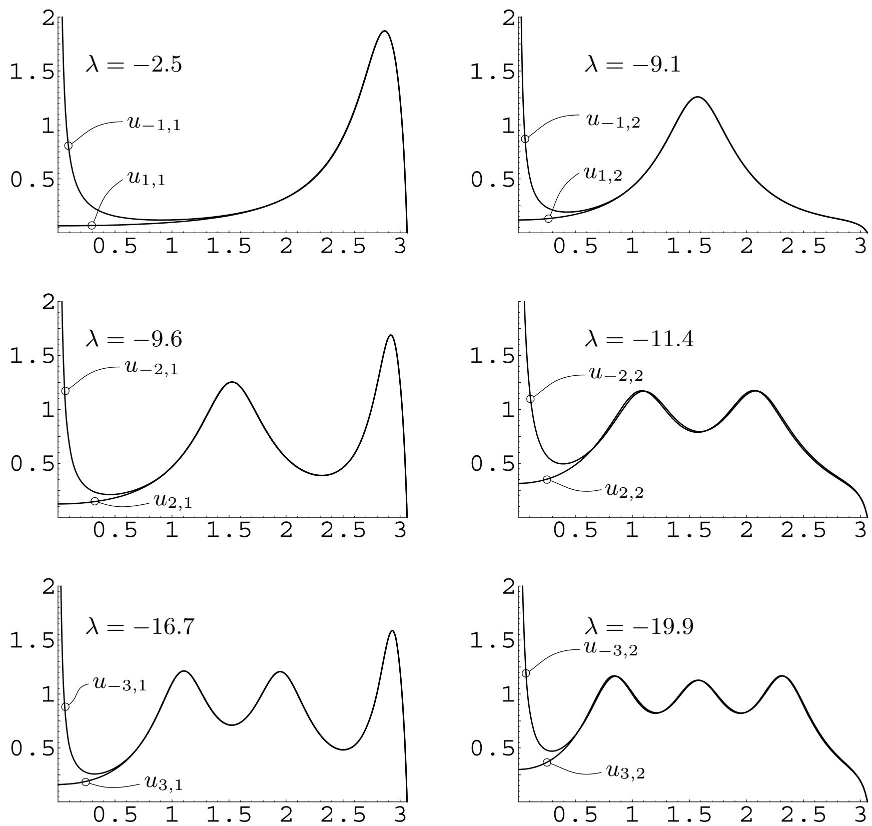

FIGURE 2. Solutions $u_{\pm k,j} \in \mathcal{B}_{\pm k}$ with $j = 1, 2$ depicted in terms of the θ-variable for $\theta_1 = 3$ and $n = 3$, scaled with $|\lambda|^{-1/(2^*-2)}$.

Remarks

1. Let

$$J[u] := \varepsilon^2 \int_D |\nabla u|^2_{\mathbf{S}^n} d\mu - \int_D u^2 d\mu - \frac{1}{2^*} \int_D |u|^{2^*} d\mu$$

be the energy associated to (7.3) and

$$I[w] := \int_{\mathbb{R}} w'^2 dx - \int_{\mathbb{R}} w^2 dx - \frac{1}{2^*} \int_{\mathbb{R}} |w|^{2^*} dx$$

be the energy associated to (1.4). Then for $u \in \mathcal{B}_k$, $k = 1, \dots$ we have $J[u] = \omega_{n-1} N \varepsilon I[w] + o(\varepsilon)$ as $\varepsilon \to 0$.

2. The Morse index of the solutions constructed in Theorem 1.1 can be computed explicitely. We state the following Theorem, whose proof is similar to the proof of (3) of Theorem 1.1 of [21] and will therefore be omitted.

Theorem 7.1. *Let $f(u) = u^p$, $p > 1$. Then the type I solution constructed in Theorem 1.1 has Morse index one, the type II of Theorem 1.1 have Morse index $2N$, and the type III solutions of Theorem 1.1 have Morse index $2N+1$.*

3. In a forthcoming paper we shall construct *non-radial* layered solutions for (1.1).

Open question An analytic proof for the existence of the branches $\mathcal{B}_{-k}$, $k = 1, \dots$ is still missing.

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