

NEW ENTIRE POSITIVE SOLUTION FOR THE NONLINEAR SCHRÖDINGER EQUATION: COEXISTENCE OF FRONTS AND BUMPS

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ABSTRACT. In this paper we construct a new kind of positive solutions of

$$\Delta u - u + u^p = 0 \text{ on } \mathbb{R}^2$$

when $p > 2$. These solutions have the following asymptotic behavior

$$u(x, z) \sim \omega(x - f(z)) + \sum_{i=1}^{\infty} \omega_0((x, z) - \xi_i \vec{e}_1)$$

as $L \rightarrow +\infty$ where ω is a unique positive homoclinic solution of $\omega'' - \omega + \omega^p = 0$ in \mathbb{R} ; ω_0 is the two dimensional positive solution and $\vec{e}_1 = (1, 0)$ and ξ_j are points such that $\xi_j = jL + \mathcal{O}(1)$ for all $j \geq 1$. This represents a first result on the *coexistence* of fronts and bumps. Geometrically, our new solutions correspond to *triunduloid* in the theory of CMC surface.

1. INTRODUCTION

1.1. Entire Solutions. Positive entire solutions of

$$(1.1) \quad \Delta u - u + u^p = 0 \text{ on } \mathbb{R}^N$$

where $1 < p < (\frac{N+2}{N-2})_+$, vanishing at infinity have been studied in many contexts. This class of problems arises in plasma and condensed-matter physics. For example, if one simulates the interaction-effect among many particles by introducing a nonlinear term, we obtain a nonlinear Schrödinger equation,

$$-i \frac{\partial \psi}{\partial t} = \Delta_x \psi - \psi + |\psi|^{p-1} \psi$$

where i is an imaginary unit and $p > 1$. Making an *Ansatz*

$$\psi(x, t) = \exp(-it)u(x)$$

one finds that a stationary wave u satisfies (1.1) ([16]).

In recent years, much effort has been devoted to the study of existence and multiplicity of positive solutions of

$$\varepsilon^2 \Delta u - V(x)u + u^p = 0; \quad u \in H^1(\mathbb{R}^N)$$

as $\varepsilon \rightarrow 0$. Floer–Weinstien [8] constructed single spike solutions concentrating around any given non-degenerate critical point of the potential V in \mathbb{R} provided $\inf_{\mathbb{R}} V > 0$, using Lyapunov-Schmidt reduction. This was later extended by Oh [27], [28] for the higher dimensional case.

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Spike layered solutions (solutions concentrating in zero dimensional sets) in bounded domain Ω with Dirichlet and Neumann boundary condition have been studied in recent years by many authors. See for example, Ni-Wei [26], Lin-Ni-Wei[17], and the review articles by Ni [24] and Wei [32]. Higher-dimensional concentration is later on studied by Malchiodi-Montenegro [18]-[19] in the Neumann case and by del Pino- Kowalczyk-Wei [6] in \mathbb{R}^2 .

In this paper, we focus on positive solutions to (1.1). The solution to (1.1) that is decaying at ∞ is well-understood: all such solutions are radially symmetric around some point (Gidas-Ni-Nirenberg [13]), and are unique modulo translations (Kwong [16]). Though solutions of (1.1) are bounded (since $p < (\frac{N+2}{N-2})_+$), not much is known about the solutions which does not decay at infinity [29]. One obvious solution of such kind is the following: if we consider a solution W_{N-1} of (1.1) in \mathbb{R}^{N-1} which decays at infinity, it induces a solution in \mathbb{R}^N which depends on $N-1$ variables and decays at infinity except for one direction. In the case $N=2$, consider solutions $u(x, z)$ to problem (1.1) which are even in z and vanish at $|x| \rightarrow \infty$,

$$(1.2) \quad u(x, z) = u(x, -z) \quad \forall (x, z) \in \mathbb{R}^2$$

and

$$(1.3) \quad \lim_{|x| \rightarrow \infty} u(x, z) = 0 \quad \forall z \in \mathbb{R}.$$

In [2], Dancer used local bifurcation arguments to obtain a class of solutions which constitute a one parameter family of solutions that are periodic in the z variable and originate from ω , where ω is the unique positive solution of

$$(1.4) \quad \omega'' - \omega + \omega^p = 0, \omega > 0, \omega(x) = \omega(-x) \text{ in } \mathbb{R}, \omega \in H^1(\mathbb{R}).$$

These solutions are called *Dancer's solutions*. They can be parameterized by a small parameter $\delta > 0$ and asymptotically

$$(1.5) \quad \omega_\delta(x, z) = \omega(x) + \delta \omega^{\frac{p+1}{2}}(x) \cos(\sqrt{\lambda_1} z) + \mathcal{O}(e^{-|x|}).$$

In a seminal paper [21], Malchiodi constructed a new kind of solutions with three rays of bumps. More precisely, the solutions constructed in [21] have the form

$$(1.6) \quad u(x, z) \approx \sum_{j=1}^3 \sum_{i=1}^{+\infty} \omega_0((x, z) - iL\vec{l}_j)$$

where $\vec{l}_j, j = 1, 2, 3$ are three unit vectors satisfying some balancing conditions (**Y**-shaped solutions, see Figure 1). Here ω_0 is the unique solution to the two dimensional entire problem

$$(1.7) \quad \begin{cases} \Delta \omega_0 - \omega_0 + \omega_0^p = 0, \omega_0 > 0, \\ \omega_0 \in H^1(\mathbb{R}^2). \end{cases}$$

On the other hand, in [4], del Pino, Kowalczyk, Pacard and Wei constructed another new kind of multi-front solutions using Dancer's solutions and Toda system. (These are even-ended front solutions. See Figure 2.) More precisely, the solutions constructed in [4] have the form

$$(1.8) \quad u(x, z) \approx \sum_{j=1}^K w_{\delta_j}(x - f_j(z), z)$$

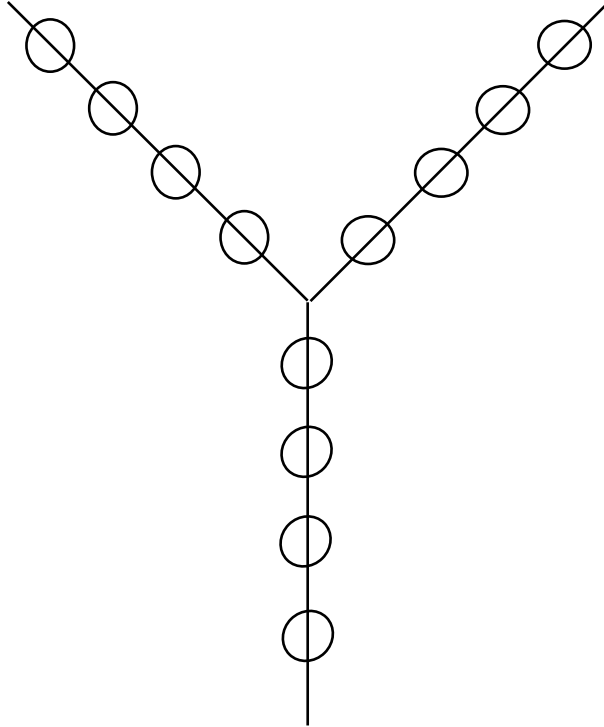


FIGURE 1. Multi-bump solutions with Y shape.

where $f_1 < f_2 < \dots < f_K$ satisfies the following Toda system

$$(1.9) \quad c_0 f_j'' = e^{f_{j-1} - f_j} - e^{f_j - f_{j+1}}, f_0 = -\infty, f_{K+1} = +\infty, c_0 > 0.$$

From now on, we call the one-dimensional solution ω as “front” solution and the two-dimensional solution ω_0 as “bump” solutions. Thus results of [4] and [21] establishes the existence of multi-front and multi-bump solutions respectively.

1.2. Main Results. In this paper we consider the nonlinear Schrödinger equation

$$(1.10) \quad \Delta u - u + u_+^p = 0 \text{ in } \mathbb{R}^2$$

where $p > 2$ and $u_\pm = \max\{\pm u, 0\}$. Our aim is to construct solutions with *both fronts and bumps*. More precisely we look for positive solutions of the form

$$(1.11) \quad u_j(x, z) = \omega(x - f(z)) + \sum_{i=1}^{\infty} \omega_0((x, z) - \xi_i \vec{e}_1)$$

for suitable large $L > 0$ and ξ_i 's are such that $\xi_1 - f(0) = L$ and

$$\xi_1 < \xi_2 < \dots < \xi_i < \dots$$

and satisfy

$$(1.12) \quad \xi_j = jL + \mathcal{O}(1)$$

for all $j \geq 1$; ω is the unique even solution to (1.4), ω_0 is the unique positive solution of (1.7) and $\vec{e}_1 = (1, 0)$. Along the line of the proof we will replace u_+ by u .

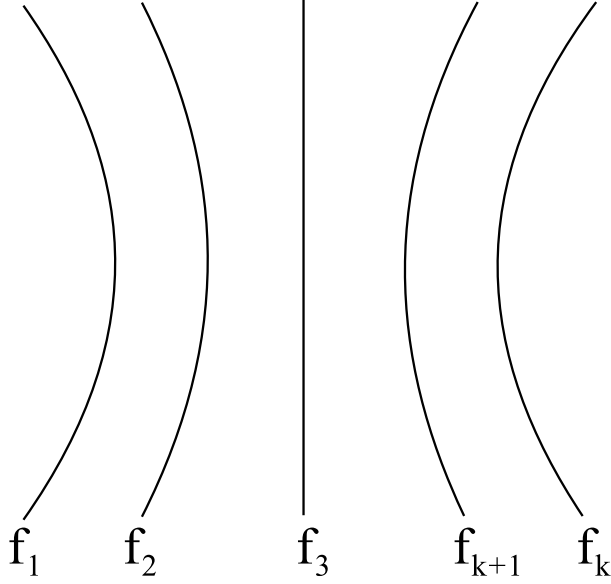


FIGURE 2. Multi-front solutions with even-ends.

Because of the new interaction between the fronts and bumps, we are led to considering the following second order ODE:

$$(1.13) \quad \begin{cases} f''(z) = \Psi_L(f, z) & \text{in } \mathbb{R} \\ f(0) = 0, \quad f'(0) = 0, \end{cases}$$

where $\Psi_L(f, z)$ is a function measuring the interactions between bumps and fronts which will be defined in Section 2. Asymptotically $\Psi_L(f, z) \sim ((f-L)^2 + z^2)^{-\frac{1}{2}} e^{-\sqrt{(f-L)^2 + z^2}}$. Let $\alpha = \int_0^{+\infty} \Psi(\sqrt{L^2 + z^2}) dz$.

The following is the main result of this paper.

Theorem 1.1. *Let $N = 2$. For $p > 2$ and sufficiently large $L > 0$, (1.10) admits a one parameter family of positive solutions satisfying*

$$(1.14) \quad \begin{cases} u_L(x, z) = u_L(x, -z) & \text{for all } (x, z) \in \mathbb{R}^2 \\ u_L(x, z) = \left(\omega_\delta(x - f(z) - h_L(z), z) + \sum_{i=1}^{\infty} \omega_0((x, z) - \xi_i \vec{e}_1) \right) (1 + o_L(1)) \end{cases}$$

where $\delta = \delta_L$ is a small constant, ω_δ is the Dancer's solution, f is the unique solution of (1.13), ξ_j satisfy (1.12) and $o_L(1) \rightarrow 0$ as $L \rightarrow +\infty$, and the function $\|h_L\|_{C_\theta^{\alpha, \mu}(\mathbb{R}) \oplus \mathcal{E}} \leq C\alpha^{1+\gamma}$ for some constant $\theta > 0, \gamma > 0$. (\mathcal{E} will be defined at Section 2.) Moreover, the solution has three ends.

Figure 3 shows graphically how the solution constructed in Theorem 1.1 looks like triunduloid type I. (This corresponds end-to-end gluing construction. See discussions at the end.)

A modification of our technique can be used to construct the following two new types of solutions: the first one is a combination of positive bump and infinitely

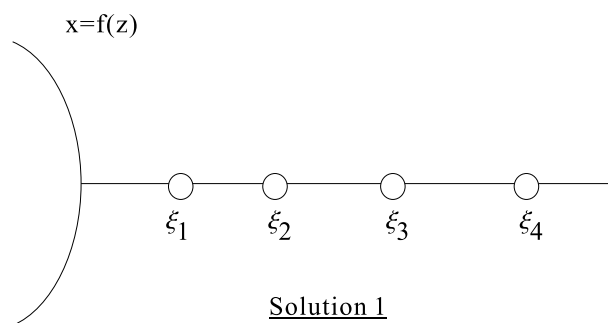


FIGURE 3. Isosceles triunduloid Type I

many sign-changing bumps—we call it Solution 2 (triunduloid type II). The second one is with two fronts and one bump (or finitely many bumps)—we call it Solution 3.

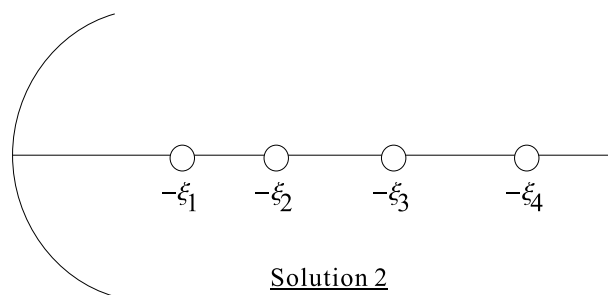


FIGURE 4. Isosceles triunduloid Type II

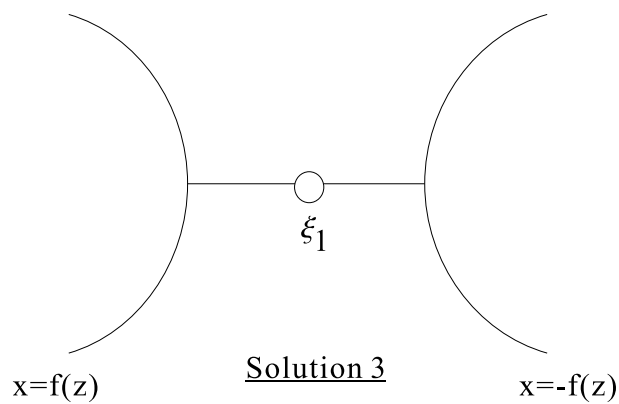


FIGURE 5. End to end gluing construction

In this paper we will only discuss the proofs of Solution 1. The modifications needed for Solution 2 and Solution 3 will be explained at the last section.

Theorem 1.1 implies that we can construct solutions which does not decay the x -axis but decay everywhere else. Though Theorem 1.1 is a purely PDE result, this result has an analogy in the theory of constant mean curvature (CMC) surface in \mathbb{R}^3 which we shall describe below.

1.3. Relation with CMC Theory. CMC surfaces in \mathbb{R}^3 are an equilibria for the area functional subjected to an enclosed volume constraint. To explain mathematically suppose an oriented surface \mathcal{S} is embedded in a manifold M and let us denote ν be the normal field compatible with the orientation. Then for any function z which is smooth small function we define a *perturbed surface* \mathcal{S}_z as the normal graph of the function of z over \mathcal{S} . Namely \mathcal{S}_z is parameterized as

$$p \in \mathcal{S} \mapsto \exp(w(p)\nu(p))$$

where \exp is the exponential map in (M, g) . Decompose z into the positive part and the negative part of z as $z = z^+ - z^-$ and define the set

$$B_{z^\pm} := \{\exp_p t\nu(p) : \pm t \in (0, z^\pm(p))\}.$$

Then the m -th volume functional

$$\mathcal{A}(z) = \int_{\mathcal{S}_z} d\text{vol}_{\mathcal{S}_z}$$

and its first and second variations at $z = 0$ are

$$D\mathcal{A}(0)(v) = \int_{\mathcal{S}} H v \, d\text{vol}_{\mathcal{S}}$$

$$D^2\mathcal{A}(0)(v, v) = \int_{\mathcal{S}} (|\nabla_g v|^2 - (\kappa_1^2 + \kappa_2^2 + \dots + \kappa_m^2)v^2 - Ric(\nu, \nu)v^2 + H^2v^2) d\text{vol}_{\mathcal{S}}$$

where κ_i are the principal curvatures of \mathcal{S} , Ric denotes the Ricci tensor on (M, g) and H is the mean curvature function and depends on \mathcal{S} . Also note that the critical points of \mathcal{A} are precisely surfaces of mean curvature zero and usually referred to as minimal surfaces. Moreover, define $(m + 1)$ th volume functional

$$\mathcal{V}(z) := \int_{B_{z^+}} d\text{vol}_M - \int_{B_{z^-}} d\text{vol}_M$$

where volumes are counted positively when $w > 0$ and negatively when $w < 0$. The first variation of \mathcal{V} is given by

$$D\mathcal{V}(0)(v) = \int_{\mathcal{S}} v d\text{vol}_{\mathcal{S}}$$

and its second variation is given by

$$D^2\mathcal{V}(0)(v, v) = - \int_{\mathcal{S}} H v^2 d\text{vol}_{\mathcal{S}}$$

Define the *shape* operator as

$$|A|^2 = \sum_{i=1}^m \kappa_i^2.$$

We see that critical points of the functional \mathcal{A} with respect to some volume constraint $\mathcal{V} = \text{constant}$ have constant mean curvature. Here the mean curvature appears as a multiple of the Lagrange multiplier associated to the constraint (and

hence it is constant). The surfaces with constant mean curvature equal to $H = \lambda$ are critical points of $\mathcal{W}(\mathcal{S}) := \mathcal{A}(\mathcal{S}) + \lambda\mathcal{V}(\mathcal{S})$. The quadratic form can be written as

$$D^2\mathcal{W}(0)(v, v) = - \int_{\mathcal{S}} v \mathcal{J}_{\mathcal{S}} v \, d\text{vol}_{\mathcal{S}}$$

where the *Jacobi* operator

$$(1.15) \quad \mathcal{J}_{\mathcal{S}} = \Delta_{\mathcal{S}} + |A|^2 + Ric_g(\nu, \nu)$$

For CMC surfaces the sign of H and its value can be changed by a reversal of orientation and homothety respectively and as a result we can normalize the surface such that $H \equiv 1$. CMC interfaces arise in many physical and variational problems. Over the past two decades there is a great deal of progress in understanding complete CMC and their *moduli* spaces. Moduli is a notion to identify invariant surfaces. In order to study the structure of moduli spaces one needs to study the properties of (1.15). The reflection technique of Alexandrov [1] shows that spheres is the only compact embedded CMC surface of *finite* topology. These are surfaces homeomorphic to a compact surface \mathcal{S} of genus g with a finite number of points removed from it say m . The neighborhood of each of these punctures are called *ends*. Mathematically, we define the ends e_j of an embedded surface \mathcal{S} in \mathbb{R}^3 with finite topology to be a non-compact connected components of the surface near infinity

$$\mathcal{S} \cap (\mathbb{R}^3 \setminus B_{R_0}(0)) = \cup_{j=1}^m e_j$$

where $B_{R_0}(0)$ denotes a ball of radius R_0 (is chosen sufficiently large so that m is constant for all $R > R_0$). Note that sphere is a zero end surface.

The theory of properly embedded CMC surfaces, was classified by Delaunay [3]. These are rotationally symmetric CMC surfaces, called *unduloids* (having genus zero and two ends). To describe these, consider the cylindrical graph

$$(1.16) \quad (t, \theta) \mapsto (h(t) \cos \theta, h(t) \sin \theta, t).$$

The CMC graph is an ordinary differential equation given by,

$$(1.17) \quad \begin{cases} h_{tt} - \frac{1}{h}(1 + h_t^2) + (1 + h_t^2)^{\frac{3}{2}} = 0 \\ \min_t h(t) = \varepsilon. \end{cases}$$

Moreover, all the positive solutions of (1.17) are periodic and may be distinguished by their minimum value $\varepsilon \in (0, 1]$, which is more often referred to as the Delaunay parameter of the surface D_{τ} where $\tau = 2\varepsilon - \varepsilon^2$. Moreover, when $\tau = 1$, D_1 is a cylinder of radius 1 and as $\tau \downarrow 0$, D_{τ} converges to an infinite array of mutually tangent spheres of radius 2 with centers along the z axis. The family D_{τ} interpolates between two extremes and ε measures the size of the neck region. Moreover, using a parameterization (1.16) and

$$(1.18) \quad t = k(s), h(t) = \tau e^{\sigma(s)},$$

we obtain the Jacobi operator for the surface D_{τ} is given by

$$(1.19) \quad \mathcal{J}_D = \frac{1}{2\tau^2 e^{2\sigma}} (\partial_s^2 + \partial_{\theta}^2 + \tau^2 \cosh 2\sigma)$$

where $\sigma'' + \frac{\tau^2}{2} \sinh 2\sigma = 0$ and $k' = \frac{\tau^2}{2} (e^{2\sigma} + 1)$.

These surfaces are periodic and interpolate between the unit cylinder and the singular surfaces formed by a string of spheres of radius 2, each tangent to the

next along a fixed axis. In particular, Delaunay established that every CMC surface of revolution is necessarily one of these “Delaunay surfaces”. Kapouleas [14] constructed numerous examples of complete embedded CMC surface in \mathbb{R}^3 (with genus $g \geq 2$ and ends $k \geq 3$) by gluing Delaunay surfaces onto spheres. In fact he produced CMC surfaces using suitably *balanced* simplicial graphs where the k edges are rays tending to infinity. By balancing condition we mean that the force vectors associated with each edge cancel at each vertex. In fact balancing condition combined with spherical trigonometry plays an important role in classifying CMC surfaces with three ends. A more flexible gluing techniques was used by Mazzeo and Pacard in [22] to explore moduli surface theory which involves several boundary value problems and then matching the boundary values across the interface.

A CMC surface \mathcal{S} of finite topology is *Alexandrov-embedded*; if \mathcal{S} is properly immersed, and if each end of \mathcal{S} is embedded; there exist a compact manifold M with boundary of dimension three and a proper immersion $F : M \setminus \{q_1, q_2, \dots, q_m\} \rightarrow \mathbb{R}^3$ such that $F|_{\partial M \setminus \{q_1, q_2, \dots, q_m\}}$ parameterizes M . Moreover, the mean curvature normal of \mathcal{S} points into M .

Then we define *triunduloid* as an Alexandrov embedded CMC surface having zero genus and three ends. Triunduloids are a basic building block for Alexandrov embedded CMC surface with any number of ends. Nonexistence of one end Alexandrov embedded CMC surface was proved by Meeks [23]. Kapouleas [14], G-Brauckmann [9] and Mazzeo-Pacard [22] established existence of triunduloid with small necksize or high symmetry. In fact G-Brauckmann [9] used conjugate surface theory construction to obtain families of symmetric embedded complete CMC surfaces. The geometry of moduli space plays an very important role for the understanding of the structure of CMC’s.

The main aim of this paper is to prove existence of *triunduloid* type of solution for (1.1) in \mathbb{R}^2 i.e. a solution having three ends. Solutions having even number of ends have been shown to exist in recent paper of del Pino, Kowalczyk, Pacard and Wei in [4]. **Y** shaped solutions of (1.1) in \mathbb{R}^3 were constructed by Malchiodi [21]. Hence Theorem 1.1 proves that the moduli space $\mathcal{M}_3(\mathbb{R}^2)$ of all 3 – *end* solutions is nonempty.

Geometrically, solutions constructed in Theorem 1.1 correspond to the so-called *end-to-end* gluing in CMC. (We are indebted to Prof. F. Pacard for this connection.) The end-to-end gluing in CMC corresponds to adding a handle to a multi-end CMC surfaces. The procedure has been done in the thesis of J. Ratzkin [30]. (A similar construction has been done for the construction of positive metrics with constant positive scalar curvature [31].) For nonlinear Schrodinger equation, adding a handle means adding a half-ray solution with infinitely many bumps. The solution in Theorem 1.1 represents first step in adding a handle. We believe that with more work it is possible to add handles to the even number ends solutions constructed in [4].

Finally we should also mention that in a recent paper [25], Musso, Pacard and Wei have constructed *nonradial finite-energy sign-changing solutions*, using geometric analogue constructions of Kapouleas [14].

1.4. **Main ideas of proof.** We sketch the main ideas of the proofs of Theorem 1.1. The solutions we construct have the form

$$(1.20) \quad u(x, z) \sim \omega_\delta(x - f(z), z) + \sum_{i=1}^{\infty} \omega_0((x, z) - \xi_i \vec{e}_1).$$

There are three main parts of proofs: firstly, we add a half-line of bumps (corresponds to $\sum_{i=1}^{\infty} \omega_0((x, z) - \xi_i \vec{e}_1)$). For this part we use the idea of Malchiodi [21]. Namely we need to use Dancer's solutions with large periods and analyze the interactions using Toeplitz matrix. Secondly, we have a front solution (corresponds to $\omega_\delta(x - f(z), z)$). This is a two-end solution and we follow the analysis by del Pino, Kowalczyk, Pacard and Wei [4]. The third part deals with the *interaction part*. Because of the exponentially decaying tails of both ω_δ and ω_0 , only the interaction of the first bump and the front is the dominating force. We have to compute the corresponding ODE which ultimately determines the curve $f(z)$. In all these three parts, we will make use of the *infinite-dimensional Liapunov-Schmidt reduction method*. For this method, we refer to [4], [5], [6], [7].

2. THE EXPONENTIAL EQUATION, TOEPLITZ MATRIX AND IT LINEARISATION

2.1. **The differential equation involving f .** In this paper the second order ODE (1.13) plays an important role. We shall study the properties of this ODE and identify the scaling parameter.

First let us define the function Ψ : let ω be the one-dimensional solution and ω_0 be the two-dimensional solution. Ψ measures the interactions between ω and ω_0 and is defined by

$$(2.1) \quad \Psi_L(f, z) = p \int_{\mathbb{R}} \omega^{p-1}(x) \omega_x(x) \omega_0(\sqrt{(x+f)^2 + z^2}) dx$$

Asymptotically

$$(2.2) \quad \Psi_L(f, z) \sim (f^2 + z^2)^{-\frac{1}{2}} e^{-\sqrt{f^2 + z^2}}.$$

We also note that

$$(2.3) \quad \frac{\partial \Psi_L(f, z)}{\partial f} < 0, \quad \frac{\partial \Psi_L(f, z)}{\partial z} < 0.$$

Let $L \gg 1$ be a fixed large number. We choose the following small parameter

$$(2.4) \quad \alpha = e^{-\frac{L}{\sqrt{2}}}$$

then $\alpha \rightarrow 0$ as $L \rightarrow \infty$.

For any $0 \leq \mu < 1$ we define $C_\theta^{l, \mu}(\mathbb{R})$ to be the space of all real-valued functions where

$$\|f\|_{C_\theta^{l, \mu}(\mathbb{R})} = \|(\cosh z)^\theta f\|_{C^{l, \mu}(\mathbb{R})} < +\infty.$$

We will fix μ later. First note that using a scaling we can consider $c_1 = 1$. Now $f'' \geq 0$ and hence f' is an increasing function. Also note as f is even, it is enough to study the behavior of f when $z > 0$. After a translation, (1.13) becomes

$$(2.5) \quad \begin{cases} f''(z) = \Psi_L(f, z) & \text{in } \mathbb{R} \\ f(0) = L \\ f'(0) = 0. \end{cases}$$

It is easy to see that (1.13) admits a global bounded solution which is also increasing. We claim the following result: there exists $C_1 > 0, a_1 > 0$ such that

$$(2.6) \quad f(z) = L + C_1 + \alpha a_1 z + \mathcal{O}(\alpha e^{-\frac{z}{\sqrt{2}}}),$$

$$(2.7) \quad f_z(z) = \alpha a_1 + \mathcal{O}(\alpha e^{-\frac{|z|}{\sqrt{2}}}),$$

$$(2.8) \quad f_{zz}(z) = \mathcal{O}(\alpha e^{-\frac{|z|}{\sqrt{2}}}).$$

Since $f' \geq 0$, it is easy to see that (2.6)-(2.7) is a consequence of (2.8). We just need to establish (2.8). To this end, we note that for all $z \in \mathbb{R}$ we have $\sqrt{L^2 + |z|^2} \geq \frac{1}{\sqrt{2}}L + \frac{1}{\sqrt{2}}|z|$. Because of our choice of α at (2.4), we have $e^{-\sqrt{L^2+z^2}} \leq \alpha e^{-\frac{1}{\sqrt{2}}|z|}$. This implies that

$$(2.9) \quad f_{zz} = \mathcal{O}(\alpha e^{-\frac{1}{\sqrt{2}}|z|})$$

which proves (2.8).

2.2. Bounded solvability of (1.13) on \mathbb{R} . In this section we study the linearized operator of (2.5), around a solution f of (2.5). Let g be an even continuous, bounded function. Consider the following linear equation

$$(2.10) \quad \mathcal{Q}(\psi) := \psi'' - \frac{\partial \Psi_L}{\partial f} \psi = g \text{ in } \mathbb{R}$$

We analyze the solvability of the linear problem in $\psi \in C_\theta^{2,\mu}(\mathbb{R})$, given $g \in C_\theta^{0,\mu}(\mathbb{R})$.

Note that asymptotically we have

$$(2.11) \quad -\frac{\partial \Psi_L}{\partial f} \sim \frac{f}{\sqrt{f^2 + z^2}} e^{-\sqrt{f^2+z^2}}$$

Remark 2.1. For the homogeneous equation, there are two fundamental solutions ψ_1 and ψ_2 satisfying

$$(2.12) \quad \begin{cases} \psi_1'' - \frac{\partial \Psi_L}{\partial f} \psi_1 = 0 & \text{in } \mathbb{R} \\ \psi_1(0) = 0 \\ \psi_1'(0) = 1, \end{cases}$$

$$(2.13) \quad \begin{cases} \psi_2'' - \frac{\partial \Psi_L}{\partial f} \psi_2 = 0 & \text{in } \mathbb{R} \\ \psi_2(0) = 1 \\ \psi_2'(0) = 0. \end{cases}$$

Note that ψ_1 is odd while ψ_2 is even. We now claim that $\psi_1'(+\infty) \neq 0$. In fact, suppose $\psi_1'(+\infty) = 0$. Since f_z satisfies

$$(2.14) \quad \begin{cases} f_z'' - \frac{\partial \Psi_L}{\partial f} f_z = \frac{\partial \Psi_L}{\partial z} < 0 & \text{in } \mathbb{R} \\ f_z(0) = 0 \\ f_{zz}(0) = \Psi_L(0,0), \end{cases}$$

and $f_z > 0$, we see that by the Maximum Principle $\psi_1 > 0$. Then if $\psi_1'(+\infty) = 0$, then we have $\int_0^{+\infty} \frac{\partial \Psi_L}{\partial z} \psi = 0$ which is impossible.

Thus ψ_1 grows like cz as $+\infty$. This implies that ψ_2 must be a constant at $+\infty$.

We define the one dimensional space called the *deficiency subspace* $\mathcal{E} = \{\chi\psi_1\}$ and χ is a smooth cut off function such that

$$(2.15) \quad \chi(z) = \begin{cases} 1 & \text{if } z > 1 \\ 0 & \text{if } z < 0. \end{cases}$$

Moreover, we define the norm on $C_a^{2,\mu}(\mathbb{R}) \oplus \mathcal{E}$ to be such that

$$\|(\psi, c\chi\psi_1)\|_{C_a^{2,\mu}(\mathbb{R}) \oplus \mathcal{E}} = \|\psi\|_{C_a^{2,\mu}(\mathbb{R})} + |c|.$$

Lemma 2.2. [*Linear Decomposition Lemma*] *Let f be the unique solution of (2.5). The mapping*

$$\mathcal{Q} : C_\theta^{2,\mu}(\mathbb{R}) \oplus \mathcal{E} \rightarrow C_\theta^{0,\mu}(\mathbb{R})$$

$$\psi \mapsto \psi'' - \frac{\partial \Psi_L}{\partial f} \psi$$

is an isomorphism.

Proof. Let $\|g\|_{C_\theta^{0,\mu}(\mathbb{R})} < +\infty$. Then it is easy to see that by the method of variation of constants the following function

$$(2.16) \quad \psi = \mathcal{R}(g) = \psi_1(z) \int_0^z \psi_2 g + \psi_2(z) \int_z^{+\infty} \psi_1 g$$

is a solution to

$$(2.17) \quad \psi \mapsto \psi'' - \frac{\partial \Psi_L}{\partial f} \psi = g, \psi'(0) = 0$$

We claim that $\psi = \mathcal{R}(g) \in C_\theta^{2,\mu}(\mathbb{R}) \oplus \mathcal{E}$. In fact, we simply write

$$(2.18) \quad \begin{aligned} \mathcal{R}(g) &= \mathcal{R}_1(g) + \mathcal{R}_2(g)\chi\psi_1 \\ &= \psi_1(z)(1 - \chi) \int_0^z \psi_2 g - \chi\psi_1 \int_z^{+\infty} \psi_2 g + \psi_2(z) \int_z^{+\infty} \psi_1 g - \psi_1(z) \\ &\quad + \int_0^{+\infty} \psi_2 g \chi\psi_1(z) \end{aligned}$$

where $\mathcal{R}_2(g) = \int_0^{+\infty} \psi_2 g$.

Clearly we have

$$(2.19) \quad \|\mathcal{R}_1(g)\|_{C_\theta^{2,\mu}(\mathbb{R})} \leq C \|g\|_{C_\theta^{0,\mu}(\mathbb{R})}, \quad |\mathcal{R}_2(g)| \leq C \|g\|_{C_\theta^{0,\mu}(\mathbb{R})}$$

□

Remark 2.3. Moreover, the space \mathcal{E} can also be described as a *parameter space* for the linear problem \mathcal{Q} , since the elements are potentially occurring parameters for the *Jacobi field* that is those elements ψ such that $\mathcal{Q}(\psi) = 0$.

2.3. Solvability of another differential equation. In an analogous way we look for even solutions of

$$(2.20) \quad e'' + \lambda_1 e = k(z)$$

where k is even with $\|k(\cosh z)^\theta\|_{C^{0,\mu}(\mathbb{R})} < +\infty$. We are interested in solution which decays to zero at $+\infty$. Since (2.20) is a resonance problem, we impose the following orthogonality condition

$$(2.21) \quad \int_0^\infty k(z) \cos(\sqrt{\lambda_1} z) dz = 0$$

to prove existence and uniqueness of solutions. Using the method of variation of parameters the solution of (2.20) can be written as $\mathcal{S}(k) = e$ where

$$(2.22) \quad \begin{aligned} \mathcal{S}(k) &= \frac{1}{\sqrt{\lambda_1}} \sin(\sqrt{\lambda_1} z) \int_z^\infty k(t) \cos(\sqrt{\lambda_1} t) dt \\ &- \frac{1}{\sqrt{\lambda_1}} \cos(\sqrt{\lambda_1} z) \int_z^\infty k(t) \sin(\sqrt{\lambda_1} t) dt \end{aligned}$$

Furthermore, we have

$$(2.23) \quad \|e(\cosh z)^\theta\|_{C^{2,\mu}(\mathbb{R})} \leq C < +\infty.$$

2.4. Location of the spikes. Let $\xi = (\xi_1, \xi_2 \dots, \dots)$ be a sequence of points satisfying

$$(2.24) \quad \xi_2 = 2\xi_1 + \mathcal{O}(1)$$

and for all $j \geq 2$

$$(2.25) \quad \xi_{j+1} - \xi_j = \xi_j - \xi_{j-1}.$$

Then we obtain for all $j \geq 1$

$$(2.26) \quad \xi_j = jL + \mathcal{O}(1).$$

2.5. Invertibility of the operator associated with the Toeplitz matrix. Let $\xi = (\xi_i)_{i \geq 1}$. We define an operator $T : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ such that $T = (T(\xi_i))_j$ where

$$(2.27) \quad (T(\xi_i))_j = \begin{cases} 2\xi_j & \text{if } j = i \\ -\xi_j & \text{if } j = i \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Our main goal is given $\chi = (\chi_1, \dots, \chi_j \dots)$ we want to solve $T(\xi) = \chi$. Using the fact that (1.4) we define a weighted norm $\xi = (\xi_i)_{i \geq 1}^\infty$ by

$$\|\xi\|_\alpha = \|(\xi_1, \xi_2, \xi_3, \dots)\|_\alpha = \max_i \alpha^{-i} |\xi_i|.$$

Let

$$\Omega = \{\xi = (\xi_1, \xi_2, \dots, \xi_i, \dots) : \|\xi\|_\alpha < +\infty\}.$$

Lemma 2.4. *The operator T has an inverse in Ω , whose norm is $\mathcal{O}(\alpha)$.*

Proof. For any $\|\chi\|_\alpha < +\infty$, we define

$$\xi_j = \sum_{k=j}^\infty (k-j) \chi_k.$$

Let I denote the operator defined by the above expression. Then I is an operator inverse of T . Clearly we have

$$(2.28) \quad |\xi_j| \leq \|\chi\|_\alpha \sum_{k=j}^{\infty} (k-j)\alpha^{k+1} \leq C\alpha^{(j+1)}\|\chi\|_\alpha$$

This implies

$$\begin{aligned} \alpha^{-j}|\xi_j| &\leq C\alpha\|\chi\|_\alpha \\ \|\xi\|_\alpha &\leq C\alpha\|\chi\|_\alpha. \end{aligned}$$

Note that C is independent of α . \square

2.6. Idea of the construction. We are actually looking for bump line solution of (1.10) whose asymptotic behavior is determined by the curve

$$\gamma = \{(x, z) : x = f(z)\}$$

which asymptotically behaves as straight lines having negative exponential growth in the second order. Then it turns out that f satisfies a second order differential equation, given by (1.13). Moreover, by (2.1) we have $f(z) = \beta + \alpha a_1|z| + \mathcal{O}_{C^\infty}(\alpha(\cosh z)^{-\frac{1}{\sqrt{2}}})$ for some $\beta > 0$. Define $\theta = \frac{1}{\sqrt{2}}$. Also note that the solution of (1.13) is unique and since $f(z), f(-z)$ are solutions to (1.13) we must have $f(z) = f(-z)$ for all $z \in \mathbb{R}$. Let Z be a positive eigenfunction of

$$(2.29) \quad \varphi_{xx} + (p\omega^{p-1} - 1)\varphi = \lambda_1\varphi$$

corresponding to the principal eigenvalue λ_1 where explicitly

$$Z(x) = \frac{\omega^{\frac{p+1}{2}}(x)}{\int_{\mathbb{R}} \omega^{p+1} dx}; \quad \lambda_1 = \frac{1}{4}(p+3)(p-1)$$

and in particular the asymptotic behavior of ω and Z at infinity are given by

$$\omega(x) \sim e^{-|x|} + \mathcal{O}_{C^\infty(\mathbb{R})}(e^{-2|x|})$$

and

$$Z(x) \sim e^{-\frac{p+1}{2}|x|} + \mathcal{O}_{C^\infty(\mathbb{R})}(e^{-(p+1)|x|}).$$

Consider the *Dancer's solution* of (1.10) as $\omega_\delta(x, z)$

$$\omega_\delta(x, z) = \omega(x) + \delta Z(x) \cos(\sqrt{\lambda_1}z) + \mathcal{O}(\delta^2)e^{-|x|}$$

where $|\delta|$ is sufficiently small.

2.7. Modified Fermi coordinates near the bump line. Let f be a solution of the (1.13). We choose $v \in \mathcal{E}$ such that

$$(2.30) \quad v = c\chi\psi_1, \quad |c| \leq \alpha^{1+k_1}$$

where k_1 is a small number to be chosen later. Now we define a model bump curve as

$$\bar{\gamma} = \{\mathbf{x} = (x, z) \in \mathbb{R}^2 : x = \bar{f}(z) = f(z) + v(z)\}$$

where f is the solution of (1.13). Then we define the local coordinate as a vector tuple (T, N) where unit tangent

$$T = \frac{1}{\sqrt{1 + (\bar{f}')^2}}(\bar{f}', 1)$$

and the unit normal to the curve

$$N = \frac{1}{\sqrt{1 + (\bar{f}')^2}}(1, -\bar{f}').$$

Let z be the arc length defined as

$$z = \int_0^z \sqrt{1 + (\bar{f}'(s))^2} ds$$

which is an increasing function of z and let $q(z)$ be the corresponding arc length parameter. Note that $q(z) \in \mathbb{R}^2$. It turns out that the asymptotic behavior of the bump line at infinity is not exactly linear but has an exponentially small correction. This correction needs to be determined and in fact this is the key step in the paper which involves the linearized operator discussed in Remark 2.1. To describe this small perturbation we consider a fixed function h

$$(2.31) \quad \|h\|_{C_o^{2,\mu}(\mathbb{R})} \leq \alpha^{1+k_2}$$

for some $k_2 > 0$ small.

A neighborhood of the curve $\bar{\gamma}$ can be parametrised in the following way

$$(2.32) \quad \mathbf{x} = X(x, z) = q(z) + (x + h(z))N(z)$$

where $t = x + h(z)$ is the *signed* distance to the curve $\bar{\gamma}$. Define a set

$$V_\zeta = \{\mathbf{x} = (x, z) : |x| \leq \zeta \sqrt{1 + z^2}\}$$

for small ζ . In fact the Fermi coordinates of the curve is defined as long as the map $(t, z) \mapsto \mathbf{x}$ is one-one. The asymptotic behavior of the curvature of $\bar{\gamma}$ as $|z| \rightarrow +\infty$ is given by

$$\kappa(z) \sim \alpha(\cosh z)^{-\theta}.$$

Furthermore, we can show that for ζ and α sufficiently small the Fermi coordinates are well defined around $\bar{\gamma}(z)$ as long as

$$(2.33) \quad |t| \leq \zeta \sqrt{1 + z^2}.$$

Also we have

$$(2.34) \quad \mathbf{x} \in V_\zeta \Rightarrow |x| = |t - h(z)| \leq \zeta \sqrt{1 + z^2}$$

where $\mathbf{x} = X(x, z)$. Moreover, we define

$$(2.35) \quad X^* f(x, z) = f \circ X(x, z).$$

Furthermore, we have

$$(2.36) \quad x = \mathbf{x}(1 + \mathcal{O}(\alpha^2)) + z\mathcal{O}(\alpha)$$

and

$$(2.37) \quad z = (1 + \mathcal{O}(\alpha^2))z.$$

2.8. Laplacian in the shifted coordinates. The curvature κ of the curve $\bar{\gamma}$ which is given by

$$(2.38) \quad \kappa = \frac{\bar{f}''(z)}{(1 + (\bar{f}'(z)))^{\frac{3}{2}}}.$$

We define A by

$$A := 1 - (x + h)\kappa.$$

Then the Laplacian in terms of the new coordinates reduces to

$$(2.39) \quad \Delta = \frac{1}{A} \left\{ \partial_x \left(\frac{A^2 + (h')^2}{A} \partial_x \right) - \partial_z \left(\frac{h'}{A} \partial_x \right) - \partial_x \left(\frac{h'}{A} \partial_z \right) + \partial_z \left(\frac{1}{A} \partial_z \right) \right\}.$$

Then (2.39) can be written as

$$(2.40) \quad \Delta = \partial_x^2 + \partial_z^2 + a_{11}\partial_x^2 + a_{12}\partial_x\partial_z + a_{22}\partial_z^2 + b_1\partial_x + b_2\partial_z^2$$

where

$$(2.41) \quad a_{11} = \frac{(h')^2}{A^2}, a_{12} = \frac{2h'}{A^2}, a_{22} = \frac{1 - A^2}{A^2}$$

$$b_1 = \frac{1}{A^3}(-\kappa A^2 - h''A + (h')^2\kappa - (x + h)h'\kappa)$$

and

$$(2.42) \quad b_2 = \frac{1}{A^3}((h + x)\kappa).$$

Note that here we have

$$\kappa = \mathcal{O}_{C_\theta^{2,\mu}(\mathbb{R})}(\alpha), \kappa' = \mathcal{O}_{C_\theta^{2,\mu}(\mathbb{R})}(\alpha^2)$$

and consequently we have

$$(2.43) \quad \begin{cases} a_{11} = \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha^2), a_{12} = \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha), a_{22} = \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha(1 + |x|)) \\ b_1 = \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha(1 + |x|)), b_2 = \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha(1 + |x|)). \end{cases}$$

2.9. Approximate solution. In this section we develop the approximate solution. Firstly we take a Dancer solution and the homoclinic solution. These two solutions need to be glued together by some cut-off function. In this way the amplitude and the phase shifts of the ends do not change but instead remain fixed. To achieve an extra degree of freedom a function whose local form is given by $e(z)Z(x)$ is added to our approximation.

Precisely, we consider $e \in C_\theta^{2,\mu}(\mathbb{R})$ such that

$$(2.44) \quad \|e\|_{C_\theta^{2,\mu}(\mathbb{R})} \leq C\alpha^{2+k_3}$$

where k_3 will be chosen later. In addition, we will use a real parameter δ such that

$$(2.45) \quad |\delta| \leq \alpha^{1+k_4}.$$

We define the following notations

$$(2.46) \quad \begin{aligned} X^*\omega_\delta(x, z) &= \omega_\delta(x, z) \\ X^*\omega(x, z) &= \omega(x) \\ X^*Z(x, z) &= Z(x) \end{aligned}$$

where ω is the homoclinic solution, ω_δ and Z being the Dancer solution and the principle eigenfunction of (2.29) respectively. Now we choose Ξ and Ξ_0 be nonnegative even cut-off function such that

$$\Xi(t) + \Xi_0(t) = 1; \forall t \in \mathbb{R}$$

with

$$\text{supp } \Xi = (-\infty, -1) \cup (1, +\infty), \text{supp } \Xi_0 = (-2, 2).$$

Also let

$$X^* \Xi(x, z) = \Xi(z), X^* \Xi_0(x, z) = \Xi_0(z).$$

Now we introduce $w = \Xi\omega + \Xi_0\omega$. Let χ be such that

$$\|\chi\|_\alpha \leq C\alpha^{k_5}$$

where k_5 is a small positive number. Define

$$\omega_j(x, z) = \omega_0(x - \xi_j - \chi_j, z) \text{ and } \omega_{j,x}(x, z) = \omega_{0,x}(x - \xi_j - \chi_j, z).$$

We thereby define the approximate solution of (1.1) in V_ζ as

$$(2.47) \quad \bar{w}(\hat{x}) = w + e(z)Z.$$

Now we intend to define a global approximation. Let η_ζ be a smooth cutoff function such that $\text{supp } \eta_\zeta \subset V_\zeta$ such that $\eta \equiv 1$ in $V_{\frac{\zeta}{2}}$ and ξ satisfying (2.26), then we define the global approximation as

$$(2.48) \quad \begin{aligned} \mathbf{w} &= \eta_\zeta(w + e(z)Z) + \sum_{j=1}^{\infty} \omega_j(x, z) \\ &= \eta_\zeta \bar{w} + \sum_{j=1}^{\infty} \omega_j(x, z). \end{aligned}$$

Notice that \mathbf{w} depends on f, v, h, δ, χ .

2.10. The key estimates. In this section we precisely derive some key estimates concerning the interaction of spikes and the interaction of the front with the spike. First note that ω_0 is radial and the asymptotic behavior of ω_0 at infinity is given by

$$\lim_{r \rightarrow \infty} e^r r^{\frac{1}{2}} \omega_0(r) = A_0 > 0; \text{ and } \lim_{r \rightarrow \infty} \frac{\omega_0'(r)}{\omega_0(r)} = -1.$$

We have the following key estimates: Let $\hat{x} = (x, z)$. Let $\vec{e}_1 = (0, 1)$, then we have

$$(2.49) \quad \begin{aligned} \int_{\mathbb{R}^2} \omega_0^p(\hat{x}) \omega_0'(|\hat{x} + L\vec{e}_1|) \frac{x+L}{|\hat{x} + L\vec{e}_1|} d\hat{x} &= -A_0 \int_{\mathbb{R}^2} |\hat{x} + L\vec{e}_1|^{-\frac{1}{2}} e^{-|\hat{x}+L\vec{e}_1|} \frac{x+L}{|\hat{x} + L\vec{e}_1|} \omega_0^p(\hat{x}) \\ &= -A_0 \int_{\mathbb{R}^2} |\hat{x} + L\vec{e}_1|^{-\frac{1}{2}} e^{-|\hat{x}+L\vec{e}_1|} \frac{x+L}{|\hat{x} + L\vec{e}_1|} \omega_0^p(\hat{x}) \\ &= -A_0 \int_{\mathbb{R}^2} L^{-\frac{1}{2}} e^{-L} e^{L-|x+L\vec{e}_1|} \frac{x+L}{|\hat{x} + L\vec{e}_1|} \omega_0^p(\hat{x}) \\ &= -A_0 L^{-\frac{1}{2}} e^{-L} \int_{\mathbb{R}^2} \omega_0^p(\hat{x}) e^{-L} e^{L-|\hat{x}+L\vec{e}_1|} \frac{x+L}{|\hat{x} + L\vec{e}_1|} \\ &= -\omega_0(L)(1 + O(L^{-1})) \int_{\mathbb{R}^2} e^{-|x|} \omega_0^p(\hat{x}) d\hat{x} \\ &= -\gamma_0 \omega_0(L) \left(1 + \mathcal{O}\left(\log \frac{1}{\alpha}\right)^{-1}\right) \end{aligned}$$

where $\gamma_0 = \int_{\mathbb{R}^2} e^{-|x|} \omega_0^p(\hat{x}) d\hat{x}$. In this case we consider L to be either $\xi_{j+1} - \xi_j$ or $\xi_j - \xi_{j-1}$ where $j \geq 2$. Similarly we can show there exists $\gamma_1 > 0$ such that

$$(2.50) \quad \int_{\mathbb{R}^2} \omega^p(x) \omega_0'(|\hat{x} + L\vec{e}_1|) \frac{x + L}{|\hat{x} + L\vec{e}_1|} d\hat{x} = -\gamma_1 e^{-|\xi_1 - f(0)|} \left(1 + \mathcal{O}\left(\log \frac{1}{\alpha}\right)^{-1}\right).$$

3. PROOF OF THEOREM 1.1

Let η and η_j be smooth cut-off function such that

$$(3.1) \quad \eta(s) = \begin{cases} 1 & \text{if } |s| \leq \frac{3}{4} \log \frac{1}{\alpha} \\ 0 & \text{if } |s| > \frac{7}{8} \log \frac{1}{\alpha} \end{cases}$$

and

$$(3.2) \quad \eta_j(s, t) = \begin{cases} 1 & \text{if } |(s, t) - \xi_j \vec{e}_1| \leq \frac{3}{4} \log \frac{1}{\alpha} \\ 0 & \text{if } |(s, t) - \xi_j \vec{e}_1| > \frac{7}{8} \log \frac{1}{\alpha} \end{cases}$$

Define $X^* \eta = \eta(x)$ and $X^* \omega' = \omega'(x)$. We are looking for solutions of (1.1) of the form $u = \mathbf{w} + \varphi$ where φ is a *small* perturbation of \mathbf{w} . Substituting the value of u in (1.1) we obtain

$$(3.3) \quad \Delta(\mathbf{w} + \varphi) - (\mathbf{w} + \varphi) + (\mathbf{w} + \varphi)^p = 0$$

where $\mathbf{w} = \mathbf{w}(\alpha, v, h, e, \delta, \chi)$ for some $\varphi \in C_{\sigma, \theta}^{2, \mu}(\mathbb{R}^2) \oplus C_{\sigma}^1(\mathbb{R}^2)$ and $v \in \mathcal{E}$. We can formally write (3.3) as

$$\mathcal{L}(\varphi) + S(\mathbf{w}) + N(\varphi) = 0 \text{ in } \mathbb{R}^2$$

where

$$\mathcal{L} := \Delta - 1 + p\mathbf{w}^{p-1}$$

and

$$N(\varphi) := (\mathbf{w} + \varphi)^p - \mathbf{w}^p - p\mathbf{w}^{p-1}\varphi$$

with

$$S(\mathbf{w}) := \Delta \mathbf{w} - \mathbf{w} + \mathbf{w}^p.$$

Hence we should write (3.3) as a fixed point problem for the nonlinear function

$$\varphi + \mathcal{L}^{-1}(S(\mathbf{w}) + N(\varphi)) = 0$$

provided \mathcal{L}^{-1} is a suitable bounded operator. But \mathcal{L} will have in general an unbounded inverse as $L \rightarrow +\infty$. Also note that near the bump line the operator $L_0 = \partial_x^2 + \partial_z^2 - p\omega^{p-1} + 1$ which has a bounded kernel spanned by ω' , $Z(x) \cos \sqrt{\lambda_1} z$ and $Z(x) \sin \sqrt{\lambda_1} z$ and near a spike the kernel of $L_1 = \Delta - 1 + p\omega_0^{p-1}$ is spanned by $\omega_{0,x}$. To get rid of this difficulty we consider a *nonlinear projected problem*

$$(3.4) \quad \mathcal{L}(\varphi) = S(\mathbf{w}) + N(\varphi) + \sum_{j=1}^{\infty} c_j \eta_j \omega_{j,x} + d(z) \eta \omega' + m(z) \eta Z$$

In the following sections we will describe:

- (1) How to solve (3.4) for unknown φ , $c = (c_1, c_2, \dots)$, d, m with the given parameters v, h, e, δ and χ .
- (2) Secondly we have to choose the parameters in such a way that c, d, m are zero.

4. LINEAR THEORY

The local structure of $\mathcal{M}_3(\mathbb{R}^2)$ near the curve γ and the spike are closely related to the determining whether \mathcal{L} is actually injective or surjective, and when \mathcal{L} is not injective, understanding the elements of its kernel. In order to do so we need to study the complex operator \mathcal{L} . We first study two simplified linear operators

$$(4.1) \quad L_0(\varphi) = \varphi_{zz} + \varphi_{xx} - \varphi + p\omega^{p-1}\varphi$$

and

$$(4.2) \quad L_1(\varphi) = \Delta\varphi - \varphi + p\omega_0^{p-1}\varphi$$

where ω is the unique solution of (1.4) and decays exponentially; and ω_0 is the unique positive solution of (1.10). Note that ω' , $Z(x) \cos \sqrt{\lambda_1}z$ and $Z(x) \sin \sqrt{\lambda_1}z$ are solutions to $L_0(\varphi) = 0$. In Lemma 4.1, we prove that indeed the converse also hold.

Lemma 4.1. *Let φ be a bounded solution of*

$$L_0(\varphi) = 0.$$

Then $\varphi \in \text{span}\{\omega'(x), Z(x) \cos \sqrt{\lambda_1}z, Z(x) \sin \sqrt{\lambda_1}z\}$.

Proof. This follows from Lemma 7.1 of [4]. □

Lemma 4.2. *Let φ be a bounded solution of*

$$L_1(\varphi) = 0.$$

satisfying $\varphi(x, z) = \varphi(x, -z)$. Then $\varphi = c\omega_{0,x}$ for some $c \in \mathbb{R}$.

Proof. Since the kernel of L_1 consists of $\omega_{0,x}$ and $\omega_{0,z}$, see [26], the result follows trivially from the fact that φ is even in z -variable. □

By Lemma 4.1 and 4.2 we define the orthogonality conditions as

$$(4.3) \quad \int_{\mathbb{R}} \varphi(x, z)\omega'(x)dx = 0 = \int_{\mathbb{R}} \varphi(x, z)Z(x)dx \quad \forall z \in \mathbb{R}$$

and

$$(4.4) \quad \int_{\mathbb{R}^2} \varphi(x, z)\omega_{0,x}(x, z)dxdz = 0.$$

Lemma 4.3. *Let φ be a bounded solution of*

$$(4.5) \quad L_0(\varphi) = k$$

satisfying (4.3). Then $\|\varphi\|_{\infty} \leq C\|k\|_{\infty}$ for some $C > 0$.

Proof. This follows from Lemma 7.2 of [4]. □

Remark 4.4. Note that Lemma 4.3 implies that $\|\varphi\|_{L^{\infty}(\mathbb{R}^2)} \leq C\|(\cosh z)^{\sigma}k\|_{L^{\infty}(\mathbb{R}^2)}$.

Lemma 4.5. *Assume that $\sigma \in (0, 1)$ be fixed. Then there exist $C > 0$ such that for any solution of $L_0(\varphi) = k$ satisfies*

$$(4.6) \quad \|(\cosh x)^{\sigma}\varphi\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\|(\cosh x)^{\sigma}k\|_{C^{0,\mu}(\mathbb{R}^2)}.$$

Proof. This is again Lemma 7.3 of [4]. □

Lemma 4.6. *Assume that $\sigma \in (0, 1)$. Then there exists $a_0 > 0$ such that for all $a \in (0, a_0]$ there exists a constant $C_a > 0$ but remains bounded as a tends to zero, such that*

$$\|(\cosh x)^\sigma (\cosh z)^a \varphi\|_{L^\infty(\mathbb{R}^2)} \leq C_a \|(\cosh x)^\sigma (\cosh z)^a k\|_{L^\infty(\mathbb{R}^2)}.$$

Proof. This follows from Lemma 7.4 of [4]. \square

4.1. Surjectivity. As far as the existence of solution of (4.5) and (4.3) is concerned we assume that

$$(4.7) \quad \int_{\mathbb{R}} k(x, z) \omega_x(x) dx = 0$$

$$(4.8) \quad \int_{\mathbb{R}} k(x, z) Z(x) dx = 0$$

for all $z \in \mathbb{R}$, we prove the following proposition.

Proposition 4.1.1. *Assume that $\sigma \in (0, 1)$ be fixed. Then there exists $a_0 > 0$ such that for all $a \in (0, a_0]$; there exists a constant $C_a > 0$ such that for all k satisfying the orthogonality conditions (4.7), (4.8) and*

$$L(\varphi) = k,$$

with

$$(4.9) \quad \|(\cosh x)^\sigma (\cosh z)^a k\|_{C^{0,\mu}(\mathbb{R}^2)} < +\infty$$

implies

$$(4.10) \quad \|(\cosh x)^\sigma (\cosh z)^a \varphi\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C_a \|(\cosh x)^\sigma (\cosh z)^a k\|_{C^{0,\mu}(\mathbb{R}^2)}.$$

Proof. The main idea is to prove the result for functions which are R periodic in the z - variable. We consider the problem

$$L_0(\varphi) = k$$

with the orthogonality conditions (4.3) and (4.4). We will apply an approximation argument. Let $\varphi(x, z)$ be a ξ periodic function in the z variable where $\xi > 0$. Define $\mathbb{R}_\xi^2 = \mathbb{R} \times \frac{\mathbb{R}}{\xi\mathbb{Z}}$. Then we have

$$\int_{\mathbb{R}_\xi^2} [|\nabla \varphi|^2 - (p\omega^{p-1} - 1)\varphi^2] \geq \frac{\lambda_1}{2} \int_{\mathbb{R}_\xi^2} \varphi^2$$

Hence given $k \in L^2(\mathbb{R}_\xi^2)$ satisfying

$$\int_{\mathbb{R}_\xi^2} k \omega_x = 0 = \int_{\mathbb{R}_\xi^2} k Z(x)$$

by Lax-Milgram lemma there exists a unique $\varphi \in H^1(\mathbb{R}_\xi^2)$ such that

$$\|\varphi\|_{H^1(\mathbb{R}_\xi^2)} \leq C \|k\|_{L^2(\mathbb{R}_\xi^2)}.$$

Moreover, by elliptic regularity, we have

$$\|\varphi\|_{L^\infty(\mathbb{R}_\xi^2)} \leq C (\|k\|_{L^2(\mathbb{R}_\xi^2)} + \|k\|_{L^\infty(\mathbb{R}_\xi^2)}).$$

Suppose in addition k satisfies (4.7) and (4.8) we obtain

$$\int_0^\xi \left(\int_{\mathbb{R}} \varphi \omega_x dx \right) \psi_{zz} dz = 0$$

and

$$\int_0^\xi \left(\int_{\mathbb{R}} \varphi Z(x) dx \right) \psi_{zz} dz = 0.$$

Hence

$$z \mapsto \int_{\mathbb{R}} \varphi \omega_x dx \text{ and } z \mapsto \int_{\mathbb{R}} \varphi Z dx$$

do not depend on z since its integral over $[0, \xi]$ is 0, we conclude that φ satisfies (4.5) and (4.3).

Hence we can apply Lemma 4.2 and 4.6 to obtain the estimate

$$\|(\cosh x)^\sigma \varphi\|_{L^\infty(\mathbb{R}_x^2)} \leq C \|(\cosh x)^\sigma k\|_{L^\infty(\mathbb{R}_x^2)}.$$

where $C > 0$ is a constant independent of ξ . Given any k satisfying the condition of the Proposition. Let

$$k_\xi = k \chi_{\mathbb{R}_x^2}$$

where χ denotes the characteristic function. Let φ_ξ be the corresponding solution to

$$L_0 \varphi_\xi = k_\xi$$

Elliptic estimates with compactness arguments yield we can pass through the limit as $\xi \rightarrow +\infty$, there exist a bounded solution φ of $L_0 \varphi = k$. \square

5. LINEAR THEORY FOR MULTIPLE INTERFACES

5.1. Gluing Procedure. In this section we decompose the nonlinear projected problem (3.4) into four coupled equations. We define

$$(5.1) \quad \rho(s) = \begin{cases} 1 & \text{if } |s| \leq \frac{7}{8} \log \frac{1}{\alpha} \\ 0 & \text{if } |s| > \frac{15}{16} \log \frac{1}{\alpha} \end{cases}$$

and

$$(5.2) \quad \rho_j(s, t) = \begin{cases} 1 & \text{if } |(s, t) - \xi_j \vec{e}_1| \leq \frac{7}{8} \log \frac{1}{\alpha} \\ 0 & \text{if } |(s, t) - \xi_j \vec{e}_1| > \frac{15}{16} \log \frac{1}{\alpha}. \end{cases}$$

Moreover, we define $X^* \rho = \rho(x)$. Using the definition we obtain $\rho_j \eta_j = \rho_j$ and $\rho_j \rho_k = 0$ for $j \neq k$. Similarly we have $\rho \eta = \rho$. Moreover, $\rho \eta_j = 0$ for every $j \in \mathbb{N}$. Note that we are looking for solutions of (3.4) of the form

$$(5.3) \quad \varphi = \sum_{j=1}^{\infty} \eta_j \phi_j + \eta \phi + \psi$$

where $\psi = \psi_1 + \psi_2$. Then for $j \in \mathbb{N}$, we have

$$(5.4) \quad \rho_j [\mathcal{L} \phi_j - (S(w) + N) - c_j \omega_{j,x}] + (\mathcal{L} - \Delta + 1) \psi_1 \rho_j = 0$$

$$(5.5) \quad \rho [\mathcal{L} \phi - (S(w) + N) - d(z) \omega' - m(z) Z] + [\mathcal{L} - \Delta + 1] \psi_2 \rho = 0$$

ψ_1 and ψ_2 satisfy the following equation,

$$(5.6) \quad \begin{aligned} (\Delta - 1)\psi_1 &= (1 - \sum_{j=1}^{\infty} \eta_j)(S(\mathbf{w}) + N) \\ &- \sum_{j=1}^{\infty} (\mathcal{L}(\phi_j \eta_j) - \eta_j \mathcal{L}(\phi_j)) - (1 - \sum_{j=1}^{\infty} \eta_j)(\mathcal{L} - \Delta + 1)\psi_1 \end{aligned}$$

and

$$(5.7) \quad \begin{aligned} (\Delta - 1)\psi_2 &= (1 - \eta)(S(\mathbf{w}) + N) \\ &- (\mathcal{L}(\phi \eta) - \eta \mathcal{L}(\phi)) - (1 - \eta)(\mathcal{L} - \Delta + 1)\psi_2 \end{aligned}$$

where $N = N(\sum_{j=1}^{\infty} \eta_j \phi_j + \eta \phi + \psi)$. This is a coupled system and the coupling terms are of the higher order in α . Note that (5.5) can be written as

$$(5.8) \quad [\partial_x^2 + \partial_z^2 - F'(\omega)]X^* \phi = X^* k + X^*(d\rho \omega') + X^*(m\rho Z)$$

where

$$(5.9) \quad \begin{aligned} X^* k &= X^*[\rho(S(\mathbf{w}) + N)] - X^*[\rho(\mathcal{L} - \Delta + 1)\psi_2] \\ &- X^* \rho(\mathcal{L}(\phi)) + X^* \rho[\partial_x^2 + \partial_z^2 - F'(\omega)]X^* \phi \end{aligned}$$

Define $\Phi = (\phi_1, \phi_2, \dots)$. Let the right hand side of the equation (5.6) and (5.7) be $Q_1 = Q_1(\Phi, \psi_1)$ and $Q_2 = Q_2(\phi, \psi_2)$ respectively. Then equation (5.6) and (5.7) reduces to

$$(5.10) \quad (\Delta - 1)\psi_1 = Q_1$$

$$(5.11) \quad (\Delta - 1)\psi_2 = Q_2$$

We will call (5.10) and (5.11) as *background system*. We will first solve the background system. Then given solution (ψ_1, ψ_2) we solve the initial equations (5.4) and (5.5).

5.2. Error of the initial approximation. For $0 < \mu \leq 1$, we define the weighted norms

$$\|\varphi\|_{C_{\sigma, \theta}^{1, \mu}(\mathbb{R}^2)} = \sup_{\hat{x} \in \mathbb{R}^2} \left((\cosh x)^\sigma (\cosh z)^\theta \|\varphi\|_{C^{2, \mu}(B_1(\hat{x}))} \right).$$

We also define the norms

$$\|\varphi\|_\sigma = \sup_{(x, z) \in \mathbb{R}^2} \left(\sum_{i=1}^{\infty} e^{-\sigma|(x, z) - \xi_i \bar{e}_1|} \right)^{-1} |\varphi(x, z)|$$

and

$$\begin{aligned} \chi &= (\chi_1, \dots, \chi_k, \dots) \\ \|\chi\| &= \max_i \alpha^{-i} |\chi_i| \end{aligned}$$

Proposition 5.2.1. For $i = 1, 2$; $S(\mathbf{w}^{(i)}) = S(\mathbf{w}, v, h^{(i)}, e^{(i)}, \delta, \chi^{(i)})$ is a continuous function of v, δ and satisfies

$$(5.12) \quad \|X^*(\rho S(\mathbf{w}))\|_{C_{\sigma, \theta}^{0, \mu}(\mathbb{R}^2)} \leq C\alpha.$$

Moreover, it is a Lipschitz function of h, e and χ ;

$$(5.13) \quad \begin{aligned} \|(X^{(1)})^* \rho^{(1)} S(\mathbf{w}^{(1)}) - (X^{(2)})^* \rho^{(2)} S(\mathbf{w}^{(2)})\|_{C_{\sigma, \theta}^{0, \mu}(\mathbb{R}^2)} &\leq C(\|h^{(1)} - h^{(2)}\|_{C_{\sigma, \theta}^{2, \mu}(\mathbb{R})} \\ &+ \|e^{(1)} - e^{(2)}\|_{C_{\sigma, \theta}^{2, \mu}(\mathbb{R})} + \alpha \|\chi^{(1)} - \chi^{(2)}\|_\alpha). \end{aligned}$$

So far we have estimated the error near the bump line. The other two propositions deal with the estimate of the norm in the complement of the set $\text{supp } \rho$ and the estimation of the error near the spikes. Note that in $\mathbb{R}^2 \setminus V_\zeta$ we have $S(\mathbf{w}) = S(\sum_{j=1}^{\infty} \omega_j)$. Let us denote

$$(5.14) \quad V_\zeta^\perp = V_\zeta \setminus \text{supp } \eta.$$

Proposition 5.2.2. *Then we have in V_ζ^\perp*

$$(5.15) \quad \|S(\mathbf{w})\|_{C_\theta^{0,\mu}(V_\zeta^\perp)} \leq C\alpha^{1+\frac{3}{4}\sigma}.$$

Moreover,

$$(5.16) \quad \begin{aligned} & \| (S(\mathbf{w}^{(1)}) - S(\mathbf{w}^{(2)})) \|_{C_{\sigma,\theta}^{0,\mu}(V_\zeta^\perp)} \leq C\alpha^{\frac{3}{4}\sigma} (\|h^{(1)} - h^{(2)}\|_{C_\theta^{2,\mu}(V_\zeta^\perp)} \\ & + \|e^{(1)} - e^{(2)}\|_{C_\theta^{2,\mu}(V_\zeta^\perp)} + \alpha \|\chi^{(1)} - \chi^{(2)}\|_\alpha). \end{aligned}$$

Proposition 5.2.3. *In $\mathbb{R}^2 \setminus V_\zeta$ we have*

$$(5.17) \quad \|S(\mathbf{w})\|_\sigma \leq C\alpha.$$

Moreover,

$$(5.18) \quad \| (S(\mathbf{w}^{(1)}) - S(\mathbf{w}^{(2)})) \|_\sigma \leq C\alpha \|\chi^{(1)} - \chi^{(2)}\|_\alpha.$$

Proof of Propositions 5.2.1, 5.2.2, 5.2.3 . We write

$$(5.19) \quad S(\mathbf{w}) = \rho S(\mathbf{w}) + \sum_{j=1}^{\infty} \rho_j S(\mathbf{w})$$

Let $U_1 := V_\zeta \cap \{x + z \geq 0\}$. Then using the approximation we have

$$(5.20) \quad \mathbf{w} = \mathbf{w} + e(z)Z(x) + \sum_{j=1}^{\infty} \omega_j$$

and using the fact $\Delta\omega_j + F(\omega_j) = 0$. As a result, we have

$$(5.21) \quad \begin{aligned} S(\mathbf{w}) &= \Delta\mathbf{w} + F(\mathbf{w}) \\ &= \underbrace{\Delta\mathbf{w} + F(\mathbf{w})}_{E_1} + \underbrace{(\Delta + F'(\mathbf{w}))e(z)Z}_{E_2} \\ &\quad + \underbrace{\{F(\mathbf{w}) - \sum_{j=1}^{\infty} F(\omega_j) - F(\mathbf{w}) - F'(\mathbf{w})eZ\}}_{E_3} \\ &= E_1 + E_2 + E_3. \end{aligned}$$

Using Taylor's expansion we obtain

$$(5.22) \quad \begin{aligned} \mathbf{w}^p &= \Xi\omega_\delta^p + \Xi_0\omega^p + (\omega_\delta + \Xi(\omega_\delta - \omega))^p - \Xi(\omega_\delta + (\omega_\delta - \omega))^p - \Xi_0\omega^p \\ &= \Xi\omega_\delta^p + \Xi_0\omega^p + \mathcal{O}_{C^{0,\mu}(U_1)}(\delta^2)(\cosh x)^{-2}(\cosh z)^{-\theta}. \end{aligned}$$

Also note that

$$(5.23) \quad \partial_z \Xi(z) = \Xi'(z), \quad \partial_z^2 \Xi(z) = \Xi''(z)$$

with $|\delta| \leq \alpha^{1+k_4}$ and since ω is not a function of z we obtain $\partial_z \omega = 0$. Moreover, if we denote the operator $S = \Delta - \partial_x^2 - \partial_z^2$ then we have

$$(5.24) \quad \begin{aligned} E_1 &= S(\Xi\omega_\delta + \Xi_0\omega) + 2[\partial_x \Xi \partial_x \omega_\delta + \partial_z \Xi_0 \partial_z \omega] \\ &\quad + 2[\partial_x^2 \Xi \omega_\delta + \partial_z^2 \Xi_0 \omega] + \mathcal{O}_{C^\infty(\mathbb{R}^2)}(|\delta|^2)(\cosh x)^{-2}(\cosh z)^{-\theta} \end{aligned}$$

Note that the first term in the above expression is of the order α due to the fact of (2.43). Hence we have

$$\|E_1\|_{C_{\sigma,\theta}^{0,\mu}(U_1)} \leq C\alpha$$

Moreover,

$$\|E_1\|_{C_{\sigma,\theta}^{0,\mu}(V_\zeta^\perp)} \leq C\alpha^{1+\frac{3}{4}\sigma}$$

and this follows due to the fact that $V_\zeta^\perp = V_\zeta \setminus V_{\frac{\zeta}{2}}$ we have $|x| \geq \frac{3}{4} \log \frac{1}{\alpha}$. The estimate for E_2 follows similarly.

Now we estimate E_3 . For $(x, z) \in V_\zeta$ we have $w \gg e(z)Z + \omega_j$ and hence

$$(5.25) \quad F(\mathbf{w}) = F(w) + F'(w)(\mathbf{w} - w) + O(w^2(\mathbf{w} - w)^{p-2})$$

Hence we have

$$\begin{aligned} E_3 &= \{F'(w)(\mathbf{w} - w) - \sum_{j=1}^{\infty} F(\omega_j)\} + O(w^{p-2}(\mathbf{w} - w)^2) \\ &= pw^{p-1} \left(\sum_{j=1}^{\infty} \omega_j \right) - \sum_{j=1}^{\infty} \omega_j^p + O(w^{p-2}(\mathbf{w} - w)^2). \end{aligned}$$

When $0 < \sigma < (p-1)$ and by (1.12) and the fact that $x \sim (x - f(z))$ and $z \sim z$ we have,

$$\begin{aligned} (p-1)|x - f(z)| + |x - \xi_i| &= \sigma|x - f(z)| + (p-1-\sigma)|x - f(z)| + |x - \xi_i| \\ &\geq \sigma|x - f(z)| + \min\{(p-1-\sigma), 1\}(|x - f(z)| + |x - \xi_i|) \\ &\geq \sigma|x - f(z)| + \min\{(p-1-\sigma), 1\}|(f(z) - \xi_i, z)| \\ &= \sigma|x - f(z)| + \min\{(p-1-\sigma), 1\}\sqrt{(f(z) - iL)^2 + z^2} \\ &\geq \sigma|x - f(z)| + \min\{(p-1-\sigma), 1\}\sqrt{L^2 + z^2} \\ (5.26) \quad &\geq \sigma|x - f(z)| + \min\{p-1-\sigma, 1\}\frac{L}{\sqrt{2}} + \theta|z| \end{aligned}$$

Also note that from (1.12) we have,

$$\begin{aligned} (p-\sigma)|(x, z) - \xi_j \vec{e}_1| &= (p-\sigma)\sqrt{(x - jL)^2 + z^2} \geq (p-\sigma)\sqrt{\frac{L^2}{4} + z^2} \\ (5.27) \quad &\geq \frac{(p-\sigma)L}{\sqrt{2}} + \theta|z|. \end{aligned}$$

Further note that

$$\begin{aligned} |w^{p-2}(\mathbf{w} - w)^2| &= w^{p-2} \left(e(z)Z + \sum_{j=1}^{\infty} \omega_j \right)^2 \\ &\leq C(\cosh x)^{-(p-2)} (\alpha^{4+2k_3} (\cosh z)^{-2\theta} (\cosh x)^{-(p+1)} + e^{-2|(x,z) - \xi_j \vec{e}_1|}) \\ (5.28) \quad &\leq C\alpha (\cosh x)^{-\sigma} (\cosh z)^{-\theta} \end{aligned}$$

This implies that

$$\|w^{p-2}(\mathbf{w} - w)^2\|_{C_{\sigma,\theta}^{0,\mu}(U_1)} \leq C\alpha$$

Hence

$$\|E_3\|_{C_{\sigma,\theta}^{0,\mu}(U_1)} \leq C\alpha.$$

Similarly we have

$$(5.29) \quad \|E_3\|_{C_{\sigma,\theta}^{\alpha,\mu}(V_\zeta^\perp)} \leq C\alpha^{1+\frac{3}{4}\sigma}.$$

Now define

$$\mathcal{A}_j = \left\{ (x, z) \in \mathbb{R}^2 : |(x - \xi_j, z)| \leq \frac{L}{2} \right\}$$

where $j \geq 1$. Then we have in $\mathbb{R}^2 \setminus V_\zeta$

$$(5.30) \quad S(\mathbf{w}) = \sum_{j \geq 1} S(\mathbf{w}) \chi_{\mathcal{A}_j}$$

and if we expand near the spike $(\xi_i, 0)$ we have using mean value theorem

$$\begin{aligned} S(\mathbf{w}) &= S\left(\sum_{j=1}^{\infty} \omega_j\right) \\ &= F\left(\sum_{j=1}^{\infty} \omega_j\right) - \sum_{j=1}^{\infty} F(\omega_j) \\ &= \left(\sum_{j=1}^{\infty} \omega_j\right)^p - \sum_{j=1}^{\infty} \omega_j^p \\ &\sim p \sum_{i \neq j} \omega_i^{p-1} \omega_j \\ &\sim p \sum_{i \neq j} e^{-(p-1-\sigma)\sqrt{(x-\xi_i)^2+z^2}} e^{-\sigma|(x,z)-\xi_j \vec{e}_1|} e^{-|\xi_j-\xi_i|} \\ (5.31) \quad &\sim p \sum_{i \neq j} e^{-(p-1-\sigma)\sqrt{(x-\xi_i)^2+z^2}} e^{-(j-i)L} e^{-\sigma|(x,z)-\xi_j \vec{e}_1|}. \end{aligned}$$

This implies

$$|S(\mathbf{w})| \leq C e^{-L} \sum_{j=1}^{\infty} e^{-\sigma|(x,z)-\xi_j \vec{e}_1|}$$

and hence we have

$$\|S(\mathbf{w})\|_\sigma \leq C e^{-L} = C\alpha.$$

□

5.3. Existence of solution for the background system. In order to solve (5.10) and (5.11) we will use the Banach fixed point theorem. Moreover, we assume that

$$(5.32) \quad \sum_{j=1}^{\infty} \|e^{\sigma|(x,z)-\xi_j \vec{e}_1|} \phi_j\|_{L^\infty(\mathbb{R}^2)} < +\infty$$

and

$$(5.33) \quad \|X^* \phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)} < +\infty.$$

Lemma 5.1. *Assume that (5.32) holds. Then there exists a unique solution of (5.10) such that*

$$(5.34) \quad \|\psi_1\|_\sigma + \|\nabla \psi_1\|_\sigma \leq C\alpha^{\frac{3}{4}\sigma} \left(\alpha + \sum_{j=1}^{\infty} (\|\phi_j\|_{\sigma,j} + \|\nabla \phi_j\|_{\sigma,j}) \right).$$

In addition ψ_1 is a continuous function of the parameter v, h, e, δ and χ and a Lipschitz function of ϕ_j and also of the parameters e, h and χ satisfies the following estimates

$$(5.35) \quad \begin{aligned} & \|\psi_1(\Phi^{(1)}) - \psi_1(\Phi^{(2)})\|_\sigma + \|\nabla\psi_1(\Phi^{(1)}) - \nabla\psi_1(\Phi^{(2)})\|_\sigma \\ & \leq C\alpha^{\frac{3}{4}\sigma} (\|\Phi^{(1)} - \Phi^{(2)}\|_\sigma + \|\nabla\Phi^{(1)} - \nabla\Phi^{(2)}\|_\sigma) \end{aligned}$$

$$(5.36) \quad \begin{aligned} \|\psi_1(h^{(1)}, e^{(1)}, \chi^{(1)}) - \psi_1(h^{(2)}, e^{(2)}, \chi^{(2)})\|_\sigma & \leq C\alpha^{\frac{3}{4}\sigma} (\|h^{(1)} - h^{(2)}\|_{C_{\theta}^{2,\mu}(\mathbb{R})} \\ & + \|e^{(1)} - e^{(2)}\|_{C_{\alpha\theta}^{2,\mu}(\mathbb{R})} + \alpha\|\chi^{(1)} - \chi^{(2)}\|_\alpha). \end{aligned}$$

Proof. Define $\|\Phi\|_\sigma = \sum_{j=1}^{\infty} \|e^{\sigma|(x,z)-\xi_j \bar{e}_1}| \phi_j\|_{L^\infty(\mathbb{R}^2)}$ and

$$\|\nabla\Phi\|_\sigma = \sum_{j=1}^{\infty} \|e^{\sigma|(x,z)-\xi_j \bar{e}_1}| \nabla\phi_j\|_{L^\infty(\mathbb{R}^2)}.$$

We have

$$(\Delta - 1)\psi_1 = Q_1.$$

For the time being we assume that $\|Q_1\|_\sigma < +\infty$ then

$$|Q_1| \leq C \sum_{j=1}^{\infty} e^{-\sigma|(x,z)-\xi_j \bar{e}_1|}.$$

Using barrier and elliptic estimates we obtain

$$|\psi_1(x, z)| + |\nabla\psi_1(x, z)| \leq C \sum_{j=1}^{\infty} e^{-\mu|(x,z)-\xi_j \bar{e}_1|}.$$

This implies that

$$\|\psi_1\|_\sigma + \|\nabla\psi_1\|_\sigma \leq C\|Q_1\|_\sigma.$$

Next we estimate the size of Q_1 and also its dependence on $\Phi = (\phi_1, \phi_2, \dots)$ and h, e, χ . We assume that

$$\|\Phi\|_\sigma + \|\nabla\Phi\|_\sigma < +\infty.$$

We now estimate Q_1 . Then we have

$$(5.37) \quad \begin{aligned} |Q_1| & \leq C \left(\alpha^{1+\frac{3}{4}\sigma} + \alpha^{\frac{3}{4}\sigma} \sum_{j=1}^{\infty} (\|\phi\|_{\sigma,j} + \|\nabla\phi\|_{\sigma,j}) + \alpha^{\frac{3}{4}\sigma} \|\psi_1\|_\sigma \right) \\ & \times \sum_{j=1}^{\infty} e^{-\sigma|(x,z)-\xi_j \bar{e}_1|} \end{aligned}$$

This implies

$$(5.38) \quad \|Q_1\|_\sigma \leq C\alpha^{\frac{3}{4}\sigma} \left(\alpha + \sum_{j=1}^{\infty} (\|\phi\|_{\sigma,j} + \|\nabla\phi\|_{\sigma,j}) \right) + \alpha^{\frac{3}{4}\sigma} \|\psi_1\|_\sigma.$$

Hence given Φ , using a standard fixed point theorem there exists $\psi_1 = \psi_1(\Phi)$ satisfying (5.10). Moreover,

$$\|\psi_1(\Phi)\|_\sigma \leq C\alpha^{\frac{3}{4}\sigma} \left(\alpha + \sum_{j=1}^{\infty} (\|\phi\|_{\sigma,j} + \|\nabla\phi\|_{\sigma,j}) \right).$$

Since $Q_1(\Phi, \cdot)$ is a uniform contraction in the second variable and it is continuous we conclude that ψ_1 is also a continuous function and we conclude that ψ_1 is continuous function of v, h, e, δ and χ . Moreover, it easily follows

$$\begin{aligned} \|\psi_1(\Phi^{(1)}) - \psi_1(\Phi^{(2)})\|_\sigma + \|\nabla\psi_1(\Phi^{(1)}) - \nabla\psi_1(\Phi^{(2)})\|_\sigma &\leq C\alpha^{\frac{3}{4}\sigma}(\|\Phi^{(1)} - \Phi^{(2)}\|_\sigma \\ &\quad + \|\nabla\Phi^{(1)} - \nabla\Phi^{(2)}\|_\sigma) \end{aligned}$$

□

Lemma 5.2. *Assume that (5.33) holds. Then there exists a unique solution of (5.11) such that*

$$(5.39) \quad \|(\cosh z)^\theta \psi_2\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}\sigma}(\alpha + \|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}).$$

In addition ψ_2 is a continuous function of the parameter v, h, e, δ and χ and a Lipschitz function of ϕ and also of the parameters e, h and χ and satisfy the following estimates

$$(5.40) \quad \|(\psi_2(\phi^{(1)}) - \psi_2(\phi^{(2)}))(\cosh z)^\theta\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}\sigma}\|X^*(\phi^{(1)} - \phi^{(2)})\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}$$

$$(5.41) \quad \begin{aligned} \|(\psi_2(h^{(1)}, e^{(1)}) - \psi_2(h^{(2)}, e^{(2)}))(\cosh z)^\theta\|_{C^{2,\mu}(\mathbb{R}^2)} &\leq C\alpha^{\frac{3}{4}\sigma}(\|h^{(1)} - h^{(2)}\|_{C_{\theta}^{2,\mu}(\mathbb{R})} \\ &\quad + \|e^{(1)} - e^{(2)}\|_{C_{\theta}^{2,\mu}(\mathbb{R})} + \alpha\|\chi^{(1)} - \chi^{(2)}\|_\alpha) \end{aligned}$$

Proof. We have

$$(\Delta - 1)\psi_2 = Q_2.$$

For the time being consider

$$(5.42) \quad \|(\cosh z)^\theta Q_2\|_{C^{0,\mu}(\mathbb{R}^2)} < +\infty.$$

Then by regularity theory we have

$$\|\psi_2\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\|Q_2\|_{C^{0,\mu}(\mathbb{R}^2)}$$

We are required to prove that

$$(5.43) \quad \|\psi_2(\cosh z)^\theta\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\|Q_2(\cosh z)^\theta\|_{C^{0,\mu}(\mathbb{R}^2)}$$

In order to so we define a barrier of the form

$$\psi_\nu = (\cosh z(z))^{-\theta} + \nu \left[\cosh \frac{x}{2} + \cosh \frac{z}{2} \right]$$

where $\nu \geq 0$ is sufficiently small. In fact we have

$$(5.44) \quad (\Delta - 1)\psi_\nu \leq -\frac{1}{4}\psi_\nu$$

and hence ψ_ν is a super solution of $\Delta - 1$. Moreover define

$$(5.45) \quad \vartheta_{\nu,M} = M\|Q_2(\cosh z)^\theta\|_{C^{0,\mu}(\mathbb{R}^2)}\psi_\nu + \psi_2$$

where $M > 0$ is large such that

$$(5.46) \quad \begin{aligned} (\Delta - 1)\vartheta_{\nu,M} &\leq -\frac{M}{4}\|Q_2(\cosh z)^\theta\|_{C^{0,\mu}(\mathbb{R}^2)}\psi_\nu + Q_2 \\ &\leq -\frac{M}{4}\|Q_2(\cosh z)^\theta\|_{C^{0,\mu}(\mathbb{R}^2)}\psi_\nu + \|Q_2(\cosh z)^\theta\|_{C^{0,\mu}(\mathbb{R}^2)}(\cosh z)^{-\theta} \\ &\leq 0 \end{aligned}$$

Letting $\nu \rightarrow 0$ we obtain

$$\psi_2(\cosh z)^\theta \leq C \|Q_2(\cosh z)^\theta\|_{C^{0,\mu}(\mathbb{R}^2)}$$

The lower estimate for (5.43) can be obtained in a similar way. Now we estimate Q_2 . Note that in *supp* Q_2 we have

$$|x| \geq \frac{3}{4} \log \frac{1}{\alpha}.$$

Note that we have already estimated the error $S(\mathbf{w})$ and hence

$$(5.47) \quad \|(\cosh z)^\theta S(\mathbf{w})\|_{C^{0,\mu}(\mathbb{R}^2)} \leq C \alpha^{1+\frac{3}{4}\sigma}.$$

Moreover, using the fact that support of $\nabla \rho$ we have

$$(5.48) \quad \|(\cosh z)^\theta (\mathcal{L}(\eta\phi) - \eta\mathcal{L}(\phi))\|_{C^{0,\mu}(\mathbb{R}^2)} \leq C \alpha^{\frac{3}{4}\sigma} \|X^*\phi\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)}$$

and

$$(5.49) \quad p\mathbf{w}^{p-1}|\psi_2| \leq C \alpha^{\frac{3}{4}\sigma} \|(\cosh z)^\theta \psi_2\|_{C^{0,\mu}(\mathbb{R}^2)} (\cosh z)^{-\theta}$$

which finally yields

$$\|(\cosh z)^\theta Q_2(\phi, \psi_2)\|_{C^{0,\mu}(\mathbb{R}^2)} \leq C \alpha^{\frac{3}{4}\sigma} \{\alpha + \|X^*\phi\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)}\} + C \alpha^{\frac{3}{4}\sigma} \|(\cosh z)^\theta \psi_2\|_{C^{0,\mu}(\mathbb{R}^2)}$$

Hence given ϕ using a standard fixed point theorem there exists $\psi_2 = \psi_2(\phi)$ satisfying (5.11). Moreover,

$$\|(\cosh z)^\theta \psi_2(\phi)\|_{C^{0,\mu}(\mathbb{R}^2)} \leq C \alpha^{\frac{3}{4}\sigma} (\alpha + \|X^*\phi\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)}).$$

Since $Q_2(\phi, \cdot)$ is a uniform contraction in the second variable and it is continuous and we conclude that ψ_2 is continuous function of v, h, e, δ and χ . Moreover, it easily follows

$$\|(\cosh z)^\theta (\psi_2(\phi^{(1)}) - \psi_2(\phi^{(2)}))\|_{C^{0,\mu}(\mathbb{R}^2)} \leq C \alpha^{\frac{3}{4}\sigma} \|X^*\phi^{(1)} - X^*\phi^{(2)}\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}.$$

□

5.4. Existence of solution for the initial problems. For the time being consider $\hat{x}_j = (x, z) - \xi_j \vec{e}_1$, $j \in \mathbb{N}$.

Lemma 5.3. *Assume that $\|k_j\|_{\sigma,j} = \|e^{\sigma|\hat{x}_j}|k_j\|_\infty < +\infty$. Then there exists $C > 0$ independent of j and $\sigma \in [0, 1)$ such that (5.54) with (5.56) satisfies*

$$(5.50) \quad \|\phi_j\|_{\sigma,j} + \|\nabla \phi_j\|_{\sigma,j} \leq C \|k_j\|_{\sigma,j}.$$

Moreover, we have

$$(5.51) \quad |c_j| \leq C \|k_j\|_{\sigma,j}.$$

Proof. We have $L(\varphi) = k_j$. Define

$$\tilde{\phi} = \frac{\phi}{\|k_j\|_\sigma}.$$

Then we have $L(\tilde{\phi}) = \tilde{k}_j$ where $\tilde{k}_j = \frac{k_j}{\|k_j\|_\sigma}$. Note that it is enough to show that the estimate holds for sufficiently large $|\hat{x}| = |(x, z)|$. Then there exists a $R > 0$ such that for $|\hat{x}| \geq R$ we have

$$p\omega_0^{p-1}(|\hat{x}|) < \frac{1 - \sigma^2}{2}.$$

Moreover, define

$$\bar{\phi}(\hat{x}) = e^{-\sigma|\hat{x}|}$$

Then

$$L_1(\tilde{\phi} - M\bar{\phi}) \geq 0$$

if $|x| > R$ and $|\tilde{\phi}| \leq M\bar{\phi}(R) = Me^{\sigma R}$ on $|\hat{x}| = R$. Hence by maximum principle we obtain $|\tilde{\phi}| \leq M|\bar{\phi}|$. Hence $|\phi| \leq M\|k_j\|_\sigma e^{-\sigma|\hat{x}|}$. As a result we obtain

$$\|\phi\|_\sigma \leq M\|k_j\|_\sigma.$$

For the gradient estimate we define $\psi = e^{-\sigma|\hat{x}|}\phi$. Then we have

$$(5.52) \quad L_1(\psi) + B(\psi) = k_j e^{\sigma|\hat{x}|}$$

where B is an operator containing terms involving gradient and zero order terms, such that $\|B\|_\infty$ is very small. Using local C^1 estimates we obtain

$$|\nabla\psi| \leq C\|k_j\|_\sigma.$$

Hence for small σ we obtain

$$e^{-\sigma|\hat{x}|}|\nabla\phi| \leq C\|k_j\|_\sigma.$$

Hence the result. Multiplying by $\omega_{i,x}$, we have on integration by parts,

$$0 = \int_{\mathbb{R}^2} L_1(\phi)\omega_{i,x} dx dz = \int_{\mathbb{R}^2} k_j(x, z)\omega_{i,x} + c_i \int_{\mathbb{R}^2} \omega_{i,x}^2 dx dz$$

Hence we have

$$\begin{aligned} |c_j| &\leq \int_{\mathbb{R}^2} |k_j(x, z)| |\omega_{i,x}| \\ &\leq C\|k_j\|_\sigma \end{aligned}$$

Hence the inequality follows easily. \square

Remark 5.4. For $p > 2$, using the inequality

$$(5.53) \quad \| |a+b|^p - |b|^p - p|a|^{p-1}b \| \leq C(p) \max\{|a|^{p-2}|b|^2, |b|^p\}.$$

5.5. Existence of solution for 5.4. As described earlier the derivation of solution of (5.4) is given by the linear theory of L_1 .

Note that we can write (5.4) as

$$(5.54) \quad (\Delta - 1 + p\omega_j^{p-1})\phi_j = k_j + c_j\omega_{j,x}$$

where

$$(5.55) \quad k_j = (S(\mathbf{w}) + N)\rho_j + p\mathbf{w}^{p-1}\psi_1\rho_j + (p\mathbf{w}^{p-1} - p\omega_j^{p-1})\phi_j\rho_j$$

with the condition of orthogonality as

$$(5.56) \quad \int_{\mathbb{R}^2} \phi_j(x, z)\omega_{j,x}(x, z) dx dz = 0$$

for all $j \in \mathbb{N}$. Hence there exists a $C > 0$ such that given $\|h\|_\sigma < +\infty$ and for some $\sigma \in (0, 1)$, there exists a unique bounded solution $\Phi = \mathcal{T}(h)$ to (5.54) and (5.56) which defines a bounded linear operator of h satisfying

$$\|\Phi\|_\sigma + \|\nabla\Phi\|_\sigma \leq C\|h\|_\sigma.$$

This follows trivially by using the Fredholm alternative. Hence we write the linear operator $\mathcal{T} = (T_1, \dots, \dots)$ such that for each $j \in \mathbb{N}$ such that $\phi_j = T_j(h)$.

Hence we can write

(5.57)

$$\phi_j = T_j(\rho_j S(\mathbf{w}) + \rho_j N(\phi) + c_j \rho_j \omega_{j,x} + (\mathcal{L} - \Delta + 1)\psi_1 \rho_j + (p\mathbf{w}^{p-1} - p\omega_j^{p-1})\phi_j)\rho_j$$

for some linear operator T_j ; $j \in \mathbb{N}$. Let $k = (k_1, k_2, \dots)$.

Lemma 5.5. *Assume that*

$$(5.58) \quad \sum_{j=1}^{\infty} (\|\phi_j\|_{\sigma} + \|\nabla \phi_j\|_{\sigma}) \leq \alpha^{\frac{3}{4}\sigma}.$$

Then we have for all $j \in \mathbb{N}$

$$(5.59) \quad \|k_j\|_{\sigma} \leq C\alpha + C\alpha^{\frac{3}{4}\sigma} \left(\sum_{j=1}^{\infty} \|\phi_j\|_{\sigma} + \|\nabla \phi_j\|_{\sigma} \right).$$

Moreover, the function k_j is a Lipschitz function of Φ and satisfy

$$(5.60) \quad \|k_j(\Phi^{(1)}) - k_j(\Phi^{(2)})\|_{\sigma} \leq C\alpha^{\frac{3}{4}\sigma} (\|\Phi^{(1)} - \Phi^{(2)}\|_{\sigma} + \|\nabla \Phi^{(1)} - \nabla \Phi^{(2)}\|_{\sigma}).$$

Furthermore, we have

$$(5.61) \quad \|c\|_{\infty} \leq C\|k\|_{\sigma}.$$

Proof. From (5.55) we have

$$\|\rho_j S(\mathbf{w})\|_{\sigma} \leq C\alpha.$$

Now

$$\rho_j N(\varphi) = \rho_j N(\eta_j \phi_j + \psi_1) = N(\rho_j \phi_j + \psi_1).$$

Hence from (5.53)

$$(5.62) \quad |\rho_j N(\varphi)| \leq C(|\phi_j|^2 + |\psi_1|^2).$$

As a result we have

$$\|\rho_j N(\varphi)\|_{\sigma} \leq C(\|\phi_j\|_{\sigma}^2 + \|\psi_1\|_{\sigma}^2).$$

and hence

$$(5.63) \quad \begin{aligned} \|\rho_j N(\varphi)\|_{\sigma} &\leq C((\|\Phi\|_{\sigma} + \|\nabla \Phi\|_{\sigma})^2 + \alpha^{\frac{3}{4}\sigma}(\alpha + (\|\Phi\|_{\sigma} + \|\nabla \Phi\|_{\sigma})^2)) \\ &\leq C\alpha + C\alpha^{\frac{3}{4}\sigma}(\|\Phi\|_{\sigma} + \|\nabla \Phi\|_{\sigma}). \end{aligned}$$

Then (5.34) implies

$$(5.64) \quad \begin{aligned} \|k_j\|_{\sigma} &\leq C\alpha + C\alpha^{1+\frac{3}{4}\sigma} \sum_{j=1}^{\infty} (\|\phi_j\|_{\sigma} + \|\nabla \phi_j\|_{\sigma}) \\ &+ \|\phi_j\|_{\sigma} \left(\sum_{j \neq i} e^{-(p-2-\sigma)|\xi_j - \xi_i|} + e^{-(p-2-\sigma)|f(z) - \xi_i|} \right). \end{aligned}$$

This implies that

$$(5.65) \quad \begin{aligned} \|k_j\|_{\sigma} &\leq C\alpha + C\alpha^{\frac{3}{4}\sigma}(\|\Phi\|_{\sigma} + \|\nabla \Phi\|_{\sigma}) + \alpha^{p-2-\sigma}(\|\Phi\|_{\sigma} + \|\nabla \Phi\|_{\sigma}) \\ &\leq C\alpha + C\alpha^{\frac{3}{4}\sigma}(\|\Phi\|_{\sigma} + \|\nabla \Phi\|_{\sigma}) \end{aligned}$$

provided we choose σ is chosen small. The Lipschitz dependence follows in a standard way. \square

Lemma 5.6. *The problem (5.57) and (4.4) has a unique solution ϕ such that*

$$(5.66) \quad \|\Phi\|_\sigma + \|\nabla\Phi\|_\sigma \leq C\alpha.$$

Moreover, the solution is a continuous function of v, h, e, δ and χ and a Lipschitz function of h, e and χ . Furthermore, for every $j \in \mathbb{N}$, there exists $C > 0$ independent of j such that

$$(5.67) \quad \begin{aligned} \|\phi_j(h^{(1)}, e^{(1)}, \chi^{(1)}) - \phi_j(h^{(2)}, e^{(2)}, \chi^{(2)})\|_\sigma \\ \leq C(\|h^{(1)} - h^{(2)}\|_{C_\theta^{2,\mu}(\mathbb{R})} + \|e^{(1)} - e^{(2)}\|_{C_\theta^{2,\mu}(\mathbb{R})} + \alpha\|\chi^{(1)} - \chi^{(2)}\|_\alpha). \end{aligned}$$

Proof. First note that from (5.58) we have

$$\sum_{j=1}^{\infty} \|\phi_j\|_\sigma + \|\nabla\phi_j\|_\sigma \leq C\alpha + \alpha^{\frac{3}{4}\sigma} \left(\sum_{j=1}^{\infty} \|\phi_j\|_\sigma + \|\nabla\phi_j\|_\sigma \right)$$

which implies

$$\sum_{j=1}^{\infty} (\|\phi_j\|_\sigma + \|\nabla\phi_j\|_\sigma) \leq C\alpha$$

which implies that the operator T_j ; $j \geq 1$ in (5.68) is a uniform contraction in the set of functions satisfying (5.58) as long as (2.30), (2.31), (2.44) and (2.45) hold. In fact ϕ_j is a continuous function of v, h, e, δ and χ and a Lipschitz function of h, e and χ which follows from Lemma (5.1) and Proposition 5.2.1. Hence by Banach fixed point theorem we obtain (5.66). \square

5.6. Existence of solution for 5.54. As described in the linear theory the derivation of solution of the (5.54) is given in the linear theory of the operator in L_0 . This problem is basically reduced to a problem of fixed point

$$(5.68) \quad X^*\phi = T(X^*k + X^*(d\rho\omega') + X^*(m\rho Z))$$

where d, m satisfy

$$(5.69) \quad d \int_{\mathbb{R}} X^*(\omega')^2 \rho dx = - \int_{\mathbb{R}} X^*\omega' k \rho dx$$

$$(5.70) \quad m \int_{\mathbb{R}} X^*Z^2 \rho dx = - \int_{\mathbb{R}} X^*Zk \rho dx$$

respectively.

Lemma 5.7. *Assume that*

$$(5.71) \quad \|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)} \leq \alpha^{\frac{3}{4}\sigma}.$$

Then we have

$$(5.72) \quad \|X^*k\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)} \leq C(\alpha + \alpha^{\frac{3}{8}\sigma} \|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}).$$

*Moreover, the function X^*k is a Lipschitz function of ϕ and satisfy*

$$(5.73) \quad \|X^*k(\phi^{(1)}) - X^*k(\phi^{(2)})\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{8}\sigma} \|X^*\phi^{(1)} - X^*\phi^{(2)}\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}.$$

Furthermore, we have

$$(5.74) \quad \|d\|_{C_\theta^{0,\mu}(\mathbb{R})} + \|m\|_{C_\theta^{0,\mu}(\mathbb{R})} \leq C\|X^*k\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)}.$$

Proof. The proof of the Lipschitz property (5.73) is quite standard and left for an interested reader. We know that

$$\|X^*(\rho S(\mathbf{w}))\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)} \leq C\alpha.$$

We need to estimate X^*k given by (5.9). We can rewrite (5.9) as

$$\begin{aligned} X^*k &= X^* \left[\rho S(\mathbf{w}) + \rho N \left(\sum \eta_j \phi_j + \eta \phi + \psi \right) \right] - X^*[\rho(\mathcal{L} - \Delta + 1)\psi_2] \\ &\quad - X^*\rho(\mathcal{L}(\phi)) + X^*\rho[\partial_x^2 + \partial_z^2 - F'(\omega)]X^*\phi \end{aligned}$$

Using Lemma 5.2 we obtain

$$(5.75) \quad \|X^*(\rho\psi_2)(\cosh z)^\theta\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{\frac{3\sigma}{4}}(\alpha + \|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}).$$

Note that using the definition of ρ and ψ we have

$$\rho N \left(\sum_{j=1}^{\infty} \eta_j \phi_j + \eta \phi + \psi \right) = \rho N(\eta \phi + \psi_2).$$

We obtain from (5.53)

$$(5.76) \quad |X^*\rho N| \leq C(|X^*\phi|^2 + |X^*(\rho\psi_2)|^2).$$

Note that

$$\text{supp}(X^*\rho) \subset \left\{ |x| \leq \frac{15}{16} \log \frac{1}{\alpha} \right\}.$$

We have from (5.75)

$$\begin{aligned} \|(\cosh x)^\sigma (\cosh z)^\theta X^*(\rho\psi_2)\|_{C^{0,\mu}(\mathbb{R}^2)}^2 &\leq C\alpha^{-\frac{15}{8}\sigma} \|(\cosh z)^\theta X^*(\rho\psi_2)\|_{C^{0,\mu}(\mathbb{R}^2)}^2 \\ &\leq \alpha^{\frac{3\sigma}{2} - \frac{15\sigma}{8}} (\alpha + \|X^*\phi\|_{C^{2,\mu}(\mathbb{R}^2)})^2 \\ &\leq C\alpha^{-\frac{3\sigma}{8}} (\alpha + \|X^*\phi\|_{C^{2,\mu}(\mathbb{R}^2)})^2 \\ (5.77) \quad &\leq C\alpha^{-\frac{3\sigma}{8}} (\alpha + \|X^*\phi\|_{C^{2,\mu}(\mathbb{R}^2)})^2. \end{aligned}$$

Hence from (5.77) we have

$$(5.78) \quad \|X^*(\rho N)\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)} \leq C(\alpha^{2-\frac{3}{8}\sigma} + \|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}^2 + \alpha^{1-\frac{3}{8}\sigma} \|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}).$$

Next we estimate the term $X^*(\rho f'(\mathbf{w})\psi_2)$. Note that $X^*(\rho \mathbf{w}^{p-1})$ decays in the x variable like $(\cosh x)^{-(p-1)}$ we obtain

$$\begin{aligned} \|X^*(\rho f'(\mathbf{w})\psi_2)\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)} &\leq C\|(\cosh z)^\theta X^*(\rho\psi_2)\|_{C^{0,\mu}(\mathbb{R}^2)} \\ (5.79) \quad &\leq \alpha^{\frac{3\sigma}{4}} (\alpha + \|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}). \end{aligned}$$

In order to estimate the last terms we use (2.40) to obtain

$$(5.80) \quad \|X^*[\rho(\Delta - \partial_x^2 - \partial_z^2)]\phi\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)} \leq C\alpha\|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}$$

and

$$(5.81) \quad \|X^*[\rho(f'(\mathbf{w}) - f'(\omega))]\phi\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)} \leq C\alpha\|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)}.$$

Orthogonality conditions (5.69) and (5.70) imply

$$\|d\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R})} + \|m\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R})} \leq C\|X^*k\|_{C_{\sigma,\theta}^{0,\mu}(\mathbb{R}^2)}.$$

The Lipschitz dependence follows in a standard way. \square

Lemma 5.8. *The problem (5.68), (5.69) and (5.70) has a unique solution ϕ such that*

$$(5.82) \quad \|X^* \phi\|_{C_{\sigma, \theta}^{2, \mu}(\mathbb{R}^2)} \leq C\alpha.$$

Proof. From (5.68) we obtain by the fixed point theorem

$$(5.83) \quad \begin{aligned} \|X^* \phi\|_{C_{\sigma, \theta}^{2, \mu}(\mathbb{R}^2)} &\leq C\|X^* k\|_{C_{\sigma, \theta}^{0, \mu}(\mathbb{R}^2)} + C\|X^* \rho d\omega'\|_{C_{\sigma, \theta}^{0, \mu}(\mathbb{R}^2)} \\ &+ C\|X^* \rho m Z\|_{C_{\sigma, \theta}^{0, \mu}(\mathbb{R}^2)} \\ &\leq C\|X^* k\|_{C_{\sigma, \theta}^{0, \mu}(\mathbb{R}^2)}. \end{aligned}$$

Using the Lemma 5.7 we obtain,

$$\|X^* \phi\|_{C_{\sigma, \theta}^{2, \mu}(\mathbb{R}^2)} \leq C(\alpha + \alpha^{\frac{3}{8}\sigma} \|X^* \phi\|_{C_{\sigma, \theta}^{2, \mu}(\mathbb{R}^2)}).$$

This implies

$$\|X^* \phi\|_{C_{\sigma, \theta}^{2, \mu}(\mathbb{R}^2)} \leq C\alpha. \quad \square$$

Lemma 5.9. *The solution of (5.68), (5.69) and (5.70) is a continuous function of v, h, e, δ and χ and a Lipschitz function of h, e and χ . Moreover, we have*

$$(5.84) \quad \begin{aligned} &\|X^* \phi(h^{(1)}, e^{(1)}, \chi^{(1)}, \cdot) - X^* \phi(h^{(2)}, e^{(2)}, \chi^{(2)}, \cdot)\|_{C_{\sigma, \theta}^{2, \mu}(\mathbb{R}^2)} \\ &\leq C\|h^{(1)} - h^{(2)}\|_{C_{\theta}^{2, \mu}(\mathbb{R})} + C\|e^{(1)} - e^{(2)}\|_{C_{\theta}^{2, \mu}(\mathbb{R})} + C\alpha\|\chi^{(1)} - \chi^{(2)}\|_{\alpha} \end{aligned}$$

Proof. First note that from Lemma 5.8 we have $\|X^* \phi\|_{C_{\sigma, \theta}^{2, \mu}(\mathbb{R}^2)} \leq \alpha$ which implies that the operator T in (5.68) is a uniform contraction in the set of functions satisfying (5.71) as long as (2.30), (2.31), (2.44) and (2.45) hold. In fact ϕ is a continuous function of v, h, e, δ and χ and Lipschitz function of h, e and χ which follows from Lemma 5.2 and Proposition 5.2.1 hence by Banach fixed point theorem we obtain (5.82). \square

6. DERIVATION OF THE REDUCED EQUATIONS

In order to finish the proof of theorem (1.1) we need to adjust the parameter in such a way that $d(z) = m(z) = c_j = 0$.

$$(6.1) \quad \int_{\mathbb{R}} X^* k \omega' d\mathbf{x} = 0.$$

$$(6.2) \quad \int_{\mathbb{R}} X^* k Z d\mathbf{x} = 0.$$

$$(6.3) \quad \int_{\mathbb{R}^2} \rho_j [N(\varphi) + S(\mathbf{w})] \omega_{j,x} + \int_{\mathbb{R}^2} [p\mathbf{w}^{p-1} - p\omega_j^{p-1}] \rho_j \phi_j \omega_{j,x} dx + p \int_{\mathbb{R}^2} \rho_j \mathbf{w}^{p-1} \psi_1 \omega_{j,x} = 0.$$

for all $j \in \mathbb{N}$. We will call (6.1), (6.2) and (6.3) as the *reduced system*. In other words our main idea is to estimate the lower order terms of (6.1), 6.2 and (6.3). We show that (6.1) and (6.2) is equivalent to a nonlocal nonlinear system of second order differential equations with in variable h, e and χ . From (6.3) we obtain an infinite dimensional Toeplitz matrix. Choose $0 < \mu < 1$. Define $\nu = \min\{k_1, k_2, k_3, k_4, k_5, \frac{3}{4}\sigma\}$.

Proposition 6.0.1. *Then (6.1) is equivalent to the following differential equation:*

$$(6.4) \quad c_1(h+v)'' - \frac{\partial \Psi_L}{\partial f}(h+v) = \mathcal{P}$$

where \mathcal{P} satisfies the following inequality

$$(6.5) \quad \|\mathcal{P}\|_{C_\theta^{0,\mu}(\mathbb{R})} \leq C\alpha^{1+\nu}.$$

Moreover, \mathcal{P} satisfies Lipschitz property

$$(6.6) \quad \begin{aligned} \|\mathcal{P}(h^{(1)}, e^{(1)}, \chi^{(1)}, \cdot) - \mathcal{P}(h^{(2)}, e^{(2)}, \chi^{(2)}, \cdot)\|_{C_\theta^{0,\mu}(\mathbb{R})} &\leq C(\|h^{(1)} - h^{(2)}\|_{C_\theta^{2,\mu}(\mathbb{R})} \\ &+ \|e^{(1)} - e^{(2)}\|_{C_\theta^{2,\mu}(\mathbb{R})} + \alpha\|\chi^{(1)} - \chi^{(2)}\|_\alpha). \end{aligned}$$

Proof. It is easy to check that the main term in the projection of X^*k on ω' is given by $X^*(\rho S(\mathbf{w}))$. We express the laplacian in the local coordinates, using the notation of (2.40), and neglecting the higher order terms in α . Then we have

$$(6.7) \quad \int_{\mathbb{R}} X^*((\rho S(\mathbf{w}))\omega') dx \sim \int_{\mathbb{R}} b_1(\partial_x \omega)^2 dx + \sum_{j=1}^{\infty} p \int_{\mathbb{R}} \omega^{p-1} \omega_j \partial_x \omega dx.$$

We will show in the later part of the proof that the difference of the left hand side and the right hand side of (6.7) is very small in terms α . We first compute the integral using (2.41) and we obtain

$$(6.8) \quad \begin{aligned} \int_{\mathbb{R}} b_1(\partial_x \omega)^2 dx &= \int_{\mathbb{R}} (\partial_x \omega)^2 \frac{1}{A^3} (-\kappa A^2 - h'' A + (h')^2 \kappa - (x+h)h' \kappa) dx \\ &= -(f'' + h'') \int_{\mathbb{R}} (\omega'(x))^2 dx + \mathcal{O}_{C_\theta^{2,\mu}(\mathbb{R})}(\|h\|_{C_\theta^{2,\mu}(\mathbb{R})}^2 + \|f\|_{C_\theta^{3,\mu}(\mathbb{R})}^2). \end{aligned}$$

Now we want to compute the terms involving the interaction between the spikes and the front. Let $j = 1$. In fact it is easy to note that for $j \geq 2$ the terms involved is of the higher order. Using the estimate in Section 2 we obtain

$$\begin{aligned} p \int_{\mathbb{R}} \omega^{p-1}(x) \omega_1(x, z) \partial_x \omega(x) dx &= - \int_{\mathbb{R}} \omega^p(x) \omega_{1,x}(x, z) dx + \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha^{1+\nu}) \\ &= \frac{\partial \Psi_L}{\partial f}(h+v) + \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha^{1+\nu}). \end{aligned}$$

Now we precisely calculate some of the terms involved in estimating

$$\int_{\mathbb{R}} X^*((\rho S(\mathbf{w}))\omega') dx.$$

We first calculate the

$$(6.9) \quad \int_{\mathbb{R}} a_{12} \Xi \rho (\partial_{x,z}^2 \omega_\delta) \omega dx \sim +\alpha \sqrt{\lambda_1} h' \delta \sin(\sqrt{\lambda_1} z) \Xi \int_{\mathbb{R}} \omega Z dx$$

Now we estimate the right hand side of (6.9). Then we have

$$\alpha |\delta| \|\sqrt{\lambda_1} h' \sin \sqrt{\lambda_1} z \Xi\|_{C_\theta^{0,\mu}(\mathbb{R})} \leq C\alpha^{2+k_2+k_4} = \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha^{2+\nu}).$$

From (5.9) we have

$$X^*(\rho(\mathcal{L} - \Delta + 1)\psi_2) \sim X^*(\rho \mathbf{w}^{p-1} \psi_2)$$

Using (5.39) we obtain

$$(6.10) \quad \left\| \int_{\mathbb{R}} X^*(\rho \mathbf{w}^{p-1} \psi_2) \omega' dx \right\|_{C_\theta^{0,\mu}(\mathbb{R})} \leq \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha^{1+\frac{3}{4}\sigma}).$$

Moreover, the last term in (5.9)

$$\int_{\mathbb{R}} [-X^*(\rho\mathcal{L}\phi) + X^*\rho[\partial_x^2 + \partial_z^2 - F'(\omega)]X^*\phi]\omega' \sim \int_{\mathbb{R}} X^*[\rho(f'(\mathbf{w}) - f'(\omega))]\phi\omega' dz.$$

Hence we have

$$\left\| \int_{\mathbb{R}} X^*[\rho(f'(\mathbf{w}) - f'(\omega))]\phi\omega' dz \right\|_{C_\theta^{0,\mu}(\mathbb{R})} \leq C|\delta| \|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)} + C\alpha \|X^*\phi\|_{C_{\sigma,\theta}^{2,\mu}(\mathbb{R}^2)} \leq C(\alpha^{2+k_4} + \alpha^2).$$

It is easy to check that the other terms are of the higher order in α . Hence we obtain

$$c_1(h+v)'' - \frac{\partial \Psi_L}{\partial f}(h+v) = \mathcal{O}_{C_\theta^{0,\mu}(\mathbb{R})}(\alpha^{1+\nu})$$

where $c_1 = \int_{\mathbb{R}} (\omega'(x))^2 dx$. The continuity and the Lipschitz property of \mathcal{P} can be obtained in a standard way using the estimate of the error in Proposition 5.2.1, the Lipschitz estimate of ψ_2 and ϕ . \square

Proposition 6.0.2. *We have (6.2) is equivalent to the following differential equation:*

$$(6.11) \quad e'' + \lambda_1 e = \mathcal{R}$$

where \mathcal{R} satisfies the following inequality

$$(6.12) \quad \|\mathcal{R}\|_{C_\theta^{0,\mu}(\mathbb{R})} \leq C\alpha^{2+\nu}.$$

Moreover, \mathcal{R} satisfies Lipschitz property

$$(6.13) \quad \begin{aligned} \|\mathcal{R}(h^{(1)}, e^{(1)}, \chi^{(1)}, \cdot) - \mathcal{R}(h^{(2)}, e^{(2)}, \chi^{(2)}, \cdot)\|_{C_\theta^{0,\mu}(\mathbb{R})} &\leq C(\|h^{(1)} - h^{(2)}\|_{C_\theta^{2,\mu}(\mathbb{R})} \\ &+ \|e^{(1)} - e^{(2)}\|_{C_\theta^{2,\mu}(\mathbb{R})} + \alpha\|\chi^{(1)} - \chi^{(2)}\|_\alpha). \end{aligned}$$

Proof. It is easy to check that the dominating term in (6.11) is given by

$$(6.14) \quad \int_{\mathbb{R}} X^*(\rho S(\mathbf{w})X^*Z)dx \sim \int_{\mathbb{R}} [\partial_x^2 + \partial_z^2 + f(\omega(x))]e(z)Z(x)\rho Z(x)dx.$$

But we know that

$$\{\partial_x^2 + f(\omega(x))\}Z = \lambda_1 Z$$

and hence we have the right hand side of (6.14) reduces to

$$(6.15) \quad \begin{aligned} \int_{\mathbb{R}} [\partial_x^2 + \partial_z^2 + f(\omega(x))]e(z)Z(x)\rho Z(x)dx &\sim \int_{\mathbb{R}} (\partial_z^2 + \lambda_1)e(z)\rho Z^2 \\ &\sim (e''(z) + \lambda_1 e) \int_{\mathbb{R}} \rho Z^2 dx. \end{aligned}$$

This gives the reduced equation for e . The Lipschitz property follows in a standard way. \square

We have from (2.20)

$$(6.16) \quad \Theta = \int_{\mathbb{R}^+} \int_{\mathbb{R}} X^*(kZ)Z(x) \cos(\sqrt{\lambda_1}z) dx dz = 0.$$

From (6.16) we deduce the reduced equation for the parameter δ .

Lemma 6.1. *Moreover,*

$$(6.17) \quad \Theta = \varsigma \sqrt{\lambda_1} \delta + \mathcal{O}(\alpha^{1+\nu}).$$

Proof. From (6.16) we have

$$\Theta = \int_{\mathbb{R}^+} \int_{\mathbb{R}} X^*(\rho S(w)) Z(x) \cos(\sqrt{\lambda_1} z) dx dz + \mathcal{O}(\alpha^{1+\nu})$$

where w is defined in (2.48). But we have from

$$(6.18) \quad X^*(\rho S(w)) \sim \partial_x^2 w + \partial_z^2 w + F(w).$$

But using the fact that $\Xi + \Xi_0 = 1$ we have

$$(6.19) \quad \begin{aligned} \partial_x^2 w + \partial_z^2 w + F(w) &\sim [\Xi'' \omega_\delta + \Xi_0'' \omega] \\ &+ 2[\Xi' \partial_z \omega_\delta + \Xi_0' \partial_z \omega] \\ &= [\Xi''(\omega_\delta - \omega_0)] + 2\Xi' \partial_z \omega_\delta \end{aligned}$$

Further we have

$$(6.20) \quad \partial_z \omega_\delta \sim -\sqrt{\lambda_1} Z \delta \sin(\sqrt{\lambda_1} z)$$

$$(6.21) \quad (\omega_\delta - \omega_0) \sim Z \delta \cos(\sqrt{\lambda_1} z),$$

where the neglected terms are of higher order $\mathcal{O}_{C^\infty(\mathbb{R})}(|\delta|^2)(\cosh x)^{-1}$ and consequently their contribution is small. Then from (6.18) we have

$$\Theta \sim \varsigma \delta \sqrt{\lambda_1} \int_{\mathbb{R}^+} \Xi' \sin 2(\sqrt{\lambda_1} z) dz = \varsigma \sqrt{\lambda_1} \delta$$

where $\varsigma = \int_{\mathbb{R}} \rho Z^2$. \square

Proposition 6.0.3. *We have (6.3) is equivalent to the following system of equations*

$$(6.22) \quad \gamma_0(e^{-|\xi_1 - f(0)|} \chi_1 - e^{-|\xi_2 - \xi_1|} (\chi_2 - \chi_1)) = \mathcal{G}_1(v, h, e, \delta, \chi)$$

and for $j \geq 2$ we have

$$(6.23) \quad \gamma_0(e^{-|\xi_{j+1} - \xi_j|} (\chi_{j+1} - \chi_j) - e^{-|\xi_j - \xi_{j-1}|} (\chi_j - \chi_{j-1})) = \mathcal{G}_j(v, h, e, \delta, \chi)$$

where $\mathcal{G} = \{\mathcal{G}_j\}_{j \geq 1}$ satisfies the following inequality

$$(6.24) \quad \|\mathcal{G}\|_\alpha = \max_i \alpha^{-i} |\mathcal{G}_i| \leq C \alpha^{1+\nu}.$$

Moreover, \mathcal{G} satisfies Lipschitz property

$$(6.25) \quad \begin{aligned} \|\mathcal{G}(h^{(1)}, e^{(1)}, \chi^{(1)}, \cdot) - \mathcal{G}(h^{(2)}, e^{(2)}, \chi^{(2)}, \cdot)\|_\alpha &\leq C(\|h^{(1)} - h^{(2)}\|_{C_\theta^{0,\mu}(\mathbb{R})} \\ &+ \|e^{(1)} - e^{(2)}\|_{C_\theta^{0,\mu}(\mathbb{R})} + \alpha \|\chi^{(1)} - \chi^{(2)}\|_\alpha). \end{aligned}$$

and continuous in the remaining variables.

Proof. Without loss of generality let $\gamma_0 = \gamma_1$. Using the estimates (2.49) and (2.50) we obtain (6.22) and (6.23). Now we estimate some of the terms involved in \mathcal{G} .

$$(6.26) \quad p \int_{\mathbb{R}^2} \mathbf{w}^{p-1} \psi_1 \omega_{j,x} dx dz = \mathcal{O}(\alpha^{1+\frac{3}{4}\sigma+j}).$$

and

$$(6.27) \quad \int_{\mathbb{R}^2} N(\varphi) \omega_{j,x} dx dz \leq C \int_{\mathbb{R}^2} |\varphi|^2 \omega_{j,x} dx dz = \mathcal{O}(\alpha^{2+j}).$$

Now we precisely calculate some of the terms involved in estimating (6.22)-(6.23)

$$(6.28) \quad \int_{\mathbb{R}^2} X^*((\rho S(\bar{w})) \omega_{j,x}) dx dz \sim \int_{\mathbb{R}^2} X^*((\rho S(w)) \omega_{j,x}) dx dz$$

thus neglecting the higher order term in α . We first calculate the lower order term in the expression

$$\int_{\mathbb{R}^2} |a_2 \Xi \rho(\partial_{x,z}^2 \omega_\delta) \omega_{j,x}| dx dz \leq \sqrt{\lambda_1} \alpha^{1+k_2} |\delta| \int_{\mathbb{R}^2} |\omega_{j,x}| dx dz = \mathcal{O}(\alpha^{2+k_2+k_4+j}).$$

□

7. SOLUTION OF THE REDUCED SYSTEMS AND PROOF OF THEOREM 1.1

7.1. Proof of Theorem 1.1. We now complete the proof of Theorem 1.1. To this end we have to solve the following system of equations

$$(7.1) \quad c_1(h+v)'' - \frac{\Psi_L(f,z)}{\partial f}(h+v) = \mathcal{P}(v, h, e, \delta, \chi)$$

$$(7.2) \quad e'' + \lambda_1 e = \mathcal{R}(v, h, e, \delta, \chi)$$

$$(7.3) \quad \sqrt{\lambda_1} s_0 \delta = \Theta(v, h, e, \delta, \chi)$$

$$(7.4) \quad \begin{cases} \gamma_0(e^{-|\xi_1-f(0)|} \chi_1 - e^{-|\xi_2-\xi_1|} (\chi_2 - \chi_1)) = \mathcal{G}_1(v, h, e, \delta, \chi) \\ \gamma_0(e^{-|\xi_{j+1}-\xi_j|} (\chi_{j+1} - \chi_j) - e^{-|\xi_j-\xi_{j-1}|} (\chi_j - \chi_{j-1})) = \mathcal{G}_j(v, h, e, \delta, \chi). \end{cases}$$

Proposition 7.1.1. *The system (7.1)-(7.4) is a one parameter family of solutions in the sense that for each choice of $\delta \in \mathbb{R}$, the system admits a solution containing δ and the functions v, h, e and the parameter χ .*

Proof. First we choose $k_i \in (0, 1), \mu \in (0, 1)$ and $0 < \sigma < \min\{p-2, 1\}$ in such a way that

$$(7.5) \quad \nu = \min\{k_1, k_2, k_3, k_4, k_5, \frac{3}{4}\sigma\}.$$

Fix δ and moreover assume that the parameter satisfy

$$(7.6) \quad |\delta| \leq \frac{1}{2} \alpha^{1+k_4}.$$

In order to complete the proof we need to go through the following steps.

- Firstly we define $\tilde{v}, \tilde{h}, \tilde{e}, \tilde{\delta}, \tilde{\chi}$. We define $\tilde{\delta} = \tilde{\delta} + \delta$ and use this parameter $\tilde{\delta}$ to calculate the right hand sides of (7.1)-(7.4). Then these functions satisfy the assertions of Propositions 6.0.1, 6.0.2 and Lemma 6.1. In particular, they are Lipschitz functions of \tilde{h}, \tilde{e} and $\tilde{\chi}$; and continuous functions of \tilde{v} and $\tilde{\delta}$.
- We now apply Banach fixed point theorem to the solve (7.1)-(7.4) for h, e and χ . Also we note that

$$\begin{aligned} \|h\|_{C_\theta^{0,\mu}(\mathbb{R})} &\leq C \|\mathcal{P}\|_{C_\theta^{0,\mu}(\mathbb{R})} \leq C \alpha^{1+\nu} \\ \|e\|_{C_\theta^{0,\mu}(\mathbb{R})} &\leq C \|\mathcal{R}\|_{C_\theta^{0,\mu}(\mathbb{R})} \leq C \alpha^{2+\nu} \end{aligned}$$

and it is easy to check that

$$\|\chi\|_\alpha \leq C \alpha^{-1} \|\mathcal{G}\|_\alpha \leq C \alpha^\nu$$

and v, δ satisfy

$$\begin{aligned} \|v\|_\mathcal{E} &\leq C \alpha^{1+\nu} \\ |\delta| &\leq C \alpha^{1+\nu}. \end{aligned}$$

- Now we define a continuous map on a finite dimensional space $\mathcal{E} \times \mathbb{R}$

$$\mathcal{F} : \mathcal{E} \times \mathbb{R} \rightarrow \mathcal{E} \times \mathbb{R}$$

given by

$$(\bar{v}, \bar{\delta}) \mapsto (v, \delta).$$

By the choice of ν , we can use Browder's fixed point theorem to obtain a fixed point of the map \mathcal{F} .

□

7.2. Final remarks on the proofs of Solution 2 and Solution 3. Finally, we show what modifications are needed for the proofs of Solution 2 and Solution 3 in Section 1.2.

For Solution 2, we use approximate solution of the following form

$$(7.7) \quad \begin{cases} u_L(x, z) = u_L(x, -z) & \text{for all } (x, z) \in \mathbb{R}^2 \\ u_L(x, z) = \left(\omega_\delta(x - f(z) - h_L(z), z) - \sum_{i=1}^{\infty} \omega_0((x, z) - \xi_i \vec{e}_1) \right) (1 + o_L(1)) \end{cases}$$

where the interaction function f satisfies

$$(7.8) \quad \begin{cases} f''(z) = -\Psi_L(f, z) & \text{in } \mathbb{R} \\ f(0) = 0, \quad f'(0) = 0. \end{cases}$$

For Solution 3, we use approximate solution of the following form

$$(7.9) \quad \begin{cases} u_L(x, z) = u_L(x, -z) & \text{for all } (x, z) \in \mathbb{R}^2 \\ u_L(x, z) = u_L(-x, z) & \text{for all } (x, z) \in \mathbb{R}^2 \\ u_L(x, z) = \left(\omega_\delta(x - f(z) - h_L(z), z) + \omega_\delta(x + f(z) + h_L(z), z) + \omega_0(x, z) \right) (1 + o_L) \end{cases}$$

where f satisfies (1.13).

The rest of the proofs remains the same.

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