# CONCENTRATING STANDING WAVES FOR THE FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We consider the semilinear equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0, \quad u > 0, \quad u \in H^{2s}(\mathbb{R}^N)$$

where $0 < s < 1$, $1 < p < \frac{N+2s}{N-2s}$, $V(x)$ is a sufficiently smooth potential with $\inf_{\mathbb{R}} V(x) > 0$, and $\varepsilon > 0$ is a small number. Letting $w_\lambda$ be the radial ground state of $(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0$ in $H^{2s}(\mathbb{R}^N)$, we build solutions of the form

$$u_\varepsilon(x) \sim \sum_{i=1}^{k} w_{\lambda_i}((x - \xi_i^\varepsilon)/\varepsilon),$$

where $\lambda_i = V(\xi_i^\varepsilon)$ and the $\xi_i^\varepsilon$ approach suitable critical points of $V$. Via a Lyapunov Schmidt variational reduction, we recover various existence results already known for the case $s = 1$. In particular such a solution exists around $k$ nondegenerate critical points of $V$. For $s = 1$ this corresponds to the classical results by Floer-Weinstein [13] and Oh [21, 22].

## 1. Introduction and main results

We consider the fractional nonlinear Schrödinger equation

$$i\psi_t = \varepsilon^{2s}(-\Delta)^s\psi + W(x)\psi - |\psi|^{p-1}\psi \tag{1.1}$$

where $(-\Delta)^s$, $0 < s < 1$, denotes the usual fractional Laplace operator, $W(x)$ is a bounded potential, and $p > 1$. We are interested in the *semi-classical limit* regime, $0 < \varepsilon \ll 1$.

We want to find *standing-wave solutions*, which are those of the form $\psi(x,t) = u(x)e^{iEt}$ with $u$ real-valued function. Letting $V(x) = W(x) + E$, equation (1.1) becomes

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

We assume in what follows that $V$ satisfies

$$V \in C^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \inf_{\mathbb{R}^N} V(x) > 0. \tag{1.3}$$

We are interested in finding solutions with a *spike pattern* concentrating around a finite number of points in space as $\varepsilon \to 0$ . This has been the topic of many works in the standard case $s = 1$, relating the concentration points with critical points of the potential, starting in 1986 with the pioneering work by Floer and Weinstein [13], then continued by Oh [21, 22]. The natural place to look for solutions to (1.2) that decay at infinity is the space $H^{2s}(\mathbb{R}^N)$, of all functions $u \in L^2(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (1 + |\xi|^{4s})\,|\hat{u}(\xi)|^2 d\xi < +\infty,$$

where $\widehat{\ }$ denotes Fourier transform. The fractional Laplacian $(-\Delta)^s u$ of a function $u \in H^{2s}(\mathbb{R}^N)$ is defined in terms of its Fourier transform by the relation

$$\widehat{(-\Delta)^s u} = |\xi|^{2s}\hat{u} \in L^2(\mathbb{R}^N).$$

We will explain next what we mean by a *spike pattern* solution of equation (1.2). Let us consider the basic problem

$$(-\Delta)^s v + v - |v|^{p-1}v = 0, \quad v \in H^{2s}(\mathbb{R}^N). \tag{1.4}$$

We assume the following constraint in $p$,

$$1 < p < \begin{cases} \frac{N+2s}{N-2s} & \text{if } 2s < N, \\ +\infty & \text{if } 2s \geq N. \end{cases} \tag{1.5}$$

Under this condition it is known the existence of a positive, radial *least energy solution* $v = w(x)$, which gives the lowest possible value for the energy

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N} v(-\Delta)^s v + \frac{1}{2}\int_{\mathbb{R}^N} v^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} |v|^{p+1}.$$

among all nontrivial solutions of (1.4). An important property, which has only been proven recently by Frank-Lenzman-Silvestre [15] (see also [2, 14]), is that there exists a radial least energy solution which is nondegenerate, in the sense that the space of solutions of the equation

$$(-\Delta)^s \phi + \phi - pw^{p-1}\phi = 0, \quad \phi \in H^{2s}(\mathbb{R}^n) \tag{1.6}$$

consists of exactly of the linear combinations of the translation-generators, $\frac{\partial w}{\partial x_j}$, $j = 1, \cdots, N$.

It is easy to see that the function

$$w_\lambda(x) := \lambda^{\frac{1}{p-1}} w(\lambda^{\frac{1}{2s}} x)$$

satisfies the equation

$$(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0 \quad \text{in } \mathbb{R}^N.$$

Therefore for any point $\xi \in \mathbb{R}^N$, taking $\lambda = V(\xi)$, the spike-shape function

$$u(x) = w_{V(\xi)}\left(\frac{x-\xi}{\varepsilon}\right) \tag{1.7}$$

satisfies

$$\varepsilon^{2s}(-\Delta)^s u + V(\xi)u - u^p = 0.$$

Since the $\varepsilon$-scaling makes it *concentrate* around $\xi$, this function constitutes a good positive approximate solution to equation (1.2), namely of

$$\begin{gathered} \varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0, \\ u > 0, \quad u \in H^{2s}(\mathbb{R}^N). \end{gathered} \tag{1.8}$$

We call a *k-spike pattern solution* of (1.8) one that looks approximately like a superposition of $k$ spikes like (1.7), namely a solution $u_\varepsilon$ of the form

$$u_\varepsilon(x) = \sum_{i=1}^{k} w_{V(\xi_i^\varepsilon)}\left(\frac{x-\xi_i^\varepsilon}{\varepsilon}\right) + o(1) \tag{1.9}$$

for points $\xi_1^\varepsilon, \ldots, \xi_k^\varepsilon$, where $o(1) \to 0$ in $H^{2s}(\mathbb{R}^N)$ as $\varepsilon \to 0$.

In what follows we assume that $p$ satisfies condition (1.5) and $V$ condition (1.3).

Our first result concerns the existence of multiple spike solution at separate places in the case of stable critical points.

**Theorem 1.** *Let $\Lambda_i \subset \mathbb{R}^N$, $i = 1, \dots, k$, $k \geq 1$ be disjoint bounded open sets in $\mathbb{R}^N$. Assume that*

$$\deg(\nabla V, \Lambda_i, 0) \neq 0 \quad \text{for all} \quad i = 1, \dots, k.$$

*Then for all sufficiently small $\varepsilon$, Problem* (1.8) *has a solution of the form* (1.9) *where $\xi_i^\varepsilon \in \Lambda_i$ and*

$$\nabla V(\xi_i^\varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

An immediate consequence of Theorem 1 is the following.

**Corollary 1.1.** *Assume that $V$ is of class $C^2$. Let $\xi_1^0, \dots, \xi_k^0$ be $k$ non-degenerate critical points of $V$, namely*

$$\nabla V(\xi_i^0) = 0, \quad D^2 V(\xi_i^0) \text{ is invertible for all } i = 1, \dots k.$$

*Then, a $k$-spike solution of* (1.8) *of the form* (1.9) *with $\xi_i^\varepsilon \to \xi_i^0$ exists.*

When $s = 1$, the result of Corollary 1.1 is due to Floer and Weinstein [13] for $N = 1$ and $k = 1$ and to Oh [21, 22] when $N \geq 1$, $k \geq 1$. Theorem 1 for $s = 1$ was proven by Yanyan Li [19].

**Remark 1.1.** As the proof will yield, Theorem 1 for $0 < s < 1$ holds true under the following, more general condition introduced in [19]. Let $\Lambda = \Lambda_1 \times \cdots \times \Lambda_k$ and assume that the function

$$\varphi(\xi_1, \dots, \xi_k) = \sum_{i=1}^{k} V(\xi_i)^\theta, \quad \theta = \frac{p+1}{p-1} - \frac{N}{2s} > 0 \tag{1.10}$$

has a *stable critical point situation* in $\Lambda$: there is a number $\delta_0 > 0$ such that for each $g \in C^1(\bar\Omega)$ with $\|g\|_{L^\infty(\Lambda)} + \|\nabla g\|_{L^\infty(\Lambda)} < \delta_0$, there is a $\xi_g \in \Lambda$ such that $\nabla\varphi(\xi_g) + \nabla g(\xi_g) = 0$. Then for all sufficiently small $\varepsilon$, Problem (1.8) has a solution of the form (1.9) where $\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_k^\varepsilon) \in \Lambda$ and $\nabla\varphi(\xi^\varepsilon) \to 0$ as $\varepsilon \to 0$.

**Theorem 2.** *Let $\Lambda$ be a bounded, open set with smooth boundary such that $V$ is such that either*

$$c = \inf_{\Lambda} V < \inf_{\partial\Lambda} V \tag{1.11}$$

*or*

$$c = \sup_{\Lambda} V > \sup_{\partial\Lambda} V$$

*or, there exist closed sets $B_0 \subset B \subset \Lambda$ such that*

$$c = \inf_{\Phi\in\Gamma} \sup_{x\in B} V(\Phi(x)) > \sup_{B_0} V. \tag{1.12}$$

*where $\Gamma = \{\Phi \in C(B, \bar\Lambda) \ / \ \Phi\Big|_{B_0} = Id\ \}$ and $\nabla V(x) \cdot \tau \neq 0$ for all $x \in \partial\Lambda$ with $V(x) = c$ and some tangent vector $\tau$ to $\partial\Lambda$ at $x$.*

*Then, there exists a* 1*-spike solution of* (1.8) *with $\xi^\varepsilon \in \Lambda$ with*
*$\nabla V(\xi_\varepsilon) \to 0$ and $V(\xi_i^\varepsilon) \to c$.*

In the case $s = 1$, the above results were found by del Pino and Felmer [7, 8]. The case of a (possibly degenerate) global minimizer was previously considered by Rabinowitz [23] and X. Wang [25]. An isolated maximum with a power type degeneracy appears in Ambrosetti, Badiale and Cingolani [1]. Condition (1.12) is called a *nontrivial linking situation* for $V$. The cases of $k$ disjoint sets where (1.11) holds was treated in [9, 17]. Multiple spikes for disjoint nontrivial linking regions were first considered in [10], see also [5, 16] for other multiplicity results.

Our last result concerns the existence of multiple spikes *at the same point*.

**Theorem 3.** *Let $\Lambda$ be a bounded, open set with smooth boundary such that $V$ is such that*

$$\sup_{\Lambda} V > \sup_{\partial\Lambda} V.$$

*Then for any positive integer $k$ there exists a $k$-spike solution of* (1.8) *with spikes $\xi_j^\varepsilon \in \Lambda$ satisfying $V(\xi_j^\varepsilon) \to \max_\Lambda V$.*

In the case $s = 1$, Theorem 3 was proved by Kang and Wei [18]. D'Aprile and Ruiz [6] have found a phenomenon of this type at a saddle point of $V$.

The rest of this paper will be devoted to the proofs of Theorems 1–3. The method of construction of a $k$-spike solution consists of a Lyapunov-Schmidt reduction in which the full problem is reduced to that of finding a critical point $\xi^\varepsilon$ of a functional which is a small $C^1$-perturbation of $\varphi$ in (1.10). In this reduction the nondegeneracy result in [15] is a key ingredient.

After this has been done, the results follow directly from standard degree theoretical or variational arguments. The Lyapunov-Schmidt reduction is a method widely used in elliptic singular perturbation problems. Some results of variational type for $0 < s < 1$ have been obtained for instance in [12] and [24]. We believe that the scheme of this paper may be generalized to concentration on higher dimensional regions, while that could be much more challenging. See [11, 20] for concentration along a curve in the plane and $s = 1$.

## 2. Generalities

Let $0 < s < 1$. Various definitions of the fractional Laplacian $(-\Delta)^s\phi$ of a function $\phi$ defined in $\mathbb{R}^N$ are available, depending on its regularity and growth properties.

As we have recalled in the introduction, for $\phi \in H^{2s}(\mathbb{R}^N)$ the standard definition is given via Fourier transform $\widehat{\ }$. $(-\Delta)^s\phi \in L^2(\mathbb{R}^N)$ is defined by the formula

$$|\xi|^{2s}\hat{\phi}(\xi) = \widehat{(-\Delta)^s}\phi. \tag{2.1}$$

When $\phi$ is assumed in addition sufficiently regular, we obtain the direct representation

$$(-\Delta)^s\phi(x) = d_{s,N} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x-y|^{N+2s}}\, dy \tag{2.2}$$

for a suitable constant $d_{s,N}$ and the integral is understood in a principal value sense. This integral makes sense directly when $s < \frac{1}{2}$ and $\phi \in C^{0,\alpha}(\mathbb{R}^N)$ with $\alpha > 2s$, or if $\phi \in C^{1,\alpha}(\mathbb{R}^N)$, $1+\alpha > 2s$. In the latter case, we can desingularize the integral representing it in the form

$$(-\Delta)^s \phi(x) = d_{s,N} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y) - \nabla\phi(x)(x-y)}{|x-y|^{N+2s}}\, dy.$$

Another useful (local) representation, found by Caffarelli and Silvestre [3], is via the following boundary value problem in the half space $\mathbb{R}^{N+1}_+ = \{(x,y) \ / \ x \in \mathbb{R}^N,\ y > 0\}$:

$$\begin{cases} \nabla \cdot (y^{1-2s} \nabla \tilde{\phi}) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ \tilde{\phi}(x,0) = \phi(x) & \text{on } \mathbb{R}^N. \end{cases}$$

Here $\tilde{\phi}$ is the $s$-harmonic extension of $\phi$, explicitly given as a convolution integral with the $s$-Poisson kernel $p_s(x,y)$,

$$\tilde{\phi}(x,y) = \int_{\mathbb{R}^N} p_s(x-z,y)\phi(z)\, dz,$$

where

$$p_s(x,y) = c_{N,s} \frac{y^{4s-1}}{(|x|^2 + |y|^2)^{\frac{N-1+4s}{2}}}$$

and $c_{N,s}$ achieves $\int_{\mathbb{R}^N} p(x,y)dx = 1$. Then under suitable regularity, $(-\Delta)^s\phi$ is the Dirichlet-to-Neumann map for this problem, namely

$$(-\Delta)^s \phi(x) \ = \ \lim_{y\to 0^+} y^{1-2s} \partial_y \tilde{\phi}(x,y). \tag{2.3}$$

Characterizations (2.1), (2.2), (2.3) are all equivalent for instance in Schwartz's space of rapidly decreasing smooth functions.

Let us consider now for a number $m > 0$ and $g \in L^2(\mathbb{R}^N)$ the equation

$$(-\Delta)^s \phi + m\phi = g \quad \text{in } \mathbb{R}^N.$$

Then in terms of Fourier transform, this problem, for $\phi \in L^2$, reads

$$(|\xi|^{2s} + m)\, \hat{\phi} = \hat{g}$$

and has a unique solution $\phi \in H^{2s}(\mathbb{R}^N)$ given by the convolution

$$\phi(x) = T_m[g] := \int_{\mathbb{R}^N} k(x-z)\, g(z)\, dz, \tag{2.4}$$

where

$$\hat{k}(\xi) = \frac{1}{|\xi|^{2s} + m}.$$

Using the characterization (2.3) written in weak form, $\phi$ can then be characterized by $\phi(x) = \tilde{\phi}(x,0)$ in trace sense, where $\tilde{\phi} \in H$ is the unique solution of

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\tilde{\phi} \nabla\varphi\, y^{1-2s} + m \int_{\mathbb{R}^N} \phi\varphi = \int_{\mathbb{R}^N} g\varphi, \qquad \text{for all} \quad \varphi \in H, \tag{2.5}$$

where $H$ is the Hilbert space of functions $\varphi \in H^1_{loc}(\mathbb{R}^{N+1}_+)$ such that

$$\|\varphi\|^2_H := \iint_{\mathbb{R}^{N+1}_+} |\nabla\varphi|^2 y^{1-2s} + m \int_{\mathbb{R}^N} |\varphi|^2 \ < \ +\infty,$$

or equivalently the closure of the set of all functions in $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ under this norm.

A useful fact for our purposes is the equivalence of the representations (2.4) and (2.5) for $g \in L^2(\mathbb{R}^N)$.

**Lemma 2.1.** *Let $g \in L^2(\mathbb{R}^N)$. Then the unique solution $\tilde\phi \in H$ of Problem (2.5) is given by the $s$-harmonic extension of the function $\phi = T_m[g] = k * g$.*

*Proof.* Let us assume first that $\hat g \in C_c^\infty(\mathbb{R}^N)$. Then $\phi$ given by (2.4) belongs to $H^{2s}(\mathbb{R}^N)$. Take a test function $\psi \in C_c^\infty(\mathbb{R}^{N+1}_+)$. Then the well-known computation by Caffarelli and Silvestre shows that

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\tilde\phi \nabla\psi\, y^{1-2s} dydx =$$

$$\int_{\mathbb{R}^N} \lim_{y\to 0} y^{1-2s}\partial_y\tilde\phi(y,\cdot)\,\psi\, dx = \int_{\mathbb{R}^N} \psi(-\Delta)^s\phi\, dx = \int_{\mathbb{R}^N} (g - m\phi)\, dx.$$

By taking $\psi = \tilde\phi\eta_R$ for a suitable sequence of smooth cut-off functions equal to one on expanding balls $B_R(0)$ in $\mathbb{R}^{N+1}_+$, and using the behavior at infinity of $\tilde\phi$ which resembles the Poisson kernel $p_s(x,y)$, we obtain

$$\iint_{\mathbb{R}^{N+1}_+} |\nabla\tilde\phi|^2\, y^{1-2s} dydx + m\int_{\mathbb{R}^N} |\phi|^2 = \int_{\mathbb{R}^N} g\phi$$

and hence $\|\tilde\phi\|_H \le C\|g\|_{L^2}$ and satisfies (2.5). By density, this fact extends to all $g \in L^2(\mathbb{R}^N)$. The result follows since the solution of Problem (2.5) in $H$ is unique. □

Let us recall the main properties of the fundamental solution $k(x)$ in the representation (2.4), which are stated for instance in [15] or in [12].

We have that $k$ is radially symmetric and positive, $k \in C^\infty(\mathbb{R}^N \setminus \{0\})$ satisfying

- $$|k(x)| + |x|\,|\nabla k(x)| \ \le\ \frac{C}{|x|^{N-2s}} \quad \text{for all} \quad |x| \le 1,$$
- $$\lim_{|x|\to\infty} k(x)|x|^{N+2s} = \gamma > 0,$$
- $$|x|\,|\nabla k(x)| \ \le\ \frac{C}{|x|^{N+2s}} \quad \text{for all} \quad |x| \ge 1.$$

The operator $T_m$ is not just defined on functions in $L^2$. For instance it acts nicely on bounded functions. The positive kernel $k$ satisfies $\int_{\mathbb{R}^N} k = \frac{1}{m}$. We see that if $g \in L^\infty(\mathbb{R}^N)$ then

$$\|T_m[g]\|_\infty \le \frac{1}{m}\|g\|_\infty.$$

We have indeed the validity of an estimate like this for $L^\infty$ weighted norms as follows.

**Lemma 2.2.** *Let $0 \le \mu < N + 2s$. Then there exists a $C > 0$ such that*

$$\|(1 + |x|)^\mu T_m[g]\|_{L^\infty(\mathbb{R}^N)} \le C\|(1 + |x|)^\mu g\|_{L^\infty(\mathbb{R}^N)}.$$

*Proof.* Let us assume that $0 \le \mu < N + 2s$ and let $\bar g(x) = \frac{1}{(1+|x|)^\mu}$. Then

$$T[\bar g](x) = \int_{|y-x|<\frac12|x|} \frac{k(y)}{(1 + |y - x|)^\mu}\, dy + \int_{|y-x|>\frac12|x|} \frac{k(y)}{(1 + |y - x|)^\mu}\, dy.$$

Then, as $|x| \to \infty$ we find

$$|x|^\mu \int_{|y-x|<\frac12|x|} \frac{k(y)}{(1 + |y - x|)^\mu}\, dy \sim |x|^{-2s} \to 0,$$

and since $k \in L^1(\mathbb{R}^N)$, by dominated convergence we find that as $|x| \to \infty$

$$\int_{|x-y|>\frac{1}{2}|x|} \frac{k(y)|x|^\mu}{(1+|x-y|)^\mu}\, dy \to \int_{\mathbb{R}^N} k(z)dz = \frac{1}{m}.$$

We conclude in particular that for a suitable constant $C > 0$, we have

$$T_m[(1+|x|)^{-\mu}] \le C(1+|x|)^{-\mu}.$$

Now, we have that

$$\pm T_m[g] \le \|(1+|x|)^\mu g\|_{L^\infty(\mathbb{R}^N)}\, T_m[(1+|x|)^{-\mu}],$$

and then

$$\|(1+|x|)^\mu T[g]\|_{L^\infty(\mathbb{R}^N)} \le C\|(1+|x|)^\mu g\|_{L^\infty(\mathbb{R}^N)}$$

as desired. □

We also have the validity of the following useful estimate.

**Lemma 2.3.** *Assume that $g \in L^2 \cap L^\infty$. Then the following holds: if $\phi = T_m[g]$ then there is a $C > 0$ such that*

$$\sup_{x \ne y} \frac{|\phi(x) - \phi(y)|}{|x-y|^\alpha} \le C\|g\|_{L^\infty(\mathbb{R}^N)} \tag{2.6}$$

*where $\alpha = \min\{1, 2s\}$.*

*Proof.* Since $\|T_m[g]\|_\infty \le C\|g\|_\infty$, it suffices to establish (2.6) for $|x-y| < \frac{1}{3}$. We have

$$|\phi(x) - \phi(y)| \le \int_{\mathbb{R}^N} |k(z+y-x) - k(z)|\, dz\, \|g\|_\infty.$$

Now, we decompose

$$\int_{\mathbb{R}^N} |k(z+y-x) - k(z)|\, dz \;=$$

$$\int_{|z|>3|y-x|} |k(z+y-x) - k(z)|\, dz + \int_{|z|<3|y-x|} |k(z+y-x) - k(z)|\, dz.$$

We have

$$\int_{|z|>3|y-x|} |k(z+(y-x)) - k(z)| \le \int_0^1 dt \int_{|z|>3|y-x|} |\nabla k(z+t(y-x))|\, dz\, |y-x|.$$

and, since $3|y-x| < 1$,

$$\int_{|z|>3|y-x|} |\nabla k(z+t(y-x))| dz \le C(1+\int_{1>|z|>3|y-x|} \frac{dz}{|z|^{N+1-2s}}) \le C(1+|y-x|^{2s-1}).$$

On the other hand

$$\int_{|z|<3|y-x|} |k(z+y-x) - k(z)|\, dz \le 2\int_{|z|<4|y-x|} |k(z)|\, dz \le C|y-x|^{2s},$$

and (2.6) readily follows. □

Next we consider the more general problem

$$(-\Delta)^s \phi + W(x)\phi = g \quad \text{in } \mathbb{R}^N \tag{2.7}$$

where $W$ is a bounded potential.

We start with a form of the weak maximum principle.

**Lemma 2.4.** *Let us assume that*

$$\inf_{x\in\mathbb{R}^N} W(x) =: m > 0$$

*and that* $\phi \in H^{2s}(\mathbb{R}^N)$ *satisfies equation* (2.7) *with* $g \geq 0$. *Then* $\phi \geq 0$ *in* $\mathbb{R}^N$.

*Proof.* We use the representation for $\phi$ as the trace of the unique solution $\tilde\phi \in H$ to the problem

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\tilde\phi\nabla\varphi y^{1-2s} + \int_{\mathbb{R}^N} W\phi\varphi = \int_{\mathbb{R}^N} g\varphi, \qquad \text{for all} \quad \varphi \in H.$$

It is easy to check that the test function $\varphi = \phi_- = \min\{\phi, 0\}$ does indeed belong to $H$. We readily obtain

$$\iint_{\mathbb{R}^{N+1}_+} |\nabla\tilde\phi_-|^2 y^{1-2s} + \int_{\mathbb{R}^N} W\phi_-^2 = \int_{\mathbb{R}^N} g\phi_-.$$

Since $g \geq 0$ and $W \geq m$, we obtain that $\phi_- \equiv 0$, which means precisely $\phi \geq 0$, as desired. □

We want to obtain a priori estimates for problems of the type (2.7) when $W$ is not necessarily positive. Let $\mu > \frac{N}{2}$, and let us assume that

$$\|(1 + |x|^\mu)g\|_{L^\infty(\mathbb{R}^N)} < +\infty.$$

The assumption in $\mu$ implies that $g \in L^2(\mathbb{R}^N)$.

Below, and in all what follows, we will say that $\phi \in L^2(\mathbb{R}^N)$ solves equation (2.7) if and only if $\phi$ solves the linear problem

$$\phi = T_m((m - W)\phi + g).$$

Similarly, we will say that

$$(-\Delta)^s\phi + W(x)\phi \geq g \quad \text{in } \mathbb{R}^N$$

if for some $\tilde g \in L^2(\mathbb{R}^N)$ with $\tilde g \geq g$ we have

$$\phi = T_m((m - W)\phi + \tilde g).$$

The next lemma provides an a priori estimate for a solution $\phi \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ of (2.7).

**Lemma 2.5.** *Let* $W$ *be a continuous function, such that for* $k$ *points* $q_i$ $i = 1, \dots, k$ *a number* $R > 0$ *and* $B = \cup_{i=1}^k B_R(q_i)$ *we have*

$$\inf_{x\in\mathbb{R}^N\setminus B} W(x) =: m > 0.$$

*Then, given any number* $\frac{N}{2} < \mu < N+2s$ *there exists a constant* $C = C(\mu, k, R) > 0$ *such that for any* $\phi \in H^{2s} \cap L^\infty(\mathbb{R}^N)$ *and* $g$ *with*

$$\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)} < +\infty$$

*that satisfy equation (2.7) we have the validity of the estimate*

$$\|\rho^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \;\leq\; C\,\left[\,\|\phi\|_{L^\infty(B)} + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}\,\right].$$

*Here*

$$\rho(x) = \sum_{i=1}^k \frac{1}{(1 + |x - q_i|)^\mu}.$$

*Proof.* We start by noticing that $\phi$ satisfies the equation

$$(-\Delta)^s\phi + \hat{W}\phi = \hat{g}$$

where

$$\hat{g} = (m - W)\chi_B\,\phi, \quad \hat{W} = m\chi_B + W(1-\chi_B).$$

Observe that

$$|\hat{g}(x)| \le M\sum_{i=1}^{k}(1+|x-q_i|)^{-\mu}, \quad M = C(\|\phi\|_{L^\infty(B)} + \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)})$$

where $C$ depends only on $R$, $k$ and $\mu$ and

$$\inf_{x\in\mathbb{R}^N}\hat{W}(x) \ge m.$$

Now, from Lemma 2.2, since $0<\mu<N+2s$ we find a solution $\phi_0(x)$ to the problem

$$(-\Delta)^s\bar{\phi} + m\bar{\phi} = (1+|x|)^{-\mu}$$

such that $\bar{\phi} = O(|x|^{-\mu})$ as $|x|\to\infty$. Then we have that

$$((-\Delta)^s + \hat{W})(\bar{\phi}) \ge M\sum_{i=1}^{k}(1+|x-q_i|)^{-\mu}$$

where

$$\bar{\phi}(x) = M\sum_{i=1}^{k}\phi_0(x-q_i).$$

Setting $\psi = (\phi - \bar{\phi})$ we get

$$(-\Delta)^s\psi + \hat{W}\psi = \tilde{g} \le 0$$

with $\tilde{g}\in L^2$. Using Lemma 2.4 we obtain $\phi\le\bar{\phi}$. Arguing similarly for $-\phi$, and using the form of $\bar{\phi}$ and $M$, the desired estimate immediately follows. □

Examining the proof above, we obtain immediately the following.

**Corollary 2.1.** *Let $\rho(x)$ be defined as in the previous lemma. Assume that $\phi\in H^{2s}(\mathbb{R}^N)$ satisfies equation (2.7) and that*

$$\inf_{x\in\mathbb{R}^N}W(x) =: m > 0.$$

*Then we have that $\phi\in L^\infty(\mathbb{R}^N)$ and it satisfies*

$$\|\rho^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \;\le\; C\,\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}\,. \tag{2.8}$$

A last useful fact is that if $f,g\in L^2(\mathbb{R}^N)$ and $W=T(f)$, $Z=T(g)$ then the following holds:

$$\int_{\mathbb{R}^N}Z(-\Delta)^sW - \int_{\mathbb{R}^N}W(-\Delta)^sZ = \int_{\mathbb{R}^N}T_m[f]g - \int_{\mathbb{R}^N}T_m[g]f = 0,$$

the latter fact since the kernel $k$ is radially symmetric.

## 3. Formulation of the problem: the ansatz

By a solution of the problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u^p = 0 \quad \text{in } \mathbb{R}^N$$

we mean a $u \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that the above equation is satisfied. Let us observe that it suffices to solve

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u - u_+^p = 0 \quad \text{in } \mathbb{R}^N \tag{3.1}$$

where $u_+ = \max\{u, 0\}$. In fact, if $u$ solves (3.1) then

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u \geq 0 \quad \text{in } \mathbb{R}^N$$

and, as a consequence to Lemma 2.4, $u \geq 0$.

After absorbing $\varepsilon$ by scaling, the equation takes the form

$$(-\Delta)^s v + V(\varepsilon x)v - v_+^p = 0 \quad \text{in } \mathbb{R}^N \tag{3.2}$$

Let us consider points $\xi_1, \ldots, \xi_k \in \mathbb{R}^N$ and designate

$$q_i = \varepsilon^{-1}\xi_i, \quad q = (q_1, \ldots, q_k).$$

Given numbers $\delta > 0$ small and $R > 0$ large, we define the configuration space $\Gamma$ for the points $q_i$ as

$$\Gamma := \{q = (q_1, \ldots, q_k) \ / \ R \leq \max_{i \neq j} |q_i - q_j|, \quad \max_i |q_i| \leq \delta^{-1}\varepsilon^{-1}\}. \tag{3.3}$$

We look for a solution with concentration behavior near each $\xi_j$. Letting $\tilde{v}(x) = v(x + \xi_j)$ translating the origin to $q_j$, Equation (3.2) reads

$$(-\Delta)^s \tilde{v} + V(\xi_j + \varepsilon x)\tilde{v} - \tilde{v}_+^p = 0 \quad \text{in } \mathbb{R}^N.$$

Letting formally $\varepsilon \to 0$ we are left with the equation

$$(-\Delta)^s \tilde{v} + \lambda_j \tilde{v} - \tilde{v}_+^p = 0 \quad \text{in } \mathbb{R}^N, \quad \lambda_j = V(\xi_j).$$

So we ask that $v(x) \approx w_{\lambda_j}(x - q_j)$ near $q_j$. We consider the sum of these functions as a first approximation. Thus, we look for a solution $v$ of (3.2) of the form

$$v = W_q + \phi, \quad W_q(x) = \sum_{j=1}^k w_j(x), \quad w_j(x) = w_{\lambda_j}(x - q_j), \quad \lambda_j = V(\xi_j),$$

where $\phi$ is a small function, disappearing as $\varepsilon \to 0$. In terms of $\phi$, Equation (3.2) becomes

$$(-\Delta)^s \phi + V(\varepsilon x)\phi - pW_q^{p-1}\phi = E + N(\phi) \quad \text{in } \mathbb{R}^N \tag{3.4}$$

where

$$N(\phi) := (W_q + \phi)_+^p - pW_q^{p-1}\phi - W_q^p,$$

$$E := \sum_{j=1}^k (\lambda_j - V(\varepsilon x))w_j + \Big(\sum_{j=1}^k w_j\Big)^p - \sum_{j=1}^k w_j^p. \tag{3.5}$$

Rather than solving Problem (3.4) directly, we consider first a projected version of it. Let us consider the functions

$$Z_{ij}(x) := \partial_j w_i(x)$$

and the problem of finding $\phi \in H^{2s}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that for certain constants $c_{ij}$

$$(-\Delta)^s\phi + V(\varepsilon x)\phi - pW_q^{p-1}\phi = E + N(\phi) + \sum_{i=1}^{k}\sum_{j=1}^{N} c_{ij}Z_{ij}, \tag{3.6}$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all} \quad i,j. \tag{3.7}$$

Let $\mathcal{Z}$ be the linear space spanned by the functions $Z_{ij}$, so that equation (3.6) is equivalent to

$$(-\Delta)^s\phi + V(\varepsilon x)\phi - pW_q^{p-1}\phi - E - N(\phi) \in \mathcal{Z}.$$

On the other hand, for all $\varepsilon$ sufficiently small, the functions $Z_{ij}$ are linearly independent, hence the constants $c_{ij}$ have unique, computable expressions in terms of $\phi$. We will prove that Problem (3.6)-(3.7) has a unique small solution $\phi = \Phi(q)$. In that way we will get a solution to the full problem (3.4) if we can find a value of $q$ such that $c_{ij}(\Phi(q)) = 0$ for all $i, j$. In order to build $\Phi(q)$ we need a theory of solvability for associated linear operator in suitable spaces. This is what we develop in the next section.

## 4. Linear theory

We consider the linear problem of finding $\phi \in H^{2s}(\mathbb{R}^N)$ such that for certain constants $c_{ij}$ we have

$$(-\Delta)^s\phi + V(\varepsilon x)\phi - pW_q^{p-1}(x)\phi + g(x) = \sum_{i=1}^{N}\sum_{i=1}^{k} c_{ij}Z_{ij} \tag{4.1}$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all} \quad i,j. \tag{4.2}$$

The constants $c_{ij}$ are uniquely determined in terms of $\phi$ and $g$ when $\varepsilon$ is sufficiently small, from the linear system

$$\sum_{i,j} c_{ij}\int_{\mathbb{R}^N} Z_{ij}Z_{lk} = \int_{\mathbb{R}^N} Z_{lk}[(-\Delta)^s\phi + V(\varepsilon x)\phi - pW_q^{p-1}(x)\phi + g]. \tag{4.3}$$

Taking into account that

$$\int_{\mathbb{R}^N} Z_{lk}(-\Delta)^s\phi = \int_{\mathbb{R}^N} \phi(-\Delta)^s Z_{lk} = \int_{\mathbb{R}^N} (pw_l^{p-1} - \lambda_l)Z_{lk}\phi,$$

we find

$$c_{ij}\int_{\mathbb{R}^N} Z_{ij}Z_{lk} = \int_{\mathbb{R}^N} gZ_{lk} + (pw_l^{p-1} - pW_q^{p-1} + V(\varepsilon x) - \lambda_l)\, Z_{lk}\phi. \tag{4.4}$$

On the other hand, we check that

$$\int_{\mathbb{R}^N} Z_{ij}Z_{lk} = \alpha_l\delta_{ijkl} + O(d^{-N})$$

where the numbers $\alpha_l$ are positive, and independent of $\varepsilon$, and

$$d = \min\{|q_i - q_j| \ / \ i \neq j\} \gg 1.$$

Then, we see that relations (4.4) define a uniquely solvable (nearly diagonal) linear system, provided that $\varepsilon$ is sufficiently small. We assume this last fact in what follows, and hence that the numbers $c_{ij} = c_{ij}(\phi, g)$ are defined by relations (4.4).

Moreover, we have that

$$|(pw_l^{p-1} - pW_q^{p-1} + V(\varepsilon x) - \lambda_l)\, Z_{lk}(x)| \,\le\, C\,(R^{-N} + \varepsilon|x-q_j|)(1+|x-q_j|)^{-N-s})$$

and then from expression (4.4) we obtain the following estimate.

**Lemma 4.1.** *The numbers $c_{ij}$ in* (4.1) *satisfy:*

$$c_{ij} = \frac{1}{\alpha_i}\int_{\mathbb{R}^N} gZ_{ij} + \theta_{ij}.$$

*where*

$$|\theta_{ij}| \le C(\varepsilon + d^{-N})\,\left[\|\phi\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^2(\mathbb{R}^N)}\right].$$

In the rest of this section we shall build a solution to Problem (4.1)-(4.2).

**Proposition 4.1.** *Given $k \ge 1$, $\frac{N}{2} < \mu < N+2s$, $C > 0$, there exist positive numbers $d_0$, $\varepsilon_0$, $C$ such that for any points $q_1, \dots q_k$ and any $\varepsilon$ with*

$$\sum_{i=1}^k |q_i| \le \frac{C}{\varepsilon}, \quad R := \min\{|q_i - q_j| \ / \ i \ne j\} > R_0, \quad 0 < \varepsilon < \varepsilon_0$$

*there exists a solution $\phi = T[g]$ of (4.1)-(4.2) that defines a linear operator of $g$, provided that*

$$\|\rho(x)^{-1}g\|_{L^\infty(\mathbb{R}^N)} < +\infty, \quad \rho(x) = \sum_{j=1}^k \frac{1}{(1+|x-q_j|)^\mu}.$$

*Besides*

$$\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \le C\|\rho(x)^{-1}g\|_{L^\infty(\mathbb{R}^N)}.$$

To prove this result we require several steps. We begin with corresponding a priori estimates.

**Lemma 4.2.** *Under the conditions of Proposition 4.1, there exists a $C > 0$ such that for any solution of (4.1)-(4.2) with $\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} < +\infty$ we have the validity of the a priori estimate*

$$\|\rho(x)^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} \le C\|\rho(x)^{-1}g\|_{L^\infty(\mathbb{R}^N)}.$$

*Proof.* Let us assume the a priori estimate does not hold, namely there are sequences $\varepsilon_n \to 0$, $q_{jn}$, $j = 1, \dots k$, with

$$\min\{|q_{in} - q_{jn}| \ / \ i \ne j\} \to \infty$$

and $\phi_n$, $g_n$ with

$$\|\rho_n(x)^{-1}\phi_n\|_{L^\infty(\mathbb{R}^N)} = 1, \quad \|\rho_n(x)^{-1}g_n\|_{L^\infty(\mathbb{R}^N)} \to 0,$$

where

$$\rho_n(x) = \sum_{j=1}^k \frac{1}{(1+|x-q_{jn}|)^\mu},$$

with $\phi_n, g_n$ satisfying (4.1)-(4.2). We claim that for any fixed $R > 0$ we have that

$$\sum_{j=1}^k \|\phi_n\|_{L^\infty(B_R(q_{jn}))} \to 0. \tag{4.5}$$

Indeed, assume that for a fixed $j$ we have that $\|\phi_n\|_{L^\infty(B_R(q_{jn}))} \ge \gamma > 0$. Let us set $\bar\phi_n(x) = \phi_n(q_{jn} + x)$. We also assume that $\lambda_j^n = V(q_{jn}) \to \bar\lambda > 0$ and

$$(-\Delta)^s \bar\phi_n + V(q_{jn} + \varepsilon_n x)\bar\phi_n + p(w_{\lambda_j^n}(x) + \theta_n(x))^{p-1}\bar\phi_n = \bar g_n$$

where

$$\bar g_n(x) = g_n(q_{jn} + x) - \sum_{l=1}^{k}\sum_{i=1}^{n} c_{ln}^i \partial_i w_{\lambda_l^n}(q_{jn} - q'_{ln} + x).$$

We observe that $\bar g_n(x) \to 0$ uniformly on compact sets. From the uniform Hölder estimates (2.6), we also obtain equicontinuity of the sequence $\bar\phi_n$. Thus, passing to a subsequence, we may assume that $\bar\phi_n$ converges, uniformly on compact sets, to a bounded function $\bar\phi$ which satisfies $\|\bar\phi\|_{L^\infty(B_R(0))} \ge \gamma$. In addition, we have that

$$\|(1 + |x|)^\mu \bar\phi\|_{L^\infty(\mathbb{R}^N)} \le 1$$

and that $\bar\phi$ solves the equation

$$(-\Delta)^s \bar\phi + \bar\lambda\bar\phi + pw_{\bar\lambda}^{p-1}\bar\phi = 0$$

Let us notice that $\bar\phi \in L^2(\mathbb{R}^N)$, and hence the nondegeneracy result in [15] applies to yield that $\bar\phi$ must be a linear combination of the partial derivatives $\partial_i w_{\bar\lambda}$. But the orthogonality conditions pass to the limit, and yield

$$\int_{\mathbb{R}^N} \partial_i w_{\bar\lambda}\bar\phi = 0 \quad \text{for all} \quad i = 1, \dots, N.$$

Thus, necessarily $\bar\phi = 0$. We have obtained a contradiction that proves the validity of (4.5). This and the a priori estimate in Lemma 2.5 shows that also, $\|\rho_n(x)^{-1}\phi_n\|_{L^\infty(\mathbb{R}^N)} \to 0$, again a contradiction that proves the desired result. □

Next we construct a solution to problem (4.1)-(4.2). To do so, we consider first the auxiliary problem

$$(-\Delta)^s\phi + V\phi = g + \sum_{i=1}^{k}\sum_{j=1}^{N} c_{ij} Z_{ij}, \tag{4.6}$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all} \quad i, j. \tag{4.7}$$

where $V$ is our bounded, continuous potential with

$$\inf_{\mathbb{R}^N} V = m > 0$$

**Lemma 4.3.** *For each $g$ with $\|\rho^{-1}g\|_\infty < +\infty$, there exists a unique solution of Problem (4.1)-(4.2), $\phi =: A[g] \in H^{2s}(\mathbb{R}^N)$. This solution satisfies*

$$\|\rho^{-1}A[g]\|_{L^\infty(\mathbb{R}^N)} \le C\|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}. \tag{4.8}$$

*Proof.* First we write a variational formulation for this problem. Let $X$ be the closed subspace of $H$ defined as

$$X = \{\tilde\phi \in H \ / \ \int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all} \quad i, j\}$$

Then, given $g \in L^2$, we consider the problem of finding a $\tilde\phi \in X$ such that

$$\langle\tilde\phi, \tilde\psi\rangle := \iint_{\mathbb{R}^{N+1}_+} \nabla\tilde\phi\nabla\tilde\psi y^{1-2s} + \int_{\mathbb{R}^N} V\phi\psi = \int_{\mathbb{R}^N} g\psi \quad \text{for all} \quad \psi \in X. \tag{4.9}$$

We observe that $\langle\cdot,\cdot\rangle$ defines an inner product in $X$ equivalent to that of $H$. Thus existence and uniqueness of a solution follows from Riesz's theorem. Moreover, we see that

$$\|\phi\|_{L^2(\mathbb{R}^N)} \le C\|g\|_{L^2(\mathbb{R}^N)}.$$

Next we check that this produces a solution in strong sense. Let $\mathcal{Z}$ be the space spanned by the functions $Z_{ij}$. We denote by $\Pi[g]$ the $L^2(\mathbb{R}^N)$ orthogonal projection of $g$ onto $\mathcal{Z}$ and by $\tilde\Pi[g]$ its natural $s$-harmonic extension. For a function $\tilde\varphi \in H$ let us write

$$\tilde\psi = \tilde\varphi - \tilde\Pi[\varphi]$$

so that $\tilde\psi \in X$. Substituting this $\tilde\psi$ into (4.9) we obtain

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\tilde\phi\nabla\tilde\varphi y^{1-2s} + \int_{\mathbb{R}^N} V\phi\varphi =$$

$$\int_{\mathbb{R}^N} g\varphi + \int_{\mathbb{R}^N} [V\phi - g]\,\Pi[\varphi] + \int_{\mathbb{R}^N} \phi(-\Delta)^s\Pi[\varphi].$$

Here we have used that $\tilde\Pi[\varphi]$ is regular and

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\phi\nabla\tilde\Pi[\varphi]y^{1-2s} = \int_{\mathbb{R}^N} \phi(-\Delta)^s\Pi[\varphi].$$

Let us observe that for $f \in L^2(\mathbb{R}^N)$ the functional

$$\ell(f) = \int_{\mathbb{R}^N} \phi(-\Delta)^s\Pi[f]$$

satisfies

$$|\ell(f)| \le C\|\phi\|_{L^2(\mathbb{R}^N)}\|\psi\|_{L^2(\mathbb{R}^N)},$$

hence there is an $h(\phi) \in L^2(\mathbb{R}^N)$ such that

$$\ell(\psi) = \int_{\mathbb{R}^N} h\psi.$$

If $\phi$ was a priori known to be in $H^{2s}(\mathbb{R}^N)$ we would have precisely that

$$h(\phi) = \Pi[(-\Delta)^s\phi].$$

Since $\Pi$ is a self-adjoint operator in $L^2(\mathbb{R}^N)$ we then find that

$$\iint_{\mathbb{R}^{N+1}_+} \nabla\tilde\phi\nabla\tilde\varphi y^{1-2s} + \int_{\mathbb{R}^N} V\phi\varphi = \int_{\mathbb{R}^N} \bar g\varphi$$

where

$$\bar g = g + \Pi[V\phi - g] + h(\phi).$$

Since $\bar g \in L^2(\mathbb{R}^N)$, it follows then that $\phi \in H^{2s}(\mathbb{R}^N)$ and it satisfies

$$(-\Delta)^s\phi + V\phi - g = \Pi[(-\Delta)^s\phi + V\phi - g] \in \mathcal{Z},$$

hence equations (4.6)-(4.7) are satisfied. To establish estimate (4.8), we use just Corollary 2.1, observing that

$$\|\rho^{-1}\Pi[(-\Delta)^s\phi + V\phi - g]\|_{L^\infty(\mathbb{R}^N)} \le C(\|\phi\|_{L^2(\mathbb{R}^N)} + \|g\|_{L^2(\mathbb{R}^N)}) \le$$

$$C\|g\|_{L^2(\mathbb{R}^N)} \le \|\rho^{-1}g\|_{L^\infty(\mathbb{R}^N)}.$$

The proof is concluded. □

**Proof of Proposition 4.1.** Let us solve now Problem (4.1)-(4.2). Let $Y$ be the Banach space

$$Y := \{\phi \in C(\mathbb{R}^N) \ / \ \|\phi\|_Y := \|\rho^{-1}\phi\|_{L^\infty(\mathbb{R}^N)} < +\infty\} \tag{4.10}$$

Let $A$ be the operator defined in Lemma 4.3. Then we have a solution to Problem (4.1)-(4.2) if we solve

$$\phi - A[pW_q^{p-1}\phi] = A[g], \quad \phi \in Y. \tag{4.11}$$

We claim that

$$B[\phi] := A[pW_q^{p-1}\phi]$$

defines a compact operator in $Y$. Indeed. Let us assume that $\phi_n$ is a bounded sequence in $Y$. We observe that for some $\sigma > 0$ we have

$$|W_q^{p-1}\phi_n| \le C\|\phi_n\|_Y \, \rho^{1+\sigma}.$$

If $\sigma$ is sufficiently small, it follows that $f_n := B[\phi_n]$ satisfies

$$|\rho^{-1} f_n| \le C\rho^\sigma$$

Besides, since $f_n = T_m((V-m)f_n + g_n)$ we use estimate (2.6) to get that for some $\alpha > 0$

$$\sup_{x\neq y} \frac{|f_n(x) - f_n(y)|}{|x-y|^\alpha} \le C.$$

Arzela's theorem then yields the existence of a subsequence of $f_n$ which we label the same way, that converges uniformly on compact sets to a continuous function $f$ with

$$|\rho^{-1} f| \le C\rho^\sigma.$$

Let $R > 0$ be a large number . Then we estimate

$$\|\rho^{-1}(f_n - f)\|_{L^\infty(\mathbb{R}^N)} \le \|\rho^{-1}(f_n - f)\|_{L^\infty(B_R(0))} + C \max_{|x|>R} \rho^\sigma(x).$$

Since

$$\max_{|x|>R} \rho^\sigma(x) \to 0 \quad \text{ as } R \to \infty$$

we conclude then that $\|f_n - f\|_\infty \to 0$ and the claim is proven.

Finally, the a priori estimate tells us that for $g = 0$, equation (4.11) has only the trivial solution. The desired result follows at once from Fredholm's alternative. □

We conclude this section by analyzing the differentiability with respect to the parameter $q$ of the solution $\phi = T_q[g]$ of (4.1)-(4.2). As in the proof above we let $Y$ be the space in (4.10), so that $T_q \in \mathcal{L}(Y)$

**Lemma 4.4.** *The map $q \mapsto T_q$ is continuously differentiable, and for some $C > 0$,*

$$\|\partial_q T_q\|_{\mathcal{L}(Y)} \le C \tag{4.12}$$

*for all $q$ satisfying constraints (3.3).*

*Proof.* Let us write $q = (q_1, \dots, q_k)$, $q_i = (q_{i1}, \dots, q_{iN})$, $\phi = T_q[g]$, and (formally)

$$\psi = \partial_{q_{ij}} T_q[g], \quad d_{lk} = \partial_{q_{ij}} c_{lk}.$$

Then, by differentiation of equations (4.1)-(4.2), we get

$$(-\Delta)^s\psi + V(\varepsilon x)\psi - pW_q^{p-1}\psi = p\partial_{q_{ij}} W_q^{p-1}\phi + \sum_{l,k} c_{lk}\partial_{q_{ij}} Z_{lk} + \sum_{l,k} d_{lk}\, Z_{lk}, \tag{4.13}$$

$$\int_{\mathbb{R}^N} \psi Z_{lk} = -\int_{\mathbb{R}^N} \phi \partial_{q_{ij}} Z_{lk} \quad \text{for all} \quad l,k. \tag{4.14}$$

We let

$$\tilde\psi = \psi - \Pi[\psi]$$

where, as before, $\Pi[\psi]$ denotes the orthogonal projection of $\psi$ onto the space spanned by the $Z_{lk}$. Writing

$$\Pi[\psi] = \sum_{l,k} \alpha_{lk} Z_{lk} \tag{4.15}$$

and relations (4.14) as

$$\int_{\mathbb{R}^N} \Pi[\psi] Z_{lk} = -\int_{\mathbb{R}^N} \phi \partial_{q_{ij}} Z_{lk} \quad \text{for all} \quad l,k, \tag{4.16}$$

we get

$$|\alpha_{lk}| \le C\|\phi\|_Y \le C\|g\|_Y. \tag{4.17}$$

From (4.13) we have then that

$$(-\Delta)^s \tilde\psi + V(\varepsilon x)\tilde\psi - pW_q^{p-1}\tilde\psi = \tilde g + \sum_{l,k} d_{lk}\, Z_{lk}, \tag{4.18}$$

or $\tilde\psi = T_q[\tilde g]$ where

$$\tilde g = p\partial_{q_{ij}} W_q^{p-1}\phi + \sum_{l,k} c_{lk}\partial_{q_{ij}} Z_{lk} - [(-\Delta)^s + V(\varepsilon x) - pW_q^{p-1}]\,\Pi[\psi]. \tag{4.19}$$

Then we see that

$$\|\tilde\psi\|_Y \le C\|\tilde g\|_Y.$$

Using (4.17) and Lemma 4.1, we see also that

$$\|\tilde g\|_Y \le C\|g\|_Y, \quad \|\Pi[\psi]\| \le C\|g\|_Y$$

and thus

$$\|\psi\| \le C\|g\|_Y. \tag{4.20}$$

Let us consider now, rigorously, the unique $\psi = \tilde\psi + \Pi[\psi]$ that satisfies equations (4.14) and (4.19). We want to show that indeed

$$\psi = \partial_{q_{ij}} T_q[g].$$

To do so, $q_i^t = q_i + te_j$ where $e_j$ is the $j$-th element of the canonical basis of $\mathbb{R}^N$, and set

$$q^t = (q_1, \dots q_{i-1}, q_i^t, \dots, q_k).$$

For a function $f(q)$ we denote

$$D_{ij}^t f = t^{-1}(f(q^t) - f(q))$$

we also set

$$\phi^t := T_{q^t}[g], \quad D_{ij}^t T_q[g] =: \psi^t = \tilde\psi^t + \Pi[\tilde\psi^t]$$

so that

$$(-\Delta)^s \tilde\psi^t + V(\varepsilon x)\tilde\psi^t - pW_q^{p-1}\tilde\psi^t = \tilde g^t + \sum_{l,k} d_{lk}^t\, Z_{lk},$$

where

$$\tilde g^t = pD_{ij}^t[W_q^{p-1}]\phi + \sum_{l,k} c_{lk} D_{ij}^t Z_{lk} - [(-\Delta)^s + V(\varepsilon x) - pW_q^{p-1}]\,\Pi[\psi^t],$$

$$d_{lk}^t = D_{ij}^t c_{lk}$$

and

$$\Pi[\psi^t] = \sum_{l,k} \alpha_{lk}^t Z_{lk},$$

where the constants $\alpha_{lk}^t$ are determined by the relations

$$\int_{\mathbb{R}^N} \Pi[\psi^t] Z_{lk} = -\int_{\mathbb{R}^N} \phi D_{ij}^t Z_{lk}.$$

Comparing these relations with (4.15), (4.16), (4.18) defining $\psi$, we obtain that

$$\lim_{t\to 0} \|\psi^t - \psi\|_Y = 0$$

which by definition tells us $\psi = \partial_{q_{ij}} T_q[g]$. The continuous dependence in $q$ is clear from that of the data in the definition of $\psi$. Estimate (4.12) follows from (4.20). The proof is concluded. □

## 5. Solving the nonlinear projected problem

In this section we solve the nonlinear projected problem

$$(-\Delta)^s \phi + V(\varepsilon x)\phi - pW_q^{p-1}\phi = E + N(\phi) + \sum_{i=1}^{k}\sum_{j=1}^{N} c_{ij} Z_{ij}, \tag{5.1}$$

$$\int_{\mathbb{R}^N} \phi Z_{ij} = 0 \quad \text{for all} \quad i, j. \tag{5.2}$$

We have the following result.

**Proposition 5.1.** *Assuming that* $\|E\|_Y$ *is sufficiently small problem* (5.1)-(5.2) *has a unique small solution* $\phi = \Phi(q)$ *with*

$$\|\Phi(q)\|_Y \le C\|E\|_Y$$

*The map* $q \mapsto \Phi(q)$ *is of class* $C^1$*, and for some* $C > 0$

$$\|\partial_q \Phi(q)\|_Y \ \le\ C[\|E\|_Y + \|\partial_q E\|_Y]. \tag{5.3}$$

*for all* $q$ *satisfying constraints* (3.3).

*Proof.* Problem (5.1)-(5.2) can be written as the fixed point problem

$$\phi = T_q(E + N(\phi)) =: K_q(\phi), \quad \phi \in Y. \tag{5.4}$$

Let

$$B = \{\phi \in Y \ / \ \|\phi\|_Y \le \rho\}..$$

If $\phi \in B$ we have that

$$|N(\phi)| \le C|\phi|^\beta, \quad \beta = \min\{p, 2\}.$$

and hence

$$\|N(\phi)\|_Y \le C\|\phi\|^2$$

It follows that

$$\|K_q(\phi)\|_Y \le C_0[\|E\| + \rho^2]$$

for a number $C_0$, uniform in $q$ satisfying (3.3). Let us assume

$$\rho := 2C_0\|E\|, \quad \|E\| \le \frac{1}{2C_0}.$$

Then

$$\|K_q(\phi)\|_Y \le C_0[\frac{1}{2C_0}\rho + \rho^2] \le \rho$$

so that $K_q(B) \subset B$. Now, we observe that

$$|N(\phi_1) - N(\phi_2)| \le C[|\phi|^{\beta-1} + |\phi|^{\beta-1}]|\phi_1 - \phi_2|$$

and hence

$$\|N(\phi_1) - N(\phi_2)\|_Y \le C\rho^{\beta-1}\|\phi_1 - \phi_2\|_Y$$

and

$$\|K_q(\phi_1) - K_q(\phi_2)\| \le C\rho^{\beta-1}\|\phi_1 - \phi_2\|_Y.$$

Reducing $\rho$ if necessary, we obtain that $K_q$ is a contraction mapping and hence has a unique solution of equation (5.4) exists in $B$. We denote it as $\phi = \Phi(q)$. We prove next that $\Phi$ defines a $C^1$ function of $q$. Let

$$M(\phi, q) := \phi - T_q(E + N(\phi))$$

Let $\phi_0 = \Phi(q_0)$. Then $M(\phi_0, q_0) = 0$. On the other hand,

$$\partial_\phi M(\phi, q)[\psi] = \psi - T_q(N'(\phi)\psi)$$

where $N'(\phi) = p[(W + \phi)^{p-1} - W^{p-1}]$, so that

$$\|N'(\phi)\psi\|_Y \le C\rho^{\beta-1}\|\psi\|_Y$$

If $\rho$ is sufficiently small we have then that $D_\phi M(\phi_0, q_0)$ is an invertible operator, with uniformly bounded inverse. Besides

$$\partial_q M(\phi, q) = (\partial_q T_q)(E + N(\phi)) + T_q(\partial_q E + \partial_q N(\phi))$$

Both partial derivatives are continuous in their arguments. The implicit function applies in a small neighborhood of $(\phi_0, q_0)$ to yield existence and uniqueness of a function $\phi = \phi(q)$ with $\phi(q_0) = \phi_0$ defined near $q_0$ with $M(\phi(q), q) = 0$. Besides, $\phi(q)$ is of class $C^1$. But, by uniqueness, we must have $\phi(q) = \Phi(q)$. Finally, we see that

$$\partial_q \Phi(q) = -D_\phi M(\Phi(q), q)^{-1}\left[(\partial_q T_q)(E + N(\Phi(q))) + T_q(\partial_q E + \partial_q N(\Phi(q)))\right]$$

$$\partial_q N(\phi) = p[(W + \phi)^{p-1} - pW^{p-1} - (p-1)W^{p-2}\phi]\partial_q W$$

and hence

$$\|(\partial_q N)(\Phi(q))\|_Y \le C\|\Phi(q)\|_Y^\beta \le C\|E\|_Y^\beta$$

From here, the above expressions and the bound of Lemma 4.4 we finally get the validity of Estimate (5.3). □

5.1. **An estimate of the error.** Here we provide an estimate of the error $E$ defined in (3.5),

$$E := \sum_{j=1}^k (\lambda_j - V(\varepsilon x))w_j + \Big(\sum_{j=1}^k w_j\Big)^p - \sum_{j=1}^k w_j^p$$

in the norm $\|\cdot\|_Y$. Here we need to take $\mu \in (\frac{N}{2}, \frac{N+2s}{2})$. We denote

$$R = \min_{i \ne j} |q_i - q_j| >> 1.$$

The first term in $E$ can be easily estimated as

$$|\rho^{-1}(x)\sum_{j=1}^k(\lambda_j - V(\varepsilon x))w_j| \le C\varepsilon^{\min(2s,1)}.$$

To estimate the interaction term in $E$, we divide the $\mathbb{R}^N$ into the $k$ sub-domains

$$\Omega_j = \{w_j \geq w_i, \forall i \neq j\}, \quad j = 1, \cdots, k.$$

In $\Omega_j$, we have

$$\begin{aligned} |(\sum_{j=1}^{k} w_j)^p - \sum_{j=1}^{k} w_j^p| &\leq Cw_j^{p-1} \sum_{i\neq j} \frac{1}{|x-q_i|^{N+2s}} \\ &\leq C\frac{1}{(1+|x-q_j|)^{(N+2s)(p-1)+\mu}} \sum_{i\neq j} \frac{1}{|q_j-q_i|^{N+2s-\mu}} \\ &\leq C\rho(x)R^{\mu-N-2s} \end{aligned}$$

In summary, we conclude that

$$\|E\|_Y \leq C\varepsilon^{2s} + CR^{\mu-N-2s} \tag{5.5}$$

As a consequence of Proposition 5.1 and the estimate (5.5), we obtain that

$$\|\Phi(q)\|_Y \leq C\varepsilon^{\min(2s,1)} + CR^{\mu-N-2s}.$$

Let us now take

$$\tau = C\varepsilon^{\min(2s,1)} + CR^{\mu-N-2s}$$

## 6. The variational reduction

We will use the above introduced ingredients to find existence results for the equation

$$(-\Delta)^s v + V(\varepsilon x)v - v_+^p = 0 \tag{6.1}$$

An energy whose Euler-Lagrange equation corresponds formally to (6.1) is given by

$$J_\varepsilon(\tilde{v}) := \frac{1}{2}\int_{\mathbb{R}^N} v(-\Delta)^s v + V(\varepsilon x)v^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} V(\varepsilon x)v^2$$

We want to find a solution of (6.1) with the form

$$v = v_q := W_q + \Phi(q)$$

where $\Phi(q)$ is the function in Proposition 5.1. We observe that

$$(-\Delta)^s v_q + V(\varepsilon x)v_q - (v_q)_+^p = \sum_{i,j} c_{ij} Z_{ij} \tag{6.2}$$

hence what we need is to find points $q$ such that $c_{ij} = 0$ for all $i, j$. This problem can be formulated variationally as follows

**Lemma 6.1.** *Let us consider the function of points $q = (q_1, \ldots, q_k)$ given by*

$$I(q) := J_\varepsilon(W_q + \Phi(q)).$$

*where $W_q + \Phi(q)$ is the unique $s$-harmonic extension of $W_q + \Phi(q)$. Then in (6.2), we have $c_{ij} = 0$ for all $i, j$ if and only if*

$$\partial_q I(q) = 0.$$

*Proof.* Let us write $v_q = W_q + \phi(q)$. We observe that

$$\partial_{q_{ij}} I(q) = \int_{\mathbb{R}^{N+1}_+} \nabla \tilde v_q \nabla(\partial_{q_{ij}} \tilde v_q) y^{1-2s} + \int_{\mathbb{R}^N} V(\varepsilon x) v_q \partial_{q_{ij}} v_q - \int_{\mathbb{R}^N} (v_q)_+^{p-1} \partial_{q_{ij}} v_q =$$

$$\int_{\mathbb{R}^N} [(-\Delta)^s v_q + V(\varepsilon x) v_q - (v_q)_+^p] \partial_{q_{ij}} v_q = \sum_{k,l} c_{kl} \int_{\mathbb{R}^N} Z_{kl} \partial_{q_{ij}} v_q. \tag{6.3}$$

We observe that

$$\partial_{q_{ij}} v_q = -Z_{ij} + O(\varepsilon\rho) + \partial_{q_{ij}} \Phi(q)$$

Since, according to Proposition (5.1)

$$\|\partial_q \Phi(q)\|_Y = O(\|E\|_Y + \|\partial_q E\|_Y)$$

and this quantity gets smaller as the number $\delta$ in (3.3) is reduced, and the functions $Z_{kl}$ are linearly independent (in fact nearly orthogonal in $L^2$), it follows that the quantity in (6.3) equals zero for all $i,j$ if and only if $c_{ij} = 0$ for all $i,j$. The proof is concluded. □

Our task is therefore to find critical points of the functional $I(q)$. Useful to this end is to achieve expansions of the energy in special situations.

**Lemma 6.2.** *Assume that the numbers $\delta$ and $R$ in the definition of $\Gamma$ in* (3.3) *is taken so small that*

$$\|E\|_Y + \|\partial_q E\| \le \tau \ll 1.$$

*Then*

$$I_\varepsilon(q) = J_\varepsilon(W_q) + O(\tau^2)$$

*and*

$$\partial_q I_\varepsilon(q) = \partial_q J_\varepsilon(W_q) + O(\tau^2)$$

*uniformly on points $q$ in $\Gamma$.*

*Proof.* Let us estimate

$$I(q) = J_\varepsilon(v_q), \quad v_q = W_q + \Phi(q).$$

We have that

$$I(\xi) = \frac{1}{2}\int_{\mathbb{R}^N} v_q(-\Delta)^s v_q + V v_q^2 - \frac{1}{p+1}\int v_q^{p+1}$$

Thus we can expand

$$I(q) = J_\varepsilon(W_q) + \int_{\mathbb{R}^N} \Phi[(-\Delta)^s v_q + V v_q - v_q^p] + \frac{1}{2}\int_{\mathbb{R}^N} \Phi(-\Delta)^s\Phi + V\Phi^2$$
$$-\frac{1}{p+1}\int_{\mathbb{R}^N} [(W_q+\Phi)^{p+1} - W_q^{p+1} - (p+1)W_q^p\Phi]$$

Since, $\|E\|_Y \le \tau$ then $\|\Phi\|_Y = O(\tau)$, and from the equation satisfied by $\Phi$, also $\|(-\Delta)^s\Phi\|_Y = O(\tau)$. This implies

$$|\frac{1}{2}\int_{\mathbb{R}^N} \Phi(-\Delta)^s\Phi + V\Phi^2| \le C\int_{\mathbb{R}^N} \rho^{2\mu}\tau^2 \le C\tau^2$$

and

$$|\int_{\mathbb{R}^N} [(W_q+\Phi)^{p+1} - W_q^{p+1} - (p+1)W_q^p\Phi]| \le C\int_{\mathbb{R}^N} \rho^{2\mu}\tau^2 \le C\tau^2$$

Here we have used the fact that $\mu \in (\frac{N}{2}, \frac{N+2s}{2})$.

On the other hand the second term in the above expansion equals 0, since by definition

$$(-\Delta)^s v_q + V v_q - v_q^p \in \mathcal{Z}$$

and $\Phi$ is $L^2$-orthogonal to that space. We arrive to the conclusion that

$$I(q) = J_\varepsilon(W_q) + O(\tau^2)$$

uniformly for $q$ in a bounded set. By differentiation we also have that

$$\partial_q I(q) = \partial_q J_\varepsilon(W_q) + \int_{\mathbb{R}^N} \partial_q \Phi(-\Delta)^s\Phi + V\Phi\partial_q\Phi +$$

$$\int_{\mathbb{R}^N} [(W_q+\Phi)^p - W_q^p - pW_q^{p-1}\Phi]\partial_q W_q + [(W_q+\Phi)^p - W_q^p]\partial_q\Phi.$$

Since we also have $\|\partial_q\Phi\|_Y = O(\tau)$, then the second and third term above are of size $O(\varepsilon^2)$. Thus,

$$\partial_q I(q) = \partial_q J_\varepsilon(W_q) + O(\rho^2).$$

uniformly on $q \in \Gamma$ and the proof is complete. □

Next we estimate $J_\varepsilon(W_q)$ and $\partial_q J_\varepsilon(W_q)$. We begin with the simpler case $k=1$. Here it is always the case that

$$\|E\|_Y + \|\partial_q E\|_Y \le \tau.$$

Let us also set $\xi = \varepsilon q$ We have now that

$$W_q(x) = w_\lambda(x-q), \quad \lambda = V(\xi).$$

We compute

$$J_\varepsilon(W_q) = J^\lambda(w_\lambda) + \frac{1}{2}\int_{\mathbb{R}^N}(V(\xi+\varepsilon x) - V(\xi))w_\lambda^2(x)\,dx$$

where

$$J^\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N} v(-\Delta)^s v + \frac{\lambda}{2}\int_{\mathbb{R}^N} v^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} v^{p+1}.$$

We recall that

$$w_\lambda(x) := \lambda^{\frac{1}{p-1}} w(\lambda^{\frac{1}{2s}} x)$$

satisfies the equation

$$(-\Delta)^s w_\lambda + \lambda w_\lambda - w_\lambda^p = 0 \quad \text{in } \mathbb{R}^N.$$

where $w = w_1$ is the unique radial least energy solution of

$$(-\Delta)^s w + w - w^p = 0 \quad \text{in } \mathbb{R}^N.$$

Then, after a change of variables we find

$$J^\lambda(w_\lambda) = \frac{1}{2}\int_{\mathbb{R}^N} w_\lambda(-\Delta)^s w_\lambda + \frac{\lambda}{2}\int_{\mathbb{R}^N} w_\lambda^2 - \frac{1}{p+1}\int_{\mathbb{R}^N} w_\lambda^{p+1} = \lambda^{\frac{p+1}{p-1}-\frac{N}{2s}} J^1(w).$$

Now since $w$ is radial, we find

$$\int_{\mathbb{R}^N} x_i w_\lambda(x)dx = 0.$$

Thus,

$$\int_{\mathbb{R}^N}(V(\xi+\varepsilon x) - V(\xi))w_\lambda^2(x)\,dx = \nabla V(\xi)\cdot\int_{\mathbb{R}^N} x w_\lambda + O(\varepsilon^2) = O(\varepsilon^2)$$

On the other hand

$$\partial_q \int_{\mathbb{R}^N} (V(\xi+\varepsilon x) - V(\xi)) w_\lambda^2(x)\, dx =$$
$$\varepsilon \int_{\mathbb{R}^N} (\nabla V(\xi+\varepsilon x) - \nabla V(\xi)) w_\lambda^2(x)\, dx +$$
$$2\int_{\mathbb{R}^N} (V(\xi+\varepsilon x) - V(\xi)) w_\lambda \partial_q w_\lambda \, dx \ = \ O(\varepsilon^2).$$

**Lemma 6.3.** *Let $\theta = \frac{p+1}{p-1} - \frac{N}{2s}$, $c_* = J_1(w)$ and $k=1$. Then the following expansions hold:*

$$I(q) = c_* V^\theta(\xi) + O(\varepsilon^{\min(4s,2)})$$
$$\nabla_q I(q) = c_* \varepsilon \nabla_\xi (V^\theta)(\xi) + O(\varepsilon^{\min(4s,2)}).$$

For the case $k>1$ and $\min_{i\neq j} |q_i - q_j| \geq R >> 1$, we observe that, also, $\|E\|_Y = O(\tau)$ and hence we also have

$$I(q) = J_\varepsilon(W_q) + O(\tau^2), \quad \partial_q I(q) = \partial_q J_\varepsilon(W_q) + O(\tau^2).$$

By expanding $I(q)$ we get the validity of the following estimate.

**Lemma 6.4.** *Letting $\xi = \varepsilon q$ we have that*

$$I(q) = c_* \sum_{i=1}^k V^\theta(\xi_i) - \sum_{i\neq j} \frac{c_{ij}}{|q_i-q_j|^{N+2s}} + O(\varepsilon^{\min(4s,2)} + \frac{1}{R^{2(N+2s-\mu)}}),$$

$$\nabla_q I(q) = c_* \varepsilon \nabla_\xi \big[ \sum_{i=1}^k V^\theta(\xi_i) - \sum_{i\neq j} \frac{c_{ij}}{|q_i-q_j|^{N+2s}} \big] + O(\varepsilon^{\min(4s,2)} + \frac{1}{R^{2(N+2s-\mu)}})$$

*where $c_*$ and $c_{ij} = c_0(V(\xi_i))^\alpha (V(\xi_j))^\beta$ are positive constants.*

*Proof.* It suffices to expand $J_\varepsilon(W_q)$. We see that, denoting $w_i(x) := w_{\lambda_i}(x-q_i)$

$$\begin{aligned} J_\varepsilon(W_q) \ &= \ J_\varepsilon\Big(\sum_{i=1}^k w_i\Big) \ = \ \sum_{i=1}^k J_\varepsilon(w_i) \ + \\ & \frac{1}{2}\sum_{i\neq j} \int_{\mathbb{R}^N} w_i (-\Delta)^s w_j + \int_{\mathbb{R}^N} V(\varepsilon x) w_i w_j \\ & -\frac{1}{p+1}\int_{\mathbb{R}^N} \Big(\sum_{i=1}^k w_i\Big)^{p+1} - \sum_{i=1}^k w_i^{p+1}. \end{aligned} \tag{6.4}$$

We estimate for $i \neq j$

$$\begin{aligned} & \int_{\mathbb{R}^N} w_i (-\Delta)^s w_j + \int_{\mathbb{R}^N} V(\varepsilon x) w_i w_j \\ = \ & \int_{\mathbb{R}^N} w_i w_j^p + \int_{\mathbb{R}^N} (V(\varepsilon x) - \lambda_j) w_i w_j \\ = \ & (c_{ij} + o(1)) \frac{1}{|q_i - q_j|^{N+2s}} + O(\frac{\varepsilon^{2s}}{R^{N+2s-\mu}}) \end{aligned} \tag{6.5}$$

where $c_{ij} = c_0 (V(\xi_i))^\alpha ((V(\xi_j))^\beta)$ and $c_0, \alpha, \beta$ are constants depending on $p, s$ and $N$ only. Indeed,

$$w_i(x) = \lambda_i^{\frac{1}{p-1}} w(\lambda_i^{\frac{1}{2s}}(x-q_i))$$

and it is known that

$$w(x) = \frac{c_0}{|x|^{N+2s}}(1+o(1)) \quad \text{as } |x| \to \infty.$$

Then, we have

$$\int_{\mathbb{R}^N} w_j^p w_i \;=\; \lambda_i^{\frac{1}{p-1}-\frac{n+2s}{2s}} \lambda_j^{\frac{p}{p-1}-\frac{n}{2s}} \left(\int_{\mathbb{R}^N} w^p\right) \frac{c_0}{|q_i-q_j|^{N+2s}},$$

and hence

$$c_{ij} = c_0 \lambda_i^{\alpha} \lambda_j^{\beta}$$

where

$$\lambda_i = V(\xi_i), \quad \lambda_j = V(\xi_j), \quad \alpha = \frac{1}{p-1} - \frac{n+2s}{2s}, \quad \beta = \frac{p}{p-1} - \frac{n}{2s}.$$

To estimate the last term we note that

$$\begin{aligned}
\int_{\mathbb{R}^N} \Big( (\sum_{i=1}^k w_i)^{p+1} - \sum_{i=1}^k w_i^{p+1} \Big)^{p+1} &= \sum_{j=1}^k \int_{\Omega_j} \Big( (\sum_{i=1}^k w_i)^{p+1} - \sum_{i=1}^k w_i^{p+1} \Big)^{p+1} \\
&= \sum_{j=1}^k \sum_{\Omega_j} \Big( (p+1) w_j^p (\sum_{i\neq j} w_i) + O(w_j^{\min(p-1,1)} (\sum_{i\neq j} w_i)^2) \Big) \\
&= \sum_{j=1}^K \sum_{i\neq j} (p+1) \int_{\mathbb{R}^N} w_j^p w_i + O(\frac{1}{R^{2(N+2s-\mu)}}) \\
&= \sum_{j=1}^K \sum_{i\neq j} (p+1) \frac{c_{ij}+o(1)}{|q_i-q_j|^{N+2s}} + O(\frac{1}{R^{2(N+2s-\mu)}}) \qquad (6.6)
\end{aligned}$$

Substituting (6.5) and (6.6) into (6.4) and using the estimate of $J_\varepsilon(w_i)$ in the proof of Lemma 6.3, we have estimated $J_\varepsilon(w_i)$, and we have proven the lemma.

## 7. The Proofs of Theorems 1–3

Based on the asymptotic expansions in Lemma 6.4, we present the proofs of Theorems 1-3.

**Proof of Theorems 1 and 2.** Let us consider the situation in Remark 1.1, which is more general than that of Theorem 1. Then, in the definition of the configuration space $\Gamma$ (3.3), we can take a fixed $\delta$ and $R \sim \varepsilon^{-1}$ and achieve that $\Lambda \subset \varepsilon\Gamma$. Then we get

$$\|E\|_Y + \|\partial_q E\|_Y = O(\varepsilon^{\min\{2s,1\}}).$$

Letting

$$\tilde{I}(\xi) := I(\varepsilon q)$$

we need to find a critical point of $\tilde{I}$ inside $\Lambda$. By Lemma 6.4, we see then that

$$\tilde{I}(\xi) - c_* \varphi(\xi) = o(1), \quad \nabla_\xi \tilde{I}(\xi) - c_* \nabla_\xi \varphi(\xi) = o(1),$$

uniformly in $\xi \in \Lambda$ as $\varepsilon \to 0$, where $\varphi$ is the functional in (1.10). It follows, by the assumption on $\varphi$ that for all $\varepsilon$ sufficiently small there exists a $\xi^\varepsilon \in \Lambda$ such that $\nabla \tilde{I}(\xi^\varepsilon) = 0$, hence Lemma 6.1 applies and the desired result follows.

Theorem 2 follows in the same way. We just observe that because of the $C^1$-proximity, the same variational characterization of the numbers $c$, for the functional

$\tilde{I}(\xi)$ holds. This means that the critical value predicted in that form is indeed close to $c$. The proof is complete. $\square$

**Proof of Theorem 3.** Finally we prove Theorem 3. Following the argument in [18], we choose the following configuration space

$$\Lambda = \{(\xi_1, ..., \xi_k) \ / \ \xi_j \in \Gamma, \min_{i\neq j} |\xi_i - \xi_j| > \varepsilon^{1-\frac{s}{4}}\} \tag{7.1}$$

with $\Gamma$ given by (3.3), and we prove the following Claim and then Theorem 3 follows from Lemma 6.1:

**Claim:** letting $\xi = \varepsilon q$, the problem

$$\max_{(\xi_1,...,\xi_k)\in\Lambda} I(q) \tag{7.2}$$

admits a maximizer $(\xi_1^\varepsilon, ..., \xi_k^\varepsilon) \in \Lambda$.

We shall prove this by contradiction. First, by continuity of $I(q)$, there is a maximizer $\xi^\varepsilon = (\xi_1^\varepsilon, ..., \xi_k^\varepsilon) \in \bar{\Lambda}$. We need to prove that $\xi \in \Lambda$. Let us suppose, by contradiction, that $\xi^\varepsilon \notin \Lambda$, hence it lies on its boundary. Thus there are two possibilities: either there is an index $i$ such that $\xi_i^k \in \partial\Gamma$, or there exist indices $i \neq j$ such that

$$|\xi_i^\varepsilon - \xi_j^\varepsilon| = \min_{i\neq j} |\xi_i - \xi_j| = \epsilon^{1-s}.$$

Denoting $q^\varepsilon = \frac{\xi^\varepsilon}{\varepsilon}$, and using Lemma 6.4, we have in the first case that

$$I(q^\varepsilon) \le c_* V^\theta(\xi_i^\varepsilon) + c_* \sum_{j\neq i} V^\theta(\xi_j^\varepsilon) + C\varepsilon^{2s} \ \le$$

$$c_* k \max_\Gamma V^\theta(x) + c_*(\max_{\partial\Gamma} V^\theta(x) - \max_\Gamma V^\theta(x)) + C\varepsilon^{2s}. \tag{7.3}$$

In the second case, we invoke again Lemma 6.4 and obtain

$$I(q^\varepsilon) \le c_* k \max_\Gamma V^\theta(x) - c_2 \varepsilon^{\frac{s}{4}} + C\varepsilon^{2s}$$

for some $c_2 > 0$. On the other hand, we can get an upper bound for $I(q^\varepsilon)$ as follows. Let us choose a point $\xi_0$ such that $V(\xi_0) = \max_\Gamma V(x)$ and let

$$\xi_j = \xi_0 + \varepsilon^{1-\frac{1}{8}s}(1, 0, ..., 0), \quad j = 1, ..., k.$$

It is easy to see that $(\xi_1, ..., \xi_k) \in \Lambda$. Now, we compute by Lemma 6.4:

$$I(q^\varepsilon) = \max_\Lambda I(q) \ge c_* k \max_\Gamma V^\theta(x) - c_3 \varepsilon^{\frac{s}{8}}. \tag{7.4}$$

For $\varepsilon$ sufficiently small, a contradiction follows immediately from (7.3)-(7.4).

$\square$

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• After completion of this work we have learned about the paper [4] in which the result of Corollary 1.1 is found for $k = 1$ under further constraints in the space dimension $N$ and the values of $s$ and $p$.

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