Course 2- Homework Assignment 3 (Due Date: Wednesday by 3:30pm)

You only need to hand in solutions of the **two problems** from the following four problems. Extra credits will be given if you hand in more problems.

You may use the following fact: consider second order ODE

$$y^{''} + py^{'} + q(x)y = g(x)$$

Let y_1, y_2 be the set of fundamental solutions. Then all solutions are given by

$$y = y_1(x) \int_a^x y_2 W - y_2(x) \int_b^x y_1 W$$

where W is the Wronskian $W = y_1 y_2^{'} - y_2 y_1^{'}$. Here a, b can be $\pm\infty$.

1. Consider the following simple ODE:

$$u^{''}(x) = f(x)$$

where $f(x)$ decays at both ends:

$$|f(x)| \leq Ce^{-|x|}$$

(a) Find out the unique solution u such that it decays at positive end:

$$|u(x)| \leq Ce^{-|x|} \text{ for } x > 0$$

Hint: integrating the equation twice, using $+\infty$ as end point.
(b) Find out the unique solution u such that it decays at negative end:

$$|u(x)| \leq Ce^{-|x|} \text{ for } x < 0$$

(c) Under what conditions on f so that we can find a bounded solution u?
(d) Under what conditions on f so that we can find a solution u such that it decays at both ends?

$$|u(x)| \leq Ce^{-|x|} \text{ for all } x$$

2. This problem concerns the 1D Liouville equation

$$(*) \qquad u^{''} + e^u = 0$$

(a) Show that a solution to (*) is given by

$$u_0(x) = -2 \log \cosh(x) + \log 2$$

(b) Show that the equation (*) is translation and scaling invariant:

$$x \to u_0(x - a), \quad x \to u_0(\lambda x) + 2 \log \lambda$$

(c) Use (b) to find the two kernels of the linearized operator

$$\phi^{''} + e^{u_0} \phi = 0$$

Hint: differentiating the two parameters a and λ.
(d) Let f be such that

$$|f| \leq Ce^{-|x|} \ \forall x$$

Find out the unique solution ϕ to

$$\phi^{''} + e^{u_0}\phi = f(x)$$

such that it decays at positive end:

$$|\phi(x)| \leq Ce^{-|x|} \text{ for } x > 0$$

Hint: use variation of parameters to find the solution formula.

(e) Under what conditions on f so that we can find a solution ϕ in (d) such that it decays at both ends?

$$|\phi(x)| \leq Ce^{-|x|} \text{ for all } x$$

3. Consider the following higher dimensional inhomogeneous Allen-Cahn in radial coordinates

$$\epsilon^2 r^{1-N}(a(r)r^{N-1}u^{'})^{'} + a(r)(u - u^3) = 0, 0 < r < +\infty$$

This is the Euler-Lagrange equation of

$$\int_0^{\infty} (\frac{\epsilon^2}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2) r^{N-1} a(r) dr$$

Find the necessary condition for which the heteroclinic solution exists:

$$u(r) \sim \tanh(\frac{r - r_0}{\epsilon\sqrt{2}})$$

What are possible sufficient conditions?

4. This problem concerns the 2D Liouville equation in radial coordinate

$$(**) \qquad u_{rr} + \frac{1}{r}u_r + e^u = 0, u^{'}(0) = 0$$

(a) Show that a solution to (**) is given by

$$u_0(r) = \log \frac{8}{(1+r^2)^2}$$

(b) Show that the equation (**) is scaling invariant:

$$r \to u_0(\lambda r) + 2\log\lambda$$

(c) Use (b) to find the kernel of the linearized operator

$$\phi^{''} + \frac{1}{r}\phi^{'} + e^{u_0}\phi = 0, \phi^{'}(0) = 0$$

(d) Let f be such that

$$|f| \leq \frac{C}{1+r^4}$$

Find out a solution to

$$(***) \quad \phi^{''} + \frac{1}{r}\phi^{'} + e^{u_0}\phi = f(r), 0 < r < +\infty, \phi^{'}(0) = 0$$

Hint: Find the solution in the form $\phi = Z_0(r)\hat{\phi}(r)$, where Z_0 is given by (c).

(e) Under what conditions on f so that we can find a solution ϕ to (***) such that it decays?

$$\phi(r) \to 0 \text{ as } r \to +\infty$$