**Course 2 - Homework Assignment 2 Solution**

1. It is easy to see that $\phi_1 = w > 0$ is the principal eigenfunction corresponding to the principal eigenvalue $\lambda_1 = 1$. We claim that for $j = 2, \ldots, N+1$, $\lambda_j = p$ and $\phi_j = \dfrac{\partial w}{\partial x_j}$.

   By Question 3 of Assignment 1, $w$ is the minimizer of the energy functional

$$E[u] = \frac{\int \left(|\nabla u|^2 + u^2\right)}{\left(\int u^{p+1}\right)^{\frac{2}{p+1}}}.$$

   Using the fact that the quadratic form is non-negative, we have, for any test function $\phi$,

$$\int \left(|\nabla \phi|^2 + \phi^2\right) \geq p \int w^{p-1}\phi^2 + (p-1)\frac{\left(\int w^p \phi\right)^2}{\int w^{p+1}}. \tag{1}$$

   Let us also observe that $\phi \perp w$ if and only if $\int w^p \phi = 0$. Indeed, this can be proved by taking the difference of $w(\Delta\phi - \phi + \lambda w^{p-1}\phi) = 0$ and $\phi(\Delta w - w + w^p) = 0$ and integrating, which gives $(\lambda - 1)\int w^p\phi = 0$. Since the principal eigenvalue $\lambda_1 = 1$ is simple, we have $\int w^p \phi = 0$.

   Hence, if $\phi \perp w$, then

$$\int \left(|\nabla \phi|^2 + \phi^2\right) \geq p \int w^{p-1}\phi^2.$$

   Therefore,

$$\begin{aligned} \lambda_2 &= \inf_{\phi \perp w} \frac{\int \left(|\nabla \phi|^2 + \phi^2\right)}{\int w^{p-1}\phi^2} \\ &\geq p \end{aligned}$$

   Since $\dfrac{\partial w}{\partial x_j}$ attains the infimum, we are done.

2. Let $h = Ce^{-\frac{|x|}{2}} + \varepsilon e^{\frac{|x|}{2}}$. Then $\Delta h - h = C\left(-\frac{3}{4} - \frac{N-1}{2|x|}\right)e^{-\frac{|x|}{2}} + \varepsilon\left(-\frac{3}{4} + \frac{N-1}{2|x|}\right)e^{\frac{|x|}{2}}$, Fix $R_1 = N$, $C = 2C_1 + \sup|\phi|\, e^{\frac{R_1}{2}}$ and $R_2 = R_2(\varepsilon) = 4\log\dfrac{\sup|\phi|}{\varepsilon}$. Then

   - on $\{R_1 \leq |x| \leq R_2\} \subset \{|x| \geq R_1\}$, we have

$$\Delta(\pm\phi - h) - (\pm\phi - h) \geq C\left(-\frac{3}{4} - \frac{N-1}{2|x|}\right)e^{-\frac{|x|}{2}} + \varepsilon\left(-\frac{3}{4} + \frac{N-1}{2|x|}\right)e^{\frac{|x|}{2}} - C_1 e^{-\frac{|x|}{2}} \geq 0;$$

   - on $\{|x| = R_1\}$, we have

$$\pm\phi - h \leq \sup|\phi| - Ce^{-\frac{R_1}{2}} - \varepsilon e^{\frac{R_1}{2}} \leq 0;$$

- on $\{|x| = R_2\}$, we have

$$\pm\phi - h \le \sup|\phi| - Ce^{-\frac{R_2}{2}} - \varepsilon e^{\frac{R_2}{2}} \le 0.$$

By comparison principle, there holds

$$\pm\phi - h \le 0 \text{ on } R_1 \le |x| \le R_2,$$

that is,

$$|\phi| \le Ce^{-\frac{|x|}{2}} + \varepsilon e^{\frac{|x|}{2}} \text{ on } R_1 \le |x| \le R_2.$$

Letting $\varepsilon \to 0$, we have $|\phi| \le Ce^{-\frac{|x|}{2}}$ on $|x| \ge R_1$. Choosing a larger $C$ if necessary, this estimate holds in the whole $\mathbb{R}^N$.

3. (a) Let

$$c = \inf_{u\in\mathcal{H}^1((0,L))} E[u] = \inf_{u\in\mathcal{H}^1((0,L))} \frac{\int_0^L (|\nabla u|^2 + u^2)}{\left(\int_0^L u^{p+1}\right)^{\frac{2}{p+1}}}$$

and let $\{w_k\}$ be a minimizing sequence. By scaling invariance, we may assume that $\int_0^L w_k^3 = 1$. By Sobolev embedding, $\mathcal{H}^1 \hookrightarrow L^\infty$ is compact, so $\{w_k\}$ has a bounded convergent subsequence. With an abuse of notation, let $w_k \to w$ and then $\int_0^L w^3 = 1$. Using $w \in \mathcal{H}^1$ as a test function and Fatou's lemma, we have

$$c \le \int_0^L \left(|\nabla w|^2 + w^2\right) \le \liminf_{k\to\infty} \int_0^L \left(|\nabla w_k|^2 + w_k^2\right) = c.$$

Hence $w$ is a minimizer. We need to show that $w$ is a non-constant function for $L > \pi$ (so that $w'(x) < 0$).

Suppose, on the contrary, that $w \equiv 1$. The relation $E''[1+t\phi] \ge 0$ gives, for any test function $\phi$, as in (1) with $w = 1$ and $p = 2$,

$$\int(\phi')^2 - \int\phi^2 + \frac{1}{L}(\int_0^L \phi)^2 \ge 0.$$

But the function $\phi(x) = \cos\left(\frac{\pi x}{L}\right)$ satisfies

$$\int(\phi')^2 - \int\phi^2 + \frac{1}{L}(\int_0^L \phi)^2 < 0,$$

a contraction.

(b) See Lecture 3.

4. See lecture 3.

5. The Contraction Mapping Principle states that a contraction mapping $T$ in a Banach space $V$ has a unique fixed point, that is there exists a unique solution $x \in V$ such that $x = Tx$. For example, it can be used to solve nonlinear equations when the nonlinearity is small. We see its use in the process of Liapunov-Schmidt reduction or the reduced problem of it.

6. The Fredholm Alternative states that if $T$ is a compact linear mapping of a normed linear space $L$ into itself, then either $x - Tx = 0$ has a nontrivial solution $x \in L$, or for each $y \in L$ the equation $x - Tx = y$ has a uniquely determined solution $x \in L$. In the second case, $(I - T)^{-1}$ is bounded. For instance, in the process of Liapunov-Schmidt reduction, this is useful when there is no explicit formula for the linearized operator, that is, we can invert the operator once we know that the homogeneous equation has no nontrivial solutions.

7. Let $x = x_0 + \varepsilon y$ and $U(y) = u(x)$. Then

$$U'' - V(x_0 + \varepsilon y)U + Q(x_0 + \varepsilon y)U^p = 0$$

$$U'' - V(x_0)U + Q(x_0)U^p = -(V(x_0 + \varepsilon y) - V(x_0))U + (Q(x_0 + \varepsilon y) - Q(x_0))U^p.$$

As $\varepsilon \to 0$, $U'' - V(x_0)U + Q(x_0)U^p = 0$ gives, by the hint,

$$U(y) = \left(\frac{V(x_0)}{Q(x_0)}\right)^{\frac{1}{p-1}} w(\sqrt{V(x_0)}y),$$

where $w'' - w + w^p = 0$. Note that $U$ and $U'$ decays exponentially to 0 at $\infty$. Multiplying both sides by $U'$ and integrating on $(-\infty, \infty)$, we see that the left hand side is 0 and the right hand side becomes, after integrating by parts,

$$\begin{aligned}
0 &= -\frac{V'(x_0)}{2}\int U^2 + \frac{Q'(x_0)}{p+1}\int U^{p+1} \\
&= -\frac{V'(x_0)}{2}\left(\frac{V(x_0)}{Q(x_0)}\right)^{\frac{2}{p-1}}\int w^2(\sqrt{V(x_0)}y)\,dy + \frac{Q'(x_0)}{p+1}\left(\frac{V(x_0)}{Q(x_0)}\right)^{\frac{p+1}{p-1}}\int w^{p+1}(\sqrt{V(x_0)}y)\,dy \\
&= -\frac{V'(x_0)}{2}\left(\frac{V(x_0)}{Q(x_0)}\right)^{\frac{2}{p-1}}\frac{1}{\sqrt{V(x_0)}}\int w^2 + \frac{Q'(x_0)}{p+1}\left(\frac{V(x_0)}{Q(x_0)}\right)^{\frac{p+1}{p-1}}\frac{1}{\sqrt{V(x_0)}}\int w^{p+1}
\end{aligned}$$

Testing the equation $w'' - w + w^p = 0$, we get $\int((w')^2 - w^2 + w^{p+1}) = 0$. If we integrate the first integral, we have $\int\left((w')^2 - w^2 + \dfrac{2}{p+1}w^{p+1}\right) = 0$. Eliminating $\int(w')^2$, we have

$$\int w^2 = \left(\frac{1}{2} + \frac{1}{p+1}\right)\int w^{p+1} = \frac{p+3}{2(p+1)}\int w^{p+1}.$$

Therefore, the above equation is simplified to

$$0 = -\frac{V'(x_0)}{2} + \frac{2Q'(x_0)}{p+3}\frac{V(x_0)}{Q(x_0)},$$

or

$$(p+3)Q(x_0)V'(x_0) = 4Q'(x_0)V(x_0).$$

Note that it can also be written as

$$\left(\frac{V^{p+3}}{Q^4}\right)'(x_0) = 0.$$

8. Let $r = r_0 + \varepsilon t$ and $U(t) = u(r)$. Then

$$U'' - V(r_0)U + U^p = -(V(r_0 + \varepsilon t) - V(r_0))U - \frac{\varepsilon(N-1)}{r_0 + \varepsilon t}U'.$$

As in the last question, when $\varepsilon = 0$,

$$U(t) = V(x_0)^{\frac{1}{p-1}} w(\sqrt{V(x_0)}y).$$

Testing the equation with $U'$, we have

$$0 = \frac{\varepsilon V'(r_0)}{2} \int U^2 - \varepsilon(N-1) \int \frac{1}{r_0 + \varepsilon t}(U')^2.$$

Note that as $\varepsilon \to 0$,

$$\frac{1}{r_0 + \varepsilon t} = \frac{1}{r_0}\left(1 - \frac{\varepsilon t}{r_0}\right) + O(\varepsilon^2)$$

so when we divide both sides by $\varepsilon$ and take $\varepsilon \to 0$, we get

$$\begin{aligned} 0 &= \frac{\varepsilon V'(r_0)}{2} \int U^2 - \frac{\varepsilon(N-1)}{r_0} \int (U')^2 \\ &= \frac{V'(r_0)}{2} V(r_0)^{\frac{2}{p-1}} \frac{1}{\sqrt{V(x_0)}} \int w^2 - \frac{N-1}{r_0} V(r_0)^{\frac{2}{p-1}} V(r_0) \frac{1}{V(r_0)} \int (w')^2 \end{aligned}$$

Using the equation for $w$, we have

$$-\int (w')^2 + \int w^2 = \frac{2}{p+1} \int w^{p+1} = \frac{2}{p+1} \int (w')^2 + \frac{2}{p+1} \int w^2$$

which gives

$$(p-1) \int w^2 = (p+3) \int (w')^2.$$

Therefore, the necessary condition is

$$\frac{(p+3)V'(r_0)}{2} = \frac{(p-1)(N-1)}{r_0} V(r_0),$$

or

$$\frac{V'(r_0)}{V(r_0)} = \frac{2(p-1)(N-1)}{(p+3)r_0}.$$

9. Referring to the paper, the Green's function is constructed as follows. Let $J_1(r)$ and $J_2(r)$ be respectively the solutions of the problem

$$J_1'' + \frac{N-1}{r} J_1' - J_1 = 0, \ J_1'(0) = 0, \ J_1(0) = 1, \ J_1 > 0$$

and

$$J_2'' + \frac{N-1}{r} J_2' - J_2 + \delta_0, \ J_2 > 0, \ J_2(+\infty) = 0.$$

They can be written in terms of modified Bessel's functions, namely,

$$J_1(r) = c_1 r^{\frac{2-N}{2}} I_\nu(r), \ J_2(r) = c_2 r^{\frac{2-N}{2}} K_\nu(r), \ \nu = \frac{N-2}{2},$$

where $c_1$, $c_2$ are two positive constants and $I_\nu$, $K_\nu$ are modified Bessel functions of order $\nu$. They are explicit when $N = 3$:

$$J_1(r) = \frac{\sinh r}{r}, \quad J_2(r) = \frac{e^{-r}}{4\pi r}.$$

By computing the Wronskian of $J_1$, $J_2$,

$$J_1'(r_0)J_2(r_0) - J_1(r_0)J_2'(r_0) = \frac{1}{c_0 r_0^{N-1}}$$

for some constant $c \neq 0$. The Green's function is then

$$G(r; r_0) = c_0 r_0^{N-1} \begin{cases} J_2(r_0)J_1(r) & \text{for } r < r_0, \\ J_1(r_0)J_2(r) & \text{for } r > r_0, \end{cases}$$

The Green's function representation formula is

$$u(r) = \int_0^\infty G(r; r_0) f(r_0)\, dr_0.$$