**Course 2- Homework Assignment 1 (Due Date: )**

The purpose of this set of problems is to study the basic properties of the two profile functions: spikes (responsible for spotty pattern), and fronts (responsible for stripe pattern).

You only need to hand in solutions of the **two problems** from the following five problems. Extra credits will be given if you hand in more problems.

1. Consider the following ODE:

$$w^{''} - w + w^p = 0, w^{'}(0) = 0, w(\infty) = 0$$

and its linearized operator

$$L_0(\phi) = \phi^{''} - \phi + pw^{p-1}\phi$$

(a) Show that the principal eigenvalue and principal eigenfunction are given by

$$\lambda_1 = \frac{(p-1)(p+3)}{4}, \quad \phi_1 = w^{\frac{p+1}{2}}$$

(b) As a consequence of (a), compute

$$L_0^{-1}(w^{r-1}), r = \frac{p+3}{2}$$

and

$$\int w^{r-1} L_0^{-1}(w^{r-1}), \ r = \frac{p+3}{2}$$

(c) Show that all eigenvalues of $L_0$ must be real.

2. Consider the following problem

$$\Delta w - w + w^p = w_{rr} + \frac{N-1}{r} w_r - w + w^p = 0, \ w^{'}(0) = 0, w(\infty) = 0$$

Show that for $p \geq \frac{N+2}{N-2}$ then $w \equiv 0$.

Hint: multiplying the equation by $r^N w^{'}$ and $r^{N-1}w$ respectively and integrating.

3. Consider the following minimization problem

$$c = \inf E[u]$$

where

$$E[u] = \frac{\int (|\nabla u|^2 + u^2)}{(\int |u|^{p+1})^{\frac{2}{p+1}}}$$

Let $u$ be a minimizer and consider the following function

$$\rho(t) := E[u + t\phi]$$

Compute

$$\rho^{'}(0)$$

and

$$\rho^{''}(0)$$

4. Now we consider the front solution. Consider the following equation

$$H^{''} + H - H^3 = 0, H(-\infty) = -1, H(+\infty) = 1$$

(a) Show that a solution is given by

$$H(t) = \tanh(\frac{t}{\sqrt{2}})$$

(b) Show that the equation in (a) is the Euler-Lagrange equation of the following energy functional

$$E(u) = \int (\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2)$$

(c) Now we consider the following Gierer-Meinhardt system with saturation

$$H^{''} - H + \frac{H^2}{1+aH^2} = 0$$

Show that for $0 < a < \frac{1}{4}$ there are two positive constant steady states $0 < h_- < h_+$. Now compute

$$\int_0^{h_+} (-H + \frac{H^2}{1+aH^2})$$

Find the equation for $a = a_0$ so that $\int_0^{h_+} (-H + \frac{H^2}{1+aH^2}) = 0$
Show that for such $a_0$ there exists a solution connecting 0 and the positive constant steady state $h_+$.

5. (a) Let $H(t) = \tanh(\frac{t}{\sqrt{2}})$. Under what conditions on $(a_1, ..., a_N, b)$ so that $u(x) = H(a_1 x_1 + ... + a_N x_N + b)$ satisfies

$$\Delta u + u - u^3 = 0 \text{ in } R^N$$

(b) Show that the equation in (b) is the Euler-Lagrange equation of the following energy functional

$$E(u) = \int (\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2)$$

(c) A solution $u$ satisfying the equation in (b) is called stable if for any function with compact support $\phi$ it holds

$$\int (|\nabla \phi|^2 + (3u^2 - 1)\phi^2) > 0$$

Show that the function defined in (a) is stable. (The converse may not be true. Counterexample is given when $N = 8$.)