AARMS SUMMER SCHOOL—LECTURE IV: INTRODUCTION TO LYAPUNOV SMICHDT REDUCTION METHODS FOR SOLVING PDE'S

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1. Allen Cahn Equation

Energy: Phase transition model.
Let $\Omega \subseteq \mathbb{R}^N$ of a "binary mixture": Two materials coexisting (or one material in two phases). We can take as an example of this: Water in solid phase $(+1)$, and water in liquid phase (-1). The configuration of this mixture in Ω can be described as a function

$$u^*(x) = \begin{cases} +1 & \text{in } \Lambda \\ -1 & \text{in } \Omega \setminus \Lambda \end{cases}$$

where Λ is some open subset of Ω. We will say that u^* is the phase function.

Consider the functional

$$\frac{1}{4}\int_\Omega (1-u^2)^2$$

minimizes if $u = 1$ or $u = -1$. Function u^* minimize this energy functional. More generally this well happen for

$$\int_\Omega W(u)dx$$

where $W(u)$ minimizes at 1 and -1, i.e. $W(+1) = W(-1) = 0$, $W(x) > 0$ if $x \neq 1$ or $x \neq -1$,$W''(+1), W''(-1) > 0$.

1.1. **The gradient theory of phase transitions.** Possible configurations will try to make the boundary $\partial\Lambda$ as nice as possible: smooth and with small perimeter. In this model the step phase function u^* is replaced by a smooth function u_ε, where $\varepsilon > 0$ is a small parameter, and

$$u_\varepsilon(x) \approx \begin{cases} +1 & \text{inside } \Lambda \\ -1 & \text{inside } \Omega \setminus \Lambda \end{cases}$$

and u_ε has a sharp transition between these values across a "wall" of width roughly $O(\varepsilon)$: the interface (thin wall).

In grad theory of phase transitions we want minimizers, or more generally, critical points u_ε of the functional

$$J_\varepsilon(u) = \varepsilon \int_\Omega \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon}\int_\Omega \frac{(1-u^2)^2}{4}$$

Let us observe that the region where $(1-u_\varepsilon^2) > \gamma > 0$ has area of order $O(\varepsilon)$ and the size of the gradient of u_ε in the same region is $O(\varepsilon^2)$ in such a way $J(u_\varepsilon) = O(1)$. We will find critical points u_ε to functionals of this type so that $J(u_\varepsilon) = O(1)$.

Let us consider more generally the case in which the container isn't homogeneous so that distinct costs are paid for parts of the interface in different locations

$$J_\varepsilon(u) = \int_\Omega \left(\varepsilon \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon}\frac{(1-u^2)^2}{4}\right) a(x)dx$$

$a(x)$ non-constant, $0 < \gamma \leq a(x) \leq \beta$ and smooth.

1.2. **Critical points of J_ε.** First variation of J_ε at u_ε is equal to zero.

$$\frac{\partial}{\partial t} J_\varepsilon(u_\varepsilon + t\varphi)\bigg|_{t=0} = DJ_\varepsilon(u_\varepsilon)[\varphi] = 0, \quad \forall \varphi \in C_c^\infty(\Omega)$$

We have

$$J_\varepsilon(u_\varepsilon + t\varphi) =$$

i.e. $\forall \varphi \in C_c^\infty(\Omega)$

$$0 = DJ_\varepsilon(u_\varepsilon)[\varphi] = \varepsilon \int_\Omega (\nabla u_\varepsilon \nabla \varphi) a + \frac{1}{\varepsilon}\int_\Omega W'(u_\varepsilon)\phi a.$$

If $u_\varepsilon \in C^2(\Omega)$

$$\int_\Omega \left(-\varepsilon \nabla \cdot (a\nabla u_\varepsilon) + \frac{a}{\varepsilon} W'(u_\varepsilon)\right)\varphi = 0, \quad \forall \varphi \in C_c^\infty(\Omega)$$

This give us the weighted Allen Cahn equation in Ω

$$-\varepsilon \nabla \cdot (a\nabla u) + \frac{a}{\varepsilon} u(1-u^2) = 0 \text{ in } \Omega.$$

We will assume in the next lectures $\Omega = \mathbb{R}^N$, where $N = 1$ or $N = 2$. If $N = 1$ weight Allen Cahn equation is

$$\varepsilon^2 u'' + \varepsilon^2 u' \frac{a'}{a} + (1-u^2)u = 0, \text{ in } (-\infty, \infty). \tag{1.1}$$

Look for u_ε that connects the phases -1 and $+1$ from $-\infty$ to ∞. Multiplying (1.1) against u' and integrating by parts we obtain

$$\int_{-\infty}^{\infty} \frac{d}{dx}\left(\varepsilon \frac{u'^2}{2} - \frac{(1-u^2)^2}{4}\right) + \int_{-\infty}^{\infty} \frac{a'}{a} u'^2 = 0 \tag{1.2}$$

Assume that $u(-\infty) = -1$, $u(\infty) = 1$, $u'(-\infty) = u'(\infty) = 0$, $a > 0$, then (1.2) implies that

$$\frac{(1-u^2)^2}{4} + \int_{-\infty}^{\infty} \frac{a'}{a} u'^2 = 0$$

from which we conclude that unless a is constant, we need a' to change sign. So: if a is monotone and $a' \neq 0$ implies the non-existence of solutions as we look for. We need the existence (if $a' \neq 0$) of local maximum or local minimum of a. We will prove that under some general assumptions on $a(x)$, given a local max. or local min. x_0 of a non-degenerate ($a''(x_0) \neq 0$), then a solution to (1.1) exists, with transition layer.

We consider first the problem with $a \equiv 1$, $\varepsilon = 1$:

$$W'' + (1 - W^2)W = 0, \quad W(-\infty) = -1, W(\infty) = 1. \tag{1.3}$$

The solution of this problem is

$$W(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$$

This solution is called "the heteroclinic solution", and it's the unique solution of the problem (1.3)up to translations.

Observation 1.1. *This solution exists also for the problem*

$$w'' + f(w) = 0, \quad w(-\infty) = -1, w(\infty) = 1 \tag{1.4}$$

where $f(w) = -W'(w)$. *Solutions satisfies* $\frac{w'^2}{2} - W(w) = E$, *where* E *is constant, and* $w(-\infty) = -1$ *and* $w(\infty) = 1$ *if and only if* $E = 0$. *This implies*

$$\int_0^w \frac{ds}{\sqrt{2w(s)}} = t$$

$t(w) \to \infty$ *if* $w \to 1$, *and* $t(w) \to -\infty$ *if* $w \to -1$, *so the previous relation defines a solution* w *such that* $w(0) = 0$, *and* $w(-\infty) = -1$, $w(\infty) = 1$.

If we wright the Hamiltonian system associated to the problem we have:

$$p' = -f(q), \quad q' = p.$$

Trajectories lives on level curves of $H(p,q) = \frac{p^2}{2} - W(q)$, *where* $W(q) = \frac{(1-q^2)^2}{4}$.

Let $x_0 \in \mathbb{R}$ (we will make assumptions on this point). Fix a number $h \in \mathbb{R}$ and set

$$v(t) = u(x_0 + \varepsilon(t+h)), \quad v'(t) = \varepsilon u'(x_0 + \varepsilon(t+h))$$

Using (1.1), we have

$$\varepsilon^2 u''(x_0+\varepsilon(t+h)) = -\varepsilon^2 \frac{a'}{a} u'(x_0+\varepsilon(t+h)) - (1-v^2(t))v(t)$$

so we have the problem

(1.5)

$$v''(t)+\varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))v'(t)+(1-v(t)^2)v(t)^2 = 0, \quad w(-\infty)=-1,\, w(\infty)=1.$$

Let us observe that if $\varepsilon = 0$ the previous problem becomes formally in (1.3), so is natural to look for a solution $v(t) = W(t) + \phi$, with ϕ a small error in ε.

Assumptions:

(1) There exists $\beta, \gamma > 0$ such that $\gamma \le a(x) \le \beta$, $\forall x \in \mathbb{R}$
(2) $\|a'\|_{L^\infty(\mathbb{R})}$, $\|a''\|_{L^\infty(\mathbb{R})} < +\infty$
(3) x_0 is such that $a'(x_0) = 0$, $a''(x_0) \neq 0$, i.e. x_0 is a non-degenerate critical point of a.

Theorem 1.1. *$\forall\varepsilon > 0$ sufficiently small, there exists a solution $v = v_\varepsilon$ to (1.5) for some $h = h_\varepsilon$, where $|h_\varepsilon| \le C\varepsilon$ and $v_\varepsilon(t) = w(t) + \phi_\varepsilon(t)$ and*

$$\|\phi_\varepsilon\| \le C\varepsilon$$

Proof. We write in (1.5) $v(t) = w(t) + \phi(t)$. From now on we write $f(v) = v(1-v^2)$. We get

$$w''+\phi''+\varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))\phi'+\varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))w'+f(w+\phi)-f(w)-f'(w)\phi+f(w)+f'(w)\phi = 0$$

$$\phi(-\infty) = \phi(\infty) = 0.$$

It can be written in the following way

$$\phi'' + f'(w(t))\phi + E + B(\phi) + N(\phi) = 0, \quad \phi(-\infty) = \phi(\infty) = 0 \tag{1.6}$$

where

$$\begin{aligned} B(\phi) =& \varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))\phi', \\ N(\phi) =& f(w+\phi) - f(w) - f'(w) = -3w\phi^2 - \phi^3, \\ E =& \varepsilon\frac{a'}{a}(x_0+\varepsilon(t+h))w'. \end{aligned}$$

We consider the problem

$$\phi'' + f'(w(t))\phi + g(t) = 0, \quad \phi \in L^\infty(\mathbb{R}), \tag{1.7}$$

and we want to know when (1.7) is solvable. We will assume $g \in L^\infty(\mathbb{R})$. Multiplying (1.7) against w' we get

$$\int_{-\infty}^{\infty}(w''' + f'(w)w')\phi + \int_{-\infty}^{\infty} gw' = 0$$

the first integral is zero because (1.4). We conclude that a necessary condition is

$$\int_{-\infty}^{\infty} gw' = 0.$$

This condition is actually sufficient for solvability. In fact, we write $\phi = w'\Psi$, we have

$$\phi'' + f'(w)\phi = g \Leftrightarrow w'\Psi + 2w''\Psi' = -g$$

Multiplying this last expression by w' (integration factor), we get

$$(w'^2\Psi')' = gw' \Rightarrow w'2\Psi'(t) = -\int_{-\infty}^{\infty} g(s)w'(s)ds$$

Let us choose

$$\Psi(t) = -\int_0^t \frac{d\tau}{w'^2(\tau)} \int_{-\infty}^{\tau} g(s)w'(s)ds$$

Then the function

$$\phi(t) = -w'(t)\int_0^t \frac{d\tau}{w'^2(\tau)} \int_{-\infty}^{\tau} g(s)w'(s)ds$$

Recall that

$$w'(t) \approx 2\sqrt{2}e^{-\sqrt{2}|t|}$$

Claim: if $\int_{-\infty}^{\infty} gw' = 0$ then we have

$$\|\phi\|_\infty \leq C\|g\|_\infty.$$

In fact, if $t > 0$

$$|\phi(t)| \leq |w'(t)| \int_0^t \frac{C}{e^{-2\sqrt{2}\tau}} \left| \int_\tau^\infty gw'ds \right| d\tau \leq C\|g\|_\infty e^{-\sqrt{2}t} \int_0^t e^{\sqrt{2}\tau} d\tau \leq C\|g\|_\infty.$$

For $t < 0$ a similar estimate yields, so we conclude

$$|\phi(t)| \leq C\|g\|_\infty.$$

□

The solution of (1.7) is not unique because if ϕ_1 is a solution implies that $\phi_2 = \phi_1 + Cw'(t)$ is also a solution. The solution we found is actually the only one with $\phi(0) = 0$. For $g \in L^\infty$ arbitrary we consider the problem

$$\phi'' + f'(w)\phi + (g - cw') = 0, \text{ in } \Re, \quad \phi \in L^\infty(\mathbb{R}) \tag{1.8}$$

where $C = C(g) = \dfrac{\int_{-\infty}^{\infty} gw'}{\int_{-\infty}^{\infty} w'^2}$.

Lemma 1.1. *$\forall g \in L^\infty(\mathbb{R})$ (1.8) has a solution which defines a operator $\phi = T[g]$ with*

$$\|T[g]\|_\infty \leq C\|g\|_\infty.$$

In fact if $\hat{T}[\hat{g}]$ is the solution find in the previous step then $\phi = \hat{T}[g - C(g)w']$ solves (1.8) and

$$\|\phi\|_\infty \leq C\|g\|_\infty + |C(g)|C \leq C\|g\|_\infty \tag{1.9}$$

Proof. Back to the original problem: We solve first the projected problem

$$\phi'' + f'(w)\phi + E + B(\phi) + N(\phi) = Cw', \quad \phi \in L^\infty(\mathbb{R})$$

where

$$C = \frac{\int_{\mathbb{R}}(E + B(\phi) + N(\phi))w'}{\int_{\mathbb{R}} w'^2}.$$

We solve first (1.9) and then we find $h = h_\varepsilon$ such that in (1.9) C=0 in such a way we find a solution to the original problem. We assume $|h| \leq 1$. It's sufficient to solve

$$\phi = T[E + B(\phi) + N(\phi)] := M[\phi].$$

We have the following remark

$$|E| \leq C\varepsilon^2, \quad \|B(\phi)\|_\infty \leq C\varepsilon\|\phi'\|_\infty, \quad \|N(\phi)\| \leq C(\|\phi^2\|_\infty + \|\phi^3\|_\infty)$$

where C is uniform on $|h| \leq 1$. We have

$$\|M\|_\infty + \|\frac{d}{dt}M\|_\infty \leq C(\|E\|_\infty + \|B(\phi)\|_\infty + \|N(\phi)\|_\infty \leq C(\varepsilon^2 + \varepsilon\|\phi'\|_\infty + \|\phi^2\|_\infty + \|\phi^3\|_\infty)$$

then if $\|\phi\|_\infty + \|\phi'\|_\infty \leq M\varepsilon^2$ we have

$$\|M\|_\infty + \|\frac{d}{dt}M\|_\infty \leq C^*\varepsilon^2.$$

We define the space $X = \{\phi \in C^1(\mathbb{R}) : \|\phi\|_\infty + \|\phi'\|_\infty \leq C^*\varepsilon^2\}$. Let us observe that $M(X) \subset X$. In addition

$$\|M(\phi_1) - M(\phi_2)\|_\infty + \|\frac{d}{dt}(M(\phi_1) - M(\phi_2))\|_\infty \leq C\varepsilon(\|\phi_1 - \phi_2\|_\infty + \|\phi_1' - \phi_2'\|_\infty).$$

So if ε is small M is a contraction mapping which implies that there exists a unique $\phi \in X$ such that $\phi = M[\phi]$. □

In summary: We found for each $|h| \leq 1$

$$\phi = \Phi(h), \text{ solution of1.7}$$

. We recall that

$$h \to \Phi(h)$$

is continuous (in $\|\|_{C^1}$) . Notice that from where we deduce that M is continuous in h.

The problem is reduced to finding h such that $C = 0$ in (1.7) for $\phi\Phi(h) =$. Let us observe that

$$C = 0 \Leftrightarrow \alpha_\varepsilon(h) := \int_{\mathbb{R}} (E_h + B[\Phi(h)]) + N[\Phi(h)])w' = 0.$$

Let us observe that if we call $\psi(x) = \frac{a'}{a}(x)$, then

$$\psi(x_0+\varepsilon(t+h)) = \psi(x_0)+\psi'(x_0)\varepsilon(t+h)+\int_0^1 (1-s)\psi''(x_0+s\varepsilon(t+h))\varepsilon^2(t+h)^2 ds$$

We add the assumption $a''' \in L^\infty(\mathbb{R})$ in order to have $\psi'' \in L^\infty(\mathbb{R})$. We deduce that

$$\int E_h w' = \varepsilon^2\psi'(x_0)\int (t+h)w'(t)^2+\varepsilon^3\int_{\mathbb{R}}(\int_0^1 (1-s)\psi''(x_0+s\varepsilon(t+h))ds)(t+h)^2 w'(t)dt$$

We recall that: $\int_{\mathbb{R}} tw'(t)^2$ and

$$|\int_{\mathbb{R}} (B[\phi(h)] + N[\phi(h)])w'| \leq C(\varepsilon\|\Phi(h)\|_{C^1} + \|\Phi(h)\|_{L^\infty}) \leq C\varepsilon^3.$$

So, we conclude that

$$\alpha_\varepsilon(h) = \psi'(x_0)\varepsilon^2(h + O(\varepsilon))$$

and the term inside the parenthesis change sign. This implies that $\exists h_\varepsilon : |h_\varepsilon| \leq M\varepsilon$ such that $\alpha_\varepsilon(h) = 0$, so $C = 0$.

Observe that

$$\overline{L}(\phi) = \phi''-2\phi+\varepsilon\psi+3(1-w^2)\phi+\frac{1}{2}f''(w+s\phi)\phi\phi+O(\varepsilon^2)e^{-\sqrt{2}|t|} = 0, \quad |t| > R$$

We consider $t > R$. Notice that $\frac{1}{2}f''(w + s\phi)\phi = O(\varepsilon^2)$. Then using $\hat{\phi} = \varepsilon e^{-|t|} + \delta e^{|t|}$. Then using maximum principle and after taking $\delta \to 0$, we obtain $\phi \leq \varepsilon e^{-|t|}$.

A property: We call

$$\mathcal{L}(\phi) = \phi'' + f'(w)\phi, \quad \phi \in H^2(\mathbb{R}).$$

We consider the bilinear form associated

$$B(\phi, \phi) = -\int_{\mathbb{R}} \mathcal{L}(\phi)\phi = \int_{\mathbb{R}} \phi'^2 - f'(w)^2\phi^2, \quad \phi \in H^1(\mathbb{R}).$$

Claim: $B(\phi, \phi) \geq 0, \forall\phi \in H^1(\mathbb{R})$ and $B(\phi, \phi) = 0 \Leftrightarrow \phi = cw'(t)$. In fact: $J''(w)[\phi, \phi] = B(\phi, \phi)$. We give now the proof of the claim:

Take $\phi \in C_c^\infty(\mathbb{R})$. Write $\phi = w'\Psi \implies \Psi \in C_c^\infty(\mathbb{R})$. Observe that $\mathcal{L}[w'\Psi] = \frac{1}{w'}(w'^2\Psi')'$ and

$$B(\phi,\phi) = -\int \frac{1}{w'}(w'^2\Psi')'w'\Psi = \int_{\mathbb{R}} w'^2\Psi'^2, \quad \forall \phi \in C_c^\infty(\mathbb{R})$$

Same is valid for all $\phi \in H^1(\mathbb{R})$, by density. So $B(\phi,\phi) = \int_{\mathbb{R}} |\phi'|^2 - f'(w)\phi^2 = \int_{\mathbb{R}} w'2|\Psi'|^2 \geq 0$ and $B(\phi,\phi) = 0 \Leftrightarrow \Psi' = 0$ which implies $\phi = cw'$.

Corollary 1.1. *Important for later porpuses There exists $r > 0$ such that if $\phi \in H^1(\mathbb{R})$ and $\int_{\mathbb{R}} \phi w' = 0$ then*

$$B(\phi,\phi) \geq \gamma \int_{\mathbb{R}} \phi^2$$

Proof. If not there exists $\phi_n \int H^1(\mathbb{R})$ such that $0 \leq B(\phi_n,\phi_n) < \frac{1}{n}\int_{\mathbb{R}} \phi_n^2$. We may assume without loss of generality $\int \phi_n^2 = 1$ which implies that up to subsequence

$$\phi_n \rightharpoonup \phi \in H^1(\mathbb{R})$$

and $\phi_n \to \phi$ uniformly and in L^2sense on bounded intervals. This implies that

$$0 = \lim_{n\to\infty} \int_{\mathbb{R}} \phi_n w' = \int_{\mathbb{R}} \phi w'$$

On the other hand

$$\int |\phi_n'|^2 + 2\int \phi_n^2 - 3\int (1-w^2)\phi_n^2 \to 0$$

and also $\int |\phi_n'|^2 + 2\int \phi_n^2 - 3\int(1-w^2)\phi_n^2 \to \int |\phi'|^2 + 2\int \phi^2 - 3\int (1-w^2)\phi^2$, so $B(\phi,\phi) = 0$, and $\int w'\phi = 0$ so $\phi = 0$. But also

$$2 \leq 3\int (1-w^2)\phi_n^2 + o(1)$$

which implies that $2 \leq 3\int(1-w^2)\phi^2$ and this means that $\phi \neq 0$, so we obtain a contradiction. □

Observation 1.2. *If we choose $\delta = \frac{\gamma}{2\|f'\|_\infty}$ then*

$$\int \phi'^2 - (1+\delta)f'(w)\phi^2 \geq 0.$$

This implies in fact that

$$B(\phi,\phi) \geq \alpha \int \phi'^2.$$

2. Nonlinear Schrödinger eqution (NLS)

$$\varepsilon i\Psi_t = \varepsilon^2\Delta\Psi - W(x)\Psi + |\Psi|^{p-1}\Psi.$$

A first fact is that $\int_{\mathbb{R}^N} |\Psi|^2 = constant$. We are interested into study solutions of the form $\Psi_(x,t) = e^{-iEt}u(x)$ (we will call this solutions standing wave solution). Replacing this into the equation we obtain

$$\varepsilon Eu = \varepsilon^2\Delta u - Wu - |u|^{p-1}u$$

whose transforms into

$$\varepsilon^2\Delta u - (W-\lambda)u + |u|^{p-1}u = 0, \quad u(x)\to 0, \text{ as } |x|\to\infty$$

choosing $E = \frac{\lambda}{\varepsilon}$. We define $V(x) = (W(x)-\lambda)$

2.1. The case of dimension 1.

(2.1)
$$\varepsilon^2 u'' - V(x)u + u^p = 0, \quad x\in\mathbb{R}, \quad 0 < u(x)\to 0, \text{ as } |x|\to\infty, p>1.$$

Assume: $V \geq \gamma > 0$, $V, V', V'', V''' \in L^\infty$, and $V \in C^3(\mathbb{R})$. Starting point

$$w'' - w + w^p = 0, \quad w > 0, \quad w(\pm\infty) = 0, p > 1 \tag{2.2}$$

There exists a homoclinic solution

$$w(t) = \frac{C_p}{\cosh\left(\frac{p-1}{2}t\right)^{\frac{2}{p-1}}}, \quad C_p = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}$$

Let us observe that $w(t) \approx 2^{2/(p-1)}C_p e^{-|t|}$ as $t\to\infty$ and also that $W(t+c)$ satisfies same equation.

Staid at x_0 with $V(x_0) = 1$ we want $u_\varepsilon(x) \approx w\left(\frac{x-x_0}{\varepsilon}\right)$ of the problem (2.1).

Observation 2.1. *Given x_0 we can assume $V(x_0) = 1$. Indeed writing*

$$u(x) = \lambda^{\frac{2}{p-1}} v(\lambda x_0 + (1-\lambda)x_0)$$

we obtain the equation

$$\varepsilon^2 v''(y) - \hat{V}(y)v + v^p = 0$$

where $y = \lambda x_0 + (1-\lambda)x_0$, and $\hat{V}(y) = V(\frac{y-(1-\lambda)x_0}{\lambda})$. Then choosing $\lambda = \sqrt{V(x_0}$, we obtain $\hat{V}(x_0) = 1$.

Theorem 2.1. *We assume $V(x_0) = 1, V'(x_0) = 0, V''(x_0) \neq 0$. Then there exists a solution to (2.1) with the form*

$$u_\varepsilon(x) \approx w\left(\frac{x-x_0}{\varepsilon}\right).$$

We define $v(t) = u(x_0 + \varepsilon(t + h))$, with $|h| \le 1$. Then v solves the problem

$$v'' - V(x_0 + \varepsilon(t+h)v + v^p = 0, \quad v(\pm\infty) = 0. \tag{2.3}$$

We define $v(t) = w(t) + \phi(t)$, so ϕ solves

(2.4)

$$\phi'' - \phi + pw^{p-1}\phi - (V(x_0+\varepsilon(t+h)) - V(x_0))\phi + (w+\phi)^p - w^p - pw^{p-1}\phi$$

$$-(V(x_0 + \varepsilon(t+h)) - V(x_0))w(t) = 0 \tag{2.5}$$

So we want a solution of

$$\phi'' - \phi + pw^{p-1}\phi + E + N(\phi) + B(\phi) = 0, \quad \phi(\pm) = 0. \tag{2.6}$$

Observe that

$$E = \frac{1}{2}V''(x_0 + \xi\varepsilon(t+h))\varepsilon^2(t+h)^2 w(t),$$

so $|E| \le C\varepsilon^2(t^2+1)e^{-|t|} \le Ce^{-\sigma t}$ for $0 < \sigma < 1$.

We won't have a solution unless V' doesn't change sign and $V \neq 0$. For instance consider $V'(x) \ge 0$, and after multiplying the equation by u' and integrating by parts, we see that $\int_{\mathbb{R}} v' \frac{u^2}{2} = 0$, which by ODE implies that $u \equiv 0$, because u and u' equals 0 on some point.

2.2. **Linear projected problem.**

$$L(\phi) = \phi'' - \phi + pw^{p-1}\phi + g = 0, \quad \phi \in L^\infty(\mathbb{R})$$

For solvability we have the necessary condition $\int L(\phi)w' = 0$. Assume g such that $\int_{\mathbb{R}} gw' = 0$. We define $\phi = w'\Psi$, but we have the problem that $w'(0) = 0$. We conclude that $(w'^2\Psi')' + w'g = 0$ for $t \neq 0$. We take for $t < 0$

$$\phi(t) = w'(t)\int_t^{-1} \frac{d\tau}{w'(\tau)^2}\int_{-\infty}^{\tau} g(s)w'(s)ds$$

and for $t > 0$

$$\phi(t) = w'(t)\int_1^t \frac{d\tau}{w'(\tau)^2}\int_\tau^\infty g(s)w'(s)ds$$

In order to have a solution of the problem we need $\phi(0^-) = \phi(0^+)$.

$$\phi(0^-) = \lim_{t\to 0^-} \frac{-\int_{-1}^t \frac{d\tau}{w'(\tau)^2}\int_{-\infty}^\tau g(s)w'(s)ds}{\frac{1}{w'(t)}} = \lim_{t\to 0^-} \frac{-\frac{1}{w'(t)^2}\int_{-\infty}^t gw'}{-\frac{1}{w'(t)^2}w''(t)} = \frac{1}{w''(0)\int_{-\infty}^0 gw'}$$

and

$$\phi(0^+) = -\frac{1}{w''(0)\int_0^\infty gw'}$$

and the condition is satisfies because of the assumption of orthogonality condition.

We get $\|\phi\|_\infty \leq C\|g\|_\infty$. In fact we get also: $\forall\, 0 < \sigma < 1, \exists C > 0:$

$$\|\phi e^{\sigma t}\|_{L^\infty} + \|\phi' e^{\sigma t}\|_{L^\infty} \leq C\|g e^{\sigma t}\|$$

Observation: We use $g = g - cw'$.(Correct this part!!!!)

2.3. **Method for solving.** In this section we consider a smooth radial cut-off function $\eta \in C^\infty(\mathbb{R})$, such that $\eta(s) = 1$ for $s < 1$ and $\eta(s) = 0$ if $s > 2$. For $\delta > 0$ small fixed, we consider $\eta_{k,\varepsilon} = \eta\left(\frac{\varepsilon|t|}{k\delta}\right)$, $k \geq 1$.

2.3.1. *The gluing procedure.* Write $\tilde{\phi} = \eta_{2,\varepsilon}\phi + \Psi$, then ϕ solves (2.5) if and only if

$$\eta_{2,\varepsilon}\left[\phi'' + (pw^{p-1} - 1)\phi + B(\phi) + 2\phi'\eta'_{2,\varepsilon}\right] \tag{2.7}$$

$$+\left[\Psi'' + (pw^{p-1} - 1)\Psi + B\Psi\right] + E + N(\eta_{2,\eta}\phi + \Psi) = 0. \tag{2.8}$$

(ϕ, Ψ) solves (2.8) if is a solution of the system

(2.9)
$$\phi'' - (1 - pw^{p-1})\phi + \eta_{1,\varepsilon}E + \eta_{3,\varepsilon}B(\phi) + \eta_{1,\varepsilon}pw^{p-1}\Psi + \eta_{1,\eta}N(\phi + \Psi) = 0$$

$$\Psi'' - (V(x_0 + \varepsilon(t + h)) - pw^{p-1}(1 - \eta_{1,\varepsilon}))\Psi \tag{2.10}$$

$$+(1 - \eta_{1,\varepsilon})E + (1 - \eta_{1,\varepsilon})N(\eta_{2,\varepsilon}\phi + \Psi) + 2\phi'\eta'_{2,\varepsilon} + \eta''_{2,\varepsilon}\phi = 0 \tag{2.11}$$

We solve first (2.11). We look first the problem

$$\Psi'' - W(x)\Psi + g = 0$$

where $0 < \alpha \leq W(x) \leq \beta$, W continuous and $g \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We claim that (2.3.1) has a unique solution $\phi \in L^\infty(\mathbb{R})$. Assume first that g has compact support and consider the well defined functional in $H^1(\mathbb{R})$

$$J(\Psi) = \frac{1}{2}\int_{\mathbb{R}} |\Psi'|^2 + \frac{1}{2}\int_{\mathbb{R}} w\Psi^2 - \int_{\mathbb{R}} g\Psi.$$

Also, this functional is convex and coercive. This implies that J has a minimizer, unique solution of (2.3.1) in $H^1(\mathbb{R})$ and it is bounded. Now we consider the problem

$$\Psi''_R - W\Psi_R + g\eta\left(\frac{|t|}{R}\right) = 0$$

Let us see that Ψ_R has a uniform bound. Take $\varphi(t) = \frac{\|g\|_\infty}{\alpha} + \rho \cosh\left(\frac{\sqrt{\alpha}}{2}|t|\right)$ for $\rho > 0$ very small. Since $\Psi_R \in L^\infty(\mathbb{R})$ we have

$$\Psi_R \le \varphi(t), \quad \text{for } |t| > t_{\rho,R}.$$

Let us observe that in $[-t_{\rho,R}, t_{\rho,R}]$

$$\varphi'' - W\varphi + g\eta\left(\frac{|t|}{R}\right) < 0.$$

From (2.3.1), we see that $\gamma = (\Psi_R - \varphi)$ satisfies

$$\gamma'' - W\gamma > 0. \tag{2.12}$$

Claim: $\gamma \le 0$ on $\mathbb{R}$. It's for $|t| > t_{\rho,R}$ if $\gamma(\bar{t}) > 0$ there is a global maximum positive $\gamma \in [-t_{\rho,R}, t_{\rho,R}]$. This implies that $\gamma''(t) \le 0$ which is a contradiction with (2.12). This implies that $\Psi_R(t) \le \frac{\|g\|_\infty}{\alpha} + \rho\cosh\left(\frac{\sqrt{\alpha}}{2}t\right)$. Taking the limit ρ going to 0 we get $\Psi_R \le \frac{\|g\|_\infty}{\alpha}$, and similarly we can conclude that

$$\|\Psi_R\|_{L^\infty} \le \frac{\|g\|_\infty}{\alpha}, \quad \forall R$$

Passing to a subsequence we get a solution $\Psi = \lim_{R\to\infty} \Psi_R$, and the convergence is uniform over compacts sets, to (2.3.1) with

$$\|\Psi\|_\infty \le \frac{\|g\|_\infty}{\alpha}$$

. Also, the same argument shows that the solution is unique (in L^∞ sense). Besides: We observe that if $\|e^{\sigma|t|}g\|_\infty < \infty$, $0 < \sigma < \sqrt{\alpha}$ then

$$\|e^{\sigma|t|}\Psi\|_\infty \le C\|e^{\sigma|t|}g\|$$

The proof of this fact is similar to the previous one. Just take as the function φ as follows

$$\varphi = M\frac{\|e^{\sigma|t|}g\|_\infty}{\alpha}e^{-\sigma|t|} + \rho\cosh\left(\frac{\sqrt{\alpha}}{2}|t|\right).$$

Observe now that Ψ satisfies (2.11) if and only if

$$\Psi = \left(-\frac{d^2}{dt^2} + W\right)^{-1}[F[\Psi, \phi]]$$

where $W(x) = V(x_0 + \varepsilon(t+h)) - pw^{p-1}(1-\eta_{1,\varepsilon})$ and $F[\phi] = (1-\eta_{1,\varepsilon})E + (1-\eta_{1,\varepsilon})N(\eta_{2,\varepsilon}\phi + \Psi) + 2\phi'\eta'_{2,\varepsilon} + \eta''_{2,\varepsilon}\phi$. The previous result tell us that the inverse of the operator $\left(-\frac{d^2}{dt^2} + W\right)$ is well define. Assume that $\|\phi\|_{C^1} := \|\phi\|_\infty + \|\phi'\|_\infty \le 1$, for some $\sigma < 1$ and $\|\Psi\|_\infty \le \rho$, where ρ

is a very small positive number. Observe that $\|(1-\eta_{1,\varepsilon})E\|_\infty \leq e^{-c\delta/\varepsilon}$. Furthermore, we have

$$|F(\Psi,\phi)| \leq e^{-c\delta/\varepsilon} + c\varepsilon\|\phi\|_{C^1} + \|\phi\|_\infty^2 + \|\Psi\|_\infty^2$$

This implies that

$$\|M[\Psi]\| \leq C_*[\mu + \|\Psi\|_\infty^2]$$

where $\mu = e^{-c\delta/\varepsilon} + c\varepsilon\|\phi\|_{C^1} + \|\phi\|_\infty^2$. If we assume $\mu < \frac{1}{4C_*2}$, and choosing $\rho = 2C_*\mu$, we have

$$\|M[\Psi]\| < \rho.$$

If we define $X = \{\Psi | \|\Psi\|_\infty < \rho\}$, then M is a contraction mapping in X. We conclude that

$$\|M[\Psi_1] - M[\Psi_2]\| \leq C_*C\|\Psi_1 - \Psi_2\|, \quad \text{where } C_*C < 1.$$

Conclusion: There exists a unique solution of (2.11) for given ϕ (small in C^1-norm) such that

$$\|\Psi(\phi)\|_\infty \leq [e^{-c\delta/\varepsilon} + \varepsilon\|\phi\|_{C^1} + \|\phi\|_\infty^2]$$

Besides: If $\|\phi\| \leq \rho$, independent of ε, we have

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_\infty \leq o(1)\|\phi_1 - \phi_2\|.$$

Next step: Solver for (2.9), with $\|\phi\|$ very small, the problem
(2.13)

$$\phi'' - (1 - pw^{p-1})\phi + \eta_{1,\varepsilon}E + \eta_{3,\varepsilon}B(\phi) + \eta_{1,\varepsilon}pw^{p-1}\Psi + \eta_{1,\eta}N(\phi+\Psi) - cw' = 0$$

where $c = \frac{1}{\int w'^2}\int_{\mathbb{R}}(\eta_{3,\varepsilon}B(\phi) + \eta_{1,\varepsilon}pw^{p-1}\Psi + \eta_{1,\eta}N(\phi+\Psi))w'$. To solve (2.13) we write it as

$$\phi = T[\eta_{3,\varepsilon}B\phi] + T[N(\phi + \Psi(\phi)) + pw^{p-1}\Psi(\phi)] + T[E] =: Q[\phi]$$

Choosing δ sufficiently small independent of ε we conclude that $Q(x) \subseteq X$, and Q is a contraction in X for $\|\cdot\|_{C^1}$. This implies that (2.13) has a unique solution ϕ with $\|\phi\|_{C^1} < M\varepsilon^2$. Also the dependence $\phi = \Phi(h)$ is continuous. Now we only need to adjust h in such a way that $c = 0$. After some calculations we obtain

$$0 = K\varepsilon^2 V''(x_0)h + O(\varepsilon^3) + O(\delta\varepsilon^2).$$

So we can find $h = h_\varepsilon$ and $|h_\varepsilon| \leq C\varepsilon$, such that $c = 0$.